On factorisation in geometric rings

Similar-looking geometric rings may differ in factorisation, if the base field is extended. We compare $\mathbb{R}[X,Y]/(X^2+Y^2-1)$ and $\mathbb{C}[X,Y]/(X^2+Y^2-1)$.

1 $\mathbb{R}[X,Y]/(X^2+Y^2-1)$

We see that $\mathbb{R}[X,Y]/(X^2+Y^2-1)$ is *not* a UFD.

We show that x, the residue class of X in $\mathbb{R}[X,Y]/(X^2+Y^2-1)$ is irreducible and does not divide $1+y$ and $1-y$, the residue classes of $1+Y$ and $1-Y$. (Note that in $\mathbb{R}[x,y]$ we have $x^2=(1+y)(1-y)$.)

If $x=z_1z_2$, then, we get $\mathrm{Nm}(x)=\mathrm{Nm}(z_1)\,\mathrm{Nm}(z_2) \Leftrightarrow Y^2-1=\mathrm{Nm}(z_1)\,\mathrm{Nm}(z_2)$. Now we have the following cases:

1. $\deg \mathrm{Nm}(z_1)=0 \Leftrightarrow z_1 \in \mathbb{R}^\times$,
2. $\deg \mathrm{Nm}(z_1)=2 \Leftrightarrow \deg \mathrm{Nm}(z_2)=0 \Leftrightarrow z_2 \in \mathbb{R}^\times$,
3. $\deg \mathrm{Nm}(z_1)=\deg \mathrm{Nm}(z_2)=1$.

If $\mathrm{Nm}(z_1)=Y-1$, then $a_1^2(Y)+b_1^2(Y)(Y^2-1)=Y-1 \Rightarrow Y-1 \mid a_1(Y) \Rightarrow \exists a_2 \in \mathbb{R}[Y]$ such that $a_1=(Y-1)a_2$ and pluggin this in the foregoing equation we get $a_2^2(Y)(Y-1)+b_1^2(Y)(Y+1)=1$. Looking now at the dominant coefficients of a_2 and b_1 we find that one of these is zero or the sum of their square is zero, both lead to a contradiction.

Assume that $x \mid 1-y$. Then $\mathrm{Nm}(x) \mid \mathrm{Nm}(1-y) \Leftrightarrow Y^2-1 \mid (Y-1)^2$, contradiction.

2 $\mathbb{C}[X,Y]/(X^2+Y^2-1)$

$\mathbb{C}[X,Y]/(X^2+Y^2-1)$ is an UFD because it is isomorphic to $\mathbb{C}[U,V]/(UV-1)$ (via the substitutions $U \mapsto X+iY$ and $V \mapsto X-iY$) and the last one is a ring of fractions of $\mathbb{C}[U]$ with respect to the multiplicative system $\{1,U,U^2,\dots\}$.

Remark 1. If R is an integral domain and $\alpha \in R$, then $R[X]/(X^2-\alpha)$ is an integral domain if and only if there are no non-zero elements $a,b \in R$ such that $b^2=\alpha a^2$.

3 $\mathbb{R}[X,Y]/(X^2+Y^2-1)$ revisited

There is another way to think about this problem.

Geometric consideration

Since $R := \mathbb{R}[x,y]/(x^2+y^2-1)$ is a smooth affine curve, it is a normal ring (i.e. integrally closed in its fraction field), and so it is factorial if and only if it has trivial class group. Here and below we will use ideas discussed in Hartshorne, Ch.II.6, in the subsection on Weil divisors.

We may consider $U := \operatorname{Spec} R$ as an affine open curve, and then consider its projective closure X. The curve X is a plane conic, and so its class group (equivalently, its Picard group) is isomorphic to $\mathbb{Z}$, generated by the class of any rational point (e.g. the class of the point $(1,0)$).

Now $Z := X \setminus U$ is irreducible (it is a single point of X, which geometrically becomes two points, namely the two points at infinity $[1\colon \pm i\colon 0]$ — note that neither of these points is individually defined over $\mathbb{R}$, but their union is, and so it corresponds to a single point on X with residue field equal to $\mathbb{C}$); this is where we use that our curve is defined over $\mathbb{R}$ rather than $\mathbb{C}$. (In the latter case Z is *not* irreducible, but is the union of the preceding two points, which are now both defined over $\mathbb{C}$.)

We now use the exact sequence of Hartshorne II.6, Prop. 6.5, namely

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \mapsto 0,$$

where the first arrow is defined by $n \mapsto$ the class of nZ.

Recalling that $\mathrm{Cl}(X) = \mathbb{Z}$, and that Z corresponds to a pair of points over $\mathbb{C}$, this exact sequence can be written more explicitly as

$$\mathbb{Z} \to \mathbb{Z} \to \mathrm{Cl}(U) \to 0,$$

where the first map is multiplication by 2.

Thus $\mathrm{Cl}(R) = \mathrm{Cl}(U) = \mathbb{Z}/2\mathbb{Z}$, and we see that R is not a UFD.

Explicitly, we see that a maximal ideal in R will be principal precisely if its residue field is equal to $\mathbb{C}$ (rather than $\mathbb{R}$). Thus e.g. the maximal ideal $(x, y-1)$, which cuts out the point $(0,1)$ and has residue field $\mathbb{R}$, is not principal.

More geometric consideration

One can think about this more geometrically:

If the maximal ideal cutting out a point P over $\mathbb{R}$ is principal, then it is generated by some real polynomial $f(x,y)$. But then the ideal (f) in R is a product of maximal ideals corresponding to the intersection of the curve $f = 0$ with the curve U. By assumption this is just the single point P, with multiplicity one, and so (now passing from the affine picture to the projective picture) all the other intersections must be with the two points in Z. By Bezout, the total number of intersections of $f = 0$ with X is even, and we are assuming the intersection of $f = 0$ with U consists of the single point P, so in fact the number of intersections with Z must be odd. But this set of intersections (counted with multiplicity) is symmetric under complex conjugation (since f has real coefficients) and so it must be even (because the two points of Z are interchanged by complex conjugation). This contradiction shows that the maximal ideal of P is *not* principal. (This is more or less a rewriting of the proof of Hartshorne's Prop. 6.5 in this particular case.)

Still more geometric consideration

It is also easy to see what happens when we extend scalars from $\mathbb{R}$ to $\mathbb{C}$, i.e. pass from R to S. The set Z now becomes the union of two points, and so for any point P of U (now over $\mathbb{C}$) we can find a generator of the maximal ideal by choosing f to be the equation of a line passing through P and one of the two points in Z. E.g. for $P = (0, 1)$, we can take a generator of the ideal $(x, y - 1)$ to be $(y - 1 \pm ix)$. (Either choice of sign will do; their ratio is a unit in S.)

In terms of the exact sequence of class groups, Z is no longer irreducible, but the union of two points each of degree one, and so the exact sequence becomes

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Cl}(X_{/\mathbb{C}}) \to \mathrm{Cl}(U_{/\mathbb{C}}) \to 0,$$

which more explicitly is

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to \mathrm{Cl}(U_{/\mathbb{C}}) \to 0,$$

with the first map being given just by $(m, n) \mapsto m + n$. Evidently this map is surjective, and so $\mathrm{Cl}(S) = \mathrm{Cl}(U_{/\mathbb{C}}) = 0$.