Chapters 0 - 3 of *Stable Homotopy Theory*
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Introduction

Transcriber's note

This is a memorandum on "Stable Homotopy Theory" by Joel M. Cohen. His book is remarkable in that the writing style (of Chapters 0, 1, and 2 at least) is very systematic, hence comprehensible: indeed, it can be used as an (informal) introduction to axiomatic homotopy theory. Considering its publishing date (1970), that is quite remarkable.

Thus the transcriber tried to "digitise" (a part of) it using LaTeX, besides fixing flaws including incorrect cross-references.

Following is the original "Introduction" by the authour.

The original introduction

These notes are essentially the lecture notes of a course I gave at the University of Chicago in the summer of 1968.[1] Most aspects of stable homotopy are touched on and some are studied in very great detail. It should, however, be emphasised that we are only concerned with finite CW complexes. Thus one never has to worry about the problems which may arise for infinite CW complexes; i.e. certain long exact sequences which are easy to get for finite dimensional CW complexes become very difficult in general unless one takes great care in defining the morphisms (as J. M. Boardman has done in his Warwick lecture notes; or see *Tierney*).

It is assumed that the reader has had a year of algebraic topology (a course which covers the equivalent of most of Spanier, say). I quote without proof some theorems from first year topology (e.g. the Hurewicz theorem) and prove others. In addition I assume the reader has some understanding of spectral sequences and what they can do. Specifically, I assume existence of the Serre spectral sequence in homology. *Spanier* covers quite adequately the necessary material.

For the computations of the stable homotopy groups of spheres in Chapter V, I quote a lot of results on the Steenrod Algebra — all of which can be found

[1]The author was partially supported by the National Science Foundation during the preparation of these notes.

in *Steenrod-Epstein* or *Mosher-Tangora*. Lack of prior knowledge of cohomology operations will not interfere with the understanding of this section, although the reader may have to accept some results on faith (or study the above-mentioned books).

This set of notes has a quite different point of view on the whole from Frank Adams' lecture notes on stable homolopy. I feel that to some degree, these complement the other. Although I do construct the Adams spectral sequence for completeness, not very much is said about it here and the reader is encouraged to pursue the subject either in Adams' notes or in Mosher-Tancrora. The present method of computing the stable homotopy groups of spheres is somewhat simpler than the Adams spectral sequence in the dimensions where it is done. (Higher up this method seems to break down and the Adams method is much neater.)

Chapter IV, on stable homotopy and category theory is entirely the work of Peter Freyd. The proofs are to some extent my own — I tried to make them more topological than category theoretical where possible; but the fact remains that the main results, which are purely topological statements, cannot be proved without using (or directly mimicking) Freyd's embedding of the stable homotopy category into an abelian category.

Thanks are due many people for the ideas incorporated in these notes. My interest in the subject was aroused by George Whitehead; much of my thinking was influenced by him and several proofs are lifted directly from him. Chapter V is an abridged version of my thesis written under Donald Anderson. I express my deep gratitude to him for many helpful suggestions during the original writing and since. In addition many parts of these notes grew out of very useful discussions with Frank Peterson, David Kraines, Gerald Porter, Peter May, Peter Freyd and Brayton Gray. I wish to thank Susan McMahon, Mary Vallery and Cecelia Ricciotti for putting up with my handwriting and typing this manuscript.

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Chapter 0

Preliminaries

To start with, we shall consider based, simply-connected spaces; i.e., every space X comes equipped with a base-point $* \in X$, X is connected and $\pi_1(X,*) = 0$. $f: X \to Y$ will always be a continuous map with $f(*) = *$. $F(X,Y)$ is the set of all such maps. We give $F(X,Y)$ the compact-open topology. Let $\mathcal{D}$ be this category.

A *homotopy* from X to Y is a continuous path H_t in $F(X,Y)$, $0 \le t \le 1$. Given such a path we say $f \sim g$, f is *homotopic to* g for $f = H_0$, $g = H_1$. (Observe that $H_t(*) = *$ for all t.) The set of homotopy classes $[f]$ of maps $f: X \to Y$ is $\pi_0(F(X,Y)) = [X,Y]$. $F(X,Y)$ has a base-point $*$ where $*(x) = *$ for all $x \in X$.

Notation 0.1. Two spaces X and Y are homotopy equivalent, $X \simeq Y$, if and only if there are maps $f: X \to Y$, $g: Y \to X$ with $fg \sim \mathrm{id}_Y$, $gf \sim \mathrm{id}_X$. We shall write $X \cong Y$ if X and Y are homeomorphic. (I.e., $fg = \mathrm{id}_Y$, $gf = \mathrm{id}_X$ for some f and g.)

If A is a subspace of X containing the base-point of X as its own (this is necessary, of course, in order to have the inclusion map base-point preserving) then X/A is X with A identified to the base-point. If it should happen that X has no base-point and $A \subset X$, then X/A still makes sense and now has a base-point.

Given spaces X and Y we form the *wedge* (essentially the one point union) $X \vee Y = X \times * \cup * \times Y \subset X \times Y$, with $* \times *$ as base-point. Then we define the *smash product* or *reduced join*

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

$\vee$ and $\wedge$ are commutative bifunctors $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and $\wedge$ distributes over $\vee$.

Observe that $\times$ is the product and $\vee$ the coproduct for all maps and also for homotopy classes; i.e.,

$$F(X, Y \times Z) \cong F(X,Y) \times F(X,Z), \quad F(X \vee Y, Z) \cong F(X,Z) \times F(Y,Z),$$
$$[X, Y \times Z] = [X,Y] \times [X,Z], \quad [X \vee Y, Z] = [X,Z] \times [Y,Z].$$

Let $\rho: F(X \wedge Y, Z) \to F(X, F(Y,Z))$ be given by $[\rho(f)(x)](y) = f(x \wedge y)$. ρ is continuous. If Y is locally compact, then ρ is, in fact, a homeomorphism. Thus, for Y locally compact, the functors $- \wedge Y$ and $F(Y, -)$ are adjoint (or by commutativity, $Y \wedge -$ and $F(Y, -)$).

We define the 1-sphere $\mathbb{S}^1 = I/\{0,1\}$ where $I =$ the unit interval $[0,1]$ with base-point 0. Let $\mathcal{S} = \mathbb{S}^1 \wedge -$ a functor $\mathcal{D} \to \mathcal{D}$. By the above, it has a right adjoint $\Omega = F(\mathbb{S}^1, -)$. $F(\mathcal{S}X, Z) \tilde{=} F(X, \Omega Z)$ so $[\mathcal{S}X, Z] = [X, \Omega Z]$. We recall that $[\mathcal{S}X, Z]$ has a group structure arising from the "pinch" map $\mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$.

If n is an integer define a function $nMod1: I \to I$ by $nMod1(t) = n \cdot t - \lceil nt \rceil$ ($\lceil x \rceil$ = greatest integer $\le x$). $nMmod1$ is not continuous but its composition with the projection $I \to \mathbb{S}^1$ is. Since $nMmod1(\{0,1\}) = * \in \mathbb{S}^1$ we in fact have $\widetilde{n}: \mathbb{S}^1 \to \mathbb{S}^1$ defined. Considering $\mathbb{S}^1$ as $\{z \in \mathbb{C} || z| = 1\}$, $\widetilde{n}(z) = z^n$.

For each integer $r > 1$ we can define $\mathbb{S}^{r+1} = \mathcal{S}(\mathbb{S}^r)$. In fact since $\mathbb{S}^1 = \mathcal{S}(\{0,1\})$ we see that $\{0,1\}$ is a good choice for $\mathbb{S}^0$ - the zero sphere. $\mathcal{S}^r$ will represent the functor $\mathbb{S}^r \wedge -$. If n is an integer, $\mathcal{S}^{r-1}(n): \mathbb{S}^r \to \mathbb{S}^r$ is defined and for confusion will be written as $\widetilde{n}: \mathbb{S}^r \to \mathbb{S}^r$.

There is one more functor that we wish to consider: Let $\widetilde{I}$ be I with the base-point 1. Then the cone functor $\mathcal{T}$ is $\widetilde{I} \vee -$. We shall embed $X \subset \mathcal{T}X$ by $x \mapsto (0, x)$. Then we see that a space X may be contracted to one point if and only if X is a retract of $\mathcal{T}X$; and a map $f: X \to Y$ is null homotopic if and only if it can be extended to a map $\mathcal{T}X \to Y$.

We recall the following basic result (cf. Spanier "Algebraic topology" for example):

Lemma 0.2. *If $f: \mathbb{S}^r \to \mathbb{S}^r$ then for some integer n, $f \sim \widetilde{n}$ and if $m \neq n$, $\widetilde{m} \not\sim \widetilde{n}$. In other words, $[\mathbb{S}^r, \mathbb{S}^r] \simeq \mathbb{Z}$, the integers, with the identity corresponding to 1. (Here of course $r \ge 1$.)*

Recall the homotopy group functor, $\pi_n = [\mathbb{S}^n,]$. We recall that a non-representable version can be defined on pairs. Now $H_n(\mathbb{S}^n) \simeq \mathbb{Z}$. For each n, choose $\iota_n in H(\mathbb{S}^n)$ a generator, so that i_{n+1} corresponds to ι_n under the natural isomorphism $H_n(\mathbb{S}^n) \simeq H_{n+1}(\mathbb{S}^{n+1})$. This defines the Hurewicz map, a natural transformation, $h_n: \pi_n \to H_n$::

$$[f] = \alpha \in \pi_n(X) \Rightarrow h(\alpha) = f_*(\iota_n) \in H_n(X).$$

For pairs of spaces satisfying a certain property, and $n > 0$, $H_n(X, A) \simeq H_n(X/A, *)$. Then we have defined $h_n(X, A): \pi_n(X, A) \to H_n(X, A)$ as the composite, for $n > 0$,

$$\pi_n(X, A) \to \pi_n(X/A, *) = \pi_n(X/A) \xrightarrow{h_n(X/A)} H_n(X/A) \tilde{\to} H_n(X, A)$$

The main theorem involving $h_n(X, A)$ is

Theorem 0.3 (Hurewicz). *If $\pi_i(X, A) = 0$, $0 \le i \le n$ then $h_n(X, A)$ is an isomorphism.*

It is important to note that we must assume X and A simply connected.

The "certain property" referred to above is this:

Definition 0.4. (X, A) has the *homotopy extension property* (*HEP* for short) if $A \subset X$, and $I \times A \cup O \times X$ is a retract of $I \times X$.

Observe that if (X, A) has the HEP then letting $r: I \times X \to (I \times A) \cup (0 \times X)$ be the retraction we have

$$r_1 = r(1, -): X \to (I \times A) \cup (0 \times X), \quad r_1(A) = 1 \times A.$$

Observe that for $X \cup \mathcal{C}A$ (this always means that $a = (0, a)$) we have

$$X \cup \mathcal{C}A = [(I \times A) \cup (0 \times X)]/(1 \times A).$$

Since $r_1(A) = 1 \times A$, r_1 induces a map $f: X/A \to X \cup \mathcal{C}A$. Defining

$$g: X \cup \mathcal{C}A \to X/A, \qquad g(x) = [x], \quad g(t, a) = *,$$

we observe that g is the homotopy inverse to f.

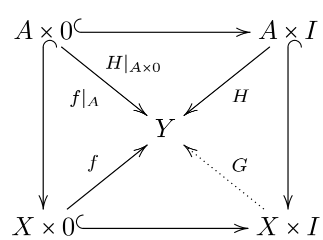
Thus

Theorem 0.5. *If (X, A) satisfies HEP, then $f: X/A \simeq X \cup \mathcal{C}A$. Furthermore*

$$\mathcal{S}A = (X \cup \mathcal{C}A)/X$$

composing f with the projection yields $p: X/A \to \mathcal{S}A$ called the canonical map*. p is unique up to homotopy and is natural once $r: I \times X \to I \times A \cup 0 \times X$ is given.*

Remark 0.6. 1. The HEP for a pair (X, A) is satisfied if and only if the following is true: given $f: X \to Y$ and a homotopy $H: A \times I \to Y$ beginning at $f|_A$, then there exists a homotopy $G: X \times I \to Y$ beginning at f and with $G|_A = H$.



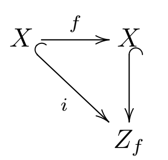
2. The HEP always holds for a pair $(A \cup e^n, A)$ where $\operatorname{int} e^n \cap A = \varnothing$ and $\operatorname{int} e^n \tilde{\equiv} \operatorname{int} I^n$; i.e. e^n is an attached n-cell. See Hu "Homotopy Theory".

Chapter 1

Homotopy and Homology Not-So-Long Exact Sequences

1.1 Basic Properties of Mapping Cones

If $f: X \to Y$ we define the *mapping cylinder* $Z_f = Y \cup (I \times X)/\sim$ where $(0, x) \sim f(x) \in Y$ and $(t, *) \sim *$. $Y \subset Z_f$ is a strong deformation retract, hence a homotopy equivalence. We include X in Z_f, $i: X \hookrightarrow Z_f$ by $i(x) = (x, 1)$. Then the following diagramme homotopy commutes:



Thus in homotopy theory we may take any map to be an inclusion by replacing the codomain by a homotopy equivalent space and the map by a homotopic one.

We now define the *mapping cone* or *cofibre* of f,

$$C_f = Z_f/X = (Y \cup \mathcal{C}X)/\sim, \quad (0, x) \sim f(x).$$

Then we have canonically the inclusion $i_f: Y \to C_f$ and the projection $\sigma_f: C_f \to C_f/Y = \mathcal{S}X$.

We now prove several basic properties.

1) If $f \sim g: X \to Y$ then $C_f \simeq C_g$: let $H: X \times I \to Y$ be such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$. Then define $\varphi: C_f \to C_g$ by $\varphi(y) = y$ and

$$\varphi(t, x) = \begin{cases} H(x, 2t) & 0 \le t \le \frac{1}{2} \\ (2t - 1, x) & \frac{1}{2} \le t \le 1. \end{cases}$$

This makes sense since $\varphi(0,x) = H(x,0) = f(x)$ and $(0,x) = \varphi(\frac{1}{2},x) = H(x,1) = g(x)$.

Similarly we define $\psi: C_g \to C_f$ and it is easy to show that ψ is the homotopy inverse of φ.

2) Let $a: X \to X'$ be a homotopy equivalence with $\widehat{a}$ its homotopy inverse. Let $f: X' \to Y$. Then there is a map $\varphi: C_{fa} \to C_f$ by $\varphi(y) = y$, $\varphi(t,x) = (t, a(x))$. Similarly there is a map $C_{fa\widehat{a}} \to C_{fa}$. But by 1), $C_{f\widehat{a}} \sim C_f$ since $a\widehat{a} \sim \mathrm{id}_{X'}$. Thus there are maps $C_f \leftrightarrows C_{fa}$, and it is not difficult to show that they are a homotopy inverses. Similarly, if $g: Z \to X$, then $C_{ag} \simeq C_g$.

3) By Theorem 0.5 we have: if $f: Y \to X$ is an inclusion and (X, Y) has the HEP, then $C_f \simeq X/Y$.

4) Putting this all together, this says that, up to homotopy type, we may replace a map by an inclusion and the cone by the quotient space in order to study the mapping cone sequence. For example

5) If $f: X \to Y$ and $g: Y \to Z$ are maps, then $\varphi: C_f \to C_{gf}$ is defined by $\varphi(y) = g(h)$, $\varphi(t,x) = (t,x)$. Then $C_\varphi \simeq C_g$: assume f and g are inclusions having the HEP. Then φ is also and

$$C_g \simeq Z/Y = (Z/X)/(Y/X) \simeq C_{gf}/C_f \simeq C_\varphi.$$

6) If $f: X \to Y$ and Z is any space, then

$$[C_f, Z] \xrightarrow{i_f^*} [Y, Z] \xrightarrow{f^*} [X, Z]$$

is exact: if $H: X \times I \to C_f$ by $H(t,x) = (t,x)$ then $H(x,0) = (i_f \circ f)(x)$ and $H(x,1) = *$. Thus $i_f \circ f \simeq *$. So $f^* \circ i_f^* = 0$. Conversely, if $g: Y \to Z$ and $g \circ f \sim *$, let $G: X \times I \to Z$ be such that $G(x,1) = *$ and $G(x,0) = g \circ f$. Then define $\widetilde{g}: C_f \to Z$ by $\widetilde{g}(y) = g(y), \widetilde{g}(t,x) = G(x,t)$. This is well-defined and $\widetilde{g} \circ i_f = g$.

7) (C_f, Y) has the HEP so that $\theta: C_{i_f} \simeq \mathcal{S}X$. The following then are homotopy commutative diagrams:

$$\begin{array}{ccc} C_f & \hookrightarrow & C_{i_f} \\ & \searrow^{\sigma_f} & \downarrow \theta \\ & & \mathcal{S}X \end{array} \qquad \begin{array}{ccc} C_{i_f} & \xrightarrow{\sigma_{i_f}} & \mathcal{S}Y \\ \downarrow \theta & \nearrow_{\pm \mathcal{S}f} & \\ \mathcal{S}X & & \end{array}$$

(the sign depends on the actual choice of θ, but usually will come out $-$) .

8) If $f: X \to Y$ and Z is some space, then form $f \wedge \mathrm{id}_Z: X \wedge Z \to Y \wedge Z$. Then there is a natural map $C_{f \wedge \mathrm{id}_Z} \to C_f \wedge Z$ which is a bijection: take f to be an inclusion; then we get the map $Y \wedge Z / X \wedge Z \to (Y/X) \wedge Z$. For Y and X compact, it is a homeomorphism. In general, it induces an isomorphism of homotopy groups.

By a *cofibration* or a *mapping cone sequence*, we mean a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

such that there is a homotopy equivalence $a\colon Z \to C_f$ and $a \circ g \sim i_f\colon Y \to C_f$. Thus if (X, A) has the HEP, $A \to X \to X/A$ is a *mapping cone sequence*.

We get the Barratt-Puppe sequence from the above constructions:

$$X \xrightarrow{f} Y \xrightarrow{i_f} C_f \xrightarrow{\sigma_f} \mathcal{S}X \xrightarrow{\mathcal{S}f} \mathcal{S}Y \xrightarrow{\mathcal{S}i_f} \mathcal{S}C_f \to \cdots \qquad (\alpha)$$

which has the property that every sequence of two maps (and three spaces) is a mapping cone sequence. Also observe that if W is any space, then $[(\alpha), W]$ is a long exact sequence.

1.2 Basic Properties of Fibres

There is an adjoint construction to that of mapping cone. Set $PY = \{\omega\colon I \to Y | \omega(0) = *\}$. If $f\colon X \to Y$ let $E_f = \{(x,\omega) \in X \times PY | \omega(1) = f(x)\}$. E_f is called the *fibre* of f. We define $j_f\colon E_f \to X$ by $jf(x,\omega) = x$ and $\varphi_f\colon \Omega Y \to E_f$ by $\varphi_f(\omega) = (*,\omega)$. (Recall that $\Omega = F(\mathbb{S}^1, -)$ and that it is left adjoint to $\mathcal{S} = \mathbb{S}^1 \wedge -$, that is, $[\mathcal{S}X, Z] = [X, \Omega Z]$ since $F(\mathcal{S}X, Z) \simeq F(X, \Omega Z)$.)

We have the dual properties to cofibres:

1') If $f \sim g$ then $E_f \simeq E_g$.

2') If $a\colon X' \to X$ and $b\colon Y \to Y'$ are homotopy equivalences and $f\colon X \to Y$, then $E_{b\circ f\circ a} \simeq E_f$.

3') (Intentionally left blank.)

4') (Intentionally left blank.)

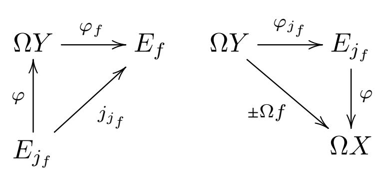
5') Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a map $\varphi\colon E_{gf} \to E_g$ given by $\varphi(x,\omega)) = (f(x),\omega)$ with fibre $E_\varphi \simeq E_f$.

6') If $f\colon X \to Y$ and Z is any space, then

$$[Z, E_f] \xrightarrow{j_{f*}} [Z, X] \xrightarrow{f_*} [Z, Y]$$

is exact.

7') There is a homotopy equivalence $\varphi\colon E_{j_f} \simeq \Omega Y$ with the following homotopy commutative:



8') If $f: X \to Y$ and Z is some space, then form $f \times \mathrm{id}_Z: X \times Z \to Y \times Z$. Then since $P(Y \times Z) = PY \times PZ$,

$$\begin{aligned} &E_{f\times \mathrm{id}_Z} \\ &= \{((x,z),(\omega_1,\omega_2)) \in (X \times Z) \times (PY \times PZ) | \omega_1(1) = f(x), \omega_2(1) = Z\} \\ &= \{(x,\omega) \in X \times PY | \omega(1) = f(x)\} \times \{(z,\omega) \in Z \times PZ | \omega(1) = z\} \times Z \\ &= E_f \times PZ \times Z. \end{aligned}$$

Since PZ is contractible, $E_{f\times \mathrm{id}_Z} \simeq E_f \times Z$.

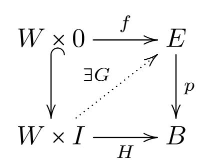
Finally, there is a Barratt-Puppe sequence

$$\cdots \to \Omega E_f \xrightarrow{\Omega j_f} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\sigma_f} E_f \xrightarrow{j_f} X \xrightarrow{f} Y \qquad (\beta)$$

such that if W is any space, then $[W, (\beta)]$ is a long exact sequence . Under certain circumstances, we shall find that $[(\beta), W]$ and $[W, (\alpha)]$ are exact. We shall investigate this in §3.2.

Remark 1.1. Care should be taken to observe that $[X, Y]$ is a pointed set and not, in general, a group unless $X = \mathcal{S}X'$ or $Y = \Omega Y'$. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of pointed sets is exact if $f(A) = g^{-1}(*)$. In particular, g is a monomorphism if $f(A) = *$; but monomorphism means only that $g^{-1}(*) = *$. It does not mean (unless g is a group homomorphism) that g is 1 - 1. Epimorphism does mean onto, however; and f is an epimorphism if and only if $g(B) = *$.

Definition 1.2. $p: E \to B$ is a *fibre map* if and only if p satisfies the covering homotopy property: if W is any space and



is a commutative diagramme, then there exists $G: W \times I \to E$ making the diagramme commute.

If p is a fibre map and $F = p^{-1}(*)$ then we call $F \xrightarrow{i} E \xrightarrow{p} B$ a *fibration*. We shall also call $F' \xrightarrow{i'} E' \xrightarrow{p'} B'$ a fibration if there exist homotopy equivalences $F \simeq F'$, $E \simeq E'$, $B \simeq B'$ making the total diagramme homotopy commute.

Theorem 1.3. *If $f: X \to Y$ is any map then*

$$E_f \xrightarrow{J_f} X \xrightarrow{f} Y$$

is a fibration.

Remark 1.4. This says that in (β) above any two consecutive maps yield a fibration.

Proof. Let $\widetilde{E}_f = \{(x,\omega) \in X \times Y^I | f(x) = \omega(1)\}$. Let $\widetilde{f}\colon \widetilde{E}_f \to Y$ be given by $\widetilde{f}(x,\omega) = \omega(0)$. We have maps $X \leftrightarrows \widetilde{E}_f$ where

$$x \mapsto (x,\omega_x) \quad x \leftarrow\!\shortmid (x,\omega).$$

with $\omega_x(t) = f(s))$. Clearly $X \to \widetilde{E}_f \to X$ is the identity and $\widetilde{E}_f \to X \to \widetilde{E}_f$ takes $(x,\omega) \mapsto (x,\omega_x)$. Let $H_t(x,\omega) = (X,\omega^t)$ where $\omega^t(1-s) = \omega(1-st)$ so that $\omega^1 = \omega$, $\omega^0 = \omega_x$.

Then $X \simeq \widetilde{E}_f$ since $H_0 = \circlearrowleft$ and $H_1 = \mathrm{id}_{\widetilde{E}_f}$. Also the following diagramme

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \widetilde{E}_f \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \widetilde{f}} \\ & & Y \end{array}$$

homotopy commutes.

Observe that $\widetilde{f}^{-1}(*) = E_f$.

Finally we need to show that f is a fibre map. Let $p\colon \widetilde{E}_f \to X$, $q\colon \widetilde{E}_f \to Y^I$ be the projections. Then given

$$\begin{array}{ccc} W \times 0 & \xrightarrow{h} & \widetilde{E}_f \\ \big\downarrow & & \big\downarrow{\scriptstyle \widetilde{f}} \\ W \times I & \xrightarrow[H]{} & Y \end{array}$$

we define $G\colon W \times I \to \widetilde{E}_f$ by $G(\omega,t) = (ph(\omega), \lambda_{\omega,t}))$ where

$$\lambda_{\omega,t} = \begin{cases} H(\omega, t-2s) & 0 \le s \le t/2 \\ qh(\omega)\left(\frac{2s-t}{2-t}\right) & t/2 \le s \le 1 \end{cases}$$

This is continuous since $H(\omega,0) = \widetilde{f}h(\omega) = qh(\omega)(0)$ so $\lambda_{\omega,t}(t/2)$ is well-defined. Since $\lambda_{\omega,t}(1) = qh(\omega)(1) = fph(\omega)$, $G(\omega,t) \in \widetilde{E}_f$.

Also $G(\omega,0) = qh(\omega) = \widetilde{f}h(\omega)$ and $\widetilde{f}G(\omega,t) = \lambda_{\omega,t}(0) = H(\omega,t)$.

So the diagramme will commute and the theorem is proved. (Except for the possibility of G being discontinuous. We simply remark that it is continuous for spaces we are interested in and we leave the exact conditions to point-set topologists.) □

Next observe that the exactness of the sequence $\pi_n(\beta)$

$$\cdots \to \Omega Y \to E_f \to X \xrightarrow{f} Y \qquad (\beta)$$

yields the following exact sequence

$$\begin{array}{ccccccccccc} \pi_n(\Omega E_f) & \longrightarrow & \pi_n(\Omega X) & \longrightarrow & \pi_n(\Omega X) & \longrightarrow & \pi_n(E_f) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(Y) \\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \sim} & \nearrow & & & & & \\ \pi_{n+1}(E_f) & \longrightarrow & \pi_{n+1}(X) & \longrightarrow & \pi_{n+1}(Y) & & & & & & \end{array}$$

(We use the fact that $\pi_{n+1}(X) = [\mathbb{S}^{n+1}, X] = [\mathcal{S}(\mathbb{S}^n), X] = [\mathbb{S}^n, \Omega X] = \pi_n(\Omega X)$.)

1.3 Some Consequences of the Serre Spectral Sequence

Under certain circumstances, a map which is not a homotopy equivalence looks like one in low dimensions. To make this more precise, observe:

Theorem 1.5. *If $f: A \to B$ then $f_*: H_i(A) \to H_i(B)$ is an isomorphism for $i < n$ and an epimorphism for $i < n$ if and only if the same is true of $f_*: \pi_i(A) \to \pi_i(B)$.*

Proof. Consider the inclusion of A into the mapping cylinder $j: A \to Z_f$. Then

$$H_i(f) \text{ is an} \begin{cases} \text{iso } i < n \\ \text{epi } i \leq n \end{cases} \Leftrightarrow H_i(j) \text{ is same } \Leftrightarrow H_i(Z_f, A) = 0 \text{ for } i \leq n$$

$$\Updownarrow \quad \text{Theorem0.3}$$

$$\pi_i(f) \quad \text{is same} \Leftrightarrow \pi_j \text{ is an} \begin{cases} \text{iso } i < n \\ \text{epi } i \leq n \end{cases} \Leftrightarrow \pi_i(Z_f, A) = 0 \text{ for } i \leq n$$

□

Corollary 1.6. *The cofibre of f is n-connected if and only if the fibre of f is $(n-1)$-connected.*

We say that f is *n-connected* in this case.

Remark 1.7. In all the above A and B are 1-connected. For A, B not 1-connected, there are examples where $C_f \simeq *$ but E_f is not 2-connected: as a group $\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2) \simeq Z[t, t^{-1}]$ (a polynomial algebra on one variable and its inverse). Let $\theta: \mathbb{S}^2 \to \mathbb{S}^1 \vee \mathbb{S}^2$ represent $2t - 1$. Then $H_2(\theta)$ is an isomorphism so $X = C_\theta$ has the same homology as $\mathbb{S}^1$. Then there is a map $f: \mathbb{S}^1 \to X$ which is a homology isomorphism and in fact $C_f/\mathbb{S}^1 \simeq *$, but $\pi_2(f)$ is not an isomorphism and E_f is not 2-connected.

We recall the Serre Spectral Sequence (e.g. as outlined in Spanier):

Theorem 1.8. *(Serre) Let $E \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with B 1-connected. Then there is a spectral sequence $\{E^r, d^r\}$ with*

$$E^2_{s,t} = H_s(B; H_t(F)) \to H_*(E)$$

and a spectral sequence $\{\widetilde{E}^r, \widetilde{d}^r\}$ with

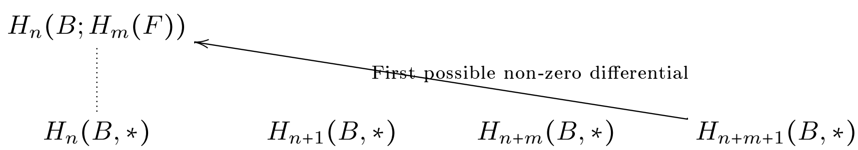
$$\widetilde{E}^2_{s,t} = H_s(B; H_t(F)) \to H_*(E, F)$$

The edge homomorphisms are those induced by p_* and i_*.

Corollary 1.9. *If B is $(n-1)$-connected and F is $(m-1)$-connected then for $\overline{p}: (E, F) \to (B, *)$,*

$$H_i(\overline{p}) \textit{ is an} \begin{cases} \textit{iso for} & i < m+n \\ \textit{epi for} & i \le m+n. \end{cases}$$

Proof. Picture the spectral sequence $\{\widetilde{E}^r, \widetilde{d}^r\}$ of $F \to E \to B$ converging to $H_*(E, F)$.



Thus since the edge homomorphism is $H_*(\overline{p})$ the result follows. □

Notice that we cannot say that $\overline{p}: (E, F) \to (B, *)$ is $(m+n)$-connected. This does not make sense in the relative case: $pi_i(\overline{p})$ is an isomorphism for all i although $H_i(\overline{p})$ is not. Conversely if $A \subset X$ satisfies HEP then $f: (X, A) \to (X/A, *)$ induces a homology isomorphism but not a homotopy isomorphism in all dimensions.

We *can*, however, put it this way:

$$\widetilde{p}: E/F \to B \text{ induces an} \begin{cases} \text{iso for} & i < m+n \\ \text{epi for} & i \le m+n \end{cases}$$

Thus $\widetilde{p}: E/F \to B$ is $(m+n)$-connected.

We now wish to look at the dual problem: If$X \xrightarrow{f} Y \xrightarrow{i} \subset C_f$ is a cofibration then there is an induced map $\rho: X \to E_i$. How close are X and E_i; i.e., how connected is ρ?

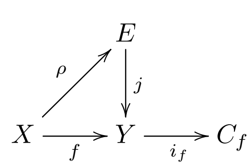
Look at Corollary 1.9 as follows:

For $i \le n+m-1$, replace $H_i(E/F)$ by $H_i(B)$ in the exact homology sequence for $F \to E \to E/F$ yielding

$$H_i(F) \to H_i(E) \to H_i(B) \to H_{i-1}(F) \to \cdots$$

exact for $i \le m+n-1$.

If $f: X \to Y$, let $E = E_{i_f} = \{(y, \omega) \in Y \times PC_f | \omega(1) = y\}$. Define $\rho: X \to E$ by $\rho(x) = (f(x), \omega_x)$ where $\omega_x(t) = (1-t, x)$ so that $\omega_x(0) = *$, $\omega_x(1) = f(x)$. Thus



is commutative. Assume that X is $(n-1)$-connected and C_f is $(m-1)$-connected. From the exact homotopy sequence for $E \to Y \to C_f$ and homology for $X \to Y \to C_f$ and the Hurewicz theorem we find that E is also $(n-1)$-connected. But from the above we have both of the following sequences exact and the diagramme commutative for $k \leq n+m-1$.

$$\begin{array}{ccccccccc}
H_k(Y) & \longrightarrow & H_k(C_f) & \longrightarrow & H_{k-1}(E) & \longrightarrow & H_k(Y) & \longrightarrow & \cdots \\
\| & & \| & & \uparrow{\scriptstyle \rho_*} & & \| & & \\
H_k(Y) & \longrightarrow & H_k(C_f) & \longrightarrow & H_{k-1}(X) & \longrightarrow & H_k(Y) & \longrightarrow & \cdots
\end{array}$$

Thus $H_k(\rho)$ is an isomorphism for $k \leq n+m-2$ (but not necessarily an epimorphism for $k = n+m-1$). Thus

Theorem 1.10. *Given $X \to Y \to C_f$ with X $(n-1)$-connected and C_f $(m-1)$-connected then the induced map $\rho: X \to E_i$ is $(n+m-2)$-connected. Hence there is an exact homotopy sequence*

$$\pi_{n+m-3}(X) \to \pi_{n+m-3}(Y) \to \pi_{n+m-3}(C_f) \to \pi_{n+m-4}(X) \to \cdots$$

1.4 Getting to the Stable Range

We shall now proceed to make great use of Theorem 1.10. This will be the essential tool in getting to a stable situation. The idea is roughly that if a space is n-connected then its properties up to dimension $2n - \varepsilon$ ($\varepsilon = 0, 1$, or 2 usually) are stable; e.g.

suspending gives an isomorphism $H_i \to H_{i+1}$ and $\pi_i \to \pi_{k+1}$;

looping gives an isomorphism $\pi_i \to \pi_{i-1}$, and $H_i \to H_{i-1}$.

Within this range, fibrations and cofibrations "look the same." These ideas will become more precise in this section. From now on, the statement $X \subset Y$ will assume that (Y, X) has the HEP.

Theorem 1.11. *(Blakers-Massey) If $X \subset Y$ (Note this means that (Y, X) has the HEP.) and X is $(n-1)$-connected and Y/X is $(m-1)$-connected then*

$$\varphi: \pi_i(Y, X) \to \pi_i(Y/X) \text{ is an} \begin{cases} \text{iso for } i < m+n-1 \\ \text{epi for } i \leq m+n-1 \end{cases}$$

Proof. We have

$$\begin{array}{ccccc}
 & & E_i & & \\
 & {\scriptstyle \rho}\nearrow & \downarrow{\scriptstyle j} & & \\
X & \longrightarrow & Y & \xrightarrow[i]{} & Y/X
\end{array}$$

where ρ is $(n+m-2)$-connected. The following is an exact diagramme (rows are exact and the diagramme commutes):

$$\begin{array}{ccccccccc}
\pi_i(E_j) & \longrightarrow & \pi_i(Y) & \longrightarrow & \pi_i(Y/X) & \longrightarrow & \pi_{i-1}(E_j) & \longrightarrow & \pi_{i-1}(Y) \\
{\scriptstyle \rho_*}\uparrow & & \| & & {\scriptstyle \varphi}\uparrow & & {\scriptstyle \rho'_*}\uparrow & & \| \\
\pi_i(X) & \longrightarrow & \pi_i(Y) & \longrightarrow & \pi_i(Y,X) & \longrightarrow & \pi_{i-1}(X) & \longrightarrow & \pi_{i-1}Y
\end{array}$$

We apply the 5-lemma:

for $i < n+m-2$ ρ'_* is $\simeq$ and ρ_* is epi so φ is $\simeq$;

for $i < n+m-1$ ρ'_* is epi so φ is epi;

□

Theorem 1.12. *If A is $(n-1)$-connected and B is $(m-1)$-connected and $n \leq m$ then*

$$\pi_i(A \vee B) \simeq \pi_i(A) \oplus \pi_i(B) \oplus \pi_{i+1}(A \wedge B), \quad i \leq m+2n-3.$$

Proof. In the sequence

$$\pi_i(A \vee B) \to \pi_i(A \times B) \to \pi_i(A \times B, A \vee B) \to \cdots$$

we observe that $\pi_i(A \times B) \simeq \pi_i(A) \oplus \pi_i(B)$, and $\pi_i(A \vee B)$ contains this as a direct summand because A and B are retracts of $A \vee B$. Thus we have split exact sequences:

$$0 \to \pi_{i+1}(A \times B, A \vee B) \to \pi_i(A \vee B) \leftrightarrows \pi_i(A) \oplus \pi_i(B) \to 0$$

so

$$\pi_i(A \vee B) \simeq \pi_i(A) \oplus \pi_i(B) \oplus \pi_{i+1}(A \times B, A \vee B).$$

But $A \vee B$ is $(n-1)$-connected and using the Künneth formula we observe that $(A \times B)/(A \vee B) = A \wedge B$ is $(m+n-1)$-connected so

$$\pi_{i+1}(A \times B, A \vee B) \simeq \pi_{i+1}(A \wedge B), \quad i+1 < m+2n-2$$

applying Theorem 1.10. □

Another place we use Theorem 1.10 is in the very important Freudenthal Suspension Theorem.

Theorem 1.13. *(Freudenthal Suspension) If X is $(n-1)$-connected then*

$$\mathcal{S}\colon \pi_i(X) \to \pi_{i+1}(\mathcal{S}X) \text{ is an} \begin{cases} \text{iso for } i < 2n-1 \\ \text{epi or } i \leq 2n-1. \end{cases}$$

Proof. Look at the cofibration $X \xrightarrow{f_*} * \xrightarrow{i} \mathcal{S}X$.

$$E_i = \Omega\mathcal{S}X, \quad \rho: X \to \Omega\mathcal{S}X$$

is the map $\rho(x)(t) = t \wedge x$ the adjoint to the identity $\mathcal{S}X \to \mathcal{S}X$. Thus the composite $\pi_i(X) \xrightarrow{\rho_*} \pi_i(\Omega\mathcal{S}X) \simeq \pi_{i+1}(\mathcal{S}X)$ is the suspension $\mathcal{S}$. But ρ is $(2n-1)$-connected by Theorem 1.10 so

$$\rho_* \text{ is an} \begin{cases} \text{iso for } i < 2n-1 \\ \text{epi for } i \leq 2n-1, \end{cases}$$

hence $\mathcal{S}$ is also. □

We now introduce the type of space which will be most convenient for studying homotopy problems.

Definition 1.14. A *CW complex* is a space X together with a sequence of subspaces X^n such that

1) for some indexing set J_n, where each

$$X^n = X^{n-1} \cup_{\alpha\in J_n} e^n_\alpha$$

where each e^n_α is an n-cell; i.e. there is an onto map $\varphi_\alpha: I^n \to e^n_\alpha$ which, restricted to the interior of I^n is a homeomorphism onto $\mathring{e}^n_\alpha$. The boundary is $\dot{e}^n_\alpha = e^n_\alpha \setminus \mathring{e}^n_\alpha$. Then

$$\mathring{e}^n_\alpha \cap (X^n \cup_{\beta\neq\alpha} e^n_\beta) = \varnothing$$

and each $\dot{e}^n_\alpha$ is contained in a finite union of cells of dimension $< n$. Since $\dot{e}^n_\alpha \simeq \mathbb{S}^{n-1}$, there is defined a family of "characteristic maps" $\mathring{\varphi}_\alpha: \mathbb{S}^{n-1} \to X^{n-1}$ and $C_{\mathring{\varphi}_{\alpha_0}} \tilde{=} X^{n-1} \cup e^n_{\alpha_0}$.

2) X^0 is discrete.

3) $X = \cup_{n=0}^{\infty} X^n$ and $O \subset X$ is open if and only if $O \cap X^n$ is open in X^n for all n.

Remark 1.15. i) 3) defines the "weak topology" on X with respect to the subspaces X^n. Observe that the topology remains the same if the word "open" is replaced by "closed."

ii) If X^{n-1} is connected, then

$$X^n / X^{n-1} = \vee_{J_n} \mathbb{S}^n$$

a wedge (or "bouquet") of n-spheres. Let

$$\varphi = \vee \mathring{\varphi}_\alpha : \vee_{J_n} \mathbb{S}^{n-1} \to X^{n-1}$$

Then $C_\varphi \simeq X^n$. Thus there is a cofibration $\vee\mathbb{S}^{n-1} \to X^{n-1} \to X^n$. We will find this particularly useful.

iii) A *subcomplex* A of a CW complex X, is a CW complex A such that $A^n \subset X^n$ and

$$A^n = A^{n-1} \cup_{\overline{J}_n} e^n_\alpha$$

where $\overline{J}_n \subset J_n$. An extension of a remark in Chapter 0 yields the fact that (X, A) has the HEP.

Definition 1.16. We define the *dimension* of a CW complex X by $\dim X \leq n$ if $X = X^n$.

1.5 CW Spaces

We call a space Y a *CW space* if and only if there is some CW complex $X \simeq Y$. We set $\dim Y = \min_{X \simeq Y} \dim X$. Since all theorems are statements only up to homotopy type any proof need involve only a CW complex X and the statement holds true for the CW space $Y \simeq X$. Observe that a CW complex may have a smaller dimension when considered as a CW space, but that will not matter because hypotheses have the form "dimension $\leq n$." For example, we have the following useful fact about CW spaces.

Lemma 1.17. *If X is a connected CW space of dimension $\leq n$ and Y is an n-connected space, then $[X, Y] = 0$.*

Proof. We proceed by induction on n. If $n = 1$, then

$$X \simeq \vee_\alpha \mathbb{S}^1 \Rightarrow [X, Y] = \vee_\alpha \mathbb{S}^1, Y] = \prod_\alpha [\mathbb{S}^1, Y] = \prod_\alpha \pi_1(Y) = 0.$$

Assume the lemma is true up to $n - l$. Let X and Y be as in the hypothesis. Then there is a map $f: \vee \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ with $C_f \simeq X$, so $X^{n-1} \to X \to \vee \mathbb{S}^n$ is a cofibration so

$$[X^{n-1}, Y] \leftarrow [X, Y] \leftarrow [\vee_\alpha \mathbb{S}^n, Y]$$

is exact since $[X^{n-1}, Y] = 0$ by induction and $\prod \pi_n(Y) = 0$ since Y is n-connected; therefore $[X, Y] = 0$. □

By more geometric means we can extend Lemma 1.17 to the infinite dimensional case.

Lemma 1.18. *Let Y be a space and X a CW complex with n cells only for those n such that $\pi_n(Y) = 0$. Then $[X, Y] = 0$.*

Proof. Let $f: X \to Y$; we wish to construct a homotopy

$$H: X \times I \to Y, \quad H(x, 0) = f(x), \; H(x, 1) = *.$$

X is obtained from $*$ by adjoining various n-cells. Well-order the procedure and construct H inductively. $H|_*$ is trivial. Let X' be a subcomplex of X and assume we are given $H|_{X'}$, let $X'' = X' \cup e^n$. Since (X'', X') has the HEP, $H|_{X'}$ can be extended to

$$\widetilde{H}: X'' \times I \to Y, \quad \widetilde{H}|_{X'} = H|_{X'}, \; \widetilde{H}(x, 0) = f(x).$$

$\widetilde{H}(x, 1) = *$ for $x \in X'$. $\widetilde{H}(x, 1) = *$ for $x \in \dot{e}^n$ so $\widetilde{H}_1: (e^n, \dot{e}^n) \to (Y, *)$ represents an element in $\pi_n(Y) = 0$, so there is a homotopy

$$G: (e^n, \dot{e}^n) \times I \to Y, \quad G_1 = *, \; G_0 = \widetilde{H}_1.$$

Now define $H|_{\overset{\circ}{e}{}^n}$ as follows: write $\overset{\circ}{e}{}^n = \mathbb{S}^{n-1} \wedge [0, 1)$ where $[0, 1)$ has base point 0. For $(x, t) \in \overset{\circ}{e}{}^n$ let

$$H((x, t), s) = \begin{cases} \widetilde{H}((x.t), \frac{1}{2}) & 0 \le s \le t, \; t > 0 \\ G((x, t), \frac{s-t}{1-t}) & t \le s \le 1. \end{cases}$$

This is continuous, and extends $H|_{X'}$ since as $t \to 1$, $G((x, t), s) \to *$ uniformly for all s. Furthermore, $H((x, t), 1) = G((x, t), 1)) = *$. Thus $H|_{X''}$ is defined. Inductively, then we have constructed a homotopy $H: f \sim *$. □

We recall a few facts about relative homotopy groups. An element $\alpha \in \pi_n(X, A)$ is represented by a map $f: (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (X, A)$ where $\mathbb{D}^n$ is some n-cell and $\mathbb{S}^{n-1}$ its boundary. $\alpha = 0$ if and only if $f \sim f' \operatorname{rel} \mathbb{S}^{n-1}$ where $f'(\mathbb{D}^n) \subset A$. Also recall the long exact sequence

$$\cdots \to \pi_{n+1}(X, A) \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \cdots$$

All are elementary facts to be found in Hu [1] or Spanier.

We can use these facts to study some properties of CW complexes. We can think of a CW complex as being built up from the empty set by adding one cell at a time. (Use the axiom of choice to well-order the procedure.) If $\theta: \mathbb{S}^{n-1} \to A$ then C_θ will often be written as $A \cup_\theta e^n$. Observe

Lemma 1.19. *If $X = A \cup_\theta e^n$ and $\pi_n(Y, B) = 0$, then any map $f: (X, A) \to (Y, B)$ is homotopic $\operatorname{rel} A$ to some f' where $f'(X) \subset B$.*

Proof. Let $i: (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (X, A)$ be the obvious map where $i|_{\mathbb{D}^n \smallsetminus \mathbb{S}^{n-1}}$ is a homeomorphism onto $e^n \smallsetminus A$. Then $[f \circ i] \in \pi_n(Y, B)$ represents 0 so there is some $h': (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (X, A)$ homotopic $\operatorname{rel} \mathbb{S}^{n-1}$ to $f \circ i$ and $h'(\mathbb{D}^n) \subset B$. Then since $h\prime|_{\mathbb{S}^{n-1}} = f \circ i|_{\mathbb{S}^{n-1}}$ we can define

$$f': (X, A) \to (Y, B), \quad f'|_A = f|_A, \; f'|_{e^n \smallsetminus A} = h' \circ i^{-1}, \quad f' \sim f \operatorname{rel} A.$$

□

Lemma 1.20. *Let X be a CW complex of dimension $N \le \infty$ such that $X^n = *$. Assume $\pi_i(Y, B) = 0$ for $n - 1 < i < N$. Then if $j: B \to Y$ is the inclusion then $j_*: [X, B] \to [X, Y]$ is injective. If $\pi_i(Y, B) = 0$ for $n < i < N + 1$ then j_* is surjective. In particular if $\pi_i(Y, B) = 0$ for $n \le i < N + 1$, then j_* is an isomorphism.*

Proof. Let E be the fibre of the inclusion $j: B \to Y$. Then the sequence $[X, E] \to [X, B] \xrightarrow{j_*} [X, Y]$ is exact. Comparing the long exact homotopy sequences, we observe that $\pi_i(E) = \pi_{i+1}(Y, B)$. Thus if $\pi_i(Y, B) = 0$ for $n - 1 < i < N$ then $\pi_i(E) = 0$ for $n < i < N + 1$. Since X has i-cells for $n < i < N + 1$, Lemma 1.18 yields the fact that $[X, E] = 0$. Thus j_* is injective.

Let $f: X \to Y$ and let $X'' = X' \cup e^i$ be one stage in the construction of X.

Assume that $f|_{X'} \sim f'|_{X'}$ where $f'(X') \subset B$. By the HEP for (X'', X') (see Chapter 0) there is some $\widetilde{f}'': X'' \to Y$ with $\widetilde{f}'' \sim f$ and $\widetilde{f}''|_{X''} = f'$. Then applying Lemma 1.19 to $\widetilde{f}''$ we get $f'': X'' \to Y$ such that $f'' \sim f' \operatorname{rel} X'$ and $f''(X'') \subset B$. Then cell by cell, we construct $\overline{f}: X \to Y$ with $\overline{f}(X) \subset B$. Since at each stage the homotopy remained fixed there is a homotopy $\overline{f} \sim f$ defined on all of X. Now let $g: X \to B$ be defined by $g(x) = \overline{f}(x)$. Then $j \circ g = \overline{f} \sim f$. Thus j_* is surjective. □

Finally we can prove a most important result on CW complexes, the Whitehead Theorem. We recall that a map $f: X \to Y$ is called a *weak homotopy equivalence* if $f_*: \pi_*(X) \to \pi_*(Y)$ is an isomorphism. A homotopy equivalence is clearly a weak homotopy equivalence. J. H. C. Whitehead has proved the converse on CW complexes:

Theorem 1.21. *Let X and Y be CW complexes. Then $f: X \to Y$ is a homotopy equivalence if and only if it is a weak homotopy equivalence. Furthermore if X and Y are 1-connected, then these conditions hold if and only if $f_*: H_*(X) \to H_*(Y)$ is an isomorphism.*

Proof. Let $i: X \subset Z_f$, $j: Y \subset Z_f$, $r: Z_f \to Y$ be the usual maps with Z_f the mapping cylinder. If f is a weak homotopy equivalence, then so is i; thus $\pi_*(Z_f, X) = 0$ so by Lemma 1.20,

$$[Z_f, X] \xrightarrow{i_*} [Z_f, Z_f]$$

is surjective. Thus there is a map $\varphi: Z_f \to X$ with $i \circ \varphi \sim \mathrm{id}_{Z_f}$. Then $g = \varphi \circ j: Y \to X$ is such that

$$f \circ g = f \circ g \circ j \sim r \circ i \circ \varphi \circ j \sim r \circ j \sim \mathrm{id}_Y .$$

Thus that f is a weak homotopy equivalence implies there is some g with $f \circ g \sim \mathrm{id}_Y$. But then g is also a weak homotopy equivalence so there is some k with $g \circ k \sim \mathrm{id}_Y$. Then

$$k \sim (f \circ g) \circ k = f \circ (g \circ k) \sim f \Rightarrow g \circ f \sim g \circ k \sim \mathrm{id}_X .$$

Thus g is a homotopy inverse to f.

The final statement on homology follows immediately from Theorem 1.5. □

It is obvious that if X and Y are CW spaces and $f: X \to Y$ then C_f is a CW space. The following is not obvious and not easy and we shall not prove it here. It will be useful to keep this in mind as we continue, although we shall not use it.

Theorem 1.22. *(Milnor) If X, Y are CW spaces and $f: X \to Y$ then E_f is a CW space. In particular if $X = *$ then $E_F = Y$ is a CW space.*

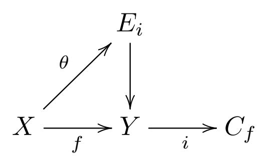
The proof for ΩY can be found in Milnor. The general case is unpublished.

Theorem 1.23. *Let $f: X \to Y$ where X is $(n-1)$-connected and C_f is $(m-1)$-connected. If W is an r-dimensional CW space where $r \leq n+m-2$, then*

$$[W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{i_{f*}} [W, C_f]$$

is exact. If Y is $(\ell-1)$-connected and $r \leq n+\ell-1$ also, there is a long exact sequence continuing to the right.

Proof. We have this diagramme



θ is $(n+m-2)$-connected by Theorem 1.10. Thus from Lemma 1.20, $\theta_*: [W, X] \to [W, E_i]$ is an epimorphism if $r \leq n+m-2$. But

$$[W, E_i] \to [W, Y] \to [W, C_f]$$

is always exact, hence if $r \leq n+m-2$ then $[W, X] \to [W, Y] \to [W, C_f]$ is exact. Applying this to the cofibration $Y \to C_f \to \mathcal{S}X$ completes the proof. □

Theorem 1.24. *Let W be a CW space of dimension r. Let $f: X \to Y$ be n-connected. Then*

$$f_*: [W, X] \to [W.Y] \text{ is a} \begin{cases} \text{monomorphism if} & r < n \\ \text{epimorphism if} & r \leq n. \end{cases}$$

Proof. $[W, E_f] \to [W, X] \xrightarrow{f_*} [W, Y]$ is exact. Since E_F is $(n-1)$-connected, $[W, E] = 0$ if $r < n$, hence f is a monomorphism. Since X is connected and C_f is n-connected $[W, X] \to [W, Y] \to [W, C_f]$ is exact for $r < n$ by Theorem 1.23 and $[W, C_f] = 0$ by Lemma 1.18. Thus f_* is an epimorphism for $r \leq n$. □

As a consequence we get theorems such as the following:

Theorem 1.25. *If A is $(n-1)$-connected and B is $(m-1)$-connected, and X is a CW space of dimension r, then*

$$[X, A \vee B] \to [X, A] \oplus [X, B] \text{ is an} \begin{cases} \text{isomorphism if} & r < m+n-1 \\ \text{epimorphism if} & r \leq m+n-1. \end{cases}$$

Proof. The map $[X, A \vee B] \to [X, A] \oplus [X, B] = [X, A \times B]$ is induced by $i: A \vee B \to A \times B$ which is $(m+n-1)$-connected since $C_i \simeq A \wedge B$. □

We make one concession to point-set topology by proving

Lemma 1.26. *Assume* $Y = \cup_{i=1}^{\infty} Y_i$ *has the weak topology. Assume* $Y_i \subset Y_{i+1}$ *and the* Y_i *are* T_1*-spaces. Then for* X *compact,* $F(X,Y) = \varinjlim F(X,Y_n)$*, the direct limit, and* $[X,Y] = \varinjlim [X,Y_n]$*.*

Proof. Since $\varinjlim F(X,Y_n) \subset F(X,Y)$, for the first part it suffices show that for any $f\colon X \to Y$ there is some n with $f(X) \subset Y_n$, if X is compact. Arguing by contradiction, assume $f(X) \not\subset Y_n$ for any n. Choose $y \in f(X) \smallsetminus Y_n$. Set $A = \{y_n\}_{n=1}^{\infty}$. For any $y \in Y$, $(A \smallsetminus \{y\}) \cap Y_n$ is finite, hence, since Y_n is T_1, closed. Thus $A \smallsetminus \{y\}$ is closed so y is not a limit point of A. So A is an infinite subset of $f(X)$ with no limit points. Thus $f(X)$ is non-compact. So X is non-compact. This contradicts the hypothesis.

We observe if X is compact then so is $X \times I$ so any homotopy takes place in some Y_n and the second part of the lemma follows. □

From this lemma it immediately follows that

$$\pi_n(\cup_{i=1}^{\infty} Y_i) = \varinjlim \pi_n(Y_i)$$

since $\mathbb{S}^n$ is compact.

We can extend Theorem 1.24 by induction to a finite wedge and then by Lemma 1.26 to a countable wedge.

Corollary 1.27. *If* $\{A_i\}_{i=1}^{\infty}$ *is a countable collection of* $(n-1)$*-connected spaces and* X *is a compact CW space of dimension* $\le 2n-2$*, then*

$$[X, \vee_{i=1}^{\infty} A_i] = \sum_i [X, A_i]$$

Proof. $\vee_{i=1}^{\infty} A_i = \cup_{j=1}^{\infty} \vee_{i=1}^{j} A_i$ so for $m - 2n - 2$

$$\pi_m(\vee_{i=1}^{\infty} A_i) = \varinjlim_j \pi_m(\vee_{i=1}^{j} A_i) = \varinjlim_j \sum_{i=1}^{j} \pi_m(A_i) = \sum_{i=1}^{\infty} \pi_m(A_i)$$

Then using the techniques of Corollary 1.27 extend to CW complexes of dimension $\le 2n-2$. □

We observe that the theorem fails for X non-compact. For example if $X = \vee_{n=1}^{\infty} \mathbb{S}^n$ then $[\mathrm{id}_X] \in [X,X]$ but $\mathrm{id}_X \notin \sum_1^{\infty} [X, \mathbb{S}^n]$.

Theorem 1.28. *(Generalised Freudenthal) If* X *is* $(n-1)$*-connected and* $\dim Y \le r$ *then*

$$\mathcal{S}\colon [Y,X] \to [\mathcal{S}Y, \mathcal{S}X] \text{ is an} \begin{cases} \text{iso if} & r < 2n-1: \\ \text{epi if} & r \le 2n-1. \end{cases}$$

Proof. As we observed in the proof of Theorem 1.13, $\rho\colon \mathcal{S}X \to \mathcal{S}\Omega X$ is $(2n-1)$-connected, where ρ is the adjoint of the identity $\mathcal{S}X \to \mathcal{S}X$. Since the composition

$$[Y,X] \xrightarrow{\rho_*} [Y.\Omega\mathcal{S}X] \simeq [SY, SX]$$

is the suspension, applying Theorem 1.24 to ρ finishes the proof. □

Two "stability" theorems we will need later are the following;

Corollary 1.29. *If X is $(n-1)$-connected and $\dim Y \le r$, then the suspension map $\mathcal{S}: F(Y,X) \to F(\mathcal{S}Y, \mathcal{S}X)$ is $(2n-r-1)$-connected.*

Proof. $\mathcal{S}$ induces

$$\begin{array}{ccc} \mathcal{S}_*: \pi_i(F(Y,X)) & \longrightarrow & \pi_i(F(\mathcal{S}Y, \mathcal{S}X)) \\ \| \wr & & \| \wr \\ \mathcal{S}: [\mathcal{S}^i Y, X] & \longrightarrow & [\mathcal{S}^{i+1} Y, \mathcal{S}X] \end{array}$$

and thus is an isomorphism for $i + r < 2n - 1$ and an epimorphism for $i + r \le 2n - 1$. ☐

Notice that $f: X \to Y$ n-connected implies $\mathcal{S}(f)$ is $(n+1)$-connected since $C_{\mathcal{S}(f)} \simeq \mathcal{S}C_f$ and $\Omega(f)$ is $(n-1)$-connected since $E_{\Omega_f} \simeq \Omega E_f$.

Corollary 1.30. *The map $X \to \Omega^r \mathcal{S}^r X$ is $(2n-1)$-connected, if X is $(n-1)$-connected.*

Proof. Taking $Y = \mathbb{S}^0$ in the above lemma yields $X \to \Omega\mathcal{S}X$ $(2n-1)$-connected. Thus $\mathcal{S}X \to \Omega\mathcal{S}^2 X$ is $(2n+1)$-connected so $\Omega\mathcal{S}X \to \Omega^2\mathcal{S}^2 X$ is $2n$-connected. Thus

$$X \to \Omega\mathcal{S}X \to \Omega^2\mathcal{S}^2 X \to \cdots \to \Omega^{r2} X$$

is $(2n-1)$-connected. ☐

Stability, for us, will refer to those cases in which $[X,Y] - \to [\mathcal{S}X, \mathcal{S}Y]$ is an isomorphism. We can now put our previous results together to "stabilise" $[X,Y]$:

Corollary 1.31. *If $\dim Y \le n$ then $[\mathcal{S}^j, \mathcal{S}^j X] \to [\mathcal{S}^{j+1} Y, \mathcal{S}^{j+1} X]$ is an isomorphism for $j \ge n+2$ (regardless of the connectivity of X) .*

Definition 1.32. $\{Y, X\} = \varinjlim [\mathcal{S}^j Y, \mathcal{S}^j X]$ is the set of $\mathcal{S}$-maps from Y to X. Observe that if $\dim Y \le n$, $\{Y,X\} = [\mathcal{S}^j Y, \mathcal{S}^j X]$ for $j \ge n+2$.

One of the most useful aspects of $\mathcal{S}$-maps comes from the following theorem.

Theorem 1.33. *If $X \to Y \to C_f$ is a cofibration, then*

$$\{W, X\} \to \{W, Y\} \to \{W, C_f\}$$

is exact, for any finite dimensional CW space W.

Proof.

$$[\mathcal{S}^n W, \mathcal{S}^n X] \to [\mathcal{S}^n W, \mathcal{S}^n Y] \to [\mathcal{S}^n W, \mathcal{S}^n C_f]$$

is exact if $\dim \mathcal{S}^n W \leq 2n-2$ since$\mathcal{S}^n X$ and $\mathcal{S}^n C_f$ are $(n-1)$-connected. Thus it is exact if $\dim W \leq n-2$. So choose $n \geq \dim W + 2$ and the sequence is exact and yields (by the above)

$$\{W, X\} \to \{W, Y\} \to \{W, C_f\}$$

exact. □

Definition 1.34. $C^{\mathcal{S}}$ is the category whose objects are finite dimensional CW spaces and whose morphisms are $\mathcal{S}$ maps $\{,\}$.

Chapter 2

Eilenberg-Mac Lane spaces and spectra

2.1 Construction of certain spaces

In this section we shall construct certain spaces having the property that their homotopy (Eilenberg - Mac Lane spaces) or homology (Moore spaces) groups in every dimension except one. Based on these spaces we will have a procedure of "dismantling" a given space to study its homology based on its homotopy or vice versa. By taking fibrations or cofibrations with these spaces we shall have means of killing off homotopy or homology groups at one at a time.

Definition 2.1. An *Eilenberg-Mac Lane space of type* (π, n) is a CW space $K(\pi, n)$ such that

$$\pi_i(K(\pi, n)) = \begin{cases} 0 & i \neq n \\ \pi & i = n. \end{cases}$$

An *Moore space of type* (π, n) is a CW space $M(\pi, n)$ such that

$$\widetilde{H}_i(M(\pi, n)) = \begin{cases} 0 & i \neq n \\ \pi & i = n. \end{cases}$$

and $\pi_1(M(\pi, n))$ is abelian.

Clearly if $n \geq 2$, the existence of a $K(\pi, n)$ requires that π be abelian. The existence of an $M(\pi, n)$ always requires π to be abelian. These conditions are almost sufficient.

Theorem 2.2. *a) If π is abelian and $n \geq 2$, then $K(\pi, n)$'s and $M(\pi, n)$'s exist and are unique up to homotopy type;*

b) If π is any group, then $K(\pi, 1)$ exists and is unique, and if π is abelian and $H^2(K(\pi, 1)) = 0$ then $M(\pi, 1)$'s exist (but are not necessarily unique);

c) $H^n(X;\pi)$ and $[X, K(\pi,n)]$ are naturally isomorphic for $n \geq 1$.

Since it will not be relevant to our work, we shall not prove b) but stick to the cases $n \geq 2$. (The case of $K(\pi,1)$ is of historical importance, however, as $H^*(K(\pi,1)) \cong H^*(\pi)$, the cohomology of the group π. Cf. Mac Lane.) For the proof of b) regarding $M(\pi,1)$ see Varadarajan. Its non-uniqueness was shown in Chapter I where we saw two examples of $M(\mathbb{Z},1)$.

Observe that $\vee_\alpha \mathbb{S}^n$ is a choice for $M(\sigma_\alpha \mathbb{Z}, n)$ (for $n \geq 2$). To construct an $M(G,n)$, $n \geq 2$, let

$$0 \to F \xrightarrow{f} H \to G \to 0$$

be a free abelian presentation of G. Let $F = \sum_{\beta \in B} \mathbb{Z}$, $H = \sum_{\alpha \in A} \mathbb{Z}$ and let f have the integral matrix form $((f_{\alpha\beta}))$, $(\alpha,\beta) \in A \times B$ for these bases. Then we have maps $f_{\alpha\beta}: \mathbb{S}^n \to \mathbb{S}^n$ (since the $f_{\alpha\beta}$ are integers). Let

$$\widetilde{f} = ((f_{\alpha\beta})): \vee_\beta \mathbb{S}^n \to \vee_\alpha \mathbb{S}^n.$$

The exact homology sequence for

$$\vee_\beta \mathbb{S}^n \xrightarrow{\widetilde{f}} \vee_\alpha \mathbb{S}^n \to C_f$$

yields the fact that C_f is an $M(G,n)$; it follows from Theorem 1.23 that C_f is 1-connected for $n \geq 2$.

For example, if $G = \mathbb{Z}/q\mathbb{Z}$, then $M(G,n) = S\mathbb{S}^n \cup_q e^{n+1}$, an n-sphere with an $(n+1)$-cell attached at its boundary (which is an n-sphere) by the map q.

As a corollary to the construction, we note that $M(G,n)$ may be taken to be a CW complex of dimension $\leq n+1$.

We can construct $K(\pi,n)$ as follows: let

$$\pi_i(K_n) = \begin{cases} \pi & \text{for } i = n \\ 0 & \text{for } i > n. \end{cases}$$

Inductively assume we have constructed $K_n \subset K_{n+1} \subset \cdots \subset K_m$ where

$$\pi_i(K_m) = \begin{cases} 0, & i < n \\ \pi, & i = n \\ 0, & n < i \leq m \end{cases}$$

and $K_r \subset K_m$ induces a homotopy isomorphism up to degree r for all $r \leq m$. Let $f_\alpha: S^{m+1} \to K_m$ be such that the $\{f_\alpha\}$ generate $\pi_{m+1}(K_m)$. Let

$$f = \vee f_\alpha: \vee \mathbb{S}^{n+1} \to K_m.$$

Then let $K_{m+1} = C_f$. By Theorem 1.23 there is an exact sequence

$$\pi_i(\vee \mathbb{S}^{m+1}) \xrightarrow{f_*} \pi_i(K_m) \xrightarrow{i_*} \pi_i(K_{m+1}) \to \pi_{i-1}(\vee \mathbb{S}^{m+1}) \to \cdots$$

for $i \le n+m-1$. Since $\pi_i(\vee\mathbb{S}^{m+1}) = 0$ for $i \le m$, i_* is an isomorphism there. But $\pi_{m+1}(f_*)$ is onto and $\pi_m(\vee\mathbb{S}^{m+1} = 0$ so $\pi_{m+1}(K_{m+1}) = 0$. Thus we have constructed K_{m+1} inductively. Then with the weak topology $K = \cup_{m=n}^{\infty} K_m$ is a CW complex and using Lemma 1.18

$$\pi_i(K) = \varinjlim_m \pi_i(K_m) = \begin{cases} 0 & i \ne n, \\ \pi & i = n. \end{cases}$$

Thus we have constructed a $K(\pi, n)$.

Next we shall show that for CW spaces X, $[X, K(\pi, n)] \simeq H^n(X; \pi)$. Now

$$H^n(K(\pi,n);\pi) \simeq \mathrm{Hom}(H^n(K(\pi,n)),\pi) \simeq \mathrm{Hom}(\pi,\pi).$$

Choose $\iota \in H^n(K(\pi,n);\pi)$ corresponding to 1_π. Define a natural transformation $T\colon[-, K(\pi,n)] \to H^n(-;\pi)$ by $T(X)[f] = f_*(\iota)$ for any $f\colon X \to K(\pi,n)$. By the choice of ι, $T(\mathbb{S}^n)$ is an isomorphism. By the triviality of both sides $T(\mathbb{S}^m)$ is an isomorphism for $m \ne n$. Since a product of isomorphisms is again an isomorphism, $T(\vee\mathbb{S}^m)$ is for all m.

Let X be a CW space. Assume that $T(\vee\mathbb{S}^i X^r)$ has been shown to be an isomorphism for all i and for some fixed r. For example, we can start off with $r = 0$. Then the Puppe sequence for the i-th suspension of the cofibration $\vee\mathbb{S}^r \to X^r \to X^{r+1}$

$$\vee\mathbb{S}^{r+i} \to \mathcal{S}^i X^r \to \mathcal{S}^i X^{r+1} \to \vee\mathbb{S}^{r+i+1} \to \mathcal{S}^{i+1} X^r$$

yields the exact diagramme

$$\begin{array}{ccccccccc}
[\vee\mathbb{S}^{r+i}, K(\pi,n)] & \longrightarrow & [\mathcal{S}^i X^r, K(\pi,n)] & \longrightarrow & [\mathcal{S}^i X^{r+1}, K(\pi,n)] & \longrightarrow & [\vee\mathbb{S}^{r+i+1}, K(\pi,n)] & \longrightarrow & [\mathcal{S}^{i+1} X^r, K(\pi,n)] \\
\downarrow{\scriptstyle T(\vee\mathbb{S}^{r+i})} & & \downarrow{\scriptstyle T(\mathcal{S}^i X^r)} & & \downarrow{\scriptstyle T(\mathcal{S}^i X^{r+1})} & & \downarrow{\scriptstyle t(\vee\mathbb{S}^{r+i+1}} & & \downarrow{\scriptstyle T(\mathcal{S}^{i+1} X^r)} \\
H^n(\vee\mathbb{S}^{r+i};\pi) & \longrightarrow & H^n(\mathcal{S}^i X^r;\pi) & \longrightarrow & H^n(\mathcal{S}^i X^{r+1};\pi) & \longrightarrow & H^n(\vee\mathbb{S}^{r+i+1};\pi) & \longrightarrow & H^n(\mathcal{S}^{i+1} X^r;\pi)
\end{array}$$

Since the first, second, fourth and fifth are isomorphisms, so is the third, hence $T(X)$ is an isomorphism for finite dimensional CW spaces X .

Observe that the above did not require $K(\pi,n)$ to be a CW space. In particular, given some $K(\pi, n+1)$,

$$\pi_i(\Omega K(\pi,n+1)) \simeq \pi_{i+1}(K(\pi,n+1)) = \begin{cases} \pi, & i = n, \\ 0, & i \ne n \end{cases}$$

so we have $T'\colon[-, K(\pi,n)] \to H^n(\pi)$ defined with $T'(X)$ an isomorphism for finite dimensional CW spaces.

Let X be an arbitrary CW complex. Then X/X^{n+1} is a CW complex with i-cells only for $i \ge n+2$ and $i = 0$. thus from lemma 1.18,

$$[X/X^{n+1}, K(\pi,n+1)] = 0 = [\mathcal{S}(X/X^{n+1}), K(\pi,n+1)]$$

Applying $[-, K(\pi, n+1)]$ to the Barratt-Puppe sequence of $\iota\colon\colon X^{n+1} \hookrightarrow X$ yields the exact sequence

$$\begin{array}{ccccccc}
0 = [X/X^{n+1}, K(\pi, n+1)] & \longleftarrow & [\mathcal{S}X^{n+1}, K(\pi, n+1)] & \longleftarrow & [\mathcal{S}X, K(\pi, n+1)] & \longleftarrow & [\mathcal{S}(X/X^{n+1}), K(\pi, n+1)] = 0 \\
 & & \downarrow \sim & & \downarrow \sim & & \\
 & & [X^{n+1}, \Omega K(\pi, n+1)] & \xleftarrow{i_\#} & [X, \Omega K(\pi, n+1)] & & \\
 & & \downarrow T'(X^{n+1}) & & \downarrow T'(X) & & \\
 & & H^n(X^{n+1}; \pi) & \xleftarrow[i^*]{} & H^n X; \pi) & &
\end{array}$$

By the exactness, $i_\#$ is an isomorphism. By the long exact cohomology sequence $i*$ is an isomorphism. By the previous part $T'(X^{n+1})$ is an isomorphism hence $T'(X){:}\,[X, \Omega K(\pi, n+1)] \to H^n(X; \pi)$ is an isomorphism.

In particular, if $K'(pi, n)$ is an Eilenberg-Mac Lane space of type (π, n) then

$$[K(\pi, n), \Omega K(\pi, n+1)] \simeq H^n(K(\pi, n); \pi) \simeq \mathrm{Hom}(\pi, \pi)$$

choosing $f{:}\, K(\pi, n) \to \Omega K(\pi, n{+}1)$ representing $\mathrm{id}_\pi{:}\, \pi \to \pi$, we see (details are in the more general lemma 2.5) that f is a weak homotopy equivalence. We may as well assume f to be an inclusion. Then from lemma 1.20 we see that for any CW space X, $f_*{:}\,[X, K(\pi, n)] \to [X, K(\pi, n{+}1)]$ is an isomorphism. Thus composing f_* with $T'(X)$ yields $T(X){:}\,[X, K(\pi, n)] \to H^n(X; \pi)$ an isomorphism.

Remark 2.3. We could have skipped some of this by using the fact (theorem 1.22, which wasn't proved) that $K(\pi, n+1)$ is a CW space, hence is a choice for $K\pi, n)$.

Next we observe the following useful property of Moore spaces.

Lemma 2.4. *If $\pi_i(X) = 0$ for $i < n \geq 2$ then there is a map $h{:}\, M(\pi_n(X), n) \to X$ which is a π_n (and hence H^n) isomorphism.*

Proof. Look at the construction of the "standard" $M(\pi, n); \pi = \pi_n(X)$: if

$$0 \to F \xrightarrow{f} G \xrightarrow{p} \pi \to 0$$

is exact where F, G are free abelian, then there is a map

$$\widetilde{f}{:}\, \vee_\alpha \mathbb{S}^n \to \vee_\beta \mathbb{S}^n$$

such that $\pi_n(\widetilde{f})$ represent f. $M(\pi, n)$ is $C_{\widetilde{f}}$. For each β, there is a map $g_\beta{:}\, \mathbb{S}^n \to X$ representing one of the generators of π. Then the map

$$g = \vee g_\beta{:}\, \vee_\beta \mathbb{S}^n \to X$$

is defined and $\pi_n(g)$ represents p. Thus $g\widetilde{f} \sim *$ (since $pf = 0$). Thus g may be lifted to a map $h : M(\pi, n) \to X$ which clearly induces a π_n (and hence H_n) isomorphism. □

In particular it follows from this and from the Whitehead theorem 1.21 that $M(\pi, n)$ are unique for $n \geq 2$. As a dual statement to lemma 2.4 we have

Lemma 2.5. *If X is an $n-1$ connected CW space then there is a map $f{:}\, X \to K(pi_n(X), n)$ which induces a π_n (and hence H_n) isomorphism.*

Proof. Let $\pi = \pi_n(X) \simeq H_n(X)$. Then

$$[X, K(\pi, n)] \cong H^n(X, \pi) \cong \mathrm{Hom}(H_n(X), \pi) \cong \mathrm{Hom}(\pi, \pi)$$

Choose $f: X \to K(\pi, n)$ corresponding to id_π.

Now let us pause a moment and consider what it means to have $\varphi \in H^n(Y, \pi)$ correspond to $g: Y \to K(\pi, n)$. In particular g induces $g_*: H_n(Y) \to H_n(K(\pi, n)) =$ pi so that $g_* \in \mathrm{Hom}(H_n(Y), \pi)$. Then the epimorphism $H^n(Y, \pi) \to \mathrm{Hom}(H_n(Y), \pi)$ sends φ to g_*. Thus in our situation

$$\begin{array}{ccc} H_n(X) & \xrightarrow{f_*} & H_n(K(\pi, n)) \\ \sim \downarrow & & \downarrow \sim \\ \pi & \xrightarrow[\mathrm{id}_\pi]{} & \pi \end{array}$$

is a commutative diagram, hence f_* is an isomorphism.

Now we can prove the uniqueness of the $K(\pi, n)$'s. If X is any Eilenberg-Mac Lane space of type (π, n) and $K(\pi, n)$ is the standard one, then by lemma 2.5 there is a map $f: X \to K(\pi, n)$ which induces a π_n-isomorphism. By the triviality in other dimensions, f is a weak homotopy equivalence hence by the Whitehead theorem a homotopy equivalence. This completes the proof of theorem 2.2. □

2.2 Properties of the Eilenberg-Mac Lane and Moore spaces

We next prove some useful properties of CW spaces. First we observe

Theorem 2.6. *Let X be a 2-connected CW space such that $H^i(X) = 0$ for $i > n$. Then for some integer r, $\mathcal{S}X \sim$ a CW complex of dimension $\leq n + r$. Conversely if $\dim X \leq n$ then $H^i(X) = 0$ for $i > n$.*

Proof. By induction on t where X is $(n - t)$-connected: For $t = 0$ or 1, X must be a wedge of spheres and thus is unique. Assume the theorem proved for $1 \leq t \leq (n - m)$ and let X be $(m - 1)$-connected. Then by Lemma 2.4 there is a map $h: M(\pi_m(X), m) \to X$ which is an H_m-isomorphism. Thus $h^*: H^*(X) \to H^*(M(\pi_m(X), m))$ is an epimorphism (by the naturality of the universal coefficient theorem). Thus $i_h^*: H^*(C_h) \to H^*(X)$ is a monomorphism so $H^i(C_h) = 0$ for $i > n$.

On the other hand C_h is m-connected. By the inductive hypothesis, then, there exists a CW complex C' of dimension $\leq (n + r)$ such that $C' \simeq \mathcal{S}^r C_h$. Now $\mathcal{S}X \simeq C_{\sigma_h}$, $\sigma_h: C_h \to M(\pi_m(X), m + 1)$ so $\mathcal{S}^{r+1} X$ is the cone of a map

$$C' \simeq \mathcal{S}^r C_h \simeq M(\pi_m(X), m + r + 1)$$

and this cone clearly is a CW complex of dimension $\leq \max(r + n + 1, m + r + 2) =$ $r + n + 1$. Thus $\mathcal{S}^{r+1} X \simeq$ a CW complex of dimension $\leq n + r + 1$.

The converse is gotten by an easy induction argument. □

(A much more geometric argument shows that r may be taken equal to 0.)

The following is an immediate consequence of this and Lemma 1.18.

Theorem 2.7. *If $H^i(Y) = 0$ for $i > n$ and $\pi_i(X) = 0$ for $i \leq n$ and Y is a CW space , then for some r $[\mathcal{S}^r Y, \mathcal{S}^r X] = 0$. Thus $[Y, X] = 0$.*

Notice that the hypotheses of the above theorem imply that $H^i(Y; \pi_i(X)) = 0$. We may wonder if that is not the most important point. In fact with much weaker hypotheses we can get a much sharper theorem by approaching the problem from the dual point of view: by fibrations and Eilenberg-Mac Lane spaces:

Theorem 2.8. *Let X and Y be CW spaces with $H^i(Y; \pi_i(X)) = 0$ for all i. If either $\pi_i(X) = 0$ for sufficiently large i or Y is finite dimensional, then $[Y, X] = 0$.*

Proof. We can prove this by induction: let $E_0 = X$. Given E_n an n-connected space with $\pi_i(E_n) \simeq \pi_i(X)$ for $i > n$, let $f: E_n \to K(\pi_{n+1}(X), n+1)$ induce $\pi_{n+1}(f)$, an isomorphism (using Lemma 2.5). Let E_{n+1}, be the fibre of f . Then E_{n+1} is $(n+1)$-connected and $\pi_n(E_{n+1}) \simeq \pi_i(E_n) \simeq (X)$ for $i > n+1$.

Now if $\pi_i(X) = 0$ for $i \geq r$ then since Y is a CW space and $\pi_*(E_r) = 0$ we have $[Y, E_r] = 0$. If on the other hand Y is r-dimensional, then $[Y, E_r] = 0$.

Now we work backwards inductively. From the fibration

$$E_{n+1} \to E_n \to -K(\pi_{n+1}(X), n+1),$$

here is an exact sequence

$$[Y, E_{n+1}] \to [Y, E_n] \to [Y, K(\pi_{n+1}(X), n+1)].$$

But $[Y, K(\pi_{n+1}(X), n+1)] = H^{n+1}(Y; \pi_{n+1}(X)) = 0$ so that $[Y, E_{n+1}] = 0$ implies that $[Y, E_n] = 0$. Thus inductively $[Y, X] = [Y, E_0] = 0$. □

We have already observed that $\Omega K(\pi, n+1) \simeq K(\pi, n)$. This is very useful. We wish to consider more general cases of sequences of spaces A_n with $A_n \simeq \Omega A_{n+1}$. We generalise this to the idea of spectra.

Definition 2.9. A *spectrum* $\underline{X}$ is a sequence of spaces X_n and maps $\epsilon: \mathcal{S}X_n \to X_{n+1}$ or equivalently, $\widetilde{\epsilon}_n: X_n \to \Omega X_{n+1}$.

Example 2.10. (1) $X_n = K(\pi, n)$, $\widetilde{\epsilon}: K(\pi, n) \simeq \Omega K(\pi, n+1)$. This gives $\underline{K}(\pi)$.

(2) W a space, $X_n = \mathcal{S}^n W$, $\varepsilon_n: \mathcal{S}(\mathcal{S}^n W) \simeq \mathcal{S}^{n+1} W$. This gives $\underline{\mathcal{S}}W$.

(3) $\underline{X}$ a spectrum, $(\underline{X} \wedge W)_n = X_n \wedge W$.

(4) $\underline{X}$ a spectrum,

$$(\underline{X}^d)_n = \begin{cases} X_{n+d} & \text{if } n+d \geq 0; \\ * & \text{if } n+d < 0. \end{cases}$$

$\underline{X}^d$ is the d-th suspension of $\underline{X}$ and is defined for *all* integers d.

(5) If W is compact and $\underline{X}$ is a spectrum we can form the spectrum $F(W, \underline{X})$ whose n-th space is $F(W, X_n)$ with the compact open topology. The maps are given by

$$\mathcal{S}F(W, X_n) \xrightarrow{\alpha} F(W, \mathcal{S}X_n) \xrightarrow{F(W,\epsilon_n)} F(W, X_{n+1})$$

where $\alpha(t \wedge f)(w) = t \wedge f(w)$.

From now on we will assume that all spectra are CW spectra i.e., each space is a CW space. We can now define some functors on spectra:

Definition 2.11.

$$\begin{aligned} H_r(\underline{X}) &= \varinjlim_n H_{r+n}(X_n), \\ \pi_r(\underline{X}) &= \varinjlim_n \pi_{r+n}(X_n), \\ H^r(\underline{X}) &= \varprojlim_n H^{r+n}(X_n). \end{aligned}$$

where the direct and inverse limit systems are given by

$$\begin{aligned} H_{r+n}(X_n) \simeq H_{r+n+1}(\mathcal{S}X_n) &\xrightarrow{\varepsilon_{n*}} H_{r+n+1}(X_{n+1}), \\ \pi_{r+n}(X_n) \to \pi_{r+n+1}(\mathcal{S}X_n) &\xrightarrow{\varepsilon_{n*}} \pi_{r+n+1}(X_{n+1}), \\ H^{r+n+1}(X_{n+1}) \xrightarrow{\varepsilon_n^*} H^{r+n+1}(\mathcal{S}X_n) &\simeq H^{r+n}(X_n). \end{aligned}$$

A *map of spectra* $f\colon \underline{X} \to \underline{Y}$ *of degree* r is a sequence of maps $f_n\colon X_n \to Y_n$, for n sufficiently large, such that the following diagramme commutes

$$\begin{array}{ccc} \mathcal{S}X_n & \xrightarrow{\mathcal{S}f_n} & \mathcal{S}Y_{n+r} \\ {\scriptstyle \varepsilon_n}\downarrow & & \downarrow{\scriptstyle \varepsilon_{n+1}} \\ X_{n+1} & \xrightarrow[f_{n+1}]{} & Y_{n+r+1} \end{array}$$

Then we can define the spectrum $\underline{C}_f$, the *cone of* f , with $(\underline{C}_f)_{n+r} = C_{f_n}$ with the map $\mathcal{S}C_{f_n} = C_{\mathcal{S}f_n} \to C_{f_{n+1}}$ induced by the commutativity of the above diagram. (We can similarly define a spectrum $\underline{E}f$, the *fibre of* f .)

The set of homotopy classes of maps of spectra of degree r will be denoted by $[\underline{X}, \underline{Y}]^r$ or $[\underline{X}, \underline{Y}]_{-r}$. We shall let $[W, \underline{Y}] = \varinjlim_n [\mathcal{S}^n W, Y_n] = [\underline{\mathcal{S}}W, \underline{Y}]$.

We shall find certain types of spectra as manageable as spaces.

Definition 2.12. (1) A spectrum $\underline{X}$ is *convergent* if and only if for some N, $\pi_i(\underline{X}) = 0$ for all $i \le N$. $\underline{X}$ will be called N-*connected*.

(2) $\underline{X}$ is *strongly convergent* if and only if for some N each X_n is $n+N$ connected for n sufficiently large (hence $\underline{X}$ is N-connected) and furthermore for all q the map

$$H_{q+k+1}(\mathcal{S}X_k) \to H_{q+k+1}(X_{k+1})$$

is an isomorphism for almost all k. This last statement says that for all q, ε_k is $q+k$ connected for almost all k.

(3) $f\colon \underline{X} \to \underline{Y}$ is a *weak homotopy equivalence* if and only if it is of degree 0 and

$$f_*\colon [W, \underline{X}] \to [W, \underline{Y}]$$

is an isomorphism for every finite CW complex W.

Theorem 2.13. *If $\underline{X}$ and $\underline{Y}$ are strongly convergent spectra then $f\colon \underline{X} \to \underline{Y}$ is a weak homotopy equivalence if and only if $f_*\colon H_*(\underline{X}) \to H_*(\underline{Y})$ is an isomorphism. For any $\underline{X}$ and $\underline{Y}$, $f\colon \underline{X} \to \underline{Y}$ is a weak homotopy equivalence if and only if $f_*\colon \pi_*(\underline{X}) \to \pi_*(\underline{Y})$ is an isomorphism.*

Proof. Let m be an integer. Choose an integer N such that

1) X_k and Y_k are 1-connected for $k \geq N$,

2) for all $j \leq m$ the natural maps

$$H_{j+k}(X_k) \to H_j(\underline{X}), \quad H_{j+k}(Y_k) \to H_j(\underline{Y})$$

are isomorphisms for all $k \geq N$,

3) For every finite CW complex of dimension $\leq m$ the maps

$$[\mathcal{S}^k P, X_k] \to [P, \underline{X}], \quad [\mathcal{S}^k P, Y_k] \to [P, \underline{Y}]$$

are isomorphisms for all $k \geq N$.

The diagrammes

$$\begin{array}{ccc} H_{j+k}(X_k) & \longrightarrow & H_j(\underline{X}) \\ \downarrow f_{k*} & & \downarrow f_* \\ H_{j+k}(Y_k) & \longrightarrow & H_j(\underline{Y}) \end{array} \qquad \begin{array}{ccc} [\mathcal{S}^k P, X_k] & \longrightarrow & [P, \underline{X}] \\ \downarrow f_{k*} & & \downarrow f_* \\ [\mathcal{S}^k P, Y_k] & \longrightarrow & [P, \underline{Y}] \end{array}$$

commute.

If $k \geq N$, $j \leq m$ and $\dim P \leq m$, then the horizontal arrows are isomorphisms.

Now if f is a weak homotopy equivalence then

$$f_{k*}\colon [\mathcal{S}^k(\mathbb{S}^r), X_k] \simeq [\mathcal{S}^k(\mathbb{S}^r), Y_k], \quad r \leq m \quad \Rightarrow \quad f_{k*}\colon \pi_{i+k}(X_k) \simeq \pi_{i+k}(Y_k), \quad i \leq m$$
$$\Rightarrow \quad f_{k*}\colon H_{i+k}(X_k) \simeq H_{i+k}(Y_k), \quad i < m \quad \Rightarrow \quad f_*\colon H_i(\underline{X}) \to H_i(\underline{Y}), \quad i < m.$$

But m was arbitrary so f_* is an isomorphism for all i. We can work backwards to prove the converse once we prove the second statement.

If $f_*\colon \pi_*(\underline{X}) \to \pi_*(\underline{Y})$ is an isomorphism, then $f_*\colon [\mathbb{S}^n, \underline{X}] \to [\mathbb{S}^n, \underline{Y}]$ is an isomorphism for every sphere $\mathbb{S}^x$. The result then follows by induction on the number of cells and by the 5-lemma in the obvious way. □

Two useful types of spectra that we have seen before are the following.

Definition 2.14. A spectrum $\underline{X}$ is an *$\mathcal{S}$-spectrum* if and only if $\mathcal{S}X_n \simeq X_{n+1}$ for almost all n. A spectrum $\underline{X}$ is an *Ω-spectrum* if and only if $X_n \simeq \Omega X_{n+1}$ for almost all n.

Theorem 2.15. *a) $\mathcal{S}$-spectra are strongly convergent.*

b) Convergent Ω-spectra are strongly convergent.

c) Given a spectrum $\underline{X}$ there exists an Ω-spectrum X' weakly homotopy equivalent to $\underline{X}$.

d) $\Leftarrow$ b) + c). If $\underline{X}$ is convergent then there exists a strongly convergent Ω-spectrum X' weakly homotopy equivalent to $\underline{X}$.

Proof. a) is trivial.
b) Assume $\pi_i(X) = 0$ for $i \leq N$. Then assume $X_n \simeq \Omega X_{n+1}$ for all $n \geq M$. Then $\pi_{n+1}(X_n) = 0$ $i \leq N$ for all $n \geq M$. Thus each X_n is $(n+N)$-connected. Then we have $\mathcal{S}X_n \simeq \mathcal{S}\Omega_{n+1} \xrightarrow{\varepsilon_n} X_{n+1}$. But

$$\begin{array}{ccc} \pi_{i+n}(X_n) & \xrightarrow{\mathcal{S}} & \pi_{i+n+1}(\mathcal{S}X_n) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \varepsilon_*} \\ \pi_{i+n}(\Omega X_{n+1}) & \xrightarrow[\sim]{} & \pi_{i+n+1}(X_{n+1}) \end{array}$$

commutes. By Theorem 1.13 $\mathcal{S}$ is an

$$\begin{cases} \text{isomorphism for} & i+n < 2(n+N+1)-1, \\ \text{epimorphism for} & i+n < 2(n+N+1)-1. \end{cases}$$

Thus ε_{n*} is an

$$\begin{cases} \text{isomorphism for} & i < n+2N+1, \\ \text{epimorphism for} & i < n+2n+1. \end{cases}$$

Thus ε_{n*} is $n+(n+2N+1)$-connected. So for all $n \geq q-2N-1$, ε_n is $(n+q)$-connected. Thus $\underline{X}$ is strongly convergent.
c) Let $X'_n = \underrightarrow{\lim}_r \Omega^r X_{n+1}$ where

$$\Omega^r X_{n+r} \xrightarrow{\Omega^r \varepsilon_{n+r}} \Omega^r(\Omega X_{n+r+1}) = \Omega^{r+1} X_{n+r+1}$$

gives the direct system.

Since $\Omega = F(\mathbb{S}^1, -)$ and $\mathbb{S}^1$ is compact, Lemma 1.26 yields

$$\Omega X'_{n+1} = \Omega \varinjlim_r \Omega^r X_{n+r+1} = \varinjlim_r \Omega^{r+1} X_{n+r} = X'_n.$$

Thus X' is an Ω-spectrum.

Define $f{:}\underline{X} \to \underline{X}'$ by $X_n = \Omega^0 X_n \to \varinjlim_r \Omega^r X_{n+r}$. Given n, choose s so that $n+s>0$. Then

$$\pi_n(\underline{X}) = \varinjlim_r \pi_{n+r}(X_r) = \varinjlim_r \pi_{n+s}(\Omega^{r-s}X_r) \simeq \pi_{n+s}(\varinjlim_r \Omega^{r-s}X_r) \simeq \pi_{n+s}(\varinjlim_r \Omega^r X_{r+s}) = \pi_{n+s}(X'_s) = \pi_n(\underline{X}')$$

and this isomorphism is that given by f_*. Using Theorem 2.13, f is a weak homotopy equivalence. □

Corollary 2.16. *$\underline{K}(G)$ and $\underline{\mathcal{S}}$ are strongly convergent.*

At this point we note that not all Ω spectra are convergent. For example, Bott periodicity says that $\Omega^2\,\mathrm{U} \simeq \mathrm{U}$ and $\Omega^8\,\mathrm{O} \simeq \mathrm{O}$ for the unitary and orthogonal groups. Thus there is a spectrum $\underline{X}$ with $X_{2n} = \Omega\,\mathrm{U}$, $X_{2n+1} = \mathrm{U}$ whence $\pi_{2m}(\underline{X}) \simeq \mathbb{Z}$ for every integer m. Similarly we make a spectrum out of O with $\pi_{4m}(\underline{X}) \simeq \mathbb{Z}$ for every integer m.

2.3 Generalised homology theories.

Let $\mathcal{C}^2$ be the category of pairs of spaces (X,A) satisfying HEP (= the homotopy extension property.) Let $\mathcal{C}^*$ be the category of based spaces. In both cases the morphisms are homotopy classes of maps. Write $(X,\varnothing)$ as X. Let $\sigma{:}\mathcal{C}^2 \to \mathcal{C}^2$ be the functor given by $\sigma(X,A) = A$. Let $\mathcal{A}$ be the category of abelian groups.

Definition 2.17. A *generalised homology theory $\mathcal{H}$ on $\mathcal{C}^2$* is a sequence of functors $H_n{:}\mathcal{C}^2 \to \mathcal{A}$ and natural transformations $\partial_n{:}H_n \to H_{n-1}\circ\sigma$ such that

(1) The following is exact for each pair (X,A):

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \to \cdots$$

(2) $H_n(X,A) \xrightarrow{p_*} H_n(X/A,*)$ is an isomorphism where p is the projection.

Note by convention $X/\varnothing = X \coprod *$ with $*$ acting as the base-point.

Definition 2.18. A *reduced generalised homology theory $\widetilde{\mathcal{H}}$ on $\mathcal{C}^*$* is a sequence of functors $\widetilde{H}_n{:}\mathcal{C}^* \to \mathcal{A}$ and natural transformations $\sigma_n{:}H_n \to H_{n+1}\circ\mathcal{S}$ such that

(1) If $A \subset X$ satisfies HEP, then $\widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A)$ is exact.

(2) $\sigma_n(X)$ is an isomorphism for every object X of $\mathcal{C}^*$.

The *coefficients* of a theory are $H_*(*)$ or $\widetilde{H}_*(\mathbb{S}^0)$. Observe that if X is a non-empty space then $x_0 \in X$ is a retract of X hence $H_*(x_0) \simeq H_*(*)$ is a summand of $H_*(X)$, so $H_*(X) \simeq H_*(x_0) \oplus H_*(X,x_0)$.

Claim: For $X \in \mathcal{C}^*$ define $\widetilde{H}_n(X) = H_n(X, *)$. Then this gives us a reduced homology theory: since $\widetilde{H}_n(X/A) = H_n(X/A, *) \simeq (X, A)$ we have the exact sequence $H_n(A) \to H_n(X) \to \widetilde{H}_n(X/A) \to H_{n-1}(A) \to H_{n-1}(X)$, but

$$\begin{array}{ccccc} H_n(A) & \simeq & H_n(*) & \oplus & \widetilde{H}_n(A) \\ \downarrow & & \downarrow \wr & & \downarrow \\ H_n(X) & \simeq & H_n(*) & \oplus & \widetilde{H}_n(X) \end{array}$$

commutes so $\widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A) \to \widetilde{H}_{n-1}(X)$ is exact,

Finally since $\mathcal{T}A/A = \mathcal{S}A$, the exactness of

$$0 = \widetilde{H}_n(TA) \to \widetilde{H}_n(TA) \xrightarrow{\partial_n} \widetilde{H}_{n-1}(A)\widetilde{H}_{n-1}(TA) = 0$$

yields an isomorphism with $\sigma_{n-1}(A) = \partial_n^{-1}$. $TA \sim *$ so $\widetilde{H}_n(TA) = 0$.

On the other hand, given $\widetilde{\mathcal{H}}$ we can define $H_n(X, A) = \widetilde{H}_n(X/A)$; as before $X/\varnothing = X \coprod *$ whence

$$H_n(X) = \widetilde{H}(X) \oplus H_n(*) = \widetilde{H}_n(X) \oplus \widetilde{H}_n(\mathbb{S}^0), \quad \mathbb{S}^0 = * \coprod *'.)$$

We define ∂_n by

$$\begin{array}{ccccc} H_n(X, A) & \overset{\partial_n}{\dashrightarrow} & H_{n-1}(A) & = & \widetilde{H}_{n-1}(A) \oplus \widetilde{H}_{n-1}(\mathbb{S}^0) \\ \| & & & & \uparrow \\ \widetilde{H}_n(X/A) & \xrightarrow[p_*]{} & \widetilde{H}_n(\mathcal{S}A) & \xleftarrow[\sim]{} & \widetilde{H}_{n-1}(A) \end{array}$$

Clearly

$$\begin{array}{ccccc} H_n(A) & \longrightarrow & H(X) & \longrightarrow & H_n(X, A) \\ \| & & \| & & \| \\ H_n(*) \oplus \widetilde{H}_n(A) & \longrightarrow & H_n(*) \oplus \widetilde{H}_n(X) & \longrightarrow & H_n(X/A) \end{array}$$

is exact.

We get the rest by observing that $X/A \to X \coprod TA$ so

$$H_n(X) \to H_n(X \coprod TA) \to H_n((X \coprod)/X)$$

yields

$$H_n(X) \to H_n(X/A) \to H_{n-1}(A) \simeq H_n(\mathcal{S}A)$$

exact hence

$$H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A)$$

is exact.

From now we on we shall deal solely with *reduced* generalised (co)homology theories. Consequently we shall neglect the word "reduced" and eliminate ~ except for ordinary cohomology.

Let A be a spectrum. For any space X let $k_n(X) = [\underline{\mathcal{S}}, X \wedge \underline{A}]_n = \pi_n(X \wedge \underline{A})$ and $k^n(X) = [\underline{\mathcal{S}}X, \underline{A}]^n = [X, \underline{A}]^n$.

Theorem 2.19. *k^* is a cohomology theory $H^*(-;\underline{A})$. If $\underline{A}$ is strongly convergent k_* is a homology theory $H_*(-;\underline{A})$.*

Proof. It is obvious that k^* is a cohomology theory. To show that k_* is a homology theory, we investigate the "stable range" of the problem. Let $\underline{A}$ be $(r-1)$-connected. Given n, choose $N \geq n - 2r + 3$ such that for all $m \geq N$, $\varepsilon_m : \mathcal{S}A_m \to A_{m+1}$, is $(m+n+2)$-connected and A_m is $(m+r-1)$-connected. Then for any space Y, $\pi_{n+m}(Y \wedge A_m) \simeq \pi_{n+m+1}(Y \wedge \mathcal{S}A_m)$ since $(n+m) \leq 2(m+r)-2$. Also $\pi_{n+m+1}(Y \wedge \mathcal{S}A_m) \simeq \pi_{n+m+1}(Y \wedge A_{m+1})$ since ε_m is $(n+m+2)$-connected. Thus $\pi_{n+m}(Y \wedge A) \simeq \pi_n(Y \wedge \underline{A}) = k_n(Y)$, and $\pi_{n+m+1}(Y \wedge A_m) \tilde{=} k_{n+1}(Y)$.

Thus we have

$$k_n(Y) \tilde{=} \pi_{n+m}(Y \wedge A_m) \tilde{=} \pi_{n+m+1}(\mathcal{S}Y \wedge A_m) \tilde{=} k_{n+1}(\mathcal{S}Y).$$

Finally if $X \subset Y$ then the cofibration $X \to Y \to Y/X$ yields the cofibration $X \wedge A_m \to Y \wedge A_m \to (Y/X) \wedge A_m$ so that

$$\pi_{n+m}(X \wedge A_m) \to \pi_{n+m}(Y \wedge A_m) \to \pi_{n+m}((Y/X) \wedge A_m)$$

is exact for $m+n < 2m+2r-2$ hence for $m \geq N$. This, then, yields the exactness of

$$k_n(X) \to k_n(Y) \to k_n(Y/X).$$

□

Observe that for the case of $\underline{A} = \underline{K}(\pi)$ we have

$$H_n(\mathbb{S}^0; \underline{K}(\pi)) = \begin{cases} \pi, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

hence $H_*(-; K(\pi))$ must be regular reduced homology. This is a special case of the following

Theorem 2.20. *Let $T: h_* \to k_*$ be a natural transformation of homology theories (i.e. of the functors and commuting with the isomorphisms σ). Then if $T(\mathbb{S}^0)$ is an isomorphism, so is $T(X)$ for any finite CW complex.*

Proof. The proof is again the usual argument by induction on the cells using the five lemma. □

There is a generalisation of the ideas of theorem 2.20 that we find particularly useful.

Definition 2.21. A *partial homology theory of bidegree* (m, M) is a sequence of functors $H_n: \mathcal{C}^* \to \mathcal{A}$ and natural transformations $\sigma_n: H_n \to H_{n+1} \circ \mathcal{S}$ such that

1) If $A \subset X$ satisfies HEP and A and X are $(m-1)$ connected then $H_n(A) \to H_n(X) \to H_n(X/A)$ is exact for $n < M$

2) $\sigma_n(X)$ is an isomorphism if X is $(m-1)$ connected and $n < M - 1$.

Remark 2.22. a) This is clearly a reduced theory.

b) For each integer $m > 1$, π_* is a homology theory of bidegree $(m, 2m-1)$.

Then we extend theorem 2.20 as follows (in fact, its proof yields a proof of theorem 2.20).

Definition 2.23. A natural transformation $T_*: h_* \to k_*$ of partial homology theories of bidegree (m, M) is called a *weak isomorphism* if $T_i(X)$ is an isomorphism for $i < M$ and an epimorphism for $i = M$ for all $m-1$ connected finite CW complexes.

Theorem 2.24. *Let $T_*: h_* \to k_*$ be a natural transformation of partial homology theories of bidegree (m, M). Assume*

$$T_i(\mathbb{S}^n): h_i(\mathbb{S}^n) \to k_i(\mathbb{S}^n)$$

is an isomorphism for all $n \geq m$ and $i < M$ and an epimorphism for $i = M$. Then T_* is a weak isomorphism.

Proof. The cofibration $X \to X \vee Y \to Y$ yields the fact that $h_i(X \vee Y) \simeq h_i(X) \oplus h_i(Y)$ if X, Y are $(m-1)$-connected and $i < M$. The same is true for k_*. Thus $T_i(\mathbb{S}^n)$ is an isomorphism for finite wedges of n-spheres, $n \geq m$, $i < M$. Given a finite $(m-1)$ connected CW complex X the long exact sequence for $X^m \subset X$ yields the fact that $X^m \simeq \vee \mathbb{S}^m$, a finite wedge. Thus $T_i(X^m)$ is an isomorphism for $i < M$. Assume $T_i(\mathcal{S}^r X^n)$ is an isomorphism for $i < M$ and all $r \geq 0$. Now from the cofibration $\vee \mathbb{S}^m \to X^n \to X^{n+1}$ we get exact sequences connecting the maps

$$T_i(\vee \mathbb{S}^{n+r}) \to T_i(\mathcal{S}^r X^n) \to T_i(\mathcal{S}^r X^{n+1}) \to T_{i-1}(\vee \mathbb{S}^{n+r}) \to T_i(\mathcal{S}^r X^n)$$

Since all but the middle map are isomorphisms for $i < M$ the 5-lemma implies that $T_i(\mathcal{S}^r X^{n+1})$ is an isomorphism for $i < M$ and all $r \geq 0$. For $i = M$ the first two are epimorphisms and the last two are isomorphisms $T_m(\mathcal{S}^r X^n)$ is an epimorphism. Since X is finite, $X = X^n$ for some n, hence $T_i(X)$ is an isomorphism for $i < M$ and $T_m(X)$ is an epimorphism. □

Chapter 3

Spanier-Whitehead duality

3.1 Duality Theorem

The central aim of this chapter is to show that corresponding to a finite CW complex X and a sufficiently large integer N there is a finite CW complex $D_N(X)$, unique up to homotopy type, having many nice properties.

First $D_{N+1}(\mathcal{S}X) \simeq D_N(X) \simeq \mathcal{S}D_{N-1}(X)$ so that we have in a natural way a spectrum $\underline{D}X$ dual to $\underline{S}X$.

Next we find that for each N where it is defined $H^i(D_N(X)) \simeq H_{N-i}(X)$. Finally we find that for two finite CW complexes X, Y, $[\underline{S}X, \underline{S}Y]$ may be identified with $[\underline{D}Y, \underline{D}X]$ in a natural way.

The first step in getting this duality is to recall certain pairings of homology and cohomology.

Observe that for any abelian groups A, B, C there is a natural evaluation map $e: A \otimes \mathrm{Hom}(A \otimes B, C) \to \mathrm{Hom}(B, C)$ given by $[e(a \otimes f)](b) = f(a \otimes b)$.

Given spaces X, Y and an abelian group G we can form the singular chain complexes $\mathcal{C}_*(X)$, $\mathcal{C}_*(Y)$ and the cochain complex $C^*(Y, G) = \mathrm{Hom}(C_*(Y), G)$. Then an evaluation map is defined

$$C_+(X) \otimes \mathrm{Hom}(C_*(X) \otimes C_*(Y), G) \to C^*(Y, G)$$

If this is composed with the map induced by the Eilenberg-Zilber map, which gives a chain equivalence $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ we have defined

$$/: C_*(X) \otimes C^*(X \times Y, G) \to C^*(Y, G).$$

Specifically if $x \in C_q(X)$, $u \in C^n(X \times Y, G)$ and $y \in C_{n-q}(Y)$ then $u/x \in C_{n-q}(Y, G)$ is given by $(u/x)(y) = u(x \times y)$. From the boundary formula on $C_*(X \times Y)$ it is immediate that

$$\delta(u/x) = (\delta u)/x - (-1)^{n-q}(u/\partial x)$$

Thus it is easy to check that we get

$$/: H_*(X) \otimes H^*(X \times Y, G) \to H^*(Y, G)$$

induced.

Then the relative Eilenberg-Zilber (Spanier, p. 234)

$$(C_*(X)\otimes C_*(Y), C_*(X)\otimes C_*(B)+C_*(A)\otimes C_*(Y)) \to (C_*(X\times Y), C_*(X\times B\cup A\times Y)$$

for pairs (X,A), (Y,B) yields the natural transformation

$$/ : H^*(X\times Y, X\times B\cup A\times Y)\otimes H_*(X,A)\to H^*(Y,B).$$

If $f:(X',A')\to(X,A)$ and $g:(Y',B')\to(Y,B)$ then

$$g^*(u/f_*(x)) = [(f\times g)^* u]/x, \quad u\in H*(X\times Y, X\times B\cup A\times Y), x\in H_*(X',A').$$

We shall now recall some facts about fibre bundles before we put the slant product to work.

Definition 3.1. (E,B,F,p) is a *fibre bundle* if and only if $p: E\to B$ is a map such that for every $b\in B$ there exists a neighbourhood U of B and a homeomorphism $\varphi: U\times F\to p^{-1}(U)$ such that $p\varphi(u,f)=u$ for all $u\in U$.

The following can be found in Spanier, p.96.

Lemma 3.2. *If (E,B,F,p) is a fibre bundle and B is paracompact and Hausdorff then p is a fibre map.*

Remark by the transcriber: The proof of Lemma 3.2 is extremely hard.

Lemma 3.3. *Let $p: E\to X$ be a fibration. Let $X'\subset X$ be such that $p' = p|_{X'}: X'\to B$ is a fibre map with fibre F'. Then $i_*:\pi_q(F,F')\xrightarrow{\sim}\pi_q(X,X')$.*

Proof. Let j be the composite isomorphism $\pi_q(X',F')\simeq\pi_q(B)\simeq\pi_q(X,F)$.

We have the triples (X,F,F') (X,X',F') leading to exact sequences (cf. Spanier, p. 378):

$$\begin{array}{ccccccccc}
\xrightarrow{h} & \pi_r(F,F') & \xrightarrow{f} & \pi_r(X,F') & \xrightarrow{g} & \pi_r(X,F) & \xrightarrow{h} & \pi_{r-1}(F,F') \\
 & & & \| & & & & \\
\xrightarrow[h']{} & \pi_r(X',F') & \xrightarrow[f']{} & \pi_r(X,F') & \xrightarrow[g']{} & \pi_r(X,X') & \xrightarrow[h']{} & \pi_{r-1}(X',F')
\end{array}$$

We wish to show that $i_* = g'f$ is an isomorphism. We know that $j = gf'$ is. Thus g is epi and f' is mono. Thus h and h' are trivial. Thus g is epi, f is mono.

i_* is mono.
Assume $i_*(x)=0$. Then $g'(f(x))=0$ so $f(x)=f'(x')$ for some x'. But then $j(x')=gf'(x')=gf(x)=0$ so $x'=0$. Then $f(x)=f'(0)=0$. Since f is mono, $x=0$. Thus i_* is mono.

i_* is epi.
Let $y\in\pi_r(X,X')$. Then $y=g'(z)$ for some $z\in\pi_r(X,F')$. $j=gf'$ is epi so for some $w\in\pi_r(X',F')$ $g(z)=j(w)=gf'(w)$. Thus $g(z-f'(w))=0$ so $f(x)=z-f'(w)$ for some x. But then $i_*(x)=g'f(x)=g'(z)-g'f'(w)=g'(z)=y$. Thus i_* is epi. $\square$

Let $\Delta = \{(x,x) | x \in \mathbb{S}^{n+1}\} \subset \mathbb{S}^{n+1} \times \mathbb{S}^{n+1}$. Let $E = \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta$ and define $p: E \to \mathbb{S}^{n+1}$ by $p(x,y) = x$. Let $F = \mathbb{S}^{n+1} \smallsetminus *$.

Proposition 3.4. *$(E, \mathbb{S}^{n+1}, F, p)$ is a fibre bundle.*

Proof. Consider
$$\mathbb{S}^{n+1} = [\{x \in \mathbb{R}^{n+1}: \|z\| \leq 3\} \times \mathbb{Z}/2\mathbb{Z}]/\sim$$
where $(z,0) \sim (z,1)$ if $\|z\| = 3$. We take x to be the point $(0,0)$ and we shall find a neighbourhood $V \ni x$ such that $p^{-1}(V) \overset{\sim}{=} V \times F$. We suppress the second coordinate and consider $\{z: \|z\| < 3\} \subset \mathbb{S}^{n+1}$ to contain x. Let $V = \{z: \|z\| < 1\}$, $D = \{z: \|z\| \leq 2\}$. If $(x', x'') \in V \times D$, $x' \neq x''$ then there is a unique point $z' \in \mathbb{R}^{n+1}$ such that $\|z'\| = 2$ and x'' belongs to the closed segment from x' to z'. If $x'' = tx' + (1-t)z'$ for $t \in [0,1)$ let $h(x', x'') = (1-t)z' \neq 0$, $h(x', x'') = 0$. Then define
$$\psi: (V \times \mathbb{S}^{n+1}, V \times \mathbb{S}^{n+1} \smallsetminus \Delta) \to V \times (\mathbb{S}^{n+1}, \mathbb{S}^{n+1} \smallsetminus x)$$
by
$$\psi(x', x'') = \begin{cases} (x', x'') & x'' \notin D, \\ (x', h(x', x'')) & x'' \in D. \end{cases}$$
Thus ψ gives a homeomorphism
$$p^{-1}(V) = V \times \mathbb{S}^{n+1} \smallsetminus \Delta \simeq V \times (\mathbb{S}^{n+1} \smallsetminus x) \overset{\sim}{=} V \times F.$$
By any rotation, what we did for x could be done for any point. □

Now using Lemma 3.2 we get

Theorem 3.5. *$p: E \to \mathbb{S}^{n+1}$ is a fibre map.*

Now we have
$$\begin{array}{ccc} \mathbb{S}^{n+1} \smallsetminus * & \subset & \mathbb{S}^{n+1} \\ \downarrow & & \downarrow \\ \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta & \subset & \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \\ \downarrow & & \downarrow \\ \mathbb{S}^{n+1} & = & \mathbb{S}^{n+1} \end{array}$$
where the vertical maps are fibre maps. Thus from Lemma 3.3
$$\pi_i(\mathbb{S}^{n+1}, \mathbb{S}^{n+1} \smallsetminus *) \simeq \pi_i(\mathbb{S}^{n+1} \times \mathbb{S}^{n+1}, \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta)$$
where
$$\pi_i(\mathbb{S}^{n+1}, \mathbb{S}^{n+1} \smallsetminus *) \simeq \pi_i(\mathbb{S}^{n+1}) = \begin{cases} 0, & i < n+1 \\ \mathbb{Z}, & i = n+1. \end{cases}$$
Thus
$$H_i(\mathbb{S}^{n+1} \times \mathbb{S}^{n+1}, \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta) = \begin{cases} 0, & i < n+1 \\ \mathbb{Z}, & i = n+1. \end{cases}$$

Choose $u \in H^{n+1}(\mathbb{S}^{n+1} \times \mathbb{S}^{n+1}, \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta)$ a generator.

Let $A \subset X$ be subcomplexes of $\mathbb{S}^{n+1}$. Choose $X' \subset A' \subset \mathbb{S}^{n+1}$ such that A' (resp. X') is a deformation retract of $\mathbb{S}^{n+1} \smallsetminus A$ (resp. $\mathbb{S}^{n+1} \smallsetminus X$).

Let

$$i\colon (X \times A', X \times X' \cup A \times A') \to (\mathbb{S}^{n+1} \times \mathbb{S}^{n+1}, \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta)$$

be the inclusion. The slant product

$$/\colon H^*(X \times A', X \times X' \cup A \times A') \otimes H_*(X, A) \to H^*(A', X')$$

is defined.

$i^*(u) \in H^{n+1}(X \times A', X \times X' \cup A \times A')$. So there is defined

$$\gamma_u\colon H_q(X, A) \to H^{n+1-q}(A', X'), \quad \gamma_u(x) = i^*(u)/x, \quad q = 0, 1, \ldots, n+1$$

Theorem 3.6. *If $A \subset X$ are subcomplexes of $\mathbb{S}^{n+1}$ for a fixed triangulation then γ_u is an isomorphism.*

The proof is quite involved.

Proof. *Case 1*: $X = *$, $A = \varnothing$. Let $A' = \mathbb{S}^{n+1}$, $X' = \mathbb{S}^{n+1} \smallsetminus *$, then

$$H^{n+q+1}(\mathbb{S}^{n+1}, \mathbb{S}^{n+1} \smallsetminus *) = \begin{cases} 0 & q \neq 0 \\ \mathbb{Z} & q = 0 \end{cases} = H_q(*).$$

We need to know the isomorphism is given by γ_u. But this follows from the fact that the map

$$H^{n+1}(\mathbb{S}^{n+1}, \mathbb{S}^{n+1} \smallsetminus *) \leftarrow H^{n+1}(\mathbb{S}^{n+1} \times \mathbb{S}^{n+1}, \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta)$$

is an isomorphism.

Assume the theorem holds for (X, A) where $\dim(X \smallsetminus A) < k$, $k > 0$.

Case 2: $X = k$-cell, $A = \dot{X}$, the boundary of X. Then $\dot{X} = E_+ \cup E_-$ a union of two hemispheres, $(k-1)$-cells. $E_+ \cap E_- = \dot{E}_+ = \dot{E}_-$ a $(k-2)$-sphere

$$\begin{array}{ccc} H_q(X, \dot{X}) & \xrightarrow{\gamma_u} & H^{n-q+1}(\dot{X}', X') \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \delta} \\ H_{q-1}(\dot{X}, E_-) & \xrightarrow[\gamma_u']{} & H^{n-q+2}(E_-', \ddot{X}') \end{array}$$

commutes up to sign by a chain level formula. By induction yγ_u is an isomorphism. ∂ is an isomorphism since $H_*(X, E_-) = 0$ since both are contractible.

We need the following lemma.

Lemma 3.7. *If A is a k-cell in $\mathbb{S}^n$, $\widetilde{H}_*(\mathbb{S}^n \smallsetminus A) = 0$.*

Proof. By induction on k. If $k = 0$, $A = *$ and $\mathbb{S}^n \smallsetminus * \,\tilde{=}\, \mathbb{R}^n$ is contractible. Assume the result for $k < m$, $m \geq 1$. Regard A as homeomorphic to $B \times I$, B an $(m-1)$-cell. $h{:}\, B \times I \to A$ the homeomorphism. Let $A' = h(B \times [0, \frac{1}{2}])$, $A'' = h(B \times [\frac{1}{2}, 1])$ then $A' \cup A'' = A$, $A' \cap A''$ is an $(m-1)$-cell. Then from the Mayer-Vietoris sequence (Spanier, p. 186) for $(\mathbb{S}^n \smallsetminus A', \mathbb{S}^n \smallsetminus A'')$ and from the inductive assumption that $\widetilde{H}_*(\mathbb{S}^n \smallsetminus (A' \cap A'')) = 0$ we get

$$\widetilde{H}_*(\mathbb{S}^n \smallsetminus A) = \widetilde{H}_*(\mathbb{S}^n \smallsetminus A') \oplus \widetilde{H}_*(\mathbb{S}^n \smallsetminus A'')$$

Thus if $0 \neq z \in \widetilde{H}_*(\mathbb{S}^n \smallsetminus A)$ then $i_*(z) \neq 0$ in $\widetilde{H}_*(\mathbb{S}^n \smallsetminus A_1$ where $A_1 = A'$ or A''. Then iterating this argument, we get a sequence of spaces

$$A \supset A_2 \supset A_2 \supset \cdots$$

and a non-zero element of $\varinjlim_i \widetilde{H}(\mathbb{S}^n \smallsetminus A_i)$. Observe that every compact set of $\mathbb{S}^n \smallsetminus \cap A_i$ is contained in some $\mathbb{S}^n \smallsetminus A_j$, hence (by an argument similar to that of Lemma 1.18, taking specific representative cycles for the homology classes) $H_*(\mathbb{S}^n \smallsetminus \cap A_i) = \varinjlim_i \widetilde{H}_*(\mathbb{S}^n \smallsetminus A_i) \neq 0$. But $\cap A_i$ is an $(m-1)$-cell so $\widetilde{H}_*(\mathbb{S}^n \smallsetminus \cap A_i) = 0$ by induction. Thus we have a contradiction unless $\widetilde{H}_*(\mathbb{S}^n \smallsetminus A) = 0$, and the lemma is proved. □

Applying this to the previous, X' and E'_- are complements of cells in $\mathbb{S}^{n+1}$ hence $\widetilde{H}_*(X') = 0 = \widetilde{H}_*(E'_-)$ so $H_*(E'_-, X') = 0$. Thus δ is an isomorphism, hence γ_u is also and *Case 2* is completed.
Case 3: $\dim X = k$, $A = X^{k-1}$. Let $E_1, \ldots, E_r$ be the k-simplices of $X \smallsetminus A$. We wish to show that $\sum_{i=1}^r H^*(\dot{E}'_i, E'_I) \simeq H^*(A', X')$ induced by the inclusion. If $r = 1$, then $(E_1, \dot{E}_1) \hookrightarrow (X, A)$ is a relative homeomorphism, hence

$$(\mathbb{S}^{n+1} \smallsetminus A, \mathbb{S}^{n+1} \smallsetminus X) \hookrightarrow (\mathbb{S}^{n+1} \smallsetminus \dot{E}_1, \mathbb{S}^{n+1} \smallsetminus E_1)$$

is also, so the isomorphism holds for $r = 1$. The Mayer-Vietoris sequence provides the inductive step proving it for general r.

Now we have the commutative diagramme

$$\begin{array}{ccc} \sum_{i=1}^r H_*(E_i, \dot{E}_i) & \xrightarrow{\varphi} & H_*(X, A) \\ \downarrow{\scriptstyle \gamma'_u} & & \downarrow{\scriptstyle \gamma_u} \\ \sum_{i=1}^r H^*(\dot{E}'_i, E'_i) & \xrightarrow[\theta]{} & H^*(A', X') \end{array}$$

γ'_u is an isomorphism by *Case 2*; φ is well known to be an isomorphism (cf. Hu [2], p. 46); θ is an isomorphism by the above; hence γ_u is an isomorphism.
Case 4: General (X, A), $\dim X = k$. Let $X_p = A \cup X^p$. By induction we will show that $\gamma_u{:}\, H_q(X_p, A) \to H^{n-q+1}(A', X')$ is an isomorphism. For $p = -1$, this is trivially true.

Look at the homology groups of the triad (X_p, X_{p-1}, A).

$$\begin{array}{ccccccccc}
H_{q+1}(X_p, X_{p-1}) & \longrightarrow & H_q(X_{p-1}, A) & \longrightarrow & H_q(X_p, A) & \longrightarrow & H_q(X_p, X_{p-1}) & \longrightarrow & H_{q-1}(X_{p-1}, A) \\
\downarrow \gamma_u^1 & & \downarrow \gamma_u^2 & & \downarrow \gamma_u^3 & & \downarrow \gamma_u^4 & & \downarrow \gamma_u^5 \\
H^{n-q}(X'_{p-1}, X'_p) & \longrightarrow & H^{n-q+1}(A', X'_{p-1}) & \longrightarrow & H^{n-q+1}(A', X'_p) & \longrightarrow & H^{n-q+1}(X'_{p-1}, X'_p) & \longrightarrow & H^{n-q+2}(A', X'_{p-1})
\end{array}$$

Inductively γ_u^2 and γ_u^5 are isomorphisms; γ_u^1 and γ_u^4 are isomorphisms by induction for $p < k$ and by *Case 3* for $p = k$. Thus γ_u^3 is an isomorphism and

$$\gamma_u : H_q(X, A) to H^{n-q+1}(A', X')$$

is an isomorphism setting $p = k$. □

In particular if $A = \varnothing$ we have

$$H_q(X) \xrightarrow{\sim\gamma_u} H^{n-q+1}(\mathbb{S}^{n+1}, X')$$

if $X \neq \varnothing$, $X' \subset \mathbb{S}^{n+1} \setminus * \subset \mathbb{S}^{n+1}$ so the inclusion map is null-homotopic so $\widetilde{H}^*(\mathbb{S}^{n+1}) \to \widetilde{H}^*(X')$ is trivial. Thus

$$H^{n-q+1}(\mathbb{S}^{n+1}, X') \simeq \begin{cases} H^{n-q}(X') & q \neq 0, n+1 \\ \mathbb{Z} \oplus H^n(X') & q = 0 \\ 0 & q = n+1 \end{cases}$$

Thus $\gamma_u : \widetilde{H}_q(X) \simeq \widetilde{H}^{n-q}(X')$.

If $X \subset \mathbb{S}^{n+1}$ and X^* is a proper deformation retract of $\mathbb{S}^{n+1} \setminus X$ then we pick $\alpha \in \mathbb{S}^{n+1} \setminus X \cup X^*$. Then we may consider $X \cup X^* \subset \mathbb{S}^{n+1} \setminus \alpha \approx \mathbb{R}^{n+1}$. Since $X \cap X^* = \varnothing$ we have

$$X \times X^* \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta, \quad \Delta = \{(x, x) | x \in \mathbb{R}^{n+1}\}$$

Then the deformation retraction

$$r : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta \to \mathbb{S}^n = \{(x, 0) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta | \|x\| = 1\}$$

given by $(x, y) \mapsto \frac{x-y}{\|x-y\|}$ composes to give a map $f : X \times X^* \to \mathbb{S}^n$.

If we make the assumption that X and X^* are connected based CW complexes then from the isomorphism $\widetilde{H}_i(X) \simeq \widetilde{H}^{n-i}(X^*)$ and the other way around, we get that $H^i(X^*) = H^i(X) = 0$ for $i \geq n$. Thus $[X^*, \mathbb{S}^n] = [X, \mathbb{S}^n] = 0$ from Theorem 2.8 so the composite $X \vee X^* \to X \times X^* \to \mathbb{S}^n$ is null-homotopic thus a map (unique if $[\mathcal{S}X^*, \mathbb{S}^n] = 0 = [\mathcal{S}X, \mathbb{S}^n]$)

$$u : X \wedge X^* \to \mathbb{S}^n$$

is defined.

Claim: For $\iota_n \in H^n(\mathbb{S}^n)$, $u^*(\iota_n)/ : \widetilde{H}^q(X) \approx \widetilde{H}^{n-q}(X^*)$.

This follows from the commutativity of the following horrible diagramme. (Represent $\mathbb{R}^{n+1}$ as $\mathbb{S}^{n+1} \smallsetminus \alpha$ as before.)

$$
\begin{array}{ccc}
H^{n+1}(\mathbb{S}^{n+1} \times \mathbb{S}^{n+1}, \mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta) & \longrightarrow & H^{n+1}(X \times \mathbb{S}^{n+1}, X \times (\mathbb{S}^{n+1} \smallsetminus X)) \\
\approx \downarrow & \nearrow & \downarrow \approx \\
H^{n+1}(\mathbb{R}^{n+1} \times \mathbb{S}^{n+1}, \mathbb{R}^{n+1} \times \mathbb{S}^{n+1} \smallsetminus \Delta) & & \\
\approx \downarrow & & \\
H^{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \smallsetminus \Delta) & \longrightarrow & H^{n+1}(X \times \mathbb{S}^{n+1}, X \times X^*) \\
\approx \uparrow \delta & & \uparrow \delta \\
H^{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \smallsetminus \Delta) & \longrightarrow & H^n(X \times X^*) \\
\approx \downarrow & \nearrow f^* & \\
H^n(\mathbb{S}^n) & &
\end{array}
$$

All unmarked maps are inclusions.

Definition 3.8. If $i: A \subset B$ then A is an S-(*deformation*) *retract* of B if and only if there exists $j \in \{B, A\}$ such that $j \circ \{i\} = \{\mathrm{id}\}$ (and $\{i\} \circ j = \{\mathrm{id}\}$).

Observe that if ß: $A \hookrightarrow B$ is an S-deformation retract, then $\underline{i}: \underline{\mathcal{S}}A \to \underline{\mathcal{S}}B$ is a homotopy equivalence. Thus $\underline{i}_*: H_*(\underline{\mathcal{S}}A) \to H_*(\underline{\mathcal{S}}B)$ is an isomorphism. But this is the same as $i_*: \widetilde{H}_*(A) \to \widetilde{H}_*(B)$. Thus i induces a homology isomorphism. Thus, in particular, if A and B are 1-connected CW complexes, i is a homotopy equivalence.

Definition 3.9. If X, X^* are based CW complexes, then X^* is a *geometric n-dual* of X if X and X^* can be embedded in $\mathbb{S}^{n+1}$ so that X^* is an S-deformation retract of $\mathbb{S}^{n+1} \smallsetminus X$.

X^* is an n-dual of X if and only if there is a map $u: X \wedge X^* \to \mathbb{S}^n$ such that $u^*(\iota_n)/: \widetilde{H}_q(X) \to \widetilde{H}^{n-q}(X^*)$ is an isomorphism, where ι_n generates $H^n(\mathbb{S}^n)$.
u is called an *n-duality map.*

Lemma 3.10. *If $u: X \wedge X^* \to \mathbb{S}^n$ is an n-duality map then*

$$
(\mathcal{S}X) \wedge X^* \to \mathcal{S}(X \wedge X^*) \xrightarrow{\mathcal{S}u} \mathcal{S}(\mathbb{S}^n) \to \mathbb{S}^{n+1}
$$

$$
X \wedge (\mathcal{S}X^*) \xrightarrow[\text{interchange}]{} \mathcal{S}(X \wedge X^*) \xrightarrow{\mathcal{S}u} \mathcal{S}(\mathbb{S}^n) \to \mathbb{S}^{n+1}
$$

are $(n+1)$-duality maps and

$$
X^* \wedge X \xrightarrow[\text{interchange}]{} X \wedge X^* \xrightarrow{u} \mathbb{S}^n
$$

is an n-duality map.

Proof. The first comes from the fact that

$$u^*(\iota_n)/ \colon \widetilde{H}_q(X) \to \widetilde{H}^{n-q}(X^*)$$

so

$$u^*(\iota_n)/ \in \mathrm{Hom}(\widetilde{H}_q(X), \widetilde{H}^{n-q}(X^*))$$

and the map

$$H^n(X \wedge X^*) \to \widetilde{H}^q(X; \widetilde{H}^{n-q}(X^*)) \to \mathrm{Hom}(\widetilde{H}_q(X), \widetilde{H}^{n-q}(X^*))$$

takes $u^*(\iota_n) \mapsto u^*(\iota_n)/$.

Then it is clear that the new maps are in fact isomorphisms. The final statement, follows from the universal coefficient theorem. ☐

3.2 Duality in Certain Spectra and Map

Let us assume that we have a duality map $u \colon X^* \wedge X \to \mathbb{S}^n$. Then we also have duality maps $\mathcal{S}^k u \colon (\mathcal{S}^k X^*) \wedge X \to \mathbb{S}^{n+k}$. This leads to maps $f_k = \widetilde{\mathcal{S}^k u} \colon \mathcal{S}^k X^* \to F(X, \mathbb{S}^{n+k})$ the adjoint. By commutativity of

$$\begin{array}{ccc} \mathcal{S}(\mathcal{S}^k X^*) & \xrightarrow{\mathcal{S} f_k} & \mathcal{S}F(X, \mathbb{S}^{n+k}) \\ \downarrow & & \downarrow \\ \mathcal{S}^{k+1} X^* & \xrightarrow[f_{k+1}]{} & F(X, \mathbb{S}^{n+k+1}) \end{array}$$

we have induced $f \colon \mathcal{S}\underline{X}^* \to \underline{F}(X, \mathbb{S})$. Reindex so that $\mathcal{S}^k X^*$ is the $(n+k)$th term of $\underline{\mathcal{S}}X^*$.

Theorem 3.11. *f is a weak homotopy equivalence.*

Proof. It suffices to prove that f induces a homology isomorphism. The idea behind the proof is represented by the following:

$H_*(\underline{F}(X, \mathbb{S}) \simeq \pi_*(\underline{F}(X, \mathbb{S}) \wedge \underline{K}) \simeq \pi_*(\underline{F}(X, \underline{K})) \simeq [\underline{\mathcal{S}}X, \underline{K}] \simeq H^*(X) \simeq H_*(X^*) \simeq H_*(\underline{\mathcal{S}}X^*)$

where $\underline{K} = \underline{K}(\mathbb{Z})$. We shall prove these isomorphisms in several stages.

Look at the map $\rho \colon F(X, Y) \wedge Z \to F(X, Y \wedge Z)$ given by $\rho(f \wedge z)(x) = f(x) \wedge z$.

Theorem 3.12. *Let X, Y and Z be CW spaces with $\dim X \leq k$, Y $(n-1)$-connected and Z $(m-1)$-connected. Then $\rho \colon F(X, Y) \wedge Z \to F(X, Y \wedge Z)$ is $(2n - 2k + m)$-connected.*

Proof. Fix X and Y. Then let

$$h_i(Z) = \pi_i(F(X, Y) \wedge Z) \quad k_i(Z) = \pi_i(F(X, Y \wedge Z)).$$

We claim

- h_* and k_* are partial homology theories of bidegree $(m, 2(m+n-k-1))$,
- $\rho: h_* \to k_*$ is an isomorphism in the appropriate degrees.

First observe that $\pi_i(F(X,Y)) = [\mathcal{S}^i, Y] = 0$ for $i + k < n - 1$ so $F(X,Y)$ is $(n-k-1)$-connected. Then if $A \subset Z$ are $m-1$)-connected spaces with the HEP, $F(X,Y) \wedge A \to F(X,Y) \wedge Z \to F(X,Y) \wedge Z/A$ is a cofibration of $(n+m-k-1)$-connected spaces. Hence there is an exact homotopy sequence (Theorem 1.10) starting with $\pi_{2(n+m-k-1)}(F(X,Y) \wedge A) = h_{2(n+m-k-1)}(A)$. The isomorphism $h_i(Z) \simeq h_{i+1}(\mathcal{S}Z)$ is evident from the exact sequence.

Next observe that $k_i(Z) \simeq [\mathcal{S}^i X, Y \wedge Z]$. Now $Y \wedge A \to Y \wedge Z \to Y \wedge Z/A$ is a cofibration of $(n+m-1)$-connected spaces, hence from Theorem 1.23, there is an exact sequence beginning $[\mathcal{S}^i X, Y \wedge A] \simeq k_i(A)$ for $i + k \le 2(n+m-1)$, hence, for $i \le 2(n+m-k-1)$. So h_* and k_* are partial homology theories of bidegree $(m, 2n-2k+m-1)$. (Actually they can both be extended a bit.)

Now let $Z = \mathbb{S}^t$, $t \ge m$. The following diagram clearly commutes up to sign.

$$\begin{array}{ccc}
h_i\mathbb{S}^t & \longrightarrow & k_i(\mathbb{S}^t) \\
\| & & \| \\
\pi_i(F(X,Y)\wedge \mathbb{S}^t) & \xrightarrow{\rho_*} & \pi_i(F(X, Y\wedge \mathbb{S}^t)) \\
\uparrow f & & \uparrow \sim \\
\pi_{i-t}(F(X,Y)) \simeq [\mathcal{S}^{i-t}X, Y] & \xrightarrow[g]{} & [\mathcal{S}^i X, \mathcal{S}^t Y]
\end{array}$$

where

$$f \text{ is } \begin{cases} \text{iso for} & (i-t) < 2(n-k) - 1 \\ \text{epi for} & (i-t) \le 2(n-k) - 1 \end{cases}$$

and

$$g \text{ is } \begin{cases} \text{iso for} & (i-t+k) < 2n - 1 \\ \text{epi for} & (i-t+k) \le 2n - 1 \end{cases}$$

Thus ρ_* is an isomorphism for $i < 2n - 2k + t$ and an epimorphism for $i = 2n - 2k + t$. Thus for $t \ge m$, ρ_* is an isomorphism for $i < 2n - 2k + m - 1$ and an epimorphism for $i = 2n - 2k + m - 1$. Thus by Theorem 2.24, for any $(m-1)$-connected Z, $\rho_*: h_i(Z) \to k_i(Z)$ is also, hence $\rho: F(X,Y) \wedge X \to F(X, Y \wedge Z)$ is $(2n - 2k + m)$-connected. □

There is a very useful principle which will be demonstrated in Corollary 3.14. It is this: if $f: Y \to Z$ is n-connected and X is a CW space of dimension m, then $F(X,f): F(X,Y) \to F(X,Z)$ is $(n-m)$ connected. This is immediate from the observation that the map $\pi_i(F(X,f)): \pi_i(F(X,Y)) \to \pi_i(F(X,Z))$ is equivalent to $[\mathcal{S}^i X, f]: [\mathcal{S}^i X, Y] \to [\mathcal{S}^i X, Z]$ which is an isomorphism for $i + m < n$ and epi for $i + m = n$.

Corollary 3.13. *Let X be $(n-1)$-connected. Then $\varphi: X \to \Omega^r \mathcal{S}^r X$, the adjoint of the identity map of $\mathcal{S}^r X$, is $(2n-1)$-connected, and $\theta: \mathcal{S}^r \Omega^r X \to X$, the adjoint of the identity map of $\Omega^r X$, is $(2n-r)$-connected.*

Proof. We have already proved that φ is $(2n-1)$-connected in Corollary 1.30. The following diagramme commutes

$$\begin{array}{ccc}
\mathcal{S}^r \Omega^r X & \xrightarrow{\theta} & X \\
\| & & \\
\mathcal{S}^r F(\mathbb{S}^r, X) & & \Big\downarrow \varphi \\
\rho \Big\downarrow & & \\
F(\mathbb{S}^r, \mathcal{S}^r X) & = & \Omega^r \mathcal{S}^r X
\end{array}$$

ρ is $(2n-r)$-connected by Theorem 3.12 and φ is $(2n-1)$-connected. Thus θ is $(2n-r)$-connected. □

Corollary 3.14. *If X is a CW space of dimension $\leq k$ then the composite*

$$F(X, \mathbb{S}^n) \wedge K(\mathbb{Z}, m) \xrightarrow{\rho} F(X, \mathcal{S}^n K(\mathbb{Z}, m)) \simeq$$
$$F(X, \mathcal{S}^n \Omega^n K(\mathbb{Z}, m+n)) \xrightarrow{F(X,\theta)} F(X, K(\mathbb{Z}, m+n))$$

is $\min(2n+m-2k, 2m+n-k)$ *connected. In particular $F(X, \mathbb{S}^m) \wedge K(\mathbb{Z}, m) \to F(X, K(\mathbb{Z}, 2m))$ is $(3m-2k)$ connected.*

Proof. ρ is $(2n-2k+m)$-connected. θ is $[2(n+m)-n]$-connected, so $F(X, \theta)$ is $(2m+n-k)$-connected; and the result is immediate. □

But $\pi_i(F(X, K(\mathbb{Z}, 2m)) \simeq [\mathcal{S}^i X, K(\mathbb{Z}, m)] \simeq H^{2m}(\mathcal{S}^i X) \simeq H^{2m-i}(X)$. Thus

Corollary 3.15. *$\pi_i(F(X, \mathbb{S}^m) \wedge K(Z, m)) \simeq H^{2m-i}(X)$ for $i < 3m-2k$ where $\dim X \leq k$.*

Theorem 3.16. *The composite*

$$\begin{array}{ccc}
\pi_{i+m}(X \wedge K(\mathbb{Z}, m)) & \xrightarrow{h} & H_{i+m}(X \wedge K(\mathbb{Z}, m)) \simeq \sum \widetilde{H}_J(X; \widetilde{H}_{i+m-j}(K(\mathbb{Z}, m))) \\
& \searrow^{\varphi_i(X)} & \Big\downarrow \\
& & \widetilde{H}_i(X)
\end{array}$$

is an isomorphism for $0 < i \leq m+t-1$ where X is a $(t-1)$-connected CW complex.

Proof. Let $h_i(X) = \pi_{i+m}(X \wedge K(\mathbb{Z}, m))$. Then for any t, h_* is a partial homology theory of bidegree $(t, m+t)$. Then $\varphi_i: h_i \to H_i(-)$ is a natural transformation of partial homology theories of bidegree $(t, m+t)$. It suffices to show that $\varphi_i(\mathbb{S}^k)$ is an isomorphism for $k \geq t$and $i \leq m+k$.

By Corollary 3.13 $\mathcal{S}^k K(\mathbb{Z}, m) \to K(\mathbb{Z}, m+k)$ is $(2m+k)$-connected. Thus for $0 < i < m+k$, $\pi_{i+m}(\mathcal{S}^k K(\mathbb{Z}, m)) = 0$ except for $i = k$ and $\pi_{m+k}(\mathcal{S}^k K(\mathbb{Z}, m)) = \mathbb{Z}$. But the same is true of $H_i(\mathbb{S}^k)$ and the Hurewicz isomorphism theorem yields the isomorphism for $i = k$. □

We proved in Corollary 3.15 that

$$\pi_{i+m}(F(X, \mathbb{S}^m) \wedge K(\mathbb{Z}, m)) \simeq \widetilde{H}^{m-i}(X), \quad i < 2m - 2k, \ \dim X \leq k.$$

Now we have that $\pi_{i+m}(F(X, \mathbb{S}^m) \wedge K(\mathbb{Z}, m)) \simeq \widetilde{H}_i(F(X, \mathbb{S}^m))$ for $i \leq m + m - k - 1$. Thus

Theorem 3.17. *If* $\dim X \leq k$ *then for* $i \leq 2m - 2k - 1$

$$\widetilde{H}_i(F(X, \mathbb{S}^m)) \simeq \widetilde{H}^{m-i}(X).$$

Recall the map $u: X^* \wedge X \to \mathbb{S}^m$ which we have assumed induces

$$u^*(\iota_n)/: \widetilde{H}_i(X^*) \simeq H^{m-i}(X).$$

It defines $\widetilde{u}: X^* \to F(X, \mathbb{S}^m)$ by $\widetilde{u}(x^*)(x) = u(x^* \wedge x)$.

Lemma 3.18. *The diagramme*

$$\begin{array}{ccc} \widetilde{H}^{m-i}(X) & \xrightarrow{\ \simeq\ } & \widetilde{H}_i(F(X, \mathbb{S}^m)) \\ \uparrow{\scriptstyle \simeq} & \nearrow_{\widetilde{u}_*} & \\ \widetilde{H}_i(X^*) & & \end{array}$$

commutes.

Proof. Recall the evaluation map $\mathrm{eval}: F(X, \mathbb{S}^m) \wedge X \to \mathbb{S}^m$. The diagramme

$$\begin{array}{ccc} X^* \wedge X & \xrightarrow{\ u\ } & \mathbb{S}^m \\ {\scriptstyle \widetilde{u}\wedge \mathrm{id}}\downarrow & \nearrow_{\mathrm{eval}} & \\ F(X, \mathbb{S}^m) \wedge X & & \end{array}$$

obviously commutes: $\mathrm{eval}(\widetilde{u} \wedge \mathrm{id})(x^* \wedge x) = \mathrm{eval}(\widetilde{u}(x^*) \wedge x) = \widetilde{u}(x^*)(x) = u(x^* \wedge x)$. Thus by naturality of the Künneth formula

$$\begin{array}{ccccc} \widetilde{H}_i(X^*) \otimes \widetilde{H}^{m-i}(X) & \xrightarrow{\ \alpha\ } & \widetilde{H}_m(X^* \wedge X) & \xrightarrow{\ u_*\ } & H_m(\mathbb{S}^m) = \mathbb{Z} \\ {\scriptstyle \widetilde{U}\otimes \mathrm{id}}\downarrow & & & \nearrow_{\mathrm{eval}_*} & \\ \widetilde{H}_i(F(X, \mathbb{S}^m)) \otimes \widetilde{H}_{m-i}(X) & \xrightarrow[\ \beta\ ]{} & \widetilde{H}_m(F(X, \mathbb{S}^m) \wedge X) & & \end{array}$$

commutes.

The map $u_*\alpha$ is the same as the composite

$$\begin{array}{ccc}
\widetilde{H}_i(X^*)\otimes\widetilde{H}_{m-i}(X) & \xrightarrow{\ \simeq\ } & \widetilde{H}^{m-i}(X)\otimes(X) \\
 & & \Big\downarrow{\scriptstyle p\otimes\mathrm{id}} \\
 & & \mathrm{Hom}(\widetilde{H}_{m-i}(X),\mathbb{Z})\otimes\widetilde{H}_{m-i}(X)\longrightarrow\mathbb{Z}
\end{array}$$

Thus in order to show that $\widetilde{u}$ is an isomorphism for i sufficiently small compared to m, it will suffice to show that the following diagramme commutes up to sign:

$$\begin{array}{ccc}
\widetilde{H}^{m-i}(X)\otimes\widetilde{H}_{m-i}(X) & \xrightarrow{\ \simeq\ } & \widetilde{H}_i(F(X,\mathbb{S}^m))\otimes\widetilde{H}_{m-i}(X) \\
{\scriptstyle p\otimes\mathrm{id}}\Big\downarrow & \searrow{\scriptstyle \theta(X)},\ {\scriptstyle \varphi(X)} & \Big\downarrow{\scriptstyle \mathrm{eval}_*\,\beta} \\
\mathrm{Hom}(\widetilde{H}_{m-i}(X),\mathbb{Z})\otimes\widetilde{H}_{m-i}(X) & \longrightarrow & \mathbb{Z}
\end{array}$$

i.e. that $\theta(X)=\pm\varphi(X)$.

First define the adjoint maps

$$\widetilde{\theta}(X),\widetilde{\varphi}(X)\colon\widetilde{H}^{m-i}(X)\to\mathrm{Hom}(H_{m-i}(X),\mathbb{Z}).$$

By the adjointness it follows that $\widetilde{\varphi}(X)$ is simply the projection

$$p\colon\widetilde{H}^{m-i}(X)\to\mathrm{Hom}(\widetilde{H}_{m-i}(X),\mathbb{Z}).$$

Thus we have $\widetilde{\theta}\colon\widetilde{H}^{m-i}(X)\to\mathrm{Hom}\,\widetilde{H}_{m-i}(X),\mathbb{Z})$ defined and natural in X and we wish to show that it is $\pm p$.

We can use the fact that $\widetilde{H}^{m-i}(-)=[-,K(\mathbb{Z},m-i)]$. But then $\widetilde{\theta}$ is determined by some element of $\mathrm{Hom}(\widetilde{H}_{m-i}(K(\mathbb{Z},m-i)),\mathbb{Z})\simeq\mathbb{Z}$. $\widetilde{\varphi}$ is obviously determined by a generator. Thus $\widetilde{\theta}=\lambda\widetilde{\varphi}$ for some integer λ. But $\widetilde{\theta}(\mathbb{S}^0)$ is clearly an isomorphism so $\lambda=\pm1$ and the lemma is proved. □

Then applying Theorem 2.13 we get

Theorem 3.11 $f\colon\underline{\mathcal{S}}X^*\to F(X,\underline{\mathbb{S}})$ is a weak homotopy equivalence. □

(End of Proof of Theorem 3.11)

Observe that for A and B finite CW complexes

$$\{A\wedge B,C\}\simeq[\mathcal{S}^NA\wedge B,\mathcal{S}^NC]\simeq[\mathcal{S}^NA,F(B,\mathcal{S}^NC)]$$

for N sufficiently large. Thus letting $N\to\infty$ we get $\{A\wedge B,C\}\simeq\{A,\underline{F}(B,\underline{\mathcal{S}}C)\}$. Let X and Y be finite CW complexes embedded in $\mathbb{S}^{m+1}$ and X^* and Y^* be their m-duals.

Thus, given duality maps $u\colon X^*\wedge X\to\mathbb{S}^m$ and $v\colon Y^*\wedge Y\to\mathbb{S}^m$ we get an isomorphism $D(u,v)\colon\{X,Y\}\simeq\{Y^*,X^*\}$ where $f\colon\mathcal{S}^rX\to\mathcal{S}^rY$ corresponds to

$g: \mathcal{S}^t Y^* \to \mathcal{S}^t X^*$ if and only if the following diagramme is stably commutative (i.e. some suspension of it commutes):

$$\begin{array}{ccc} \mathcal{S}^t Y^* \wedge \mathcal{S}^r X & \xrightarrow{g \wedge \mathrm{id}} & \mathcal{S}^t X^* \wedge \mathcal{S}^r X \\ {\scriptstyle \mathrm{id} \wedge f}\downarrow & & \downarrow {\scriptstyle {}^t u^r} \\ \mathcal{S}^t Y^* \wedge \mathcal{S}^r Y & \xrightarrow[{}^t v^r]{} & \mathbb{S}^{m+t+r} \end{array}$$

where ${}^t u^r$ and ${}^t v^r$ are the appropriate suspensions of u and v.

Thus duality is almost an isomorphism of the stable category with itself. Unfortunately it isn't quite because of the choices involved. For most practical purposes, however, we can regard it as a functor assigning to spaces X embeddable in $\mathbb{S}^n$, the space $D_n X$ in such a way that $D_n D_n X = X$ and $\{X, Y\} = \{D_n Y, D_n X\}$.

Recall that if $X \wedge D_n X \to \mathbb{S}^n$ is a duality map then so are $(\mathcal{S}^m X) \wedge D_n X \to \mathbb{S}^{n+m}$ and $X \wedge \mathcal{S}^m D_n X \to \mathbb{S}^{n+m}$. Thus $D_{n+m}(\mathcal{S}^m X) = D_n X$ and $D_{n+m} X = \mathcal{S}^m D_n X$.

We can make the category $\mathcal{C}$ of finite CW complexes and $\mathcal{S}$-maps a graded category by defining

$$\{X, Y\}_n = \{\mathcal{S}^n X, Y\} = [\underline{S}X, \underline{S}Y]_n.$$

Then

$$\{X, Y\}_n = \{\mathcal{S}^n X, Y\} \simeq \{D_{m+n} Y, D_{m+n} X\} = \{\mathcal{S}^n D_m Y, D_m X\} = \{D_m Y, D_m X\}_n.$$

Thus D_m preserves grading,

Observe that if $X \to Y \to C_f$ is cofibration then we get long exact sequences

$$\begin{aligned} &\{W, X\}_n \to \{W, Y\}_n \to \{W, C_f\}_n \to \{W, X\}_{n-1} \\ &\{X, W\}_n \leftarrow \{Y, W\}_n \leftarrow \{C_f, W\}_n \leftarrow \{X, W\}_{n+1} \end{aligned}$$

Also if $X \to Y \to C_f$ is cofibration, then

$$D_n C_f \to D_n Y \to D_n X$$

acts like one in terms of the long exact sequences.

This leads to being able to consider certain problems by only looking at dual problems.

Example 3.19. Freyd conjectures (Freyd [2]) the following: If X and Y are finite CW complexes and $f \in \{X, Y\}$ is such that $f_*: \pi_*^S(X) \to \pi_*^S(Y)$ is zero then $f = 0$. ($t\pi_*^s(-) = H_*(-; \underline{\mathcal{S}})$.)

A dual conjecture replaces f_* by $f^*: \{Y, \mathbb{S}^0\}_* \to \{X, \mathbb{S}^0\}_*$.

They are equivalent: Assume the former conjecture true. Pick $f \in \{X, Y\}$ with $f^*: \{Y, \mathbb{S}^0\} \to \{X, \mathbb{S}^0\}$ zero; then for N large we have $D_N f \in \{D_n Y, D_N X\}$

with $(D_N f)_*: \{\mathbb{S}^N, Y\} \to \{\mathbb{S}^N, X\}$ zero hence $D_N f = 0$ hence $f = 0$. Similarly the second conjecture implies the first.

We now consider the following category $\mathscr{S}$: the objects are pairs (X, n) where X is a finite CW space and a given embedding of it into some sphere (and hence all higher spheres) and n is an integer and we set $(X, n) = (\mathcal{S}X, n-1)$. The morphisms are $\mathscr{S}((X,n),(Y,m)) = \{\mathcal{S}^{r+n}X, \mathcal{S}^{r+m}Y\}$ where $r+n$, $r+m \geq 0$. Observe that we can use the fact that

1. This is independent of r.

2. This is unaffected by replacement of (X, n) by $(\mathcal{S}X, n-1)$.

Roughly, the objects of $\mathscr{S}$ are finite CW complexes and their formal de-suspensions. We write $(X, n) = \mathcal{S}^n X$ for any integer n and this makes sense in $\mathscr{S}$. We can always de-suspend objects in $\mathscr{S}$. Call an object of $\mathscr{S}$ "real" if it is equal to some $(X, 0)$.

$\mathscr{S}$ has an advantage when it comes to Spanier-Whitehead duality: we can talk about *the* dual as follows:

Given a real object X choose n sufficiently large that $D_n X$ exists. Then as usual $D_{n+1}X = \mathcal{S}D_n X$ so if we define $DX = \mathcal{S}^{-n}D_n X$ we get a unique object of $\mathscr{S}$ (in general, it is not real), independent of n. For any object of $\mathscr{S}$ we extend by setting $D\mathcal{S}^r X = \mathcal{S}^{-r}DX$. This also is unique up to homotopy type since

$$\mathcal{S}D_n\mathcal{S}X \simeq D_n X$$

whenever $D_n X$ is defined. Observe also that the maps $\mathrm{eval}^n_X: X \wedge D_n X \to \mathbb{S}^n$ will yield in $\mathscr{S}$ a natural map $\mathrm{eval}_X: X \wedge DX - \to \mathbb{S}^0$ by taking de-suspension.

Theorem 3.20. *D acts like a contravariant functor which is an anti-automorphism and involution on $\mathscr{S}$. I.e.,*

1) $D^2 X \simeq X$.

2) $\mathscr{S}(X, Y) \simeq \mathscr{S}(DX, SY)$.

We observe that $D\mathcal{S}^n \simeq \mathcal{S}^{-n}$ for all n. Also observe that since $D_n X \wedge D_m Y = D_{n+m}(X \wedge Y)$ for spaces we get $DX \wedge DY = D(X \wedge Y)$ in $\mathscr{S}$.

Theorem 3.21. *$\mathscr{S}(X \wedge Y, Z)$ is naturally equivalent to $\mathscr{S}(X, DY \wedge X)$. Thus $- \wedge Y$ and $DY \wedge -$ are adjoint functors.*

Proof. First observe that it will suffice to prove this for real objects Y since if $Y = \mathcal{S}^r Y'$ then

$$\mathscr{S}(X \wedge Y, Z) = \mathscr{S}(\mathcal{S}^r X \wedge Y', Z) = \mathscr{S}(X \wedge Y', \mathcal{S}^{-r}Z)$$
$$\mathscr{S}(X, DY \wedge Z) = \mathscr{S}(X, D\mathcal{S}^r Y' \wedge Z) = \mathscr{S}(X, (\mathcal{S}^{-r}DY') \wedge Z) = \mathscr{S}(X, DY' \wedge \mathcal{S}^{-r}Z).$$

Fixing X and Z define $k^n(Y) = \mathscr{S}_{-n}(X, DY \wedge Z)$ and $k^{-n}(Y) = \mathscr{S}_{-n}(X \wedge Y, Z)$. Observe that k^* and k^{-*} are cohomology theories. Define $\alpha: k^* \to k^{-*}$ as the composite

$$\begin{array}{ccc}
\mathscr{S}_{-n}(X, DY \wedge Z) & \xrightarrow{Y\wedge -} & \mathscr{S}_{-n}(Y \wedge X, Y \wedge DY \wedge Z) \\
\downarrow{\scriptstyle \alpha(Y)} & & \downarrow{\scriptstyle \mathscr{S}_{-n}(Y\wedge X, \mathrm{eval}_Y \wedge Z)} \\
\mathscr{S}_{-n}(X \wedge Y, Z) & \xleftarrow{T^*} & \mathscr{S}_{-n}(Y \wedge X \wedge Y, \mathbb{S}^0 \wedge Z)
\end{array}$$

where $T: X \wedge Y \to Y \wedge X$ is the twist and $\mathbb{S}^0 \wedge Z$ is identified with Z. It suffices to show that $\alpha(\mathbb{S}^0)$ is an isomorphism, but this is trivial since each of the three maps is then an isomorphism. □

Observe that reduced homology and cohomology can be defined on by setting $\widetilde{H}_k(\mathcal{S}^r X) = \widetilde{H}^k(X)$, $\widetilde{H}^k(\mathcal{S}^r) = \widetilde{K}^{k-r}(X)$ and we have for $\widetilde{H} = \widetilde{H}(-, \mathbb{Z})$

$$\widetilde{H}_{-k}(DX) = \widetilde{H}_{-k}(\mathcal{S}^{-n} D_n X) = \widetilde{H}_{n-k}(D_n X) \simeq \widetilde{H}^k(X)$$

for X real and hence by suspension for all X.

Observe that Theorem 3.21 makes sense and holds for any space Z. Setting $X = \mathbb{S}^n$ yields $\mathscr{S}_n(Y, Z) = \mathscr{S}(\mathcal{S}^n Y, Z) \simeq \mathscr{S}(\mathbb{S}^n, DY \wedge Z) = \pi_n^{\mathcal{S}}(DY \wedge Z)$.

Next observe that if $\underline{A}$ is a convergent spectrum then

$$H_k(W; \underline{A}) = \varinjlim_r \pi_{k+r}(W \wedge A_r) = \varinjlim_r \pi_{k+r}^{\mathcal{S}}(W \wedge A_r).$$

Thus

$$\begin{aligned}
H_k(DY; \underline{A}) &= \varinjlim_r \pi_{k+r}^{\mathcal{S}}(DY \wedge A_r) = \varinjlim_r \mathscr{S}(\mathcal{S}^{k+r} Y, A_r) \\
&= \varinjlim_r [\mathcal{S}^{k+r} Y, A_r] = \varinjlim_r [\mathcal{S}^r Y, A_{r-k}] = H^{-k}(Y; \underline{A}).
\end{aligned}$$

Thus duality holds for homology with coefficients in a convergent spectrum. Also if k^* is any (reduced) cohomology theory, we get a dual homology theory by defining $k_n(X) = k^{-n}(DX)$.

Finally let us observe that if we take the category of spectra, take the full subcategory $\mathscr{S}'$ of all $(\underline{\mathcal{S}}X)^d$ where X is a finite CW space then $\mathscr{S}' \simeq \mathscr{S}$ ($\mathcal{S}^n X \in \mathscr{S}$ corresponds to $(\underline{\mathcal{S}}X)^n$).

Incidentally we may ask about homology and cohomology theories defined in $\mathscr{S}$ in general and ask if they come from spectra.

Theorem 3.22. *If $\mathscr{H}$ is a generalised homology theory defined on $\mathscr{S}$ and $H_n(\mathbb{S}^0)$ is countable for all n, then there exists a spectrum $\underline{A}$ such that $\mathscr{H} = H_*(-; \underline{A})$.*

This theorem is due to E. H. Brown, Jr. (cf. Brown) and we shall not prove it.