Some remarks on topology

Proposition 0.1. *Let $f, g\colon X \to Y$ be continuous maps with Y Hausdorff and $A \subset X$ be dense. If $f \mid A = g \mid A$, then $f = g$.*

Proof. Since A is dense in X, $\mathrm{Cl}(A) = X$. Let $E = \{x \in X \mid f(x) = g(x)\}$. Then $A \subset E \subset X$. Note that $X = \mathrm{Cl}(A) \subset \mathrm{Cl}(E) \subset X$, thus $\mathrm{Cl}(E) = X$. Define $\varphi\colon X \to Y \times Y$ by $\varphi(x) = (f(x), g(x))$.

$$
\begin{tikzcd}
X \arrow[r, "\varphi=(f,g)"] & Y \times Y \arrow[d, "\Delta"] \\
E \arrow[u, hook] \arrow[r] & \Delta(Y \times Y) \\
A \arrow[u, hook] \arrow[r] & \varphi(A) \arrow[u, hook]
\end{tikzcd}
$$

Then φ is continuous as f and g are continuous, and $\Delta(Y \times Y)$ is closed in $Y \times Y$ as Y is Hausdorff. So $E = \varphi^{-1}(\Delta(Y \times Y))$, being the inverse image of a continuous map, is closed: $\mathrm{Cl}(E) = E$. Thus $\{x \in X \mid f(x) = g(x)\} = E = \mathrm{Cl}(E) = X$, or $f = g$. $\square$

Proposition 0.2. *Let $\mathbb{R}, \mathbb{Q}$ denote the field of real numbers and field of rational numbers, respectively. Then $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.*

Proof. This is immediate since $\mathbb{Q}$ is not closed in $\mathbb{R}$. $\square$