Every group of order p^2, where p is a prime, is abelian.

The following theorem is famous and many proofs are known, but most of them use contradiction.

Theorem 1. *Every group of order p^2, where p is a prime, is abelian.*

Proof. Let G be a group of order p^2. First, recall that every element $x \in G$ has order either 1, p, or p^2, since $\operatorname{ord} x || G|$.

If $\operatorname{ord} x = p^2$, then $G = \langle x \rangle$ is cyclic, thus abelian and we are done.

So, let us assume that no element of G has order p^2. By the class formula,

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

where $C_G(x_i)$ is the centraliser of the element x_i. As $|G| = p^2$ and $[G : C_G(x_i)]|p$, $|Z(G)| > 1$. Suppose $x \in Z(G)$ is an element other than id_G. Then $\operatorname{ord} x = p$, as no element has order p^2. Now, $N = \langle x \rangle$ is a cyclic group, and $|N| = p$. Pick a $y \in G \setminus N$. Then for $C_G(y)$, $N \subsetneq C_G(y)$ and $y \in C_G(y)$. Since $y \notin N$, $|C_G(y)| > p$, and thus $|C_G(y)| = p^2$, i.e., $C_G(y) = G$. Thus $C_G(G) = G$, i.e., G is abelian. □