On $\mathrm{Gal}(\mathbb{R}, \mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$.

The following is an exercise (with no solution) in "Fields and Rings" by Kaplansky.

Theorem 1. *Let* $\mathbb{R}$ *and* $\mathbb{Q}$ *be the real number field and the rational number field, respectively. Then* $\mathrm{Gal}(\mathbb{R}, \mathbb{Q}) = \mathrm{id}$

Proof. Let $f \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$.

First, note that f preserves the order in $\mathbb{R}$, i.e, for all $a, b \in \mathbb{R}$, $a \leq b \Rightarrow f(a) \leq f(b)$. In fact, $f(b) - f(a) = f(b - a) = f((\sqrt{b-a})^2) = f(\sqrt{b-a})^2 \geq 0$.

Next, let $x \in \mathbb{R}$ be any real number. Let B be a subset of $\mathbb{Q}$ with $b \leq x$ for all $b \in B$. Then $b = f(b) \leq f(x)$, and by definition $x = f(\sup_{b \in B} b) = \sup_{b \in b} f(b)$. So $x = \sup_{b \in b} f(b) \leq f(x)$. Let U be a subset of $\mathbb{Q}$ with $u \geq x$ for all $u \in U$. Then by a simlar argument we have $x = \inf_{u \in U} f(u) \geq f(x)$.

Combining the above two results, we have $x \leq f(x) \leq x$ for every $x \in \mathbb{R}$, that is, $f(x) = x$, or $f = \mathrm{id}_{\mathbb{R}}$. Hence $\mathrm{Gal}(\mathbb{R}, \mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(\mathbb{R}) = \mathrm{id}$. □