Correspondence Theorem for Rings

This article, taken from http://math.stackexchange.com/questions/813978/correspondence-theorem-for-rings, gives a full proof for the Correspondence theorem for rings.

Proposition 1. *Let A be a multiplicative ring with identity and I an ideal of A. Then there is a one-to-one correspondence between*

the ideals of A that contain I

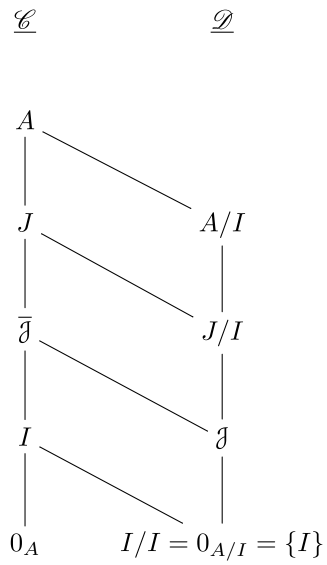
and

the ideals of the factor ring A/I.

Proof. Let $\mathscr{C}$ and $\mathscr{D}$ denote, respectively, the collection of ideals of A containing I, and the collection of ideals of A/I and define

$$f\colon \mathscr{C} \to \mathscr{D} \text{ by } f(J) = J/I = \{a + I \mid a \in J\} \subset A/I$$
$$g\colon \mathscr{D} \to \mathscr{C} \text{ by } g(\mathfrak{J}) = \overline{\mathfrak{J}} = \{a \mid a + I \in \mathfrak{J}\} \subset A.$$

The following diagram epitomises the situation.



If $\mathfrak{J} \in \mathscr{C}$, then we have

$$(f \circ g)(\mathfrak{J}) = \{a + I \mid a \in g(\mathfrak{J})\} = \{a + I \mid a + I \in \mathfrak{J}\} = \mathfrak{J}.$$

If $J \in \mathscr{C}$, then

$$\begin{aligned}(g \circ f)(J) &= \{a \mid a + I \in f(J)\} = \{a \mid a + I = b + I \text{ for some } b \in J\} \\ &= \{a \mid a \in b + I \text{ for some } b \in J\}.\end{aligned}$$

This last set clearly contains J. Now,

$$a \in b + I \implies (a - b) \in I \subset J \Rightarrow a = b + J \Rightarrow a \in J.$$

So $(g \circ f)(J) = J$, and we have shown that f and g establish a bijection between $\mathscr{C}$ and $\mathscr{D}$. $\square$