First Course in Topology

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Foreword

Preface by the original authours (with some modification by the transcriber)

This book is the result of reworking part of a rather lengthy course of lectures of which we delivered several versions at the Leningrad and Moscow Universities. In these lectures we presented an introduction to the fundamental topics of topology: homology theory, homotopy theory, theory of bundles, and topology of manifolds. The structure of the course was well determined by the guiding term *elementary topology*, whose main significance resides in the fact that it made us use a rather simple apparatus. In this book we have retained those sections of the course where algebra plays a subordinate role. We plan to publish the more algebraic part of the lectures as a separate book.

Reprocessing the lectures to produce the book resulted in the profits and losses inherent in such a situation: the rigour has increased to the detriment of the intuitiveness, the geometric descriptions have been replaced by formulae needing interpretations, etc. Nevertheless, it seems to us that the book retains the main qualities of our lectures: their elementary, systematic, and pedagogical features. The preparation of the reader is assumed to be limited to the usual knowledge of set theory, algebra, and calculus which mathematics students should master after the first year and a half of studies. The exposition is accompanied by examples and exercises. We hope that the book can be used as a topology textbook.

The most essential difference between the book and the corresponding part of our lectures is the arrangement of the material: here we have followed a much more orderly succession of topics. However, from our experience, a lecture course in elementary topology which exaggerates in the last respect is rather tedious and less efficient than one which mixes geometry with algebra and applications. This remark may serve as a warning to the teacher who would like to use our book as a guide. In fact, it is by no means necessary to read the book in its order; a reader who is interested in getting to the homotopy groups or to any other topic sooner, can easily do so.

Concerning the terminology and notation, we have tried to stick to standard usage, and have permitted ourselves only a few reforms. For example, we do not use the terms “simplicial complexes” or “CW-complexes”, but *simplicial spaces* and *cellular spaces*; not "cofibrations", but *Borsuk pairs*; not "fibre bundles" (or "fibred products"), but *Steenrod bundles*. There is even one term which we do not use in the generally accepted way: for us, a *connected space* refers to what usually is called a linearly connected (or path-connected) space (we do not have a special name for the spaces which are usually called connected). Furthermore, we have avoided using non-standardised notations for standard objects. In fact, in the majority of cases, our notation is just an abbreviation of the corresponding term and can be understood by itself: for example, proj stands for projection, incl - for inclusion, dim - for dimension, skel - for skeleton, bs - for base, etc.

Topology requires a very precise set-theoretic language, and this compelled us to devote a special attention to this language; this is illustrated in *Set-Theoretical Terms and Notations Used in this Book, but not Generally Adopted* below. We emphasise that on these pages we only list the terms and notations, assuming that the objects themselves are known.

In this book we rarely refer to the history of topology. We have even departed from the tradition that some theorems bear the names of their real or imaginary authors. In return, we willingly have used names of topologists in the terminology and notations.

The organisation of the text and the system of references may be briefly described as follows. Each Chapter is divided into Sections, each Section - into Subsections, each Subsection - into Numbers. The chapters, sections and subsections have numbers and titles, while the numbers are denoted by either (Remark, Definition, Theorem, Lemma or Corollary preceding) their numbers. Each fact announced without proof is called Information, and is distinguished from the rest of the text by this title. To refer to a section, subsection, or number within the same chapter, we do not indicate the number of the chapter, and references within a section or subsection are similarly abbreviated. Examples: the entries §1.2 (Section 2 of Chapter 1), Subsection 1.2.3 (Subsection 3 of Section 2 of Chapter 1), and (Remark, say) 1.2.3.4 (No. 4 of Subsection 3 of Section 2 of Chapter 1) are abbreviated, within Chapter 1, as §2 , Subsection 2.3, and 2.3.4, respectively; the second of these entries is abbreviated within §1.2 as Subsection 3; the third entry is abbreviated within §1.2 and Subsection 1.2.3 as 3.4 and 4, respectively.

The Authours

Review in MathScinet by J.F. Adams

The Russian original has been reviewed ["Nauka", Moscow, 1977; MR0645388]. Chapter 1 is a very good summary of "general topology" for the non-specialist user. Chapters 2, 3 and 4 give comprehensive groundwork in (respectively) CW-complexes and simplicial complexes, smooth manifolds, and fibrations. Chapter 5 gives some homotopy theory, for the non-specialist user. All this is done without any homology theory, because the authors "plan to publish the more algebraic part of the lectures as a separate book". For example, the notion of "degree" is obtained from the work on differential topology. The English translation is thus a useful addition to the available textbooks; probably, as the review of the original edition suggests, it is most suited to readers "oriented towards the advanced theory of differentiable manifolds".

Remark by the transcriber

This textbook is, as far as the transcriber knows, the only "introductory homotopy theory" covering smooth manifolds and bundle theory in the homotopy-theoretic context. Their treatment avoids functorial treatments, which may be suitable for the beginning level. It is quite regrettable that this wonderful textbook is less known than the (now classic) Hu and (more modern) Arkovitz.

The transcriber is sure that upon completing this textbook, the reader will be ready to study axiomatic or categorical homotopy theory.

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Chapter 0

Set-Theoretical Terms and Notations Used in this Book, but not Generally Adopted

Mathematicians manage with a surprisingly modest collection of set-theoretic terms and notations, which can be roughly divided into three groups.

- The first contains terms and notations which have attained general recognition.
- The terms and notations in the second group are equally well-known, but can be understood differently or have varying connotations.
- The third group consists of terms and notations used less frequently.

There is no need to define terms from the first group. For example, the notations $X \cup Y$, $X \cap Y$, and $X_1 \times \cdots \times X_n$ for the union, intersection, and product of sets, or the notations $f\colon X \to Y$, $\operatorname{im} f$ and $f|_A\colon A \to Y$ for a map, its image, and its restriction are understood in the same way by all people. The same is true for the notation $x \in X$ and the terms *one-to-one map* (*injective map*) and *map onto* (*surjective map*).

For the sake of precision, we must say a few words about our usage of terms and notations from the second group. We denote the empty set by $\varnothing$. We understand the notation $X \subset Y$ in the broadest sense, i.e., the equality $X = Y$ is not excluded. The same is true for the term *countable set*: we use it both for infinite countable and finite sets. The identity map of the set X is denoted by id_X, or, when there is no ambiguity about X, simply by id. We shall say that a map is invertible if it has an inverse, i.e., it is simultaneously injective and surjective. We let $\{x \in X | \dots\}$ denote the set of points x of the set X which satisfy the condition appearing instead of the three dots. A *family* $\{X_\mu\}_{\mu \in M}$ is a map a set M onto a set of objects X_μ with $\mu \in M$, defined by the formula $\mu \mapsto X_\mu$.

Next our main task is to list the terms and notations appearing in this book and belonging to the third group.

0.1 Maps

If A is a subset of a set X, then the inclusion of A in X may be considered as the map defined by the formula $x \mapsto x$. We denote it by $\operatorname{incl}\colon A \to X$. If there is no ambiguity about A and X, we simply write incl.

If A is a subset of X and B is a subset of Y, then each map $f\colon X \to Y$ such that $f(A) \subset B$ induces a map $\operatorname{abr} f\colon A \to B$, $x \mapsto f(x)$, and called here the *abridgement* (or *compression*) *of the map f to A, B*. When there is no ambiguity about A and B, one can write $\operatorname{abr} f$ instead of $\operatorname{abr} f\colon A \to B$. If $B = Y$, then $\operatorname{abr} f$ is just the usual restriction of f to A.

By a map of a sequence $(X, A_1, \dots, A_n)$ into a sequence $(Y, B_1, \dots, B_n)$, where $(B_1, \dots, B_n)$ are subsets of X (respectively, Y), we mean a sequence of maps

$$(\varphi\colon X \to Y, \varphi_1\colon A_1 \to B_1, \dots, \varphi_n\colon A_n \to B_n)$$

such that $\varphi_i = \operatorname{abr} \varphi$. We denote such a map by

$$(\varphi, \varphi_1, \dots, \varphi_n)\colon (X, A_1, \dots, A_n) \to (Y, B_1, \dots, B_n).$$

If the subsets $A_1, \dots, A_n$ and $B_1, \dots, B_n$ are fixed, then the map

$$f = (\varphi, \varphi_1, \dots, \varphi_n)$$

and its first component φ are uniquely determined by each other and usually we do not distinguish between them. For example, the notation

$$f\colon (X, A_1, \dots, A_n) \to (Y, B_1, \dots, B_n)$$

may be also used to say that f is a map of X into Y such that

$$f(A_1) \subset B_1, \dots, f(A_n) \subset B_n.$$

When we wish to emphasise explicitly this relationship between f and φ, we shall write:

$$f = \operatorname{rel} \varphi, \quad \varphi = \operatorname{abrs} f.$$

Sometimes we simply write rel instead of rel id.

0.2 Quotients

We denote the quotient (or factor) set of a set X by a partition $\mathfrak{p}$ of X by $X/\mathfrak{p}$. The map $X \to X/p$ which takes each point into the element of $\mathfrak{p}$ containing it is called *projection* and is denoted by proj. A subset of X which is a union of elements of the partition is said to be *saturated*. The smallest saturated set containing a given subset A of X (i.e., $\operatorname{proj}^{-1}(\operatorname{proj}(A))$) is called the *saturation* of A.

If p and q are partitions of X and Y, respectively, then each map $f\colon X \to Y$ which transforms elements of p into elements of q induces a map $X/p \to Y/q$, which takes each element A of p into the element of q which contains $f(A)$. This map is denoted by $\operatorname{fact} f$. In particular, $\operatorname{fact} f$ is defined when q is the partition of Y into its points, and f is constant on the elements of p. Thus, for each map $f\colon X \to Y$ constant on the elements of the partition p of X we have the corresponding map $\operatorname{fact} f\colon X/p \to Y$.

Given a map $f\colon X \to Y$, the partition of X into the non-empty pre-images of the points of Y is denoted by $\operatorname{zer}(f)$. The corresponding map

$$\operatorname{fact} f\colon X/\operatorname{zer}(f) \to Y$$

is injective and is called the *injective factor* (or *injective quotient*) of the map f.

0.3 Sums

The *sum of the family of sets* $\{x_\mu\}_{\mu\in M}$ is the union of disjoint copies of the sets X_μ, i.e., the set of pairs (x,μ) such that $x_\mu \in X_\mu$. Notation: $\coprod_{\mu\in M} X_\mu$. The map of X_ν ($\nu \in M$) into $\coprod_{\mu\in M} X_\mu$, defined by the formula $x \mapsto (x,\nu)$, is denoted by incl_ν.

We note that the maps incl_ν are injective and their images $\operatorname{incl}_\nu(X_\nu)$ are pairwise disjoint and cover $\coprod_{\mu\in M} X_\mu$ Therefore, for any family $\{Y_\mu\}_{\mu\in M}$ indexed by the same set M, and each family of maps $\{f_\mu\colon X_\mu \to Y_\mu\}_{\mu\in M}$, there is a unique map $f\colon \coprod_{\mu\in M} X_\mu \to \coprod_{\mu\in M} Y_\mu$ which satisfies the relations $f \circ \operatorname{incl}_\nu = \operatorname{incl}_\nu \circ f$. f is called the *sum of the maps* f and is denoted by $\coprod_{\mu in M} f_\mu$.

If M consists of the numbers $1,\dots,n$, we use, along with $\coprod_\mu X_\mu$ and $\coprod f_\mu$, the notations $X_1 \coprod \dots \coprod X_n$ and $f_1 \coprod \dots \coprod f_n$.

0.4 Products

The product $X_1 \times \dots \times X_n$ is mapped naturally onto its factor X_i following the rule $(x_1,\dots,x_n) \mapsto x_i$. This mas is called the *i-th projection* and is denoted by proj_i.

Given maps $f_1\colon X_1 \to Y_1, \dots, f_n\colon X_n \to Y_n$, the rule

$$(x_1,\dots x_n) \mapsto (f_1(x_1),\dots f_n(x_n))$$

defines a map of the product $X_1 \times \dots \times X_n$ into the product $Y_1 \times \dots \times Y_n$, called the *product of the maps* $f_1,\dots,f_n$ and denoted by $f_1 \times \dots \times f_n$.

If p is a partition of the set X and q is a partition of the set Y, we let $p \times q$ denote the partition of the product $X \times Y$ into the sets $A \times B$, where A is an element of p and B is an element of q.

There is a natural map of X into $X \times X$, given by $x \mapsto (x, x)$. This is the *diagonal* map and is denoted by diag or Δ. Its image

$$\Delta(X) = \mathrm{diag}(X \times X) \subset X \times X$$

is called the *diagonal* of $X \times X$.

Chapter 1

TOPOLOGICAL SPACES

1.1 FUNDAMENTAL CONCEPTS

1.1.1 Topologies

Definition 1.1.1.1. We say that a *topological structure* or, simply, a *topology*, is defined on a set X if there is given a class of subsets of X which contains

(i) the union of any collection in the class, and

(ii) the intersection of any finite collection in the class.

A set endowed with a topological structure is called a *topological space*, its elements - *points*, and the sets of the given class - *open sets*.

A collection of sets for which we take the union or the intersection may be empty. The union of the empty collection is $\varnothing$, while the intersection of the empty collection of subsets of X is the entire set X. Hence $\varnothing$ and X are open sets (in any topology).

Two examples of topological structures are:

(a) the trivial topology, whose only open sets are $\varnothing$ and X, and

(b) the discrete topology, in which all the subsets of X are open.

If X has more than one element, it is possible to define other topologies on X. For example, if X consists of two elements, a and b, then it will admit two topologies aside from the trivial and discrete ones. In one, the open sets are $\varnothing$, a, and X, while in the second the open sets are $\varnothing$, b, and X. More serious examples will appear in the sequel.

Definition 1.1.1.2. A subset of a topological space is *closed* if its complement is open. The class of closed sets contains the intersection of any collection of sets from the class, and it contains the union of any finite collection of sets from the class. Moreover, given any class of subsets of a set X with these properties, there exists a unique topology on X such that the given class is the class of all closed sets.

Definition 1.1.1.3. A *neighbourhood* of a point of a topological space is any open set containing the given point. A neighbourhood of a subset of a topological space is any open set containing the given subset.

Derived Concepts

Definition 1.1.1.4. If we consider the open sets contained in a given subset A of a topological space X, there is one which is the largest, namely the union of all such sets. It is called the *interior part* or, simply, the *interior* of the set A. We denote it by $\operatorname{int} A$ or $Int_X A$. Similarly, among the closed sets containing A, there is one which is the smallest, namely the intersection of all such sets. It is called the *closure* of the set A and is denoted by $\operatorname{Cl} A$ or $\operatorname{Cl}_X A$. The difference $\operatorname{Cl} A \setminus \operatorname{int} A$ can be represented as the intersection of the closed sets $\operatorname{Cl} A$ and $X \setminus \operatorname{int} A$, and is therefore closed. This set is called the *boundary* or *frontier* of the set A and is denoted by $\operatorname{Fr} A$ or $\operatorname{Fr}_X A$. We remark that $X \setminus \operatorname{int} A = \operatorname{Cl}(X \setminus A)$ and that A and $X \setminus A$ have the same boundary.

Definition 1.1.1.5. Relative to the set A, the points of the sets $\operatorname{int} A$, $\operatorname{Cl} A$, $\operatorname{Fr} A$, and $X \setminus \operatorname{Cl} A = \operatorname{int}(X \setminus A)$ are called *interior*, *adherent boundary* (or *frontier*) and *exterior points*, respectively. They can be characterised more explicitly in terms of neighbourhoods. A point is:

(i) an interior point if it has a neighbourhood entirely contained in A;

(ii) an adherent point if each of its neighbourhoods intersects A;

(iii) a boundary point if each of its neighbourhoods intersects both A and $X \setminus A$; and

(iv) an exterior point if it has a neighbourhood which does not intersect A.

Clearly, a set is open (closed) if and only if it coincides with its interior (respectively, closure), i.e., if it consists only of interior points (respectively, if it contains all its boundary points).

Definition 1.1.1.6. A subset A of a topological space X is said to be *dense in* X (or *everywhere dense*) if $\operatorname{Cl} A = X$, i.e., if A intersects any non-empty open set in X. A set A is *nowhere dense* if $X \setminus \operatorname{Cl} A$ is everywhere dense.

Bases and Prebases

Definition 1.1.1.7. A *base of a topological space* is a collection of open sets such that any open set can be represented as a union of sets from this collection. Equivalently, a collection Γ of open sets is a base if for any open set U and any point $x \in U$ there is $V \in \Gamma$ such that $x \in V \subset U$.

A base completely determines the topology: the open sets are exactly those which can be expressed as union of elements of the base.

The following Proposition provides us with a standard method of introducing a topology on a set.

Proposition 1.1.1.8. *Let* Γ *be a collection of subsets of a set* X. *Then there exists a topology on* X *with base* Γ *if and only if the intersection of every finite sub-collection of sets from* Γ *can be expressed as a union of sets from* Γ.

The intersection of a finite collection of sets belonging to a base is open; hence the necessity of the condition. Sufficiency follows from the fact that the class of subsets of X which are representable as unions of sets from Γ satisfies the conditions of Definition 1.1.1.1. The previous Theorem can be reformulated in the following useful way.

Theorem 1.1.1.9. *There is a topology on* X *with base* Γ *if and only if the sets of* Γ *cover* X, *and for any* $U, V \in \Gamma$ *and any point* $x \in U \cap V$, *there exists* $W \in \Gamma$ *such that* $x \in W \subset U \cap V$.

Definition 1.1.1.10. A collection of subsets of a topological space is said to be a *prebase* of the space if the intersections of finite sub-collections of sets from the given collection form a base.

Proposition 1.1.1.8 shows that *any collection* Γ *of subsets of a set* X *is the prebase of a unique topology on* X.

Definition 1.1.1.11. A *base at the point* x of a topological space X is a collection of neighbourhoods of x such that any neighbourhood of x contains a neighbourhood from this collection. A *prebase at the point* x is a collection of sets such that the intersections of finite sub-collections form a base at x.

Covers

Definition 1.1.1.12. As a rule, the covers we shall encounter will be either covers of a topological space by some of its subsets, or covers of a subset of a topological space by other subsets of this space. If we need to emphasise that a certain cover of a subset A of a topological space X consists of subsets of X which are not necessarily included in A, we shall refer to it as a *cover of the set* A *in* X.

A cover Γ is a *refinement* of a cover Δ if any element of Γ is contained in an element of A.

A cover is *locally finite* if any point of the space has a neighbourhood which intersects only a finite number of elements of the cover.

A cover is *open* (*closed*) if all its elements are open (respectively, closed) sets.

Remark 1.1.1.13. Every open cover of a topological space X has a refinement whose sets belong to a given base of X.

For example, the sets of the base contained in the sets of the given cover yield such a refinement.

1.1.2 Metrics

Definition 1.1.2.1. A non-negative real function ρ defined on the square $X \times X$ of the set X is a *metric* on X if it satisfies three conditions:

(i) $\rho(x, y) = 0$ if and only if $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for any $x, y, z \in X$.

A *metric space* is a set equipped with a metric. We use the symbol dist as the standard notation for metrics.

The values taken by a metric are called *distances*, and the *inequality* they satisfy according to the definition is the *triangle inequality.*

Example 1.1.2.2. The standard n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 0)$ is the fundamental example of metric space. $\mathbb{R}^n$ is the set of all sequences $\{x_i\}_1^n$ of real numbers, where the distance between two sequences $\{x_i\}_1^n$ and $\{y_i\}_1^n$ is defined as $[\sum_1^n (x_i - y_i)^2]^{1/2}$. The line $\mathbb{R}^1$ is usually identified with the field of real numbers, denoted by $\mathbb{R}$.

We obtain the definition of the standard Hilbert space ℓ_2 by replacing the n-term sequences $\{x_i\}_1^n$ with infinite sequences $\{x_i\}_1^\infty$ satisfying the condition $\sum_1^\infty x_i^2 < \infty$ and writing $\sum_1^\infty$ instead of $\sum_1^n$ in the distance formula.

Definition 1.1.2.3. The *ball with centre* $x_0 \in X$ *and radius* $r > 0$ in the metric space X is the set of points $x \in X$ such that $\operatorname{dist}(x_0, x) \leq r$. If we write $<$ (respectively $=$) instead of $\leq$, we obtain the definition of the *open ball* (respectively, of the *sphere*). The unit ball and the unit sphere of $\mathbb{R}^n$, i.e., the ball and sphere with centre $(0, \dots, 0)$ and radius 1, are simply called the *n-dimensional ball* $\mathbb{D}^n$ and the $(n-1)$-*dimensional sphere* $\mathbb{S}^{n-1}$. In particular, $\mathbb{D}^0$ is just a point, $\mathbb{S}^0$ - a pair of points, and $\mathbb{S}^{-1} = \varnothing$. Moreover, we set $\mathbb{D}^n = \varnothing$ for $n < -1$ and $\mathbb{S}^n = \varnothing$ for $n \leq -2$.

Definition 1.1.2.4. By definition, the *distance between two sets*, A and B, is the number $\inf_{x \in A, y \in B} \operatorname{dist}(x, y)$, and we denote it by $\operatorname{Dist}(A, B)$. In particular, if a is a point, $\operatorname{Dist}(a, B) = \inf_{y \in B} \operatorname{dist}(a, y)$.

The *diameter* of the set A is the number $\sup_{x,y \in A} \operatorname{dist}(x, y)$, denoted by $\operatorname{diam} A$. A set is *bounded* if its diameter is finite.

The Metric Topology

Definition 1.1.2.5. As a consequence of the triangle inequality, if the open ball with centre at x_0 and radius r contains a point x_1, then it also contains the open ball with centre at x_1 and radius $r - \operatorname{dist}(x_0, x_1)$. Therefore, in any metric space, the intersection of two open balls contains, together with each point, some open ball centred at that point. Moreover, since the open balls cover the space, they constitute the base of a certain topology (see Proposition 1.1.1.9). In this way, every metric space becomes a topological one.

The resulting topology is called the *metric topology*. From now on we shall tacitly regard the metric spaces as topological spaces, having in mind the metric topology. In particular, this refers to $\mathbb{R}^n$ and ℓ_2.

A topological space whose topology is the metric topology relative to some metric is said to be *metrisable*.

Remark 1.1.2.6. Clearly, the open balls centred at a given point of a metric space constitute a base at that point. The part of this base consisting of the balls of radii $1/n$ $(n = 1, 2, \dots)$ is also a base.

Definition 1.1.2.7. If A is a subset of a metric space X, its *metric neighbourhood of radius* $r > 0$ is, by definition, the set of all points $x \in X$ such that $\operatorname{Dist}(A, x) < r$. Since this set is the union of all open balls of radius r centred at the points of A, it is open, i.e., a genuine neighbourhood of A.

1.1.3 Subspaces

Definition 1.1.3.1. We shall now discuss the *relative topology*, which transforms any subset A of a topological space X into an independent topological space. This topology is defined by taking its open sets to be those of the form $A \cap B$, where B is an open subset of X. It is evident that all the conditions of Definition 1.1.1.1 are satisfied. Moreover, the closed subsets of A in this topology are exactly the intersections $A \cap B$ where B is a closed subset of X. The subsets of the space X, equipped with the relative topology, are called *subspaces* of X.

If X is a topological space and A is a subspace of X, the pair (X, A) is called a *topological pair*. A *topological triple* is a triple (X, A, B) consisting of a topological space X and two subspaces A, B of X, such that $B \subset A$.

Remark 1.1.3.2. Let A be a subspace of X. It is clear that any subset of A which is open or closed in X has the same property in A. If A is open, then every set open in A is also open in X. If A is closed, then any set closed in A is also closed in X. In any case, if $B \subset A \subset X$, then $\operatorname{Cl}_A B = (\operatorname{Cl}_X B) \cap A$.

Obviously, if Γ is a base (pre-base) of the space X, then the sets $A \cap B$ with $B \in \Gamma$ yield a base (respectively, pre-base) of the space A.

As a direct consequence of its definition, the relative topology is transitive: if B is a subset of the subspace A of X, the topologies induced on B by the inclusions $B \subset A$ and $B \subset X$ coincide.

Remark 1.1.3.3. If X is a metric space and A is a subset of X, then the restriction of the function dist to $A \times A$ is clearly a metric on A. Consequently, any subset of a metric space is itself a metric space. In addition, it is obvious that the metric topology of the latter coincides with the relative topology induced on A by the metric topology of the ambient space X.

Example 1.1.3.4. The previous constructions greatly increase the supply of non trivial examples of topological spaces: we can now include all the subsets of $\mathbb{R}^n$ and ℓ_2. In particular, the balls and spheres of $\mathbb{R}^n$ are topological spaces.

Here we shall add only the *cubes* of $\mathbb{R}^n$, defined by inequalities of the form $\alpha_i \leq x_i \leq \alpha_i + a$ $(i = 1, \ldots, n)$, with real numbers $\alpha_1, \ldots, \alpha_n, a$, and $a > 0$. If $\alpha_1 = \cdots = \alpha_n = 0$ and $a = 1$, the cube is called the *unit cube* I^n. The unit segment I^1 is also denoted by I.

Fundamental Covers

Definition 1.1.3.5. A cover Γ of a topological space X is *fundamental* if each subset A of X such that $A \cap B$ is open in B for all $B \in \Gamma$ is itself open. Equivalently, Γ is fundamental if each subset A of X such that $A \cap B$ is closed in B for all $B \in \Gamma$ is itself closed.

Obviously, a cover which admits a fundamental refinement is itself fundamental.

Theorem 1.1.3.6. *All open covers, and all finite or locally finite closed covers are fundamental.*

Proof. Clearly, the claim is true for both open and finite closed covers. Now suppose that Γ is a locally finite closed cover of a space X. Consider a cover Δ of X consisting of open sets which intersect only a finite number of elements of Γ. Since Δ is fundamental, it is enough to check that, given any set $U \in \Delta$, the cover of U with elements $U \cap B$, $B \in \Gamma$, is fundamental. But this results from the fact that the latter cover is finite and closed. $\square$

Definition 1.1.3.7. A triple (X, A, B), where X is a topological space and A and B are subsets of X which constitute a fundamental cover of X, is termed a *triad.* If $\operatorname{int} A \cup \operatorname{int} B = X$, or if $A \cup B = X$ with A and B closed, then (X, A, B) is a triad.

1.1.4 Continuous Maps

Definition 1.1.4.1. A map f of a topological space X into a topological space Y is *continuous* if the preimage of each open subset of Y is open in X. Equivalently, f is continuous if the pre-image of each closed set is closed.

A map

$$f\colon (X, A_1, \ldots, A_n) \to (Y, B_1, \ldots, B_n),$$

where $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are subsets of the spaces X and Y, respectively, is said to be continuous, if the map $\operatorname{abrs} f\colon X \to Y$ is continuous.

A useful comment: in order for a map $X \to Y$ to be continuous, it is enough that the pre-images of the sets comprising some pre-base of Y be open.

Remark 1.1.4.2. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then the composition $g \circ f\colon X \to Z$ is obviously continuous. Trivially, the identity map $\mathrm{id}_X\colon X \to X$ is continuous for any topological space X.

According to the definition of the relative topology, if $f\colon X \to Y$ is continuous and $A \subset X$, $B \subset Y$ are subsets with $f(A) \subset B$, then the map $\operatorname{abr} f\colon A \to B$ is also continuous. In particular, the restriction $f|_A\colon A \to Y$ of a continuous map $f\colon X \to Y$ to an arbitrary subset A of X is continuous. For example, the inclusion of a subspace into its ambient space is always continuous.

If a compression abr f of the map $f\colon X \to Y$ is defined on the entire space X, then its continuity is equivalent to the continuity of f. In particular, $f\colon X \to Y$ is continuous if and only if the map $f\colon X \to f(X)$ is continuous.

It is clear that if γ is a fundamental cover of X, then a map $f\colon X \to Y$ is continuous whenever all the restrictions $A \in \Gamma$, are continuous. An equivalent formulation:

Theorem 1.1.4.3. *let Γ be a fundamental cover of the topological space X, and assume that for each $A \in \Gamma$ there is a continuous map $f\colon A \to Y$, such that $f_A(x) = f_B(x)$ for all $x \in A \cap B$ $(A, B \in \Gamma)$; then the map $f\colon X \to Y$ defined by*

$$f(x) = f_A(x) \quad \textit{for} \quad x \in A \quad (A \in \Gamma)$$

is continuous.

Definition 1.1.4.4. A continuous map is *open* if the images of the open sets are open, and *closed* if the images of the closed sets are closed.

Obviously, a composition of open maps is open, and a composition of closed maps is closed.

Here we note one useful sufficient condition for a map to be open:

Theorem 1.1.4.5. *$f\colon X \to Y$ is certainly open if for each $x \in X$ there is a neighbourhood U_x of $f(x)$ and a continuous map $g_x\colon U_x \to X$ such that the composition $f \circ g$ coincides with* incl: $U_x \to Y$.

Proof. If this is the case, then $f(A) = U_{x \in} g_x^{-1}(A)$ for any subset A of X. ☐

Continuity at a Point

Definition 1.1.4.6. A map $f\colon X \to Y$ is *continuous at the point* $x \in X$ if for any neighbourhood V of the point $f(x)$ there is a neighbourhood U of x such that $f(U \subset V$.

One can reformulate this definition using fewer open sets. In fact, let us assume that, along with the map $f\colon X \to Y$, we are given an arbitrary prebase at the point $x \in X$, Δ, and an arbitrary pre-base at the point $f(x) \in Y$, E. Then one readily sees that f is continuous at x if and only if each neighbourhood $V \in E$ contains the image of some neighbourhood $U \in A$.

When X and Y are metric spaces, and Δ and E consists of open balls centred at the points x and $f(x)$, respectively, the last statement reduces to the usual numerical formulation given in calculus: the map $f\colon X \to Y$ is continuous at the point $x \in X$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{dist}_X(x, x') < \delta$ implies $\operatorname{dist}_Y(f(x), f(x')) < \varepsilon$.

Theorem 1.1.4.7. *A map $f\colon X \to Y$ is continuous if and only if it is continuous at each point of X.*

Proof. If f is continuous and V is a neighbourhood of the point $f(x)$, then $f^{-1}(V)$ is a neighbourhood of the point x, and $f(f^{-1}(V)) \subset V$.

If f is continuous at each point and V is open in Y, then each point of the set $f^{-1}(V)$ is an interior point, since it has a neighbourhood whose image lies in V. □

Homeomorphisms and Embeddings

Remark 1.1.4.8. If $f\colon X \to Y$ is an invertible continuous map, the inverse map $f^{-1}\colon Y \to X$ is not necessarily continuous. For example, consider the identity map of a set with the discrete topology onto the same set, but equipped with a different topology; its inverse is not continuous.

An invertible map f such that both f and f^{-1} are continuous is a *homeomorphism.* If there is a homeomorphism $X \to Y$, the space Y is said to be *homeomorphic* to the space X.

The following maps are obviously homeomorphisms: the identity transformation of a space, the map inverse to a homeomorphism, and the composition of two homeomorphisms. Thus, the homeomorphism of spaces is an equivalence relation.

Example 1.1.4.9. The open ball $\operatorname{int} \mathbb{D}^n$ is homeomorphic to $\mathbb{R}^n$. The standard homeomorphism $\mathbb{R}^n \to \operatorname{int} \mathbb{D}^n$ is given by the formula

$$x \mapsto \begin{cases} 2x \arctan(\operatorname{dist}(0,x))/\pi \operatorname{dist}(0,x), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Example 1.1.4.10. The cube I^n is homeomorphic to $\mathbb{D}^n$; its interior $\operatorname{int} I^n$ is homeomorphic to $\operatorname{int} \mathbb{D}^n$, and its boundary $\operatorname{Fr} I^n$ is homeomorphic to $\operatorname{Fr} \mathbb{D}^n$, i.e., to $\mathbb{S}^{n-1}$. The standard homeomorphisms $\mathbb{D}^n \to I^n$, $\operatorname{int} \mathbb{D}^n \to \operatorname{int} I^n$, and $\operatorname{Fr} \mathbb{D}^n \to \operatorname{Fr} I^n$ are realised by translation with the vector

$$(\mathrm{ort}_1 + \cdots + \mathrm{ort}_n)/2,$$

followed by central projection
(here $\mathrm{ort}_1, \ldots, \mathrm{ort}_n$ denote the vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$.

Example 1.1.4.11. The punctured sphere $\mathbb{S}^n$ (i.e., $\mathbb{C}^n$ with one point removed) is homeomorphic to $\mathbb{R}^n$. A homeomorphism $\mathbb{R}^n \to \mathbb{S}^n \setminus \mathrm{ort}_1$ is given by the composition of the homeomorphism $\{x_1, \ldots, x_n)\} \mapsto \{(0, x_1, \ldots, x_n)\}$ of $\mathbb{R}^n$ onto a subspace of $\mathbb{R}^{n+1}$ with the stereographic projection, i.e., the central projection of this subspace onto $\mathbb{S}^n \setminus \mathrm{ort}_1$ from the point ort_1.

Definition 1.1.4.12. A map $f\colon X \to Y$ is an *embedding* or, more specifically, a *topological embedding,* if abr $f\colon X \to f(X)$ is a homeomorphism. For example, the inclusion of a subspace in its ambient space is an embedding.

Retractions

Definition 1.1.4.13. A *retraction* is a continuous map of a space X onto a subspace A which is the identity map on A. A subset onto which a space can be retracted is a *retract* of the space.

Each point of a topological space is a retract of this space. However, a pair of points is already not necessarily a retract. For example, a segment cannot be retracted onto its boundary since any such retraction would be a real continuous function taking two values but no intermediate ones.

Theorem 1.1.4.14. *A subspace A of a topological space X is a retract of X if and only if every continuous map $A \to Y$ can be extended to a continuous map $X \to Y$, for any topological space Y.*

Proof. If $\rho\colon X \to A$ is a retraction and $f\colon A \to Y$ is continuous, then the composition $f \circ \rho$ extends f to X.

$$\begin{tikzcd}
X \arrow[d, "\rho"'] \arrow[dr, dotted, "f\circ\rho"] & \\
A \arrow[r, "f"'] & Y
\end{tikzcd}$$

If every continuous map $A \to Y$ extends to a continuous map $X \to Y$, then extending the identity map $A \to A$ to a continuous map $X \to A$ yields a retraction.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\subset"'] & A \\
X \arrow[ur, "\exists r"'] &
\end{tikzcd}$$

□

Numerical Functions

Remark 1.1.4.15. The well-known theorem of calculus asserting that the arithmetic operations performed upon continuous functions again produce continuous functions is obviously true for the numerical functions defined on an arbitrary topological space. Similarly, the theorem asserting the continuity of the limit of a uniformly convergent sequence of continuous functions holds for numerical function on a topological space.

Theorem 1.1.4.16. *If X is a metric space and A is a subset of X, then the function $X \to \mathbb{R}$, $x \mapsto \mathrm{Dist}(x, A)$, is continuous.*

Proof. Let $x, y \in X$ and $z \in A$. Then

$$\mathrm{Dist}(x, A) \leqq \mathrm{dist}(x, z) \leq \mathrm{dist}(x, y) + \mathrm{dist}(y, z).$$

Hence $\mathrm{Dist}(x, A) \leq \mathrm{dist}(x, y) + \mathrm{Dist}(y, A)$ for any $x, y \in X$, and since x and y appear symmetrically, we obtain $|\mathrm{Dist}(x, A) - \mathrm{Dist}(y, A)| \leq \mathrm{dist}(x, y)$. □

Definition 1.1.4.17. A subset A of a topological space X is said to be *distinguishable* if there is a continuous function $f\colon X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \in X \setminus A$. Any function with this property is said to *distinguish* the set A.

A distinguishable set is obviously closed. It is also clear that any closed subset of a metric space is distinguishable: for example, the function

$$x \mapsto \min(1, \mathrm{Dist}(x, A))$$

distinguishes the closed subset A.

1.1.5 Separation Axioms

Remark 1.1.5.1. In this subsection and the two that follow, we formulate additional restriction which are often imposed on a topological structure in order to bring the properties of the corresponding topological space closer to those that characterize the subsets of the spaces $\mathbb{R}^n$.

Definition 1.1.5.2. More than ten "separation axioms" are known. We need the following four.

T1. Given two arbitrary points a and b, $a \neq b$, there is a neighbourhood of a which does not contain b. Equivalent formulations: each point is a closed set; finite sets are closed.

T2. Two arbitrary distinct points have disjoint neighbourhoods.

T3. Any point and any closed set not containing this point have disjoint neighbourhoods. An equivalent formulation: every neighbourhood of an arbitrary point contains the closure of a neighbourhood of this point.

T4. Any two disjoint closed sets have disjoint neighbourhoods. An equivalent formulation: every neighbourhood of an arbitrary closed set contains the closure of a neighbourhood of this set. Another equivalent formulation: given an arbitrary finite collection of pairwise disjoint closed sets, there are neighbourhoods of these sets with pairwise disjoint closures.

Definition 1.1.5.3. Axiom T_1 is a consequence of T_2, but simple examples show that it is not a consequence of T_3 or zT_4. Spaces which satisfy axiom T_2 are called *Hausdorff*, those which satisfy the axioms T_1 and T_3 - *regular*, and those which satisfy the axioms T_1 and T_4 - *normal*.

Every normal space is regular, and every regular space is Hausdorff.

Obviously, every subspace of a Hausdorff space is Hausdorff, every subspace of a regular space is regular, and every closed subspace of a normal space is normal.

Information 1.1.5.4. A non-closed subspace of a normal space is not necessarily normal; see [11].

Theorem 1.1.5.5. *Every retract of a Hausdorff space is closed.*

Proof. Let A be a retract of a topological space X and let $\rho\colon X \to A$ be a retraction. If $b \in X \setminus A$, then since X is Hausdorff and ρ is continuous, the points b and $\rho(b)$ have disjoint neighbourhoods, U and V, such that $\rho(U) \subset V$. This implies that $\rho(x) \neq x$ for $x \in U$, i.e., $U \cap A = \varnothing$. Thus any point which is not contained in A is an exterior point for A. □

Remark by the transcriber:
The following proof may be more comprehensible. First, we need a lemma.

Lemma 1.1.5.6. *Let X be a topological space. Then X is Hausdorff if and only if the diagonal of $X \times X$: $\Delta = \{(x, x) \in X \times X | x \in X\}$ is closed in $X \times X$.*

Proof. Let X be Hausdorff, then if $x \neq y$ there are neighbourhoods V_x and V_y such that $V_x \cap V_y = \varnothing$. Therefore $V_x \times V_y \cap \Delta = \varnothing$ and thus the complement of Δ is open.

Now, assume that Δ is closed in $X \times X$. Then, for any point (x, y), $x \neq y$, there is an open set around it that does not intersect Δ. Therefore, there are two sets $x \in V_x$ and $y \in V_y$ such that $V_x \times V_y$ doesn't intersect Δ, hence $V_x \cap V_y = \varnothing$. □

Now we return to

Proof. Let $f\colon X \to X \times X$ by $f(x) = (r(x), x)$ where $r\colon X \to A$ is the retraction. Since each "coordinate" is continuous, f is a continuous map (notice that the first map is just the composition $i \circ r$, where i inclusion of A in X). Since X is Hausdorff, the diagonal $\Delta = \{(x, y) \in X \times X | x = y\} \subset X \times X$ is closed. Hence, by continuity of f,

$$f^{-1}(\Delta) = \{x \in X | f(x) \in \Delta\} = \{x \in X | r(x) = x\} = \{x \in X | x \in A\} = A$$

is closed. □

Theorem 1.1.5.7. *Every metric space is normal.*

Proof. Clearly, every metric space satisfies axiom T_1. Let us verify T_4. Suppose A and B are disjoint closed subsets of a metric space, and set

$$U = \{x | \operatorname{Dist}(x, A) < \operatorname{Dist}(x, B)\}, \quad V = \{x | \operatorname{Dist}(x, B) < \operatorname{Dist}(x, A)\}.$$

Since $\operatorname{Dist}(x, A)$ and $\operatorname{Dist}(x, B)$ depend continuously on x (see Theorem 1.1.4.16), U and V are open. Trivially, $U \cap V = \varnothing$, $A \subset U$, and $B \subset V$. □

Urysohn Functions

Lemma 1.1.5.8. *Let A and B be two closed subsets of a topological space X. Let Γ be the collection of all neighbourhoods of A which do not intersect B, and let Δ be the set of dyadic rational numbers of the interval I (i.e., the numbers $m/2^q$ with arbitrary non-negative integers m, q satisfying $m \leq 2^q$). If X is normal, then there exists a mapping $\varphi\colon \Delta \to \Gamma$ such that*

$$\operatorname{Cl} \varphi(r_1) \subset \varphi(r_2) \quad \text{for} \quad r_1 < r_2. \tag{1.1.5.9}$$

Proof. Set $\varphi(1) = X \setminus B$ and let $\varphi(0)$ be any neighbourhood of A which is contained, together with its closure, in $X \setminus B$ (see the second formulation of axiom). If $\varphi(r)$ is already defined in such a way that (1.1.5.9) holds for the numbers $r = m/2^q \in \Delta - A$ with $q = n$, we can extend the definition to

$$r = m/2^{n+1} \in A,$$

keeping (1.1.5.9) valid: if $r = m/2^{n+1} \in A$ with odd $m = 2k+1$, we take $\varphi(r)$ to be any open set containing $\operatorname{Cl} \varphi(k/2^n)$ and contained along with its closure in $\varphi((k+1)/2^n)$. This induction yields a mapping $\varphi\colon \Delta \to \Gamma$ with property (1.1.5.9) . □

Theorem 1.1.5.10. *Given two arbitrary disjoint closed subsets A and B of a normal space X, there is a continuous function $X \to I$, equal to 0 on A and equal to 1 on B.*

Proof. Using Lemma 1.1.5.8 and its notations, define a function $f\colon X \to I$ by the formula

$$f(x) = \begin{cases} \inf\{r | \varphi(r) \ni x\}, & \text{if} \quad x \in \varphi(1), \\ 1, & \text{if} \quad x \in X \setminus \varphi(1). \end{cases}$$

It is evident that f is equal to 0 on A and to 1 on B. To show that f is continuous, note that the intervals $[0, r)$ and $(r, 1]$ with $r \in A$ constitute a prebase of the segment I, and that $f^{-1}([0, r)) = \cap_{r'<r} \varphi(r')$, while $f^{-1}([0, r]) = \cap_{r'>r} \varphi(r')$. Using property (1.1.5.9), we see that the last intersection is just $\cap_{r'>r} \operatorname{Cl} \varphi(r')$; hence intervals $[0, r)$ and $(r, 1]$ have open pre-images, and f is continuous. □

Definition 1.1.5.11. A continuous function $f\colon X \to I$ such that $f(x) = 0$ for $x \in A \subset X$ and $f(x) = 1$ for $x \in B \subset X$ is referred to as a *Urysohn function for the pair* A, B.

A Urysohn function for a pair A, B may also take the value 0 outside of A. However, if A is distinguishable, f is any Urysohn function for the pair A, B, and g distinguishes A, then $x \mapsto \min(f(x) + g(x), 1)$ provides a Urysohn function for A, B which is positive outside A.

One may note that the proof of Theorem 1.1.5.10 does not use axiom T_4 and conclude that this theorem is true for any T_4-space. The converse is also true: if any pair of disjoint closed subsets of X admits a Urysohn function, then X satisfies axiom T_4.

Finally, we remark that one can prove a slightly stronger version of Theorem 1.1.5.10, by composing the Urysohn function with a linear transformation

$$t \mapsto a + (b-a)t,$$

where a and b are arbitrary real numbers. The composition is a continuous function $X \mapsto [a,b]$, equal to a on A and to b on B.

Extension Theorems

Lemma 1.1.5.12. *Let F be a closed subset of the topological space X, and let $\varphi\colon F \to \mathbb{R}$ be a continuous function bounded in absolute value by a number $L > 0$. If X is normal, then there exists a continuous function $\psi\colon X \to \mathbb{R}$ such that*

$$\begin{cases} |\psi(x)| \le L/3 & \textit{for} \quad x \in X, \\ \textit{and} & \\ |\psi(x) - \varphi(x)| \le 2L/3 & \textit{for} \quad x \in F. \end{cases} \tag{1.1.5.13}$$

Proof. The subsets of F determined by the inequalities $\varphi(x) \le -L/3$ and $\varphi(x) \ge L/3$ are closed in F, and hence in X, and disjoint. Therefore, there is a continuous function $\psi\colon X \to [-L/3, L/3]$, equal to $-L/3$ on the first set, and equal to $L/3$ on the second (see Definition 1.1.5.11 and the comment below). It is clear that ψ satisfies the requirements (1.1.5.11) . ☐

Theorem 1.1.5.14. *If A is a closed subset of the normal space X, then every continuous function $A \to \mathbb{R}$ extends to a continuous function $X \to \mathbb{R}$. This claim remains true if one takes an interval instead of the real line $\mathbb{R}$.*

Proof. First, let us show that every continuous map f of A into an interval can be extended to a continuous map g of X into the same interval. Without loss of generality, we may take the interval $[-1,1]$. Define g as the sum of a series of continuous functions $g_k\colon X \to \mathbb{R}$ which satisfy the conditions

$$|g_k(x)| \le 2^{k-1}/3^k, \quad \text{if} \quad x \in X \tag{1.1.5.15}$$

and

$$|f(x) - \sum_0^k g_i(x)| \le (2/3)^k, \quad \text{if} \quad x \in A \tag{1.1.5.16}$$

The functions g_k are constructed inductively: take $g_0 = 0$ and, assuming that $g_0, \dots, g_n$ are already constructed and satisfy (**??**) and (1.1.5.16) for $k \le n$, define g_{n+1} to be the function obtained when one applies the previous lemma to $\varphi = f - \sum_0^n (g_i|_A)$, $F = A$, and $L = (2/3)^n$. Inequality (**??**) shows that the series $\sum_0^\infty g_k$ converges uniformly on X, and hence its sum g is a continuous function (see Remark 1.1.2.6). Inequality (1.1.5.16) implies that $g|_A = f$.

To prove the first part of the theorem, notice that $\mathbb{R}$ is homeomorphic to the open interval $(-1, 1)$. We have just showed that given any continuous function $g\colon A \to (-1, 1)$, there exists a continuous $g\colon X \to [-1, 1]$ such that $g(x) = f(x)$ for all $x \in A$. Let $B = g^{-1}(-1) \cup g^{-1}(1)$. The sets A and B are closed and disjoint; hence the pair A, B has a Urysohn function. If we multiply the latter by g, we get the desired extension $X \to (-1, 1)$ of the function f. ☐

Theorem 1.1.5.17. *If A is a closed subset of the normal space X, then every continuous map $A \to \mathbb{R}^n$ extends to a continuous map $X \to \mathbb{R}^n$. This claim remains true if one takes a cube instead of $\mathbb{R}^n$.*

To see this, it suffices to apply Theorem 1.1.5.14 to the coordinate functions of the given map $A \to \mathbb{R}^n$.

1.1.6 Countability Axioms

Definition 1.1.6.1. A topological space is said to satisfy the *second axiom of countability* (or to be a *second countable space*) if it has a countable base. A topological space is said to satisfy the *first axiom of countability* (or to be a *first countable space*) if it has a countable base at each point. A topological space is *separable* if it has a countable dense subset.

Theorem 1.1.6.2. *The second axiom of countability implies the first axiom of countability and the separability. A metric space is always first countable, and is second countable if and only if it is separable.*

It is immediate that a second countable space is first countable, and Remark 1.1.2.6 shows that every metric space is first countable. To produce a countable dense set in a space with countable base, just pick a point in each set of a given countable base. Given a separable metric space, the open balls centred at the points of a countable dense set and with radii $1/n$ ($n = 1, 2, \dots$) constitute a countable base.

Theorem 1.1.6.3. *$\mathbb{R}^n$ and ℓ_2 are separable, and hence have a countable base.*

Proof. The collection of all sequences $\{x_i\}_1^n$ with rational x_i's is a countable dense set in $\mathbb{R}^n$.
The set of all finitely supported (i.e., having only a finite number of non-zero terms) sequences $\{x_i\}_1^\infty$ with rational x_i's is countable and dense in ℓ_2. ☐

Theorem 1.1.6.4. *Every subspace of a second countable space is second countable. In particular, all subspaces of $\mathbb{R}^n$ and ℓ_2 have countable base.*

Proof. Indeed, a countable base of the space induces a countable base of each of its subspaces; see Theorem 1.1.6.2. ☐

Theorem 1.1.6.5. *In a separable space, every collection of pairwise disjoint open subsets is countable.*

Proof. In fact, let S be a countable dense subset of the given space. For any set in the given collection, pick a point of S contained in this set. This yields an injective mapping of the given collection into S. $\square$

Remark 1.1.6.6. Obviously, a continuous surjective map of topological spaces carries every dense set into a dense set. It is also clear that an open map between two topological spaces transforms each base into a base, and each base at a point into a base at the image of this point. Therefore, the image of a separable space under a continuous map is separable, and the image of a first or a second countable space under an open map is first, respectively second countable.

Theorem 1.1.6.7. *Every regular second countable space is normal.*

Proof. Let A and B be closed disjoint subset of a regular second countable space. According to the second formulation of axiom T_3, each point of any of the sets A and B has a neighbourhood whose closure does not intersect the other set. Picking such neighbourhoods, we get open covers of A and B, and we may assume that these covers are countable; if not, we may refine them by covers made of sets belonging to a countable base (see Remark 1.1.1.13). Let us index these two covers, writing them as $U_1, U_2, \dots$ and $V_1, V_2, \dots$, and then set $U'_n = U_n \setminus \cap_1^n \operatorname{Cl} V_i$ and $V'_n = V \setminus \cup_1^n \operatorname{Cl} U_i$. The sets $U = \cup_1^n U'_n$ and $V = \cup_1^n V'_n$ are open and clearly disjoint. Since $\operatorname{Cl} U_i \cap B = \varnothing$ and $\operatorname{Cl} V_i \cap A = \varnothing$, we have $U \supset A$ and $V \supset B$. $\square$

Embedding and Metrisation Theorems

Theorem 1.1.6.8. *Every regular second countable space can be embedded in* ℓ_2.

Proof. Let X be a regular space with countable base Γ. We index the pairs (U, V), $U, V \in \Gamma$, satisfying $\operatorname{Cl} U \subset V$, writing them as a sequence

$$(U_1, V_1), (U_2, V_2), \dots .$$

Now define $f\colon X \to \ell_2$ by the rule $f(x) = \{k^{-1}\varphi_k(x)\}_1^\infty$, where φ_k is an arbitrary Urysohn function for the pair $\operatorname{Cl} U_k, X \setminus V_k$ (see Definition 1.1.5.11). If $x \neq y$, then there exists an index k such that $x \in U_k$, $y \in X \setminus V_k$ (indeed, X is regular), and so f is injective. We show next that f is an embedding.

Since the φ'_k's are continuous, given $x_0 \in X$, $\varepsilon > 0$, and n, there is a neighbourhood U of x_0 such that $\sum_1^n (|\varphi_k(x) - \varphi_k(x_0)|/k)^2 < \varepsilon^2/2$ for all $x \in U$. Choosing n such that $\sum_{n+1}^\infty k^{-2} < \varepsilon^2/2$, we see that $\operatorname{dist}(f(x_0), f(x)) < \varepsilon$, and so f is continuous.

Let g denote the inverse of $\operatorname{abr} f\colon X \to f(X)$. Given a point $y_0 \in f(X)$ and a neighbourhood U of $g(y_0)$, find n such that $g(y_0) \in_U$n and $V_n \in U$. If $y = f(X)$ and $\operatorname{dist}(y_0, y) < 1/n$, then clearly $|\varphi_n(g(y)) - \varphi_n(y_0))| < 1$, which in turn implies that $g(y) \in V$. In conclusion, for $y \in f(X)$ and $\operatorname{dist}(y_0, y) < 1/n$, we have $g(y) \in U$, proving the continuity of g. $\square$

Theorem 1.1.6.9. *A second countable topological space is metrisable if and only if it is regular.*

Proof. The necessity of this condition is contained in 1.1.5.7, and its sufficiency - in 1.1.6.8. □

1.1.7 Compactness

Definition 1.1.7.1. A topological space is *compact* if any of its open covers contains a finite cover. For example, a finite set endowed with an arbitrary topology is compact, whereas an infinite set endowed with the discrete topology is not compact.

It is clear that a subspace A of a topological space X is compact if only if from each open cover of A in X one can extract a finite cover.

Theorem 1.1.7.2. *Every closed subset of a compact space is compact.*

Proof. Let A be a closed subset of the compact space X, and let Δ be a cover of A in X. We add the set $X \setminus A$ (which is of course open) to Δ, extract a finite cover from the resulting open cover of X, and then delete the set $X \setminus A$ from the latter, if it still remains. This obviously yields a finite cover of A in X which is contained in Δ. □

Theorem 1.1.7.3. *In a Hausdorff space, any two compact disjoint sets have disjoint neighbourhoods.*

Proof. Let A and B denote the given sets. If B is a point, then for each point $x \in A$ consider disjoint neighbourhoods and V of x and B, and extract a finite cover $U_{x_1}, \ldots, U_{x_s}$ from the open cover of A given by the neighbourhoods U_x; then $\cup_1^s U_{x_i}$ and $\cap_1^s V_{x_i}$ are disjoint neighbourhoods of the set A and the point B.

In the general case, pick for each $x \in B$ disjoint neighbourhoods U_x and V_x of A and of x, and then extract a finite cover $V_{x_1}, \ldots, V_{x_s}$ from the resulting cover of B by neighbourhoods V_x; then $\cap_1^s U_{x_i}$ and $\cup_1^s V_{x_i}$ are disjoint neighbourhoods of A and B. □

Theorem 1.1.7.4. *Every compact subset of a Hausdorff space is closed.*

Proof. Indeed, from Theorem 1.1.7.3 we see that a point which is not contained in a given compact subset of a Hausdorff space has a neighbourhood which does not intersect this subset. □

Theorem 1.1.7.5. *Every compact Hausdorff space is normal.*

Proof. This is a consequence of Theorems 1.1.7.3 and 1.1.7.3. □

Compactness and Fundamental Covers

Theorem 1.1.7.6. *Suppose A is a compact subset of a T_1-space X. Then from every countable fundamental cover of X one can extract a finite cover of A.*

Proof. Let $U_1, U_2, \ldots$ be the given cover. If none of the sets $\cup_1^m U_i$ covers A, pick a point from each set $A \setminus \cup_1^m U_i$, and denote the set of all these points by Y. It is obvious that Y is infinite and that each intersection $Y \cap U_i$ is finite. The latter shows that Y and all its subsets are closed; hence Y is at the same time compact (see Proposition 1.1.7.2) and discrete. But this contradicts the fact that Y is infinite. □

Remark 1.1.7.7. The countability assumption in Theorem 1.1.7.6 is essential. For example, the cover of a segment by all its countable subsets is fundamental, but one cannot extract from it a countable cover.

Compactness and Maps

Theorem 1.1.7.8. *The image of a compact space under a continuous map is compact.*

Proof. Let f be a continuous map of the compact space X onto a topological space Y, and let Δ be an open cover of Y. The sets$f(V)$, $V \in f^{-1}(\Delta)$, form an open cover of X, and clearly a sub-cover $U_1, \ldots, U_s$ of this cover yields a sub-cover $f(U_1), \ldots, f(U_s)$ of Δ. □

Theorem 1.1.7.9. *Every continuous map of a compact space into a Hausdorff space is closed.*

Proof. This is a corollary of Propositions 1.1.7.2, 1.1.7.8, and 1.1.7.4. □

Theorem 1.1.7.10. *Every invertible continuous map of a compact space onto a Hausdorff space is a homeomorphism. Every injective continuous map of a compact space into a Hausdorff space is an embedding.*

Proof. These are consequences of Theorem 1.1.7.9 and of the obvious fact that a closed invertible map is a homeomorphism. □

Compactness and Metrics

Theorem 1.1.7.11. *Every compact subset of a metric space can be covered by a finite number of open balls having radius ε, for any positive ε.*

Proof. Such a cover can be extracted from any cover consisting of balls of radius ε. □

Theorem 1.1.7.12. *Every compact metric space has a countable base.*

Proof. To obtain such a base, it suffices to construct, for each positive integer n, a finite cover of open balls of radius $1/n$, and then take the union of these covers. □

Theorem 1.1.7.13. *Every compact metric space is bounded.*

Proof. This is a consequence of 1.1.7.11. ☐

Theorem 1.1.7.14. *Let X be a compact topological space. Then every continuous function $X \to \mathbb{R}$ attains its absolute maximum and absolute minimum.*

Proof. Theorems 1.1.7.8 and 1.1.7.13 show that the image of X in $\mathbb{R}$ is bounded. Theorem 1.1.7.9 shows that this image is closed, which in turn implies that it contains its adherent points, including its greatest lower bound and its least upper bound. ☐

Theorem 1.1.7.15. *Let A and B be disjoint subsets of a metric space. If A is compact and B is closed, then* $\operatorname{Dist}(A, B) > 0$*.*

Proof. Since A is compact and $\operatorname{Dist}(x, B)$ depends continuously on $x \in A$ (see Theorem 1.1.4.16), there exists $a \in A$ such that

$$\operatorname{Dist}(a, B) = \inf_{x \in A} \operatorname{Dist}(x, B) = \operatorname{Dist}(A, B)$$

(see Theorem 1.1.7.14). Since B is closed and $a \notin B$, $\operatorname{Dist}(a, B) > 0$, and thus $\operatorname{Dist}(A, B) > 0$. ☐

Theorem 1.1.7.16. *Suppose that f is a continuous map of a metric space X into a topological space Y and Δ is an open cover of Y. If X is compact, then there is $\varepsilon > 0$ such that for any set $A \subset X$ with diameter* $\operatorname{diam} A < \varepsilon$*, $f(A)$ is contained in some element of Δ.*

Proof. It is enough to show that there is an $\varepsilon > 0$ such that any two points $x, y \in X$ with $\operatorname{dist}(x, y) < \varepsilon$ are both contained in one of the sets of the cover $\Gamma = f^{-1}(\Delta)$. For each $x \in X$, pick a ball centred at x and contained in one of the sets of Γ, and let U_x be the concentric ball with half the radius. Now extract a finite cover $U_{x_1}, \dots, U_{x_s}$ from the cover of X by the balls U_x. Let ε_i denote the radius of U_{x_i}, and let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_s\}$. If $x, y \in X$ and $\operatorname{dist}(x, y) < \varepsilon$, then $\operatorname{dist}(x_i, x) < \varepsilon_i$ and $\operatorname{dist}(x_i, y) < \operatorname{dist}(x_i, x) + \operatorname{dist}(x, y) < 2\varepsilon_i$ for some i. Therefore, x and y belong to one and the same set of the cover Γ. ☐

Compactness in Euclidean Space

Theorem 1.1.7.17. *The cubes of $\mathbb{R}^n$ are compact.*

Proof. Obviously, any cube in $\mathbb{R}^n$ can be divided into 2^n cubes of half the edge, and if some cover Γ of the original cube by open subsets of $\mathbb{R}^n$ does not contain a finite sub-cover, then it retains the same property as a cover of one of the smaller cubes. An iteration of this argument yields a decreasing sequence of cubes $Q_1, Q_2, \dots$, each of them being half the size of the preceding one, and such that none of them is covered by a finite collection of sets from Γ. But the point common to all these cubes is certainly covered by some set from Γ, which must also cover all the cubes Q_k with k large enough. ☐

Theorem 1.1.7.18. *A subset of* $\mathbb{R}^n$ *is compact if and only if it is bounded and closed.*

Proof. The necessity of these conditions is implied in Theorems 1.1.7.13 and 1.1.7.4. The sufficiency is a consequence of Propositions 1.1.7.17 and 1.1.7.2, since any bounded subset of $\mathbb{R}^n$ is contained in some cube. ☐

Local Compactness

Definition 1.1.7.19. A topological space is *locally compact* if each of its points has a neighbourhood with compact closure.

Compact spaces are obviously locally compact. The most important examples of non-compact, locally compact spaces are $\mathbb{R}^n$ with $n > 0$.

Theorem 1.1.7.20. *Every closed subset of a locally compact space is locally compact.*

Proof. Indeed, if a is a point of a closed subset A of the locally compact space X, and U is a neighbourhood of a in X with compact closure $\operatorname{Cl}_X U$, then $U \cap A$ is a neighbourhood of a in A with compact closure $\operatorname{Cl}_A(U \cap A)$. [$\operatorname{Cl}_A(U \cap A)$, being closed in X, is closed in the compact subset $\operatorname{Cl}_X U$ of X, and hence is compact; see Theorem 1.1.7.2]. ☐

Theorem 1.1.7.21. *Every open subset of a locally compact Hausdorff space is locally compact.*

Proof. Let a be a point of the open subset A of the locally compact space X, and let U be a neighbourhood of a in X with compact closure $\operatorname{Cl}_X U$. Since the space $\operatorname{Cl}_X U$ is regular (see Proposition 1.1.7.5), a has a neighbourhood V in $\operatorname{Cl}_X U$ such that $\operatorname{Cl}_{\operatorname{Cl}_X U} V \subset U \cap A$. We show that V is a neighbourhood of a in A with compact closure $\operatorname{Cl}_A V$.

The set V is open in $\operatorname{Cl}_X U$, hence in $U \cap A$, which in turn implies that V is open in A. To verify that $\operatorname{Cl}_A V$ is compact, note that the closure $\operatorname{Cl}_{\operatorname{Cl}_X U} V$ is compact (see Proposition 1.1.7.2) and contained in $U \cap A$. This implies that it equals $\operatorname{Cl}_{U \cap A} V$ and that the latter is closed in A (see Proposition 1.1.7.4), which finally shows that $\operatorname{Cl}_{U \cap A} V = \operatorname{Cl}_A V$. ☐

Theorem 1.1.7.22. *Let U be a neighbourhood of the point a of the locally compact space X. If X is Hausdorff, then a has a neighbourhood whose closure is compact and contained in U .*

Proof. Since U is a locally compact space (see Theorem 1.1.7.21), a has a neighbourhood V in U with compact closure. Since U is open, V is open in U. Finally, since $\operatorname{Cl}_U V$ is compact and X is Hausdorff, $\operatorname{Cl}_U V$ is closed in X (see Theorem 1.1.7.4) and thus it coincides with c_XV. We see that V is the desired neighbourhood of a. ☐

Theorem 1.1.7.23. *Locally compact Hausdorff spaces are regular.*

Proof. This is a consequence of Theorem 1.1.7.22. ☐

Information: Paracompactness

Definition 1.1.7.24. A Hausdorff space is *paracompact* if each of its open covers has a locally finite refinement. The compact Hausdorff spaces are (obviously) paracompact, and so are all the metric spaces. All paracompact spaces are normal. For details, see [11].

One can show that a paracompact space which can be covered by open metrisable sets is metrisable; see [13].

1.2 CONSTRUCTIONS

1.2.1 Sums

The sum $\coprod_{\mu \in M} X_\mu$ of a family $\{x_\mu\}$ of topological spaces becomes a topological space if we declare a subset to be open if its pre-images under all maps $\mathrm{incl}_\nu \colon X_\nu \to \coprod X_\mu$ are open. Equivalently, a subset of $\coprod X_\mu$ is closed if its pre-images under all maps incl are closed. It is evident that each map $\mathrm{incl}_\nu \colon X_\nu \to \coprod_\mu$ is an embedding and that the images $\mathrm{incl}_\nu(X_\nu)$ are both open and closed in $\coprod X_\mu$.

Let $\{Y_\mu\}$ be another family of topological spaces, indexed by the same set M, and let $f_\mu \colon X_\mu \to Y_\mu$ be continuous maps. Then the map $\coprod f_\mu \colon \coprod X_\mu \to \coprod Y_\mu$ is obviously continuous.

Remark 1.2.1.1. If all spaces X_μ satisfy one of the axioms T_1, T_2, T_3, or T_4, then their sum satisfies the same axiom. The same hold for the first axiom of countability, and also for the properties of local compactness and metrisability if X_μ are metric spaces, one can define a metric on $\coprod X_\mu$ by the formulae:

$$\mathrm{dist}(\mathrm{incl}_\nu(x), \mathrm{incl}_{\nu'}(x')) = \begin{cases} 1 & \text{if} \quad \nu \neq \nu' \\ & \text{or if} \quad \nu = \nu' \quad \text{and} \quad \mathrm{dist}(x, x') \geq 1; \\ \mathrm{dist}(x, x') & \text{if} \quad \mathrm{dist}(x, x') < 1. \end{cases}$$

If each X_μ has a countable base and M is countable, then $\coprod X_\mu$ has a countable base too. Similarly, when M is countable and each X_μ is separable, $\coprod X_\mu$ is separable too. Finally, $\coprod X_\mu$ is compact whenever all X_μ are compact and M is finite.

1.2.2 Products

Remark 1.2.2.1. Let $X_1, \dots, X_n$ be topological spaces. We define a topology on $X_1 \times \cdots \times X_n$ by taking as a base the collection of all sets

$$U_1 \times \cdots \times U_n \subset X_1 \times \cdots \times x_n,$$

where U_i is open in X_i, $i = 1, \dots, n$. The conditions of Proposition 1.1.1.8 are satisfied by virtue of the relation

$$(U_1' \times \cdots \times U_n') \cap (U_1'' \times \cdots \times U_n'') = (U_1' \cap U_1'') \times \cdots \times (U_n' \cap U_n'').$$

The resulting topological space is called the *product of the spaces* $X_1 \times \cdots \times X_n$.

If $A_1, \dots, A_n$ are subspaces of $X_1 \times \cdots \times X_n$ then the topology of the product $A_1 \times \cdots \times A_n$ is obviously identical with the topology induced by the inclusion $A_1 \times \cdots \times A_n \subset X_1 \times \cdots \times X_n$.

Actually, we have met with some products already. Indeed, $\mathbb{R}^n$ is the product of n copies of the real line $\mathbb{R}$, while I^n is the product of n copies of the unit segment I.

The product $X \times I$, where X is a topological space, is known as the *cylinder over* X.

Remark 1.2.2.2. Forming the product of spaces is a commutative and associative operation: there are obvious canonical homeomorphisms

$$\begin{aligned}&X_1 \times X_2 \to X_2 \times X_1\\&(X_1 \times X_2) \times X_3 \to X_1 \times (X_2 \times X_3),\\&(X_1 \times \cdots \times X_{n-1}) \times X_n = X_1 \times \cdots \times X_n.\end{aligned}$$

Moreover, sums and products of spaces satisfy the distributive law: there is a canonical homeomorphism $X \times (\coprod_{\mu \in M} Y_\mu) \to \coprod_{\mu \in M}(X \times Y_\mu)$.

Theorem 1.2.2.3. *If the sets $A_1, \ldots, A_n$ are open in $X_1, \ldots, X_n$, then*

$$A = A_1 \times \cdots \times A_n$$

is open in $X_1 \times \cdots \times X_n$. If $A_1, \ldots, A_n$ are closed, then A is closed. In all cases, $\operatorname{Cl} A = \operatorname{Cl} A_1 \times \cdots \times \operatorname{Cl} A_n$.

Proof. The first statement is a direct result of the definition of the topology of $X_1 \times \cdots \times X_n$, while the second is a consequence of the third; so let us verify the third statement. A point $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ is an adherent point of A if and only if each of its base neighbourhoods $U_1 \times \cdots \times U_n$ has a non-empty intersection with A, i.e., if and only if for any neighbourhoods $U_1, \ldots, U_n$ of the points $x_1, \ldots, x_n$, $U_i \cap A_i \neq \varnothing$, $i = 1, \ldots, n$. That is to say, $(x_1, \ldots, x_n)$ is an adherent point of A if and only if x_i is an adherent point of A_i, $i = 1, \ldots, n$. Therefore, $\operatorname{Cl} A = \operatorname{Cl} A_1 \times \cdots \times \operatorname{Cl} A_n$. □

Remark 1.2.2.4. We note that the projections $\operatorname{proj}_i c\colon X_1 \times \cdots \times X_n \to X_i$ are continuous and open for any topological spaces $X_1, \ldots, X_n$.

The sets of the form

$$x_1^0 \times \cdots \times x_{i-1}^0 \times X_i \times x_{i+1}^0 \times \cdots \times x_n^0, \quad x_j^0 \in X_j \quad (j \neq i)$$

are called the *fibres* of the product $X_1 \times \cdots \times X_n$. Clearly, the restriction of $\operatorname{proj}_i\colon X_1 \times \cdots \times X_n \to X_i$ to any fibre $x_1^0 \times \cdots \times x_{i-1}^0 \times X_i \times x_{i+1}^0 \times \cdots \times x_n^0$ is a homeomorphism. Hence the fibres are canonically homeomorphic to the factors of the product.

For any map $f\colon Y \to X_1 \times \cdots \times X_n$, where $Y, X_1, \ldots, X_n$ are arbitrary sets, we have the corresponding maps $\operatorname{proj}_i \circ f\colon Y \to X_i$. Conversely, given arbitrary $f\colon Y \to X_i$, there is a unique map $f\colon Y \to X_1 \times \cdots \times X_n$ such that $\operatorname{proj}_i \circ f = f_i$. Clearly, if $Y, X_1, \ldots, X_n$ are topological spaces, then f is continuous if and only if all the maps $\operatorname{proj}_i \circ f$ are continuous.

In particular, it follows that the map $\operatorname{diag}\colon X \to X \times X$ is continuous for every topological space X. We make the (obvious) remark that the diagonal $\operatorname{diag}(X)$ is closed if and only if X is Hausdorff (see Lemma 1.1.5.6).

Remark 1.2.2.5. Obviously, every product

$$f_1 \times \cdots \times f_n\colon X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n$$

of continuous maps $f_1\colon X_1 \to Y_1, \ldots, f_n\colon X_n \to Y_n$ is continuous. Moreover, $f_1 \times \cdots \times f_n$ is open whenever $f_1, \ldots, f_n$ are open.

Remark 1.2.2.6. If X is a metric space, then for any points $x_1, x_2, x'_1, x'_2 \in X$

$$|\operatorname{dist}(x'_1, x'_2) - \operatorname{dist}(x_1, x_2)| \leq \operatorname{dist}(x'_1, x_1) + \operatorname{dist}(x'_2, x_2)$$

clearly holds. This inequality shows that the function dist: $X \times X \to \mathbb{R}$ is continuous.

Properties of Products

Theorem 1.2.2.7. *Every product of* T_1*-spaces is a* T_1*-space. Every product of Hausdorff spaces is Hausdorff. Every product of regular spaces is regular.*

Proof. The first and second assertions are immediate. We show that a product of T_3-spaces $X_1, \ldots, X_n$ is a T_3-space. Let U be a neighbourhood of $(x_1, \ldots, xn) \in X_1 \times \cdots \times X_n$. Pick neighbourhoods $U_1 \times \cdots \times U_n$ of the points $x_1, \ldots, x_n$, such that $U_1 \times \cdots \times U_n \subset U$, and fix neighbourhoods $V_1, \ldots, V_n$ of the same points with $\operatorname{Cl} V_1 \subset U_1, \ldots \operatorname{Cl} V_n \subset U_n$. Since $\operatorname{Cl}(V_1 \times \times V_n) = \operatorname{Cl} V_1 \times \cdots \times \operatorname{Cl} V_n$ (see Theorem 1.2.2.3), one has $\operatorname{Cl}(V_1 \times \times V_n) \subset U$. □

INFORMATION. There are product of normal spaces which are not normal; see [14].

Remark 1.2.2.8. If $S_1, \ldots, S_n$ are dense sets in the spaces $X_1, \ldots, X_n$, then $S_1 \times \cdots \times S_n$ is obviously dense in $X_1 \times \cdots \times X_n$. Consequently, *a product of separable spaces is separable.*

If $\Gamma_1, \ldots, \Gamma_n$ are bases of $X_1, \ldots, X_n$ then the sets $U_1 \times \cdots \times U_n$ with

$$U_1 \in \Gamma_1, \ldots, U_n \in \Gamma n$$

form a base of the space $X_1 \times \cdots \times X_n$. Consequently, *a product of second countable spaces is second countable.*

If now $\Gamma_1, \ldots, \Gamma_n$ are bases of $X_1, \ldots, X_n$ at the points $(x_1, \ldots, x_n)$, then the sets $U_1 \times \cdots \times U_n$ with $U_1 \in \Gamma_1, \ldots, U_n \in \Gamma_n$ form a base of $X_1 \times \cdots \times X_n$ at the point $(x_1, \ldots, x_n)$. Consequently, *a product of first countable spaces is first countable.*

Theorem 1.2.2.9. *Every product of metrisable spaces is metrisable.*

Proof. In fact, we can say more: if $X_1, \ldots, X_n$ are metric spaces, then the formula

$$\operatorname{dist}((x_1, \ldots, x_n), (x'_1, \ldots, x'_n)) = [\sum_{i=1}^{n} (\operatorname{dist}(x_i, x'_i))^2]^{1/2}$$

defines a canonical metric on the product $X_1 \times \cdots \times X_n$. □

Theorem 1.2.2.10. *Every product of compact spaces is compact.*

Proof. It suffices to consider a product of two spaces. So let X and Y be compact topological spaces, and let Γ be an open cover of $X \times Y$. Consider an arbitrary refinement Δ of Γ, consisting of open sets of the form $U \times V$ (see Remark

1.1.1.13). Since the fibres $x \times Y$ are homeomorphic to Y and Y is compact, one can find, for each point $x \in X$, a finite collection $\Delta_x = \{U_i(x) \times V_i(x)\}_{i=1}^{n(x)}$ of elements of Δ which covers $X \times Y$ (see Theorem 1.1.7.2); one may assume that $x \in U_i(x)$ for all $i = 1, \dots, n(x)$. Since the sets $U_x = \cap_{i=1}^{n(x)}$ are open and cover the compact space X, there exists a finite collection $U_{x_1}, \dots, U_{x_m}$ covering X. It is clear that $\Delta' = \cup_{j=1}^{m} \Delta_{x_j}$ is a cover of $X \times Y$. Finally, replacing each set $W \in \Delta'$ by a set of Γ containing W, we produce a finite subcover of Γ. □

Theorem 1.2.2.11. *Every product of locally compact spaces is locally compact.*

Proof. Let $U_1, \dots, U_n$ be neighbourhoods of the points $x_1 \in X_1, \dots, x_n \in X_n$. Then $U_1 \times \cdots \times U_n$ is a neighbourhood of $(x_1, \dots, x_n)$ in $X_1 \times \cdots \times X_n$. Furthermore, its closure $\mathrm{Cl}(U_1 \times \cdots \times U_n)$ is just $\mathrm{Cl}\, U_1 \times \cdots \times \mathrm{Cl}\, U_n$ (see Theorem 1.2.2.3), and so is compact whenever $\mathrm{Cl}\, U_1, \dots, \mathrm{Cl}\, U_n$ are compact (see Theorem 1.2.2.10). □

An Application: A Method for Constructing Continuous Maps

Remark 1.2.2.12. Theorem 1.2.2.14 below allows us to establish continuity a map in some situations similar to those treated by Theorem 1.1.4.3, but where the latter is not applicable.

Lemma 1.2.2.13. *Suppose that the map $f\colon X \times Q \to Z$ is continuous and transforms the fibre $x_0 \times Q$ into a point. If the space Q is compact, then given any neighbourhood W of the point $f(x_0 \times Q)$, there is a neighbourhood U of x_0 such that $f(U \times Q) \subset W$.*

Proof. Given any point $q \in Q$, fix a neighbourhood of U_q and a neighbourhood V_q of q with $f(U_q \times V_q) \subset W$. Since Q is compact, one can cover it with a finite collection $V_{q_1}, \dots, V_{q_s}$. Now set $U = \cap_{i=1}^{s} U_{q_i}$. □

Theorem 1.2.2.14. *Suppose that X, Y, Z and Q are topological spaces, A is a subset of X, B is a closed subset of Y, and $f\colon X \times Q \to Z$ and $g\colon Y \to X$ are continuous maps such that $f(x \times Q)$ reduces to a point for each $x \in A$, and $g(B) \subset A$. If Q is compact, then for each continuous map $\varphi/colon Y \setminus B \to Q$, the map $h\colon Y \to Z$ given by*

$$h(y) = \begin{cases} f(g(y)), \varphi(y)), & \text{for} \quad y \in Y \setminus B, \\ f(g(y) \times Q), & \text{for} \quad y \in B, \end{cases}$$

is continuous.

Proof. The map h is clearly continuous at the points of $Y \setminus B$; let us verify its continuity at the points $y \in B$. By virtue of Lemma 1.2.2.13, given any neighbourhood W of the point $h(y)$, there is a neighbourhood U of the point $g(y)$ such that $f(U \times Q) \subset W$. The last inclusion shows that $h(g^{-1}(U)) \subset W$, and finally note that $g^{-1}(U)$ is a neighbourhood of y. □

Information

Information 1.2.2.15. The notion of a product of an infinite number of topological spaces can be defined in a natural way; in this case too a product of compact spaces remains compact; see [3] for details.

1.2.3 Quotients

Definition 1.2.3.1. The quotient set $X/\wp$ of a topological space X by any of its partitions $\wp$ is equipped with a natural topology: a subset of $X/\wp$ is open if its pre-image under the map $\operatorname{proj}\colon X \to X/\rho$ is open. Equivalently, a subset of $X/\wp$ is closed if its pre-image is closed. This topology is called the *quotient topology*, and the set $X/\wp$ with the quotient topology is the *quotient space of the space X by its partition $\wp$*.

It is clear that $\operatorname{proj}\colon X \to X/\wp$ is continuous.

In the special case of a partition $\wp$ whose elements are a single set A and the points of $X \setminus A$, $X/\wp$ is called the *quotient of the space X by A* and is denoted by X/A.

Remark 1.2.3.2. Given two topological spaces X and Y with respective partitions $\wp$ and q, and a continuous map $f\colon X \to Y$ which takes the elements of $\wp$ into elements of q, the map $\operatorname{fact} f\colon X/\wp \to Y/q$ is continuous. This is a straightforward consequence of the definition of the quotient topology. Indeed, if $U \to Y/q$ is open, then the set $f^{-1}(\operatorname{proj}^{-1}(U))$ is open in X, and so the identity $f^{-1}(\operatorname{proj}^{-1}(U)) = \operatorname{proj}^{-1}((\operatorname{fact} f)^{-1}(U))$ implies that $(\operatorname{fact} f)^{-1}(U)$ is open in $X/\wp$.

If q is the partition of Y into single points, then $Y/q = Y$ and $\operatorname{proj}\colon Y \to Y/q$ is the identity map. In this case, $f \rightarrowtail \operatorname{fact} f$ defines a one-to-one correspondence between continuous maps $X \to Y$ which are constant on the elements of the partition $\wp$, and continuous maps $X/\wp \to Y$.

Remark 1.2.3.3. In particular, the discussion above shows that given a continuous map $f\colon X \to Y$, its injective factor $\operatorname{fact} f\colon X/\operatorname{zer}(f) \to Y$ is continuous too. The converse is also true: every map $f\colon X \to Y$ can be represented as the composition $X \xrightarrow{\operatorname{proj}} X/\operatorname{zer}(f) \xrightarrow{\operatorname{fact} f} Y$, and so f is continuous whenever $\operatorname{fact} f$ is continuous.

Remark 1.2.3.4. A continuous map whose injective factor is a homeomorphism will be referred to as a *factorial map* (or a *quotient map*).

An equivalent definition: a map f of a topological space X into a topological space Y is factorial if $f(X) = Y$, and the preimage $f^{-1}(B)$ of a set $B \subset Y$ is open if and only if B is open. If we substitute closed sets for open ones, we obtain another equivalent definition.

Obviously, the composition of two factorial maps is factorial, and any injective factorial map is a homeomorphism. Moreover, it is plain that if f is factorial and the composition $g \circ f$ is continuous, then the map g is continuous. Also, f continuous and $g \circ f$ factorial imply g factorial.

The projections onto quotient spaces form the main class of factorial maps. A crude necessary condition for a map $f\colon X \to Y$ with $f(X) = Y$ to be factorial is that f be an open or a closed map.

Remark 1.2.3.5. Taking quotients is a transitive operation: if $\wp$ is a partition of X and $\wp'$ is a partition of $X/\wp$, then the quotient space $(X/\wp)/\wp'$ is canonically homeomorphic to X/q , where q partitions X into the pre-images of the elements of $\wp'$ under the projection $X \to X/\wp$. This canonical homeomorphism is defined as the injective factor of the composite map $X \to X/\wp \to (X/\wp)/\wp'$, and is truly a homeomorphism, because this composition is factorial (see Remark 1.2.3.4).

Theorem 1.2.3.6. *If the sets A and B constitute a fundamental cover of the space X , then*

$$\text{fact}[\text{incl}\colon A \to X]\colon A/A \cap B \to X/B$$

is a homeomorphism.

Proof. Given an open subset U of the quotient $A/A \cap B$, it is enough to show that $V = [\text{proj}\colon X \to X/B](\text{fact}\,\text{incl}(U))$ is open in X. But this is a consequence of the equalities

$$V \cap A = [\text{proj}\colon A \to A/A \cap B]^{-1}(U)$$

and

$$V \cap B = \begin{cases} B, & \text{if} \quad \text{proj}(A \cap B) \in U, \\ \varnothing, & \text{if} \quad \text{proj}(A \cap B) \notin U. \end{cases}$$

□

Properties of Quotients

Remark 1.2.3.7. Obviously, a quotient space $X/\wp$ satisfies axiom T_1 if and only if the elements of the partition $\wp$ are closed. Also, $X/\wp$ is Hausdorff if and only if any two distinct elements of $\wp$ have disjoint saturated neighbourhoods. Similarly, $X/\wp$ is a T_3-space (T_4-space) if and only if for any element A of $\wp$ and any saturated closed subset B of X (respectively, for any saturated, closed subsets A and B of X) such that $A \cap B = \varnothing$, A and B have disjoint saturated neighbourhoods in X.

Moreover, it is readily seen that $X/\wp$ is second countable if and only if there is a countable collection of open saturated sets in X such that any saturated set can be expressed as the union of one of its subcollections.

It is immediate from Remark 1.1.6.6 that a quotient of a separable space is separable.

Similarly, Theorem 1.1.7.8 implies that a quotient of a compact space is compact.

Closed Partitions

Definition 1.2.3.8. A partition $\wp$ of the space X is *closed* if $\text{proj}\colon X \to X/\wp$ is a closed map. An equivalent condition: saturations of closed sets are closed.

Obviously, a partition which has only one element that is not reduced to a point is closed if and only if this element is closed.

Theorem 1.2.3.9. *The quotient of a* T_1*-space by a closed partition is a* T_1*-space. The quotient of a normal space by a closed partition is normal.*

Proof. Since the first assertion is straightforward, all we have to show is that the quotient $X/\wp$ of a T_4-space X by a closed partition $\wp$ is a T_4-space. Let F_1 and F_2 be disjoint, saturated, closed subsets of X. Since X is normal, F_1 and F_2 have disjoint neighbourhoods. Furthermore, since $\wp$ is a closed partition, the saturations of the complements of these neighbourhoods are closed, and now it is clear that the complements of these saturations are disjoint saturated neighbourhoods of F_1 and F_2. □

Open Partitions

Remark 1.2.3.10. A partition $\wp$ of the space X is *open* if $\mathrm{proj}\colon X \to X/\wp$ is an open map. An equivalent condition: saturations of open sets are open.

If $\wp$ is an open partition and A is a saturated set, then the saturation of $\operatorname{int} A$ is open, and hence equals $\operatorname{int} A$; passing to complements, we see that the saturation of $\operatorname{Cl} A$ is just $\operatorname{Cl} A$. Therefore, in the case of an open partition, the interior and the closure of a saturated set are saturated.

As it follows from Remark 1.1.6.6, the quotient of a first countable (second countable) space by an open partition is first countable (respectively, second countable).

Theorem 1.2.3.11. *Let* $\wp$ *and* q *be open partitions of the respective spaces* X *and* Y*. The product* $(X/\wp) \times (Y/q)$ *is canonically homeomorphic to the quotient* $(X \times Y)/(\wp \times q)$.

Proof. The injective factor of the map $\mathrm{proj} \times \mathrm{proj}\colon X \times Y \to (X \times Y)/(\wp \times q)$ defines this canonical map, which is a homeomorphism because $\mathrm{proj} \times \mathrm{proj}$ is open (see Remark 1.2.2.5). □

1.2.4 Glueing

Remark 1.2.4.1. *Glueing* (or *pasting*) topological spaces is a composite operation which consists of taking a sum and subsequently passing to a quotient. More precisely, suppose that $\{x_\mu\}_{\mu \in M}$ is a family of topological spaces and $\wp$ is a partition of the space $X = \coprod X_\mu$. Then we say that *the quotient space* $X/\wp$ *is obtained by glueing the spaces* X_μ (or *according to*) $\wp$. The composite map

$$X_\nu \xrightarrow{\mathrm{incl}_\nu} X \xrightarrow{\mathrm{proj}} X/\wp$$

is termed the ν-th immersion and is denoted by Imm_ν. Clearly, the sets $\mathrm{Imm}_\nu(X_{\neq})$ yield a fundamental cover of $X/\wp$, and a map $f\colon X/\wp \to Y$, where Y is an arbitrary topological space, is continuous if and only if all the compositions $f \circ \mathrm{Imm}_\nu$ are continuous.

Unions

Definition 1.2.4.2. Let $\{X_\mu\}_{\mu\in M}$be a family of topological spaces. Suppose that for each pair $(\mu,\mu')\in M\times M$ there is given a subset $A_{\mu\mu'}\subset X_\mu$. In addition, suppose that for each pair $(\mu,\mu')\in M\times M$ there is given an invertible map $\varphi_{\mu\mu'}\colon A_{\mu\mu'}\to A_{\mu'\mu'}$ such that:

(i) $A_{\mu\mu}=X_\mu$ and $\varphi_{\mu\mu}=\operatorname{id}X_\mu$, for any $\mu\in M$;

(ii) $\varphi_{\mu\mu'}(A_{\mu\mu'}\cap A_{\mu\mu''})=A_{\mu'\mu}\cap A_{\mu'\mu''}$ and the diagramme

$$\begin{tikzcd}
A_{\mu\mu'}\cap A_{\mu\mu''} \arrow[rr, "\operatorname{abr}\varphi_{\mu\mu'}"] \arrow[dr, "\operatorname{abr}\varphi_{\mu\mu''}"'] & & A_{\mu'\mu}\cap A_{\mu'\mu''} \arrow[dl, "\operatorname{abr}\varphi_{\mu'\mu''}"] \\
& A_{\mu''\mu}\cap A_{\mu''\mu'} &
\end{tikzcd}$$

is commutative for every $\mu,\mu',\mu''\in M$. For $x\in\coprod X_\mu$, denote by B_x the subset of $\coprod X_\mu$ consisting of all the points $\operatorname{incl}_\mu(\varphi_{\nu\mu}(x))$, where $X\in A_{\nu\mu}$. The sets B_x are pairwise disjoint and define a partition of $\coprod X_\mu$. The corresponding quotient space is called the *union of the spaces X_μ by (or along) the maps $\varphi_{\mu\mu'}$*.

This construction is a special case of glueing, when all the immersions Imm_μ are injective. Moreover, assuming that all the maps $\varphi_{\mu\mu'}$ are homeomorphisms and that the sets $A_{\mu\mu'}$ are all open or all closed, we see at once that all the maps Imm_μ are embeddings.

In the general case, a union of T_1-spaces is clearly a T_1-space.

Remark 1.2.4.3. Often the union construction is employed when all the spaces X_μ are subsets of a set X and cover X, while $A_{\mu\mu'}$ and $\varphi_{\mu\mu'}$ are given by $A_{\mu\mu'}=X_\mu\cap X_{\mu'}$ and $\varphi_{\mu\mu'}=\mathrm{id}$. In this situation, conditions 1.2.4.2 (i) and 1.2.4.2 (ii) are automatically fulfilled, and one may describe the union of the X_μ's simply as the set X equipped with the following topology: a set $C\subset X$ is open (closed) if and only if the intersection $C\cap X_\mu$ is open (respectively, closed) in X_μ for any $\mu\in M$.
Memo by the transcriber: This topology is the one exactly used in the definition of CW complexes, under the (somewhat misleading) name "weak topology".

We devote some special attention to the case where the topology of each set X_μ is induced by some topology already given on X. Then our construction produces a new topology on X. It is clear that the sets open (closed) in the old topology remain open (respectively, closed) in the new topology. Moreover, if all the intersections $X_\mu\cap X_{\mu'}$, are open in their sets X_μ (endowed with the initial topology), then the new topology on X induces the initial topology back on each set X_μ; the same holds whenever all the intersections $X_\mu\cap X_{\mu'}$, are closed in their sets X_μ.

Limits and Filtrations

Remark 1.2.4.4. Let $X_0, X_1, \dots$ be topological spaces, and let $\varphi_0 \colon X_0 \to X1$, $\varphi_1 \colon X_1 \to X2, \dots$ be embeddings. Set

$$A_{kk'} = \begin{cases} \varphi_{k-1} \circ \cdots \circ \varphi_{k'}(X_{k'}), & \text{if} \quad k' < k, \\ X_{k'} & \text{if} \quad k' \geq k \end{cases}$$

and

$$\varphi_{kk'} = \begin{cases} \operatorname{abr}(\varphi_{k-1} \circ \cdots \circ \varphi_{k'}), & \text{if} \quad k' < k, \\ \operatorname{id}_{X_k}, & \text{if} \quad k' = k, \\ \operatorname{abr}(\varphi_{k'-1} \circ \cdots \circ \varphi_k)^{-1}, & \text{if} \quad k' \geq k \end{cases}$$

The union of the spaces X_s is well-defined because conditions 1.2.4.2 (i) and 1.2.4.2 (ii) are obviously satisfied. This union is called the *limit of the sequence* $\{x_k\}$ and denoted by $\lim(X_k, \varphi_k)$ or $\lim X_k$.

A specific property of the limit construction is that the maps

$$\operatorname{Imm}_k \colon X_k \to \lim X_k$$

are embeddings: indeed, every closed subset A of X_k is the pre-image under Imm_k of some closed subset of $\lim X_k$, for example, of

$$\bigcup_{k'=k+1}^{\infty} \operatorname{Imm}_{k'}(\operatorname{Cl}_{X_k}(\varphi_{k'-1} \circ \cdots \circ \varphi_k(A)))$$

Obviously, if $\varphi_k(X_k)$ is open (closed) in X_{k+1} for all k, then all the sets $\operatorname{Imm}_k(X_k)$ are open (respectively, closed) in $\lim(X_k, \varphi_k)$.

Suppose that $\{X'_{k'}, \varphi'_k \colon X'_k \to X'_{k+1}\}$ another sequence of topological spaces and embeddings, and that for each k there is given a continuous map

$$f_k \colon X_k \to X'_k,$$

so that all the diagrammes

$$\begin{tikzcd} X_k \arrow[r, "f_k"] \arrow[d, "\varphi_k"'] & X'_k \arrow[d, "\varphi'_k"] \\ X_{k+1} \arrow[r, "f_{k+1}"'] & X'_{k+1} \end{tikzcd}$$

are commutative. Then the rule $f(\operatorname{Imm}_k(x)) = \operatorname{Imm}_k(f_k(x))$ defines a continuous map $f \colon \lim(X_k, \varphi_k) \to \lim(X'_k, \varphi'_k)$ (see Remark 1.2.3.2); f is called the *limit of the sequence* $f_0, f_1, \dots$ and is denoted by $\lim f_k$.

Theorem 1.2.4.5. *If $X_0, X_1, \dots$ are T_1-spaces, then every compact subset of $\lim X_k$ is contained in one of the sets $\operatorname{Imm}_k(X_k)$.*

Proof. This is a consequence of Theorem 1.1.7.6. □

Theorem 1.2.4.6. *If $X_0, X_1 < \dots$ are normal spaces and $\varphi_k(X_k)$ is closed in X_{k+1} for every k, then $\lim(X_k, \varphi_k)$ is normal.*

Proof. Since we already know that $\lim(X_k, \varphi_k)$ is a T_1-space, it suffices to show that there exists a Urysohn function for any pair A, B of closed disjoint subsets of this space (see Definition 1.1.5.11). To see this, we merely have to produce a sequence $f_k\colon \mathrm{Imm}_k(X_k) \to I$, such that each f_k is a Urysohn function for the pair $A \cap \mathrm{Imm}_k(X_k)$, $B \cap \mathrm{Imm}_k(X_k)$, and $f_{k+1}|_{\mathrm{Imm}_K(X_k)} = f_k$, for each k.

As f_0 take any Urysohn function for the pair $A \cap \mathrm{Imm}_0(X_0)$, $B \cap \mathrm{Imm}_0(X_0)$. Given f_k, we define $\mathrm{Imm}_{k+1}\colon \mathrm{Imm}_k(X_{k+1}) \to I$ as the (continuous) extension of the function

$$g_k\colon [\mathrm{Imm}_k(X_k)] \cup [A \cap \mathrm{Imm}_k(X_{k+1})] \cup [B \cap \mathrm{Imm}_{k+1}(X_{k+1})] \to I,$$

defined by the formula

$$g_k(x) = \begin{cases} f_k(x), & \text{if} \quad x \in \mathrm{Imm}_k(X_k), \\ 0, & \text{if} \quad x \in A \cap \mathrm{Imm}_{k+1}(X_{k+1}), \\ 1, & \text{if} \quad x \in B \cap \mathrm{Imm}_{k+1}(X_{k+1}) \end{cases}$$

(see Theorem 1.1.5.14). The functions g_k are continuous because the sets $\mathrm{Imm}_k(X_k)$, $A \cap \mathrm{Imm}_{k+1}(X_{k+1})$, and $B \cap \mathrm{Imm}_{k+1}(X_{k+1})$ are closed (see Theorem 1.1.4.3), which in turn is a consequence of the fact that $\varphi_k(X_k)$, $\varphi_{k+1}(X_{k+1})$ are closed. □

Definition 1.2.4.7. A sequence $X_0, X_1, \dots$ of subsets of a topological space X is a *filtration* of X if, firstly, $X_0 \subset X_1 \subset \cdots$, and, secondly, the sets X_k form a fundamental cover of X.

The first condition shows that the inclusions $\mathrm{incl}\colon X_k \to X_{k+1}$ and the limit $\lim(X_k, \mathrm{incl})$ are meaningful, while the second condition is equivalent to the following: the map $X \to \lim(X_k, \mathrm{incl})$, which equals $\mathrm{Imm}_k\colon X_k \to \lim(X_k, \mathrm{incl})$ on each X_k, is a homeomorphism. Using this canonical homeomorphism, we may identify $\lim(X_k, \mathrm{incl})$ with X.

Attaching

Remark 1.2.4.8. Let X_1, X_2, C, and $\varphi\colon C \to X_2$ be two topological spaces, a subset of X_1, and a continuous map, respectively. Denote by $\wp$ the partition of $X_1 \coprod X_2$ into the points of $\mathrm{incl}_1(X_1 \setminus C)$ and $\mathrm{incl}_2(X_2 \setminus \varphi(C))$, and the sets $\mathrm{incl}_1((\varphi^{-1}(x)) \cup \mathrm{incl}_2(x)$ with $x \in \varphi(C)$. The quotient space $(X_1 \coprod X_2)/\wp$ is written $X_2 \cup_\varphi X_1$. We say that $X_2 \cup_\varphi X_1$ is obtained by *attaching the space* X_1 *to the space* X_2 *by* (or *along*) φ.

This construction is clearly a special kind of glueing, and it is plain that $\mathrm{Imm}\,2\colon X_2 \to X_2 \cup_\varphi X_1$ is an embedding.

If X_2 reduces to a point, then $X_2 \cup_\varphi X_1$ is canonically homeomorphic to the quotient space X_1/C; the canonical homeomorphism is just

$$\operatorname{fact}[\mathrm{Imm}_1 \colon X_1 \to X_2 \cup_\varphi X_1].$$

Theorem 1.2.4.9. *If X_1 and X_2 are normal and C is closed, then $X_2 \cup_\varphi X_1$ is normal.*

Proof. Since we already know that $X_2 \cup_\varphi X_1$ is a T_1-space (see Definition 1.2.4.2), it suffices to show that there exists a Urysohn function for any pair of closed disjoint subsets A, B of $X_2 \cup_\varphi X_1$. Let $f_2 \colon X_2 \to I$ be a Urysohn function for the pair $\mathrm{Imm}_2^{-1}(A)$, $\mathrm{Imm}_2^{-1}(B)$. Define $g \colon C \cup [\mathrm{Imm}_1^{-1}(A)] \cup [\mathrm{Imm}_1^{-1}(B)] \to I$ by

$$g(x) = \begin{cases} f_2(\varphi(x)), & \text{if} \quad x \in C, \\ 0, & \text{if} \quad x \in \mathrm{Imm}_1^{-1}(A), \\ 1, & \text{if} \quad x \in \mathrm{Imm}_1^{-1}(B), \end{cases}$$

and extend it to a continuous function $f_1 \colon X_1 \to I$ (see Theorem 1.1.5.14). The function $X_2 \cup_\varphi X_1 \to I$, defined as

$$y \mapsto \begin{cases} f_1(x), & \text{if} \quad y \in \mathrm{Imm}_1(x) \quad [x \in X_1], \\ f_2(x), & \text{if} \quad y \in \mathrm{Imm}_2(x) \quad [x \in X_2], \end{cases}$$

is obviously a Urysohn function for the pair A, B. □

1.2.5 Projective Spaces

Remark 1.2.5.1. In this subsection we shall describe the real, complex, quaternionic, and Cayley projective spaces. These may be considered as examples illustrating the previous definitions, but are also important spaces in their own right.

We denote the field of complex numbers by $\mathbb{C}$, the field of quaternions by $\mathbb{H}$, and the algebra of Cayley numbers by $(\mathbb{C}a$. The corresponding n-dimensional spaces, i.e., the products of n copies $\mathbb{C} \times \dots \times \mathbb{C}$, $\mathbb{H} \times \dots \times \mathbb{H}$, and $\mathbb{C}a \times \dots \times \mathbb{C}a$, are denoted by $\mathbb{C}^n$, $\mathbb{H}^n$, and $\mathbb{C}a^n$. Since every complex number is a pair of real numbers, every quaternion a quadruplet of real numbers, and every Cayley number an octuplet of real numbers, one can naturally identify $\mathbb{C}^n$, $\mathbb{H}^n$, and $\mathbb{C}a^n$ with $\mathbb{R}^{2n}$, $\mathbb{R}^{3n}$, and $\mathbb{R}^{8n}$, respectively. In particular, the former are endowed with natural topologies and metrics. The vector operations in $\mathbb{C}^n$, $\mathbb{H}^n$, and $\mathbb{C}a^n$ (addition of vectors and left or right multiplication by scalars) are continuous in these topologies.

Definition 1.2.5.2. The n-dimensional real projective space $\mathbb{R}P^n$ is defined as the quotient space of $\mathbb{S}^n$ by its partition into pairs of diametrically opposed (= antipodal) points. One may equivalently describe $\mathbb{R}P^n$ as the quotient space of $\mathbb{D}^n$ by its partition into the points of $\operatorname{int} \mathbb{D}^n$ and the pairs of diametrically opposed points of $\operatorname{Fr}\mathbb{D}^n = \mathbb{S}^{n-1}$. The canonical homeomorphism permitting us to identify these two quotient spaces is fact f, where $f\colon \mathbb{D}^n \to \mathbb{S}^n$ is defined by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, \sqrt{1 - x_1^2 - \cdots - x_n^2}).$$

One may also identify the points of $\mathbb{R}P^n$ with the lines of $\mathbb{R}^{n+1}$ which pass through the point $0 = (0, \dots, 0)$ (the line passing through the points x and $-x$ corresponds to the pair of points $x, -x \in \mathbb{S}^n$. The set of all these lines, equipped with the angular metric (i.e., the distance between two lines is defined as the angle between them which is less than $\pi/2$), is a metric space, and the above natural map of $\mathbb{R}P^n$ onto this space is clearly a homeomorphism. This provides a third description of the real projective space. The fourth, a coordinate description, can be obtained if one remarks that every line passing through 0 is uniquely determined by any of its non-zero points, and that the coordinates of any two non-zero points of such a line are proportional. This enables us to interpret the points of $\mathbb{R}P^n$ as classes of proportional non-zero real sequences $(x_1, \dots, x_{n+1})$; the point determined by the sequence $(x_1, \dots, x_{n+1})$ is denoted by $(x_1 : \cdots : x_{n+1})$, and the numbers $x_1, \dots, x_{n+1}$ are called its *homogeneous coordinates*. The description of the topology of $\mathbb{R}P^n$ in terms of the homogeneous coordinates is plain.

Definition 1.2.5.3. All the above discussion of the space $\mathbb{R}P^n$ can be repeated naturally for the complex case, leading to four equivalent description of the n-*dimensional complex projective space* $\mathbb{C}P^n$.
First description: $\mathbb{C}P^n$ is the quotient space of the unit sphere $\mathbb{S}^{2n+1}$ of $\mathbb{C}$ by its partition into the circles obtained by intersecting $\mathbb{S}^{2n+1}$ with the (complex) lines of $\mathbb{C}$ which pass through the point 0.
The second description: $\mathbb{C}P^n$ is the quotient space of the unit ball $\mathbb{D}^{2n}$ of $\mathbb{C}$ by its partition into the points of $\operatorname{int}\mathbb{D}^{2n}$ and the circles on $\operatorname{Fr}\mathbb{D}^{2n} = \mathbb{S}^{2n^1}$ obtained by intersecting $\mathbb{S}^{2n^1}$ with the lines of $\mathbb{C}$ passing through 0. The canonical homeomorphism between these two quotient spaces is fact f , where $f\colon \mathbb{D}^{2n} \to \mathbb{S}^{2n+1}$ is given by

$$f(x_1, \dots, x_{2n}) = (x_1, \dots, x_{2n}, \sqrt{1 - x_1^2 - \cdots - x_{2n}^2}).$$

The third description: $\mathbb{C}P^n$ is the set of lines of $\mathbb{C}^{n+1}$ passing through the point 0, equipped with the topology induced by the angular metric.
The fourth description: $\mathbb{C}P^n$ is the space of classes of (complex) proportional non-zero complex sequences $(x_1, \dots, x_{n+1})$.

The notation $(x_1 : \cdots : x_{n+1})$ introduced in Definition 1.2.5.2 also extends to the complex case, and the numbers $x_1, \dots, x_{n+1}$ are known, as in the real case, as the homogeneous coordinates of the point $(x_1, \dots, x_{n+1})$.

Definition 1.2.5.4. Since the field of quaternions is not commutative, one has to distinguish between the left and the right lines in $\mathbb{H}^n$. But as soon as we have chosen one type of lines, we can automatically repeat the discussion in Definition

1.2.5.3 for the quaternionic case, and so obtain four equivalent descriptions of the corresponding (left or right) *n-dimensional quaternionic projective space* $\mathbb{H}P^n$. Since the anti-automorphism $x \mapsto x^{-1}$ of the multiplicative group of the field $\mathbb{H}$ takes left lines into right ones, the right space $\mathbb{H}P^n$. Since the anti-automorphism $x \mapsto x^{-1}$ of the multiplicative group of the field $\mathbb{H}$ takes left lines into right ones, the right space $\mathbb{H}P^n$ is homeomorphic to the left one.

Henceforth, we shall regard $\mathbb{H}^n$ as a left vector space, and accordingly $\mathbb{H}P^n$ will be regarded as the left projective space.

Definition 1.2.5.5. Since the Cayley algebra is not associative, we cannot successfully define lines in $\mathbb{C}a^n$ for $n > 2$. In the Cayley plane $\mathbb{C}a^2$ one can define a line passing through the point $(0,0)$ as a set $\{(x_1, x_2) | x_2 = cx_1\}$, where $c \in \mathbb{C}a$; in addition, there is the vertical coordinate line $\{(x_1, x_2) | x_1 = 0\}$. If one identifies $\mathbb{C}a^2$ with $\mathbb{R}^{16}$, it becomes clear that these lines are 8-dimensional subspaces. Moreover, every point different from $(0,0)$ of $\mathbb{C}a$ sits on exactly one of these lines, and each line intersects $\mathbb{S}^{15}$ along a 7-dimensional sphere. One can define the projective Cayley line $\mathbb{C}aP^1$ as the quotient space of $\mathbb{S}^{15}$ by its partition into these 7-dimensional spheres. Of course, there are three more description of this projective line, which are appropriately modified versions of those given in Definitions 1.2.5.2, 1.2.5.3, and 1.2.5.4. In addition, we can define the projective Cayley plane $\mathbb{C}aP^2$ as the quotient of $\operatorname{int}\mathbb{D}^{16}$ by its partition into the points of $\operatorname{int}\mathbb{D}^{16}$ and the 7-dimensional spheres just described. However, an attempt to describe the projective plane $\mathbb{C}aP$ in the spirit of the alternatives of Definitions 1.2.5.2, 1.2.5.3, and 1.2.5.4 fails. Projective Cayley spaces of higher dimensions are not defined.

Remark 1.2.5.6. The spaces $\mathbb{R}P^1$, $\mathbb{C}P^1$, $\mathbb{H}P^1$, and $\mathbb{C}aP^1$ are canonically homeomorphic to $\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^4$, and $\mathbb{S}^8$. The homeomorphism $\mathbb{R}P^1 \to \mathbb{S}^1$ transforms the line $x_1 = 0$ into ort_1, and each line $x_2 = cx_1$ into the point of the punctured sphere $\mathbb{S}^1 \setminus \mathrm{ort}_1$ which corresponds to c via the homeomorphism $\mathbb{R}^1 \to \mathbb{S}^1 \setminus \mathrm{ort}_1$ described in Example 1.1.4.9. The homeomorphisms $\mathbb{C}P^1 \to \mathbb{S}^2$, $\mathbb{H}P^1 \to \mathbb{S}^4$, and $\mathbb{C}aP^1 \to \mathbb{S}^8$ are similarly defined, when we substitute the homeomorphisms $\mathbb{C}^1 = \mathbb{R}^2 \to \mathbb{S}^2 \setminus \mathrm{ort}_1$, $\mathbb{H}^1 = \mathbb{R}^4 to \mathbb{S}^4 \setminus \mathrm{ort}_1$, and $\mathbb{C}a^1 = \mathbb{R}^8 \to \mathbb{S}^1 \setminus \mathrm{ort}_1$ for the homeomorphism $\mathbb{R}^1 \to \mathbb{S}^1 \setminus \mathrm{ort}_1$.

Remark 1.2.5.7. The canonical embedding $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0)$ of $\mathbb{R}^k$ in $\mathbb{R}^{k+1}$ permits identification of $\mathbb{R}^k$ with the subspace $x_{k+1} = 0$ of $\mathbb{R}^{k+1}$ and may be regarded as an inclusion. This map induces inclusions $\mathbb{D}^k \to \mathbb{D}^{k+1}$, $\mathbb{S}^{k-1} \to \mathbb{S}^k$, and $\mathbb{R}P^{k-1} \to \mathbb{R}P^k$. Similarly, the inclusions $\mathbb{C}^k \to \mathbb{C}^{k+1}$ and $\mathbb{H}^k \to \mathbb{H}^{k+1}$ induce inclusions $\mathbb{C}P^{k-1} \to \mathbb{C}P^k$ and $\mathbb{H}P^{k-1} \to \mathbb{H}P^k$.

Set

$$\mathbb{R}^\infty = \lim \mathbb{R}^k, \quad \mathbb{C}^\infty = \lim \mathbb{C}^k, \quad \mathbb{H}^\infty = \lim \mathbb{H}^k,$$
$$\mathbb{D}^\infty = \lim \mathbb{D}^k, \quad \mathbb{S}^\infty = \lim \mathbb{S}^k,$$

and

$$\mathbb{R}P^\infty = \lim \mathbb{R}P^k, \quad \mathbb{C}P^\infty = \lim \mathbb{C}P^k, \quad \mathbb{H}P^\infty = \lim \mathbb{H}P^k.$$

The points of the spaces $\mathbb{R}^\infty$, $\mathbb{C}^\infty$, and $\mathbb{H}^\infty$ can be naturally identified with the real, complex, and quaternionic finitely-supported (i.e., having only a finite number of non-zero terms) sequences $\{x_k\}_1^\infty$. The sphere $\mathbb{S}^\infty$ is included in the ball $\mathbb{D}^\infty$, which in turn is included in the space $\mathbb{R}^\infty$. The projective spaces $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$ and $\mathbb{H}P^\infty$ are constructed from $\mathbb{S}^\infty$ and $\mathbb{D}^\infty$ by taking quotients that are limits of the quotients described in Definitions 1.2.5.2, 1.2.5.3, and 1.2.5.4.

Remark 1.2.5.8. The previous description of $\mathbb{C}P^n$ and $\mathbb{H}P^n$, and also of $\mathbb{C}aP^1$ as quotient spaces of spheres, define projections $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$, $\mathbb{S}^{4n+3} \to \mathbb{H}P^n$, and $\mathbb{S}^{15} \to \mathbb{C}aP^1$, which play a distinguished role. They are called *Hopf maps*. The most important Hopf maps are $\mathbb{S}^3 \to \mathbb{C}P^1 = \mathbb{S}^2$, $\mathbb{S}^7 \to \mathbb{H}P^1 = \mathbb{S}^4$, and $\mathbb{S}^{15} \to \mathbb{C}aP^1 = \mathbb{S}^8$.

1.2.6 More Special Constructions

Definition 1.2.6.1. Let X be a topological space. The quotient space

$$(X \times I)/(X \times 0)$$

is called the *cone over* X and is denoted by $\operatorname{cone} X$. The point $\operatorname{proj}(X \times 0)$ is the vertex of the cone, the set $\operatorname{proj}(X \times 1)$ is the base of the cone, and each set $\operatorname{proj}(x \times I)$, $x \in X$, is a *generatrix* of the cone. The base of $\operatorname{cone} X$ is canonically homeomorphic to X and is usually identified with X. The generatrices are obviously canonically homeomorphic to I.

For each map f from X into another topological space Y, we have the map $\operatorname{fact}(f \times \mathrm{id}_I)\colon \operatorname{cone} X \to \operatorname{cone} Y$, which is continuous whenever f is so. This map is denoted by $\operatorname{cone} f$.

Definition 1.2.6.2. The quotient of the product $X \times I$ by its partition whose elements are the sets $X \times 0$ and $X \times 1$, and the points of the set

$$(X \times I) \setminus [(X \times 0) \cup (X \times 1)],$$

is called the *suspension of* X and is denoted by $\operatorname{sus} X$. The points $\operatorname{proj}(X \times 0)$ and $\operatorname{proj}(X \times 1)$ are the *vertices* of the suspension, the set $\operatorname{proj}(X \times \frac{1}{2})$ is its base, and the sets $\operatorname{proj}(x \times I)$, $x \in X$, are its *generatrices*. The base of $\operatorname{sus} X$ is canonically homeomorphic to X, while each of its generatrices is canonically homeomorphic to I.

For each map $f\colon X \to Y$ there is the corresponding map

$$\operatorname{fact}(f \times \mathrm{id}_I)\colon \operatorname{sus} X \to \operatorname{sus} Y,$$

which we denote by $\operatorname{sus} f$; $\operatorname{sus} f$ is continuous whenever f is so.

Notice that the suspension $\operatorname{sus} X$ can be alternatively described as $\operatorname{cone} X/X$.

Remark 1.2.6.3. Let X_1 and X_2 be topological spaces. The quotient space of $X_1 \times X_2 \times I$ by its partition into the sets $x_1 \times X_2 \times 0$ (with $x_1 \in X_1$), $X_1 \times x_2 \times 1$ (with $x_2 \in X_2$), and the points of

$$X_1 \star X_2 := (X_1 \times X_2 \times I) \setminus [(X_1 \times X_2 \times 0) \cup (X_1 \times X_2 \times 1)],$$

is called the *join* of X_1 and X_2. The sets $\operatorname{proj}(X_1 \times X_2 \times 0)$ and $\operatorname{proj}(X_1 \times X_2 \times 1)$ are the *bases* of the join, and the sets $\operatorname{proj}(x_1 \times x2 \times I)$ with $x_1 \in X_1$ and $x_2 \in X_2$ are its *generatrices*. The bases are obviously canonically homeomorphic to X_1 and X_2, and are usually identified with X_1 and X_2. The generatrices are canonically homeomorphic to I.

For each pair of maps $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ we have the map $\operatorname{fact}(f_1 \times f_2 \times \mathrm{id}_I)\colon X_1 \star X_2 \to Y1 \star Y_2$, denoted by $f_1 \star f_2$; $f_1 \star f_2$ is continuous whenever both f_1 and f_2 are so.

The $\star$-operation is commutative, i.e., there exists a canonical homeomorphism $X_2 \star X_1 \to X_1 \star X_2$.

We remark that the join $X_1 \star X_2$ may be alternatively defined as

$$(X_1 \coprod X_2) \cup_\varphi (X_1 \times X_2 \times I),$$

where $\varphi\colon X_1 \times X_2 \times (0 \cup 1) \to X_1 \coprod X_2$ is given by $\varphi(x_1, x_2, O) = \mathrm{incl}_1(x_1)$, $\varphi(x_1, x_2, 1) = \mathrm{incl}_2(x_2)$. It is also clear that the quotient space of $X_1 \star X_2$ by its partition whose elements are the bases X_1 and X_2, and the points of the set $(X_1 \star X_2) \setminus (X_1 \cup X_2)$ is the suspension $\operatorname{sus}(X_1 \times X_2)$.

Remark 1.2.6.4. The iterated join $(\cdots((X_1 \star X_2) \star X_3) \cdots) \star X_n$ maybe canonically embedded in the product $\operatorname{cone} X_1 \times \cdots \times \operatorname{cone} X$. This embedding is denoted by jc or, more precisely, by $\mathrm{jc}_{X_1,\dots,X_n}$, and is defined inductively: for $n = 1$ it takes $x_1 \in X_1$ into $\operatorname{proj}(x_1, 1) \in \operatorname{cone} X_1$, while for $n \geq 2$ it is given by

$$\mathrm{jc}_{X_1,\dots,X_n}(\operatorname{proj}(x, x_n, t)) = ((1-t)\,\mathrm{jc}_{X_1,\dots,X_{n-1}}(x), \operatorname{proj}(x_n, t)),$$

where $x \in (\cdots((X_1 \star X_2) \star X_3) \cdots) \star X_{n-1}$, $x_n \in X_n$, and $t \in I$; the multiplication of a point of $\operatorname{cone} X_1 \times \cdots \times \operatorname{cone} X_{n-1}$ by $1 - t$ is defined by the rule

$$\begin{aligned}&(1-t)(\operatorname{proj}(x_1, t_1), \dots, \operatorname{proj}(x_{n-1}, t_{n-1})) =\\ &(\operatorname{proj}(x_1, (1-t)t_1), \dots, \operatorname{proj}(x_{n-1}, (1-t)t_{n-1})).\end{aligned}$$

Clearly, the image of the embedding $\mathrm{jc}_{X_1,\dots,X_n}(\operatorname{proj}(x, x_n, t))$ is precisely

$$\{(\operatorname{proj}(x_1, t_1), \dots, \operatorname{proj}(x_n, t_n)) \in \operatorname{cone} X_1 \times \cdots \times \operatorname{cone} X_n | t_1 + \cdots + t_n = 1\},$$

which allows us to identify the iterated join $(\cdots((X_1 \star X_2) \star X_3) \cdots) \star X_n$ with this set.

Remark 1.2.6.5. The $\star$-operation is associative, meaning, as usual, that the two joins $(X_1 \star X_2) \star X_3$ and $X_1 \star (X_2 \star X_3)$ are canonically homeomorphic. The canonical homeomorphism

$$(X_1 \star X_2) \star X_3 \to X_1 \star (X_2 \star X_3)$$

is the composition of the canonical homeomorphism

$$(X_1 \star X_2) \star X_3 \to (X_2 \star X_3) \star X_1$$

with the suitable compression of the canonical homeomorphism

$$\operatorname{cone} X_2 \times \operatorname{cone} X_3 \times \operatorname{cone} X_1 \to \operatorname{cone} X_1 \times \operatorname{cone} X_2 \times \operatorname{cone} X_3.$$

A consequence of the associativity of the $\star$-operation is that the multiple join $X_1 \star \cdots \star X_n$ is meaningful for any topological spaces $X_1, \dots, X_n$.

Theorem 1.2.6.6. *The product* $\operatorname{cone} X_1 \times \cdots \times \operatorname{cone} X_n$ *is canonically homeomorphic to* $\operatorname{cone}(X_1 \star \cdots \star X_n)$.

Proof. The canonical homeomorphism

$$\operatorname{cone}(X_1 \star \cdots \star X_n) \to \operatorname{cone} X_1 \times \cdots \times \operatorname{cone} X_n$$

is defined as

$$\begin{gathered}\operatorname{proj}(\operatorname{jc}^{-1}_{X_1,\dots X_n}(\operatorname{proj}(x_1, t_1), \dots, \operatorname{proj}(x_n, t_n)), t) \mapsto \\ (\operatorname{proj}(x_1, tt_1/\max(t_1, \dots, t_n)), \dots, \operatorname{proj}(x_n, tt_n/\max(t_1, \dots, t_n))).\end{gathered}$$

□

Theorem 1.2.6.7. $\operatorname{cone} \mathbb{S}^m$ *and* $\operatorname{sus} \mathbb{S}^m$ *are are canonically homeomorphic to* $\mathbb{D}^{m+1}$ *and* $\mathbb{S}^{m+1}$.

Proof. The canonical homeomorphisms $\operatorname{cone} \mathbb{S}^m \to \mathbb{D}^{m+1}$ and $\operatorname{sus} \mathbb{S}^m \to \mathbb{S}^{m+1}$ are defined by the formulae

$$\operatorname{proj}((x_l, \dots, x_{m+1}), t) \mapsto (tx_l, \dots, tx_{m+1})$$

and

$$\operatorname{proj}((x_1, \dots, x_{m+1}), t) \mapsto (x_1 \sin \pi t, \dots, x_{m+1} \sin \pi t, \cos \pi t).$$

□

Theorem 1.2.6.8. *The join* $X \star \mathbb{D}^0$ *is canonically homeomorphic to* $\operatorname{cone} X$. *The join* $X \star \mathbb{S}^0$ *is canonically homeomorphic to* $\operatorname{sus} X$. *The join* $X \star \mathbb{S}^k$ *is canonically homeomorphic to the iterated suspension* $\operatorname{sus}^{k+1} X$*; in particular,* $\mathbb{S}^{m_1} \star \mathbb{S}^{m_2}$ *is canonically homeomorphic to* $\mathbb{S}^{m_1+m_2+1}$.

Proof. The canonical homeomorphism $X \star \mathbb{D}^0 \to \operatorname{cone} X$ is given by

$$\operatorname{proj}(x, 0, t) \mapsto \operatorname{proj}(x, t).$$

The canonical homeomorphism $X \star \operatorname{sus} X$ is given by the formulae

$$\begin{aligned}\operatorname{proj}(x, 1, t) &\mapsto \operatorname{proj}(x, (1 + t)/2), \\ \operatorname{proj}(x, -1, t) &\mapsto \operatorname{proj}(x, (1 - t)/2).\end{aligned}$$

Finally, the canonical homeomorphism $X \star \mathbb{S}^k \to \operatorname{sus} X$ is the composite map

$$X \star \mathbb{S}^k \to \operatorname{sus} X \star \mathbb{S}^{k-1} \to \cdots \to \operatorname{sus}^k X \star S \to \operatorname{sus}^{k+1} X,$$

where the last arrow denotes the canonical homeomorphism, and the r-th arrow, with $r \leq k$, denotes the composite canonical homeomorphism

$$\begin{aligned}\operatorname{sus}^{r-1} X \star \mathbb{S}^{k-r+1} &\to \operatorname{sus}^{r-1} X \star \operatorname{sus} \mathbb{S}^{k-r} \to \operatorname{sus}^{r-1} X \star \mathbb{S}^{k^r} \star \mathbb{S}^0 \\ &\to \operatorname{sus}^{r-1} X \star \mathbb{S}^0 \star \mathbb{S}^{k-r} \to \operatorname{sus}^r X \star \mathbb{S}^{k-r}.\end{aligned}$$

□

Remark 1.2.6.9. Combining the canonical homeomorphisms constructed in 1.2.6.6, 1.2.6.7 and 1.2.6.8, we obtain the composite homeomorphisms

$$\begin{aligned}&\mathbb{S}^{m_1} \star \cdots \star \mathbb{S}^{m_n} \to \\ &\mathbb{S}^{m_1} \star \cdots \star \mathbb{S}^{m_{n-2}} \star \mathbb{S}^{m_{n-1}+m_n+1} \to \cdots \to \\ &\mathbb{S}^{m_1} \star \mathbb{S}^{m_2+\cdots m_n+n-2} \to \mathbb{S}^{m_1+m_n+n-1}, \\ &\mathbb{D}^{m_1} \times \cdots \times \mathbb{D}^{m_n} \to \operatorname{cone} \mathbb{S}^{m_1-1} \times \cdots \times \operatorname{cone} \mathbb{S}^{m_n-1} \to \\ &\operatorname{cone}(\mathbb{S}^{m_1-1} \star \cdots \star \mathbb{S}^{m_n-1}) \to \operatorname{cone} \mathbb{S}^{m_1+m_n+n-1} \to \\ &\mathbb{D}^{m_1+\ldots+m_n},\end{aligned}$$

and

$$\begin{aligned}&\mathbb{D}^{m_1} \star \cdots \star \mathbb{D}^{m_n} \to \operatorname{cone} \mathbb{S}^{m_1-1} \star \mathbb{D}^0 \cdots \star \operatorname{cone} \mathbb{S}^{m_n-1} \star \mathbb{D}^0 \to \\ &\operatorname{cone}(\mathbb{S}^{m_1-1} \star \mathbb{D}^0 \star \cdots \star \mathbb{S}^{m_{n-1}-1} \star \mathbb{D}^0 \star \mathbb{S}^{m_n-1}) \to \\ &\operatorname{cone} \mathbb{S}^{m_1-1} \times \operatorname{cone} \mathbb{D}^0 \times \cdots \times \operatorname{cone} \mathbb{S}^{m_n-1} \times \operatorname{cone} \mathbb{D}^0 \times \mathbb{S}^{m_n-1} \to \\ &\mathbb{D}^{m_1} \times I \times \cdots \times \mathbb{D}^{m_{n-1}} \times I \times \mathbb{D}^{m_n} \to \\ &\mathbb{D}^{m_1} \times \mathbb{D}^1 \times \cdots \times \mathbb{D}^{m_{n-1}} \times \mathbb{D}^1 \times \mathbb{D}^{m_n} \to \\ &\mathbb{D}^{m_1+\cdots+m_n+n-1}\end{aligned}$$

Therefore, the join $\mathbb{S}^{m_1} \star \cdots \star \mathbb{S}^{m_n}$, the product $\mathbb{D}^{m_1} \times \cdots \times \mathbb{D}^{m_n}$, and the join $\mathbb{D}^{m_1} \star \cdots \star \mathbb{D}^{m_n}$ are canonically homeomorphic to the sphere $\mathbb{S}^{m_1+m_n+n-1}$, the ball $\mathbb{D}^{m_1+\ldots+m_n}$, and the ball $\mathbb{D}^{m_1+\cdots+m_n+n-1}$, respectively.

The Mapping Cylinder and the Mapping Cone

Definition 1.2.6.10. Let $f\colon X_1 \to X_2$ be a continuous map. The result of attaching the product $X_1 \times I$ to X_2 by the map $X_1 \times X_2$, $(x, 1) \mapsto f(x)$, is called the *mapping cylinder* of f, and is denoted by Cyl_f. The sets $\operatorname{Imm}_1(X_1 \times 0)$ and $\operatorname{Imm}_2(X_2)$ are the *lower* and *upper bases* of Cyl_f, and the sets $\operatorname{Imm}_1(x \times I)$ with $x \in X_1$ are its *generatrices*.

Clearly, the bases are canonically homeomorphic to X_1 and X_2, and they are usually identified with these two spaces; the generatrices are canonically homeomorphic to I. Moreover, there is a canonical retraction $\operatorname{rt} f\colon \operatorname{Cyl}_f \to X_2$, defined on $\operatorname{Imm}_1(X \times I)$ as $\operatorname{rt} f(\operatorname{Imm}_1(x, t)) = \operatorname{Imm}_1(x, 1)[= f(x)]$.

It is evident that the composite map

$$X_1 \xrightarrow{\text{incl}} \operatorname{Cyl}_f \xrightarrow{\operatorname{rt} f} X_2$$

equals f.

If $X_2 = X_1$ and $f = \mathrm{id}_{X_1}$, then Cyl_f is canonically homeomorphic to the cylinder over X_1, $X_1 \times I$.

Definition 1.2.6.11. The *mapping cone* of the continuous map $f\colon X_1 \to X_2$ is the space $X_2 \cup \operatorname{cone} X$, denoted by Cone_f (do not confuse it with cone f, which is a map, defined in Definition 1.2.6.11). Equivalent definition: $\operatorname{Cone}_f = \operatorname{Cyl}_f / X_1$.

1.2.7 Spaces of Continuous Maps

Definition 1.2.7.1. Let $\mathcal{C}(X, Y)$ be the set of all continuous maps of a topological space X into a topological space Y. The set of all maps $\varphi \in \mathcal{C}(X, Y)$ such that $\varphi(A_1) \subset B_1, \dots, \varphi(A_n) \subset B_n$, where $A_1, \dots, A_n$ and $B_1, \dots, B_n$ are given subsets of X and Y, respectively, is denoted by $\mathcal{C}(X, A_1, \dots, A_n; Y, B_1, \dots, B_n)$. It may be interpreted as the set of all continuous maps

$$(X, A_1, \dots, A_n) \to (Y, B_1, \dots, B_n).$$

We equip $\mathcal{C}(X, Y)$ with the *compact-open topology*: by definition, this is the topology with the pre-base consisting of all sets $\mathcal{C}(X, A; Y, B)$ with A compact and B open. Together with $\mathcal{C}(X, Y)$, all the sets $\mathcal{C}(X, A_1, \dots, A_n; Y, B_1, \dots, B_n)$ become topological spaces.

If Y is a point, then $\mathcal{C}(X, Y)$ reduces to a point. If X is discrete and consists of the points $x_1, \dots, x_n$, then $\mathcal{C}(X, Y)$ is canonically homeomorphic to the product $Y \times \dots \times Y$ of n copies of the space Y; this homeomorphism is given by $\varphi \mapsto (\varphi(x_1, \dots, \varphi(x_n))$.

To each pair of continuous maps $f\colon X' \to X$ and $g\colon Y \to Y'$ there corresponds a mapping $\mathcal{C}(X, Y) \to \mathcal{C}(X', Y')$, given by the rule $\varphi \mapsto g \circ \varphi \circ f$. This mapping is continuous, and we shall denote it by $\mathcal{C}(f, g)$.

$$\begin{tikzcd}
X \arrow[r, "\varphi"] & Y \arrow[d, "g"] \\
X' \arrow[u, "f"] \arrow[r, "\mathcal{C}(f,g):\varphi\mapsto g\circ\varphi\circ f"'] & Y'
\end{tikzcd}$$

Theorem 1.2.7.2. *If Y is a Hausdorff space, then so is $\mathcal{C}(X, Y)$.*

Proof. Indeed, if $\varphi, \psi \in \mathcal{C}(X, Y)$ and $\varphi \neq \psi$, then there is $x \in X$ such that $\varphi(x) \neq \psi(x)$. Let U and V be disjoint neighbourhoods of the points $\varphi(x)$ and $\psi(x)$. Then $\mathcal{C}(X, x; Y, U)$ and $\mathcal{C}(X, x; Y, V)$ are disjoint neighbourhoods of the points φ and ψ. □

Theorem 1.2.7.3. *If X is compact and Y is metrisable, then $\mathcal{C}(X,Y)$ is metrisable. Moreover, if Y is equipped with a metric, then*

$$\operatorname{dist}(\varphi,\psi) = \sup_{x\in X} \operatorname{dist}(\varphi(x), psi(x))$$

defines a metric on $\mathcal{C}(X,Y)$, compatible with its topology.

Proof. Given $\varphi \in \mathcal{C}(X,Y)$, the set $\varphi(X)$ can be covered by a finite number of balls of an arbitrarily small radius ε (see Theorem 1.1.7.11). It is clear that $\omega = \cap_{i=1}^{s}\mathcal{C}(X,\varphi^{-1}(U_i);Y,U_i)$ is a neighbourhood of the point φ, contained in the ball of radius 2ε centred at φ. Therefore, every ball in $\mathcal{C}(X,Y)$ contains a neighbourhood of its centre.

On the other hand, if $A \subset X$ is compact and $B \subset Y$ is open, with $\varphi(A) \subset B$, then $\mathcal{C}(X,A;Y,B)$ contains the ball with radius $\operatorname{Dist}(\varphi(A), Y\setminus B)$ centred at φ (see Theorem 1.1.7.15). Therefore, every neighbourhood of φ belonging to the pre-base considered in Definition 1.2.7.1 contains a ball centred at φ. □

Theorem 1.2.7.4. *For any topological spaces X and $Y_1,\dots,Y_n$, the space $\mathcal{C}(X,Y_1\times\cdots\times Y_n)$ is canonically homeomorphic to the product*

$$\mathcal{C}(X,Y_1)\times\cdots\times\mathcal{C}(X,Y_n).$$

Proof. This canonical homeomorphism takes each $\varphi \in \mathcal{C}(X,Y_1\times\cdots\times Y_n)$ into $(\operatorname{proj}_1\circ\varphi,\dots,\operatorname{proj}_n\circ\varphi) \in \mathcal{C}(X,Y_1)\times\cdots\times\mathcal{C}(X,Y_n)$ (cf. Remark 1.2.2.4). □

Theorem 1.2.7.5. *Let $\wp$ be a closed partition of the compact Hausdorff space X, and let Y be an arbitrary topological space. Then*

$$\mathcal{C}(\operatorname{proj},\operatorname{id}_Y)\colon \mathcal{C}(X/\wp,Y)\to\mathcal{C}(X,Y)$$

is an embedding.

Proof. It suffices to show that given a compact subset A of $X/\wp$ and an open subset B of Y, the set

$$\mathcal{C}(\operatorname{proj},\operatorname{id}_Y)\colon\ [\mathcal{C}(X/\wp,A;Y,B)]$$

is open in $\mathcal{C}(\operatorname{proj},\operatorname{id}_Y)[\mathcal{C}(X/\wp,Y)]$. Since $X/\wp$ is Hausdorff (see Theorem 1.2.3.9), A is closed. It follows that $\operatorname{proj}(A)$ is closed, and hence compact. Consequently, $\mathcal{C}(X,\operatorname{proj}^{-1}(A);Y,B)$ is open in $\mathcal{C}(X,Y)$, and it remains to note that

$$\mathcal{C}(\operatorname{proj},\operatorname{id}_Y)[\mathcal{C}(X/\wp,A;Y,B)] = \mathcal{C}(X,\operatorname{proj}^{-1}(A);Y,B)\cap\mathcal{C}(\operatorname{proj},\operatorname{id}_Y)[\mathcal{C}(X/\wp;Y)].$$

□

The Mappings $X \times Y \to Z$ and $X \to \mathcal{C}(Y, Z)$

Theorem 1.2.7.6. *Suppose that X, Y and Z are topological spaces, and*

$$\varphi \colon X \times Y \to Z$$

is continuous. Then the formula $[\varphi^\vee(x)](y) = \varphi(x, y)$ defines a continuous mapping $\varphi^\vee \colon X \to \mathcal{C}(Y, Z)$.

Let $\psi \colon X \to \mathcal{C}(Y, Z)$ be a continuous mapping, and suppose that Y is Hausdorff and locally compact. Then the formula $\psi^\wedge(x, y) = [\psi(x)](y)$ defines a continuous mapping $\psi^\wedge \colon X \times Y \to Z$.

Proof. To prove the first assertion, pick a point $X_0 \in X$, a compact set $B \subset Y$, and an open set $C \subset Z$. Then it is enough to exhibit a neighbourhood U of x_0 such that $\varphi^\vee(U) \subset C(Y, B; Z, C)$. For each point $y \in B$ fix neighbourhoods U_y and V_y of x_0 and y such that $\varphi(U_y \times V_y) \subset C$, and then extract a finite cover $V_{y_1}, \dots, V_{y_S}$ of B from the collection $\{V_y\}_{y \in B}$. It is clear that $U = \cup_{i=1}^{s} U_{y_i}$ is a neighbourhood of x_0 and that $\varphi(U \times B) \subset \cup_{i=1}^{s} \varphi(U_{y_i} \times V_{y_i}) \subset C$. It remains to remark that the inclusion $\varphi(U \times B) \subset C$ is equivalent to $\varphi^\vee(U) \subset \mathcal{C}(Y, B; Z, C)$.

To prove the second assertion, pick a point $(x_0, y_0) \in X \times Y$ and a neighbourhood W of the point $\varphi^\wedge(x_0, y_0)$. Now let us find a neighbourhood V of y_0 with compact closure $\operatorname{Cl} V$ satisfying $\operatorname{Cl} V \subset [\psi(x_0)]^{-1}(W)$ (see Theorem 1.1.7.22), and then a neighbourhood U of x_0 satisfying $\psi(U) \subset \mathcal{C}(Y, \operatorname{Cl} V; Z, W)$. Obviously, $U \times V$ is a neighbourhood of the point (x_0, y_0) and $\psi^\wedge(U \times V) \subset W$. $\square$

Theorem 1.2.7.7. *The mapping $\mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$ defined by the rule $\varphi \mapsto \varphi^\vee$ (see Theorem 1.2.7.6) is continuous for any topological spaces X, Y and Z. If X is Hausdorff and Y is Hausdorff and locally compact, then this mapping is a homeomorphism, and its inverse is given by the rule $\psi \mapsto \psi^\wedge$.*

Proof. The continuity of the mapping $\varphi \mapsto \varphi^\vee$ results from the fact that the pre-image of $\mathcal{C}(X, A; \mathcal{C}(Y, Z), \mathcal{C}(Y, B; Z, C))$ under this mapping is just

$$\mathcal{C}(X \times Y, A \times B; Z, C).$$

Assume that X is Hausdorff and Y is Hausdorff and locally compact. Consider a point $\psi_0 \in \mathcal{C}(X, \mathcal{C}(Y; Z))$, a compact subset Q of $X \times Y$, a neighbourhood W of the set $\psi_0^\wedge(Q)$, and a point $q \in Q$. Now find a neighbourhood $U_q \times V_q$ of q such that $\psi_0^\wedge(U_q \times \operatorname{Cl} V_q) \subset W$. Since Q is compact, its images $\operatorname{proj}_1(Q)$ and $\operatorname{proj}_2(Q)$ in X and Y are also compact (see Proposition 1.1.7.8). Moreover, they are Hausdorff spaces together with X and Y, and hence normal (see Proposition 1.1.7.5). Consequently, there exist open subsets U_q^{prime} of $\operatorname{proj}_1(Q)$ and V'_q of $\operatorname{proj}_2(Q)$ such that

$$\begin{aligned} \operatorname{proj}_1(q) \in U'_{q'}, &\quad \operatorname{Cl}_{\operatorname{proj}_1(Q)} U'_q \subset U_{q'}, \\ \operatorname{proj}_2(q) \in V'_{q'}, &\quad \operatorname{Cl}_{\operatorname{proj}_2(Q)} V'_q \subset V_{q'} \end{aligned}$$

and it is plain that the intersection $(U'_q \times V'_q) \cap Q$ is open in Q. Being compact, Q can be covered by a finite number of such intersections, say

$$U'_{q_1} \times V'_{q_1}, \dots, U'_{q_s} \times V'_{q_s}.$$

Now set

$$T = \bigcap_{i=1}^{s} \mathcal{C}(X, \mathrm{Cl}_{\mathrm{proj}_1(Q)} U_{q_i}; \mathcal{C}(Y, Z), \mathcal{C}(Y, \mathrm{Cl}_{\mathrm{proj}_2(Q)} V_{q_i}; Z, W)).$$

It is clear that T is a neighbourhood of q and that the image of T under the mapping $\psi \mapsto \psi^\wedge$ is contained in $\mathcal{C}(X \times Y, Q; Z, W)$. We conclude that $\psi \mapsto \psi^\wedge$ is continuous. It is readily seen that the mappings $\varphi \mapsto \varphi^\vee$ and $\psi \mapsto \psi^\wedge$ are inverses of one another. ☐

A Surprising Application

Theorem 1.2.7.8. *Let $f\colon X \to X'$ be a factorial map. If the space Y Hausdorff and locally compact, then the map $f \times \mathrm{id}_Y\colon X \times Y \to X' \times Y$ is factorial.*

Proof. One can assume that $X' = X/\operatorname{zer}(f)$ and that f is the projection

$$X \to X/\operatorname{zer}(f).$$

Consider the projection $\mathrm{proj}\colon X \times Y \to (X \times Y)/(\operatorname{zer}(f) \times \operatorname{zer}(\mathrm{id}_Y))$. The mapping $\mathrm{Pr}^\vee\colon X \to \mathcal{C}(Y, (X \times Y)/(\operatorname{zer}(f) \times \operatorname{zer}(\mathrm{id}_Y))$ is constant on the elements of the partition $\operatorname{zer}(f)$, and hence it induces continuous mappings

$$\begin{gathered}\operatorname{fact}\mathrm{proj}^\vee\colon X' \to \mathcal{C}(Y, (X \times Y)/(\operatorname{zer}(f) \times \operatorname{zer}(\mathrm{id}_Y)) \\ (\operatorname{fact}\mathrm{proj}^\vee)^\wedge\colon X' \times Y \to (X \times Y)/(\operatorname{zer}(f) \times \operatorname{zer}(\mathrm{id}_Y)).\end{gathered}$$

It it clear that the second of these mappings is the inverse of the injective factor of $f \times \mathrm{id}_Y\colon X \times Y \to X' \times Y$. Thus the injective factor of $f \times \mathrm{id}_Y\colon X \times Y \to X' \times Y$ is a homeomorphism. ☐

Theorem 1.2.7.9. *Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be factorial maps. If X' and Y are Hausdorff and locally compact, then the map $f \times g\colon X \times Y \to X' \times Y'$ is factorial.*

Proof. In fact, one can express $f \times g$ as the composition

$$X \times Y \xrightarrow{f \times \mathrm{id}_Y} X' \times Y \xrightarrow{\mathrm{id}_{X'} \times g} X' \times Y'$$

and recall that a composition of factorial maps is again factorial. ☐

1.2.8 The Case of Pointed Spaces

Definition 1.2.8.1. In the sequel, the class of topological spaces equipped with a simple additional structure - a distinguished point (i.e, topological pairs (X, x_0), where x_0 is a point) will play an important role; we call these spaces *pointed spaces*, and call the distinguished point a *base point.* The constructions described in the previous subsections must be naturally modified when applied to such spaces. For some of these construction, the modification entails merely the addition of a base point to the resulting space: for example, the quotient space of pointed space (X, x_0) has the natural base point $\mathrm{proj}(x_0)$, the product of the pointed spaces $(X_1, x_1), \dots, (X_n, x_n)$ has the natural base point $(x_1, \dots, x_n)$, and the space of continuous maps from X into a pointed topological space (Y, y_0) contains the constant map $\mathrm{const}\colon X \to Y$, $x \mapsto y_0$, and hence has the natural base point const. Other constructions such as the sum, suspension, and join need more serious modifications.

We shall describe these modified constructions below, and also introduce a new one - the *tensor product of pointed spaces.* In every case, pointed spaces produce pointed spaces, and base point-preserving maps again produce base point-preserving maps. We remark that the maps fact f, $\mathcal{C}(f, g)$, and $f_1 \times \cdots \times f_n$ preserve base points whenever the initial maps have this property.

We use the symbol bp as a general notation for the base points.

Bouquets and Tensor Products

Definition 1.2.8.2. The construction below replaces the sum construction for pointed spaces.

Let $\{X_\mu\}_{\mu \in M}$ be a family of topological spaces with base points. The quotient space of the sum $\coprod_{\mu \in M} X_\mu$ by the subset consisting of all points $\in_\mu (x_\mu)$ is called the *bouquet* (or the *wedge*) of the spaces, and is denoted by $\bigvee_{\mu \in M}(X_\mu, x_\mu)$. If M consists of the numbers $1, \dots, n$, we also write $(X_1, x_1) \bigvee \cdots \bigvee (X_n, x_n)$. The point $\mathrm{proj} \circ \mathrm{incl}_\nu(x_\nu) \in \bigvee(X_\mu, x_\mu)$ does not depend on ν; it is called the *centre of the bouquet* $\bigvee(X_\mu, x_\mu)$, and is taken as its base point.

The bouquet $\bigvee(X_\mu, x_{mu})$ is obviously a union of the spaces X_μ (see Definition 1.2.4.2), and so there exist the embeddings $\mathrm{Imm}_\nu\colon X_\nu \to \bigvee(X_\mu, x_\mu)$. The maps $\mathrm{proj}_\nu\colon \bigvee(X_\mu, x_\mu) \to X_\nu$, defined by

$$\mathrm{proj}_\nu(\mathrm{Imm}_{\nu'}, (x) = \begin{cases} X_{\nu'}, & \text{if} \quad \nu' \neq \nu, \\ x, & \text{if} \quad \nu' = \nu, \end{cases}$$

are specific to the bouquet construction. Clearly, $\mathrm{proj}_\nu \circ \mathrm{Imm} = \mathrm{id}_{X_\nu}$ and $\mathrm{proj}_\nu \circ \mathrm{Imm}_{\nu'} = \mathrm{const}$ if $\nu' \neq \nu$.

If M also indexes another family of pointed spaces (Y_μ, y_ν) and a family of continuous maps $f_\mu\colon X_\mu \to Y_\mu$ such that $f_\mu(x_\mu) = y_\mu$, then the map $\mathrm{fact}(\coprod f_\mu)\colon \bigvee(X_\mu, x_\mu) \to \bigvee(Y_\mu, y_\mu)$ is well-defined and continuous; we denote it by $\bigvee f_\mu$.

Definition 1.2.8.3. Let $(X_1, x_1), \dots, (X_n, x_n)$ be pointed spaces. The rules

$$x \mapsto (x, x_2, \dots, x_n)\ [x \in X_1], \dots, x \mapsto (x_1, \dots, x_{n-1}, x)\ [x \in X_n],$$

define canonical embeddings

$$X_1 \to X_1 \times \cdots \times X_n, \dots, X_n \to X_1 \times \cdots \times X_N,$$

denoted by $\mathrm{incl}_1, \dots, \mathrm{incl}_n$. Moreover, the rule $x \mapsto (\mathrm{proj}_1(x), \dots, \mathrm{proj}_n(x))$ defines a canonical embedding

$$(X_1, x_1) \bigvee \cdots \bigvee (X_n, x_n) \to X_1 \times \cdots \times X_n,$$

which allows us to regard the bouquet $(X_1, x_1) \bigvee \cdots \bigvee (X_n, x_n)$ as a subspace of $X_1 \times \cdots \times X_n$. Clearly, $\mathrm{incl}_i \colon X_i \to X_1 \times \cdots \times X_n$ is the composition of the embedding $\mathrm{Imm}_i \colon X_i \to (X, x_1) \bigvee \cdots \bigvee (X_n, x_n)$ with the inclusion

$$(X, x_1) \bigvee \cdots \bigvee (X_n, x_n) \to X_1 \times \cdots \times X_n$$

while the projection $\mathrm{proj}_i \colon (X, x_1) \bigvee \cdots \bigvee (X_n, x_n) \to X_i$ is the restriction of $\mathrm{proj}_i \colon X_1 \times \cdots \times X_n \to X_i$.

The quotient space

$$(X, x_1) \otimes \cdots \otimes (X_n, x_n) := (X_1 \times \cdots \times X_n)/[(X, x_1) \bigvee \cdots \bigvee (X_n, x_n)]$$

is called the *tensor product of the spaces* $X_1, \dots, X_n$. The point

$$\mathrm{proj}[(X_1, x_1) \bigvee \cdots \bigvee (X_n, x_n)] \in (X, x_1) \otimes \cdots \otimes (X_n, x_n)$$

is called the *centre of the tensor product* $(X, x_1) \otimes \cdots \otimes (X_n, x_n)$ and is taken as its base point.

The tensor product is a commutative and associative operation: there are obvious canonical homeomorphisms

$$(X_1, x_1) \otimes (X_2, x_2) \to (X_2, x_2) \otimes (X_1, x_1)$$
$$(X_1, x_1) \otimes [(X_2, x_2) \otimes (X_3, x_3), \mathrm{bp}] \to [(X_1, x_1) \otimes (X_2, x_2), \mathrm{bp}] \otimes ((X_3, x_3);$$

this is also the way we understand the more general equality

$$[(X_1, x_1) \otimes \cdots \otimes (X_{n-1}, x_{n-1}), \mathrm{bp}] \otimes (X_n, x_n) = (X_1, x_1) \otimes \cdots \otimes (X_n, x_n).$$

If $(Y_1, y_1), \dots, (Y_n, y_n)$ are other pointed spaces and

$$f_1 \colon X_1 \to Y_1, \dots, f_n \colon X_n \to Y_n$$

are continuous, base point-preserving maps, then the map

$$\mathrm{fact}(f_1 \times \cdots \times f_n) \colon (X_1, x_1) \otimes \cdots \otimes (X_n, x_n) \to (Y_1, y_1) \otimes \cdots \otimes (Y_n, y_n)$$

is well defined and continuous; we denote it by $f_1 \otimes \cdots \otimes f_n$.

Cones, Suspensions, and Joins

Definition 1.2.8.4. The *cone over the pointed space* (X, x_0) is defined as quotient of the usual cone $\operatorname{cone} X$ by its generatrix $\operatorname{proj}(x_0 \times I)$, and is denoted by $\operatorname{cone}(X, x_0)$. The image of $\operatorname{proj}(x_0 \times I)$ under the projection

$$\operatorname{cone} X \to \operatorname{cone}(X, x_0)$$

is the *vertex* of $\operatorname{cone}(X, x_0)$, and is taken as its base point. The image of the base of $\operatorname{cone} X$ under the projection $\operatorname{cone} X \to \operatorname{cone}(X, x_0)$ is the *base* of $\operatorname{cone}(X, x_0)$; this projection carries the first base onto the second one, and thus allows us to identify the base of $\operatorname{cone}(X, x_0)$ with X.

If (Y, y_0) is another pointed space and $f \colon X \to Y$ is continuous, with

$$f(x_0) = y_0,$$

then the map $\operatorname{fact} \operatorname{cone} f \colon \operatorname{cone}(X, x_0) \to \operatorname{cone}(Y, y_0)$ is well defined and continuous, and we denote it simply by $\operatorname{cone} f$.

Equivalently, one may describe $\operatorname{cone}(X, x_0)$ as the quotient space of the cylinder $X \times I$ by $(X \times 0) \cup (x_0 \times I)$.

Definition 1.2.8.5. The *suspension of the pointed space* (X, x_0) is defined as the quotient of the usual suspension $\operatorname{sus} X$by its generatrix $\operatorname{proj}(x_0 \times I)$, and is denoted by $\operatorname{sus}(X, x_0)$. The image of this generatrix under the projection $\operatorname{sus} X \to \operatorname{sus}(X, x_0)$ is the *vertex* of $\operatorname{sus}(X, x_0)$ and is taken as its base point.

If (Y, y_0) is another pointed space and $f \colon X \to Y$ is continuous, with $f(X_0) = y_0$, then the map $\operatorname{fact} \operatorname{sus} f \colon \operatorname{sus}(X, x_0) \to \operatorname{sus}(Y, y_0)$ is well defined and continuous, and we denote it simply by $\operatorname{sus} f$.

Equivalently, we may describe $\operatorname{sus}(X, x_0)$ as the quotient space of the cylinder $X \times I$ by $(X \times (0 \cup 1)) \cup (x_0 \times I)$, i.e., as

$$(X, x_0) \otimes (I/(0 \cup 1), \mathrm{bp}) = (X, x_0) \otimes (\mathbb{S}^1, \mathrm{ort}_1).$$

Another equivalent description: $\operatorname{sus}(X, x_0) = \operatorname{cone}(X, x_0)/X$.

Definition 1.2.8.6. The *join of the pointed spaces* (X_1, x_1) and (X_2, x_2) is defined as the quotient space of the usual join $X_1 \star X_2$ by its generatrix

$$\operatorname{proj}(x_1 \times x_2 \times I),$$

and is denoted by $(X_1, x_1) \star (X_2, x_2)$. The image of $\operatorname{proj}(x_1 \times x_2 \times I)$ under the projection $X_1 \star X_2 \to (X_1, x_1) \star (X_2, x_2)$ is the *centre* of $(X_1, x_1) \star (X_2, x_2)$, and is taken as its base point.

If (Y_1, y_1) and (Y_2, y_2) are another pointed spaces, and $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ are continuous maps such that $f_1(x_1) = y_1$ and $f_2(x_2) = y_2$, then the map

$$\operatorname{fact}(f_1 \star f_2) \colon (X_1, x_1) \star (X_2, x_2) \to (Y_1, y_1) \star (Y_2, y_2))$$

is well-defined and continuous, and we denote it simply by $f_1 \star f_2$.

Theorem 1.2.8.7. *For any two pointed spaces* (X_1, x_1) *and* (X_2, x_2)*, the bouquet of suspensions is canonically homeomorphic to the suspension of bouquets:*

$$(\operatorname{sus}(X_1, x_1), \mathrm{bp}) \bigvee (\operatorname{sus}(X_2, x_2), \mathrm{bp}) \approx \operatorname{sus}((X_1, x_1) \bigvee (X_2, x_2), \mathrm{bp}).$$

Proof. The canonical homeomorphism

$$(\operatorname{sus}(X_1, x_1), \mathrm{bp}) \bigvee (\operatorname{sus}(X_2, x_2), \mathrm{bp}) \to \operatorname{sus}((X_1, x_1) \bigvee (X_2, x_2), \mathrm{bp})$$

is given by

$$\operatorname{proj}(\operatorname{Imm}_i(x), t) \mapsto \operatorname{Imm}_i(\operatorname{proj}(x, t)) \quad [x \in X_i, \quad i = 1, 2].$$

☐

Theorem 1.2.8.8. $\operatorname{cone}(\mathbb{S}, \mathrm{ort}_1)$*,* $\operatorname{sus}(\mathbb{S}, \mathrm{ort}_1)$*, and* $(\mathbb{S}^m, \mathrm{ort}_1) \star (\mathbb{S}^n, \mathrm{ort}_1)$ *are canonically homeomorphic to* $\mathbb{D}^{m+1}$*,* $\mathbb{S}^{m+1}$*, and* $\mathbb{S}^{m+n+1}$*, respectively.*

Proof. The canonical homeomorphism $\operatorname{sus}(\mathbb{S}^m, \mathrm{ort}_1) \to \mathbb{D}^{m+1}$ is defined as

$$\operatorname{proj}((x_1, \dots, x_{m+1}), t) \mapsto (tx_1 + (1 - t), tx_2, \dots, tx_{m+1}).$$

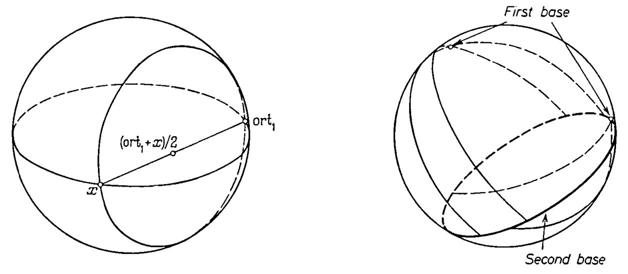

Figure 1.1: Suspensions of spheres

The canonical homeomorphism $\operatorname{sus}(\mathbb{S}^m, \mathrm{ort}_1) \to \mathbb{S}^{m+1}$ transforms the generatrix passing through the point $x \in \mathbb{S}^m$ onto the circle on $\mathbb{S}^{m+1}$ with centre $(\mathrm{ort}_1 + x)/2$ (which degenerates to the point ort_1 if $x = \mathrm{ort}_1$); as t varies from 0 to 1, the image of the point $\operatorname{proj}(x, t)$ moves uniformly on this circle, starting from ort_1, and continuing into the half space $x_{m+1} \leq 0$ (see Fig. 1.1, left).

Finally, the canonical homeomorphism $(\mathbb{S}^m, \mathrm{ort}_1) \star (\mathbb{S}^n, \mathrm{ort}_1) \to \mathbb{S}^{m+n+1}$ is defined on the bases $\mathbb{S}^m$ and $\mathbb{S}^n$ by the formulae

$$(x_1, \dots, x_{m+1}) \mapsto \left(\frac{3x_1+1}{4}, \frac{x_2^{\sqrt{3}}}{2}, \dots, \frac{x_{m+1}^{\sqrt{3}}}{2}, 0, \dots, 0, \frac{\sqrt{3}(1-x_1)}{4} \right)$$
$$(x_1, \dots, x_{n+1}) \mapsto \left(\frac{3x_1+1}{4}, 0, \dots, 0, \frac{x_2^{\sqrt{3}}}{2}, \dots, \frac{x_{n+1}^{\sqrt{3}}}{2}, \frac{\sqrt{3}(x_1-1)}{4} \right)$$

and maps the generatrix joining the points $x \in \mathbb{S}^m$ and $x' \in \mathbb{S}^n$ onto the arc of the great circle on $\mathbb{S}^{m+n+1}$ which joins the images of x and $'$, in such a way that the lengths are linearly transformed (see Fig. 1.1, right). □

Definition 1.2.8.9. Since

$$\mathrm{cone}(\mathbb{S}^m, \mathrm{ort}_1) = (\mathbb{S}^m, \mathrm{ort}_1) \otimes (I, 0) \quad \text{(see Definition1.2.8.4), and}$$
$$\mathrm{sus}(\mathbb{S}^m, \mathrm{ort}_1) = (\mathbb{S}^m, \mathrm{ort}_1) \otimes (\mathbb{S}^1, \mathrm{ort}_1) \quad \text{(see Definition1.2.8.5),}$$

the homeomorphisms $\mathrm{cone}(\mathbb{S}^m, \mathrm{ort}_1) \to \mathbb{D}^{m+1}$ and $\mathrm{sus}(\mathbb{S}^m, \mathrm{ort}_1) \to \mathbb{S}^{m+1}$, defined in Proposition 1.2.8.8, lead for $n \geq 1$ to the canonical homeomorphisms

$$\mathbb{S}^n = \overbrace{(\mathbb{S}^1, \mathrm{ort}_1) \otimes \cdots \otimes (\mathbb{S}^1, \mathrm{ort}_1)}^{n},$$
$$\mathbb{D}^n = \underbrace{(\mathbb{S}^1, \mathrm{ort}_1) \otimes \cdots \otimes (\mathbb{S}^1, \mathrm{ort}_1)}_{n-1} \otimes (I, 0).$$

Now one can define the maps

$$\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \mathrm{proj} \colon \mathbb{D}^n = (\mathbb{S}^1, \mathrm{ort}_1) \otimes \cdots \otimes (\mathbb{S}^1, \mathrm{ort}_1) \otimes (I, 0) \to$$
$$(\mathbb{S}^1, \mathrm{ort}_1) \otimes \cdots \otimes (\mathbb{S}^1, \mathrm{ort}_1) \otimes (I/(0 \cup 1), \mathrm{bp}) = \mathbb{S}^n,$$
$$(\mathrm{proj} \otimes \cdots \otimes \mathrm{proj} \otimes \mathrm{id}_I) \circ \mathrm{proj} \colon I^n = I \times \cdots \times I \to$$
$$(I/(0 \cup 1), \mathrm{bp}) \otimes \cdots \otimes (I/(0 \cup 1), \mathrm{bp}) \otimes (I, 0) = \mathbb{D}^n.$$

We denote these by $\mathbb{DS}$ and $I\mathbb{D}$. It is clear that $\mathbb{DS}$ takes $\mathrm{int}\,\mathbb{D}^n$ homeomorphically onto $\mathbb{S}^n \setminus \mathrm{ort}_1$, while $I\mathbb{D}$ takes homeomorphically $\mathrm{int}\, I^n$ onto $\mathrm{int}\,\mathbb{D}^n$, and $\mathrm{int}\, I^{n-1}$ onto $\mathbb{S}^{n-1} \setminus \mathrm{ort}_1$, and carries $\mathrm{Fr}\, I^n \setminus \mathrm{int}\, I^{n-1}$ into ort_1.

Since the map $\mathbb{DS}$ is closed (see Theorem 1.1.7.9), its injective factor

$$\mathrm{fact}\,\mathbb{DS} \colon \mathbb{D}/\mathbb{S}^{n-1} \to \mathbb{S}^n$$

is a homeomorphism. Consequently, *for $n \geq 1$, the quotient space $\mathbb{D}^n/\mathbb{S}^{n-1}$ is canonically homeomorphic to $\mathbb{S}^n$.*

The Mappings $(X, x_0) \otimes (Y, y_0) \to Z$ and $X \to \mathcal{C}(Y, y_0; Z, z_0)$

Theorem 1.2.8.10. *Let (X, x_0), $(Y, y0)$ and (Z, z_0) be pointed topological spaces. If $\varphi \colon (X, x_0) \otimes (Y, y_0) \to Z$ is continuous and preserves base points, then the* *formula* $[\varphi^{\cup}(x)](y) = \varphi(\text{proj}(x, y))$ *defines a continuous, base point-preserving mapping* $(\varphi^{\cup}\colon X \to \mathcal{C}(Y, y_0; Z, z_0)$.

Let $\psi\colon X \to \mathcal{C}(Y, y_0; Z, z_0)$ *be a continuous, base point-preserving mapping, and suppose that* Y *is Hausdorff and locally compact. Then the formula* $\psi^{\cap}(\text{proj}(x, y)) = [\psi(x)](y)$ *defines a continuous, base point-preserving mapping* $\psi^{\cap}\colon (X, x_0) \otimes (Y, y_0) \to Z$.

Proof. Indeed,

$$\varphi^{\cup} = \text{abr}[(\varphi \circ \text{proj}\colon X \times y \to Z)^{\vee}],$$

while

$$\begin{aligned} &\psi^{\cap} \circ (\text{proj}\colon X \times Y \to (X, x_0) \otimes (Y, y_0)) \\ =&[\psi \circ (\text{incl}\colon \mathcal{C}(Y, y_0; Z, z_0) \to \mathcal{C}(Y, Z))]^{\wedge}. \end{aligned}$$

□

Theorem 1.2.8.11. *Given arbitrary pointed topological spaces* (X, x_0), (Y, y_0), *and* (Z, z_0), *the mapping*

$$\mathcal{C}((X, x_0) \otimes (Y, y_0), \text{bp}; Z, z_0) \to \mathcal{C}(X, x_0; \mathcal{C}(Y, y_0; Z, z_0), \text{const})$$

given by the formula $\varphi \mapsto \varphi^{\cup}$ *(see Theorem 1.2.8.10) is continuous. If* X *and* Y *are Hausdorff and compact, then this mapping is a homeomorphism and its inverse is given by the formula* $\psi \mapsto \psi^{\cap}$.

Proof. The preimage of $\mathcal{C}(X, A, x_0; \mathcal{C}(Y, y_0; Z, z_0), \mathcal{C}(Y, B, y_0; Z, C, z_0), \text{const})$ under the mapping $\varphi \mapsto \varphi^{\cup}$ is just $\mathcal{C}((X, x_0) \otimes (Y, y_0), \text{proj}(A \times B), \text{bp}; Z, C, z_0)$, which shows that $\varphi \mapsto \varphi^{\cup}$ is continuous. Assume now that X and Y are Hausdorff and compact, and consider the mapping

$$\mathcal{C}(\text{proj}_{,i}\, d_Z)\colon \mathcal{C}((X, x_0) \otimes (Y, y_0), Z) \to \mathcal{C}(X \times Y, Z).$$

By Theorem 1.2.7.5, this mapping is an embedding. Consider the diagramme

$$\begin{tikzcd} \mathcal{C}(X, A, x_0; \mathcal{C}(Y, y_0; Z, z_0), \mathcal{C}(Y, B, y_0; Z, C, z_0), \text{const}) \arrow[r] \arrow[d] & \mathcal{C}((X, x_0) \otimes (Y, y_0), \text{bp}; Z, z_0) \arrow[d] \\ \mathcal{C}X, \mathcal{C}(Y, Z) \arrow[r] & \mathcal{C}(X \times Y, Z) \end{tikzcd}$$

where the horizontal arrows denote the mappings i$\psi \mapsto \psi^{\cap}$ and $\psi \mapsto \psi^{\wedge}$, and the vertical arrows the composite mappings

$$\begin{aligned} \mathcal{C}(X, x_0; \mathcal{C}(Y, y_0; Z, z_0), \text{const}) &\xrightarrow{\text{incl}} \mathcal{C}(X, \mathcal{C}(Y, y_0; Z, z_0)) \\ &\xrightarrow{\mathcal{C}(\text{id}_X, \text{incl})} \mathcal{C}(X, \mathcal{C}(Y, Z)) \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}((X, x_0) \otimes (Y, y_0), \text{bp}; Z, z_0) &\xrightarrow{\text{incl}} \mathcal{C}((X, x_0) \otimes (Y, y_0, Z)) \\ &\xrightarrow{\mathcal{C}(\text{proj}, \text{id}_Z)} \mathcal{C}(X \times Y, Z). \end{aligned}$$

Since this diagramme is commutative, the fact that $\mathcal{C}(\mathrm{proj}, \mathrm{id}_Z)$ is an embedding implies the continuity of $\psi \mapsto \psi^{\cap}$. That the mappings $\varphi \mapsto \varphi^{\cup}$ and $\psi \mapsto \psi^{\cap}$ are inverses of one another is plain. □

1.2.9 Exercises

Exercise 1.2.9.1. Show that for each topological space X and each compact topological space Y the map $\mathrm{proj}_1\colon X \times Y \to X$ is closed.

Exercise 1.2.9.2. Show that the subset of $\mathbb{S}^n$ defined by the inequality

$$x_1^2 + \cdots + x_k^2 \leq x_{k+1}^2 + \cdots + x_n^2$$

in the standard coordinates of $\mathbb{R}$, is homeomorphic to $\mathbb{D} \times \mathbb{S}^{n-k}$.

Exercise 1.2.9.3. Let M, X_μ, and $\varphi_{\mu\mu'}$, be as in Definition 1.2.4.2, and let X denote the union of the spaces X_μ defined by the homeomorphisms $\varphi_{\mu\mu'}$. Show that the maps $\mathrm{Imm}_\mu\colon X_\mu \to X$ are topological embeddings whenever M has only two elements, but when M has three elements this is not necessarily so.

Exercise 1.2.9.4. Show that for $n \geq 1$ the spaces $\mathcal{C}(I, 0, 1; \mathbb{S}^n, \mathrm{ort}_1, \mathrm{ort}_2)$ and $\mathcal{C}(I, 0, 1; \mathbb{S}^n, \mathrm{ort}_1, \mathrm{ort}_1)$ are homeomorphic.

Exercise 1.2.9.5. Let $\mathcal{T}$ be the set of all real sequences $\{x_i\}_1^\infty$, with the topology defined by the prebase consisting of all sets of the form $\{\{x_i\}_1^\infty | a < x_s < b\}$. Further, let $\mathcal{S}$ be the quotient space of $\mathcal{T} \setminus 0$ (where $0 = \{x_i = 0\}_1^\infty$) by its partition into rays, i.e, into the sets $\{\{tx_i^0\} | 0 < t < \infty\}$ with $\{x_i^0 \in \mathcal{T} \setminus 0$. Show that $\mathcal{T}$ is metrisable, while $\mathcal{S}$ is regular, but has the peculiar property that every continuous map $\mathcal{S} \to \mathbb{R}$ is constant.

1.3 HOMOTOPIES

1.3.1 General Definitions

Definition 1.3.1.1. A continuous map $f'\colon X \to Y$ is *homotopic* to the continuous map $f\colon X \to Y$ if there is a continuous map $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$, for all $x \in X$. Every such map F is called a *homotopy from f to f'* (or *connecting f to f'*). One says also that F is a *homotopy of f*. A map homotopic to a constant map is also said to be *null homotopic*.

Often a homotopy $F\colon X \times I \to Y$ is interpreted as a family of continuous maps $f_t\colon X \to Y$, related to F via $f_t(x) = F(x,t)$ $(0 \leq t \leq 1)$. According to Theorem 1.2.7.6, the continuity of F implies that this family is continuous as a map of the segment I into $\mathcal{C}(X,Y)$. Moreover, if X is Hausdorff, then the continuity of the family f_t is equivalent to that of the map F.

Obviously, the *constant* homotopy F of a continuous map $f\colon X \to Y$, given by $F(x,t) = f(x)$, connects f to f; if the homotopy F connects f to f', then the *inverse* homotopy, F', defined by $F'(x,t) = F(x, 1-t)$, connects f' to f; if the homotopy F connects f to f' and the homotopy F' connects f' to f'', then their *product* F'', defined as

$$F''(x,t) = \begin{cases} F(x,2t), & \text{for} \quad t \leq 1/2, \\ F'(x,2t-1), & \text{for} \quad t \geq 1/2, \end{cases}$$

is a homotopy connecting f to f''. Thus *homotopy is an equivalence relation*, which yields a partition of $\mathcal{C}(X,Y)$ into equivalence classes, called *homotopy classes*. We denote the set of these classes by $\pi(X,Y)$.

Example 1.3.1.2. An example is the *rectilinear* homotopy. Namely, let f and f' be continuous maps of a space X into a subspace Y of $\mathbb{R}^n$. If for each $x \in X$ the segment joining $f(x)$ to $f'(x)$ is entirely contained in Y, then

$$F(x,t) = (1-t)f(x) + tf'(x)$$

defines a homotopy from f to f', referred to as rectilinear.

Obviously, any two maps of an arbitrary space into $\mathbb{R}^n$ or $\mathbb{D}^n$ are rectilinearly homotopic.

Theorem 1.3.1.3. *Let the maps $f, f'\colon X \to Y$ be homotopic. Then given any continuous maps $g\colon Y \to Y'$ and $h\colon X' \to X$, the maps $g \circ f \circ h$ and $g \circ f' \circ h$ are homotopic.*

Proof. In fact, let $F\colon X \times I \to Y$ be a homotopy from f to f'. Then

$$g \circ F \circ (h \times \mathrm{id}_I)$$

is a homotopy from $g \circ f \circ h$ to $g \circ f' \circ h$. □

Remark 1.3.1.4. As Theorem 1.3.1.3 shows, the mapping

$$\mathcal{C}(h,g)\colon \mathcal{C}(X,Y) \to \mathcal{C}(X',Y')$$

induced by two continuous maps $h\colon X' \to X$ and $g\colon Y \to Y'$ transforms homotopy classes into homotopy classes. The resulting mapping

$$\operatorname{fact}\mathcal{C}(h,g)\colon \pi(X,Y) \to \pi(X',Y')$$

is denoted by $\pi(h,g)$, and Theorem 1.3.1.3 implies that it depends only on the homotopy classes of h and g.

Stationary Homotopies

Definition 1.3.1.5. Let A be a subset of the space X. A homotopy

$$F\colon X \times I \to Y$$

is said to be *stationary on* A or, simply, to be an A-homotopy if $F(x,t) = F(x,0)$ for all $x \in A$ and $t \in I$. Two maps which can be connected by an A-homotopy are *A-homotopic*. (Transcriber's note: In the Western(?) world, this notion is called a *homotopy relative to* A.)

As with usual homotopy, A-homotopy defines an equivalence relation, dividing the set of continuous maps $X \to Y$ which coincide on A with a given map $f\colon A \to Y$, into equivalence classes. The latter are called A-homotopy classes or, in full, *homotopy classes of continuous extensions of the map f to X*. We denote the set of these classes by $\pi(X,A;f)$.

Notice that a rectilinear homotopy from f to f' (see Example 1.3.1.2) is stationary on the set of points where f and g agree.

If one wants to specify that a certain homotopy is ordinary, i.e., not stationary, then one says that it is *free*.

Homotopy Equivalence of Spaces

Definition 1.3.1.6. A continuous map $g\colon Y \to X$ is a *homotopy inverse* of the continuous map $f\colon X \to Y$ if the composition $g \circ f$ is homotopic to id_X and the composition $f \circ g$ is homotopic to id_Y.
(Transcriber's note: this means that there are continuous maps $GF\colon X \times I \to Y$ and $FG\colon Y \times I \to Y$such that

$$\begin{aligned} &GF\colon X \times I \to X, \quad GF(x,0) = g \circ f(x), \quad GF(x,1) = x \\ &FG\colon Y \times I \to Y, \quad FG(y,0) = f \circ g(y), \quad FG(y,1) = y \end{aligned}$$

).
A continuous map which has a homotopy inverse is called a *homotopy equivalence*. If there is a homotopy equivalence $X \to Y$, then one says that the space Y is *homotopy equivalent* to the space X.

The following are obviously homotopy equivalences: the identity map of any space, a map which is a homotopy inverse of a homotopy equivalence, and the composition of two homotopy equivalences. Thus, homotopy equivalence among topological spaces is an equivalence relation. It divides the topological spaces into classes called *homotopy types* (instead of saying that Y is homotopy equivalent to X, one says also that X and Y *have the same homotopy type*).

Every homeomorphism is clearly a homotopy equivalence.

Theorem 1.3.1.7. *If one of the continuous maps* $f\colon X \to Y$ *and* $g\colon Y \to Z$, *and their composition* $g \circ f\colon X \to Z$ *are homotopy equivalences, then the other map is also a homotopy equivalence.*

Proof. Indeed, let h be a homotopy inverse of $g \circ f$, and suppose that f is a homotopy equivalence. Then $f \circ h$ is a homotopy inverse of g. Similarly, if g is a homotopy equivalence, then $h \circ g$ is a homotopy inverse of f.

$$\begin{tikzcd}
X \arrow[rr, shift left, "g\circ f"] \arrow[dr, shift left, "f"] & & Z \arrow[ll, shift left, "h"] \\
& Y \arrow[ul, shift left] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, shift left, "g\circ f"] \arrow[dr, "f"] & & Z \arrow[ll, shift left, "h"] \arrow[dl, shift left] \\
& Y \arrow[ur, shift left, "g"] &
\end{tikzcd}$$

☐

Theorem 1.3.1.8. $\pi(X, Y)$ *is a homotopy invariant. That is to say, if*

$$g\colon Y \to Y', \quad f\colon X \to X'$$

are homotopy equivalences, then $\pi(f, g)\colon \pi(X, Y) \to \pi(X', Y')$ *is invertible.*

Proof. Evidently, if f' (respectively, g') is a homotopy inverse of f (respectively, of g), then the map $\pi(f', g')$ is the inverse of $\pi(f, g)$. ☐

Contractible Spaces

Definition 1.3.1.9. A space X is *contractible* if the map id_X is homotopic to a constant map.

$\mathbb{R}$ and $\mathbb{D}$n are examples of contractible spaces (see Example 1.3.1.2).

Theorem 1.3.1.10. *A space is contractible if and only if it is homotopy equivalent to a point.*

Proof. If id_X is homotopic to a constant map φ, then the map $f\colon \mathbb{D}^0 \to X$ taking the value $\varphi(X)$, and the map $g\colon X \to \mathbb{D}^0$ are homotopy inverses of one another: indeed, $f \circ g = \varphi$ and $g \circ f = \mathrm{id}_{\mathbb{D}^0}$.

If now $f\colon \mathbb{D}^0 \to X$ and $g\colon X \to \mathbb{D}^0$ are homotopy inverses of one another, then id_X is homotopic to the constant map $f \circ g$. ☐

Theorem 1.3.1.11. *If X is contractible, then any two continuous maps of an arbitrary topological space into X are homotopic. In particular,* id_X *is homotopic to any constant map $X \to X$.*

Proof. This is a straightforward consequence of Theorems 1.3.1.10 and 1.3.1.8. □

Deformation Retractions

Definition 1.3.1.12. A retraction ρ of a topological space X onto one of its subspaces A (see Definition 1.1.4.13) is called a *deformation* (*strong deformation*) *retraction* if the composition $X \xrightarrow{\rho} A \xrightarrow{\text{incl}} X$ is homotopic (respectively, A-homotopic) to id_X. If the space X admits a deformation retraction (a strong deformation retraction) onto A, then A is called a *deformation retract* (respectively, a *strong deformation retract*) of X.

Obviously, if $\rho\colon X \to A$ is a deformation retraction, then ρ and the inclusion $A \hookrightarrow X$ are homotopy equivalences, each being a homotopy inverse of the other. It is clear also that any space which admits a deformation retraction onto one of its points is contractible, and that every point of a contractible space is a deformation retract of the ambient space.

Relative Homotopies

Remark 1.3.1.13. Let X (respectively Y) be a space with a distinguished sequence of subsets $A_1, \dots, A_n$ (respectively, $B_1, \dots, B_n$). A map

$$F\colon (X \times I, A_1 \times I, \dots, A_n \times I) \to (Y, B_1, \dots, B_n)$$

is called a *homotopy connecting the continuous maps*

$$f, f'\colon (X, A_1, \dots, A_n) \to (Y, B_1, \dots, B_n)$$

if $\operatorname{abrs} F$ is a homotopy connecting the maps $\operatorname{abrs} f$ and $\operatorname{abrs} f'$. In this case, it is evident that $\operatorname{abr}\operatorname{abrs} F\colon A_i \times I \to B_i$ is a homotopy connecting the maps $\operatorname{abr}\operatorname{abrs} f$, $\operatorname{abr}\operatorname{abrs} f'\colon A_i \to B_i$. Moreover, it is readily seen that the homotopies

$$(X \times I, A_1 \times I, \dots, A_n \times I) \to (Y, B_1, \dots, B_n)$$

yield an equivalence relation. This relation divides

$$\mathcal{C}(X, A_1, \dots, A_n; Y, B_1, \dots, B_n)$$

into homotopy classes forming a set denoted by $\pi(X, A_1, \dots, A_n; Y, B_1, \dots, B_n)$. We may give an analogous definition of the map $\pi(h, g)$ from Remark 1.3.1.4.

A continuous map $g\colon (Y, B_1, \dots, B_n) \to (X, A_1, \dots, A_n)$ is said to be a *homotopy inverse* of the continuous map $f\colon (X, A_1, \dots, A_n) \to (Y, B_1, \dots, B_n)$ if $g \circ f$ is homotopic to $\operatorname{rel} \mathrm{id}_X$ and $f \circ g$ is homotopic to $\operatorname{rel} \mathrm{id}_Y$. A continuous map possessing a homotopy inverse is called a *homotopy equivalence.* Two sequences $(X, A_1, \dots, A_n)$ and $(Y, B_1, \dots, B_n)$ are said to be *homotopy equivalent*, or *to have the same homotopy type*, if they are related by a homotopy equivalence.

Theorems 1.3.1.7 and 1.3.1.8, as they stand, apply to the case of relative homotopy.

Remark 1.3.1.14. The situation discussed in Remark 1.3.1.13 encompasses the case when X and Y are pointed spaces (in this case A_1 and B_1 are points, $n = 1$, and the homotopies defined in Remark 1.3.1.13 are just the homotopies stationary at A_1). Moreover, the definition of a contractible space given in Definition 1.3.1.9 extends to pointed spaces (however, the homotopy from id_X to a constant map must be stationary at the base point). The same is true for Theorems 1.3.1.10 and 1.3.1.11, as well as for the definitions of a deformation retraction and deformation retract, given in Definition 1.3.1.12 (X and A must have the same base point, and the homotopy from the composition

$$X \xrightarrow{\rho} A \xrightarrow{\text{incl}} X$$

to id_X must be stationary at this point). Also, the remarks in Definition 1.3.1.12 remain valid, while the definition of strong deformation retraction is entirely unaffected by the presence of a base point.

1.3.2 Paths

Definition 1.3.2.1. A *path* in a topological space X is any continuous map of the closed unit interval I into X. The points $s(0)$ and $s(1)$ are called the *origin* and the *end* of the path s. Closed path (in which $s(0) = s(1)$) are also termed *loops*.

Given a path s, the formula $t \mapsto s(1-t)$ defines a new path, called the *inverse* of s and denoted by s^{-1}. Given two paths s_1 and s_2 with $s_1(1) = s_2(0)$, the formula

$$t \mapsto \begin{cases} s_1(2t), & \text{for} \quad t \leq 1/2, \\ s_2(2t-1), & \text{for} \quad t \geq 1/2 \end{cases}$$

defines a path, called the *product* of the paths s_1 and s_2, and denoted by $s_1 s_2$. Obviously, $(s^{-1})^{-1} =$ and $(s_1 s_2)^{-1} = s_2^{-1} s_1^{-1}$.

Remark 1.3.2.2. Since $I = D^0 \times I$ if any path can be considered as a homotopy of a map $D^0 \to X$. If one adopts such an interpretation, then the inverse path becomes the inverse homotopy, while the product of paths becomes the product of homotopies.

On the other hand, every homotopy between two continuous maps

$$f, f' \colon X \to Y$$

defines a path in $\mathcal{C}(X, Y)$, joining f and f' (see Proposition 1.2.7.6), and again the inverse path corresponds to the inverse homotopy, and the product of paths to the product of homotopies. If X is Hausdorff and locally compact, then a homotopy connecting two maps $f, f' \colon X \to Y$ may be even defined as a path in $\mathcal{C}(X, Y)$ joining f and f'.

Remark 1.3.2.3. Since any path is a continuous map, it can be also subjected to homotopies. Unfortunately, the generally accepted terminology for such homotopies is not in complete agreement with our definitions in subsection 1.3.1.1 (which are also generally accepted). More precisely, when we consider paths, the homotopies and the homotopy relation are understood always as $(0 \cup 1)$-homotopies (i.e., homotopies stationary at the extremities of the interval I) and $(0 \cup 1)$-homotopy relation, respectively. Moreover, a free homotopy of a loop is understood always as a usual free homotopy whereby the path remains a loop all the time (i.e., as a continuous map $F\colon I \times I \to X$ such that $F(0,t) = F(1,t)$ for all $t \in I$).

1.3.3 Connectedness and k-Connectedness

Remark 1.3.3.1. The properties of topological spaces we study in this subsection represent weaker versions of the contractibility in the absolute case, and of deformation retractability in the relative case.

Definition 1.3.3.2. A topological space is *connected* (see the Preface) if each pair of its points can be joined by a path. (Note by the transcriber: this is important in that we can avoid considering pathological cases like Topologist's sine curve.) Equivalently, X is connected if the set $\pi(D^0, X)$ contains just an element; see Remark 1.3.2.2.

Since $\pi(D^0, X)$ is a homotopy invariant, connectedness is a homotopically invariant property. In particular, all contractible spaces are connected. For example, $\mathbb{R}^n$ and $\mathbb{D}^n$ are connected for every n.

For $n > 0$, $\mathbb{S}^n$ is also connected: any two points of $\mathbb{S}^n$can be joined by a path, which in fact is contained in $\mathbb{S}^n \setminus p$, where p is a third point (recall that the punctured sphere $\mathbb{S}^n \setminus p$ is homeomorphic to $\mathbb{R}^n$. S^0 is not connected: a path joining -1 and 1 would be a continuous function on $[0, 1]$, taking two distinct values but no intermediate ones.

The only connected subsets of the real line $\mathbb{R}$ are the empty set, the finite or infinite intervals, the finite or infinite semi-intervals, and the closed intervals. Indeed, if α and β are the exact lower and upper bounds of a connected subset A of $\mathbb{R}$, then A contains the interval (α, β).

Remark 1.3.3.3. Given an arbitrary topological space X, the property of being joined by a path defines a relation between its points, which obviously satisfies all the requirements for an equivalence relation . This relation defines a partition of X into subsets which are the maximal connected subsets of X, and are called the *components* of X. Clearly, the set of components may be identified with $\pi(D^0, X)$. We denote it by $\operatorname{comp} X$.

Every continuous map $f\colon X \to Y$ induces the map

$$\operatorname{fact} f = \pi(\operatorname{id} D^0, f)\colon \operatorname{comp} X \to \operatorname{comp} Y.$$

This map does not change when we replace f by an arbitrary homotopic map, and $\operatorname{fact} f$ is invertible whenever f is a homotopy equivalence (see Remark 1.3.1.4

and Theorem 1.3.1.8). It is also plain that if $f(X) = Y$, then

$$\operatorname{fact} f(\operatorname{comp} X) = \operatorname{comp} Y.$$

In particular, the image of a connected space under a continuous map is connected.

Theorem 1.3.3.4. *If X can be written as the union of two connected subsets A_1 and A_2 with $A_1 \cap A_2 \neq \varnothing$, then X is connected.*

Proof. Indeed, a component of X which contains a point $x_0 \in A_1 \cap A_2$ contains also A_1 and A_2, i.e., contains X. ☐

Theorem 1.3.3.5. *Consider a partition of X into open sets. Then every connected subset of X is contained in one of the elements of this partition. In particular, every subset of a connected space which is both open and closed is either empty or the whole space X.*

Proof. Let A be a connected subset of X, and let U be an element of the partition, such that $U \cap A \neq \varnothing$. Consider the map $f\colon X \to \mathbb{S}^0$ which takes U into 1 and $X \setminus U$ into -1. Since f is continuous, $f(A)$ is connected, whence $f(A) = 1$ and $A \subset U$. ☐

k-Connectedness

Theorem 1.3.3.6. *The following properties of a continuous map $f\colon \mathbb{S}^r \to X$ with $r \geq 0$ are equivalent:*

(i) *f is homotopic to a constant map;*

(ii) *f extends to a continuous map $\mathbb{D}^{r+1} \to X$;*

(iii) *the compositions $f \circ \mathbb{DS}_+, f \circ \mathbb{DS}_-\colon \mathbb{D}^r \to X$ are $\mathbb{S}^{r-1}$-homotopic, where $\mathbb{DS}_+$ and $\mathbb{DS}_-$ are the embeddings of $\mathbb{D}^r$ in $\mathbb{S}^r$, defined by*

$$\mathbb{DS}_+(x_1, \dots, x_r) = \left(x_1, \dots, x_r, \sqrt{1 - x_1^2 - \cdots - x_r^2}\right) \quad \textit{and}$$
$$\mathbb{DS}_-(x_1, \dots, x_r) = \left(x_1, \dots, x_r, -\sqrt{1 - x_1^2 - \cdots - x_r^2}\right)$$

(iv) *f is ort_1-homotopic to a constant map.*

Proof. The proof follows the following scheme:

$$\begin{tikzcd}
(i) \arrow[dr, Rightarrow] \arrow[dd, dotted] & & \\
& (ii) \arrow[r, Leftrightarrow] & (iii) \\
(iv) \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

(i) → *(ii)*. A homotopy $F\colon \mathbb{S}^r \times I \to X$ from f to a constant map takes the upper base of the cylinder $\mathbb{S}^r \times I$ into one point. Consequently, F may be expressed as the composition of the map $\mathbb{S}^r \times I \to \mathbb{D}^{r+1}$, defined by

$$((x_1, \dots, x_{r+1}), t) \mapsto (x_1(1-t), \dots, x_{r+1}(1-t)),$$

and a continuous map $g\colon \mathbb{D}^{r+1} \to X$ (see Remark 1.2.3.4 and Proposition 1.1.7.9), and it is clear that $g|_{\mathbb{S}^r} = f$.
(ii) → *(iii)* and *(ii)* → *(iv)*. Suppose $g\colon \mathbb{D}^{r+1} \to X$ is a continuous extension of f. Then the formulae

$$(x_1, \dots, x_{r+1}), t) \mapsto g\left(x_1, \dots, x_{r+1}, (1-2t)\sqrt{1 - x_1^2 - \dots - x_r^2}\right) \quad \text{and}$$

$$(x_1, \dots, x_{r+1}), t) \mapsto g(t + (1-t)x_1, (1-t)x_2, \dots, (1-t)x_{r+1})$$

define an $\mathbb{S}^{r-1}$-homotopy $\mathbb{D}^r \times I \to X$ from $f \circ \mathbb{DS}_+$ to $f \circ \mathbb{DS}_-$, and an ort$_1$-homotopy $\mathbb{S}^r \times I \to X$ from f to a constant map.
(iii) → *(ii)*. An $\mathbb{S}^r$-homotopy $f\colon \mathbb{D}^r \times I \to X$ from $f \circ \mathbb{DS}_+$ to $f \circ \mathbb{DS}_-$ takes every generatrix of the cylinder $\mathbb{S}^r \times I$ into one point. Consequently, F can be expressed as the composition of the map $D^r \times I \to \mathbb{D}^{r+1}$, defined by

$$((x_1, \dots, x_r), t) \mapsto \left(x_1, \dots, x_r, (2t-1)\sqrt{1 - x_1^2 - \dots - x_r^2}\right),$$

and some continuous map $g\colon \mathbb{D}^{r+1} \to X$ (see Remark 1.2.3.4 and Proposition 1.1.7.9), and it is clear that $g|_{\mathbb{S}^r} = f$.
(iv) → *(i)*. This implication is trivial. □

Definition 1.3.3.7. A non-empty space X is said to be *k-connected* ($0 \le k \le \infty$), if any continuous map $\mathbb{S}^r \to X$ with $r \le k$ is homotopic to a constant map, i.e., satisfies condition (i) of Proposition 1.3.3.6. Theorem 1.3.3.6 shows that this definition has three more equivalent formulations, based on conditions (ii), (iii) and (iv. Moreover, since for any continuous maps $f_1, f_2\colon \mathbb{D}^r \to X$ which agree on $\mathbb{S}^{r-1}$ there is a continuous map $f\colon \mathbb{S}^r \to X$ such that $f \circ \mathbb{DS}_+ = f_1$ and $f \circ \mathbb{DS}_- = f_2$, we conclude that a non-empty space X is k-connected if and only if any continuous maps $f_1, f_2\colon \mathbb{D}^r \to X$, $r \le k$, which agree on $\mathbb{S}^{r-1}$ are $\mathbb{S}^{r-1}$-homotopic.

Obviously, for non-empty spaces 0-connectedness is nothing else but connectedness. The 1-connected spaces are usually called *simply connected*. Note that a 0-connected space is simply connected if and only if any two paths with common extremities are homotopic.

The homotopy invariance of the sets $\pi(\mathbb{S}^r, X)$ implies that a space which is homotopy equivalent to a k-connected space is itself k-connected. In particular, every contractible space is ∞-connected.

The Relative Case

Theorem 1.3.3.8. *The following properties of a continuous map*

$$f\colon (\mathbb{D}^r, \mathbb{S}^{r-1}) \to (X, A), \quad r > 0$$

are equivalent:

(i) f is homotopic to a constant map;

(ii) abrs f *is $\mathbb{S}^{r-1}$-homotopic to a map which carries $\mathbb{D}^r$ into a subset of A;*

Proof. (i) ⇒ (ii): If $F\colon (\mathbb{D}^r \times I, \mathbb{S}^r \times I) \to (X, A)$ is a homotopy from f to a constant map, then the formula

$$(x,t) \mapsto \begin{cases} F(x/\operatorname{dist}(0,x), 2(1-\operatorname{dist}(0,x))), & \text{if} \quad \operatorname{dist}(0,x) \geq (2-t)/2, \\ F(2x/(2-t), t), & \text{if} \quad \operatorname{dist}(0,x) \leq (2-t)/2, \end{cases}$$

defines an $\mathbb{S}^r$-homotopy $\mathbb{D} \times I \to X$ from abrs f to a map which carries $\mathbb{D}^r$ into a subset of A.
(ii) ⇒ (i): If $G\colon \mathbb{D}^r \times I \to X$ is a homotopy stationary on $\mathbb{S}^{r-1}$ from abrs f to a map which carries $\mathbb{D}^r$ into a subset of A, consider the map $F\colon \mathbb{D}^r \times I \to X$ given by

$$F((x_1, \ldots, x_r), t) = \begin{cases} G(x_1, \ldots, x_r), 2t), & \text{if} \quad t \leq 1/2, \\ G((2x_1(1-t), \ldots, 2x_r(1-t), 1), & \text{if} \quad t \geq 1/2. \end{cases}$$

Then rel $F\colon (\mathbb{D}^r \times I, \mathbb{S}^{r-1} \times I) \to (X, A)$ is a homotopy from f to a constant map. □

Remark 1.3.3.9. A pair (X, A) is *k-connected* ($0 \leq k \leq \infty$) if for any map $f\colon (\mathbb{D}^r, \mathbb{S}^{r-1}) \to (X, A)$ with $r \leq k$, abrs f is $\mathbb{S}^{r-1}$-homotopic to a map whose image is contained in A.

It is clear that the pair (X, A) is 0-connected if and only if each component of the space X intersects A. If $k > 0$, then (X, A) is k-connected if and only if every continuous map $f\colon (\mathbb{D}^r, \mathbb{S}^{r-1}) \to (X, A)$ with $r \leq k$ is homotopic to a constant map; see Theorem 1.3.3.8.

A pair which is homotopy equivalent to a k-connected pair is k-connected. As a consequence, we see that when A is a strong deformation retract of X, the pair (X, A) is ∞-connected; indeed, (X, A) is homotopy equivalent to (X, X). It will be clear later that the pair (X, A) is already ∞-connected if A is a deformation retract of X, or even when the inclusion $A \to X$ is a homotopy equivalence; see Remark 5.1.6.7.

1.3.4 Local Properties

Definition 1.3.4.1. A topological space X is *locally contractible at the point* $x_0 \in X$ if each neighbourhood U of x_0 contains another neighbourhood V of x_0 such that the inclusion $V \hookrightarrow U$ is homotopic to the constant map $V \to x_0$. A topological space is *locally contractible* if it is locally contractible at any of its points.

If we replace in these definitions the homotopies by x_0-homotopies, the we get the definitions of a space X which is *strongly locally contractible at the point* x_0, and of a *strongly locally contractible space* X.

$\mathbb{R}^n$, $\mathbb{D}^n$ and $\mathbb{S}^n$ are examples of strongly locally contractible spaces.

Definition 1.3.4.2. A topological space X is *locally connected at the point* $x_0 \in X$ if each neighbourhood U of x_0q contains another neighbourhood V of x_0, such that any two points in V can be joined by a path in U. A topological space is *locally connected* if it is locally connected at any of its points.

It is clear that a locally contractible space is locally connected. As an example of a connected space which is not locally connected we may take the subset of $\mathbb{R}^2$ consisting of the lines $m_1x_1 + m_2x_2$, with $m_1, m_2 \in \mathbb{Z}$.

Theorem 1.3.4.3. *A space is locally connected if and only if the components of its open sets are open. In particular, in a locally connected space every neighbourhood of an arbitrary point contains a connected neighbourhood of this point.*

Proof. Suppose that X is locally connected, U is an open subset of X, A a component of U, and $x_0 \in A$ an arbitrary point. Then, by definition, U contains a neighbourhood V of x_0 such that any two points in V can be joined by a path in U. Hence $V \subset A$ and $x_0 \in \operatorname{int} A$. This proves that in a locally connected space the components of the open sets are open. ☐

1.3.5 Borsuk Pairs

Definition 1.3.5.1. A topological pair (X, A) is a *Borsuk pair* if given any topological space YE, any continuous map $f\colon X \to Y$, and any homotopy $F\colon A \times I \to Y$ of the map $f|_A$, there is a homotopy $X \times I \to Y$ of f which extends F.

$$\begin{tikzcd}
A \times 0 \arrow[rr, "\subset"] \arrow[dr, "f|_{A\times 0}"] \arrow[dd, hook] & & X \times 0 \arrow[dl, "f"'] \arrow[dd, hook] \\
& Y & \\
A \times I \arrow[ur, "F"] \arrow[rr, "\subset"'] & & X \times I \arrow[ul, dotted]
\end{tikzcd}$$

(Transcriber's note: As stated in the Preface, this is called a "cofibration" in the western literature.)

If (X, A, B) is a topological triple such that (X, A) and (A, B) are Borsuk pairs, then (X, B) is obviously a Borsuk pair.

Theorem 1.3.5.2. *Let (X, A) be a topological pair. Then in order for (X, A) to be a Borsuk pair it is necessary that $(X \times 0) \cup (A \times I)$ be a retract of the cylinder $X \times I$. When A is closed, this condition is also sufficient.*

Proof. THE NECESSITY. Any homotopy of the map

$$\text{incl}\colon X = X \times 0 \to (X \times 0) \cup (A \times I)$$

which extends the homotopyincl: $A \times I \to (X \times 0) \cup (A \times I)$ is a retraction of the cylinder $X \times I$ onto $(X \times 0) \cup (A \times I)$.

THE SUFFICIENCY. Let $\rho\colon X \times I \to (X \times 0) \cup (A \times I)$ be a retraction. Then given any topological space Y, any continuous map $f\colon X \to Y$, and any homotopy $F\colon A \times I \to Y$ of the map $f|_A$, the composition

$$X \times I \xrightarrow{\rho} (X \times 0) \cup (A \times I) \xrightarrow{G} Y,$$

where G is defined by

$$G(x,t) = \begin{cases} f(x), & \text{if} \quad t = 0, \\ F(x,t), & \text{if} \quad x \in A, \end{cases}$$

is a homotopy of f which extends F. ☐

Remark 1.3.5.3. The following statement completes Theorem 1.3.5.22 in an essential way. If X is Hausdorff, then the assumption that $(X \times 0) \cup (A \times I)$ is a retract of the cylinder $X \times I$ implies automatically that A is closed.

Indeed, it suffices to note that the above hypothesis implies that $(X \times 0) \cup (A \times I)$ is closed in $X \times I$ (see Proposition 1.1.5.5), and that A is the pre-image of this set under the map $X \to X \times I$, $x \mapsto (x, 1)$.

Theorem 1.3.5.4. *If the sets A and B form a closed cover of the space X and $(A, A \cap B)$ is a Borsuk pair, then (X, B) is also a Borsuk pair.*

Proof. This is a consequence of Proposition 1.3.5.2; in fact, any retraction

$$\rho\colon A \times I \to [A \times 0] \cup [(A \cap B) \times I]$$

defines a retraction $X \times I \to (X \times 0) \cup (B \times I)$ by

$$(x,t) \mapsto \begin{cases} \rho(x,t), & \text{if} \quad x \in A, \\ (x,t), & \text{if} \quad x \in B. \end{cases}$$

☐

Theorem 1.3.5.5. *If (X, A) is a Borsuk pair and A is closed, then*

$$(Z \times X, Z \times A)$$

is a Borsuk pair for every topological space Z.

Proof. If ρ is a retraction of the cylinder $X \times I$ onto $(X \times 0) \cup (A \times I)$, then $\mathrm{id}_Z \times \rho$ is a retraction of the cylinder $(Z \times X) \times I = Z \times (X \times l)$ onto

$$[(Z \times X) \times 0] \cup [(Z \times A) \times I] = Z \times [(X \times 0) \cup (A \times I)].$$

☐

Borsuk Pairs and Deformation Retractions

Theorem 1.3.5.6. *If (X, A) is a Borsuk pair and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.*

Proof. Let $pi\colon X \to A$ be a homotopy inverse of the inclusion $A \hookrightarrow X$. Extend the homotopy from $\pi|_A = \pi \circ \mathrm{incl}\colon A \to A$ to id_A to a homotopy of the map π; this yields a homotopy from π to a retraction of X onto A, which we denote by ρ. Since the composition $X \xrightarrow{\pi} A \xrightarrow{\mathrm{incl}} X$ is homotopic to id_X, the composition $X \xrightarrow{\rho} A \xrightarrow{\mathrm{incl}} X$ is also homotopic to id_X, and thus ρ is a deformation retraction. □

Theorem 1.3.5.7. *If A is a deformation retract of X and*

$$(X \times I, (X \times 0) \cup (A \times I) \cup (X \times 1))$$

is a Borsuk pair, then A is a strong deformation retract of X.

Proof. Let $\rho\colon X \to A$ be a deformation retraction, and let $f\colon X \times I \to X$ be a homotopy from id_X to the composite map $X \xrightarrow{\rho} A \xrightarrow{\mathrm{incl}} X$. Define a homotopy

$$g\colon [(X \times 0) \cup (A \times I) \cup (X \times 1)] \times I \to X$$

by

$$g((x, t_1), t_2) = \begin{cases} x, & \text{if} \quad t_1 = 0, \\ f(x, (1 - t_2)t_1), & \text{if} \quad x \in A, \\ f(\rho(x), 1 - t_2), & \text{if} \quad t_1 = 1, \end{cases}$$

and extend it to some homotopy $G\colon (X \times I) \times I \to X$ of the map $f\colon X \times I \to X$. It is clear that $(x, t) \mapsto G((x, t), 1)$ yields an A-homotopy $X \times I \to X$ from id_X to $\mathrm{incl} \circ \rho$. □

Theorem 1.3.5.8. *If (X, A) is a Borsuk pair and B is a strong deformation retract of the space A, then the map* rel: $(X, B) \to (X, A)$ *is a homotopy equivalence.*

Proof. Consider a B-homotopy from id_A to the composition of a strong deformation retraction $A \to B$ and the inclusion $B \hookrightarrow A$. Now extend it to a homotopy G of id_X. It is clear that the map $(X, A) \to (X, B)$, $x \mapsto G(x, 1)$, is a homotopy inverse of rel. □

Local Characteristics of Borsuk Pairs

Theorem 1.3.5.9. *Suppose that (X, A) is a Borsuk pair with X normal, Y is any topological space, and $f\colon X \to Y$ is any continuous map. Then given any homotopy F of the map $f|_A$ and any neighbourhood U of A, there is an $(X \setminus U)$-homotopy of f extending F.*

Proof. Let G be a homotopy of f extending F, and let φ be any Urysohn function for the pair $(X \setminus U, A)$. Then the formula $(x, t) \mapsto G(x, t\varphi(x))$ defines an $(X \setminus U)$-homotopy of f extending F. □

Theorem 1.3.5.10. *If (X, A) is a Borsuk pair, then there exists a neighbourhood U of A such that the inclusion $U \hookrightarrow X$ is A-homotopic to a map which takes U into a subset of A. If X is normal and A is distinguishable (in particular, if X is metrisable and A is closed), then this condition is also sufficient, i.e., the converse of the above statement is valid.*

Proof. *THE SUFFICIENCY*: Let $F\colon U \times I \to X$ be an A-homotopy such that $F(x, 0) = x$ and $f(x, 1) \in A$ for all $x \in U$, and let $\varphi\colon X \to I$ be a Urysohn function for the pair $(A, X \setminus U)$, which distinguishes A (see definition 1.1.5.11). The formula

$$G(x,t) = \begin{cases} F(x, \min(t/\varphi(x), 1)), & \text{if} \quad x \in U \setminus A, \\ x, & \text{if} \quad x \in A, \end{cases}$$

defines a map $G\colon U \times I \to X$, and Theorem 1.2.2.14 shows that G is continuous. This in turn implies the continuity of the map $H\colon X \times I \to X \times I$ defined by

$$H(x,t) = \begin{cases} (G(x, \max(0, t - \varphi(x))), \max(0, t - 2\varphi(x))) & \text{if} \quad x \in U, \\ (x, 0) & \text{if} \quad x \in X \setminus U. \end{cases}$$

It is readily seen that $H(X \times I) = (X \times 0) \cup (A \times I)$ and that

$$\operatorname{abr} H\colon X \times I \to (X \times 0) \cup (A \times I)$$

is a retraction. □

Theorem 1.3.5.11. *Let (X, A) be a topological pair such that A is a strong deformation retract of one of its neighbourhoods. If X is normal and A is distinguishable (in particular, if X is metrisable and A is closed), then (X, A) is a Borsuk pair.*

Proof. This is a corollary of Theorem 1.3.5.10. □

Theorem 1.3.5.12. *If (X, A) is a Borsuk pair, then given any neighbourhood V of A, there is another neighbourhood W of A, such that $W \subset V$ and the inclusion $W \hookrightarrow V$ is A-homotopic to a map which takes W into a subset of A.*

Proof. By Theorem 1.3.5.10, there exists a neighbourhood U of A and an A-homotopy $F\colon U \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) \in A$ for all $x \in U$. Now Lemma 1.2.2.13 shows that every point $x \in U$ has a neighbourhood W_x in U with $F(W_x \times I) \subset V$. Set $W = \cup_{x \in A} W_x$. It is clear that $W \subset V$, $F(W \times I) \subset V$, and that $\operatorname{abr} F\colon W \times I \to V$ is an A-homotopy from the inclusion $W \to I$ to a map which takes W into A. □

Theorem 1.3.5.13. *If X is a topological space and $x \in X$ is such that (X, x) is a Borsuk pair, then X is strongly locally contractible at x. If X is normal and locally contractible at a distinguishable point x, then (X, x) is a Borsuk pair.*

Proof. This is a consequence of Theorems 1.3.5.12 and 1.3.5.10. □

1.3.6 CNRS-spaces

Definition 1.3.6.1. A subset A of a topological space X is said to be a *neighbourhood retract of* X if A is a retract of one of its neighbourhoods in X.

The retracts and the open sets are trivial examples of neighbourhood retracts.

Theorem 1.3.6.2. *If* A *is a neighbourhood retract of* X *and* B *is a neighbourhood retract of* A*, then* B *is a neighbourhood retract of* X*.*

Proof. Indeed, let $\rho\colon\ : U \to A$ be a neighbourhood retraction for A in X, and let $\sigma\colon V \to B$ be a neighbourhood retraction for B in A. Then $\sigma \circ (\rho|_W)\colon W \to B$, where $W = \rho^{-1}(V)$, is a neighbourhood retraction for B in X. ☐

Definition 1.3.6.3. A topological space is a *CNRS-space* or, simply, a *CNRS*, if it is compact and can be embedded in a Euclidean space (of a certain dimension) as a neighbourhood retract; CNRS is the abbreviation of *compact neighbourhood retract of a sphere.*

$\mathbb{D}^n$ and $\mathbb{S}^n$ are obvious examples of CNRS's.

Theorem 1.3.6.4. *A compact neighbourhood retract of a CNRS is a CNRS.*

Proof. This is a result of Theorem 1.3.6.2. ☐

Theorem 1.3.6.5. *The image of any embedding of a CNRS in a normal space is a neighbourhood retract.*

Proof. Let $f\colon X \to Y$ be an embedding of the CNRS X in the normal space Y, and let $g\colon X \to \mathbb{R}^n$ be an embedding of X such that $g(X)$ is a neighbourhood retract of $\mathbb{R}^n$. Further, consider

$$f_1 = [\operatorname{abr} f\colon X \to f(X)], \quad g_1 = [\operatorname{abr} g\colon X \to g(X)],$$

and let $\rho\colon V \to g(X)$ be a neighbourhood retraction. Since $f(X)$ is closed (see Proposition 1.1.7.9), $g \circ f_1^{-1}\colon f(X) \to \mathbb{R}^n$ extends to a continuous map

$$h\colon Y \to \mathbb{R}^n$$

(see Theorem 1.1.5.17). It is clear that $U = h^{-1}(V)$ is a neighbourhood of $f(X)$, and that $f_1 \circ g_1^{-1} \circ \rho \circ [\operatorname{abr} h\colon U \to V]$ is a retraction of U onto $f(X)$. ☐

Theorem 1.3.6.6. *Given any compact neighbourhood retract* X *of* $\mathbb{R}^n$*, there is a number* $\varepsilon > 0$ *such that any two maps,* f *and* g*, of an arbitrary space* Y *into* X *which satisfy*

$$\sup_{y \in Y} \operatorname{dist}(f(y), g(y)) < \varepsilon$$

are homotopic. Moreover, one may choose a homotopy stationary on the set where f *and* g *agree.*

Proof. Let $\sigma\colon U \to X$ be a neighbourhood retraction. We show that one may take ε to be the distance between X and $\mathbb{R}^n \setminus U$ (which is positive by Theorem 1.1.7.15).

Let $f, g\colon Y \to X$ be continuous and satisfy $\sup_{y\in Y} \operatorname{dist}(f(y), g(y)) < \varepsilon$. For any point $y \in Y$, the segment with the extremities $f(y)$ and $g(y)$ lies in U. Consequently, the composite maps $Y \xrightarrow{f} X \xrightarrow{\text{incl}} U$ and $Y \xrightarrow{g} X \xrightarrow{\text{incl}} U$ can be connected by a rectilinear homotopy $F\colon Y \times I \to U$, and it is plain that $\sigma \circ F\colon Y \times I \to X$ is a homotopy from f to g. Furthermore, $\sigma \circ F$ is is stationary on the set where f and g agree. □

Theorem 1.3.6.7. *if A is a neighbourhood retract of a CNRS X, then (X, A) is a Borsuk pair.*

Proof. Let $\sigma\colon U \to A$ be a neighbourhood retraction. Consider X as a neighbourhood retract of $\mathbb{R}^n$ and pick ε as in Theorem 1.3.6.6. Denote by V the neighbourhood of A in X consisting of all the point $x \in U$ for which $\operatorname{dist}(x, \sigma(x)) < \varepsilon$, and let φ be the composition

$$V \xrightarrow{\sigma|_V} A \xrightarrow{\text{incl}} X.$$

Then $\operatorname{dist}(\varphi(x), x) < \varepsilon$ for $x \in V$, and $\varphi(x) = x$ for $x \in A$. Hence the inclusion $V \to X$ is A-homotopic to φ, and since $\varphi(V) = A$, we can apply Theorem 1.3.5.10. □

Theorem 1.3.6.8. *Every CNRS is strongly locally contractible.*

Proof. This is a consequence of Definition 1.3.5.1, and Theorem 1.3.5.13. □

Information

Information 1.3.6.9. The converse of Theorem 1.3.6.8 is also true: every locally contractible compact subspace of a Euclidean space is a neighbourhood retract of this space. For a proof, see [14].

1.3.7 Homotopy Properties of Topological Constructions

Remark 1.3.7.1. In this subsection we establish the homotopy invariance of some of the constructions described in §1.2 and study the homotopy properties of the resulting spaces.

Products

Remark 1.3.7.2. Obviously, two continuous maps $f, g\colon Y \to X_1 \times \cdots \times X_n$ are homotopic if and only if $\operatorname{proj}_i \circ f, \operatorname{proj}_i \circ g\colon Y \to X_i$ are homotopic for all $i = 1, \ldots, n$ (see Remark 1.2.2.4). In particular, $X_1 \times \cdots \times x_n$ is k-connected if and only if all the X_i's are k-connected $(0 \leq k \leq \infty)$.

It is also clear that if the maps $g_1\colon X_1 \to Y_1, \dots, g_n\colon X_n \to Y_n$ are homotopic to the maps $f_1\colon X_1 \to Y_1, \dots, f_n\colon X_n \to Y_n$, then

$$\begin{aligned} &g_1 \times \cdots \times g_n\colon X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n \quad \text{and} \\ &f_1 \times \cdots \times f_n\colon X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n \end{aligned}$$

are homotopic; moreover, if $f_1, \dots, f_n$ are homotopy equivalences, then so is $f_1 \times \cdots \times f_n$.

Remark 1.3.7.3. If A is a deformation retract (strong deformation retract) of X, then $A \times UY$ is a deformation retract (respectively, a strong deformation retract) of the product $X \times Y$, for any space Y. In particular, if X is contractible, then the fibres $x \times Y$ of the product $X \times Y$ are deformation retracts of $X \times Y$.

Quotients

Remark 1.3.7.4. Since the projection $X \to X/\rho$ is continuous, the quotient space of a connected space is connected (see Remark 1.3.3.3). Moreover, in order that the quotient X/A of X by its subspace A be connected, it is even enough that the pair (X, A) be 0-connected; if the components of X are open, then the connectedness of X/A implies the 0-connectedness of (X, A).

We shall see later that neither the k-connectedness with $k > 0$, nor the contractibility are, generally speaking, preserved when one takes quotients.

Theorem 1.3.7.5. *Let $\wp$ and q be partitions of the spaces X and Y. If the maps $f_t\colon X \to Y$ form a homotopy and take the elements of $\wp$ into elements of q, then the maps* $\operatorname{fact} f_t\colon X/\wp \to Y/q$ *also form a homotopy.*

Proof. We have to verify that the map

$$G\colon (X/\wp) \times I \to Y/q, \quad (x, t) \mapsto (\operatorname{fact} f_t)(x),$$

is continuous. To do this, it suffices to note (see Remark 1.2.3.4) that the composition

$$X \times I \xrightarrow{\operatorname{proj} \times \operatorname{id}_I} (X/\wp) \times I \xrightarrow{G} Y/q$$

is continuous, and that the map $\operatorname{proj} \times \operatorname{id}_I$ is factorial. The first is a consequence of the commutativity of the diagram

$$\begin{tikzcd} X \times I \arrow[r, "\operatorname{proj} \times \operatorname{id}_I"] \arrow[d, "F"'] & (X/\wp) \times I \arrow[d, "G"] \\ Y \arrow[r, "\operatorname{proj}"'] & Y/q \end{tikzcd}$$

where F is $F(x, t) = f_t(x)$, while the second follows from Theorem 1.2.7.8. $\square$

Theorem 1.3.7.6. *If $f, f'\colon (X, A) \to (Y, B)$ are homotopic, then the maps* $\operatorname{rel}\operatorname{fact} f, \operatorname{rel}\operatorname{fact} f'\colon (X/A, \operatorname{proj}(A)) \to (Y/B, \operatorname{proj}(B))$ *are also homotopic. If f is a homotopy equivalence, then so is* $\operatorname{rel}\operatorname{fact} f$.

Proof. The first assertion is a corollary of Theorem 1.3.7.5, while the second is a consequence of the first: if $g\colon (Y,B) \to (X,A)$ is a homotopy inverse of f, then $\operatorname{rel}\operatorname{fact} g\colon (Y/B, \operatorname{proj}(B)) \to (X/A, \operatorname{proj}(A))$ is a homotopy inverse of $\operatorname{rel}\operatorname{fact} f$. □

Theorem 1.3.7.7. *If (X,A) is a Borsuk pair and A is contractible, then $\operatorname{rel}\operatorname{proj}\colon (X,A) \to (X/A, \operatorname{proj}(A))$ is a homotopy equivalence.*

Proof. Let $F\colon A \times I \to A$ be a homotopy from id_A to a constant map, and let $G\colon X \times I \to X$ be a homotopy of id_X extending F. Denote by $g\colon X \to X$ the map $x \mapsto G(x,1)$ which is homotopic to id_X. Since g is constant on A, the map $\operatorname{fact} g\colon X/A \to X$ is meaningful; moreover, since $\operatorname{fact} g(\operatorname{proj}(A)) \subset A$, the map $\operatorname{rel}\operatorname{fact} g\colon (X/A; \operatorname{proj}(A)) \to (X,A)$ is also meaningful. Let us check that $\operatorname{rel}\operatorname{fact} g$ is a homotopy inverse of $\operatorname{rel}\operatorname{proj}$. Consider the homotopies

$$\begin{aligned}&\operatorname{rel} G\colon (X \times I, A \times I) \to (X,A) \quad \text{and}\\ &\operatorname{rel}\operatorname{fact} G\colon ((X/A) \times I, \operatorname{proj}(A) \times I) \to (X/A, \operatorname{proj}(A)).\end{aligned}$$

The first connects the maps

$$\operatorname{rel}\mathrm{id}_X, \operatorname{rel} g\colon (X,A) \to (X,A)$$

while the second connects the maps

$$\operatorname{rel}\mathrm{id}_{(X/A)}, \operatorname{rel}\operatorname{fact} g\colon (X/A, \operatorname{proj}(A)) \to (X/A, \operatorname{proj}(A)).$$

It is clear that

$$\begin{aligned}&[\operatorname{rel} g\colon (X,A) \to (X,A)] =\\ &[\operatorname{rel}\operatorname{fact} g\colon (X/A, \operatorname{proj}(A)) \to (X,A)] \circ [\operatorname{rel}\operatorname{proj}\colon (X,A) \to (X/A, \operatorname{proj}(A))]\end{aligned}$$

and

$$\begin{aligned}&[\operatorname{rel}\operatorname{fact} g\colon (X/A, \operatorname{proj}(A)) \to (X/A, \operatorname{proj}(A))] =\\ &[\operatorname{rel}\operatorname{proj}\colon (X,A) \to (X/A, \operatorname{proj}(A))] \circ [\operatorname{rel}\operatorname{fact} g\colon (X/A, \operatorname{proj}(A)) \to (X,A)].\end{aligned}$$

□

Attachings

Theorem 1.3.7.8. *If (X_1, C) is a Borsuk pair and $\varphi, \varphi'\colon C \to X_2$ are homotopic, then the spaces $X_2 \cup_\varphi X_1$ and $X_2 \cup_{\varphi'} X_1$ are homotopy equivalent. Moreover, there is a homotopy equivalence $f\colon X_2 \cup_\varphi X_1 \to X_2 \cup_{\varphi'} X_1$ such that the following diagram is commutative:*

$$\begin{tikzcd} & X_2 \arrow[dl, "\mathrm{Imm}_2"'] \arrow[dr, "\mathrm{Imm}_2"] & \\ X_2 \cup_\varphi X_1 \arrow[rr, "f"'] & & X_2 \cup_{\varphi'} X_1 \end{tikzcd} \tag{1.3.7.9}$$

Proof. Let $\varphi\colon C \times i \to X_2$ be a homotopy from φ to φ', and let

$$\sigma\colon X_1 \times I \to (x_1 \times 0) \cup (C \times I)$$

be a retraction. Define the maps

$$h\colon (X_1 \times 0) \cup (C \times I) \to X_2 \cup_\varphi X_1, \quad h'\colon (X_1 \times 0) \cup (CxI) \to X_2 \cup_{\varphi'} X_1$$

by

$$h(x,t) = \begin{cases} \mathrm{Imm}_1(x), & \text{if} \quad t = 0, \\ \mathrm{Imm}_2 \circ \varphi(x,t), & \text{if} \quad x \in C, \end{cases}$$

and

$$h'(x,t) = \begin{cases} \mathrm{Imm}_1(x), & \text{if} \quad t = 0, \\ \mathrm{Imm}_2 \circ \psi(x,1-t), & \text{if} \quad x \in C, \end{cases}$$

Now define two more maps,

$$f\colon X_2 \cup_\varphi X \to X_2 \cup_{\varphi'} X_1, \quad g\colon X_2 \cup_{\varphi'} X_1 \to X_2 \cup_{\varphi'} X_1$$

via

$$\begin{aligned} &f \circ \mathrm{Imm}_1(x) = h' \circ \sigma(x,1), \quad f \circ \mathrm{Imm}_2(x) = \mathrm{Imm}_2(x), \quad \text{and} \\ &g \circ \mathrm{Imm}_1(x) = h \circ \sigma(x,1), \quad g \circ \mathrm{Imm}_2(x) = \mathrm{Imm}_2(x). \end{aligned}$$

It is clear that all these maps are continuous and that the diagram (1.3.7.9) is commutative. Moreover, it is readily seen that the map $X_2 \cup_\varphi X_1 \to X_2 \cup_{\varphi'} X_1$ given by

$$(\mathrm{Imm}_1(x), t) \mapsto h \circ \sigma \circ \psi(\sigma(x,t), t), \quad (\mathrm{Imm}_2(x), t) \mapsto \mathrm{Imm}_2(x),$$

where $\psi\colon (X_1 \times I) \times I \to X \times I$, $\psi((x,u),t) = (x, \max(0, t-u))$, is a homotopy from $\mathrm{id}(X_2 \cup_\varphi X_1)$ to $g \circ f$. Thus, $g \circ f$ is homotopic to $\mathrm{id}_{(X_2 \cup_\varphi X_1)}$ and, similarly, $f \circ g$ is homotopic to $\mathrm{id}_{(X_2 \cup_{\varphi'} X_1)}$. □

Lemma 1.3.7.10. *If the maps $f\colon Y \to Y'$ and $f'\colon Y' \to Y$ are homotopy inverses of one another, then given any homotopy $F'\colon Y \times I \to Y$ from id_Y to $f' \circ f$, there is a homotopy $F'\colon Y' \times I \to Y'$ from $f \circ f'$, such that the maps $f \circ F, F' \circ (f \times \mathrm{id}_I)\colon Y \times I \to Y'$ are $[(Y \times 0) \cup (Y \times 1)]$-homotopic.*

Proof. Let $G\colon Y' \times I \to Y'$ be an arbitrary homotopy from $\mathrm{id}_{Y'}$ to $f \circ f'$, and let F' be the product of the following three homotopies: $G, f \circ F \circ (f' \times \mathrm{id}_I)$, and the inverse of the homotopy $G \circ (f \times \mathrm{id}_I) \circ (f' \times \mathrm{id}_I)$. Divide the square I^2 into eight pieces, as shown in Fig. 1.2, left:
the points A_1, A_2, A_3, A_4 have the abscissae $0, 1/2, 3/4, 1$, and ordinate 0,
the points $B_1, \dots, B_6$ the abscissae $0, 1/8, 1/4, 1/2, 3/4, 1$, and ordinate $1/2$, and
the points C_1, C_2 the abscissae $0, 1$, and ordinate 1.

Now define affine maps $\alpha_1 \colon I^2 \to P_1$ and $\alpha_2 \colon I^2 \to P_2$ with the following properties:

$$\alpha_1(0,0) = B_3, \quad \alpha_1(1,0 = B_4, \quad \alpha_1(0,1) = A_2,$$
$$\alpha_2(0,0) = B_5, \quad \alpha_2(1,0) = B_4, \quad \alpha_2(0,1) = A_4.$$

Further, for $y \in Y$ define a map $\varphi_y \colon I^2 \to Y'$ through the conditions:

$$\varphi_y(\alpha_1(t_1,t_2)) = f(F(F(y,t_2),t_1) \quad \text{and}$$
$$\varphi_y(\alpha_2(t_1,t_2)) = G(f \circ F(y,t_2),t_1);$$

if $0 \leq t \leq 1/2$, then $\varphi_y(t,0) = G(f(y),2t)$; the restriction
$\varphi_y|_{Q_1}$ is constant on all segments parallel to the line A_2B_2;
$\varphi_y|_{Q_2}$ is constant on all segments passing through the point D_1;
$\varphi_y|_{Q_3}$ is constant on the vertical segments;
$\varphi_y|_{Q_4}$ is constant on all segments parallel to the line C_1B_5;
$\varphi_y|_{Q_5}$ is constant on all segments passing through the point D_2; and, at last,
$\varphi_y|_{Q_6}$ is constant on the vertical segments
those segments on which φ_y is constant are depicted in Fig. 1.2, right.

It is clear that the formula $((y,t_1),t_2) \mapsto \varphi_y(t_1,t_2)$ defines a $[(Y \times 0) \cup (Y \times 1)]$-homotopy $(Y \times I) \times I \to Y'$ from $f \circ F$ to $F' \circ (f \times \mathrm{id}_I)$. □

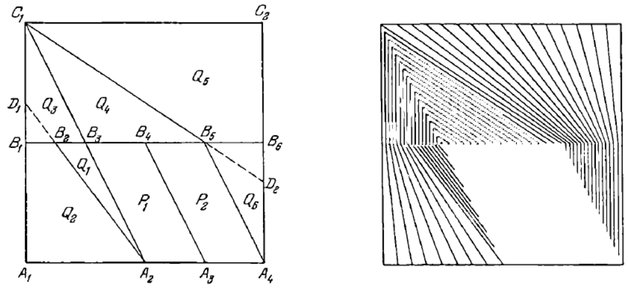

Figure 1.2: Left and right.

Theorem 1.3.7.11. *Let (X,C) be a Borsuk pair with C closed. Let $\varphi \colon C \to Y$ be a continuous map. If $f \colon Y \to Y'$ is a homotopy equivalence, then*

$$\mathrm{fact}(\mathrm{id}_X \amalg f) \colon Y \cup_\varphi X \to Y' \cup_{f \circ \varphi} X$$

is also a homotopy equivalence.

Proof. Let us fix:

(i) homotopy inverse of f, $f'\colon Y' \to Y$;

(ii) a homotopy $F\colon Y \times I \to Y$ from id_Y to $f' \circ f$;

(iii) a homotopy $F'\colon Y' \times I \to Y'$ from $\mathrm{id}_{Y'} \to f \circ f'$,
together with an $[(Y \times 0) \cup (Y \times 1)]$-homotopy $G\colon (Y \times I) \times I \to Y'$ from $f \circ F$ to $F' \circ (f \times \mathrm{id}_I)$ (see Lemma 1.3.7.10) ;

(iv) a retraction $\rho\colon X \times I \to (X \times 0) \cup (C \times I)$.

Further, define

$$g\colon (X \times 0) \cup (C \times I) \to U \cup_\varphi X \quad \text{and}$$
$$g'\colon [(X \times 0) \cup (C \times I)] \times I \to Y' \cup_{f\circ\varphi} X$$

by

$$g(x,t) = \begin{cases} \mathrm{Imm}_1(x), & \text{if} \quad t = 0, \\ \mathrm{Imm}_2 \circ F(\varphi(x), t), & \text{if} \quad x \in C, \end{cases}$$

and

$$g'((x,t_1),t_2) = \begin{cases} \mathrm{Imm}_1(x), & \text{if} \quad t_1 = 0, \\ \mathrm{Imm}_2 \circ G((\varphi(x), t_1), t_2), & \text{if} \quad x \in C. \end{cases}$$

Together, these maps yield a map $h\colon Y' \sup_{f\circ\varphi} X \to Y \cup_\varphi X$ which satisfies

$$h(\mathrm{Imm}_1(x)) = g \circ \rho(x,1), \quad h(\mathrm{Imm}_2(y')) = \mathrm{Imm}_2(f'(y')).$$

Now it is clear that the formulae

$$H(\mathrm{Imm}_1(x),t) = g \circ \rho(x,t), \quad H(\mathrm{Imm}_2(y'),t) = \mathrm{Imm}_2(F(y,t)).$$

define a homotopy $H\colon (Y \cup_\varphi X) \times I \to Y \cup_\varphi X$ from $\mathrm{id}_{(Y\cup_\varphi X)}$ to $h \circ \mathrm{fact}(\mathrm{id}_X \amalg f)$. Also, we see that the formulae

$$\begin{aligned} H_1'(\mathrm{Imm}_1(x),t) &= g'(\rho(x,t),1), \quad H_1'(\mathrm{Imm}_2(y'),t) = \mathrm{Imm}_2(F'(y',t)), \quad \text{and} \\ H_2'(\mathrm{Imm}_1(x),t) &= g'(\rho(x,t),1), \quad H_2'(\mathrm{Imm}_2(y'),t) = \mathrm{Imm}_2(F'(f \circ f'(y)), \end{aligned}$$

define two homotopies $H_1', H_2'\colon (Y' \cup_{f\circ\varphi} X) \times I \to Y' \cup_{f\circ\varphi} X$, whose product is a homotopy from $id_{Y'\cup_{f\circ\varphi} X)}$ to $\mathrm{fact}(id_X \amalg f) \circ h$. Therefore, h is a homotopy inverse of $\mathrm{fact}(\mathrm{id}_X \amalg f)$. □

More Special Constructions

Theorem 1.3.7.12. $\mathrm{cone}\, X$ *is contractible for any* X. $\mathrm{sus}\, X$ *is connected for any* X. $X_1 \star X_2$ *is connected for any* X_1 *and* X_2.

Proof. The contractibility of the cone is obvious. The connectedness of the suspension is a consequence of the fact that it is a quotient space of the cone. To prove that the join $X_1 \star X_2$ is connected, we may check that any two points $\operatorname{proj}(x_1, x_2, t), \operatorname{proj}(x'_1, x'_2, t') \in X_1 \star X_2$, $[(x_1, x_2, t), (x'_1, x'_2, t') \in X_1 \times X_2 \times I]$ can be joined by the path

$$\tau \mapsto \begin{cases} \operatorname{proj}(x_1, x_2, 3\tau(1-t)+t), & \text{if } \tau \le 1/3, \\ \operatorname{proj}(x'_1, x_2, 2-3\tau), & \text{if } 1/3 \le \tau \le 2/3, \\ \operatorname{proj}(x'_1, x'_2, t'(3\tau-2)), & \text{if } \tau \ge 2/3. \end{cases}$$

□

Remark 1.3.7.13. Given two homotopic maps $f, g\colon X \to Y$, a straightforward application of Proposition 1.3.7.5 shows that the maps

$$\operatorname{sus} f, \operatorname{sus} g\colon \operatorname{sus} X \to \operatorname{sus} Y$$

are also homotopic. As usual, we may deduce that $\operatorname{sus} f$ is a homotopy equivalence whenever f is one.

Moreover, the same Proposition 1.3.7.5 shows that given homotopic maps $f_1, g_1\colon X_1 \to Y_1$ and homotopic maps $f_2, g_2\colon X_2 \to Y_2$ the maps

$$f_1 \star f_2, g_1 \star g_2\colon X_1 \star X_2 \to Y_1 \star Y_2$$

are also homotopic. Similarly, we conclude that $f_1 \star f_2$ a homotopy equivalence whenever f_1 and f_2 are homotopy equivalences.

Remark 1.3.7.14. Given a continuous map $f\colon X_1 \to X_2$, $\operatorname{rt} f\colon \operatorname{Cyl} f \to X_2$ (see Definition 1.2.6.10) is obviously a strong deformation retraction of the mapping cylinder $\operatorname{Cyl} f$ onto X_2. Hence X_2 is a strong deformation retract of $\operatorname{Cyl} f$.

The inclusion $X_1 \to \operatorname{Cyl} f$ is a homotopy equivalence if and only if f is a homotopy equivalence. In fact, the composition

$$X_1 \xrightarrow{\text{incl}} \operatorname{Cyl} f \xrightarrow{\operatorname{rt} f} X_2$$

coincides with f, and $\operatorname{rt} f$ is a homotopy equivalence.

The Case of Pointed Spaces

Remark 1.3.7.15. The results obtained in Remarks 1.3.7.2, 1.3.7.3, Theorems 1.3.7.6, 1.3.7.7, 1.3.7.11, 1.3.7.12, and Remark 1.3.7.13 have obvious analogues for pointed spaces. Let us add that those theorems which are the analogues of Remark 1.3.7.13 are valid for both bouquets and tensor products:

if $f_\mu, g_\mu\colon (X_\mu, x_\mu) \to (Y_\mu, y_\mu)$ are homotopic for each μ, then

$$\bigvee f_\mu, \bigvee g_\mu\colon (\bigvee (X_\mu, x_\nu), \mathrm{bp}) \to (\bigvee (Y_\mu, y_\nu), \mathrm{bp})$$

are homotopic, and if all the f_μ's are homotopy equivalences, then so is $\bigvee f_\mu$.

Similarly, if

$$f_1, g_1 \colon (X_1, x_1) \to (Y_1, y_1)$$

and

$$f_2, g_2 \colon (X_2, x_2) \to (Y_2, y_2)$$

are homotopic, then

$$f_1 \otimes f_2, g_1 \otimes g_2 \colon ((X_1, x_1) \otimes (X_2, x_2), \mathrm{bp}) \to ((Y_1, y_1) \otimes (Y_2, y_2), \mathrm{bp})$$

are homotopic; if f_1 and f_2 are homotopy equivalences, then so is $f_1 \otimes f_2$.

1.3.8 Exercises

Exercise 1.3.8.1. Show that the sphere $\mathbb{S}^\infty$ is contractible.

Exercise 1.3.8.2. Show that if (X, A) is a Borsuk pair and X is contractible, then the quotient X/A is homotopy equivalent to $\operatorname{sus} A$.

Exercise 1.3.8.3. Suppose that the product of two topological spaces is homeomorphic to the suspension of some other topological space. Prove that either both factors of the product are contractible, or one of them reduces to a point.

Exercise 1.3.8.4. Let $f_1 \colon X_1 \to X_2$ be a homotopy equivalence. Show that X_1 is a strong deformation retract of $\operatorname{Cyl} f$.

Exercise 1.3.8.5. Show that if X is metrisable and $x \in X$ is such that (X, x) is a Borsuk pair, then the projection $\operatorname{sus} X \to \operatorname{sus}(X, x)$ is a homotopy equivalence.

Exercise 1.3.8.6. Given an arbitrary connected topological space X and two arbitrary points $x, y \in X$, show that the subset of $\mathcal{C}(I, 0; X, x)$ consisting of all paths passing through the point y is contractible.

Chapter 2

CELLULAR SPACES

2.1 CELLULAR SPACES AND THEIR TOPOLOGICAL PROPERTIES

2.1.1 Fundamental Concepts

Definition 2.1.1.1. A decomposition $\wp$ of a topological space X is called *cellular* if there is a function d taking the set of elements of $\wp$ into the non-negative integers, such that for every element e of $\wp$ there exists a continuous map $\mathbb{D}^{d(e)} \to X$ with the following two properties:

(i) it maps $\operatorname{int} D^{d(e)}$ homeomorphically onto e;

(ii) it maps $\mathbb{S}^{d(e)-1}$ onto a union of elements of $\wp$ on which d takes values smaller than $d(e)$.

The elements of a cellular decomposition and their closures are called *cells* and *closed cells*, respectively. The number $d(e)$ is the *dimension of the cell* e and is usually denoted by $\dim e$; the n-dimensional cells are also termed *n-cells*. Any continuous map $\mathbb{D}^{d(e)} \to X$ with the properties (i) and (ii) is said to be *characteristic* for e; we use the symbol char_e as a standard notation for such a map.

Obviously, $\operatorname{char}_e(\mathbb{D}^{\dim e}) \subset \operatorname{Cl} e$, and if X is Hausdorff, then

$$\operatorname{char}_e(\mathbb{D}^{\dim e}) = \operatorname{Cl} e.$$

In particular, *every closed cell in a cellular decomposition of a Hausdorff space is compact.* Moreover, in the case of a Hausdorff space, given any cell e, $\operatorname{Cl} e \setminus e$ is covered by cells of dimension lower than $\dim e$.

Definition 2.1.1.2. A cellular decomposition is said to be *rigged* (or *equipped*) if for each of its cells there is fixed a characteristic map. The resulting family

$\{\mathrm{char}_e\colon \mathbb{D}^{\dim e} \to X\}$ is called a *rigging* (or an *equipment*) of the decomposition, and the map

$$\mathrm{char}\colon \coprod_{e\in X/\wp} \mathbb{D}^{\dim e} \to X$$

defined by the relations $char \circ \mathrm{incl}_e = \mathrm{char}_e$ is called the *total characteristic map.*

Definition 2.1.1.3. According to the general definitions of Chapter 1 (see Remark 1.2.4.3), the cover of the space X by the closed cells of a cellular decomposition $\wp$ defines a new topology on X. A subset of X is closed in this topology if and only if its intersections with the closed cells of the decomposition $\wp$ are closed in the initial topology of X. The new topology on X is called the *weak* or *cellular topology*, and the process by which we pass from the initial to the cellular topology of X is called a *cellular weakening* of the initial topology. The weakening of a topology can only enlarge the supply of open and closed sets; in particular, a Hausdorff space remains Hausdorff. In any case, it does not affect the topology of the closed cells, and so the decomposition $\wp$ remains cellular and retains its characteristic maps.

If X is Hausdorff and the decomposition $\wp$ is endowed with a rigging $\{\mathrm{char}_e\}$, then the cellular topology can be described effectively in terms of the corresponding total characteristic map: a subset A of X is open (closed) if and only if its pre-image $\mathrm{char}^{-1}(A)$ is open (respectively, closed). In other words, the cellular topology is exactly that topology on X which transforms the injective factor of the map char into a homeomorphism. Recall that the injective factor of char is a map of the quotient space of the sum $\coprod_{e\in X/\wp} \mathbb{D}^{\dim e}$ by its partition zer(char) onto X. The equivalence of these two definitions of the cellular topology results from the fact that the maps char are closed (see Proposition 1.1.7.9).

Definition 2.1.1.4. A *cellular space* is a Hausdorff topological space endowed with a cellular decomposition which satisfies the following two conditions:

(C) every closed cell intersects only a finite number of cells;

(W) the closed cells constitute a fundamental cover of the space.

Obviously, condition (**W**) implies that the cellular topology coincides with the initial one. The notations (**C**) and (**W**) are generally accepted, and originate from the terms *closure finiteness* and *weak topology.*

Property (**C**) is clearly preserved under the cellular weakening of the topology. Therefore, a Hausdorff topological space having a cellular decomposition satisfying (**C**) becomes a cellular space via cellular weakening.

Usually the terminology specific to cellular decompositions is applied to cellular spaces too. In particular, a cellular space may be finite, countable, and rigged. Thus, a finite cellular space is one that has a finite number of cells, but not necessarily a finite number of points.

The *dimension* of a cellular space is the supremum of the dimensions of its cells; the dimension of the empty space (which is not excluded from the family of cellular spaces) is taken equal to -1. The dimension - finite or infinite - of the cellular space X is denoted by $\dim X$.

We let $\operatorname{cell}_r X$ denote the set of all r-cells of X.

Example 2.1.1.5. The simplest cellular spaces are the discrete spaces decomposed into 0-cells (isolated points). It is clear that all O-dimensional cellular spaces are of this type:
Caveat a decomposition of a non-discrete Hausdorff space into 0-cells does not satisfy condition (**W**).

The decomposition of the ball $\mathbb{D}^n$ $(n > 1)$ into the n-cell $\operatorname{int} \mathbb{D}^n$ and the 0-cells which cover $\operatorname{Fr} \mathbb{D}^n = \mathbb{S}^{n-1}$ is an example of cellular decomposition which satisfies condition (**W**), but not condition (**C**).

The Locally Finite Case

Remark 2.1.1.6. In agreement with the general definitions given in Definition 1.1.1.12, a cellular decomposition is *locally finite* if every point of the given space has a neighbourhood which intersects only a finite number of cells. An equivalent condition: every point has a neighbourhood which intersects only a finite number of closed cells.

Clearly, in a space possessing a locally finite cellular decomposition, every compact subset has a neighbourhood which intersects only a finite number of cells. As a consequence, every locally finite cellular decomposition of a Hausdorff space satisfies condition (**C**). Theorem 1.1.3.6 shows that condition (**W**) is satisfied by *any* locally finite cellular decomposition. We conclude that *every Hausdorff space endowed with a finite or locally finite cellular decomposition is a cellular space.*

Theorem 2.1.1.7. *A cellular space is locally finite if and only if every cell intersects only a finite number of closed cells.*

Proof. In a locally finite cellular space the closure of an arbitrarily given cell has a neighbourhood which intersects only a finite number of cells (see Remark 2.1.1.6). It is clear that this neighbourhood, and hence the given cell, do not intersect the closure of other cells.

Conversely, if every cell of a cellular space intersects only a finite number of closed cells, then the axioms (**C**) and (**W**) imply that the union of any collection of closed cells is closed. Consequently, the complement of the union of all closed cells which do not intersect an arbitrarily given cell e is a neighbourhood of e. Since this complement cannot intersect the closed cells which do not intersect e, it intersects only a finite number of closed cells. ☐

Subspaces

Lemma 2.1.1.8. *Let A be a subset of the cellular space X with the following property: if $x \in A$, then A contains the closure of the cell in which x lies. Then any part of A whose intersections with all the closed cells contained in A are closed is itself closed.*

Proof. Indeed, if $B \subset A$ is such a part, and e is an arbitrary cell, then one can write the intersection $B \cap \operatorname{Cl} e$ as $\cup_{i=1}^{s}[(B \cap \operatorname{Cl} e_i) \cap \operatorname{Cl} e]$, where $e_1, \dots, e_s$ are all the cells of A which intersect $\operatorname{Cl} e$, and this shows that $B \cap \operatorname{Cl} e$ is closed. $\square$

Definition 2.1.1.9. A subset of a cellular space which contains together with each point the closure of its cell is called a *subspace* of the given cellular space. Every subspace is a cellular space, with the cellular decomposition induced by the cellular decomposition of the ambient space. By Lemma 2.1.1.8, this decomposition satisfies condition (**W**), and it obviously satisfies condition (**W**).

As another consequence of Lemma 2.1.1.8, we see that every subspace of a cellular space is closed. Notice also that the union and the intersection of any collection of subspaces are again subspaces, and that every cover of a cellular space by subspaces is fundamental.

A pair consisting of a cellular space and one of its subspaces is called a *cellular pair*. *Cellular triples* and *cellular triads* are defined similarly.

Warning: a closed cell is not necessarily a subspace. For an example, consider the bouquet $(\mathbb{D}^1, 0) \vee (\mathbb{S}^2, \operatorname{ort}_1)$, with the decomposition into four cells: the 0-cells $\operatorname{Imm}_1(-1)$ and $\operatorname{Imm}_1(1)$, the 1-cell $\operatorname{Imm}_1(\operatorname{int} \mathbb{D}^1)$, and the 2-cell $\operatorname{Imm}_2(\mathbb{S}^2 \setminus \operatorname{ort}_1)$. This is obviously a cellular space; however, the closure of the 2-cell touches the 1-cell, but does not contain it.

Remark 2.1.1.10. The most important subspaces of a cellular space X are its skeleta

$$\operatorname{skel}_0 X, \operatorname{skel}_1 X, \dots, \operatorname{skel}_r X, \dots,$$

defined as $\operatorname{skel}_r X = \cup_{\dim e \le r} e$. If $X \neq \varnothing$, then all the skeleta are non-empty (since the presence of cells of a certain positive dimension implies the presence of cells of lower dimension). For formal reasons, we add the empty skeleton $\operatorname{skel}_{-1} X$ and the skeleton $\operatorname{skel}_\infty X = X$. The sequence $\{\operatorname{skel}_r X\}_{0 \le r < \infty}$ is clearly a filtration of X.

We remark that any map $\mathbb{D}^{\dim e} \to X$ which is characteristic for the cell e takes $\mathbb{S}^{\dim e - 1}$ into $\operatorname{skel}_{\dim e - 1} X$ (in fact, we saw this already in Definition 2.1.1.1). If X carries a rigging $\{\operatorname{char}_e\}$, then the map

$$\operatorname{abr} \operatorname{char}_e \colon \mathbb{S}^{\dim e - 1} \to \operatorname{skel}_{\dim e - 1} X$$

is called an *attaching map* for e, and is denoted by att_e.

Theorem 2.1.1.11. *Every cell of a cellular space is included in a finite subspace.*

Proof. Use induction on the dimension of the cell. A 0-cell is itself a subspace. If e is a cell of positive dimension, then $\operatorname{Cl} e \setminus e$ may be covered by a finite number of lower dimensional cells, and the union of e with a collection of finite subspaces which contain these cells is a finite subspace containing e. $\square$

Compact Subsets

Theorem 2.1.1.12. *A compact subset of a cellular space intersects only a finite number of cells.*

Proof. Every subset A of a cellular space contains a part B which intersects at only one point each cell intersecting A. Since B intersects any closed cell at a finite number of points, B and all its subsets are closed. Therefore, B must be discrete. When A is compact, B as a discrete, closed, and compact set is finite. ☐

Theorem 2.1.1.13. *Every compact subset of a cellular space is contained in a finite subspace.*

Proof. For each cell intersected by the given subset, pick a finite subspace containing this cell. The union of these subspaces is the desired finite subspace. ☐

Theorem 2.1.1.14. *Every compact subset of a locally finite cellular space is contained in the interior of a finite subspace.*

Proof. Indeed, such a subset has a neighbourhood which intersects only a finite number of cells (cf. Remark 2.1.1.6). For each such cell, pick a finite subspace which contains it; the union of these subspace is again a finite subspace and contains the above neighbourhood. ☐

Cellular Maps

Remark 2.1.1.15. A map of a cellular space X into a cellular space Y is said to be *cellular* if it is continuous and maps the skeleton $\operatorname{skel}_r X$ into $\operatorname{skel}_r Y$, for each r.

A cellular map obviously transforms 0-cells into 0-cells. However, a cell of positive dimension is not necessarily transformed into a single cell: for example, consider the identity map of the segment $\mathbb{D}^1$, decomposed into the 0-cells -1, 1 and the 1-cell $(-1,1)$, onto the same segment, but decomposed now into the 0-cells -1, 0, 1 and the 1 -cells $(-1,0)$, $(0,1)$; this cellular map takes the 1-cell $(-1,1)$ into the union of a 0-cell and two 1 -cells.

Definition 2.1.1.16. A cellular map is a *cellular equivalence* if it is invertible and its inverse is also cellular. An equivalent formulation: a cellular equivalence is a homeomorphism which transforms the cellular decomposition of the domain space exactly into the cellular decomposition of the image space. If there is a cellular equivalence between two cellular spaces, then they are said to be *cellular equivalent*. Two rigged cellular spaces related by a cellular equivalence which transforms one rigging into the other are said to be *rigged-equivalent*.

If X and Y are cellular spaces, a map $f\colon X \to Y$ is a cellular embedding if $f(X)$ is a subspace of Y (defined as in Definition 2.1.1.9) and $\operatorname{abr} f\colon X \to f(X)$ is a cellular equivalence.

Warning: there are cellular homeomorphisms which are not cellular equivalences. An example is the homeomorphism described in Remark 2.1.1.15.

2.1.2 Glueing Cellular Spaces From Balls

Theorem 2.1.2.1. *If $r \geq 0$, then the skeleton $\operatorname{skel}_r X$ of the rigged cellular space X is canonically homeomorphic to the space*

$$(\operatorname{skel}_{r-1} X) \cup_\varphi (\coprod_{e \in M_r} \mathbb{D}_e = \mathbb{D}^r)),$$

where $M_r = \operatorname{cell}_r X$ and the map $\varphi\colon \coprod_{e\in M_r}(\mathbb{S}_e = \mathbb{S}^{r-1}) \to \operatorname{skel}_{r-1} X$ is defined by $\varphi \circ \operatorname{incl}_e = \operatorname{att}_e (e \in M_r)$.

Proof. This canonical homeomorphism between the above spaces is the injective factor of the map

$$(\operatorname{skel}_{r-1} X) \amalg (\coprod_{e \in M_r} \mathbb{D}_e) \to \operatorname{skel}_r X$$

defined by the inclusion

$$\operatorname{skel}_{r-1} X \to \operatorname{skel}_r X$$

and the maps

$$\operatorname{abr}\operatorname{char}_e : \mathbb{D}_e \to \operatorname{skel}_r X.$$

□

Remark 2.1.2.2. The description in Definition 2.1.1.3 of the weak topology in terms of the total characteristic map shows that one can glue any cellular space from balls, and even do it in a nice way. Theorem 2.1.2.1 above reduces this glueing to a sequence of attaching processes: the r-th process transforms $\operatorname{skel}_{r-1} X$ into $\operatorname{skel}_r X$ $(r = 0, 1, \dots)$, and X is defined, starting from the sequence $\{\operatorname{skel}_r X\}$, as $X = \lim \operatorname{skel}_r X$ (see Remark 2.1.1.10).

The following formal procedure transforms this description of cellular spaces into a useful inductive method of constructing such spaces. First of all, note that if we are given a topological space A with a rigged cellular decomposition into cells of dimensions $< q$, and to A we attach a sum $\coprod_{\mu\in M}(\mathbb{D}_\mu = \mathbb{D}^q)$ of q-dimensional balls by some continuous map $\varphi colon \coprod_{\mu \in M}(\mathbb{S}_\mu = \mathbb{S}^{q-1}) \to A$, then we obtain a space endowed with an obvious rigged cellular decomposition into cells of dimensions $< q + 1$. This space satisfies condition (**W**) whenever A satisfies it, and Proposition 1.2.4.9 shows that it is normal if A is normal. Moreover, it follows from Theorem 2.1.1.12 that this space satisfies condition (**C**) provided A is cellular. Finally, we conclude that if A is a normal rigged cellular space,then $A \cup_\varphi (\coprod_{\mu\in M} \mathbb{D}_\mu)$ is a normal rigged cellular space too.

These observations form the basis of our inductive construction. We start with $q = 0$, i.e., take $A = \varnothing$, and at the r-th step we attach the space $\coprod_{\mu \in M_r}(\mathbb{D}_\mu = \mathbb{D}^r)$ to the previously constructed normal rigged cellular space X_{r-1}, $\dim X_{r-1} \leq r - 1$, by a continuous map

$$\varphi_r\colon \coprod_{\mu \in M_r} (\mathbb{S}_\mu = \mathbb{S}^{r-1}) \to X_{r-1}.$$

The r-th step yields a normal rigged cellular space $X_r = X_{r-1} \cup_\varphi (\coprod_{\mu \in M_r} \mathbb{D}_\mu)$, with $\dim X_r \leq r$. The result of the whole process is a sequence $\varnothing = X_{-1}, X_0, X_1, \ldots,$ with natural cellular embeddings $X_r \to X_{r+1}$, and limit space $X = \lim X_r$. According to Theorem 1.2.4.6, X is normal and is endowed with an obvious cellular decomposition satisfying properties (**C**) and (**W**). Therefore, X is a normal rigged cellular space, and clearly $\operatorname{skel}_r X = X_r$.

We say that X is an *inductively glued cellular space.* The discussion above demonstrates that every rigged cellular space is rigged-equivalent to an inductively glued cellular space.

Corollary 2.1.2.3. *Every cellular space is normal.*

2.1.3 The Canonical Cellular Decompositions of Spheres, Balls, and Projective Space

Remark 2.1.3.1. The spheres, balls, and projective spaces admit canonical cellular decompositions making them into cellular spaces. These are all rigged cellular decompositions and will be described in the present subsection. It will be evident in each case that properties (**C**) and (**W**) are satisfied.

Remark 2.1.3.2. The canonical cellular decomposition of the sphere $\mathbb{S}^n$ with $0 \leq n < \infty$, consists of the 0-cell ort_1 and the n-cell $\mathbb{S}^n \setminus \operatorname{ort}_1$. As the characteristic map of the cell $\mathbb{S}^n \setminus \operatorname{ort}_1$ we take $\mathbb{DS}\colon \mathbb{D}^n \to \mathbb{S}^n$.

Remark 2.1.3.3. The canonical cellular decomposition of the ball $\mathbb{D}^n$ with $1 \leq n < \infty$, is given by the 0-cell ort_1, the $(n-l)$-cell $\mathbb{S}^{n-1} \setminus \operatorname{ort}_1$, and the n-cell $\operatorname{int} \mathbb{D}^n$. For the characteristic maps of the cells $\mathbb{S}^{n-1} \setminus \operatorname{ort}_1$ and $\operatorname{int} \mathbb{D}^n$ we take the composite map

$$\mathbb{D}^{n-1} \xrightarrow{\mathbb{DS}} \mathbb{S}^{n-1} \xrightarrow{\text{incl}} \mathbb{D}^n$$

and $\operatorname{id}_{\mathbb{D}^n}$, respectively.

Remark 2.1.3.4. The canonical cellular decomposition of the real projective space $\mathbb{R}P^n$ $(0 \leq n \leq \infty)$ consists of the r-cells $e_r = \mathbb{R}P^r \setminus \mathbb{R}P^{r-1}$, where $0 \leq r \leq n$ for $n < \infty$, and $0 \leq r < \infty$ for $n = \infty$. The composition

$$\mathbb{D}^r \xrightarrow{\text{proj}} \mathbb{R}P^r \xrightarrow{\text{incl}} \mathbb{R}P^n$$

(where proj is the projection arising from the characterisation of $\mathbb{R}P^r$ as a quotient space of $\mathbb{D}^r$, given in Definition 1.2.5.2) is taken as the characteristic map of the cell e_r.

It is clear that att_{e_r} is simply $\operatorname{proj}\colon \mathbb{S}^{r-1} \to \mathbb{R}P^{r-1}$, and that $\operatorname{skel}_r \mathbb{R}P^n = \mathbb{R}P^r$, $r \leq n$.

Remark 2.1.3.5. The canonical cellular decomposition of the complex projective space $\mathbb{C}P^n$ $(0 \leq n \leq \infty)$ consists of the $2r$-cells $e_r = \mathbb{C}P^r \setminus \mathbb{C}P^{r-1}$, where $0 \leq r \leq n$ for $n < \infty$, and $0 \leq r < \infty$ for $n = \infty$. The composition

$$\mathbb{D}^{2r} \xrightarrow{\text{proj}} \mathbb{C}P^r \xrightarrow{\text{incl}} \mathbb{C}P^n$$

is taken as the characteristic map of the cell e_{2r}. Obviously, $\operatorname{skel}_r \mathbb{C}P^n = \mathbb{C}P^{[}r/2]$, for $r \leq 2n$.

The canonical rigged cellular decompositions of the projective spaces $\mathbb{H}P^n$ $(0 \leq n \leq \infty)$ and $(\mathbb{C}aP^n$ $(0 \leq n \leq 2)$ are defined similarly. The decomposition of $\mathbb{H}P^n$ is given by the $4r$-cells $e_r = \mathbb{H}P^r \setminus \mathbb{H}P^{r-1}$ where $0 \leq r \leq n$ for $n < \infty$, and $0 \leq r < \infty$ for $n = \infty$. For $\mathbb{C}aP^n$ the cells are $e_r = \mathbb{C}aP^r \setminus \mathbb{C}aP^{r-1}$ with $0 \leq r \leq n$ and $\dim e_r = 8r$.

2.1.4 More Topological Properties of Cellular Spaces

Remark 2.1.4.1. Our task in this subsection is to examine what connections exist between properties of cellular spaces such as compactness, local compactness, separability, second countability, and metrisability, and properties of their cellular decompositions such as finiteness, countability, and local finiteness. Incidentally, we prove that cellular spaces are CNRS's. Moreover, conditions for the connectedness of a cellular space, as well as its partition into components are studied.

Compactness and Local Compactness

Theorem 2.1.4.2. *A cellular space is compact if and only if it is finite.*

Proof. The necessity of this condition is a result of Theorem 2.1.1.12. Since every finite cellular space can be covered by a finite number of closed cells, the condition is also sufficient. ☐

Theorem 2.1.4.3. *A cellular space is locally compact if and only if it is locally finite.*

Proof. By Theorem 2.1.1.12, every neighbourhood with compact closure of an arbitrarily given point intersects only a finite number of cells, and hence the condition is necessary. It is also sufficient, because the closure of a neighbourhood which intersects only a finite number of cells is contained in the union of the closures of these cells. ☐

Embedding Theorems

Theorem 2.1.4.4. *Every cellular space can be embedded in a Euclidean space of sufficiently high dimension.*

Proof. We shall proceed by induction. Given a finite cellular space X of dimension $n \geq 0$, and an embedding $j\colon \operatorname{skel}_{n-1} X \to \mathbb{R}^q$, we shall construct an embedding $J\colon X \to \mathbb{R}^{2q+n+1}$ (since a cellular space of dimension -1 is empty, the first step of the induction is trivial). Pick a rigging of X and arrange the n-cells of X in a sequence $e_1, \dots, e_n$.

Now let $\rho = \sqrt{y_1^2 + \cdots + y_n^2}$ and define the maps $\varphi_1, \ldots, \varphi_s \colon \mathbb{D} \to \mathbb{R}^{q+n+1}$ by

$$\varphi_k(y_1, \ldots, y_k) = \begin{cases} (0, \ldots, 0, y_1, \ldots, y_{n-1}, y_n + 2k, 1), & \text{if} \quad \rho \le 1/2, \\ (2\rho - 1) j \circ \operatorname{char}_{e_k}\left(\frac{y_1}{\rho}, \ldots, \frac{y_n}{\rho}\right) + \\ (2 - 2\rho)\left(0, \ldots, 0, \frac{y_1}{2\rho}, \ldots, \frac{y_{n-1}}{2\rho}, \frac{y_n}{2\rho} + 2k, 1\right), & \text{if} \quad \rho \ge 1/2 \end{cases}$$

and then set

$$J(x) = \begin{cases} j(x), & \text{if} \quad x \in \operatorname{skel}_{n-1} X, \\ \varphi_k(y), & \text{if} \quad x \in \operatorname{Cl} e_k \quad \text{and} \quad x = \operatorname{char}_{e_k}(y). \end{cases}$$

This yields a continuous map $J \colon X \to \mathbb{R}^{q+n+1}$ and since the plane $y_1 = 0, \ldots, y_q = 0, y_{q+n+1} = 1$ contains no parallel to $\mathbb{R}^q$, J is injective. Hence, J is an embedding (see Theorem 1.1.7.10). □

Theorem 2.1.4.5. *Every finite cellular space is a CNRS.*

Proof. The proof is a continuation of the previous one and also requires an induction on n. Namely, we show that $\mathbb{R}^q \cup J(X)$ is a neighbourhood retract of $\mathbb{R}^{q+n+1}$. This is enough: if we assume that $j(\operatorname{skel}_{n-1} X)$ is a neighbourhood retract of $\mathbb{R}^q$, then $J(X)$ is obviously a neighbourhood retract of $\mathbb{R}^q \cup J(X)$.

Set

$$A_k = \varphi_k(\operatorname{int} \mathbb{D}^n) \quad \text{and} \quad B_k = A_k \cap \{(x_1, \ldots, x_{q+n+1}) : X_{q+n+1} \ne 1\}.$$

Since the sets $A_1, \ldots, A_s$ are pairwise disjoint and closed in $\mathbb{R}^{q+n+1} \backslash \cup_{k=1}^s \varphi_k(\mathbb{S}^{n-1})$, they have pairwise disjoint neighbourhoods $U_1, \ldots, U_s$ in $\mathbb{R}^{q+n+1} \backslash \cup_{k=1}^s \varphi_k(\mathbb{S}^{n-1})$, and clearly $U_1, \ldots, U_s$ are open in $\mathbb{R}^{q+n+1}$. Moreover, since $A_1, \ldots, A_s$ are all homeomorphic to $\mathbb{R}^n$, the identity maps $A_1 \to A_1, \ldots, A_s \to A_s$ extend to continuous maps $\psi_1 \colon U_1 \to A_1, \ldots, \psi_s \colon U_s \to A_s$ (see Theorem 1.1.5.17).

Now let

$$\begin{aligned} V_k &= \{x \in U_k |\, \operatorname{dist}(x, \psi_k(x)) < \operatorname{Dist}(x, \mathbb{R}^q)\}, \\ V &= \{x \in \mathbb{R}^{q+n+1} |\, \operatorname{Dist}(x, \mathbb{R}^q) < 1/2\}, \end{aligned}$$

and denote by ψ the orthogonal projection $V \to \mathbb{R}^q$. It is clear that $W = V \cup V_1 \cup \cdots \cup V_s$ is a neighbourhood of $\mathbb{R}^q \cup J(X)$ in $\mathbb{R}^{q+n+1}$ and that the sets

$$\left\{ \begin{array}{l} \{[(\cup_{k=1}^s \operatorname{Cl} V_k) \cap \operatorname{Fr} V] \cup (\cup_{k=1}^s A_k)\} \cap W, \\ \{[(\cup_{k=1}^s \operatorname{Fr} V_k) \cap \operatorname{Cl} V \cap W] \setminus \cup_{k=1}^s \varphi_k(\mathbb{S}^{n-1})\}, \end{array} \right. \tag{2.1.4.6}$$

are disjoint and closed in $Y \setminus \cup_{k=1}^s \varphi_k(\mathbb{S}^{n-1})$, where

$$Y = (\cup_{k=1}^s \operatorname{Cl} V_k) \cap \operatorname{Cl} V \cap W.$$

Let $f \colon Y \setminus \cup_{k=1}^s \varphi_k(\mathbb{S}^{n-1}) \to I$ be a Urysohn function for the pair (2.1.4.6) . Now given $y \in \operatorname{Cl} B_k$, and $z \in \mathbb{R}^q$, let u_{yz} be the path in $\mathbb{R}^q \cup J(X)$ defined as the product of the rectilinear path between y and $\varphi_k(\varphi_{k^{-1}})/\operatorname{dist}(0, \varphi_k^{-1}(y)))$ with he rectilinear path between $\varphi_k(\varphi_{k^{-1}})/\operatorname{dist}(0, \varphi_k^{-1}(y)))$ and z. Define

$$\sigma\colon (U_{k=1}^{s} \operatorname{Cl} B_k) \times \mathbb{R}^q \times I \to \mathbb{R}^q \cup J(X)$$

by $\sigma(y, z, t) = u_{yz}(t)$. According to Theorem 1.2.2.14, the map $\tau\colon Y \to \mathbb{R}^q \cup J(X)$,

$$\tau(x) = \begin{cases} \sigma(\psi_k(x), \psi(x), f(x)), & \text{if } \ x \in (\operatorname{Cl} V_k \cap \operatorname{Cl} V \cap W) \setminus \varphi_k(\mathbb{S}^{n-1}), \\ x, & \text{if } \ x \in \cup_{k=1}^{s} \varphi_k(\mathbb{S}^{n-1}), \end{cases}$$

is continuous. By Theorem 1.1.4.3, the map $W \to \mathbb{R}^q \cup J(X)$ defined by the formula

$$x \mapsto \begin{cases} \psi_k(x), & \text{if } \ x \in \operatorname{Cl} V_k \cap (W \setminus V), \\ \psi(x), & \text{if } \ x \in W \setminus \cup_{k=1}^{s} V_k, \\ \tau(x), & \text{if } \ x \in Y, \end{cases}$$

is also continuous, and it is clearly a retraction. □

Connectedness. Components

Theorem 2.1.4.7. *The components of a cellular space are open subspaces.*

Proof. Every closed cell is connected, as the image of a ball under a characteristic map. Therefore, a component of a cellular space contains along with each point the closure of the cell in which the point lies, i.e., it is a subspace. The complement of a component is the union of the remaining components and so it is a subspace too. Consequently, this complement is closed, and hence the component is open. □

Theorem 2.1.4.8. *If $r \geq 1$, then the r-th skeleton $\operatorname{skel}_r A$ of a component A of the cellular space X is a component of $\operatorname{skel}_r X$. In particular, a cellular space X is connected if and only if $\operatorname{skel}_1 X$ is connected.*

Proof. Obviously, $\operatorname{skel}_r A = A \cap \operatorname{skel}_r X$ for all r, and if B is another component of X, then $\operatorname{skel}_r A$ and $\operatorname{skel}_r B$ sit in different components of $\operatorname{skel}_r X$. Therefore, all we have to show is that the skeletons $\operatorname{skel}_r A$ with $r \geq 1$ are connected or, equivalently, that given a connected cellular space X, all its skeletons $\operatorname{skel}_r X$ ($r \geq 1$) are connected. But this is plain if one notes that the construction of an inductively glued cellular space described in Remark 2.1.2.2 cannot result in a connected space if one of the spaces X_r with $r \geq 1$ is not connected. □

Theorem 2.1.4.9. *Every connected locally finite cellular space is countable.*

Proof. Given a connected, locally finite cellular space X, fix an arbitrary point $x_0 \in X$ and consider the set A_m of all points of X that can be joined to x_0 by a path intersecting at most m cells. Since any path intersects only a finite number of cells (see Theorem 2.1.1.12), $X = \cup_{m=1}^{\infty} A_m$, and it is clear that each

A_m consists of whole cells. Therefore, all we have to verify is that each set A_m with $m \geq 1$ contains only a finite number of cells, and we do it by induction on m.

It is readily seen that the closure of any cell contained in A_{m+1} intersects the closure of some cell from A_m. On the other hand, since X is locally finite, the closure of any cell from A_m (being a compact subset of X) can intersect only a finite number of closed cells. Thus the number of cells in A_{m+1} is finite provided that the number of cells in A_m is so, and to complete the proof note that A_1 is just one cell. ☐

Countability Axioms and Metrisability

Theorem 2.1.4.10. *A cellular space is separable if and only if it is countable.*

Proof. If the cellular space X is countable, pick a countable dense set in each cell and then take the union of all these sets to produce a countable dense set in X.

Now suppose that X is separable. Then every point of X lies in a finite subspace (see Theorem 2.1.1.11). Pick such a subspace for each point of a fixed countable dense set in X. The union of these subspaces is a countable subspace and actually coincides with X. ☐

Lemma 2.1.4.11. *If the cellular space X has a countable base at a point $x_0 \in X$, then this base contains a neighbourhood of x_0 which intersects only a finite number of cells of X .*

Proof. Suppose that this is not true. Write the elements of the given base in a sequence and $U_1, U_2 \dots$ and (using a trivial induction), select a sequence of points $x_1, x_2, \dots$ in $X \setminus x_0$, such that:

(i) $x_i \in U_i$;

(ii) if $i \neq j$, then x_i and x_j sit in distinct cells.

Since any closed cell contains only a finite number of the x_i's, the set of all x_{i*}'s is closed, and its complement is a neighbourhood of x_0. We reached a contradiction, because this complement contains none of the neighbourhoods $U_1, U_2 \dots$. ☐

Theorem 2.1.4.12. *A cellular space is first countable if and only if it is locally finite.*

Proof. The necessity of this condition is a corollary of Lemma 2.1.4.11. To prove its sufficiency, use Proposition 2.1.1.14 to deduce that every point x_0 of a locally finite cellular space has a neighbourhood U contained in a finite subspace. By Theorem 2.1.4.4, such a subspace, and hence U, are first countable spaces, and it is obvious that a countable base of U at x_0 is also a countable base of the given space at x_0. ☐

Theorem 2.1.4.13. *A cellular space is second countable if and only if it is countable and locally finite.*

Proof. The necessity of this condition is a corollary of 2.1.4.10 and 2.1.4.12. To prove its sufficiency, for each closed cell we fix a neighbourhood contained in a finite subspace (see Theorem 2.1.1.14). As a result of Theorem 2.1.4.4, these neighbourhoods are second countable, and it is clear that the union of their countable bases is a countable base of the given space. □

Theorem 2.1.4.14. *A cellular space is metrisable if and only if it is locally finite.*

Proof. The necessity of this condition is a corollary of Theorem 2.1.4.12. The metrisability of a connected locally finite cellular space follows from Propositions 2.1.4.9, 2.1.4.13, 2.1.2.3, and 1.1.6.9. Since every locally finite cellular space is homeomorphic to the sum of its components, and these are metrisable, the space is metrisable too (see Theorem 2.1.4.7 and Remark 1.2.1.1). □

2.1.5 Cellular Constructions

Remark 2.1.5.1. When applied to cellular spaces, the constructions described in §2.1.2 need to be appropriately modified. For certain constructions the modification consists merely of observing that the resulting space is endowed with a cellular decomposition and becomes cellular; an obvious example is the sum. For other examples, such as the product, the modification also affects the topology of the resulting space.

Below we describe the main modifications of both types. We emphasise that all these constructions, when applied to rigged cellular spaces, produce again rigged cellular spaces.

Cellular Product

Remark 2.1.5.2. Let X_1 and X_2 be topological spaces with cellular decompositions $\wp_1$ and $\wp_2$. Then the product $X_1 \times X_2$ has a natural cellular decomposition, namely $\wp_1 \times \wp_2$ with $\dim(e_1 \times e_2) = \dim e_1 + \dim e_2$. As a characteristic map for the cell $e_1 \times e_2$ one may take the composition of the canonical homeomorphism

$$\mathbb{D}^{\dim e_1 + \dim d_2} \to \mathbb{D}^{\dim e_1} \times \mathbb{D}^{\dim d_2}$$

with the product

$$\mathrm{char}_{e_1} \times \mathrm{char}_{e_2} : \mathbb{D}^{\dim e_1} \times \mathbb{D}^{\dim d_2} \to X_1 \times X_2$$

of arbitrary characteristic maps

$$\mathrm{char}_{e_1} : \mathbb{D}^{\dim e_1} \to X_1 \quad \text{and} \quad \mathrm{char}_{e_2} : \mathbb{D}^{\dim d_2} \to X_2.$$

When the decompositions $\wp_1$ and $\wp_2$ are rigged, $\wp_1 \times \wp_2$ takes on a canonical rigging.

If $\wp_1$ and $\wp_2$ fulfil property (**C**), then (**C**z) holds for $\wp_1 \times \wp_2$ too. However, there are situations where $\wp_1 \times \wp_2$ does not have property (**W**), even when X_1 and X_2 are cellular spaces; see Exercise 2.1.6.6. The cellular space arising from the product $X_1 \times X_2$ of the cellular spaces X_1 and X_2 through cellular weakening of its topology is called the *cellular product* of X_1 and X_2, and is denoted by $X_1 \times_{\mathbf{C}} X_2$.

We note that the cellular weakening does not alter the topology of the compact parts of the space $X_1 \times X_2$. Indeed, every compact subset of $X_1 \times X_2$ has compact images under the projections $X_1 \times X_2 \to X_1$ and $X_1 \times X_2 \to X_2$, and hence it can be covered by a finite number of cells.

Theorem 2.1.5.3. *If X_1 is locally finite, then $X_1 \times_{\mathbf{C}} X_2 = X_1 \times X_2$ for any cellular space X_2.*

Proof. Let char^1 and char^2 be the total characteristic maps corresponding to some riggings of the cellular decompositions $\wp_1$ and $\wp_2$ of X_1 and X_2. It is clear that the total characteristic map corresponding to $\wp_1 \times \wp_2$ can be expressed as the composition

$$\begin{aligned}&\coprod \mathbb{D}^{\dim(e_1 \times e_2)} \to \coprod (\mathbb{D}^{\dim e_1} \times \mathbb{D}^{\dim e_2}) = \\ &(\coprod \mathbb{D}^{\dim e_1}) \times (\coprod \mathbb{D}^{\dim e_2}) \xrightarrow{\operatorname{char}^1 \times \operatorname{char}^2} X_1 \times X_2\end{aligned} \tag{2.1.5.4}$$

where the first map is the sum of the canonical homeomorphisms $\mathbb{D}^{\dim(e_1 \times e_2)} \to \mathbb{D}^{\dim e_1} \times \mathbb{D}^{\dim e_2}$. Since $\wp_1$ and $\wp_2$ satisfy condition (**W**), the maps char^1 and char^2 are factorial (see Definition 2.1.1.3). Furthermore, since $\coprod \mathbb{D}^{\dim e_2}$ and X_1 are locally compact (see Theorem 2.1.4.3), the map $\operatorname{char}^1 \times \operatorname{char}^2$ is factorial too (see Theorem 1.2.7.9), which in turn implies that the composite map (2.1.5.4) is factorial. Therefore, the decomposition $\wp_1 \times \wp_2$ has property (**W**). ☐

Information 2.1.5.5. If every point of each of the cellular spaces X_1 and X_2 has a neighbourhood which intersects only a countable family of cells, then $X_1 \times_{\mathbf{C}} X_2 = X_1 \times X_2$; see [6] for a proof.

Attaching

Remark 2.1.5.6. Consider two cellular spaces X_1 and X_2, a subspace C of X_1, and a cellular map $\varphi \colon C \to X_2$. According to Remark 1.2.4.8, $X_2 \cup_\varphi X_1$ is a well-defined topological space, while Corollary 2.1.2.3 and Proposition 1.2.4.9 imply that $X_2 \cup_\varphi X_1$ is normal. Now decompose $X_2 \cup_\varphi X_1$ into the sets $\operatorname{Imm}_1 e_1$ and $\operatorname{Imm}_2 e_2$, where e_1 and e_2 run over the cells in $X_1 \setminus C$ and X_2 respectively, and put $\dim(\operatorname{Imm}_1 e_1) = \dim e_1$ and $\dim(\operatorname{Imm}_2 e_2) = \dim e_2$. This is a cellular decomposition: as a characteristic map for $\operatorname{Imm}_i e_i$ one may take the composition of an arbitrary characteristic map $\operatorname{char}_{e_i}$ with Imm_i. Clearly, the only cells that the closure of the cell $\operatorname{Imm}_1 e_1$ intersects are either $\operatorname{Imm}_1 \varepsilon_1$, where ε_1 is a cell in X_1 intersecting $\operatorname{Cl} e_1$, or $\operatorname{Imm}_2 \varepsilon_2$ where ε_2 is a cell in X_2 intersecting $\varphi(\operatorname{Cl} e_1 \cap C)$. Moreover, we see that$\operatorname{Cl} \operatorname{Imm}_2 e_2$ intersects only the cells $\operatorname{Imm}_2 \varepsilon_2$, where ε_2 is a cell in X_2 intersecting $\operatorname{Cl} e_2$. Consequently, our decomposition has property (**C**).

To see that it has property (**W**) too, let F be a subset of having closed intersections with all the cells in $X_2 \cup_\varphi X_1$. The equality

$$\operatorname{Imm}_2^{-1}(F) \cap \operatorname{Cl} e_2 = \operatorname{Imm}_2^{-1}(F \cap \operatorname{Cl}(\operatorname{Imm}_2 e_2))$$

shows that $\operatorname{Imm}_2^{-1}(F)$ is closed, and now the equality

$$\operatorname{Imm}_1^{-1}(F) \cap \operatorname{Cl} e_1 = \begin{cases} \varphi^{-1}(\operatorname{Imm}_2^{-1}(F) \cap \varphi(\operatorname{Cl} e_1), & \text{if} \quad e_1 \subset C, \\ \operatorname{Imm}_1^{-1}(F \cap \operatorname{Cl}(\operatorname{Imm}_1 e_1)), & \text{if} \quad e_1 \subset X_1 \setminus C, \end{cases}$$

proves that $\operatorname{Imm}_1^{-1}(F)$ is closed too. We conclude that the space $X_2 \cup_\varphi X_1$ is cellular. It is immediate that $\operatorname{Imm}_2(X_2)$ is a subspace of $X_2 \cup_\varphi X_1$, that Imm_2 is a cellular embedding, and that the map Imm_1 is cellular.

If $X_2 = \mathbb{D}^0$, then $\varphi\colon C \to X_2$ is a cellular map for any cellular pair (X_1, C), and $X_2 \cup_\varphi X_1 = X_1/C$. Thus, the previous definition implies that the quotient space of a cellular space by a subspace is cellular.

Limits

Remark 2.1.5.7. Suppose that $X_0, X_1, \dots,$ are cellular spaces and $\varphi_0\colon X_0 \to X_1, \varphi_1\colon X_1 \to X_2, \dots,$ are cellular embeddings. By Remark 1.2.4.4, the limit $\lim(X_k, \varphi_k)$ is a well-defined topological space, which is also normal (see Corollary 2.1.2.32.3 and Theorem 1.2.4.6). Now consider the decomposition of $\lim(X_k, \varphi_k)$ into the sets $\operatorname{Imm}_k e_k$, where e_k is a cell in $X_k \setminus \varphi_{k-1}(X_{k-1})$, $k = 0, 1, \dots,$ and put $\dim(\operatorname{Imm}_k e_k) = \dim e_k$. If we take the composition of an arbitrary characteristic map $\operatorname{char}_{e_k}$ with Imm_k as a characteristic map for the cell $\operatorname{Imm}_k e_k$, we see that this decomposition is cellular. Since it obviously satisfies conditions (**C**) and (**W**), $\lim(X_k, \varphi_k)$ becomes a cellular space, and Imm_k become cellular embeddings.

Notice that this definition of the limit includes as a special case the inductive process of glueing a cellular space from balls that we discussed in subsection 2.1.2.

More Special Constructions

Remark 2.1.5.8. Since decomposing the segment I into the cells 0, 1, and $\operatorname{int} I$ makes I into a finite cellular space, the cylinder $X \times I$ is cellular for any cellular space X; see Remark 2.1.5.2 and Theorem 2.1.5.3. The bases of $X \times I$ are cellular subspaces (in the sense of Definition 2.1.1.9); hence when we pass to the quotient space $\operatorname{cone} X$ of $X \times I$, and then to the quotient space $\operatorname{sus} X$ of $\operatorname{cone} X$, we find ourselves in the situation covered by the construction in Remark 2.1.5.6. Therefore, the cone and the suspension over a cellular space are also cellular spaces.

If $f\colon X_1 to X_2$ is a cellular map, then the attaching processes which transform $X_1 \times I$ into $\operatorname{Cyl} f$, and $\operatorname{cone} X_1$ into $\operatorname{Cone} f$ fall again into the category described in Remark 2.1.5.6 Therefore, the mapping cylinder and the mapping cone of a cellular map are cellular spaces.

Definition 2.1.5.9. The *cellular join* $X_1 \star_{\mathbf{C}} X_2$ of two cellular spaces X_1 and X_2 is defined as

$$X_1 \star_{\mathbf{C}} X_2 = (X_1 \coprod X_2) \cup_\varphi [(X_1 \times_{\mathbf{C}} X_2) \times I],$$

where

$$\varphi\colon [(X_1 \times_{\mathbf{C}} X_2) \times 0] \cup [(X_1 \times_{\mathbf{C}} X_2) \times 1] \to X_1 \coprod X_2$$

is given by

$$\varphi(x_1, x_2, 0) = \mathrm{incl}_1(x_1), \quad \varphi(x_1, x_2, 1) = \mathrm{incl}_2(x_2);$$

cf. Remark 1.2.6.3. Since φ is cellular, the space $X_1 \star_{\mathbf{C}} X_2$ is cellular.

According to Theorem 2.1.5.3, when X_1 is locally finite, $X_1 \star_{\mathbf{C}} X_2$ is topologically the same as $X_1 \star X_2$. In general, the cellular decomposition of $X_1 \star_{\mathbf{C}} X_2$ is cellular for $X_1 \star X_2$ too, and so cellular weakening of the topology of $X_1 \star X_2$ yields $X_1 \star_{\mathbf{C}} X_2$. However, this process does not affect the topology of the compact sets of $X_1 \star X_2$; cf. Remark 2.1.5.2.

The Case of Pointed Spaces

Remark 2.1.5.10. Suppose that X is a cellular space and x_0 is a 0-cell that we take as a base point. The cone $\mathrm{cone}(X, x_0)$ and the suspension $\mathrm{sus}(X, x_0)$ are quotients of $\mathrm{cone}\, X$ and $\mathrm{sus}\, X$ by subspaces, and as such they are cellular spaces. Similarly, the bouquet of a family of cellular spaces with 0-cells as base points is the quotient of the sum of this family by a subspace, and hence is a cellular space.

Finally, we define the *cellular tensor product* and the *cellular join* of the cellular spaces X_1 and X_2 with the 0-cells x_1 and x_2 taken as base points, as the quotient spaces

$$X_1 \times_{\mathbf{C}} X_2/[(X_1 \times x_2) \cup (x_1 \times X_2)] \quad \text{and} \quad (X_1 \star_{\mathbf{C}} X_2)/(X_1 \star X_2)$$

respectively. These are cellular spaces, denoted by

$$(X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2) \quad \text{and} \quad (X_1, x_1) \star_{\mathbf{C}} (X_2, x_2).$$

If X_1 is locally finite, then they are identical with

$$(X_1, x_1) \otimes (X_2, x_2) \quad \text{and} \quad (X_1, x_1) \star (X_2, x_2).$$

as topological spaces. In the general case, the cellular decompositions of

$$(X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2) \quad \text{and} \quad (X_1, x_1) \star_{\mathbf{C}} (X_2, x_2).$$

are cellular for

$$(X_1, x_1) \otimes (X_2, x_2) \quad \text{and} \quad (X_1, x_1) \star (X_2, x_2)$$

too. Thus

$$(X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2) \quad \text{and} \quad (X_1, x_1) \star_{\mathbf{C}} (X_2, x_2).$$

arise from the cellular weakening of the topologies of

$$(X_1, x_1) \otimes (X_2, x_2) \quad \text{and} \quad (X_1, x_1) \star (X_2, x_2)$$

respectively, and it is clear that this process does not affect the topology of the compact subsets of

$$(X_1, x_1) \otimes (X_2, x_2) \quad \text{and} \quad (X_1, x_1) \star (X_2, x_2).$$

2.1.6 Exercise

Exercise 2.1.6.1. Show that given an arbitrary cellular space X and an arbitrary point $x \in X$, there exists a cellular space Y together with a cellular homeomorphism $f\colon X \to Y$ such that $f(x) \in \operatorname{skel}_0 Y$.

Exercise 2.1.6.2. Show that the sphere $\mathbb{S}^\infty$ and the ball $\mathbb{D}^\infty$ are homeomorphic to cellular spaces.

Exercise 2.1.6.3. Show that every connected, locally finite cellular space can be topologically embedded in $\mathbb{R}^\infty$.

Exercise 2.1.6.4. Show that every connected, finite dimensional, locally finite cellular space can be embedded in $\mathbb{R}^q$, for sufficiently large q.

Exercise 2.1.6.5. Show that every finite cellular space admits a cellular embedding in a cellular space homeomorphic to $\mathbb{D}^q$, for sufficiently large q.

INFORMATION. Every finite cellular space of dimension n can be embedded in a cellular space homeomorphic to $\mathbb{D}^{2n+1}$.

Exercise 2.1.6.6. Consider the bouquet $B = V_{t \in \mathbb{R}}(I_t = I, 0)$ as a cellular space (see Remark 2.1.5.10) and show that the map $\operatorname{id}\colon B \times_{\mathbf{C}} B \to B \times B$ is not a homeomorphism.

2.2 SIMPLICIAL SPACES

2.2.1 Euclidean Simplices

Remark 2.2.1.1. Let A be a subset of $\mathbb{R}^n$ consisting of $r+1$ points ($r \geq 0$) which are not contained in any $(r-1)$-dimensional plane. The convex hull of A (i.e., the smallest convex set containing A) is called the *Euclidean simplex spanned by* A, and is denoted by $\operatorname{Esi} A$. The points of A are the *vertices* of the simplex $\operatorname{Esi} A$, and the number r is its *dimension.* $\operatorname{Esi} A$ is also called a *Euclidean r-simplex.*

Obviously, a point of $\operatorname{Esi} A$ is a vertex if and only if $\operatorname{Esi} A$ contains no non-degenerate segment whose midpoint falls on the given point. Therefore, the set A is uniquely determined by $\operatorname{Esi} A$.

Every simplex spanned by a subset of A is called a *face* of the simplex $\operatorname{Esi} A$. It is clear that $\operatorname{Esi} A_1 \cap \operatorname{Esi} A_2 = \operatorname{Esi}(A_1 \cap A_2)$, for any $A_1, A_2 \subset A$.

Two faces spanned by complementary subsets A_1 and A_2 of A are said to be *opposite.* In this case, the formula

$$\operatorname{proj}(x_1, x_2, t) \mapsto (1-t)x_1 + tx_2 \quad (x_1 \in \operatorname{Esi} A_1, \quad x_2 \in \operatorname{Esi} A_2, \quad t \in I)$$

defines a homeomorphism of the join $\operatorname{Esi} A_1 \star \operatorname{Esi} A_2$ onto $\operatorname{Esi} A$. Thus, every Euclidean simplex is canonically homeomorphic to the join of any of the pairs of its opposite faces.

Since $\mathbb{D}^r$ is canonically homeomorphic to any join with $p+q=r-1$ (see Remark 1.2.6.9), a trivial induction proves that both the spaces $\operatorname{Esi} A$ and $\mathbb{D}^r$ are homeomorphic to a join of $r+1$ points. We conclude that every Euclidean r-simplex is homeomorphic to $\mathbb{D}^r$.

It is clear that the boundary of the simplex $\operatorname{Esi} A$ in the r-plane that it determines is precisely the union of its $(r-1)$-faces. Usually, this boundary and its complement in $\operatorname{Esi} A$ are simply referred to as the *boundary* and the *interior* of the simplex $\operatorname{Esi} A$.

Remark 2.2.1.2. We may equivalently describe the simplex $\operatorname{Esi} A$ as the set of all sums $\sum_{a \in A} t_a a$, where $t_a \geq 0$ and $\sum_{a \in A} t_a = 1$. Since there is no $(r-1)$-plane containing A, the numbers t_a are determined uniquely for any point $x = \sum_{a \in A} t_a a$; $t_a$is called *a-th barycentric coordinates* of x and is denoted by $\operatorname{bary}_a(x)$. Obviously, a face $\operatorname{Esi} B$ of the simplex $\operatorname{Esi} A$ is defined in the barycentric coordinates of $\operatorname{Esi} A$ by the equations $\operatorname{bary}_a(x) = 0$ for $a \in A \setminus B$. Moreover, if $x \in \operatorname{Esi} B$, then the coordinates $\operatorname{bary}_a(x)$ computed in $\operatorname{Esi} A$ and $\operatorname{Esi} B$ coincide for all $a \in B$.

The point of $\operatorname{Esi} A$ having all barycentric coordinates equal, i.e, equal to $1/(r+1)$, is the *centre* of the simplex $\operatorname{Esi} A$.

Remark 2.2.1.3. A map $\operatorname{Esi} A \to \operatorname{Esi} B$ is called *simplicial* if it is affine and takes A into B. It is clear that such a map takes each face of $\operatorname{Esi} A$ simplicially into a face of $\operatorname{Esi} B$, and takes the interior of $\operatorname{Esi} A$ onto the interior of the simplex which is its image.

Obviously, every map $A \to B$ extends uniquely to a simplicial map $\operatorname{Esi} A \to \operatorname{Esi} B$. If the given map $A \to B$ is injective (invertible), then its simplicial extension $\operatorname{Esi} A \to \operatorname{Esi} B$ is an embedding (respectively, a homeomorphism).

Definition 2.2.1.4. $\operatorname{Esi} A$ is said to be an *ordered* simplex if the set A is ordered. Since the subsets of an ordered set inherit a natural order, all the faces of an ordered simplex are ordered simplices.

If $\operatorname{Esi} A$ and $\operatorname{Esi} B$ are ordered r-simplices, then the orders of A and B define an invertible map $A \to B$, and hence a simplicial homeomorphism $\operatorname{Esi} A \to \operatorname{Esi} B$. Consequently, all ordered Euclidean simplices of the same dimension are canonically simplicial homeomorphic.

Remark 2.2.1.5. The simplex spanned by the points $\operatorname{ort}_1, \dots, \operatorname{ort}_r$ of $\mathbb{R}^{r+1}$ is called the *unit r-simplex* and is denoted by T^r. This simplex is notable due to the fact that its barycentric coordinates are the usual coordinates in $\mathbb{R}^{r+1}$. The given order of its vertices transforms T^r into an ordered simplex, and thus every ordered Euclidean r-simplex is canonically simplicial homeomorphic to T^r.

Note that given an ordered simplex $\operatorname{Esi} A$, the homeomorphism $\operatorname{Esi} A \to \mathbb{D}^r$ discussed in Remark 2.2.1.1 is now canonical. The canonical homeomorphism $T^r \to \mathbb{D}^r$ and its inverse are denoted by $T\mathbb{D}$ and $\mathbb{D}T$, respectively. That $T\mathbb{D}$ maps the boundary (the interior) of T^r onto $\mathbb{S}^{r-1}$ (respectively, onto $\operatorname{int} \mathbb{D}^r$) is plain.

Topological Simplices

Remark 2.2.1.6. A topological space X is an *ordered topological simplex of dimension r* (or an *ordered topological r-simplex*) if there exists a homeomorphism $T^r \to X$; this is called a *characteristic homeomorphism of the simplex X*, while X is sometimes referred to as the *support* of the simplex. For example, all ordered Euclidean r-simplices and the ball $\mathbb{D}^r$ are ordered topological r-simplices; see Remark 2.2.1.5.

The standard way to destroy an order is to introduce simultaneously all possible orders. Accordingly, we say that the topological space X is a *topological simplex of dimension r* (or a *topological r-simplex*) if there are given $(r+1)!$ homeomorphisms $T^r \to X^r$, which can be transformed into each other by simplicial homeomorphisms $T^r \to T^r$. The terms *characteristic homeomorphism* and *support* are employed in this situation too; however, now we have at our disposal $(r+1)!$ equally rightful characteristic homeomorphisms.

If X is a topological r-simplex (an ordered topological r-simplex), and Y is a topological space, then every homeomorphism $X \to Y$ transforms Y into a topological r-simplex (respectively, into an ordered topological r-simplex). Consequently, every homeomorphic image of a Euclidean r-simplex (ordered Euclidean r-simplex) is a topological r-simplex (respectively, an ordered topological r-simplex).

The vertices, faces, boundary, interior, barycentric coordinates, centre, and simplicial maps are defined in an obvious fashion for topological simplices. The faces of a topological simplex (ordered topological simplex) are topological simplices (respectively, ordered topological simplices). As with a Euclidean simplex, a topological simplex becomes an ordered one as soon as we fix an order of its vertices.

2.2.2 Simplicial Spaces and Simplicial Maps

Definition 2.2.2.1. A *triangulation* of a set X is a cover Δ of X by topological simplices such that:

(i) every face of an arbitrary simplex in A is again a simplex in Δ;

(ii) if a simplex in Δ is contained in another simplex of A, then the first is a face of the second;

(iii) the intersection of the supports of two overlapping simplices of Δ is again the support of a simplex in Δ.

A set X endowed with a triangulation is known as a *simplicial space*; the simplices of the triangulation are called *simplices of the space*, and the O-simplices are its *vertices*. The smallest simplex in the triangulation which contains a given point $x \in X$ is denoted by $\operatorname{simplex}_x$.

According to Remark 1.2.4.3, a triangulation transforms the given set into a topological space, and Remark 1.2.4.1 shows that the supports of the simplices of the triangulation yield a fundamental cover of this space. Since the intersection of two simplices in the triangulation is closed in each of them, the simplices in the triangulation keep the same topology when considered as subspaces of this topological space (see Definition 1.2.4.2).

Let a be a vertex of the simplicial space X. Then the a-th barycentric coordinate $\operatorname{bary}_a(x)$ is well defined for any point x belonging to any simplex which has a as one of its vertices (see Remarks 2.2.1.2 and 2.2.1.6), and we obtain a continuous function $\operatorname{bary}_a \colon X \to \mathbb{R}$ if we set $bar_a(x) = 0$ for those points $x \in X$ contained in simplices which do not have a as a vertex. bary_a is called the *a-th barycentric function*. Given two arbitrary distinct points $x, y \in X$, there obviously is a vertex a such that $\operatorname{bary}_a(x) \neq \operatorname{bary}_a(y)$. Consequently, *every simplicial space is Hausdorff.*

When a set X endowed with a triangulation already has a topology, it is useful to find conditions ensuring that the topology defined by the triangulation is identical with the initial one. We have an immediate necessary and sufficient condition: the topology of each simplex in the triangulation coincides with the topology induced by the initial topology of X, and the cover of X by the supports of these simplices is fundamental in the initial topology. If this condition is satisfied, then the given triangulation is said to be a *triangulation of the initial topological space* X. Example: the cover of a topological simplex by all its faces is a triangulation of this simplex.

A simplicial space is *ordered* if its simplices are ordered in such a way that the orders of the faces of any simplex agree with the order of the simplex itself.

In particular, this holds whenever the order of the simplices is induced by some order on the set of all vertices of the given space, which incidentally shows that a simplicial space can be always ordered.

Remark 2.2.2.2. We shall presently describe a fundamental class of simplicial spaces. Given an arbitrary non-empty set A, we let $\operatorname{Si} A$ denote the set of all non-negative, finitely supported functions $\varphi\colon A \to \mathbb{R}$ such that $\sum_{a \in A} \varphi(a) = 1$. If $B \subset A$, then we identify $\operatorname{Si} B$ with the subset of $\operatorname{Si} A$ consisting of all functions $\varphi \in \operatorname{Si} A$ such that $\varphi(x) = 0$ for $x \in A \setminus B$. If A is finite and has $r + 1$ elements, then $\operatorname{Si} A$ is obviously a topological simplex: indeed, $\operatorname{Si} A$ is a subset of the $(r + 1)$-dimensional Euclidean space of all functions $A \to \mathbb{R}$. Moreover, corresponding to the $(r + 1)!$ orders on A there are $(r + 1)!$ homeomorphisms $T^r \to \operatorname{Si} A$, each transforming the point $(x_1, \dots, x_{r+1})$ into a function taking the values $x_1, \dots, x_{r+1}$; one may transform one homeomorphism into another by composition with a simplicial homeomorphism $T^r \to T^r$. In the general case, $\operatorname{Si} A$ is covered by the topological simplices $\operatorname{Si} B$ corresponding to all finite subsets B of A, and it is clear that this yields a triangulation of $\operatorname{Si} A$. $\operatorname{Si} A$ is therefore a simplicial space, and we call it the *simplex spanned by A*. Ordering $\operatorname{Si} A$ is equivalent to ordering the set A.

Definition 2.2.2.3. The interiors of the simplices of a simplicial space X constitute a decomposition of the set X. If we define the dimension of the interior e of the simplex s by $\dim e = \dim s$, and take as a characteristic map for e the composition

$$\mathbb{D}^{\dim s} \xrightarrow{\mathbb{D}T} T^{\dim s} \xrightarrow{\varphi} s \xrightarrow{\text{incl}} X, \tag{2.2.2.4}$$

where φ is any characteristic homeomorphism for the simplex s, then the above decomposition becomes cellular. Since conditions (**C**) and (**W**) are clearly satisfied in this situation, and we already know that every simplicial space is Hausdorff, we see that this cellular decomposition transforms X into a cellular space. Thus, *every simplicial space, decomposed into the interiors of its simplices, is a cellular space.*

Since an r-simplex has $(r + 1)!$ characteristic homeomorphisms, formula (2.2.2.4) distinguishes $(r + 1)!$ privileged maps in the family of all maps that are characteristic for an r-cell in a simplicial space; we call them *simplicial characteristic maps* and note that they are topological embeddings. Fixing a simplicial characteristic map is equivalent to fixing an order of the simplex s; hence every ordered simplicial space is canonically rigged. It is clear that the skeleton $\operatorname{skel}_r X$ of a simplicial space X is simply the union of all its simplices of dimension $\leq r$, and that $\dim(\operatorname{skel}_r X) = r$ for $r \leq \dim X$. In particular, $\operatorname{skel}_0 X$ is the set of vertices of X. Of course, X is finite if and only if $\operatorname{skel}_0 X$ is finite as a set. A simplicial space is locally finite if and only if each of its vertices is contained in only a finite number of simplices (see Theorem 2.1.1.7).

Subspaces

Definition 2.2.2.5. A *subspace* of a simplicial space is any subset which is a union of whole simplices. Every subspace has a natural triangulation, and hence is a simplicial space. The subspaces of an ordered simplicial space are also ordered simplicial spaces.

Obviously, a subset of a simplicial space X is a subspace of X if and only if it is a cellular subspace of the cellular space X.

A subspace of a simplicial space X is *complete* if its intersection with the support of any simplex of X is either the support of some simplex of X or empty. An equivalent formulation: a subspace is complete if it contains, along with the vertices of a simplex, the simplex itself.

Of course, the simplices of a simplicial space are complete subspaces. A subspace of $\operatorname{Si} A$ is complete if and only if it is of the form $\operatorname{Si} B$, where $B \subset A$.

Simplicial Maps

Remark 2.2.2.6. A map $X \to Y$, where X and Y are simplicial spaces, is called *simplicial* if it transforms every simplex of X simplicially into a simplex of Y. It is clear that such a map is also cellular and maps X onto a subspace of Y.

The following facts are also immediate.

An invertible simplicial map is a homeomorphism, and its inverse is also simplicial. Every injective simplicial map is a topological embedding. A simplicial map $f\colon X \to Y$ is uniquely defined by the map $\operatorname{abr} f\colon \operatorname{skel}_0 X \to \operatorname{skel}_0 Y$ from the set of vertices of X into the set of vertices of Y. A map $\operatorname{skel}_0 X \to \operatorname{skel}_0 Y$ extends to a simplicial map $X \to Y$ if and only if it carries the vertices of each simplex of X into the vertices of a simplex of Y. A simplicial map $f\colon XY$ is injective (invertible) if and only if $\operatorname{abr} f\colon \operatorname{skel}_0 X \to \operatorname{skel}_0 Y$ is injective (respectively, invertible).

Two simplicial spaces which can be transformed one into another by a simplicial homeomorphism are said to be *simplicial homeomorphic*.

Definition 2.2.2.7. A simplicial map $f\colon X \to Y$, where X and Y are ordered simplicial spaces, is *monotone* if $f(a) \preceq f(b)$ for any pair of vertices, a and b, of X which belong to the same simplex and satisfy $a < b$.

Every simplicial map between simplicial spaces can be made monotone by suitably ordering the spaces. Moreover, if X and Y are simplicial spaces and Y is ordered, then one can transform a given simplicial map $f\colon X \to Y$ into a monotone one my suitably ordering X; indeed, it suffices to order arbitrarily the pre-image of each vertex of Y, and then order the simplices of X by the rule: $a \prec b$ whenever $f(a) \prec f(b)$ or $f(a) = f(b)$ and $a \prec b$ in $f^{-}(f(a))$.

2.2.3 Simplicial Schemes

Definition 2.2.3.1. A *simplicial scheme* (or *schema*) is a pair (M, S), where M is a set and S is a cover of M by finite subsets, such that S contains, along with each set $A \in S$, all the parts of A.

A *map of the simplicial scheme* (M, S) into the simplicial scheme (M', S') is a pair of maps, $\varphi\colon M \to M'$ and $\varphi\colon S \to S'$, such that $\varphi(A) = \varphi(A)$ for all $A \in S$. The last condition shows that the map (φ, φ) of (M, S) into (M', S')

is actually uniquely determined by φ. Obviously, $\varphi\colon M \to M'$ defines a map of the scheme (M, S) into the scheme (M', S') if and only if $\varphi(A) \in S'$ for all $A \in S$. If φ and φ are invertible, i.e., φ is invertible and $\varphi(S) = S'$, then the map (φ, φ) is called an *isomorphism*. Two simplicial schemes which can be related by an isomorphism are *isomorphic*.

A simplicial scheme (M, S) is a *subscheme* of the simplicial scheme (M', S') if $M \subset M'$ and $S \subset S'$. The subscheme (M, S) is *complete* if $A \in S'$ and $A \subset M$ imply $A \in S$.

Definition 2.2.3.2. The simplicial scheme given by the skeleton $\operatorname{skel}_0 X$ of a simplicial space X and the cover of $\operatorname{skel}_0 X$ by the 0-skeletons of the simplices of X is termed the *scheme of the space* X and is denoted by $\operatorname{sch} X$. For example, the scheme of $\operatorname{Si} A$ (see Remark 2.2.2.2 consists of the set A and of the cover of A by all its finite subsets.

The map $\operatorname{abr} f\colon \operatorname{skel}_0 X \to \operatorname{skel}_0 X'$ induced by a simplicial map $f\colon X \to X'$ takes the 0-skeleton of each simplex of X into the 0-skeleton of a simplex of X'. Hence it defines a map of $\operatorname{sch} X$ into $\operatorname{sch} X'$, called the *scheme of the map* f and denoted by $\operatorname{sch} f$. The discussion in Remark 2.2.2.6 implies that a simplicial map is uniquely determined by its scheme, that every map of $\operatorname{sch} X$ into $\operatorname{sch} X'$ is the scheme of some simplicial map $X \to X'$, for any simplicial spaces X and X', and that a simplicial map is invertible if and only if its scheme is an isomorphism. In particular, *two simplicial spaces X and X' are simplicial homeomorphic if and only if their schemes $\operatorname{sch} X$ and $\operatorname{sch} X'$ are isomorphic.*

Remark 2.2.3.3. If X is a subspace of the simplicial space X', then $\operatorname{sch} X$ is a subscheme of $\operatorname{sch} X'$, and $\operatorname{sch} X$ is complete if and only if X is complete. Moreover, it is clear that every subscheme of $\operatorname{sch} X'$ is the scheme of a subspace of X'.

In particular, let (M, S) be an arbitrary simplicial scheme, and consider the simplex $\operatorname{Si} M$. Obviously, (M, S) is a subscheme of $\operatorname{sch} \operatorname{Si} M$, and so (M, S) is the scheme of a subspace of $\operatorname{Si} M$. Thus, *every simplicial scheme is the scheme of a simplicial space.* Moreover, given an arbitrary simplicial space X, we may take (M, S) to be the scheme of X and conclude that every simplicial space X can be simplicially embedded in $\operatorname{Si} \operatorname{skel}_0 X$.

Definition 2.2.3.4. A simplicial scheme (M, S) is *ordered* if the sets of S are ordered and the order of each set $A \in S$. S is compatible with the orders of the subsets of A. A map (φ, φ) between ordered simplicial schemes (M, S) and (M', S') is *monotone* if $\varphi(a) \preceq \varphi(b)$ whenever $a \preceq b$. Therefore, ordering the scheme of a simplicial space is equivalent to ordering the space itself, and the scheme of a simplicial map between two ordered simplicial spaces is monotone if and only if the map itself is monotone.

2.2.4 Polyhedra

Remark 2.2.4.1. A *polyhedron* is a subset of Euclidean space which admits a finite triangulation by Euclidean simplices. Of course, the simplest polyhedra are the Euclidean simplices.

A subspace of a polyhedron is obviously a polyhedron.

Now any simplicial space can be simplicially embedded in the simplex spanned by its 0-skeleton (see Remark 2.2.3.3), and when the initial space is finite and has q vertices, this skeleton is simplicial homeomorphic to T^{q-1} Therefore, every finite simplicial space admits a simplicial embedding in a Euclidean simplex with the same number of vertices. Hence every finite simplicial space is simplicial homeomorphic to a polyhedron.

Theorem 2.2.4.2. *Every finite n-dimensional simplicial space is simplicial homeomorphic to a polyhedron contained in $\mathbb{R}^{2n+1}$.*

Proof. Since every finite n-dimensional simplicial space can be simplicially embedded in $\operatorname{skel}_n T^q$ for q large enough (see Remark 2.2.4.1), it suffices to construct, for arbitrarily given q and n, a linear mapping $f\colon \mathbb{R}^{q+1} \to \mathbb{R}^{2n+1}$ which is injective on $\operatorname{skel}_n T^q$.

If $q \le 2n$, we may take f to be the inclusion $\mathbb{R}^{q+1} \to \mathbb{R}^{2n+1}$. If $q > 2n$, define f by

$$f(\{x_j\}_{j=1}^{q+1}) = \left\{ \sum_{j=1}^{q+1} j^i x_j \right\}_{i=1}^{2n+1} .$$

All that remains is to verify that if $x = (x_1, \dots, x_{q+1})$ and $x' = (x'_1, \dots, x'_{q+1})$ belong to $\operatorname{skel}_n T^q$ and $f(x) = f(x')$, then $x = x'$. Since each of the points x and x' lies in an n-dimensional face of T^q, at most $n+1$ of the numbers $x_1, \dots, x_{q+1}$, and $n+1$ of the numbers $x'_1, \dots, x'_{q+1}$ are different from zero. Consequently, no more than $2n+2$ numbers $x_1 - x'_1, \dots, x_{q+1} - x'_{q+1}$ are different from zero, i.e., there are positive integers $j_1, \dots, j_{2n+2}$ such that $j_1 < \dots < j_{2n+2} \le 2q+1$ and $x_j = X'_j$ for $j \ne j_1, \dots, j_{2n+2}$. Since

$$\sum_{j=1}^{q+1} x_j = \sum_{j=1}^{q+1} x'_j (= 1) \quad \text{and} \quad \sum_{j=1}^{q+1} j^i x_j = \sum_{j=1}^{q+1} j^i x'_j$$

for $i = 1, \dots, 2n+1$, we have

$$\sum_{r=1}^{2n+2} j_r^i (x_{j_r} - x'_{j_r}) = 0, \quad i = 0, \dots, 2n+1.$$

The determinant of the matrix $\{j_r^i\}_{i=0,r=1}^{i=2n+1,r=2n+2}$ does not vanish, and so $x_{j_r} = x'_{j_r}$ for $r = 1, \dots, 2n+2$ and finally $x = x'$. ☐

Information 2.2.4.3. For any n there are n-dimensional polyhedra which cannot be topologically embedded in $\mathbb{R}^{2n}$. An example is $\operatorname{skel}_n T^{2n+2}$; see [10] for a proof.

2.2.5 Simplicial Constructions

Remark 2.2.5.1. Many of the topological and cellular constructions described in §1.2 and Subsection 2.1.5 can be replaced by parallel constructions which produce simplicial spaces out of simplicial ones. The simplest examples are the $\coprod$ and $\bigvee$ operations: a sum of simplicial spaces and a bouquet of pointed simplicial spaces with vertices as base points are obviously simplicial spaces. There are also more elaborate constructions, the more important ones being discussed below. The main one is the barycentric subdivision construction, which refines triangulations and has no analogues in §1.2 and Subsection 2.1.5.

Lemma 2.2.5.2. *Let Γ be a fundamental cover of the topological space X by triangulated subspaces. Suppose that for any $A, B \in \Gamma$ the intersection $A \cap B$ is a complete subspace of both A and B (considered as simplicial spaces) and inherits from A and B the same triangulation. Then there exists a unique triangulation of X relative to which the elements of Γ become simplicial subspaces.*

Proof. This triangulation of X is simply the union of the triangulations of the elements of Γ. One may check directly that this union satisfies conditions (i), (ii), and (iii) in Definition 2.2.2.1 [the completeness of the intersections $A \cap B$ is necessary for (iii)]. Uniqueness is also evident. ☐

Barycentric Subdivision

Definition 2.2.5.3. The construction below produces a new simplicial space, $\operatorname{bary} X$, from any simplicial space X, such that $\operatorname{bary} X$ is identical to X as a topological space, but has a finer triangulation, called the *barycentric subdivision* of the initial triangulation.

Consider first a Euclidean simplex X. For an arbitrary numeration $a_0, \dots, a_r$ of the vertices of X, form the set

$$\{x \in X \,|\, \operatorname{bary}_{a_0}(x) \leq \operatorname{bary}_{a_1}(x) \leq \cdots \leq \operatorname{bary}_{a_r}(x)\}. \tag{2.2.5.4}$$

It is readily seen that (2.2.5.4) is the Euclidean simplex whose vertices are the centres of the simplices $\operatorname{Esi} A_0, \dots, \operatorname{Esi} A_r$, where $A_i = a_0, \dots, a_r$. Furthermore, the simplices of the form (2.2.5.4) corresponding to all possible numerations of the vertices of X and their faces clearly yield a triangulation of X. This is precisely the barycentric subdivision of the standard triangulation of the simplex X, and it transforms X into $\operatorname{bary} X$. An obvious property of this construction is that the inclusion $\operatorname{bary} X \hookrightarrow \operatorname{bary} X'$ is a simplicial embedding whenever X is a face of the simplex X'.

Now if X is a topological simplex, we define the barycentric subdivision of its standard triangulation as the image of the barycentric subdivision of the standard triangulation of the unit simplex T^r, $r = \dim X$, under a simplicial homeomorphism $T^r \to X$. This is clearly a correct definition, i.e., the triangulation of X thus obtained does not depend on the choice of the simplicial homeomorphism $T^r \to X$ among the $(r+1)!$ available ones.

Finally, let X be an arbitrary simplicial space, and consider the cover of X by its simplices, each subdivided as above. It is easy to verify that this cover satisfies the conditions of Lemma 2.2.5.2, and hence we obtain a new triangulation of X, which is precisely the barycentric subdivision of the initial triangulation of X.

We note that the barycentric subdivision transforms a finite (locally finite) simplicial space into a finite (respectively, locally finite) one. Moreover, if X is a polyhedron, then so is $\operatorname{bary} X$.

Remark 2.2.5.5. The set of vertices of the space $\operatorname{bary} X$ equals exactly the set of centres of the simplices of X. The centres of the simplices $s_1, \dots, s_m$ of X are the vertices of a simplex of $\operatorname{bary} X$ if and only if $s_1, \dots, s_m$ Can be re-indexed to form an increasing sequence. This observation enables us to give a concise description of the barycentric subdivision in the language of schemes: if $\operatorname{sch} X = (M, S)$, then $\operatorname{sch} \operatorname{bary} X = (S, \operatorname{bary} S)$, where $\operatorname{bary} S$ is precisely the collection of those finite parts of S that can be ordered by inclusion. At the same time, we obtain a canonical order of $\operatorname{bary} X$: if $a, a' \in \operatorname{skel}_0 \operatorname{bary} X$, then $a \prec a'$ whenever the simplex (of X) with centre a is contained in the simplex with centre a'.

In particular the above description of $\operatorname{sch} \operatorname{bary} X$ shows that $\operatorname{bary} X$ is a complete subspace of $\operatorname{bary} X'$ whenever X is a subspace of X'. Indeed, $\operatorname{sch} \operatorname{bary} X$ is clearly a complete subscheme of $\operatorname{sch} \operatorname{bary} X'$.

In general, given a simplicial map $f\colon X \to X'$, the map $f\colon \operatorname{bary} X \to \operatorname{bary} X'$ is not simplicial (the simplest example: take $X = T^2$, $X' = T^1$, $f(\operatorname{ort}_1) = \operatorname{ort}_1$, $f(\operatorname{ort}_2) = f(\operatorname{ort}_3) = \operatorname{ort}_2$). However, the map $\operatorname{sch} f\colon \operatorname{sch} X \to \operatorname{sch} X'$ naturally induces a map $\operatorname{sch} \operatorname{bary} X \to \operatorname{sch} \operatorname{bary} X'$, and hence a simplicial map $\operatorname{bary} X \to \operatorname{bary} X'$. The latter is denoted by $\operatorname{bary} f$ and is clearly always monotone.

Theorem 2.2.5.6. *If X is a polyhedron, then the maximal diameter of the simplices of the polyhedron $\operatorname{bary} X$ does not exceed the maximal diameter of the simplices of X times $n/(n+1)$, where $n = \dim X$.*

Proof. It is enough to show that if X is the Euclidean simplex with vertices $a_0, \dots, a_r$, then the diameter of the simplex (2.2.5.4) is no bigger than $[r/(r+1)] \operatorname{diam} X$. Consider the part X' of X defined by the inequality $\operatorname{bary}_{a_r}(x) \geq r/(r+1)$. X' is the Euclidean simplex obtained by contracting X towards the vertex a_r by a factor of $r/(r+1)$. Consequently, $\operatorname{diam} X' \leq [r/(r+1)] \operatorname{diam} X$, and we finally note that X' contains the simplex (2.2.5.4). ☐

Corollary 2.2.5.7. *For any polyhedron X and any $\varepsilon > 0$ there is a positive integer m such that every simplex of the polyhedron $\operatorname{bary}^m X$ has diameter $< \varepsilon$.*

Simplicial Products

Definition 2.2.5.8. If X_1 and X_2 are simplicial spaces with $\dim X_1 > 0$ and $\dim X_2 > 0$, then it is readily seen that their cellular product $X_1 \times_{\mathbf{C}} X_2$ does not admit a triangulation such that the interiors of its simplices are products of interiors of simplices of X_1 and X_2. However, we shall presently show that $X_1 \times_{\mathbf{C}} X_2$ admits triangulations, and we shall construct a canonical triangulation when X_1 and X_2 are ordered. This construction produces a simplicial space out of $X_1 \times_{\mathbf{C}} X_2$, called the *simplicial product* of X_1 and X_2, and denoted by $X_1 \times_{\mathbf{S}} X_2$.

To begin with, let X_1 be the Euclidean simplex in $\mathbb{R}^m$ with vertices $a_0, \dots \dots, a_q$, and let X_2 be the Euclidean simplex in $\mathbb{R}^n$ with vertices $b_0, \dots \dots, b_r$. Set for $x_1 \in X_1$ and $x_2 \in X_2$:

$$\alpha_i(x_1) = \sum_{k=0}^{i} \operatorname{bary}_{a_k}(x_1), \quad \beta_j(x_2) = \sum_{l=0}^{j} \operatorname{bary}_{b_l}(x_2),$$

and arrange the numbers $\alpha_0(x_1), \dots, a_{q-1}(x_1), \beta_0(x_2), \dots, \beta_{r-1}(x_2)$ in a non-decreasing sequence $\gamma_1(x_1, x_2), \dots, \gamma_{q+r}(x_1, x_2)$. Further, let M_{qr} denote the collection of subsets with q elements of $\{1, \dots, q+r\}$, and let $s(\mu)$, where $\mu \in M_{qr}$, denote the set of all points $(x_1, x_2) \in X_1 \times X_2$ such that each of the numbers $\gamma_p(x_1, x_2)$, $p \in \mu$ is equal to one of the numbers $\gamma_0(x_1), \dots, \gamma_{q-1}(x_1)$. One may check directly that there is no $(q+r-1)$-dimensional plane containing the $q+r+1$ points

$$\begin{array}{l} (a_0, b_0), \dots, (a_0, b_{j_1-1}); \\ (a_1, b_{j_1-1}), \dots, (a_1, b_{j_2-2}); \\ \dots\dots\dots\dots\dots\dots \\ (a_{q-1}, b_{j_q-(q-1)}), \dots, (a_{q-1}, b_{j_q-q}); \\ (a_q, b_{j_q-q}), \dots, (a_q, b_r), \end{array} \tag{2.2.5.9}$$

where $j_1, \dots j_q \in \mu$, $j_1 < \cdots j_q$. Also, one may verify that

$$\sum_{k=0}^{q} \sum_{l=j_k-k}^{j_{k+1}-(k+1)} [\gamma_{k+l+1}(x_1, x_2) - \gamma_{k+l}(x_1, x_2)](a_k, b_l) = (x_1, x_2)$$

for $(x_1, x_2) \in s(\mu)$, $j_0 = 0$, $j_{q+1} = q+r+1$, $\gamma_0(x_1, x_2) = 0$ and $\gamma_{q+r+1}(x_1, x_2) = 1$. Moreover,

$$\sum_{k=0}^{q} \sum_{l=j_k-k}^{j_{k+1}-(k+1)} [\gamma_{k+l+1}(x_1, x_2) - \gamma_{k+l}(x_1, x_2)] = \sum_{p=0}^{q+r} [\gamma_{p+1}(x_1, x_2) - \gamma_p(x_1, x_2)] = 1$$

and $\gamma_{p+1}(x_1, x_2) - \gamma_p(x_1, x_2) \geq 0$. Consequently, the set $s(\mu)$ is contained in the Euclidean simplex spanned by the points (2.2.5.9), and since $s(\mu)$ is obviously convex and contains all the points (2.2.5.9), it equals this simplex. Now it is clear that the sets $s(\mu)$ cover $X_1 \times X_2$, and that $s(\mu_1) \cap s(\mu_2)$ is just the Euclidean simplex spanned by the vertices common to the simplices $s(\mu_1)$ and $s(\mu_2)$, for any $\mu_1, \mu_2 \in M_{qr}$. As a result, the simplices $s(\mu)$ and their faces constitute a triangulation of the product $X_1 \times X_2$, and this triangulation transforms $X_1 \times X_2$ into $X_1 \times_{\mathbf{S}} X_2$. Moreover it is readily seen that if X_1 and X_2 are faces of the ordered Euclidean simplices X_1' and X_2', then the inclusion $X_1 \times_{\mathbf{S}} X_2 \to X_1' \times_{\mathbf{S}} X_2'$ is a simplicial embedding.

Next let $X_1 \times X_2$ be ordered topological simplices. To define the simplicial product $X_1 \times_{\mathbf{S}} X_2$, use the previous prescription to triangulate the product of the unit simplices T^q and T^r, with $q = \dim X_1$ and $r = \dim X_2$, and then employ the product $T^q \times T^r \to X_1 \times X_2$ of canonical simplicial homeomorphisms $T^q \to X_1$ and $T^r \to X_2$ to carry this triangulation to $X_1 \times X_2$. With the resulting triangulation, $X_1 \times X_2$ becomes $X_1 \times_{\mathbf{S}} X_2$.

Finally, let X_1 and X_2 be arbitrary ordered simplicial spaces. It is readily verified that the cover of $X_1 \times_{\mathbf{C}} X_2$ by the product $s_1 \times s_2$ of simplices s_1 of X_1 and s_2 of X_2, where $s_1 \times s_2$ is triangulated as above, satisfies the conditions of Lemma 2.2.5.2. This lemma yields a triangulation which transforms $X_1 \times_{\mathbf{C}} X_2$ into the simplicial product $X_1 \times_{\mathbf{S}} X_2$.

We remark that each cell e of $X_1 \times_{\mathbf{C}} X_2$ can be represented as the union of a finite number of cells of $X_1 \times_{\mathbf{S}} X_2$, having dimensions $\leq \dim e$. In particular, the map $\mathrm{id}\colon X_1 \times_{\mathbf{C}} X_2 \to X_1 \times_{\mathbf{S}} X_2$ is cellular.

Remark 2.2.5.10. A straightforward corollary of the simplicial product construction is that the product $f_1 \times f_2\colon X_1 \times_{\mathbf{S}} X_2 \to X_1' \times_{\mathbf{S}} X_2'$ of two monotone simplicial maps, $f_1\colon X_1 \to X_1'$ and $f_2\colon X_2 \to X_2'$, is also simplicial. It is also plain that if X_1 and X_2 are subspaces of the ordered simplicial spaces X_1' and X_2', then $X_1 \times_{\mathbf{S}} X_2$ is a subspace of $X_1' \times_{\mathbf{S}} X_2'$.

Let us conclude with a description of the simplicial product in terms of schemes. Suppose $\operatorname{sch} X_1 = (M_1, S_1)$ and $\operatorname{sch} X_2 = (M_2, S_2)$. Then $\operatorname{sch}(X_1 \times_{\mathbf{S}} X_2) = (M_1 \times M_2, S)$, where S is the collection of sets $A \subset X_1 \times X_2$ such that:

(i) $\mathrm{proj}_1(A) \in S_1$, $\mathrm{proj}_2(A) \in S_2$;

(ii) if $(a_1, a_2) \in A$, $(a_1', a_2') \in A$, and $a_1 \preceq a_1'$, then $a_2 \preceq a_2'$.

Limits

Remark 2.2.5.11. Let $X_0, X_1, X_2, \dots$ be simplicial spaces, together with simplicial embeddings $\varphi_0\colon X_0 \to X_1, \varphi_1\colon X_1 \to X_2, \dots$. Consider the cover of $\lim(X_k, \varphi_k)$ by the sets $\mathrm{Imm}_k(s_k)$, where s_k is a simplex of X_k $(k = 0, 1, \dots)$. If we take for a characteristic homeomorphism of $\mathrm{Imm}_k(s_k)$ the composition of a characteristic homeomorphism $T^{\dim s_k} \to s_k$ with the homeomorphism $\operatorname{abr}\mathrm{Imm}_k\colon s_k \to \mathrm{Imm}_k(s_k)$, then $\mathrm{Imm}_k(s_k)$ becomes a topological simplex. It is clear that in this way the cover $\{\mathrm{Imm}_k(s_k)\}$ becomes a triangulation of $\lim(X_k, \varphi_k)$; hence $\lim(X_k, \varphi_k)$ is a simplicial space. As a cellular space, this space coincides with the limit defined in Remark 2.1.5.7. Moreover, Imm_k are simplicial embeddings.

If X_k are ordered spaces and φ_k are monotone, then the space $\lim(X_k, \varphi_k)$ is ordered, and the embeddings Imm_k are monotone.

Joins, Cones, and Suspensions

Remark 2.2.5.12. Let X_1 and X_2 be topological simplices. Then the join $X_1 \star X_2$ naturally becomes a topological simplex if we define the characteristic homeomorphisms as

$$T^{\dim X_1 + \dim X_2 + 1} = T_1 \star T_2 \xrightarrow{\varphi_1 \star \varphi_2} X_1 \star X_2,$$

where T_1 and T_2 are opposite faces of the simplex $T^{\dim X_1 + \dim X_2 + 1}$ and φ_1 and φ_2 are simplicial homeomorphisms (the equality $T^{\dim X_1 + \dim X_2 + 1} = T_1 \star T_2$ stands for the simplicial homeomorphism established in Remark 2.2.1.1). Now one may canonically triangulate the cellular join $X_1 \star_{\mathbf{C}} X_2$ of two arbitrary simplicial spaces X_1 and X_2: its simplices are the images of the simplices of X_1 and X_2 under the inclusions $X_1 \to X_1 \star_{\mathbf{C}} X_2$ and $X_2 \to X_1 \star_{\mathbf{C}} X_2$, and also the images of the simplices $s_1 \star s_2$ under the inclusions $\operatorname{incl} \star \operatorname{incl}\colon s_1 \star s_2 \to X_1 \star X_2$, where s_1 is a simplex of X_1 and s_2 a simplex of X_2. The resulting simplicial space is called the *simplicial join* of X_1 and X_2, and is denoted by $X_1 \star_{\mathbf{S}} X_2$. As a cellular space, $X_1 \star_{\mathbf{S}} X_2$ is identical to $X_1 \star_{\mathbf{C}} X_2$.

If $\operatorname{sch} X_1 = (M_1, S_1)$ and $\operatorname{sch} X_2 = (M_2, S_2)$, then clearly $X_1 \star_{\mathbf{S}} X_2 = (M_1 \amalg M_2, S)$, where S is the collection of non-empty subsets A of $M_1 \amalg M_2$ such that $\operatorname{incl}_1^{-1}(A) \in S_1$ or $\operatorname{incl}_1^{-1}(A) = \varnothing$, while $\operatorname{incl}_2^{-1}(A) \in S_2$ or $\operatorname{incl}_2^{-1}(A) = \varnothing$. Here $\operatorname{incl}_1\colon M_1 \to M_1 \amalg M_2$ and $\operatorname{incl}_2\colon M_2 \to M_1 \amalg M_2$ are the canonical mappings.

In particular, since

$$\operatorname{cone} X = X \star \mathbb{D}^0 \quad \text{and} \quad \operatorname{sus} X = X \star \mathbb{S}^0$$

for any topological space X (see Theorem 1.2.6.8), we see that the simplicial join construction transforms the cone and the suspension over an arbitrary simplicial space into simplicial spaces.

Simplicial Mapping Cylinders

Remark 2.2.5.13. To a given monotone simplicial map $f\colon X_1 \to X_2$, this construction associates a simplicial space $\operatorname{Scyl} f$, called the *simplicial mapping cylinder* of f. Generally speaking, $\operatorname{Scyl} f$ is not homeomorphic to the usual mapping cylinder $\operatorname{Cyl} f$ of f (see Chapter 4 Exercise 4.6.6.12), but has similar properties.

The most suitable language for describing the space $\operatorname{Scyl} f$ is that of schemes. Thus, let $\operatorname{sch} X_1 = (M_1, S_1)$, $\operatorname{sch} X_2 = (M_2, S_2)$, and $\operatorname{sch} f = (\varphi, \Phi)$. We define $\operatorname{Scyl} f$ by the formula $\operatorname{sch} \operatorname{Scyl} f = (M_1 \amalg M_2, S)$, where S is the collection of (finite) subsets $A \subset M_1 \amalg M_2$ such that:

(i) $\operatorname{incl}_1^{-1}(A) \in S_1$ or $\operatorname{incl}_1^{-1}(A) = \varnothing$;

(ii) $\varphi(\operatorname{incl}_1^{1}(A)) \cup \operatorname{incl}_2^{-1}(A) \in S_2$;

(iii) if $a_1 \in \operatorname{incl}_1^{-1}(A) \neq \varnothing$, then $\operatorname{incl}_2^{-1}(A) \in \varphi(S_1)$;

(iv) if $a_1 \in \operatorname{incl}_1^{-1}(A)$ and $a_2 \in \operatorname{incl}_2^{-1}(A)$, then $a_2 \preceq \varphi(a_1)$.

The maps incl_1 and incl_2 define two maps $\mathrm{sch}\, X_1 \to \mathrm{sch}\,\mathrm{Scyl}\, f$ and $\mathrm{sch}\, X_2 \to \mathrm{sch}\,\mathrm{Scyl}\, f$, and hence two simplicial embeddings, $X_1 \to \mathrm{Scyl}\, f$ and $X_2 \to \mathrm{Scyl}\, f$. The images of these embeddings are called the (lower and upper) *bases* of the cylinder $\mathrm{Scyl}\, f$ and can be identified with X_1 and X_2. Moreover, the map $M_1 \times (0 \cup 1) \to M_1 \amalg M_2$ defined by $(a, 0) \mapsto \mathrm{incl}_1(a)$, $(a, 1) \mapsto \mathrm{incl}_2\, \varphi(a)$, induces a certain map $\mathrm{sch}(X_1 \times_{\mathbf{S}} I) \to \mathrm{sch}\,\mathrm{Scyl}\, f$, and hence a simplicial map $X_1 \times_{\mathbf{S}} I \to \mathrm{Scyl}\, f$. Clearly, together with the inclusion $X_2 \to \mathrm{Scyl}\, f$, this simplicial map yields a continuous map $(X_1 \times_{\mathbf{S}} I) \amalg X_2 \to \mathrm{Scyl}\, f$, which in turn induces a continuous map $\mathrm{csc}\, f\colon \mathrm{Cyl}\, f \to \mathrm{Scyl}\, f$. Moreover, we see that $\mathrm{csc}\, f(\mathrm{Cyl}\, f) = \mathrm{Scyl}\, f$, and that the canonical retraction $\mathrm{rt}\, f\colon \mathrm{Cyl}\, f \to X_2$ (see Definition 1.2.6.10) is constant on the elements of the partition $\mathrm{zer}(\mathrm{csc}\, f)$. Also, the canonical X_2-homotopy from $\mathrm{id}(\mathrm{Cyl}\, f)$ to the composite map

$$\mathrm{Cyl}\, f \xrightarrow{\mathrm{rt}\, f} X_2 \xrightarrow{\mathrm{incl}} \mathrm{Cyl}\, f$$

is constant on the elements of the partition $\mathrm{zer}(\mathrm{csc}\, f) \times \mathrm{zer}(\mathrm{id}\, I)$. Consequently, $\mathrm{rt}\, f$ defines a strong deformation retraction $\mathrm{Scyl}\, f \to X_2$ and the composition of the inclusion $X_1 \to \mathrm{Scyl}\, f$ with this retraction obviously equals f. We conclude that the inclusion $X_2 \to \mathrm{Scyl}\, f$ is always a homotopy equivalence, whereas the inclusion $X_1 \to \mathrm{Scyl}\, f$ is a homotopy equivalence if and only if f is a homotopy equivalence.

2.2.6 Stars. Links. Regular Neighbourhoods

Remark 2.2.6.1. The *star* of a simplex s in a simplicial space X is the union of all simplices of X which contain s. Notation: $\mathrm{Star}\, s$ or $\mathit{Star}(s, X)$. Clearly, $\mathrm{Star}\, s$ is a subspace of X.

The *open star* of the simplex s is the union of the interiors of all simplices containing s. Notation: $\mathrm{star}\, s$ or $\mathrm{star}(s, X)$. It is readily seen that $\mathrm{star}\, s$ is the open set defined by the inequalities

$$\mathrm{bary}_{a_0}(x) > 0, \dots, \mathrm{bary}_{a_q}(x) > 0,$$

where $a_0 \dots, a_q$ are the vertices of s. Moreover, $\mathrm{Cl}\,\mathrm{star}\, s = \mathrm{Star}\, s$.

The *link* of the simplex s is the union of all simplices in $\mathrm{Star}\, s$ which do not intersect s. Notation: $\mathrm{link}\, s$ or $\mathrm{link}(s, X)$. Clearly, $\mathrm{link}\, s$ is a subspace of the spaces X and $\mathrm{Star}\, s$.

The following are obvious facts.

- If s' is a face of s, then $\mathrm{Star}\, s' \subset \mathrm{Star}\, s$, $\mathrm{star}\, s' \subset \mathrm{star}\, s$, $\mathrm{link}\, s' \subset \mathrm{link}\, s$.
- If $a_0, \dots, a_q$ are vertices which do not sit in the same simplex, then the intersection $\cap_{i=0}^{q} \mathrm{star}\, a_i$ is empty. However, if $a_0, \dots, a_q$ are vertices of a simplex s, then $\cap_{i=0}^{q} \mathrm{star}\, a_i = \mathrm{star}\, s$.
- If X is a subspace of X', then $\mathrm{star}(s, X) = \mathrm{star}(s, X') \cap X$ for any simplex s of X.

- Moreover, if X is complete, then $\operatorname{Star}(s, X) = \operatorname{Star}(s, X')\cap X$ and $\operatorname{link}(s, X) = \operatorname{link}(s, X') \cap X$.

- Finally, if s' is a simplex of $\operatorname{link}(s, X)$, then $\operatorname{link}(s', 1k(s, X)) = \operatorname{link}(s'', X)$, where s'' is the smallest simplex containing s and s'.

Definition 2.2.6.2. We can extend the definition of the star, open star, and link to points of a simplicial space X: for $x \in X$, the star $\operatorname{Star} x = \operatorname{Star}(x, X)$, the open star $\operatorname{star} x = \operatorname{star}(x, X)$, and the link $\operatorname{link} x = \operatorname{link}(x, X)$, are defined as

$$\operatorname{Star} x = \operatorname{Star} \operatorname{simplex} x, \quad \operatorname{star} x = \operatorname{star} \operatorname{simplex} x, \quad \text{and} \quad \operatorname{link} x = \operatorname{Star} \operatorname{simplex} x \backslash \operatorname{star} \operatorname{simplex} x.$$

Obviously, $\operatorname{star} x$ is a neighbourhood of x, and $\operatorname{link} x = \operatorname{Fr} \operatorname{Star} x = \operatorname{Fr} \operatorname{star} x$. In addition, the star $\operatorname{Star} x$ is homeomorphic to the cone over *linkx*. In fact, the formula

$$\operatorname{proj}(y, t) \mapsto \varphi((1 - t)\varphi^{-1}(x) + t\varphi^{-1}(y)),$$

where $y \in \operatorname{link} x$ and $t \in I$, defines a canonical homeomorphism $\operatorname{cone} \operatorname{link} x \to \operatorname{Star} x$; here φ is any characteristic homeomorphism of any simplex containing x and y.

Warning: the equality $\operatorname{link} x = \operatorname{link} \operatorname{simplex} x$ holds only when x is a vertex.

If s' is a simplex of $\operatorname{link}(s, X)$, then by Remarks 2.2.1.1 and 2.2.5.12, the join $s \star s'$ is canonically simplicial homeomorphic to the smallest simplex of X containing both s and s'. It is clear that all these simplicial homeomorphisms together yield a simplicial homeomorphism $s \star \operatorname{link}(s, X) \to \operatorname{Star}(s, X)$. If follows that the star $\operatorname{Star}(x, X)$ of any point $x \in X$ is canonically simplicial homeomorphic to $\operatorname{simplex} x \star \operatorname{link}(\operatorname{simplex} x, X)$, and we readily see that this simplicial homeomorphism maps the join of the boundary of six with the link $\operatorname{link}(\operatorname{simplex} x, X)$ onto $\operatorname{link}(x, X)$. Moreover, since $\operatorname{Fr} \operatorname{simplex} x$ is homeomorphic to $S^{\dim \operatorname{simplex} x - 1}$ and the join $S^{\dim \operatorname{simplex} x - 1} \star \operatorname{link}(\operatorname{simplex} x, X)$ is homeomorphic to the iterated suspension $\operatorname{sus}^{\dim \operatorname{simplex} x} \operatorname{link}(\operatorname{simplex} x, X)$, we conclude that $\operatorname{link}(x, X)$ is homeomorphic to $\operatorname{sus}^{\dim \operatorname{simplex} x} \operatorname{link}(\operatorname{simplex} x, X)$.

The link of a point is a homotopy invariant

Lemma 2.2.6.3. *Let A and B be retracts of a topological space Y. If the inclusions $i\colon A \to Y$ and $j\colon B \to Y$ are homotopic to some maps $f\colon A \to Y$ and $g\colon B \to Y$ such that $f(A) \subset B$ and $g(B) \subset A$, then A and B are homotopy equivalent.*

Proof. Consider two arbitrary retractions, $\sigma\colon Y \to B$ and $\rho\colon Y \to A$. Then the restrictions $\sigma|_A$ and $\rho|_B$ are homotopy equivalences $A \to B$ and $B \to A$, and inverses of one another. Indeed, $\sigma|_A = \sigma \circ i$, $\rho|_B = \rho \circ j$, and the composition $\rho \circ j \circ \sigma \circ i$ is homotopic to $\rho \circ j \circ \sigma \circ f = \rho \circ f$, which in turn is homotopic to $\rho \circ i = \operatorname{id}_A$. Therefore, $\rho|_B \circ \sigma|_A$ is homotopic to id_A, and a similar argument proves that $\sigma|_A \circ \rho|_B$ is homotopic to id_B. □

Theorem 2.2.6.4. *Let T_1 and T_2 be subspaces of the topological space X, and assume that both T_1 and T_2 and are endowed with finite triangulations. If $x_0 \in X$ is an interior point for both T_1 and T_2, then the links* $\operatorname{link}(x_{0,1}\,)$ *and* $\operatorname{link}(x_0, T_2)$ *have the same homotopy type.*

Proof. Let F_i denote a homotopy $\operatorname{Star}(x_0, T_i) \times I \to X$ from the inclusion $\operatorname{Star}(x_0, T_i) \to X$ to the constant map $\operatorname{Star}(x_0, T_i) \to x_0 \in X$, such that F_i is rectilinear on each simplex of $\operatorname{Star}(x_0, T_i)$ $(i = 1, 2)$. Set $C_i(t) = F_i(\operatorname{Star}(x_0, T_i) \times t)$ $(i = 1, 2)$. Since $\operatorname{star}(X_0, T_i)$ is open and $\operatorname{Star}(x_0, T_i)$ is compact $(i = 1, 2)$, there exists $\varepsilon > 0$ and $\delta > 0$ such that $C_1(\varepsilon) \subset \operatorname{Star}(x_0, T_2)$, $C_2(\varepsilon) \subset \operatorname{Star}(x_0, T_1)$, $C_1(\delta) \subset C_2(\varepsilon)$, and $C_2(\delta) \subset C_1(\varepsilon)$. Moreover, since $C_i(\varepsilon) \setminus x_0$ is a retract of $\operatorname{Star}(x_0, T_i) \setminus x_0$ $(i = 1, 2)$, $C_1(\varepsilon) \setminus x_0$ and $C_2(\varepsilon) \setminus x_0$ are retracts of $Y = [\operatorname{Star}(x_0, T_1) \cup \operatorname{Star}(x_0, T_2)] \setminus x_0$. Finally, the formulae

$$(y, t) \mapsto F_1(y, \delta t/\varepsilon) \quad \text{and} \quad (y, t) \mapsto F_2(y, \delta t/\varepsilon)$$

define homotopies $(C_1(\varepsilon) \setminus x_0) \times I \to Y$ and $(C_2(\varepsilon) \setminus x_0) \times I \to Y$ from the inclusions $C_1(\varepsilon) \setminus x_0 \to Y$ and $C_2(\varepsilon) \setminus x_0 \to Y$ to maps whose images lie in $C_2(\varepsilon) \setminus x_0$ and $C_1(\varepsilon) \setminus x_0$ respectively. Consequently, $C_2(\varepsilon) \setminus x_0$ and $C_1(\varepsilon) \setminus x_0$ have the same homotopy type (see Lemma 2.2.6.3), and it remains to note that $C_i(\varepsilon) \setminus x_0$ has the same homotopy type as $\operatorname{link}(x_0, T_i)$ $(i = 1, 2)$. □

Regular Neighbourhoods

Definition 2.2.6.5. The *regular neighbourhood* of the subspace A of a simplicial space X is the union of the open stars $\operatorname{star}(a, X)$ with $a \in A$ or, equivalently, the union of the open stars $\operatorname{star}(a, X)$ with $a \in \operatorname{skel}_0 A$.

Theorem 2.2.6.6. *If the subspace A is complete, then A is a deformation retract of its regular neighbourhood U.*

Proof. In fact, there is even a canonical A-homotopy $h\colon U \times I \to U$ from id_U to the composition of a retraction $U \to A$ with the inclusion $A \to U$. This homotopy is given by

$$\operatorname{bary}_a(h(x, t)) = \begin{cases} \frac{\{1 - t \sum_{b \in \operatorname{skel}_0 A} \operatorname{bary}_b(x)\} \operatorname{bary}_a(x)}{1 - \sum_{b \in \operatorname{skel}_0 A} \operatorname{bary}_b(x)}, & \text{if} \quad a \in \operatorname{skel}_0 A, \\ t \operatorname{bary}_a(x), & \text{if} \quad a \in \operatorname{skel}_0 X \setminus \operatorname{skel}_0 A. \end{cases}$$

□

In particular, this shows that every subspace of a simplicial space X is a deformation retract of its regular neighbourhood in $\operatorname{bary} X$ (see Remark 2.2.5.5).

Barycentric Stars and Barycentric Links

Definition 2.2.6.7. The *barycentric star* of the simplex s of a simplicial space X is the union of all simplices of $\operatorname{bary} X$ which have as their first vertex the centre of s. Notation: barstr or $\operatorname{barstr}(s, X)$. An equivalent description: $\operatorname{barstr} s$ is the set of all points $x \in X$ such that

$$\operatorname{bary}_a(x) \begin{cases} = \operatorname{bary}_b(x), & \text{if } \quad a, b \in s \cap \operatorname{skel}_0 X, \\ \geq \operatorname{bary}_b(x), & \text{if } \quad a \in \operatorname{skel}_0 X, \quad b \in (X \setminus s) \cap \operatorname{skel}_0 X. \end{cases}$$

It is clear that the barycentric stars of the simplices of X cover X and are subspaces of $\operatorname{bary} X$. Moreover, $\operatorname{barstr} s \neq \operatorname{barstr} s'$ whenever $s \neq s'$, and $\operatorname{barstr} s \subset \operatorname{barstr} s'$ whenever $s \subset s'$.

Definition 2.2.6.8. The union of those simplices of the barycentric star $\operatorname{barstr} s$ which do not contain the centre of s is the *barycentric link* of the simplex s, and is denoted by $\operatorname{barlk} s$. The $\star \operatorname{barstr} s$ is clearly simplicial homeomorphic to the cone over $\operatorname{barlk} s$. Moreover, the rectilinear projection from the centre of s induces a homeomorphism of $\operatorname{barlk} s$ onto the link $\operatorname{link} s$ of the simplex s in X (and the barycentric subdivision of $\operatorname{link} s$ transforms this homeomorphism into s simplicial one). Therefore, the pairs $(\operatorname{barstr} s, \operatorname{barlk} s)$ and $(\operatorname{cone} \operatorname{link} s, \operatorname{link} s)$ are homeomorphic.

2.2.7 Simplicial Approximation of Continuous Maps

Definition 2.2.7.1. Let $f \colon X \to Y$ be a continuous map, where X and Y are simplicial spaces. A simplicial map $g \colon X \to Y$ is a *simplicial approximation of* f if $g(x) \in \operatorname{simplex} f(x)$ for any point $x \in X$.

Theorem 2.2.7.2. *Every simplicial approximation g of the map $f \colon X \to Y$ is canonically homotopic to f.*

Proof. The canonical homotopy $X \times I \to Y$ from f to g is an affine mapping from each generatrix $x \times I$ of the cylinder $X \times I$ onto the (possibly degenerate) rectilinear segment joining $f(x)$ and $g(x)$. It is clear that this homotopy is stationary on the set of the points $x \in X$ where $g(x) = f(x)$. ☐

Theorem 2.2.7.3. *A simplicial map $g \colon X \to Y$ is a simplicial approximation of the continuous map $f \colon X \to Y$ if and only if $f(\operatorname{star} a) \subset \operatorname{star} g(a)$ for every vertex a of X.*

Proof. Assume first that g is a simplicial approximation of f, and let $x \in \operatorname{star} a$. Recalling that $g(x) \in \operatorname{simplex} f(x)$, that g is simplicial, and that x lies in the interior of a simplex with vertex a, we conclude that $g(x)$ lies in the interior of a simplex with vertex $g(a)$ (see 2.2.1.3). Thus, $g(a)$ is a vertex of $\operatorname{simplex} f(x)$, and hence $f(x) \subset \operatorname{star} g(a)$.

Now suppose that $f(\operatorname{star} a) \subset \operatorname{star} g(a)$ for every vertex a of X. Pick $x \in X$; if $a_0, \dots, a_q$ are the vertices of $\operatorname{simplex} f(x)$, then $x \in \cap_{i=0}^{q} \operatorname{star} a_i$, whence

$$f(x) \in f(\cap_{i=0}^{q} \operatorname{star} a_i) \subset \cap_{i=0}^{q} f(\operatorname{star} a_i) \subset \cap_{i=0}^{q} \operatorname{star} g(a_i).$$

Therefore, the points $g(a_0), \dots, g(a_q)$ are among the vertices of the simplex $\operatorname{simplex} f(x)$, and since $g(x)$ lies in the simplex with vertices $g(a_0), \dots, g(a_q)$, $g(x) \in \operatorname{simplex} f(x)$. ☐

Theorem 2.2.7.4. *A continuous map $f\colon X \to Y$ of simplicial spaces has a simplicial approximation if and only if for each vertex a of X there is a vertex b of Y such that $f(\operatorname{star} a) \subset \operatorname{star} b$.*

Proof. The necessity of this condition is an immediate consequence of Proposition 2.2.7.3. To prove its sufficiency, fix a map $\varphi\colon \operatorname{skel}_0 X \to \operatorname{skel}_0 Y$ such that $f(\operatorname{star} a) \subset \operatorname{star}(\varphi(a)$ for every vertex $a \in \operatorname{skel}_0 X$. If $a_0, \dots, a_q$ are the vertices of X, then $\cap_{i=0}^{q} \operatorname{star} a_i \neq \varnothing$, and the inclusions

$$\cap_{i=0}^{q} \operatorname{star} \varphi(a_i) \supset \cap_{i=0}^{q} f(\operatorname{star} a_i) \supset f(\cap_{i=0}^{q} \operatorname{star} a_i)$$

demonstrate that $\cap_{i=0}^{q} \operatorname{star} \varphi(a_i) \neq \varnothing$ too. This in turn implies that $\varphi(a_0), \dots, \varphi(a_q)$ are among the vertices of a simplex of Y (see Remark 2.2.6.1). Therefore, φ extends to a simplicial map $X \to Y$ (see Remark 2.2.2.6) and, applying Proposition 2.2.7.3, this extension is a simplicial approximation of f. $\square$

Theorem 2.2.7.5. *For each continuous map f of a finite simplicial space X into a simplicial space Y there is a positive integer m such that the map $f\colon \operatorname{bary}^m X \to Y$ admits a simplicial approximation.*

Proof. Without loss of generality, we may assume that X is a polyhedron (see Remark 2.2.4.1). Since the open stars of the vertices of Y constitute an open cover, there is $\varepsilon > 0$ such that, given any subset A of X with $\operatorname{diam} A < \varepsilon$, $f(A)$ is contained in one of these open stars (see Theorem 1.1.7.16). Let m be large enough so that the simplices of $\operatorname{bary}^m X$ have diameters less than $\varepsilon/2$ (see Corollary 2.2.5.7). Then given any vertex of $\operatorname{bary}^m X$, the diameter of its star is less than ε, and Theorem 2.2.7.4 shows that $f\colon \operatorname{bary}^m X \to Y$ has a simplicial approximation. $\square$

2.2.8 Exercise

Exercise 2.2.8.1. Let X be a simplicial space. Show that the formula

$$\operatorname{dist}(x, y) = [\sum_{a \in \operatorname{skel}_0 X} (\operatorname{bary}_a(y) - \operatorname{bary}_a(x))^2]^{1/2}$$

defines a metric on X, and verify that the resulting metric topology coincides with the initial topology if and only if X is locally finite.

Exercise 2.2.8.2. Show that for every polyhedron $X \subset \mathbb{R}^n$ there is a triangulation of $\mathbb{R}^n$ by Euclidean simplices, relative to which X becomes a simplicial subspace of $\mathbb{R}^n$.

Exercise 2.2.8.3. Show that every connected, locally finite, n-dimensional simplicial space can be simplicially embedded in $\mathbb{R}^{2n+1}$ triangulated by Euclidean simplices.

Exercise 2.2.8.4. Let $f\colon X \to Y$ be continuous, where X and Y are simplicial spaces. Produce a new triangulation of X with the following two properties:

(a) each of its simplices is contained in one of the simplices of the original triangulation;

(b) f has a simplicial approximation relative to the new triangulation.

2.3 HOMOTOPY PROPERTIES OF CELLULAR SPACES

2.3.1 Cellular Pairs

Definition 2.3.1.1. Suppose that X is a rigged cellular space and A is subspace of X. Let $h_0\colon A\cup \operatorname{skel}_0 X \to I$ denote the function equal to zero on A and equal to 1 on $(A\cup \operatorname{skel}_0 X)\setminus A$, and define inductively a sequence of functions $h_r\colon A\cup \operatorname{skel}_r X\to I$ $(r=1,2,\dots)$, such that

$$h_r(x)=\begin{cases} h_{r-1}(x), & \text{if} \quad x\in A\cup \operatorname{skel}_{r-1} X,\\ 1-\tau[1-h_{r-1}(\operatorname{att}_e(y)], & \text{if} \quad x=\operatorname{char}_e(\tau y)\end{cases}$$

where $e\in \operatorname{cell}_r X\setminus \operatorname{cell} A$, $\tau\in I$, and $y\in \mathbb{S}^{r-1}$. Since the functions h_r are continuous and each of them extends the preceding one, together they yield a continuous function $X\to I$. This function is called the *characteristic function of the pair* (X,A), and the neighbourhood of A consisting of all points of X where the characteristic function is less than 1 is called the *neat neighbourhood* of the subspace A.

Obviously, the characteristic function of the pair (X,A) vanishes on A, and only on A; hence, every subspace of a cellular space is distinguishable.

If X is a simplicial space, then we may construct a characteristic function starting with a simplicial rigging of X, and it is readily seen that such a function does not depend upon the choice of the rigging. In this case, the neat neighbourhood of a subspace A is simply the regular neighbourhood of A in $\operatorname{bary} X$.

Theorem 2.3.1.2. *Every subspace A of a rigged cellular space X is a strong deformation retract of its neat neighbourhood.*

Proof. Let U denote the neat neighbourhood of A in X. Since the products $(A\cup \operatorname{skel}_r X)\times I$ are subspaces of the cylinder $X\times I$ and cover it, they constitute a fundamental cover of $X\times I$. Therefore, their intersections with $U\times I$, i.e., the cylinders $U_r\times I$, where $= U_r = U\cap(A\cup \operatorname{skel}_r X)$, constitute a fundamental cover of $U\times I$. Let G_0 be the constant homotopy of the inclusion $A\to X$, and define homotopies $F_r\colon U_r\times I\to U$, $r\geq 1$, by the formula

$$F_r(x,t)=\begin{cases} x, & \text{if} \quad x\in U_{r-1},\\ \operatorname{char}_e(((1-t)\tau+t)y), & \text{if} \quad x=\operatorname{char}_e(\tau y),\end{cases}$$

where $e\in \operatorname{cell}_r X\setminus \operatorname{cell}_r A$, $i\in(0,1]$, and $y\in\mathbb{S}^{r-1}$. Now construct homotopies $G_r\colon U_r\times I$, $r\geq 1$, by

$$G_r(x,t)=\begin{cases} x, & \text{if} \quad 0\leq t\leq 2^{-r},\\ F_r(x,2^rt-1), & \text{if} \quad 2^{-r}\leq t\leq 2^{-r+1},\\ G_{r-1}(F_r(x,1),t), & \text{if} \quad 2^{-r+1}\leq t\leq 1.\end{cases}$$

Each homotopy G_r extends the preceding one, and together they yield an A-homotopy $U \times I \to U$ from id_U to a map which takes U into A. The compression of the last map to a map $U \to A$ is the desired strong deformation retraction. ☐

Theorem 2.3.1.3. *Every cellular pair is a Borsuk pair (i.e, has the homotopy extension property with respect to any space).*

Proof. This is a consequence of Proposition 2.3.1.2 combined with Proposition 1.3.5.11, because cellular spaces are normal, and their subspaces are distinguishable. ☐

Theorem 2.3.1.4. *If (X, A) is a cellular pair and the inclusion $A \subset X$ is a homotopy equivalence, then A is a strong deformation retract of X.*

Proof. In order to prove this, first apply Propositions 2.3.1.3 and 1.3.5.6 to the pair (X, A), then apply Proposition 2.3.1.3 to the pair $(X \times I, (X \times 0) \cup (A \times I) \cup (X \times 1))$ and, finally, apply Theorem 1.3.5.7 to the pair (X, A). ☐

Cellular Pairs and k-Connectedness

Theorem 2.3.1.5. *Let k be a non-negative integer or ∞. Suppose that (X, A) is a cellular pair such that all the cells in $X \setminus A$ have dimension at most k, and let (Y, B) be an arbitrary k-connected topological pair. Then every continuous map $f\colon X \to Y$ such that $f(A) \subset B$ is A-homotopic to a map which takes X into a subset of B. In particular, every continuous map of a k-dimensional cellular space into a k-connected topological space is homotopic to a constant map.*

Proof. We exhibit a sequence of A-homotopies $\{F_r\colon (A \cup \mathrm{skel}_r X) \times I \to Y\}_{r=-1}^{\infty}$, each extending the preceding one, and satisfying the conditions:

(i) $F_r(x, 0) = f(x)$ for all $x \in A \cup \mathrm{skel}_0 X$;

(ii) $F_r((A \cup \mathrm{skel}_r X) \times (1 - 2^{-r-1}) \subset B$;

(iii) $F_r(x, t)$ does not depend upon t for $t \geq 1 - 2^{-r-1}$.

Then the map $F\colon X \times I \to Y$ which equals F_r on $(A \cup \mathrm{skel}_r X) \times I$ will be a homotopy from f to a map which takes X into a subset of B.

We proceed by induction. Define F_{-1} as the constant homotopy of $f|_A$, and assume that homotopies $F_{-1}, \dots, F_{q-1}$, each extending its predecessor and satisfying (i)-(iii), are already constructed. If $q > k$, then $A \cup \mathrm{skel}_q X = X$ and we simply take $F_q = F_{q-1}$. So suppose now that $q \leq k$. Since the pair $((A \cup \mathrm{skel}_q X) \times I, (A \cup \mathrm{skel}_{q-1} X) \times I)$ is Borsuk (see Theorem 2.3.1.3), there is a homotopy G of the map $f|_(A \cup \mathrm{skel}_q X)$, such that $G|_{(A \cup \mathrm{skel}_{q-1} X) \times I} = F_{q-1}$. Using the fact that $F_{q-1}((A \cup \mathrm{skel}_{q-1} X) \times (1 - 2^{-q})) \subset B$, the formula $h_e(y) = G(\mathrm{char}_e(y), 1 - 2^{-q})$ defines a map $h_e\colon \mathbb{D}^q \to Y$ which takes $\mathbb{S}^{q-1}$ into B, for each cell $e \in \mathrm{cell}_q X \setminus \mathrm{cell}_q A$. Now take advantage of the k-connectedness of the pair (Y, B) to deduce that, given any cell $e \in \mathrm{cell}_q X \setminus \mathrm{cell}_q A$, there is an

$\mathbb{S}^{q-1}$-homotopy $H_e \colon \mathbb{D}^q \times I \to Y$ from h_e to a map whose image is a subset of B. We put

$$F_q(x,t) = \begin{cases} F_{q-1}(x,t), & \text{if } x \in A \cup \operatorname{skel}_{q-1} X, \\ G(x,t), & \text{if } 0 \leq t \leq 1 - 2^{-q}, \\ H_e(y, s^{q+1}(t - (1 - 2^{-q})), & \text{if } x = \operatorname{char}_e(y) \text{ and } 1 - 2^{-q} \leq t \leq 1 - 2^{-q-1}, \\ H_e(y,1) & \text{if } x = \operatorname{char}_e(y) \text{ and } 1 - 2^{-q-1} \leq t \leq 1. \end{cases}$$

Then it is immediate that the map F_q is continuous, extends F_{q-1}, and fulfils properties (i)-(iii) with $r = q$. ☐

Theorem 2.3.1.6. *Let k be a non-negative integer or ∞. If the cellular pair (X, A) is k-connected and every cell in $X \setminus A$ is of dimension at most k, then A is a strong deformation retract of X. In particular, every k-connected k-dimensional cellular space is contractible.*

Proof. Indeed, id_X is A-homotopic to a map $g \colon X \to X$ such that $g(X) \subset A$ (see Theorem 2.3.1.5), and hence $\operatorname{abr} g \colon X \to A$ is a strong deformation retraction. ☐

2.3.2 Cellular Approximation of Continuous Maps

Lemma 2.3.2.1. *Let $X = A \cup_\varphi [\amalg_{\nu \in M}(D_\mu = D^{k+1})]$, where A is a topological space and φ is a continuous map $\amalg_{\mu \in M}(S_\mu = \mathbb{S}^k) \to A$. Let $f \colon \mathbb{D}^{r+1} \to X$ be a continuous map such that $f(\mathbb{S}^r) \subset A' = \mathrm{Imm}_2(A)$. Then:*

(I) if $r < k$, f is $\mathbb{S}^r$-homotopic to a map g such that $g(\mathbb{D}^{r+1}) \subset A'$;

(II) if $r = k$, f is $\mathbb{S}^r$-homotopic to a map g such that there are affine maps $\alpha_1, \dots, a_s : \mathbb{D}^{k+1} \to D^{k+1}$ with four properties:

  (i) the images $d_i = \alpha_i(\mathbb{D}^{k+1})$ are pairwise disjoint balls lying in $\operatorname{int} \mathbb{D}^{k+1}$;

  (ii) each of the compositions $g \circ \alpha_i$ coincides with one of the composite maps

$$\mathbb{D}^{k+1}_\mu \xrightarrow{\mathrm{incl}_\mu} \amalg_{\nu \in M} \mathbb{D}_\nu \xrightarrow{\mathrm{Imm}_2} X; \qquad (2.3.2.2)$$

  (iii) $g(\mathbb{D}^{k+1} \setminus U^s_{i=1} \operatorname{int} d_i) \subset A'$;

  (iv) for $k \geq 1$, the point of the ball d_i having the largest value of the first coordinate is just $\alpha_i(\mathrm{ort}_1)$, and the segment joining this point with ort_1 is entirely contained in $\mathbb{D}^{k+1} \setminus \cup^s_{j=1}$ $(i = 1, \dots, s)$.

(Part (I) of this lemma, i.e., the case $r < k$, merely asserts that the pair (X, A') is k-connected, and this is the only information that we shall actually use in the present section; part (II) is needed in §5.3.)

Proof. Denote the composite map (2.3.2.2) by h , fix an arbitrary Euclidean $(k+1)$-simplex σ in $\mathbb{D}^{k+1}$, and then fix in $\operatorname{int}_\sigma$ an arbitrary $(k+1)$-simplex τ of the triangulation $\operatorname{bary}^2\sigma$. The sets $h_\mu(\operatorname{int}\sigma)$, $\mu \in M$, and $X \setminus \cup_{\mu\in M} h_\mu(\tau)$ constitute an open cover of X, and so there is a triangulation of $\mathbb{D}^{r+1}$ which is fine enough to ensure that the image of any of its simplices under f lies in one of the sets of this cover (see Corollary 2.2.5.7 and Theorem 1.1.7.6). Let K_μ (respectively, L) be the union of those simplices whose images are contained in $h_\mu(\operatorname{int}\sigma)$ (respectively, in $X \setminus \cup_\mu h_\mu(\tau)$). Obviously, the sets K_μ are pairwise disjoint, only a finite number of them are non-empty, K_μ and L are simplicial subspaces of the simplicial space $\mathbb{D}^{r+1}$, and $L \cup (\cup_\mu K_\mu) = \mathbb{D}^{r+1}$.

Now apply Proposition 2.2.7.5 to the composite maps

$$K_\mu \xrightarrow{\operatorname{abr} f} h_\mu(\sigma) \xrightarrow{(\operatorname{abr} h_\mu)^{-1}} \operatorname{bary}^2\sigma. \tag{2.3.2.3}$$

This theorem tells us that there is an m such that the maps (2.3.2.3) admit simplicial approximations when one replaces K_μ by $\operatorname{bary}^m K_\mu$. We let F_μ denote the canonical homotopy from (2.3.2.3) to the above simplicial approximation. Since τ is not a face of any other simplex, $F_\mu((L \cap K_\mu) \times I) \cap \operatorname{int}\tau = \varnothing$, and so together the homotopies F_μ define a homotopy $F\colon (L \cap (\cup_\mu K_\mu)) \times I \to X \setminus \cup_\mu h_\mu(\operatorname{int}\tau)$. By Propositions 2.3.1.3 and 1.3.5.9, F extends to a homotopy $G\colon L \times I \to X \setminus \cup_\mu X \setminus \cup_\mu h_\mu(\operatorname{int}\tau)$ of the map $\operatorname{abr} f\colon L \to X \ \setminus \cup_\mu h_\mu(\operatorname{int}\tau)$, stationary on $B = \mathbb{D}^{r+1} \setminus \cup_\mu f^{-1}(h_\mu(\operatorname{int}\tau))$. It is evident that the composite maps

$$K_\mu \times I \xrightarrow{F_\mu} \sigma \xrightarrow{h_\mu|_\sigma} X$$

and

$$L \times I \xrightarrow{G} X \setminus \cup_\mu h_\mu(\operatorname{int}\tau) \xrightarrow{\text{incl}} X$$

yield together a B-homotopy of f. This homotopy connects f to a map $f_1\colon \mathbb{D}^{r+1} \to X$ such that, for every μ, the composition

$$\operatorname{bary}^m K \xrightarrow{\operatorname{abr} f_1} h_\mu(\sigma) \xrightarrow{(\operatorname{abr} h_\mu)^{-1}} \operatorname{bary}^2\sigma$$

is simplicial and $h_\mu(\tau) \subset f_1(K_\mu) \subset h_\mu(\sigma)$. Since $B \supset \mathbb{S}^r$, f_1 is $\mathbb{S}^r$-homotopic to f, and to complete the proof of our lemma it suffices to examine (I) and (II) for f_1 rather than f.

Consider an arbitrary ball $\delta \subset \operatorname{int}\tau$, and let ψ denote the homeomorphism $\delta \to \mathbb{D}^{k+1}$, $\psi(x) = (x-a)/\rho$, where a and ρ are the centre and the radius of δ. Moreover, let $\Psi\colon \mathbb{D}^{k+1} \times I \to \mathbb{D}^{k+1}$ be defined as

$$\Psi(x,t) = \frac{x - t_0 a}{1 - t_0(1-\rho)}$$

where t_0 is the largest of the numbers $\theta \in [0,t]$ such that $(x-\theta a)/(1-\theta(1-\rho)) \in \mathbb{D}^{k+1}$. Now the family of mappings $p_t\colon\ \colon X \to X$, given by

$$p_t(x) = \begin{cases} x, & \text{if} \quad x \in A', \\ h_\mu \circ \psi(h_\mu^{-1}(x), t)) & \text{if} \quad x \in h_\mu(\operatorname{int}\mathbb{D}^{k+1}), \end{cases}$$

obviously yields an A'-homotopy of id_X such that $p_1(X \setminus \cup_\mu h_\mu(\delta)) \subset A'$ and $p_1 \circ h_\mu = h_\mu \circ \psi$.

If $r < k$, then $f_1(\mathbb{D}^{r+1}) \subset X \setminus \cup_\mu h_\mu(\delta)$, and to complete the proof in this case we only need to note that $(y,t) \mapsto p_t(f_1(y))$ is an $\mathbb{S}^r$-homotopy from f_1 to a map whose image is included in A'. If $r = k$, then $f_1^{-1}(\cup_\mu h_\mu(\delta))$ can be decomposed into pairwise disjoint ellipsoids $\delta_1, \dots, \delta_s$, each being affinely mapped onto one of the sets $h_\mu(\delta)$ by f_1.

Let $\{q_t\}$ be an $\mathbb{S}^r$-homotopy of $\mathrm{id}_{\mathbb{D}^{k+1}}$, with the following properties:

- the preimages $d_i = q_1^{-1}(\delta_i)$ are balls;
- the maps $\operatorname{abr} q_1 \colon d_i \to \delta_i$ are affine; and
- for $k \geq 1$ the point of the ball d_i having the largest value of the first coordinate is carried by $f_1 \circ q_1$ into one of the points $h_\mu(\psi^{-1} \operatorname{ort}_1))$, and
- the segment joining this point with ort_1 is entirely contained in $\mathbb{D}^{k+1} \setminus \cup_{j=1}^{s} \operatorname{int} d_j$.

Obviously, the formula $(y,t) \mapsto (f_1 \circ q_t(y), t)$ defines an $\mathbb{S}^r$-homotopy from f_1 to a map g satisfying conditions (IIi)-(IIiv). ☐

Theorem 2.3.2.4. *Every cellular pair (X,A) with $A \supset \operatorname{skel}_k X$ is k-connected ($0 \leq k \leq \infty$). In particular, every cellular space whose k-skeleton reduces to a point is k-connected.*

Proof. Since any continuous map of a ball into X takes the ball into a subset of one of the skeletons $\operatorname{skel}_r X$, it suffices to show that all the pairs $(A \cup \operatorname{skel}_r X, A)$ with r >k are k-connected. But this is an immediate consequence of the k-connectedness of the pairs $(A \cup \operatorname{skel}_{q+1} X, A \cup \operatorname{skel}_q X)$ with $q \geq k$, which in turn follows from Lemma 2.3.2.1 (see Theorem 2.1.2.1). ☐

Corollary 2.3.2.5. *$\mathbb{S}^r$ is $(n-1)$-connected ($n \geq 1$). $\mathbb{C}P^n$ is simply connected ($0 \leq n \leq \infty$). $\mathbb{C}aP^2$ is 7-connected.*

Theorem 2.3.2.6. *Every continuous map f from a cellular space X into cellular space Y is homotopic to a cellular map. If, in addition, f is cellular on a subspace A of X, then f is A-homotopic to a cellular map.*

Proof. Given $f \colon X \to Y$, continuous on X and cellular on A, we shall construct a sequence of maps $\{f_r \colon X \to Y\}_{r=-1}^{\infty}$ and a sequence of homotopies $\{F_r \colon X \times I \to Y)_{r=0}^{\infty}$, such that:

(i) $f_{-1} = f$;

(ii) f_r is cellular on $A \cup \operatorname{skel}_r X$;

(iii) F_r is a homotopy from f_{r-1} to f_r, stationary on $A \cup \operatorname{skel}_{r-1} X$. Then the formula

$$(x,t) \mapsto \begin{cases} F_r(x, 2 - 2^{r+1}(1-t)), & \text{if} \quad 1 - 2^{-r} \leq t \leq 1 - 2 \ -r - 1, \\ F_r(X, 1), & \text{if} \quad x \in A \cup \operatorname{skel}_r X \quad \text{and} \quad t = 1, \end{cases}$$

will define an A-homotopy from f to a cellular map.

We proceed by induction. If f_r and F_r are already constructed for $r < k$ and satisfy (i)-(i), then by Theorems 2.3.2.4 and 2.3.1.5 there is a $(\operatorname{skel}_k A \cup \operatorname{skel}_{k-1} X)$-homotopy from $f_{k-1}|_{\operatorname{skel}_k X}$ to a map whose image is contained in $\operatorname{skel}_k Y$. This homotopy together with the constant homotopy of $f_{k-1}|_{A \cup \operatorname{skel}_{k-1} X}$ yield some $(A \cup \operatorname{skel}_{k-1} X)$-homotopy of $f_{k-1}|_{A \cup \operatorname{skel}_k X}$. Applying Theorem 2.3.1.3, the last homotopy extends to some $(A \cup \operatorname{skel}_{k-1} X)$-homotopy of the map f_{k-1}, which we take as F_K. Finally, set $= F_k(x, 1)$, $x \in X$. ☐

Theorem 2.3.2.7. *Two homotopic cellular maps $f, g\colon X \to Y$ are cellular homotopic. If, in addition, f and g are A-homotopic and A is a subspace of X, then f and g are cellular A-homotopic.*

Proof. Surely, every A-homotopy from f to g is a continuous map of the cylinder $X \times I$ into Y, and is cellular on $(X \times (0 \cup 1)) \cup (A \times I)$. By Proposition 2.3.2.6, this map is $[(X \times (0 \cup 1)) \cup (A \times I)]$-homotopic to a cellular map. ☐

2.3.3 k-Connected Cellular Pairs

Theorem 2.3.3.1. *Every k-connected cellular pair (X, A) $(0 \leq k \leq \infty)$ is homotopy equivalent to a cellular pair (Y, B) such that $B \supset \operatorname{skel}_k Y$.*

Proof. If $k = \infty$, then, by Theorem 2.3.1.6, A is a strong deformation retract of X, and hence the pair (X, A) is homotopy equivalent to (X, X).

Turning now to the case $k < \infty$, may assume that $A \supset \operatorname{skel}_{k-1} X$; indeed, one reduces to this case by induction on k, because for $k \geq 1$ every k-connected pair is also $(k-1)$-connected, while the condition $A \supset \operatorname{skel}_{-1} X$ is trivially fulfilled. According to Theorems 2.3.1.5 and 2.3.2.7, there is a cellular $\operatorname{skel}_k A$-homotopy $f\colon \operatorname{skel}_k X \times I \to X$ from the inclusion $\operatorname{skel}_k X \to X$ to a map which takes $\operatorname{skel}_k X$ into A. Define $F\colon \operatorname{skel}_k X \times I \times I \to X$ by $F(x, t_1, t_2) = f(f, t_1)$, and set

$$C = (\operatorname{skel}_k X \times I \times 0) \cup (\operatorname{skel}_k X \times (0 \cup 1) \times I) \cup (\operatorname{skel}_k A \times I \times I),$$
$$D = \operatorname{skel}_k X \times I \times 1.$$

Obviously, C and D are subspaces of the cellular space $\operatorname{skel}_k X \times I \times I$ and the map F is cellular. Now define Y and B by

$$Y = X \cup_{F|_C} (\operatorname{skel}_k X \times I \times I), \quad B = \operatorname{Imm}_2(A) \cup \operatorname{Imm}_1(D).$$

By Remark 2.1.5.6, Y is a cellular space, and it is clear that B is a subspace of Y containing $\operatorname{skel}_k Y$. To verify that (X, A) and (Y, B) have the same homotopy type, note that $\operatorname{Imm}_2(A)$ is a strong deformation retract of B and $\operatorname{Imm}_2(X)$ is a strong deformation retract of Y. In fact, the formula

$$\begin{cases} (\operatorname{Imm}_1(x, t_1, 1), t) & \mapsto \operatorname{Imm}_1(x, tt_1, 1) \quad [x \in \operatorname{skel}_k X, t, t_1 \in I], \\ (\operatorname{Imm}_2(x), t) & \mapsto \operatorname{Imm}_2(x) \quad [x \in A, t \in I], \end{cases}$$

defines a homotopy $B\times I\to B$, stationary on $\mathrm{Imm}_2(A)$, from id_B to a retraction $B\to \mathrm{Imm}_2(A)$. Similarly, the formula

$$\begin{cases}(\mathrm{Imm}_1(x,t_1,t_2),t) & \mapsto \mathrm{Imm}_1(x,t_1,tt_2) \quad [x\in \operatorname{skel}_k X, t,t_1,t_2\in I],\\ (\mathrm{Imm}_2(x),t) & \mapsto \mathrm{Imm}_2(x) \quad [x\in X, t\in I],\end{cases}$$

defines a homotopy $Y\times I\to Y$, stationary on $\mathrm{Imm}_2(Y)$, from id_Y to a retraction $Y\to \mathrm{Imm}_2(X)$. Consequently, the pair $(Y,\mathrm{Imm}_2(A))$ is homotopy equivalent to both the pairs (Y,B) (see Theorem 1.3.5.8) and $(\mathrm{Imm}_2(X),\mathrm{Imm}_2(A))$, and it remains to observe that the pairs $(\mathrm{Imm}_2(X),\mathrm{Imm}_2(A))$ and (X,A) are homeomorphic. ☐

Theorem 2.3.3.2. *Every k-connected cellular space ($0\le k\le\infty$) is homotopy equivalent to a cellular space whose k-skeleton reduces to a point.*

Proof. Let X be a k-connected cellular space and choose a 0-cell x_0 in X. The pair (X,x_0) is homotopy equivalent to a cellular pair (X,A) such that $A\supset \operatorname{skel}_k X$ (see Theorem 2.3.3.1). Set $Y=X/A$. Since A is contractible, Y has the same homotopy type as X (see Theorems 1.3.7.7 and 2.3.1.3), and it is clear that $\operatorname{skel}_k Y$ is just a point. ☐

Remark 2.3.3.3. Theorem 2.3.3.2 says nothing about the dimension of the space Y which replaces the given space X. However, its proof demonstrates that one can always choose Y to satisfy $\dim Y\le \max(\dim X,k+2)$. Our next task is to prove that for $k=0$ the last equality may be sharpened to $\dim Y-\dim X$ (see Theorem 2.3.3.6).

Lemma 2.3.3.4. *Let Y be a topological space, and let $\{Y_k\}_{k=0}^{\infty}$ be a fundamental cover of Y such that $Y_k\cap Y_l=\varnothing$ whenever $k-l>1$. If $Y_{k-1}\cap Y_k$ is a strong deformation retract of Y_k for all $k\ge 1$, then Y_0 is a strong deformation retract of Y.*

Proof. If $F_k\colon Y_k\times I\to Y_k$ is a homotopy, stationary on $Y_{k-1}\cap Y_k$, from id_{Y_k} to a map which takes Y_k into $Y_{k-1}\cap Y_k$, then the formula

$$(y,t)\mapsto\begin{cases} y, & \text{if } \ y\in Y_k \ \text{ and}\\ & 0\le t\le 2^{-k},\\ F_l(F_{l+1}(\dots F_k(x,1)\dots,1),2^lt-1), & \text{if } \ y\in Y_k \ \text{ and}\\ & 2^{-l}\le t\le 2^{-l+1} \quad (l\le k)\end{cases}$$

defines an y_0-homotopy $Y\times I\to Y$ from id_Y to a map which takes Y into y_0. ☐

Theorem 2.3.3.5. *Given any connected cellular space X, there is a contractible one-dimensional subspace of X containing all the 0-cells.*

Proof. Fix an arbitrary 0-cell x_0 in X and let A_k be the set of all 0-cells that can be joined to x_0 by a path $I \to \operatorname{skel}_1 X$ which touches at most k 1-cells. Since $\operatorname{skel}_1 X$ is connected (see Theorem 2.1.4.8), and a path can touch only a finite number of cells, $\cup_{k=0}^{\infty} A_k = \operatorname{skel}_0 X$. Now given any 0-cell $x \in A_k \setminus A_{k-1}$ with $k \geq 1$, pick a closed 1 -cell $c(x)$ joining x to some cell in $A_{k-1} \setminus A_{k-2}$ and set

$$Y_k = \begin{cases} x_0, & \text{if} \quad k = 0, \\ \cup_{y \in A_k \setminus A_{k-1}} c(y), & \text{if} \quad k > 0, \end{cases}$$

and $Y = \cup_{k=0}^{\infty} Y_k$. Obviously, Y is a one-dimensional subspace of X containing $\operatorname{skel}_0 X$, and the cover $\{Y_k\}$ of Y satisfies the conditions of Lemma 2.3.3.4. Therefore, Y_0 is a strong deformation retract of Y, i.e, Y is contractible. ☐

Theorem 2.3.3.6. *Every connected n-dimensional cellular space is homotopy equivalent to a cellular space of dimension at most n, and having only one 0-cell. In particular, every connected one-dimensional cellular space is homotopy equivalent to a bouquet of circles.*

Proof. This results from Theorems 2.3.3.5, 2.3.1.3, and 1.3.7.7. ☐

Applications to Cellular Constructions

Theorem 2.3.3.7. *If the cellular space X is k-connected, then* $\operatorname{sus} X$ *and* $\operatorname{sus}(X, x_0)$*, where x_0 is a 0-cell, are $(k+1)$-connected.*

Proof. The proof reduces to three remarks.

First, since $\operatorname{sus} X$ and $\operatorname{sus}(X, x_0)$ have the same homotopy type (see Theorems 2.1.4.5, 1.3.6.8, and 1.3.7.7), the $(k+1)$-connectedness of one is equivalent to the $(k+1)$-connectedness of the other.

Secondly, according to Theorem 2.3.2.4 and Remark 1.3.7.13, it is enough to verify that $\operatorname{sus}(X, x_0)$ is $(k+1)$-connected when $\operatorname{skel}_k X = x_0$.

And thirdly, if $\operatorname{skel}_k X = x_0$, then the $(k+1)$-connectedness of $\operatorname{sus}(X, x_0)$ is a corollary of Theorem 2.3.2.4, because under this assumption $\operatorname{skel}_{k+1} \operatorname{sus}(X, x_0)$ also reduces to a point. ☐

Theorem 2.3.3.8. *Suppose X_i is a k_i-connected cellular space and x_i is a 0-cell of X_i, $i = 1, 2$. Then the tensor products $(X_1, x_1) \otimes (X_2, x_2)$ and $(X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2)$ are $(k_1 + k_2 + 1)$-connected.*

Proof. Again, the proof reduces to three remarks.

First, since $(X_1, x_1) \otimes (X_2, x_2)$ induces on its compact subsets topologies which are identical to those induced by the topology of $(X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2)$, the $(k_1 + k_2 + 1)$-connectedness of one of these spaces implies the $(k_1 + k_2 + 1)$-connectedness of the other.

Secondly, using Theorem 2.3.3.1 and Remark 1.3.7.13, it is enough to verify that $(X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2)$ is $(k_1 + k_2 + 1)$-connected when $\operatorname{skel}_{k_1} X = x_1$ and $\operatorname{skel}_{k_2} X = x_2$.

Thirdly, under these circumstances, the (k_1+k_2+1)-connectedness of $(X_1, x_1)\otimes_{\mathbf{C}} (X_2, x_2)$ follows from Proposition 2.3.3.2, because $\operatorname{skel}_{k_1+k_2+1}((X_1, x_1)\otimes_{\mathbf{C}}(X_2, x_2))$ also reduces to a point. □

Lemma 2.3.3.9. *For any cellular spaces X_1 and X_2 with 0-cells x_1 and x_2 taken as base points, the cellular join $(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)$ is homotopy equivalent to $\operatorname{sus}((X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2), \mathrm{bp})$.*

Proof. By definition, the spaces

$$(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2), \mathrm{bp}), \quad \operatorname{sus}((X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2), \mathrm{bp})$$

are obtained from projection $(X_1 \times_{\mathbf{C}} X_2) \times I$ by taking quotients two times, and the projection

$$(X_1 \times_{\mathbf{C}} X_2) \times I \to \operatorname{sus}((X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2), \mathrm{bp})$$

is constant on the elements of the partition

$$\operatorname{zer}(\operatorname{proj}\colon (X_1 \times_{\mathbf{C}} X_2) \times I \to (X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2)).$$

The resulting map

$$\begin{gathered} f = \operatorname{fact}[\operatorname{proj}\colon (X_1 \times_{\mathbf{C}} X_2) \times I \to (X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)]\colon \\ \operatorname{sus}((X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2), \mathrm{bp}) \to (X_1, x_1) \star_{\mathbf{C}} (X_2, x_2) \end{gathered}$$

is factorial (see Remark 1.2.3.4). Since the only element of the partition $\operatorname{zer}(f)$ which does not reduce to a point is $f^{-1}(\mathrm{bp})$, we see that

$$\operatorname{sus}((X_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2), \mathrm{bp}) = [(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2]/f^{-1}(\mathrm{bp}).$$

Finally, note that

$$f^{-1}(\mathrm{bp})) = [(X_1, x_1) \otimes_{\mathbf{C}} (x_2, x_2)] \cup [(x_1, x_1) \otimes_{\mathbf{C}} (X_2, x_2)],$$

and since this union is contractible, the quotient space $[(X_1, x_1)\star_{\mathbf{C}}(X_2, x_2)]/f^{-1}(\mathrm{bp})$ is homotopy equivalent to $(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)$. □

Theorem 2.3.3.10. *Let the cellular spaces X_1 and X_2 be k_1- and respectively k_2-connected. Then the joins $X_1 \star X_2$, $X_1 \star_{\mathbb{C}} X_2$, $(X_1, x_1) \star (X_2, x_2)$, and $(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)$, where x_1 and x_2 are 0-cells, are $(k_1 + k_2 + 2)$-connected.*

Proof. The proof reduces to four remarks.

First, since $(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)$ is a quotient of $X_1 \star_{\mathbf{C}} X_2$ by a contractible space (the closed 1 -cell $x_1 \star x_2$), $(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)$ and $X_1 \star_{\mathbf{C}} X_2$ have the same homotopy type.

Secondly, $X_1 \star_{\mathbf{C}} X_2$ induces on its compact subsets the same topologies as does $X_1 \star X_2$, and hence the $(k_1 + k_2 + 2)$-connectedness of one of these spaces implies the $(k_1 + k_2 + 2)$-connectedness of the other.

Thirdly, and from the same reason, $(X_1, x_1) \star (X_2, x_2)$ is $(k_1 + k_2 + 2)$-connected if and only if $(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)$ is $(k_1 + k_2 + 2)$-connected.

Fourthly, the $((k_1 + k_2 + 2)$-connectedness of $(X_1, x_1) \star_{\mathbf{C}} (X_2, x_2)$ is an immediate consequence of Lemma 2.3.3.9, Theorems 2.3.3.8, and 2.3.3.7. □

2.3.4 Simplicial Approximation of Cellular Spaces

Lemma 2.3.4.1. *Suppose that X and Y are cellular spaces and $\{X_r\}_{r=0}^{\infty}$ and $\{Y_r\}_{r=0}^{\infty}$ are filtrations of X and Y by subspaces. Let $f\colon X \to Y$ be a cellular map such that $f(X_r) \subset Y_r$ $(0 \leq r \leq \infty)$. If all the maps* $\operatorname{abr} f\colon Xr \to Y_r$ *are homotopy equivalence, then so is f.*

Proof. Since the $Z_r = \operatorname{Cyl}(\operatorname{abr} f\colon X_r \to Y_r)$ are cellular subspaces of $Z = \operatorname{Cyl} f$ and satisfy the conditions $Z_r \subset Z_{r+1}$ and $cup_{r=0}^{\infty} Z_r = Z$, they yield a filtration of Z (see Definition 2.1.1.9). Thus, the image of any continuous map $\mathbb{D}^k \to Z$ is contained in one of the sets Z_r (see Theorem 1.2.4.5), and so the pair (Z, X) is ∞-connected provided that all the pairs (Z_r, X_r) are ∞-connected. Now note that (Z_r, X_r) is ∞-connected if and only if $\operatorname{abr} f\colon X_r \to Y_r$ is a homotopy equivalence; similarly, (Z, X) is ∞-connected if and only if f is a homotopy equivalence (see Theorems 2.3.1.3, 2.3.1.6, Remarks 1.3.3.9, and 1.3.7.14). ☐

Theorem 2.3.4.2. *Given any cellular space X, there is a simplicial space which has the same homotopy type and the same dimension as X, and is finite or countable together with X.*

Proof. The proof consists of producing three sequences:

(1) one of simplicial spaces $\{Y_r\}_{r=0}^{\infty}$,

(2) one of simplicial embeddings $\{i_r\colon Y_r \to Y_{r+1}\}_{r=0}^{\infty}$,

(3) and one of cellular homotopy equivalences $\{f_r\colon \operatorname{skel}_r \to Y_{r+1}\}_{r=0}^{\infty}$,

with the following four properties:

(i) $f_r|_{\operatorname{skel}_{r-1} X} = i_{r-1} \circ f_{r-1}$;

(ii) $\dim Y_r = \dim \operatorname{skel}_r X$;

(iii) if $\operatorname{skel}_r X$ is finite (countable), then Y_r is finite (respectively, countable);

(iv) if $\operatorname{skel}_r X = \operatorname{skel}_{r-1} X$, then $Y_r = Y_{r-1}$ and $i_r = \mathrm{id}_{Y_{r-1}}$.

This will enable us to define the simplicial space $\lim(Y_r, i_r)$ having dimension $\dim X$, and finite or countable together with X, as well as a cellular map $f\colon X \to \lim(Y_r, i_r)$ such that $f|_{\operatorname{skel}_r X} = \operatorname{Imm}_r \circ f$. Finally, we shall use Lemma 2.3.4.1 to show that f is a homotopy equivalence.

Define Y_0 and f_0 as $\operatorname{skel}_0 X$ and $\mathrm{id}_{\operatorname{skel}_0 X}$, and assume that simplicial spaces Y_r, cellular homotopy equivalences f_r, and simplicial embeddings i_{r-1}, satisfying (i)-(iv), are already constructed for $r < q$. By Theorem 2.1.2.1, we may represent $\operatorname{skel}_q X$ as $\operatorname{skel}_{q-1} \cup_{\varphi} \triangle$, where $\triangle = \amalg_{e \in \operatorname{cell}_q X}(\mathbb{D}_e = \mathbb{D}^q)$ and φ is a continuous map of $\Sigma = \amalg_{e \in \operatorname{cell}_q X}(\mathbb{S}_e = \mathbb{S}^q)$ into $\operatorname{skel}_{q-1} X$.

Next triangulate $\triangle$ so that Σ becomes a complete subspace and the map $f_{q-1} \circ \varphi\colon \Sigma \to Y_{q-1}$ admits a simplicial approximation $g\Sigma \to Y_{q-1}$. Further, order Y_{q-1} and Σ in such a manner that the map g becomes monotone. Applying successively Propositions 1.3.7.11, 1.3.7.8, and then again 1.3.7.11, we obtain three homotopy equivalences:

- a homotopy equivalence $\operatorname{skel}_q X \to Y_{q-1} \cup_{f_{q-1}\circ\varphi} \triangle$ which agrees with f_{q-1} on $\operatorname{skel}_{q-1} X$;
- a homotopy equivalence $Y_{q-1} \cup_{f_{q-1}\circ\varphi} \triangle \to Y_{q-1} \cup_g \triangle$ which is the identity on Y_{q-1};
- and a homotopy equivalence $Y_{q-1} \cup_g \triangle \to (\operatorname{Scyl} g) \cup_{\text{incl}} \triangle$, where $\text{incl} = [\text{incl}\colon \Sigma \to \operatorname{Scyl} g]$ (see Remark 2.2.5.13), which agrees with the inclusion $Y_{q-1} \to \operatorname{Scyl} g$ on Y_{q-1}.

At last, we may define Y_q as $(\operatorname{Scyl} g) \cup_{\text{incl}} \triangle$, f_q as the composition of the three homotopy equivalences above, and i_{q-1} as the composite embedding $Y_{q-1} \to \operatorname{Scyl} g \to Y_q$. The triangulations of $\triangle$ and of the cylinder $\operatorname{Scyl} g$ yield together a triangulation of Y_q (see Lemma 2.2.5.2). It is plain that i_{q-1} is a simplicial embedding and that Y_q, f_q, and i_{q-1} satisfy conditions (i)-(iv) for $r = q$. $\square$

Theorem 2.3.4.3. *Let X and Y be cellular spaces with X finite and Y countable. Then the set $\pi(X, Y)$ is countable.*

Proof. By Propositions 2.3.4.2 and 1.3.1.8, we need only consider the case when X and Y are simplicial spaces. Under this assumption, Theorem 2.2.7.5 shows that the cardinal of $\pi(X, Y)$ does not exceed the cardinal of the set of all simplicial mappings $\operatorname{bary}^m X \to Y$ $(m = 0, 1, \dots)$, and the latter is obviously countable. $\square$

2.3.5 Exercises

Exercise 2.3.5.1. Suppose that the cellular spaces X_i and X_i' are homotopy equivalent, $i = 1, 2$. Show that $X_1 \times_{\mathbf{C}} X_2$ and $X_1' \times_{\mathbf{C}} X_2'$ are homotopy equivalent, and that the same is true for the spaces $X_1 \star_{\mathbf{C}} X_2$ and $X_1' \star_{\mathbf{C}} X_2'$.

Exercise 2.3.5.2. Show that every cellular space is homotopy equivalent to a locally finite cellular space.

Exercise 2.3.5.3. Show that every cellular pair is homotopy equivalent to a simplicial pair, and that every finite cellular pair is homotopy equivalent to a finite simplicial pair.

Exercise 2.3.5.4. Show that there is no cellular space having the same homotopy type as the subspace of the real line consisting of the points 0 and $1/n$, $n = 1, 2, \dots$.

Chapter 3

SMOOTH MANIFOLDS

3.1 FUNDAMENTAL CONCEPTS

3.1.1 Topological Manifolds

Remark 3.1.1.1. This chapter comprises an elementary introduction to differential topology. The basic objects of this theory are the smooth manifolds. They are defined in the next subsection and represent (as do cellular and simplicial spaces) topological spaces with an additional structure. The present subsection is devoted to topological manifolds, which occupy an intermediate position between smooth manifolds and topological spaces, and do not carry an additional structure.

Locally Euclidean Spaces

Remark 3.1.1.2. A topological space is said to be a an *n-dimensional locally Euclidean space* if each of its points has a neighbourhood homeomorphic to the space $\mathbb{R}^n$ or to the half space $\mathbb{R}^n_-$, where $\mathbb{R}^n_-$ is the set of all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ with $x_1 \leq 0$. The half space $\mathbb{R}^n_-$ is defined for $n \geq 1$; we do not define it for $n = 0$ and, accordingly, a 0-dimensional locally Euclidean space is simply a topological space such that each of its points has a neighbourhood homeomorphic to $\mathbb{R}^0$, i.e., a discrete space.

In a locally Euclidean space X, the points having a neighbourhood homeomorphic to $\mathbb{R}^n$ are called *interior points*, while the remaining ones are called *boundary points*. The interior (boundary) points form the *interior* (respectively, the *boundary*) of the locally Euclidean space X, denoted by $\operatorname{int} X$ (respectively ∂X). (The difference between the notations $\operatorname{int}, \partial$ and $\operatorname{int}, \operatorname{Fr}$ should prevent us, in each context, from confusing the interior and boundary points, and the interior part and boundary defined here with the interior and boundary points and the interior part and boundary of a set in a topological space.) Clearly, the interior of X is a dense open set, whereas the boundary of X is closed.

If each point of a topological space has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$ or $\mathbb{R}^n_-$, then obviously it already is an n-dimensional locally Euclidean space. Consequently, every open subset of an n-dimensional locally Euclidean space is also an n-dimensional locally Euclidean space. In particular, the interior of an n-dimensional Euclidean space is an n-dimensional locally Euclidean space without boundary. Moreover, the interior and boundary of an open subset U of a locally Euclidean space X are given by $\operatorname{int} U = U \cap \operatorname{int} X$ and $\partial U = U \cap \partial X$.

Since a locally Euclidean space is locally connected, its components are open (see Theorem 1.3.4.3), and hence also closed.

Obvious examples of n-dimensional locally Euclidean spaces are $\mathbb{R}^n$, $\mathbb{R}^n_-$, $\mathbb{S}^n$, and $\mathbb{D}^n$. It is clear that $\partial\mathbb{R}^n = \varnothing$ and $\partial\mathbb{S}^n = \varnothing$. Furthermore, all the boundary points of the half space $\mathbb{R}^n_-$ lie in the limiting hyperplane $\mathbb{R}^{n-1}_1$, consisting of the points $(x_1, \ldots, xn)$ such that $x_1 = 0$, and all the boundary points of the ball $\mathbb{D}^n$ lie in the limiting sphere $\mathbb{S}^{n-1}$.

Remark 3.1.1.3. Since the product $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is homeomorphic to $\mathbb{R}^{n_1+n_2}$, we see that the product $X_1 \times X_2$ of two locally Euclidean spaces X_1 and X_2 of dimensions n_1 and n_2, and without boundary, is an (n_1+n_2)-dimensional locally Euclidean space. This is true in general, i.e, a product $X_1 \times \cdots \times X_s$ of boundary-less locally Euclidean spaces $X_1, \ldots, X_s$ of dimensions $n_1, \ldots, n_s$, is an $(n_1 + \cdots + n_s)$-dimensional boundary-less locally Euclidean space. Turning to locally Euclidean spaces with boundary, note that the formula

$$((x_1, \ldots, x_{n_1}), (y_1, \ldots, y_{n_2})) \mapsto (x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2})$$

which gives the canonical homeomorphism $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1+n_2}$ also defines a homeomorphism $\mathbb{R}^{n_1}_- \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1+n_2}_-$ for $n_1 > 0$. Similarly, the formula

$$((x_1, \ldots, x_{n_1}), (y_1, \ldots, y_{n_2})) \mapsto (y_1, \ldots, y_{n_2}, x_1, \ldots, x_{n_1})$$

defines a homeomorphism $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_- \to \mathbb{R}^{n_1+n_2}_-$ for $n_2 > 0$, and the formula

$$((x_1, \ldots, x_{n_1}), (y_1, \ldots, y_{n_2})) \mapsto (-2x_1y_1, x_1^2 - y_1^2, x_2, \ldots, x_{n_1}, y_2, \ldots, y_{n_2})$$

defines a homeomorphism $\mathbb{R}^{n_1}_- \times \mathbb{R}^{n_2}_- \to \mathbb{R}^{n_1+n_2}_-$ for $n_1 > 0$, $n_2 > 0$. Thus, each of the products $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_-$, $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_-$, and $\mathbb{R}^{n_1}_- \times \mathbb{R}^{n_2}$ is homeomorphic to $\mathbb{R}^{n_1+n_2}_-$. We conclude that given locally Euclidean spaces X_1 and X_2 of dimensions n_1 and n_2, the product $X_1 \times X_2$ is an (n_1+n_2)-dimensional locally Euclidean space. In general, the product $X_1 \times \cdots \times X_s$ of arbitrary locally Euclidean spaces of dimensions $n_1, \ldots, n_s$ is an $(n_1 + \cdots + n_s)$-dimensional locally Euclidean space.

Remark 3.1.1.4. The discussion in Remark 3.1.1.2 raises two non-trivial questions.

The first one is whether a non-empty topological space can be a locally Euclidean space of dimension n and, simultaneously, a locally Euclidean space of a different dimension $n' : n \neq n'$? In Chapter 4 this question is answered negatively (see Theorem 4.6.5.11). The answer is obvious when $n = 1$, $n' > 1$ or $n' = 1$, $n > 1$. In fact, any connected subset of a one-dimensional locally Euclidean space becomes disconnected after one removes two suitably chosen points (for example, two points belonging to an open subset which is homeomorphic to $\mathbb{R}^1$); in contrast, every non-empty locally Euclidean space of dimension $n' > 1$ contains a non-empty open subset which cannot be disconnected by removing two points (any open subset homeomorphic to $\mathbb{R}^{n'}$ has this property). The picture is crystal clear when $n = 0$ or $n' = 0$. However, when $n > 1$, $n' > 1$, the proof requires a technique which we will develop only later.

The second question is whether we can formulate more efficient definitions of the interior and boundary points, which would permit us to actually recognise them. For example, consider the half space $\mathbb{R}^n_-$. At this point we can show only the trivial inclusion $\partial\mathbb{R}^n_- \subset \mathbb{R}^{n-1}_1$ (see Remark 3.1.1.2), and we are forced to settle for one of the extreme equalities $\partial\mathbb{R}^n_- = \mathbb{R}^{n-1}_1$ or $\partial\mathbb{R}^n_- = \varnothing$ (which obviously are the only possible ones). We shall prove in Chapter 4 that $\partial\mathbb{R}^n_- = \mathbb{R}^{n-1}_1$ (see Theorem 4.6.5.13). This equality is plain for $n = 1$ (assuming that the point 0 has a neighbourhood in $\mathbb{R}^1_-$ which is homeomorphic to $\mathbb{R}^1$, then by removing 0 we would disconnect this neighbourhood; this is absurd, because the latter cannot happen to a connected neighbourhood of 0 in $\mathbb{R}^1_-$). But for $n > 1$, we again need techniques which are to be developed. The equality $\partial\mathbb{R}^n_- = \mathbb{R}^{n-1}_1$ settles satisfactorily the general problem of recognising the interior and boundary points too. Indeed, it follows that

Theorem 3.1.1.5. *a point x of the n-dimensional locally Euclidean space X which has a neighbourhood U with a homeomorphism $U \to \mathbb{R}^n$, is a boundary point of U, and hence, of X, if and only if this homeomorphism takes x into a point of the hyperplane $\mathbb{R}^{n-1}_1$.*

For $\mathbb{D}^n$ this theorem asserts that $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$.

Finally, we note that the alternative equality $\partial\mathbb{R}^n_- = \varnothing$ would obviously imply that $\partial X = \varnothing$ for any n-dimensional locally Euclidean space.

Remark 3.1.1.6. In general, it would be more prudent not to use the theorems formulated in Remark 3.1.1.4, i.e, the theorem on dimensions and the equality $\partial\mathbb{R}^n_- = \mathbb{R}^{n-1}_1$ as long as they have not been proven. This indeed is the way we shall deal with the theorem on dimensions - the only exception is a harmless remark in 3.1.2.3. However, we have already used the equality $\partial\mathbb{R}^n_- = \mathbb{R}^{n-1}_1$ and we will take advantage of it again, before its proof, in Theorem 3.1.1.8 and in Remarks 3.1.2.6, 3.1.2.7. But these are the only instances where these theorems and their corollaries will be used before their proofs.

Theorem 3.1.1.7. *The boundary of an n-dimensional locally Euclidean space is an $(n-1)$-dimensional locally Euclidean space without boundary.*

Proof. Let x be a boundary point of the locally Euclidean space X and let U be a neighbourhood of x, homeomorphic to $\mathbb{R}^n_-$. Then the boundary ∂U is a neighbourhood of x in ∂X, since $\partial U = U \cap \partial X$, and is homeomorphic to $\mathbb{R}^{n-1}$, since $\partial\mathbb{R}^n_- = \mathbb{R}^n_1$ (here the reference to Chapter 4 for the equality $\partial\mathbb{R}^n_- = \mathbb{R}^n_1$ is unnecessary: the alternative $\partial\mathbb{R}^n_- = \varnothing$ is excluded, because $\partial X \neq \varnothing$). ☐

Theorem 3.1.1.8. *For any locally Euclidean spaces* $X_1, \dots, X_s$

$$\begin{aligned} \operatorname{int}(X_1 \times \cdots \times X_s) &= \operatorname{int} X_1 \times \cdots \times \operatorname{int} X_s, \quad \textit{and} \\ \partial(X_1 \times \cdots \times X_s) &= (\partial X_1 \times \cdots \times X_s) \cup \cdots \cup (X_1 \times \cdots \times \partial X_s). \end{aligned}$$

Proof. It is enough to prove the statement for $s = 2$. Let $x_i \in X_i$ and let φ_i be a homeomorphism of a neighbourhood U_i of x_i onto $\mathbb{R}^{n_i}$ or $\mathbb{R}^{n_i}_-$, $i = 1, 2$. Then $\varphi_1 \times \varphi_2$ is a homeomorphism of the neighbourhood $U_1 \times U_2$ of the point (x_1, x_2) onto one of the products

(1) $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

(2) $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_-$,

(3) $\mathbb{R}^{n_1}_- \times \mathbb{R}^{n_2}$, or

(4) $\mathbb{R}^{n_1}_- \times \mathbb{R}^{n_2}_-$,

and composing it with one of the homeomorphisms exhibited in Remark 3.1.1.3, we obtain a homeomorphism of $U_1 \times U_2$ onto $\mathbb{R}^{n_1+n_2}$ or $\mathbb{R}^{n_1+n_2}_-$. We denote this composition by φ and analyse the four possible cases.

(1): If $\varphi_1(U_1) = \mathbb{R}^{n_1}$ and $\varphi_2(U_2) = \mathbb{R}^{n_2}$, then $\varphi_1(U_1 \times (U_2) = \mathbb{R}^{n_1+n_2}$ and x_1, x_2 and (x_1, x_2) are all interior points.

(2): If $\varphi_1(U_1) = \mathbb{R}^{n_1}$ and $\varphi_2(U_2) = \mathbb{R}^{n_2}_-$, then $\varphi_1(U_1 \times (U_2) = \mathbb{R}^{n_1+n_2}_-$ and $\varphi(x_1, x_2) \in \mathbb{R}^{n_1+n_2}_1$ if and only if $\varphi(x_2) \in \mathbb{R}^{n_2-1}_1$. Thus (x_1, x_2) is an interior (boundary) point if and only if x_2 is an interior (respectively, boundary) point, while x_1 is an interior point.

(3): Similarly, if $\varphi_1(U_1) = \mathbb{R}^{n_1}_-$ and $\varphi_2(U_2) = \mathbb{R}^{n_2}$, then (x_1, x_2) is an interior or boundary point simultaneously with x_1, while x_2 is an interior point.

(4): Finally, if $\varphi_1(U_1) = \mathbb{R}^{n_1}_-$ and $\varphi_2(U_2) = \mathbb{R}^{n_2}_-$, then $\varphi_1(U_1 \times U_2) = \mathbb{R}^{n_1+n_2-1}_1$ if and only if $\varphi(x_1) \in \mathbb{R}^{n_1-1}_1$ or $\varphi(x_2) \in \mathbb{R}^{n_2-1}_1$. That is to say, (x_1, x_2) is a boundary point if and only if at least one of the points x_1, x_2 is boundary.

Our conclusion is that in all cases (x_1, x_2) is an interior (boundary point) if x_1 and x_2 are interior points (respectively, if x_1 and x_2 are boundary points). □

Theorem 3.1.1.9. *A locally Euclidean space is connected if and only if its interior is connected.*

Proof. This condition is obviously sufficient. Now let us show that it is also necessary. Let X be a connected locally Euclidean space, let A be a component of $\operatorname{int} X$, and let B be the union of the remaining components. Since the closed sets $\operatorname{Cl} A$ and $\operatorname{Cl} B$ cover X and X is connected, $\operatorname{Cl} A \cap \operatorname{Cl} B \neq \varnothing$ whenever $B \neq \varnothing$, and obviously $\operatorname{Cl} A \cap \operatorname{Cl} B \subset \partial X$. Let $x \in \operatorname{Cl} A \cap \operatorname{Cl} B$, and let U be a neighbourhood of x homeomorphic to $\mathbb{R}^n_-$. Since $\operatorname{int} \mathbb{R}^n_-$ is connected, its homeomorphic image $U \cap \operatorname{int} X = \operatorname{int} U$ is also connected, which is impossible if $B \neq \varnothing$. Consequently, $B = \varnothing$ and $\operatorname{int} X$ is connected. □

Definition 3.1.1.10. The topological space $X \cup_{\mathrm{incl}\colon \partial X \to X} X$ constructed from a locally Euclidean space X is called the *double* of X and is denoted by $\operatorname{dopp} X$. The double of an n-dimensional locally Euclidean space is an n-dimensional locally Euclidean space without boundary.

From now on, we shall identify $\mathrm{Imm}_1(X)$ with X and we shall denote the map $\operatorname{abr} \mathrm{Imm}_2 \colon X \to \mathrm{Imm}_2(X)$ by cop, and $\mathrm{Imm}_2(X)$ - by $\operatorname{cop} X$. Note that X and $\operatorname{cop} X$ are closed in $\operatorname{dopp} X$.

Manifolds

Remark 3.1.1.11. A locally Euclidean space is called a *topological manifold* or, briefly, a *manifold* if it is a Hausdorff topological space with countable base. A manifold is *closed* if it is compact and has no boundary and *open* if it has no compact components.

Comparing what was said in Remarks 3.1.1.2, 3.1.1.3, Theorem 3.1.1.7, and Definition 3.1.1.10 with the corresponding properties of Hausdorff, second countable, and compact spaces we see that:

- every open subset of an n-dimensional manifold is an n-dimensional manifold;
- the interior of an n-dimensional manifold is an n-dimensional manifold without boundary;
- the boundary of an n-dimensional manifold is an $(n-1)$-dimensional manifold without boundary;
- the boundary of a compact manifold is a closed manifold;
- the product of s manifolds of dimensions $n_1, \dots, n_s$ is an $(n_1 + \dots + n_s)$-dimensional manifold;
- the double of an n-dimensional manifold is an n-dimensional manifold without boundary;
- and the double of a compact manifold is a closed manifold.

Since the components of a manifold constitute an open cover, their number is finite in the compact case and countable in general (see Theorem 1.1.6.5).

Clearly $\mathbb{R}^n$, $\mathbb{R}^n_-$, $\mathbb{S}^n$ and $\mathbb{D}^n$, which we gave above as examples of locally Euclidean spaces, are manifolds.

Theorem 3.1.1.12. *Manifolds are locally compact.*

Proof. Let x be a point of an n-dimensional manifold X. Fix a homeomorphism φ of a neighbourhood U of x onto $\mathbb{R}^n$ or $\mathbb{R}^n_-$, and a neighbourhood V of $\varphi(x)$ in $\varphi(U)$ with compact closure $\operatorname{Cl} V$. Let $U' = \varphi^{-1}(V)$. Then obviously U' is a neighbourhood of x in X, $U' \subset \varphi^{-1}(\operatorname{Cl} V)$, and $\varphi^{-1}(\operatorname{Cl} V)$ is compact. Thus $\varphi^{-1}(\operatorname{Cl} V)$ is closed and contains both the neighbourhood U' and its closure $\operatorname{Cl} U'$. We conclude that $\operatorname{Cl} U'$ is compact. $\square$

Theorem 3.1.1.13. *Manifolds are metrisable.*

Proof. Every locally compact Hausdorff space is regular (see Theorem 1.1.7.23), and every regular second countable space is metrisable (see Theorem 1.1.6.9). □

Example 3.1.1.14. The following example shows that for $n \geq 1$ there are n-dimensional locally Euclidean spaces with countable base which are *not* Hausdorff. Consider $X = \mathbb{R}^n cup_i \mathbb{R}^n$, where $i = [\text{incl}\colon \mathbb{R}^n \setminus \mathbb{R}^n_{-}] \to \mathbb{R}^n]$. Then X is obviously an n-dimensional, second countable, locally Euclidean space, but for $n \geq 1$ and $x \in \mathbb{R}^{n-1}_1$, any two neighbourhoods of the points $\mathrm{Imm}_1(x)$ and $\mathrm{Imm}_2(x)$ in X intersect.

Information 3.1.1.15. For $n \geq 1$ there are connected, Hausdorff, n-dimensional locally Euclidean spaces that are not second countable. A two-dimensional example can be found in [5], and a one-dimensional one - in [11], p. 164 (the transfinite line, or "Alexandrov's line"). Higher-dimensional examples can be constructed from these by taking direct products with Euclidean spaces.

One-dimensional Manifolds

Remark 3.1.1.16. A zero-dimensional connected manifold obviously reduces to a point. Theorems 3.1.1.18 and 3.1.1.20 below provide the topological classification of connected one-dimensional manifolds. The two-dimensional case will be analysed in §3.5 (see Subsection 3.5.3). The topological classification of manifolds of higher dimensions is a very difficult problem.

Lemma 3.1.1.17. *If a connected Hausdorff space X can be represented as the union of two open subsets homeomorphic to $\mathbb{R}^1$, then X is homeomorphic to either $\mathbb{R}^1$ or $\mathbb{S}^1$.*

Proof. Let $X = U \cup V$ be the above representation and $\varphi\colon U \to \mathbb{R}^1$, $\psi\colon V \to \mathbb{R}^1$ - the corresponding homeomorphisms. We exclude the trivial cases $U \subset V$ and $V \subset U$, where X is homeomorphic to $\mathbb{R}^1$, and examine the sets $\varphi(U \cap V)$ and $\psi(U \cap V)$.

Since the intersection $U \cap V$ is open in both U and V, $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open in $\mathbb{R}^1$ and their components are intervals. None of these intervals is bounded: indeed, suppose that $\varphi(U \cap V)$ contains a bounded interval (a, b). Then $\varphi^{-1}((a, b))$ is both closed in V (as the intersection of the compact, and hence closed set $\varphi^{-1}([a, b])$ with V) and open in V, which implies $V = \varphi^{-1}((a, b)) \subset U$, a contradiction. Moreover, $\varphi(U \cap V) \neq \mathbb{R}^{-1}$ because, if not, $U \subset V$. Similarly, $\psi(U \cap V) \neq \mathbb{R}^{-1}$. Finally, we are left with only two possible cases:

(i) each of the sets $\varphi(U \cap V)$ and $\psi(U \cap V)$ is an open half line;

(ii) each of the sets $\varphi(U \cap V)$ and $\psi(U \cap V)$ the union of two disjoint open half lines.

Since we may multiply both φ and ψ by -1, we may assume that:

Case (i), $\varphi(U \cap V)$ has the form $(-\infty, a)$, and $\psi(U \cap V)$ - the form (b, ∞). Consider the composition

$$(-\infty, a) = \varphi(U \cap V) \xrightarrow{\operatorname{abr} \varphi^{-1}} U \cap V \xrightarrow{\operatorname{abr} \psi} \psi(U \cap V) = (b, \infty).$$

This map is injective and continuous, and hence monotone and obviously increasing (if it were to decrease, the points $\varphi^{-1}(a)$ and $\psi^{-1}(b)$ would have no disjoint neighbourhoods in X). Thus,

$$X = \psi^{-1}((-\infty, \psi(x_Q]) \cup \varphi - -1([(\varphi(x_0), \infty)),$$

for some point $x_0 \in U \cap V$, and so X is homeomorphic to $\mathbb{R}^1$.

Case (ii), $\varphi(U \cap V) = (\infty, a_1) \cup (a_2, \infty)$ and $\psi(U \cap V) = (\infty, b_1) \cup (b_2, \infty)$, for some a_1, a_2, b_1, b_2 $(a_1 < a_2, b_1 < b_2)$, and we may assume that the composite homeomorphism

$$\varphi(U \cap V) \xrightarrow{\operatorname{abr} \varphi^{-1}} U \cap V \xrightarrow{\operatorname{abr} \psi} \psi(U \cap V)$$

maps $(-\infty, a_1)$ onto (b_2, ∞), and (a_2, ∞) onto $(-\infty, b_1)$. Both functions $(-\infty, a_1) \mapsto (b_2, \infty)$, and $(a_2, \infty) \mapsto (-\infty, b_1)$, which represent compressions of this composite homeomorphism, are increasing (if, for example, the first were to decrease, then the points $\varphi^{-1}(a_1)$ and $\psi^{-1}(b_2)$ would have no disjoint neighbourhoods in X). We can thus write

$$X = \psi^{-1}([\psi(x_2), \psi(x_1)]) \cup \varphi^{-1}([\varphi(x_1), \varphi(x_2)]),$$

with some points $x_1 \in \varphi^{-1}((-\infty, a_1)) = \psi^{-1}((b_2, \infty))$ and $x_2 \in \varphi^{-1}((a_2, \infty)) = \psi^{-1}((-\infty, b_2))$. Therefore, X is homeomorphic to $\mathbb{S}^1$. □

Theorem 3.1.1.18. *Every compact, connected, one-dimensional manifold is homeomorphic to either* $\mathbb{S}^1$ *or* $\mathbb{D}^1$*.*

Proof. For a start, assume that the given manifold is closed. Then it can be covered by a finite number of open subsets homeomorphic to $\mathbb{R}^1$, and we may arrange these in a sequence $U_1, \ldots, U_s$ such that each union $V_k = U_1 \cup \cdots \cup U_k$ is connected. According to Lemma 3.1.1.17, the first of the sets $V_1, \ldots, V_s$ not homeomorphic to $\mathbb{R}^1$ is homeomorphic to $\mathbb{S}^1$, and being both open and closed, it is the entire manifold, which is thus homeomorphic to $\mathbb{S}^1$.

Assume now that the manifold has a boundary. Then its double is a closed, connected, one-dimensional manifold, and as such is homeomorphic to $\mathbb{S}^1$. Therefore, the original manifold is homeomorphic to a subset of $\mathbb{S}^1$. Since this subset is connected, closed, non-empty, different from $\mathbb{S}^1$, and not reduced to a point, it is homeomorphic to $\mathbb{D}^1$. □

Lemma 3.1.1.19. *If a topological space* X *can be represented as the union of a non-decreasing sequence of open subsets, all homeomorphic to* $\mathbb{R}^1$*, then* X *is homeomorphic to* $\mathbb{R}^1$*.*

Proof. Let $X = \cup V_i$ be the given representation. Clearly, any homeomorphism of onto some interval (a, b) extends to a homeomorphism of $V_{i+1}|$ onto one of the intervals (a, b), $(a-1, b)$, $(a, b+1)$, or $(a-1, b+1)$. Hence one can construct inductively a sequence of intervals and a sequence of homeomorphisms $\varphi_1 \colon V_1 \to \Delta_1$, $\varphi_2 \colon V_2 \to \Delta_2$, $\ldots$ such that $\varphi_i = \operatorname{abr} \varphi_{i+1}$. The map of X onto the interval $\cup \Delta_i$, which agrees with φ_i on V_i, is obviously a homeomorphism. $\square$

Theorem 3.1.1.20. *Every non-compact, connected, one-dimensional manifold is homeomorphic to either* $\mathbb{R}^1$ *or* $\mathbb{R}^1_-$.

Proof. First, assume that the given manifold X has no boundary. Then X can be covered by a countable family of open subsets, all homeomorphic to $\mathbb{R}^1$, and we can arrange these in a sequence $U_1, U_2, \ldots$, such that all unions $U_1 \cup \cdots \cup U_k$ are connected. Then all these unions are homeomorphic to $\mathbb{R}^1$. Indeed, if not, the first of them not homeomorphic to $\mathbb{R}^1$ is, according to Lemma 3.1.1.17, homeomorphic to $\mathbb{S}^1$, and being open and closed must coincide with X, a contradiction. Therefore, one can apply Lemma 3.1.1.19 to our manifold and deduce that it is homeomorphic to $\mathbb{R}^1$.

Now assume that X has a boundary. Then $\operatorname{dopp} X$ is a non-compact, connected, one-dimensional, boundary-less manifold, and must be homeomorphic to $\mathbb{R}^1$. It follows that X is homeomorphic to a connected, closed, non-compact subset of $\mathbb{R}$, different from $\mathbb{R}$; as such, it is homeomorphic to $\mathbb{R}^1_-$. $\square$

3.1.2 Differentiable Structures

Remark 3.1.2.1. Recall that a real function defined on an open subset of $\mathbb{R}^n$ is of *class* $\mathcal{C}^r$ (or a $\mathcal{C}^r$*-function*) if it has continuous partial derivatives of all orders up to and including r. The definition implies that $0 \leq r \leq \infty$, that $\mathcal{C}^0$ is the class of all continuous functions, and that C^∞ is the class of all functions which have continuous partial derivatives of all orders. In addition, we say that the real analytic functions are of class $\mathcal{C}^a$ (or $\mathcal{C}^a$*-functions*). It is convenient to consider $a > \infty$ and thus encompass all the classes listed above by the inequality $0 \leq r \leq a$.

Obviously, these definitions can be extended to real functions defined on an open subset of the half space $\mathbb{R}^n_-$. To do this, we consider the derivatives with respect to the first coordinate at the points of the boundary hyperplane $\mathbb{R}^n_-$ to be left derivatives, and analyticity at such points is understood as the existence of an analytic continuation to an open set in $\mathbb{R}^n$. We further extend the definitions to maps of an open subset of $\mathbb{R}^n$ or of $\mathbb{R}^n_-$, into any subset of $\mathbb{R}^q$: such a map is of *class* $\mathcal{C}^r$, or, simply, a $\mathcal{C}^r$*-map*, if its coordinate functions are of class $\mathcal{C}^r$.

Remark 3.1.2.2. A map f of an open subset of $\mathbb{R}^n$ or $\mathbb{R}^n_-$ into an open subset of $\mathbb{R}^n$ or $\mathbb{R}^n_-$ is a *diffeomorphism* if it is invertible and both f and f^{-1} are of class $\mathcal{C}^1$. Two sets which can be transformed into each other by a diffeomorphism are said to be *diffeomorphic.*

The following facts are contained in well-known theorems of calculus:

(i) If an open subset of $\mathbb{R}^p$ or $\mathbb{R}^p_-$ is diffeomorphic to an open subset of $\mathbb{R}^n$ or $\mathbb{R}^n_-$, then $p = n$.

(ii) An open subset of $\mathbb{R}^n_-$ which is diffeomorphic to an open subset of $\mathbb{R}^n$ is open in $\mathbb{R}^n$.

(iii) A diffeomorphism which is the inverse of a diffeomorphism of class $\mathcal{C}^r$ is itself of class $\mathcal{C}^r$.

C^r-structures and C^r-spaces

Remark 3.1.2.3. The definitions below refer to a given set X.

A *chart of dimension* n on X is an invertible map of a subset of X onto an open subset of $\mathrm{I}\mathbb{R}^p$ or $\mathbb{R}^p\_0$ The domain of a chart φ is called the *support* of φ and is denoted by $\operatorname{supp}\varphi$.

Two charts, φ and ψ, are $\mathcal{C}^r$-compatible (or have a $\mathcal{C}^r$-overlap) $(0 \leq r \leq a)$ if the set $\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi)$ is open in $\operatorname{im}\varphi$, the set $\psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi)$ is open in $\operatorname{im}\psi$, and the maps

$$\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \xrightarrow{\operatorname{abr}\varphi^{-1}} \operatorname{supp}\varphi \cap \operatorname{supp}\psi \xrightarrow{\operatorname{abr}\psi} \psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \quad \text{and}$$

$$\psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \xrightarrow{\operatorname{abr}\psi^{-1}} \operatorname{supp}\varphi \cap \operatorname{supp}\psi \xrightarrow{\operatorname{abr}\varphi} \varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi),$$

which are inverses of one another, are of class $\mathcal{C}^r$ (i.e, $\mathcal{C}^r$-diffeomorphisms for $r \geq 1$ and homeomorphisms for $r = 0$). This condition is trivially satisfied whenever $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \varnothing$. If $\operatorname{supp}\varphi \cap \operatorname{supp}\psi \neq \varnothing$, then the $\mathcal{C}^1$-compatibility of the charts φ and ψ implies the equality of their dimensions. In fact, this equality results also from the $\mathcal{C}^0$-compatibility of φ and ψ, as shown by Remark 3.1.1.4.

A collection of charts is an *n-dimensional $\mathcal{C}^r$-atlas of the set X* if these charts cover X, are n-dimensional, and each two of them are $\mathcal{C}^r$-compatible. Two $\mathcal{C}^r$-atlases of X are *$\mathcal{C}^r$-equivalent* if their union is again a $\mathcal{C}^r$-atlas. This is clearly an equivalence relation, and the equivalence classes of n-dimensional $\mathcal{C}^r$-atlases of the set X are called n-dimensional $\mathcal{C}^r$-structures. The $\mathcal{C}^r$-structures with $r > 0$ are called *differentiable structures.*

Clearly, if $0 \leq q \leq r$, then each n-dimensional $\mathcal{C}^r$-atlas is also an n-dimensional $\mathcal{C}^q$-atlas, and two equivalent $\mathcal{C}^r$-atlases are also $\mathcal{C}^q$-equivalent. Thus, when $0 \leq q \leq r$ every n-dimensional $\mathcal{C}^r$-structure uniquely extends to a $\mathcal{C}^r$-structure.

Every $\mathcal{C}^r$-structure contains a maximal atlas, namely the union of all its atlases. The latter is called the *complete atlas of the structure*, and its charts are called the *charts of the structure.* When we pass from a $\mathcal{C}^r$-structure to its $\mathcal{C}^q$-extension, the complete atlas extends too.

Remark 3.1.2.4. A set endowed with an n-dimensional $\mathcal{C}^r$-structure is called an n-*dimensional $\mathcal{C}^r$-space.* The charts and atlases of the structure are refereed to as the charts and atlases of the space. The complete atlas of a $\mathcal{C}^r$-space X is denoted by $\operatorname{Atl} X$.

The coordinate functions of a chart φp of the $\mathcal{C}^r$-space X are called *coordinates on* $\operatorname{supp}\varphi$ or, alternatively, *local coordinates in* X.

We shall denote by $\mathcal{C}^q X$ the $\mathcal{C}^q$-space obtained from the $\mathcal{C}^q$-space X by extending its $\mathcal{C}^r$-structure to a $\mathcal{C}^q$-structure, $0 \leq q \leq r$.

The $\mathcal{C}^r$-spaces with $r \geq q$ are also termed *$\mathcal{C}^{\geq q}$-spaces.*

For examples of $\mathcal{C}^r$-spaces we may look at all the open subsets X of $\mathbb{R}^n$ or $\mathbb{R}^n_-$, with the $\mathcal{C}^r$-structure defined by the atlas reduced to the single chart $\mathrm{id}\colon X \to X$. In particular, for any r, the charts $\mathrm{id}_{\mathbb{R}^n}$ and $\mathrm{id}_{\mathbb{R}^n_-}$ transform $\mathbb{R}^n$ and $\mathbb{R}^n_-$ into n-dimensional $\mathcal{C}^r$-spaces.

Remark 3.1.2.5. Every n-dimensional locally Euclidean space has an obvious $\mathcal{C}^0$-structure: its complete atlas consists of all possible homeomorphisms $U \to U'$, where U is an open subset of the space and U' is an open subset of $\mathbb{R}^n$ or $\mathbb{R}^n_-$. On the other hand, applying the "union of topological spaces" construction (see Remark 1.2.4.3) to the complete atlas of a given n-dimensional $\mathcal{C}^0$-space, we obtain an n-dimensional locally Euclidean space, and this transition is the inverse of the previous one. Therefore, $\mathcal{C}^0$-spaces are just locally Euclidean spaces.

Since any differentiable structure extends uniquely to a $\mathcal{C}^0$-structure, every $\mathcal{C}^r$-space with $r > 0$ is also a locally Euclidean space. Its topology may be described in a more direct fashion as the topology of the union constructed from any atlas of the structure.

Remark 3.1.2.6. Obviously, every point of an n-dimensional $\mathcal{C}^r$-space X can be covered by a chart φ of X such that $\operatorname{im}\varphi = \mathbb{R}^n$ or $\mathbb{R}^n_-$. The points with $\operatorname{im}\varphi = \mathbb{R}^n$ are called *interior points* and form an open dense set, called the *interior* of the space X, denoted by $\operatorname{int} X$. The remaining points are called *boundary points* and they form a closed set, called the boundary of X, denoted by ∂X. These notations are in agreement with those introduced at Remark1 3.1.1.2. In fact, when $r = 0$, the previous and present definitions of the interior and boundary points coincide.

When $r > 0$, we use Remark 3.1.2.2 (ii) in order to recognise the interior and boundary points. According to this remark, when $r > 0$ a point X of an n-dimensional $\mathcal{C}^r$-space is a boundary point if and only if $\operatorname{im}\varphi \subset \mathbb{R}^n_-$ and $\varphi(x) \in \mathbb{R}^{n-1}_1$, where φ is a chart on this space with $x \in \operatorname{supp}\varphi$. In particular, if we regard $\mathbb{R}^n_-$ as a $\mathcal{C}^r$-space with $r > 0$, then $\partial\mathbb{R}^n_- = \mathbb{R}^{n-1}_1$. Recall that the corresponding statement for $r = 0$ appeared in Remark 3.1.1.4 and its proof was postponed until Chapter 4.

The above characterisation of the boundary points shows that the interior and the boundary of a $\mathcal{C}^r$- space do not change when we extend its $\mathcal{C}^r$-structure to a $\mathcal{C}^q$-structure, for any $q \leq r$. In other words, for $0 \leq q \leq r$, $\operatorname{int}(\mathcal{C}^q X) = \operatorname{int} X$ and $\partial(\mathcal{C}^q X) = \partial X$. We emphasise that the equalities $\operatorname{int}(\mathcal{C}^0 X) = \operatorname{int} X$ and $\partial(\mathcal{C}^0 X) = \partial X$ were proved by a reference to Remark 3.1.1.4, i.e., they depended upon results from Chapter 4, whereas the equalities $\operatorname{int}(\mathcal{C}^q X) = \operatorname{int} X$ and $\partial(\mathcal{C}^q X) = \partial X$ for $q > 0$ need no such reference.

Using the relation $\partial X = \partial(\mathcal{C}^0 X)$, we see that Proposition 3.1.1.9 is valid for $\mathcal{C}^r$-spaces with $r > 0$ too. That is to say, a $\mathcal{C}^r$-space is connected if and only if its interior is connected. However, this $\mathcal{C}^r$-variant of Proposition 3.1.1.9 can be proved by merely repeating the proof of the original theorem, and therefore we can eliminate the reference to Chapter 4.

Remark 3.1.2.7. Suppose A is an open subset of an n-dimensional $\mathcal{C}^r$-space X. Then the charts of X whose supports are included in A yield a $\mathcal{C}^r$-atlas of the set A, and define an n-dimensional $\mathcal{C}^r$-structure on A. In this way, any open subset of an n-dimensional $\mathcal{C}^r$-space is an n-dimensional $\mathcal{C}^r$-space. In particular, the interior of any n-dimensional $\mathcal{C}^r$-space is an n-dimensional $\mathcal{C}^r$-space without boundary. Moreover, the interior and the boundary of an open subset U of the $\mathcal{C}^r$-space X are obviously given by $\operatorname{int} U = U \cap \operatorname{int} X$ and $\partial U = U \cap \partial X$.

Suppose φ is a chart on an n-dimensional $\mathcal{C}^r$-space X. Then $\operatorname{abr}\varphi\colon \partial X \cap \operatorname{supp}\varphi \to \varphi(\partial X \cap \operatorname{supp}\varphi)$ is an $(n-1)$-dimensional chart on ∂X. In this way we may construct a $\mathcal{C}^r$-atlas of the set ∂X, and so define a $\mathcal{C}^r$-structure on ∂X. Thus, the boundary of an n-dimensional $\mathcal{C}^r$-space is an $(n-1)$-dimensional $\mathcal{C}^r$-space without boundary.

If $\varphi_1(\varphi_2)$ is a chart on the $\mathcal{C}^{r_1}$-space X_1 ($\mathcal{C}^{r_1}$-space X_2) such that $\operatorname{im}\varphi_1 = \mathbb{R}^{n_1}$ or $\mathbb{R}^{n_1}_-$ (respectively, $\operatorname{im}\varphi_2 = \mathbb{R}^{n_2}$ or $\mathbb{R}^{n_2}_-$), then the composition of $\varphi_1 \times \varphi_2$ with one of the homeomorphisms $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1+n_2}$ and $\mathbb{R}^{n_1}_- \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1+n_2}_-$, defined in Remark 3.1.1.3, provides an (n_1+n_2)-dimensional chart on $X_1 \times X_2$. If $\partial X_2 = \varnothing$ then the charts constructed as above form a $\mathcal{C}^r$-atlas of the set $X_1 \times X_2$, with $r = \min(r_1, r_2)$, and hence define a $\mathcal{C}^r$-structure on $X_1 \times X_2$. Thus the product of the n_1-dimensional $\mathcal{C}^{r_1}$-space X_1 and the n_2-dimensional $\mathcal{C}^{r_2}$-space X_2 with $\partial X_2 = \varnothing$ is an $(n_1 + n_2$-dimensional $\mathcal{C}^r$-space, where $r = \min(r_1, r_2)$.

In general, the product of the n_i-dimensional $\mathcal{C}^{r_i}$-spaces $X_i, i = 1, \dots, s$, such that at most one of them has a boundary, is an $(n_1 + \dots + n_s)$-dimensional $\mathcal{C}^r$-space, where $r = \min(r_1, \dots, r_s)$. Moreover,

$$\operatorname{int}(X_1 \times \dots \times X_s) = \operatorname{int} X_1 \times \dots \times X_s,$$

and if X_i is the only space having a boundary, then

$$\partial(X_1 \times \dots \times X_s) = X_1 \times \cdots X_{i-1} \times \partial X_i \times \dots \times X_s.$$

For $r = 0$ both formulae can be found in Theorem 3.1.1.8; for $r > 0$, they are plain.

It is clear that when we extend the $\mathcal{C}^r$-structure of the $\mathcal{C}^r$-space X to a $\mathcal{C}^q$-structure, the $\mathcal{C}^r$-structures induced on the open subsets of X and on its boundary X also extend to $\mathcal{C}^q$-structures, and that $\mathcal{C}^q(X_1 \times \dots \times X_s) = \mathcal{C}^q X_1 \times \dots \times \mathcal{C}^q X_s$. In particular, the topology defined by the above induced $\mathcal{C}^r$-structures coincide with the relative topology, and the product of the $\mathcal{C}^{r_i}$-spaces $X_i, i = 1, \dots, s$, considered as a topological space, is just the product of the topological spaces $X_1, \dots, X_s$.

Smooth Maps

Definition 3.1.2.8. A continuous map f of a $\mathcal{C}^{\geq r}$-space X into a $\mathcal{C}^{\geq r}$-space Y is of *class* $\mathcal{C}^r$, or a $\mathcal{C}^r$*-map*, if for any chart φ on X and any chart on Y, the composite map

$$\varphi(\operatorname{supp}\varphi \cap f^{-1}(\operatorname{supp}\psi)) \xrightarrow{\operatorname{abr}\varphi^{-1}} \operatorname{supp}\varphi \cap f^{-1}(\operatorname{supp}\psi) \xrightarrow{\operatorname{abr} f} \operatorname{supp}\psi \xrightarrow{\psi} \operatorname{im}\psi$$

is of class $\mathcal{C}^r$ (see Remark 3.1.2.1). Such a composite map is called a *local representative* of the map f; we use the notation $\operatorname{loc}(\varphi, \psi)f$.

Obviously, a map $f\colon X \to Y$ of $\mathcal{C}^{\geq r}$-spaces is of class $\mathcal{C}^r$ if and only if the local representatives of f constructed for all the charts of of some atlases of X and Y are of class $\mathcal{C}^r$.

Note that this general definition of $\mathcal{C}^r$-maps contains the definition given in Remark 3.1.2.1. Now, as before, a $\mathcal{C}^0$-map is just a continuous map. The maps of class $\mathcal{C}^1$ are called *smooth*, and the maps of class $\mathcal{C}^a$ - *(real) analytic*.

The composition of two $\mathcal{C}^0$-maps is obviously a $\mathcal{C}^0$-map. If A is an open subset or the boundary of a $\mathcal{C}^r$-space X, then the inclusion $A \hookrightarrow X$ is a $\mathcal{C}^r$-map. If A is open in X or $A = \partial X$, and B is open in Y or $B = \partial Y$, then the compression $A \to B$ of any $\mathcal{C}^r$-map $X \to Y$ is a $\mathcal{C}^r$-map.

Remark 3.1.2.9. A map f of a $\mathcal{C}^{\geq 1}$-space X into a $\mathcal{C}^{\geq 1}$-space Y is a *diffeomorphism* if it is invertible and both f and f^{-1} are smooth. The space Y is said to be *diffeomorphic* to the space X if there is a diffeomorphism $X \to Y$, and *$\mathcal{C}^r$-diffeomorphic* to X, if there is a $\mathcal{C}^r$-diffeomorphism $X \to Y$.

Of course, the identity map of a $\mathcal{C}^{\geq 1}$-space with $r \geq 1$ is a $\mathcal{C}^r$-diffeomorphism. Also, the composition of two $\mathcal{C}^r$-diffeomorphisms is a $\mathcal{C}^r$-diffeomorphism, and the inverse of a $\mathcal{C}^r$-diffeomorphism is a $\mathcal{C}^r$-diffeomorphism itself, as we may easily see from Remark 3.1.2.2 (iii). Therefore, the property of being $\mathcal{C}^r$-diffeomorphic is an equivalence relation.

Using Remark 3.1.2.2 (i), we conclude that non-empty diffeomorphic spaces have the same dimension.

Remark 3.1.2.10. Let $f_1\colon X_1 \to Y_1, \dots, f_1\colon X_m \to Y_m$ be $\mathcal{C}^r$-maps with $r \geq 1$ where no more than one of the spaces $X_1, \dots, X_m$, and no more than one of the spaces $Y_1, \dots, Y_m$, has a boundary. Then

$$f_1 \times \cdots \times f_m\colon X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_m$$

is obviously a $\mathcal{C}^r$-map. If $f_1, \dots, f_m$ are diffeomorphisms, then so is $f_1 \times \cdots \times f_m$.

The canonical homeomorphism $X_1 \times X_2 \to X_2 \times x_1$ is a $\mathcal{C}^r$-diffeomorphism for any two $\mathcal{C}^r$-spaces X_1 and X_2 with $r \geq 1$, such that one of them has no boundary. The canonical homeomorphisms

$$\begin{gathered}(X_1 \times \cdots \times X_{m-1}) \times X_m \to X_1 \times \cdots \times X_m \quad \text{and} \\ X_1 \times (X_2 \times \cdots \times X_m) \to X_1 \times \cdots \times X_m\end{gathered}$$

are $\mathcal{C}^r$-diffeomorphisms for any $\mathcal{C}^r$-spaces $X_1 \times \cdots \times X_m$ with $r \geq 1$, such that no more that one of them has a boundary.

Subspaces

Remark 3.1.2.11. A subset A of an n-dimensional $\mathcal{C}^r$-space X with $r \geq 1$ is a k-d*dimensional subspace of* X if each point of A is covered by a chart φ on X such that the pair $(\operatorname{im}\varphi, \varphi(A \cap \operatorname{supp}\varphi))$ coincides with one of the pairs $(\mathbb{R}^n, \mathbb{R}^k)$, $(\mathbb{R}^n, \mathbb{R}^k_-)$, or $(\mathbb{R}^n_-, \mathbb{R}^k_-)$. If $k > 0$, then this condition is obviously equivalent to the following one: each point of A is covered by a chart φ of the space X such that $\varphi(A \cap \operatorname{supp}\varphi) = \operatorname{im}\varphi \cap \mathbb{R}^k$ or $\operatorname{im}\varphi \cap \mathbb{R}^k_-$.

Suppose A is a k-dimensional subspace of an n-dimensional $\mathcal{C}^r$-space X, and consider the maps abr$t\varphi$: $A \cap \operatorname{supp}\varphi \to \varphi(A \cap \operatorname{supp}\varphi)$ corresponding to all charts φ on X such that $\varphi(A \cap \operatorname{supp}\varphi) = \operatorname{im}\varphi \cap \mathbb{R}^k$ or $\operatorname{im}\varphi \cap \mathbb{R}^k_-$. Each such map is a k-dimensional chart on A, and together they yield a k-dimensional $\mathcal{C}^r$-atlas of A. The $\mathcal{C}^r$-structure defined by this atlas transforms A into a k-dimensional $\mathcal{C}^r$-space. The topology of this $\mathcal{C}^r$-space obviously coincides with the relative topology.

When we extend the $\mathcal{C}^r$-structure of a space to a $\mathcal{C}^q$-structure with $q \geq 1$, its subspaces remain subspaces. The subspaces of $\mathcal{C}^q X$ are called *$\mathcal{C}^q$-subspaces* of the original $\mathcal{C}^r$-space X.

The *codimension* of a subspace is the difference between the dimension of the ambient space and that of the subspace.

It is evident that the open subsets of a $\mathcal{C}^r$-space (with $r \geq 1$) are among its subspaces; in particular, we cite its interior and its components. Warning: if not empty, the boundary of a $\mathcal{C}^r$-space is not a subspace.

It is readily seen that if A is a subspace of codimension 0 of a $\mathcal{C}^r$-space X, then $\operatorname{int} A = \operatorname{int} A \cap \operatorname{int} X$.

A subspace of a $\mathcal{C}^r$-space is *neat* if it is closed as a subset and its boundary is contained in the boundary of the space. Note that every neat subspace of codimension 0 is made of whole components of the ambient space.

Remark 3.1.2.12. The definition of a subspace given in Remark 3.1.2.11 contains implicitly the generally used method of defining subspaces through equations and inequalities. Namely, according to the second variant of our definition, a subset A of an n-dimensional $\mathcal{C}^r$-space X ($r \geq 1$) is a subspace of X of positive dimension k if and only if each point of A has a neighbourhood U with coordinates $\varphi_1, \dots, \varphi_n$, such that the intersection $A \cap U$ is defined in U either by the equations $\varphi_{k+1} = 0, \dots, \varphi_n = 0$, or by the equations $\varphi_{k+1} = 0, \dots, \varphi_n = 0$ and the inequality $\varphi_1 \leq 0$.

As a procedure for defining subspaces, this formulation has an obvious disadvantage; namely, we must assume from the beginning that $\varphi_1, \dots, \varphi_n$ are local coordinates. We make use the implicit function theorem to make it more efficient.

Let X be a $\mathcal{C}^r$-space ($r \geq 1$), $x_0 \in X$, and let $f_1, \dots, f_m$ be real $\mathcal{C}^1$-functions defined on a neighbourhood U_0 of X_0. We say that $f_1, \dots, f_m$ are *independent at the point* x_0 if there is a chart φ on X with $x_0 \in \operatorname{supp}\varphi$ and such that the functions $g_1, \dots, g_m \colon \varphi(U_0 \cap \operatorname{supp}\varphi) \to \mathbb{R}$ defined by $g_i(y) = f_i(\varphi^{-1}(y))$ have linearly independent gradients at the point $\varphi(x_0)$. The following statements are consequences of the implicit function theorem:

(i) if the $\mathcal{C}^r$-functions $f_1, \dots, f_m$ are independent at the interior point x_0 of the $\mathcal{C}^r$-space X, then they can be completed to a system of coordinates in a neighbourhood of x_0;

(ii) if the $\mathcal{C}^r$-functions $f_1, \dots, f_m$ are independent at the boundary point x_0 of the $\mathcal{C}^r$-space X, and the function f_1 is negative at the interior points and zero on the boundary, then $f_1, \dots, f_m$ can be completed to a system of coordinates in a neighbourhood of x_0 having f_1 as the first coordinate.

Comparing with the previous, coordinate description of the subspaces of a $\mathcal{C}^r$-space, we see that a subset A of the n-dimensional $\mathcal{C}^r$-space X is a subspace of positive dimension k of X if:

(i) each point $x_0 \in A$ which is an interior point for X has a neighbourhood U where the intersection $A \cap U$ is defined either by the equations $\varphi_{k+1} = 0, \dots, \varphi_n = 0$, where $\varphi_{k+1}, \dots, \varphi$ are $\mathcal{C}^r$-functions independent at x_0, or by the equations $\varphi_{k+1} = 0, \dots, \varphi_n = 0$ and the inequality $\varphi_1(x) \leq 0$, where $\varphi_1, \varphi_{k+1}, \dots, \varphi$ are $\mathcal{C}^r$-functions independent at x_0;

(ii) each point $x_0 \in A$ which lies on the boundary of X has a neighbourhood U such that the intersections $\operatorname{int} X \cap U$, $\partial X \cap U$, and $A \cap U$ are defined in U by the inequality $\varphi_1(x) < 0$, by the equation $\varphi_1(x) = 0$, and by the equations $\varphi_{k+1} = 0, \dots, \varphi_n = 0$, respectively, where $\varphi_1, \varphi_{k+1}, \dots, \varphi$ are $\mathcal{C}^r$-functions independent at x_0.

Remark 3.1.2.13. An obvious consequence of the definition of a subspace is that the interior $\operatorname{int} A$ of a subspace A of the $\mathcal{C}^r$-space X is contained in $\operatorname{int} X$, and that the intersections $\partial A \cap \operatorname{int} X$ and $\partial A \cap \partial X$ are open in ∂A, i.e, they consist of whole components of ∂A. Moreover, it is clear that $\operatorname{int} A$ is a subspace of both X and $\operatorname{int} X$, while $\partial A \cap \operatorname{int} X$ is a subspace of X. In particular, if A is a neat subspace of X, then $\partial A = A \cap \partial X$ and $\operatorname{int} A$ is a neat subspace of $\operatorname{int} X$, while ∂A is a neat subspace of ∂X.

The inclusion $A \hookrightarrow X$ of a subspace A of a $\mathcal{C}^r$-space X is obviously a $\mathcal{C}^r$-map. The compression $A \to B$ of any $\mathcal{C}^q$-map $X \to Y$, where A (B) is a subspace of X (respectively, Y) is a $\mathcal{C}^q$-map.

Let A_i be a subspace of the $\mathcal{C}^r$-space $X_i, i = 1, \dots, s$, and assume that at most one of the spaces $X_1, \dots, X_s$ has a boundary, the same being true for the subspaces $A_1, \dots, A_s$. Then $A_1, \times \cdots \times A_s$ is a subspace of $X_1, \times \cdots \times X_s$, and is a neat subspace if each A_i is neat. For example, the fibres of the product $X_1, \times \cdots \times X_s$ are neat subspaces.

Let A be a subspace of the $\mathcal{C}^r$-space X, and let B be a subspace of A. Then, using the description of subspaces in Remark 3.1.2.12, we see that B is a subspace of X; in particular, if B is a neat subspace of A and A is a neat subspace of X, then B is also a neat subspace of X. In a similar fashion we conclude that a neat subspace B of the $\mathcal{C}^r$-space X which is contained in a neat subspace A of X is also a neat subspace of A.

C^r-manifolds

Definition 3.1.2.14. A $\mathcal{C}^r$-space is a *$\mathcal{C}^r$-manifold*, or a *manifold of class $\mathcal{C}^r$*, if it is a topological manifold, i.e., a second-countable, Hausdorff space. A manifold of class $\mathcal{C}^0$ is simply a topological manifold; see Remark 3.1.2.5. The manifolds of class $\mathcal{C}^r$ with $r \geq 1$ are called *smooth*, or differentiable. The manifolds of class $\mathcal{C}^a$ are called *(real) analytic.*

Since any $\mathcal{C}^r$-structure is defined by one of its atlases, it is interesting to discover those properties of an atlas of a $\mathcal{C}^r$-space which guarantee that the space is Hausdorff and second-countable. Here we formulate only two obvious conditions: if each pair of points is covered by a chart of the atlas or by two disjoint charts of the atlas, then the space is Hausdorff; if the atlas is countable, then the space is second-countable.

There is no need to check that a space is Hausdorff and second-countable if the differentiable structure is introduced on a set which is already a topological manifold, and the topology defined by the differentiable structure coincides with the initial one. If the differentiable structure is defined by an atlas $\{\varphi_{\alpha\}}$, then the two topologies agree if and only if $\operatorname{supp} \varphi_\alpha$ are open and $\mathrm{c}\varphi_\alpha$ are homeomorphisms; see Remark 3.1.2.5.

A smooth manifold is *closed* if it is compact and has no boundary; cf. Remark 3.1.1.11. Warning: at the present time the equality $\partial X = \partial(\mathcal{C}^0 X)$ is not proven (see Remark 3.1.2.6). Therefore, we should be careful to distinguish between the smooth manifold X being closed and the topological manifold $\mathcal{C}^0 X$ being closed.

Definition 3.1.2.15. Reconsidering the statements made in Remarks 3.1.2.7 and 3.1.2.11 in the light of the corresponding properties of Hausdorff spaces, second-countable spaces, and compact spaces, we may deduce the following:

- every open subset of an n-dimensional $\mathcal{C}^r$-manifold is an n-dimensional $\mathcal{C}^r$-manifold;
- the interior of an n-dimensional $\mathcal{C}^r$-manifold is an n-dimensional $\mathcal{C}^r$-manifold without boundary;
- the boundary of an n-dimensional $\mathcal{C}^r$-manifold is an $(n-1)$-dimensional $\mathcal{C}^r$-manifold without boundary;
- the boundary of a compact C -manifold is a closed C -manifold;
- the product of the $\mathcal{C}^r$-manifolds $X_1, \ldots, X_s$ of dimensions $n_1, \ldots, n_s$, such that no more than one of them has a boundary, is an $(n_1 + \cdots + n_s)$-dimensional $\mathcal{C}^r$-manifold;
- a subspace of a $\mathcal{C}^r$-manifold is a $\mathcal{C}^r$-manifold.

The subspaces of smooth manifolds are called *submanifolds*, and the neat subspaces - *neat submanifolds.*

Remark 3.1.2.16. The basic examples of n-dimensional $\mathcal{C}^r$-manifolds with $r \geq 1$ are again $\mathbb{R}^n$ and $\mathbb{R}^n_-$ (see Remark 3.1.2.4). The submanifolds of these spaces provide an unlimited supply of examples of $\mathcal{C}^r$-manifolds. The simplest are the submanifolds of $\mathbb{R}^n$ defined by one system of equations $\varphi_{k+1}(x) = 0, \ldots, \varphi_n(x) = 0$, where $\varphi_{k+1}, \ldots, \varphi_n$ are $\mathcal{C}^r$-functions defined on an open subset of $\mathbb{R}^n$ and having linearly independent gradients on the set of their common zeros. We may add to such a system the inequality $\varphi_1(x) \leq 0$, where φ_1 is any $\mathcal{C}^r$-function defined in a neighbourhood of the set of common zeros of $\varphi_{k+1}, \ldots, \varphi_n$, zero on this set, and such that the gradients of $\varphi_1, \varphi_{k+1}, \ldots, \varphi_n$ are linearly independent on the same set. For example, the sphere $\mathbb{S}^{n-1}$ is defined, in standard coordinates, by the equation $x_1^2 + \cdots + x_n^2 - 1 = 0$, and the ball $\mathbb{D}^n$ - by the inequality $x_1^2 + \cdots + x_n^2 - 1 \leq 0$. Hence $\mathbb{S}^{n-1}$ and $\mathbb{D}^n$ are submanifolds of $\mathbb{R}^n$ and, in particular, $\mathcal{C}^a$-manifolds.

The following facts are clear: $\mathbb{R}^k$, for $k \leq n$, and $\mathbb{S}^k$, for $k < n$, are neat submanifolds of $\mathbb{R}^n$; $\mathbb{R}^k_-$ and $\mathbb{D}^k$ are not neat submanifolds of $\mathbb{R}^n$ for $k \leq n$; $\mathbb{R}^k_-$ is a neat submanifold of $\mathbb{R}^n_-$ for $k \leq n$; $\mathbb{S}^k$ is a neat submanifold of $\mathbb{S}^n$ for $k \leq n$; and $\mathbb{D}^k$ is a neat submanifold of $\mathbb{D}^n$ for $k \leq n$.

Remark 3.1.2.17. Finally, we note that every real, n-dimensional vector space has a natural $\mathcal{C}^r$-structure for any r $(0 \leq r \leq a)$, which makes it into an n-dimensional $\mathcal{C}^r$-manifold. This structure is defined by the linear charts i.e, by the linear maps onto $\mathbb{R}^n$.

3.1.3 Orientations

Remark 3.1.3.1. Consider a $\mathcal{C}^r$-manifold X, and let $\operatorname{Catl} X$ denote the atlas of X consisting of all the charts with connected support. If φ and ψ are two charts on the smooth manifold X, we denote by $J(\varphi, \psi)$ the Jacobian of the map $\operatorname{loc}(\varphi, \psi)\,\mathrm{id}$, i.e, of the composite map

$$\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \xrightarrow{\operatorname{abr}\varphi^{=1}} \operatorname{supp}\varphi \cap \operatorname{supp}\psi \xrightarrow{\operatorname{abr}\psi} \psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi).$$

An *orientation* of the smooth manifold X is a function $\omega\colon \operatorname{Catl} X \to \mathbb{S}^0$ such that $\omega(\varphi) = [\operatorname{sgn} J(\varphi, \psi)](y)\omega(\psi)$ for each two charts $\varphi, \psi \in \operatorname{Catl} X$, where y is an arbitrary point of $\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi)$ [therefore, the function $\operatorname{sgn} J(\varphi, \psi)$ must be constant on $\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi)$] . A smooth manifold endowed with an orientation is said to be *oriented*. A smooth manifold which can be oriented is *orientable*. Clearly, an orientation $\operatorname{Catl} X \to \mathbb{S}^0$ is determined by its values on any subatlas of $\operatorname{Catl} X$. Moreover, every function which carries a subatlas of $\operatorname{Catl} X$ into $\mathbb{S}^0$ and satisfies the previous compatibility condition, i.e, its values on two charts φ, ψ of the given subatlas are obtained one from another multiplying by $[\operatorname{sgn} J(\varphi, \psi)](y)$ with $y \in \varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi)$, extends to an orientation $\operatorname{Catl} X \to \mathbb{S}^0$. When one extends the $\mathcal{C}^r$-structure of a $\mathcal{C}^r$-manifold X to a $\mathcal{C}^q$ structure with $q < r$, $\operatorname{Catl} X$ becomes a subatlas of $\operatorname{Catl} \mathcal{C}^q X$. If $q \geq 1$, this establishes a one-to-one correspondence between the orientations of the manifolds X and $\mathcal{C}^q X$.

Remark 3.1.3.2. For each orientation $\omega\colon \operatorname{Catl} X \to \mathbb{S}^0$ there exists the opposite orientation $-\omega$, and thus every non-empty orientable manifold has at least two orientations.

Given two arbitrary orientations of the manifold X, the set covered by the charts on which they agree, and the set covered by the charts on which they do not agree are open, disjoint, and together they cover X; hence, each of them is a union of components of the manifold X. Consequently, the orientation of a connected manifold is uniquely determined by its value at one chart. Moreover, every smooth, connected, orientable manifold has exactly two orientations, whereas an orientable smooth manifold with s components has 2^s orientations. The standard way of describing an orientation of a smooth connected manifold is to indicate the charts where it is positive. For example, $\mathbb{R}^n$ has a natural orientation, positive on the chart $\mathrm{id}_{\mathbb{R}^n}$.

Remark 3.1.3.3. Since any point of a zero-dimensional manifold X is covered by only one chart of $\operatorname{Catl} X$, it follows that every zero dimensional manifold is orientable, and has actually a natural orientation, identically equal to $+1$. We shall see later (in Subsection 5.3.1 and Remark 5.6.3.4) that all one-dimensional manifolds are orientable, whereas the manifolds of dimension ≥ 2 are not necessarily so.

In Chapter 5 we shall give effective sufficient conditions for the orientability of a manifold of arbitrary dimension (see Subsection 5.6.3). The crudest of them is that the manifold be simply connected.

Remark 3.1.3.4. Let A be an open subset of a smooth manifold X. Then $\operatorname{Catl} A \subset \operatorname{Catl} X$, and every orientation of X induces an orientation of A; in particular, A is orientable whenever X is. If A intersects all the components of X, then the orientation of X is determined by the orientation induced on the submanifold A.

In the case $A = \operatorname{int} X$, we may say more: not only does each orientation of X restrict to an orientation of the manifold $\operatorname{int} X$, but also each orientation of $\operatorname{int} X$ extends to an orientation of X. Indeed, for each connected subset U of X, $U \cap \operatorname{int} X$ is connected (see Remarks 3.1.2.6 and 3.1.2.7) and so the compression

$$\operatorname{abr} \varphi\colon \operatorname{supp} \varphi \cap \operatorname{int} X \to \varphi(\operatorname{supp} \varphi \cap \operatorname{int} X)$$

belongs to $\operatorname{Catl}(\operatorname{int} X)$ for any $\varphi \in \operatorname{Catl} X$. This enables us to extend any orientation $\omega\colon \operatorname{Catl}(\operatorname{int} X) \to \mathbb{S}^0$ to an orientation $\operatorname{Catl} X$ through the formula $\varphi \mapsto \omega(\operatorname{abr} \varphi)$. Thus, if we associate to each orientation $\operatorname{Catl} X \to \mathbb{S}^0$ its restriction $\operatorname{Catl}(\operatorname{int} X) \to \mathbb{S}^0$, we obtain a one-to-one correspondence between the orientations of the manifolds X and $\operatorname{int} X$. In particular, X is orientable whenever $\operatorname{int} X$ is such. According to Remark 3.1.2.7, for each chart φ on the smooth manifold X we have the corresponding chart

$$\operatorname{abr} \varphi\colon \operatorname{supp} \varphi \cap \partial X \to \varphi(\operatorname{supp} \varphi \cap \partial X)$$

on its boundary ∂X. It is clear that every chart in $\operatorname{Catl} \partial X$ is of the form $\operatorname{abr} \varphi$, where $\varphi \in \operatorname{Catl} X$, and that for each orientation $\omega\colon \operatorname{Catl} X \to \mathbb{S}^0$ of the manifold

X we have an orientation of its boundary, defined by the rule $\operatorname{abr}\varphi \mapsto \omega(\varphi)$. In particular, ∂X is orientable whenever X is such. If all the components of X have a boundary, then the orientation of X is uniquely determined by the orientation it induces on ∂X.

The only submanifolds of a smooth manifold X which inherit a natural orientation when X is oriented are those of codimension zero. This orientation has been implied in the discussion of the open subsets of X. If A is an arbitrary submanifold of X of codimension zero, then the induced orientation is defined by the orientation of its interior $\operatorname{int} A$, which is open in X. In particular, A is orientable if X is such. If A intersects all the components of X, then the orientation of X is uniquely determined by the orientation it induces on A.

Since the manifold $\mathbb{R}^n$ has a natural orientation, all its n-dimensional submanifolds inherit a natural orientation too. In particular, $\mathbb{D}^n$ carries a natural orientation, which in turn induces an orientation of its boundary $\mathbb{S}^{n-1}$. Warning: the orientation of $\mathbb{S}^0$ induced by the orientation of $\mathbb{D}^1$ does not coincide with the canonical orientation of $\mathbb{S}^0$, considered as a zero-dimensional manifold (see Remark 3.1.3.3).

Orientations and Diffeomorphisms

Definition 3.1.3.5. Every diffeomorphism $f\colon X \to Y$ establishes a one-to-one correspondence between the orientations of the manifolds X and Y. If both manifolds are oriented and f transforms the orientation of X into the orientation of Y (into the opposite orientation of Y), then we say that f *preserves* (*reverses*) *the orientation* or that f is *orientation preserving* (respectively, *orientation reversing*).

To determine whether a diffeomorphism $f\colon X \to Y$ is orientation preserving or not, we may look at its local representatives; if X and Y are connected, it is enough to analyse only one local representative. Namely, suppose that ω_X and ω_Y are orientations of the connected manifolds X and Y. Let $\varphi \in \operatorname{Catl} X$ and $\psi \in \operatorname{Catl} Y$ be two charts and pick $x \in \operatorname{supp}\varphi \cap f^{-1}(\operatorname{supp}\psi)$. If the sign of the Jacobian of $\operatorname{loc}(\varphi,\psi)f$ at the point $\varphi(x)$ coincides with (is opposite to) the sign of the product $\omega_X(\varphi)\omega_Y(\psi)$, then f is clearly orientation preserving (respectively, reversing).

Remark 3.1.3.6. Of special interest is the case where $X = Y$ is a connected manifold. It is readily seen that in this situation a diffeomorphism which preserves (reverses) one orientation of X, will preserve (respectively, reverse) all orientations of X. Therefore, in this case one can talk about an orientation preserving (reversing) diffeomorphism without fixing an orientation. In particular, every (auto)diffeomorphism of a smooth, connected, orientable manifold is either orientation preserving or orientation reversing.

As an example, consider a non-singular linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$. It is clear that f is a diffeomorphism and that f is orientation preserving (reversing) if $\det f > 0$ (respectively, $\det f < 0$). If the transformation f is orthogonal, then its compressions $\operatorname{abr} f\colon \mathbb{D}^n \to \mathbb{D}^n$ and $\operatorname{abr} f\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ are meaningful, are diffeomorphisms, and preserve (reverse) orientation if $\det f = 1$ (respectively, $\det f = -1$) [here $\mathbb{S}^0$ is oriented as $\mathbb{D}^1$]. If f is given by $f(x) = -x$, then $\det f = (-1)^n$. Hence this diffeomorphism and the induced antipodal map $\operatorname{abr} f \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ are orientation preserving for n even and orientation reversing for n odd.

Orientations and Products of Manifolds

Remark 3.1.3.7. We have shown in Remark 3.1.2.7 that every product $\varphi_1 \times \cdots \times \varphi_s$ of charts $\varphi_1, \dots, \varphi_s$ on the smooth manifolds $X_1, \dots, X_s$ without boundary of dimensions $n_1, \dots, n_s$, may be regarded, using the canonical identification $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_s} = \mathbb{R}^{n_1 + \cdots + n_s}$ defined by the formula

$$((x_{11}, \dots, x_{1n_1}), \dots, (x_{s1}, \dots x_{sn_s})) \mapsto (x_{11}, \dots, x_{1n_1}, \dots, x_{s1}, \dots, x_{sn_s})$$

as a chart on the product $X_1 \times \cdots \times X_s$. Given orientations $\omega_1, \dots, \omega_s$ of the manifolds $X_1, \dots, X_s$, let us define a mapping from the collection of all charts $\varphi_1 \times \cdots \times \varphi_s$ with $\varphi_1 \in \operatorname{Catl} X_1, \dots, \varphi_s \in \operatorname{Catl} X_s$, into $\mathbb{S}^0$ by the formula

$$\varphi_1 \times \cdots \times \varphi_s \mapsto \omega_1(\varphi_1), \dots, \omega_s(\varphi_s).$$

It is obvious that the above collection of charts is a subatlas of $\operatorname{Catl}(X_1, \dots, X_s)$, and that this mapping satisfies the compatibility condition introduced in Remark 3.1.3.1. Therefore, it extends to an orientation of the manifold $X_1 \times \cdots \times X_s$, called the *product of the orientations*, $\omega_1, \dots, \omega_s$. The latter can be defined even if one of the manifolds $X_1, \dots, X_s$ has a boundary: it is the orientation induced by the orientation of the interior $\operatorname{int} X_1 \times \cdots \times \operatorname{int} X_s$. Thus, a product of smooth, oriented manifolds (such that no more than one of them has a boundary) is oriented, and a product of smooth, orientable manifolds is orientable.

We note that the orientability of $X_1 \times \cdots \times X_s$ implies the orientability of each factor $X_1, \dots, X_s$. In fact, if ω is an orientation of the product and $\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_s$ are fixed charts of

$$\operatorname{Catl}(\operatorname{int} X_1), \dots, \operatorname{Catl}(\operatorname{int} X_{i-1}), \operatorname{Catl}(\operatorname{int} X_{i+1}), \dots, \operatorname{Catl}(\operatorname{int} X_s),$$

then the mapping $\operatorname{Catl}(\operatorname{int} X_i) \to \mathbb{S}^0$ given by

$$\varphi \mapsto \omega(\varphi_1 \times \cdots, \varphi_{i-1} \times \varphi \times \varphi_{i+1} \times \cdots \times \varphi_s)$$

is an orientation of the manifold $\operatorname{int} X_i$.

Theorem 3.1.3.8. *Suppose X_1 and X_2 are smooth oriented manifolds of dimensions n_1 and n_2, such that one of them has no boundary. The canonical diffeomorphism $X_1 \times X_2 \to X_2 \times x_1$ preserves orientation if $n_1 n_2$ is even and reverses orientation if $n_1 n_2$ is odd.*

Proof. Let $\varphi_1 \in \operatorname{Catl}(\operatorname{int} X_1)$ and $\varphi_2 \in \operatorname{Catl}(\operatorname{int} X_2)$ be charts with $\operatorname{im} \varphi_1 = \mathbb{R}^{r_1}$ and $\operatorname{im} \varphi_2 = \mathbb{R}^{r_2}$, respectively. Clearly,

$$\varphi_1 \times \varphi_2 \in \operatorname{Catl}(\operatorname{int}(X_1 \times x_2)), \quad \varphi_2 \times \varphi_1 \in \operatorname{Catl}(\operatorname{int}(X_2 \times x_1)),$$

and the local representative $\mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2+n_1}$ of the canonical diffeomorphism $X_1 \times X_2 \to X_2 \times X_1$ relative to these charts is given by

$$(x_1, \dots, x_{n_1+n_2}) \mapsto (x_{n_1+1}, \dots, x_{n_1+n_2}, x_1, \dots, x_{n_1}).$$

Therefore, the Jacobian of this local representative is $(-1)^{n_1 n_2}$. □

Theorem 3.1.3.9. *Suppose $X_1, \dots, X_s$ are smooth oriented manifolds of dimensions $n_1, \dots, n_s$ such that only one, say X_i, has a boundary. Then the canonical orientation of the product*

$$X_1 \times \cdots \times X_{i-1} \times \partial X_i \times X_{i+1} \times \cdots \times X_s$$

and the natural orientation that this product receives as $\partial(X_1 \times \cdots \times X_s)$ differ by a factor of $(-1)^{n_1+\cdots+n_{i-1}}$.

Proof. Obviously, the above orientations agree if $i = 1$. The case $i > 1$ reduces to $i = 1$ with the aid of the diffeomorphisms

$$X_1 \times \cdots \times X_s \to X_i \times X_1 \times \cdots X_{i-1} \times X_{i+1} \times \cdots \times X_s$$
$$X_1 \times \cdots \times X_{i-1} \times \partial X_i \times X_{i+1} \times \cdots \times X_s \to \partial X_i \times X_1 \times \cdots X_{i-1} \times X_{i+1} \times \cdots \times X_s.$$

The first of them is the product of the canonical diffeomorphism

$$(X_1 \times \cdots \times X_{i-1}) \times X_i \to X_i \times (X_1 \times \cdots X_{i-1})$$

with $\mathrm{id}_{X_{i+1} \times \cdots \times X_s}$, while the second is the product of the canonical diffeomorphism

$$(X_1 \times \cdots \times X_{i-1}) \times \partial X_i \to \partial X_i \times (X_1 \times \cdots X_{i-1})$$

with $\mathrm{id}_{X_{i+1} \times \cdots \times X_s}$. The first diffeomorphism preserves (reverses) the orientation if the product $(n_1 + \cdots + n_{i-1}) n_i$ is even (respectively, odd), while the second has the same property if the product $(n_1 + \cdots + n_{i-1})(n_i - 1)$ is even (respectively, odd); see Theorem 3.1.3.8. This explains the factor $(-1)^{n_1+\cdots+n_{i-1}}$. □

Orientations of Vector Spaces

Remark 3.1.3.10. Real vector spaces are smooth manifolds (see Remark 3.1.2.17) and hence the definition of orientation given in Remark 3.1.3.1 applies. On the other hand, there is a well-known, purely vectorial definition of an orientation of a real vector space: an orientation is a mapping from the set of all bases of the space into $\mathbb{S}^0$, which takes the same value on two bases if and only if the matrix transforming one basis into the other has a positive determinant. This vectorial definition clearly agrees with the definition given in Remark 3.1.3.1 and is often more convenient.

In particular, the vectorial definition makes obvious the following remark. Let f be a linear map of a real vector space V into another real vector space. Fix a subspace $V' \subset V$ that is mapped isomorphically by f onto $\operatorname{im} f$. Then we may represent V as the direct sum of the spaces $\operatorname{im} f$ and $\ker f$. Consequently, the orientations of any two of the three spaces V, $\operatorname{im} f$, and $\ker f$ determine the orientation of the third. As an immediate result of our definition, we see that this connection between the orientations of V, $\operatorname{im} f$, and $\ker f$ does not depend upon the choice of V', but only on the map f.

(Transcriber's note: I have no idea what the above paragraph is referring to. Chances are the authours are talking about the following fact.)
We have $V/\ker f \simeq \operatorname{im} f$ by the homomorphism theorem. Since V is a vector space, we have $V \simeq \operatorname{im} f \oplus \ker f$.

3.1.4 The Manifold of Tangent Vectors

Remark 3.1.4.1. Suppose that X is an n-dimensional $\mathcal{C}^r$-manifold, $r \geq 1$. For each point $x \in X$, let $\operatorname{Atl}_x X$ be the collection of all charts $\varphi \in \operatorname{Atl} X$ such that $x \in \operatorname{supp} \varphi$; recall that $\operatorname{Atl} X$ denotes the complete atlas of X. If $\varphi, \psi \in \operatorname{Atl}_x X$, then at the point $\varphi(x)$ the differential of the diffeomorphism

$$\operatorname{loc}(\varphi, \psi)\operatorname{id} \colon \varphi(\operatorname{supp} \varphi \cap \operatorname{supp} \psi) \to \psi(\operatorname{supp} \varphi \cap \operatorname{supp} \psi)$$

is meaningful [and is the linear map $\mathbb{R}^n \to \mathbb{R}^n$ whose matrix is the Jacobi matrix of the map $\operatorname{loc}(\varphi, \psi)\operatorname{id}$ at the point $\varphi(x)$]. We denote this differential by $d_x(\varphi, \psi)$.

Now consider the real vector space of all the maps $\operatorname{Atl}_x X \to \mathbb{R}^n$ (with the natural operations). It is clear that those maps $v \colon \operatorname{Atl}_x X \to \mathbb{R}^n$ which satisfy the relation $v(\psi) = d_x(\varphi, \psi)v(\varphi)$ for any two charts $\varphi, \psi \in \operatorname{Atl}_x X$ yield a subspace of this vector space, i.e, they form a real vector space. The latter is called the *tangent space to X at the point x* and is denoted by $\operatorname{Tang}_x X$. The maps in $\operatorname{Tang}_x X$ are called *tangent vectors at the point x (on X)*, or *vectors tangent to X at x.* Obviously, a tangent vector is completely defined by its value on an arbitrary chart of $\operatorname{Atl}_x X$, and for each chart $\varphi \in \operatorname{Atl}_x X$ and each vector $u \in \mathbb{R}^n$ there is a vector $v \in \operatorname{Tang}_x X$ with $v(\varphi) = u$. Therefore, the mapping $\varphi_\# \colon \operatorname{Tang}_x X \to \mathbb{R}^n$, $\varphi_\#(v) = v(\varphi)$, defined for a chart $\varphi \in \operatorname{Atl}_x X$, is invertible. Moreover, $\varphi_\#$ being linear, it is an isomorphism; in particular, $\dim Tang_x X = n$. The isomorphism $\varphi_\#^{-1}$ takes the canonical basis $\operatorname{ort}_1, \ldots, \operatorname{ort}_n$ of $\mathbb{R}^n$ into a basis of $\operatorname{Tang}_x X$, which we shall call the *φ-basis.* The coordinates of a vector $v \in \operatorname{Tang}_x X$ relative to the φ-basis coincide with the usual coordinates of the vector $v(\varphi)$ and are called the *φ-coordinates* of v.

Remark 3.1.4.2. We denote the union $\cup_{x \in X} \operatorname{Tang}_x X$, i.e., the space of all vectors tangent to X, by $\operatorname{Tang} X$. The map $\operatorname{Tang} X \to X$ transforming $\operatorname{Tang}_x \to X$ into x is called *projection* and is denoted by proj. Thus $\operatorname{proj}^{-1}(x) = \operatorname{Tang}_x X$.

The set $\operatorname{Tang} X$ has a natural topology which makes it a topological manifold. Furthermore, for $r \geq 2$, $\operatorname{Tang} X$ has a natural differentiable structure. In order to describe these structures, we define, for each chart $\varphi \in \operatorname{Atl} X$, the map

$$\operatorname{tn}\varphi\colon\ \operatorname{proj}^{-1}(\operatorname{supp}\varphi) \to \operatorname{im}\varphi \times \mathbb{R}^n, \quad \operatorname{tn}\varphi(v) = (\varphi \circ \operatorname{proj}(v), \varphi_{\#}(v)).$$

If we identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{R}^{2n}$, and consider $\operatorname{im}\varphi \times \mathbb{R}^n$ to be an open subset of $\mathbb{R}^{2n}$ or $\mathbb{R}_{-}{}^{2n}$, then we may interpret $\operatorname{tn}\varphi$ as a $2n$-dimensional chart on $\operatorname{Tang} X$. Obviously, if $\psi \in \operatorname{Atl} X$ is another chart, the composition

$$\begin{aligned}
&\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \times \mathbb{R}^n = \\
&\operatorname{tn}\varphi(\operatorname{proj}^{-1}(\operatorname{supp}\varphi) \cap \operatorname{proj}^{-1}(\operatorname{supp}\psi) \xrightarrow{\operatorname{abr}(\operatorname{tn}\varphi)^{-1}} \\
&\operatorname{proj}^{-1}(\operatorname{supp}\varphi) \cap \operatorname{proj}^{-1}(\operatorname{supp}\psi) \xrightarrow{\operatorname{abr}(\operatorname{tn}\varphi)} \\
&\operatorname{tn}\psi(\operatorname{proj}^{-1}(\operatorname{supp}\varphi) \cap \operatorname{proj}^{-1}(\operatorname{supp}\psi)) = \psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \times \mathbb{R}^n
\end{aligned} \tag{3.1.4.3}$$

is given by the formula $(a, u) \mapsto (\operatorname{loc}(\varphi, \psi)\operatorname{id}_{\varphi^{-1}(a)}(\varphi, \psi)u)$. Equivalently, writing this composite map in coordinates, we have

$$(a_1, \dots, a_n; u_1, \dots, u_n) \mapsto (b_1, \dots, b_n; v_1, \dots, v_n),$$

where

$$\left.\begin{aligned} b_j &= \ell_j(a_1, \dots, a_n), \\ v_j &= \textstyle\sum_{i=1}^{m} D_i\ell_j(a_1, \dots, a_n)u_i, \end{aligned}\right\} j = 1, \dots, n. \tag{3.1.4.4}$$

and $\ell_1, \dots, \ell_n$ are the coordinate functions of the map $\operatorname{loc}(\varphi, \psi)\operatorname{id}$ (D_i denotes the partial derivative with respect to the i-th coordinate). Formula (3.1.4.4) shows that the charts $\operatorname{tn}\varphi$ and $\operatorname{tn}\psi$ are $\mathbb{C}^{r-1}$ compatible (we set $\mathbb{C}^{\infty-1} = \mathbb{C}^{\infty}$ and $\mathbb{C}^{a-1} = \mathbb{C}^{a}$). Moreover, the charts $\operatorname{tn}\varphi$, $\varphi \in \operatorname{Atl} X$, cover $\operatorname{Tang} X$, and thus yield a $\mathbb{C}^{r-1}$-atlas of the set $\operatorname{Tang} X$. This atlas has a countable subatlas (since $\operatorname{Atl} X$ has such a subatlas). Furthermore, for any two vectors of $\operatorname{Tang} X$, it has either a chart which contains both of them, or a pair of disjoint charts, each containing one of the vectors (indeed, recall that $\operatorname{Atl} X$ contains, for any two points of X, either a chart containing both of them, or a pair of disjoint charts, each containing one of the points). Therefore, it makes $\operatorname{Tang} X$ into a $2n$-dimensional $\mathbb{C}^{r-1}$-manifold, which we call the *total manifold of vectors tangent to the manifold* X.

Clearly, the projection $\operatorname{Tang} X \to X$, the inclusions $\operatorname{Tang}_x X \to \operatorname{Tang} X$, and the natural map $X \to \operatorname{Tang} X$ which takes each point x into the zero vector of the space $\operatorname{Tang}_x X$, are all $\mathbb{C}^{r-1}$ maps.

Formula (3.1.4.4) shows that for $r \geq 2$ the Jacobian of the composite map (3.1.4.3) at the point (a, u) is equal to the square of the Jacobian of the map $\operatorname{loc}(\varphi,_p si)\operatorname{id}$ at the point a. We deduce that for $r \geq 2$ the manifold $\operatorname{Tang} X$ is always orientable and even carries a canonical orientation, namely the one which is positive on the charts $\operatorname{tn}\varphi$ with $\varphi \in \operatorname{Catl} X$.

One more remark: let X_1 and X_2 be two arbitrary smooth manifolds such that $\partial X_2 = \varnothing$. Then for any two points $x_1 \in X_1$ and $x_2 \in X_2$, $\operatorname{Tang}_{x_1,x_2}(X_1 \times X_2)$ and $\operatorname{Tang}_{x_1} X_1 \oplus \operatorname{Tang}_{x_2} X_2$ are isomorphic as vector spaces, and the isomorphism is natural. In addition, the isomorphisms corresponding to all pairs (x_1, x_2) yield a diffeomorphism of $\operatorname{Tang}(X_1 \times X_2)$ onto $\operatorname{Tang} X_1 \times \operatorname{Tang} X_2$.

The Differential of a Smooth Map

Remark 3.1.4.5. Let f be a $\mathbb{C}^r$-map of an m-dimensional $\mathbb{C}^{\geq r}$-manifold X into an n-dimensional $\mathbb{C}^{\geq r}$-manifold Y, $r \geq 1$. For a point $x \in X$ and two charts $\varphi \in \operatorname{Atl}_x X$ and $\psi \in \operatorname{Atl}_{f(x)} Y$, we let $d_x(f; \varphi, \psi)$ denote the differential of the map $\operatorname{loc}(\varphi, \psi)f$ at the point $\varphi(x)$, regarded as the linear map $\mathbb{R}^m \to \mathbb{R}^n$ whose matrix is the Jacobi matrix of $\operatorname{loc}(\varphi, \psi)f$ at $\varphi(x)$. If $\varphi' \in \operatorname{Atl}_x X$ and $\psi' \in \operatorname{Atl}_{f(x)} Y$ are two other charts, then

$$d_x(f; \varphi', \psi') = d_{f(x)}(\psi, \psi')) \circ d_x(f; \varphi, \psi) \circ d_x(\varphi', \varphi).$$

Combining this relation with the equalities $d_x(\varphi', \varphi = \varphi_\# \circ (\varphi'_\#)^{-1}$ and $d_{f(x)}(\psi, \psi')) = \psi'_\# \circ \psi^{-1}$ we see that

$$(\psi'_\#)^{-1} \circ d_x(f; \varphi', \psi') \circ \varphi'_\# = \psi_\#^{-1} \circ d_x(f; \varphi, \psi) \circ \varphi_\#,$$

i.e., the linear map $\psi_\#^{-1} \circ d_x(f; \varphi, \psi) \circ \varphi_\# \colon \operatorname{Tang}_x X \to \operatorname{Tang}_{f(x)} Y$ does not depend upon the choice of the charts φ and ψ. This linear map is called the *differential of the map f at the point x* and is denoted by $d_x f$. The map $\operatorname{Tang} X \to \operatorname{Tang} Y$ which equals $d_x f$ on $\operatorname{Tang}_x X$ for all $x \in X$ is called the *differential of the map f* and is denoted by df. The resulting diagramme

$$\begin{tikzcd}
\operatorname{Tang} X \arrow[r, "df"] \arrow[d, "\operatorname{proj}"'] & \operatorname{Tang} Y \arrow[d, "\operatorname{proj}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is clearly commutative.

Let $\ell_1, \ldots, \ell_n$ be the coordinate functions of the map $\operatorname{loc}(\varphi, \psi)f$. Then the local representative $\operatorname{loc}(\operatorname{tn}\varphi, \operatorname{tn}\psi)df$ of the differential df is given in coordinates by the formula

$$(a_1, \ldots, a_m; u_1, \ldots, u_m) \mapsto (b_1, \ldots, b_n; v_1, \ldots, v_n),$$

where

$$\left.\begin{aligned} b_j &= \ell_j(a_1, \ldots, a_m), \\ v_j &= \textstyle\sum_{i=1}^m D_i \ell_j(a_1, \ldots, a_m) u_i, \end{aligned}\right\} j = 1, \ldots, n.$$

Therefore, df is of class $\mathcal{C}^{r-1}$.

Of course, $d(h \circ f) = dh \circ df$ for any smooth map h of Y into a third smooth manifold, and $df = \operatorname{id}(\operatorname{Tang} X)$ when $X = Y$ and $f = \operatorname{id} X$. Also, if f is a diffeomorphism, then df is a diffeomorphism for $r \geq 2$ and a homeomorphism for $r = 1$.

In the special case when X is an open subset of $\mathbb{R}^m$ or of $\mathbb{R}^m_-$, and Y is an open subset of $\mathbb{R}^n$ or of $\mathbb{R}^n_-$, in addition to the differential $d_x f \colon \operatorname{Tang} X_x \to \operatorname{Tang}_{f(x)} Y$ one has the classical differential of the map f, i.e, the linear map $\mathbb{R}^m \to \mathbb{R}^n$ whose matrix is the Jacobi matrix of f at the point x. Let us identify the spaces $\operatorname{Tang}_x X$ and $\mathbb{R}^m$ ($\operatorname{Tang}_{f(x)} Y$ and $\mathbb{R}^n$) via the linear isomorphism $(\mathrm{id}_X)_\#\colon \operatorname{Tang}_x \to \mathbb{R}^m$ (respectively, $(\mathrm{id}_Y)_\#\colon \operatorname{Tang}_{f(x)} Y \to \mathbb{R}^n$). Then it is clear that $d_x f$ becomes the classical differential.

Remark 3.1.4.6. If A is a submanifold or the boundary of a smooth manifold X, then the differential $d_x \operatorname{incl}\colon \operatorname{Tang}_x A \to \operatorname{Tang}_x X$ of inclusion $\operatorname{incl}\colon A \to X$ is a monomorphism for each point $x \in A$. Thus, we can identify the tangent space $\operatorname{Tang}_x A$ with the subspace $d_x \operatorname{incl}(\operatorname{Tang}_x A)$ of $\operatorname{Tang}_x X$, and $\operatorname{Tang} A$ - with $d \operatorname{incl}(\operatorname{Tang} A)$.

If A is a submanifold of $\mathbb{R}^n$, then along with the identification $\operatorname{Tang}_x A = d_x \in (\operatorname{Tang}_x A)$ one has the identification $\operatorname{Tang}_x \mathbb{R}^n = \mathbb{R}^n$ via the canonical (linear) isomorphism $(\mathrm{id}_{\mathbb{R}^n})\colon \operatorname{Tang}_x \mathbb{R}^n \to \mathbb{R}^n$, and so $\operatorname{Tang}_x A$ becomes a subspace of $\mathbb{R}^n$.

It is easy to describe this subspace explicitly when A is defined in a neighbourhood of x by the independent functions $\varphi_{k+1}, \dots \varphi_n$ as in Remark 3.1.2.12. In this situation, $\operatorname{Tang}_x A$ consists of all the vectors of $\mathbb{R}^n$ which are orthogonal to the $n-k$ vectors $\operatorname{grad} \varphi_{k+1}, \dots \operatorname{grad} \varphi_n$. For example, $\operatorname{Tang}_x \mathbb{S}^{n-1}$ is the subspace of $\mathbb{R}^n$ composed of all vectors orthogonal to the vector x.

Vector Fields

Definition 3.1.4.7. A *vector field* on a smooth manifold X is a continuous map $X \to \operatorname{Tang} X$ which takes each point $x \in X$ into a vector tangent to X at x. A trivial example is the *zero* vector field, whose value at each point $x \in X$ is the zero vector of the space $\operatorname{Tang}_x X$ (see Remark 3.1.4.2).

A smooth, n-dimensional, $\mathcal{C}^r$-manifold X is *$\mathcal{C}^r$-parallelisable* if there exist n $\mathcal{C}^r$-vector fields $f_1, \dots, f_n\colon X \to \operatorname{Tang} X$ such that, at each point $x \in X$, the vectors $f_1(x), \dots, f_n(x)$ yield a basis of the space $\operatorname{Tang}_x X$. For example, $\mathbb{R}^n$ (regarded as a $\mathcal{C}^a$-manifold) is $\mathcal{C}^a$-parallelisable: a parallelisation is given by the vector fields which associate to each point $x \in \mathbb{R}^n$ the $(\mathrm{id}_{\mathbb{R}^n})$-basis of the space $\operatorname{Tang}_x \mathbb{R}^n$.

$\mathcal{C}^0$-parallelisability is simply called *parallelisability*. It will be shown in Chapter 4 (see Remark 4.6.4.3) that the parallelisability of a compact $\mathcal{C}^{\geq r+1}$-manifold with $r \leq \infty$ implies its $\mathcal{C}^r$-parallelisability.

INFORMATION. The $\mathcal{C}^r$-parallelisability is a consequence of the parallelisability of a $\mathcal{C}^{\geq r+1}$-manifold even if the manifold is not compact, or if $r = a$.

Theorem 3.1.4.8. *Suppose that the smooth manifold X is parallelised by the $\mathcal{C}^r$-vector fields $f_1, \dots, f_n\colon X \to \operatorname{Tang} X$. Then the formula $(x, (y_1, \dots, y_n)) \mapsto y_1 f_1(x) + \dots + y_n f_n(x)$ defines a $\mathcal{C}^r$-diffeomorphism of the product $X \times \mathbb{R}^n$ onto $\operatorname{Tang} X$. Indeed, the formula $v \mapsto (\operatorname{proj}(v), (y_1, \dots, y_n))$, where $y_1, \dots, y_n$ are the coordinates of the vector v relative to the basis $f_1(\operatorname{proj}(v)), \dots, f_n(\operatorname{proj}(v))$ of $\operatorname{Tang}_{\operatorname{proj}(v)} X$, defines the inverse map $\operatorname{Tang} X \to X \times \mathbb{R}^n$, and obviously both maps are of class $\mathcal{C}^r$. Thus the total manifold of vectors tangent to a $\mathcal{C}^r$-parallelisable n-dimensional smooth manifold X is $\mathcal{C}^r$-diffeomorphic to the product $X \times \mathbb{R}^n$.*

In particular, we deduce that every $\mathcal{C}^1$-parallelisable smooth manifold is orientable.

Proof. (of the second statement)
In fact, $\operatorname{Tang} X$ is orientable for any $\mathcal{C}^r$-manifold X with $r \geq 2$ (see Remark 3.1.4.2); hence, in our case, the product $X \times \mathbb{R}^n$ is orientable, which in turn implies the orientability of X (see Remark 3.1.3.7). □

Example 3.1.4.9. For each odd n, there is a vector field with no zeros on the sphere $\mathbb{S}^n$. For example, such a vector field $\mathbb{S}^n \to \mathbb{R}^{n+1}$ is given by $x = (x_1, \ldots, x_{n+1}) \mapsto (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$, $x \in \mathbb{S}^n$, where $y_{2k-1} = -x_{2k}$, $y_{2k} = x_{2k-1}$ $(k = 1, \ldots, (n+1)/2)$; here we consider $(y_1, \ldots, y_{n+1})$ as a vector in $\operatorname{Tang}_x \mathbb{S}^n$ (see Remark 3.1.4.6).

Note that the same vector field can be defined in a more concise fashion as the map $x \mapsto xi$, regarding $\mathbb{R}^{n+1}$ as $\mathbb{C}^{(n+1)/2}$. If $n+1$ is divisible by 4, $\mathbb{R}^{n+1}$ may be regarded as $\mathbb{H}^{(n+1)/4}$, and the formulae $x \mapsto \operatorname{ort}_2$, $x \mapsto \operatorname{ort}_3$, and $x \mapsto \operatorname{ort}_4$ (ort_2, ort_3, and ort_4 are considered here as the imaginary quaternion units) define three vector fields on $\mathbb{S}^n$ which are linearly independent at each point. If $n+1$ is divisible by 8, $\mathbb{R}^{n+1}$ may be regarded as $\mathbb{C}a^{(n+1)/8}$, and the formulae $x \mapsto \operatorname{ort}_2, \ldots, \operatorname{ort}_8$ ($\operatorname{ort}_2, \ldots, \operatorname{ort}_8$ are considered here as the imaginary Cayley units) define seven vector fields on $\mathbb{S}^n$ which are linearly independent at each point. Since all the above vector fields are analytic, this construction shows, in particular, that $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^8$ are $\mathcal{C}a$-parallelisable.

Information 3.1.4.10. For $n \neq 0, 1, 3, 7$ the sphere $\mathbb{S}^n$ is not parallelisable. For a proof, see [1] and [2].

3.1.5 Embeddings, Immersions, and Submersions

Remark 3.1.5.1. A map $f \colon X \to Y$ of smooth manifolds is a *$\mathcal{C}^r$-embedding* if $f(X)$ is a $\mathcal{C}^r$-submanifold of Y and $\operatorname{abr} f \colon X \to f(X)$ is a $\mathcal{C}^r$-diffeomorphism. For example, the inclusion of a submanifold into its ambient $\mathcal{C}^r$-manifold is a $\mathcal{C}^r$-embedding. Since every map $f \colon X \to Y$ can be written as the composition of its compression $\operatorname{abr} f \colon X \to f(X)$ with the inclusion $f(X) \to Y$, a $\mathcal{C}^r$-embedding is really a map of class $\mathcal{C}^r$.

The $\mathcal{C}^1$-embeddings are also termed *differentiable embeddings*. Using Theorem 3.1.5.3 we shall prove below, we can see that a differentiable embedding which is of class $\mathcal{C}^r$ is a $\mathcal{C}^r$-embedding. Moreover, it is evident that differentiable embeddings are topological embeddings. The latter are sometimes called $\mathcal{C}^0$-embeddings

A differentiable embedding $f \colon X \to Y$ is *neat* if $f(X)$ is a neat submanifold of Y. For example, the inclusion of a neat submanifold into its smooth ambient manifold is such an embedding. Clearly, if $\dim X = \dim Y$, Y is connected, and $X \neq \varnothing$, then every neat differentiable embedding $X \to Y$ is a diffeomorphism.

It is obvious that for each $\mathcal{C}^r$-embedding $f \colon X \to Y$ and each point $x \in X$, there are charts $\varphi \in \operatorname{Atl}_x \mathcal{C}^r X$ and $\psi \in \operatorname{Atl}_{f(x)} \mathcal{C}^r Y$ such that $\operatorname{loc}(\varphi, \psi) f$

coincides with one of the inclusions

$$\mathbb{R}^m \to \mathbb{R}^n, \quad \mathbb{R}^m_- \to \mathbb{R}^n, \quad \text{or} \quad \mathbb{R}^m_- \to \mathbb{R}^n_-,$$

where $m = \dim X$ and $n = \dim Y$. If f is neat, then the second case must be excluded.

Immersions

Remark 3.1.5.2. A smooth map $f\colon X \to Y$ of smooth manifolds is an *immersion* if

(i) $d_x f\colon \operatorname{Tang}_x X \to \operatorname{Tang}_{f(x)} Y$ is a monomorphism for any $x \in X$;

(ii) $(df)^{-1}(\operatorname{Tang} \partial Y) \subset \operatorname{Tang} \partial X$.

We remark that condition (i) implies that $\dim X \leq \dim Y$, and condition (ii) - that $f(\operatorname{int} X) \subset \operatorname{int} Y$. If $\partial Y = \varnothing$, then (ii) is automatically fulfilled.

Trivially, the composition of two immersions is an immersion.

The differentiable embeddings are examples of immersions.

Theorem 3.1.5.3. *If $f\colon X \to Y$ is an immersion of class $\mathcal{C}^r$, then each point of the manifold X has a neighbourhood N such that the restriction of f to N is a $\mathcal{C}^r$-embedding.*

Proof. Let $x_0 \in X$ be an arbitrary point, and put $m = \dim X$, $n = \dim Y$. Now pick some charts $\varphi \in \operatorname{Atl} X_{x_0} X$, $\psi \in\in \operatorname{Atl}_{f(x_0)} Y$, and denote by $\ell_1, \dots, \ell_n$ the coordinate functions of the map $\operatorname{loc}(\varphi, \psi) f$. According to condition 3.1.5.2 (i), the Jacobi matrix of $\operatorname{loc}(\varphi, \psi) f$ at $\varphi(x_0)$ has rank m. Using condition 3.1.5.2 (ii, one can assume that the minor M of this matrix, constructed from its first m rows, is not zero.

[If $f(x_0) \in \operatorname{int} Y$, 3.1.5.2 (ii is not necessary: one can achieve $M \neq 0$ by re-indexing the local coordinates of the chart i.e, by permuting the rows of the matrix. When $f(x_0) \in \partial Y$, 3.1.5.2 (ii ensures that all the elements of the first row of the Jacobi matrix, starting with the second, are zero; this allows to achieve $M \neq 0$ by re-indexing the coordinates $\psi_2, \dots, \psi_n$.]

Next apply the implicit function theorem to deduce the existence of a neighbourhood W of the point $(\psi_1 f(x_0)), \dots, \psi_m(f(x_0))$ in $\mathbb{R}^m$ in $\mathbb{R}^m$ and of a $\mathcal{C}^r$-embedding $h\colon W \to \operatorname{im} \varphi$, such that, for $i = 1, \dots, m$, one has:

$$\begin{aligned} &h_i(\psi_1 f(x_0)), \dots, \psi_m(f(x_0)) = \varphi_i(x_0) \quad \text{and} \\ &\ell_i(h_1(y_1, \dots, y_m), \dots, (h_1(y_1, \dots, y_m)) = y_i \quad [(y_1, \dots, y_m) \in W], \end{aligned}$$

where $h_1, \dots, h_m$ are the coordinate functions of h. Let $N = \varphi^{-1}(h(W))$. Evidently, N is a neighbourhood of x_0. Its image $f(N)$ in $\varphi^{-1}(W \times \mathbb{R}^{n-m})$ is defined by the equations

$$\psi_j - \ell_j(h_1(\psi_1, \dots, \psi_m), \dots, h_m(\psi_1, \dots, \psi_m)) = 0,$$

and, possibly, by the additional inequality $h_1(\psi_1, \ldots, \psi_m) \leq 0$. Therefore, $f(N)$ is a $\mathcal{C}^r$-submanifold of Y. The composite map

$$f(N) \xrightarrow{\operatorname{abr}\psi} \psi(f(N))) \to W \xrightarrow{h} h(W) \xrightarrow{(\operatorname{abr}\varphi)^{-1}} N,$$

(where the second map is the compression of the orthogonal projection $\mathbb{R}^n \to \mathbb{R}^m$) is of class $\mathcal{C}^r$ and is the inverse of the map $\operatorname{abr} f\colon N \to f(N)$. We conclude that $f|_N$ is a $\mathcal{C}^r$-embedding. □

Corollary 3.1.5.4. *If an immersion of class $\mathcal{C}^r$ is a topological embedding, then it is a $\mathcal{C}^r$-embedding. In particular, every injective $\mathcal{C}^r$-immersion of a compact manifold is a $\mathcal{C}^r$-embedding.*

Theorem 3.1.5.5. *Let $f\colon X \to Y$ be a smooth map such that $(df)^{-1}(\operatorname{Tang}\partial Y) \subset \operatorname{Tang}\partial X$, and let A be a compact subset of the manifold X. If $f|_A$ is injective and the differential $d_x f$ is non-degenerate for each point $x \in A$, then f is a differentiable embedding on some neighbourhood of A. In particular, if we require, in addition to the previous conditions, that $\dim X = \dim Y$ and $f(\partial X) \subset \partial Y$, then f carries a neighbourhood of A diffeomorphically onto a neighbourhood of $f(A)$.*

Proof. Fix for each point $x \in A$ a neighbourhood such that $f|_{U_x}$ is a $\mathcal{C}^r$-embedding (see Theorem 3.1.5.3), and then cover A by a finite number of such neighbourhoods, say $U_{x_1}, \ldots, U_{x_s}$. Since the set $Y \times Y \setminus \operatorname{diag} Y$ is open in $Y \times Y$ (see Remark 1.2.2.4), its pre-image W under the map $f \times f\colon X \times X \to Y \times Y$ is also open. But f is injective on A; whence $W \cup [\cup_{i=1}^{s}(U_{x_i} \times U_{x_i})]$ contains $A \times A$, and is actually a neighbourhood of $A \times A$. Next introduce a metric on X (see Theorem 3.1.1.13, get the corresponding metric on $X \times X$ (see Theorem 1.2.2.9), and then set

$$B = \{x \in X |\operatorname{Dist}(A, x) < \operatorname{Dist}((X \times X) \setminus W, A \times A)/2\}.$$

Since B is open and contains A (see Theorem 1.1.7.15), the intersection $B \cap (\cup_{i=1}^{s} U_{x_i})$ is also open and contains A. This intersection contains a relatively compact neighbourhood U of A, because A is compact. Moreover, $B \times B \subset W$, and so f is injective on B. We conclude that $f|_{\operatorname{Cl} U}$ is a topological embedding and $f|_U$ - a differentiable embedding (see Corollary 3.1.5.4). □

Submersions

Remark 3.1.5.6. A smooth map $f\colon X \to Y$ of smooth manifolds is a *submersion* if:

(i) $d_x f\colon \operatorname{Tang}_x X \to \operatorname{Tang}_{f(x)} Y$ is an epimorphism for any $x \in X$;

(ii) $d_x f(\operatorname{Tang}_x \partial X) = \operatorname{Tang}_{f(x)} Y$ for any point $x \in \partial X \cap f^{-1}(\operatorname{int} X)$;

(iii) $\partial X \cap f^{-1}(\operatorname{int} Y)$ is a union of whole components of the manifold ∂X.

We remark that condition (i) implies $\dim X \geq \dim Y$, and that conditions (ii) and (iii) are automatically fulfilled whenever $\partial X = \varnothing$.

As examples, consider smooth real functions $f \colon X \to \mathbb{R}$. As in the classical calculus, a point $x \in X$ and the corresponding value $f(x)$ are said to be *critical* for f if $d_x f = 0$. It it clear that f is a submersion if and only if the functions f and $f|_{\partial X}$ have no critical points.

As additional examples of submersions we cite the projections of the product of two smooth manifolds (one of them being without boundary) onto its factors.

Theorem 3.1.5.7. *A map f of a smooth m-dimensional manifold X into a smooth n-dimensional manifold Y is a submersion of class $\mathcal{C}^r$ if and only if for each point $x \in X$ there are charts $\varphi \in \mathrm{Atl}_x \, \mathcal{C}^r X$ and $\psi \in \mathrm{Atl}_{f(x)} \, \mathcal{C}^r Y$ such that the following holds:*

- $f(\operatorname{supp} \varphi) \subset \operatorname{supp} \psi$;
- *the pair* $(\operatorname{im} \varphi, \operatorname{im} \psi)$ *coincides with one of the pairs*
  - $(\mathbb{R}^m, \mathbb{R}^n)$,
  - $(\mathbb{R}^m_-, \mathbb{R}^n)$, *or*
  - $(\mathbb{R}^m_-, \mathbb{R}_-)$,

  and in each case $\varphi(x) = 0$ and $\psi(f(x)) = 0$; the corresponding local representative $\mathrm{loc}(\varphi, \psi) f$ can be described,
  - *in the first case, as the projection of the product $\mathbb{R}^n \times \mathbb{R}^{m-n}$ onto its first factor,*
  - *in the second case - as the projection of the product $\mathbb{R}^{m-n}_- \times \mathbb{R}^n$ onto its second factor,*
  - *and in the third case - as the projection of the product $\mathbb{R}^n_- \times \mathbb{R}^{m-n}$ onto its first factor.*

Proof. The sufficiency of this condition is obvious. Let us prove its necessity. Let $\varphi^1 \in \mathrm{Atl}_X X$ and $\psi^1 \in \mathrm{Atl}_{f(x)} Y$ be arbitrary charts such that $f(\operatorname{supp} \varphi^1) = \operatorname{supp} \psi^1$, $\varphi^1(x) = 0$, and $\psi^1(f(x)) = 0$. We denote by $\varphi^1_1, \ldots, \varphi^1_m$ and $\psi^1_1, \ldots, \psi^1_m$ the corresponding coordinate functions, and by $\ell_1, \ldots, \ell_n$ the coordinate functions of the map $\mathrm{loc}(\varphi^1, \psi^1) f$. We consider three distinct cases:

a) $x \in \operatorname{int} X$, $f(x) \in \operatorname{int} Y$;

b) $x \in \partial X$, $f(x) \in \operatorname{int} Y$; and

c) $f(x) \in \partial Y$.

Condition 3.1.5.6 (i) says that the Jacobi matrix of the map $\mathrm{loc}(\varphi^1, \psi^1) f$, computed at the point 0, has rank n in each of the three cases. Condition 3.1.5.6 (ii) says that in case b) this rank does not decrease when we remove from the matrix the first column. Therefore, in cases a) and c) we may assume that the minor constructed from the first n columns does not vanish, while in case b) the same is true for the minor constructed from the last n columns. Moreover, in case c), 0 is a boundary point of $\operatorname{im}\varphi^1$ in $\mathbb{R}^m$, and the function ℓ^1 vanishes on $\varphi^1(\partial X \cap \operatorname{im}\varphi^1)$ in a neighbourhood of this point. Indeed, the first part of the last assertion follows from the fact that is a non-positive function, vanishes at 0, and has non-zero gradient at 0; now the second part of the assertion is seen to be a consequence of 3.1.5.6 (iii).

We pass from φ^1, ψ^1 to the required charts φ, ψ through the intermediary charts $\varphi^2 \in \operatorname{Atl}_x \mathcal{C}^r X$ and $\psi^2 \in \operatorname{Atl}_{f(x)} \mathcal{C}^r Y$. In cases a) and c), φ^2 is the chart whose local coordinates are the restrictions of the functions

$$\ell_1 \circ \varphi^1, \dots, \ell_n \circ \varphi^1, \varphi^1_{n+1}, \dots \varphi^1_m$$

to a small enough neighbourhood U_2 of the point x, while ψ^2 is the chart $\operatorname{abr}\psi^1 \colon V_2 \to \psi^1(V_2)$, where $V_2 = f(U_2)$. In case b), φ^2 and ψ^2 are similarly defined,

$$\ell_1 \circ \varphi^1, \dots, \ell_n \circ \varphi^1, \varphi^1_{n+1}, \dots \varphi^1_m$$

by the functions

$$\varphi^1_1, \dots \varphi^1_{m-n}, \ell_1 \circ \varphi^1, \dots, \ell_n \circ \varphi^1.$$

In all cases $f(U_2) = V_2$ and obviously in cases a) and c), the map $\operatorname{loc}(\varphi^2, \psi^2)f$ is given in the new coordinates $\varphi^2_1, \dots \varphi^2_m$ and $\psi^2_1, \dots \psi^2_n$ by the formulae $\psi^2_1 = \varphi^2_1, \dots, \psi^2_n = \varphi^2_n$, whereas in case b) the corresponding formulae are $\psi^2_1 = \varphi^2_{m-n+1}, \dots, \psi^2_n = \varphi^2_m$. Fix a positive ε and define the subsets U and V of U_2 and V_2 by the inequalities

$$|\varphi^2_i| < \varepsilon \quad (i = 1, \dots, m) \quad \text{and} \quad |\psi^2_j| < \varepsilon \quad (j = 1, \dots, n)$$

respectively. It is clear that for ε small enough, the charts φ, ψ with $\operatorname{supp}\varphi = U$, $\operatorname{supp}\psi = V$, and local coordinates

$$\varphi_i(y) = \frac{\varphi^2_i(y)}{\varepsilon - |\varphi^2_i|} \quad (i = 1, \dots, m), \quad \psi_j(z) = \frac{\psi^2_j(y)}{\varepsilon - |\psi^2_j|} \quad (j = 1, \dots, n)$$

have the desired properties. □

Corollary 3.1.5.8. *If $f \colon X \to Y$ is a submersion, then $f(\operatorname{int} X) \subset \operatorname{int} Y$, $f^{-1}(\partial Y) \subset \partial X$, and the maps $\operatorname{abr} f \colon \operatorname{int} X \to \operatorname{int} Y$ and $\operatorname{abr} f \colon f^{-1}(\partial Y) \to \partial X$ are submersions.*

If $f \colon X \to Y$ is a submersion of class $\mathcal{C}^r$, then $f^{-1}(y)$ is a neat $\mathcal{C}^r$-submanifold of X for $y \in \operatorname{int} Y$, and a neat $\mathcal{C}^r$-submanifold of ∂X for $y \in \partial Y$.

Every submersion is an open map.

The composition of two submersions is a submersion.

Theorem 3.1.5.9. *A $\mathcal{C}^r$-map $f \colon X \to Y$ satisfying condition 3.1.5.6 (iii) is a submersion if and only if for each point $x_0 \in X$ there is a neighbourhood V of the point $f(x_0)$ and a $\mathcal{C}^r$-map $g \colon V \to X$, such that $f(g(y)) = y$ for all $y \in V$ and $g(f(x_0)) = x_0$.*

Proof. The sufficiency of this condition is clear, its necessity results from Theorem 3.1.5.7. ☐

Theorem 3.1.5.10. *Let $f\colon X \to Y$ be a submersion of class $\mathcal{C}^r$ such that $f(X) = Y$, and let h be a map of Y into a third manifold. If the composition $h \circ f$ is of class $\mathcal{C}^r$, then h is of class $\mathcal{C}^r$ too.*

Proof. According to Theorem 3.1.5.9, one can find for each point of Y a neighbourhood V and a $\mathcal{C}^r$-map $g\colon V \to X$ such that $f \circ g = [\text{incl}\colon V \to Y]$, and hence $h|_V = (h \circ f) \circ g$. ☐

3.1.6 Complex Structures

Remark 3.1.6.1. Recall that a map of an open subset of $\mathbb{C}^m$ into a subset of $\mathbb{C}^n$ is *holomorphic* if its coordinate functions are holomorphic, and *biholomorphic* if it is invertible and both the map and its inverse are holomorphic. Obviously, in the last case we must have $m = n$; cf. Remark 3.1.2.2.

If we regard $\mathbb{C}^m$ and $\mathbb{C}^n$ as $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$, respectively, then the holomorphic maps become $\mathbb{C}^a$-maps, and the biholomorphic ones - $\mathbb{C}^a$-diffeomorphisms. A smooth map of an open subset of $\mathbb{R}^{2m}$ into a subset of $\mathbb{R}^{2n}$ is holomorphic relative to the complex structures on $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$ resulting from the identifications $\mathbb{R}^{2m} = \mathbb{C}^m$ and $\mathbb{R}^{2n} = \mathbb{C}^n]$ if and only if it satisfies the Cauchy-Riemann conditions. Evidently, a map which is the inverse of a diffeomorphism satisfying the Cauchy-Riemann conditions also satisfies these conditions. Consequently, every holomorphic diffeomorphism is a biholomorphic map.

Remark 3.1.6.2. Suppose that a holomorphic map between subsets of $\mathbb{C}^n$ has a non-degenerate differential at some point. Then it retains the same property when considered as a $\mathcal{C}^a$-map, and hence it maps a neighbourhood of the given point diffeomorphically onto its image. Thus, a holomorphic map whose Jacobian does not vanish at a point maps a neighbourhood of the point biholomorphically onto its image.

This statement is the exact analogue, and also the result of the theorem concerning local inversion of a smooth map in the real case. In a similar fashion, one can translate a more general theorem from the real calculus into the complex language - the implicit function theorem. Namely, let f be a holomorphic map of an open subset A of $\mathbb{C}^m \times \mathbb{C}^n$ into $\mathbb{C}^n$, and let $(z^0, w^0) \in A$ be such that $f(z^0, w^0) = 0$. Suppose that the Jacobian of f with respect to the second variable does not vanish at (z^0, w^0). The complex implicit function theorem states the existence of a neighbourhood U of the point z^0 in $\mathbb{C}^m$, of a neighbourhood V of the point w^0 in $\mathbb{C}^n$, and of a holomorphic map $g\colon U \to V$, such that $U \times V \subset A$ and the pre-image of 0 under $f|_{U\times V}$ is precisely the graph of g.

Remark 3.1.6.3. A linear transformation of $\mathbb{C}^n$ is also linear as a transformation of $\mathbb{R}^{2n}$; hence to each complex $n \times n$-matrix C one can associate a real $2n \times 2n$-matrix R. If $C = A + iB$ and $z = x + iy$ are the decompositions of the matrix C and the vector $z \in \mathbb{C}^n$ into real and imaginary parts, then $Cz = (Ax - By) + i(Bx + Ay)$, and

$$R = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

In particular, $\det R = |\det C|$. Indeed, if we add to the first block-row of R the second, multiplied by i, and then add to the second block-column the first, multiplied by $-i$, we obtain the matrix

$$\begin{bmatrix} C & 0 \\ B & \overline{C} \end{bmatrix},$$

which has the same determinant as R.

Complex Manifolds

Definition 3.1.6.4. An n-dimensional *complex chart* on the set X is an invertible mapping of a subset of X onto an open subset of $\mathbb{C}^n$. Two complex charts, φ and ψ, are compatible if the set $\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi)$ is open in $\operatorname{im}\varpi$, the set $\psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi)$ is open in $\operatorname{im}\psi$, and the composite maps

$$\varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \xrightarrow{\operatorname{abr}\varphi^{-1}} \operatorname{supp}\varphi \cap \operatorname{supp}\psi \xrightarrow{\operatorname{abr}\psi} \psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \quad \text{and}$$
$$\psi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi) \xrightarrow{\operatorname{abr}\psi^{-1}} \operatorname{supp}\varphi \cap \operatorname{supp}\psi \xrightarrow{\operatorname{abr}\varphi} \varphi(\operatorname{supp}\varphi \cap \operatorname{supp}\psi),$$

which are inverses of each other, are holomorphic (here supp is defined as in the real case). If two overlapping charts of dimensions m and n are compatible, then $m = n$.

A collection of complex charts is an *n-dimensional holomorphic atlas of the set X* if these charts cover X, are n-dimensional, and are pairwise compatible. Two holomorphic atlases are ***holomorphically equivalent*** if their union is again an atlas. The family of n-dimensional holomorphic atlases of X is divided into disjoint classes of holomorphically equivalent atlases. These classes are called *n-dimensional complex structures on X.*

Remark 3.1.6.5. Any n-dimensional complex chart may be regarded as a $2n$-dimensional real chart, i.e., a $2n$-dimensional chart in the sense of Remark 3.1.2.3. Furthermore, compatible complex charts yield $\mathcal{C}^a$-compatible charts, holomorphic atlases yield $\mathcal{C}^a$-atlases, and holomorphically equivalent atlases yield $\mathcal{C}^a$-equivalent atlases. Therefore, an n-dimensional complex structure on the set X induces a $2n$-dimensional $\mathcal{C}^a$-structure on X. For us, the most important case occurs when this $\mathcal{C}^a$-structure makes X into a manifold, i.e, when it defines a Hausdorff, second countable topology. A set X equipped with an n-dimensional complex structure enjoying this property is an *n-dimensional complex manifold.*

We let $\operatorname{Atl} X$ denote the complete atlas of the complex manifold X, i.e., the collection of all charts of all atlases of its complex structure.

Definition 3.1.6.6. A continuous map $f\colon X \to Y$ between complex manifolds is *holomorphic* if all its local representatives are holomorphic, i.e., the maps $\varphi(\operatorname{supp}\varphi \cap f^{-1}(\operatorname{supp}\psi)) \to \operatorname{im}\psi$, $x \mapsto \psi(f(\varphi^{-1}(x)))$, constructed by means of the charts $\varphi \in \operatorname{Atl} X$ and $\psi \in \operatorname{Atl} Y$, are holomorphic. A map $f\colon X \to Y$ is *biholomorphic* if it is holomorphic, invertible, and its inverse is also holomorphic. Two complex manifolds which can be transformed one into another by a biholomorphic map are said to be *biholomorphically equivalent.*

If complex manifolds are considered as $\mathcal{C}^a$-manifolds, the holomorphic maps become $\mathcal{C}^a$-maps, and the biholomorphic maps - $\mathcal{C}^a$-diffeomorphisms.

Definition 3.1.6.7. Let A be a subset of an n-dimensional complex manifold X. A is a *k-dimensional submanifold* of X if for each point $x \in A$ there is a chart $\varphi \in \operatorname{Atl} X$ such that $x \in \operatorname{supp}\varphi$ and $\varphi(\operatorname{supp}\varphi \cap A) = \operatorname{im}\varphi \cap \mathbb{C}^k$. The charts $\operatorname{abr}\varphi\colon \operatorname{supp}\varphi \cap A \to \operatorname{im}\varphi \cap \mathbb{C}^k$ derived from the charts $\varphi \in \operatorname{Atl} X$ form a k-dimensional holomorphic atlas of the set A, thus transforming A into a complex manifold.

The notion of independent functions defined in Remark 3.1.2.12 makes sense for the complex case too. Therefore, using the implicit function theorem, we deduce that a subset A of the complex n-dimensional manifold X is a k-dimensional submanifold of X if and only if for each point $x_0 \in A$ there are a neighbourhood U of x_0 in X and holomorphic functions $\varphi_{k+1}, \dots, \varphi_n\colon U \to \mathbb{C}$, independent at x_0, and such that the intersection $A \cap U$ is defined in U by the equations $\varphi_{k+1}(x) = 0, \dots, \varphi_n(x) = 0$; cf. Remark 3.1.2.12.

If the complex manifold X is regarded as a $\mathcal{C}^a$-manifold, then a submanifold remains a submanifold and its $\mathcal{C}^a$-structure induced from the $\mathcal{C}^a$-structure of X is identical to the $\mathcal{C}^a$-structure induced by its own complex structure.

If A is a submanifold of the complex manifold X, then the inclusion $A \to X$ is holomorphic.

A map $f\colon X \to Y$ between complex manifolds is a *holomorphic embedding* if $\operatorname{abr} f\colon X \to f(X)$ is a biholomorphic map of X onto a submanifold of Y. In this case f is the composition of the biholomorphic map $X \to f(X)$ and the inclusion $f(X) \to Y$. We conclude that every holomorphic embedding is a holomorphic map.

Remark 3.1.6.8. Suppose $X_1, \dots, X_s$ are complex manifolds of dimensions $n_1, \dots, n_s$. The products $\varphi_1 \times \cdots \times \varphi_s$ of all charts $\varphi_1 \in \operatorname{Atl} X_i$ form an $(n_1 + \cdots + n_s)$-dimensional holomorphic atlas of the set $X_1 \times \cdots \times X_s$, transforming it into an $(n_1 + \cdots + n_s)$-dimensional complex manifold. Considered as a $\mathcal{C}^a$-manifold, the latter is just the product of the $\mathcal{C}^a$-manifolds $X_1, \dots, X_s$.

Remark 3.1.6.9. The same definitions of tangent vectors, tangent vector spaces, total manifold of tangent vectors, and differential of a map (see Subsection 3.1.4) apply in the complex case. The space $\operatorname{Tang}_x X$ tangent to the n-dimensional complex manifold X at the point x is an n-dimensional complex vector space; the total manifold $\operatorname{Tang} X$ is a $2n$-dimensional complex manifold, and the projection $\operatorname{Tang} X \to X$ is holomorphic. The differential $d_x f$ of a holomorphic map $f\colon X \to Y$ at the point $x \in X$ is a linear mapping $\operatorname{Tang}_x X \to \operatorname{Tang}_{f(x)} Y$

of complex vector spaces, and the differential df is a holomorphic mapping $\operatorname{Tang} X \to \operatorname{Tang} Y$.

Again, the tangent spaces $\operatorname{Tang}_x X$ and the manifold $\operatorname{Tang} X$ may be considered as real vector spaces and as a $\mathcal{C}^a$-manifold, respectively. As one may guess, they coincide with the tangent space and the total manifold of tangent vectors to X, regarded as a $\mathcal{C}^a$-manifold. The differential of a holomorphic map f, regarded as a $\mathcal{C}^a$-map, coincides with the differential of the $\mathcal{C}^a$-map f.

The simplest examples of complex manifolds are the spaces $\mathbb{C}^n$ themselves. We obtain an unlimited supply of additional examples by defining submanifolds of $\mathbb{C}^n$ through systems of equations; cf. Remark 3.1.2.16. However, this method will never produce compact manifolds of positive dimension. In fact,

Theorem 3.1.6.10. *Every compact submanifold of* $\mathbb{C}^n$ *has dimension zero.*

Proof. To convince ourselves that this is true, it is enough to show that on a compact manifold the only holomorphic functions are the constants. This is a straightforward consequence of the well-known theorem stating that a function holomorphic on an open subset of the complex line $\mathbb{C}$ which attains its maximum modulus is constant. Now suppose X is a compact, connected, complex manifold, and $f\colon X \to \mathbb{C}$ is holomorphic. Let c be a value of f such that $|c| = \max|f(w)|$, and let $x \in X$ be such that $f(x) = c$. Then for each chart $\varphi \in \operatorname{Atl} X$ with $x \in \operatorname{supp}\varphi$ and $\operatorname{im}\varphi = \operatorname{int}\mathbb{D}^{2\dim X}$, and for each point $y \in \operatorname{supp}\varphi$, the set of complex numbers z such that $(1-z)\varphi(x) + z\varphi(y) \in \operatorname{im}\varphi$ is an open disc with centre 0 and radius greater than 1. The formula $z \mapsto f(\varphi^{-1}((1-z)\varphi(x)+z\varphi(y)))$ defines a holomorphic function on this disc, which attains its maximum modulus as 0. Since such a function is necessarily a constant, we see that $f(y) = c$ by setting $z = 0$ and $z = 1$. Therefore, the set $f^{-1}(c)$ is open. Because $f^{-1}(c)$ is also closed and non-empty, it is all of X. □

Examples of compact complex manifolds will appear in §3.2.

Manifolds of Complex Origin

Definition 3.1.6.11. Every complex manifold gives rise to a $\mathcal{C}^a$-manifold when we pass from complex to real numbers, as described in Remark 3.1.6.5. A $\mathcal{C}^a$-manifold arising in this way is called a *manifold of complex origin.*

Clearly, the manifolds of complex origin are even-dimensional and have no boundary. They are orientable, and if the original complex structure is known, they receive a canonical orientation, namely that orientation which is positive on the real connected charts which arise from the charts of the complete atlas of the complex structure when we pass to the reals. (Using Remark 3.1.6.3 we see that the compatibility condition required in Remark 3.1.3.1 is satisfied.) It is also clear that the products of manifolds of complex origin are again manifolds of complex origin.

3.1.7 Exercises

Exercise 3.1.7.1. Show that a non-empty, closed, smooth manifold of dimension $n > 0$ cannot be immersed in $\mathbb{R}^n$.

Exercise 3.1.7.2. Show that the equation $z_1^2 + \cdots + z_n = 1$ defines a submanifold of $\mathbb{C}^n$ which is $\mathcal{C}^a$-diffeomorphic to $\operatorname{Tang} \mathbb{S}^{n-1}$.

Exercise 3.1.7.3. Show that the map $f \colon \mathbb{S}^2 \to \mathbb{R}^4$, defined by the formula

$$f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1 x_2, x_1 x_3, x_2 x_3),$$

is an immersion and that $f(\mathbb{S}^2)$ is a submanifold of $\mathbb{R}^4$ diffeomorphic to $\mathbb{R}P^2$.

Exercise 3.1.7.4. Show that if one of the numbers $n_1, \ldots, n_s$ is odd and $s > 1$, then the manifold $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_s}$ is parallelisable.

3.2 STIEFEL AND GRASSMANN MANIFOLDS

3.2.1 Stiefel Manifolds

Remark 3.2.1.1. We denote by $\mathbb{R}V(n,k)$, or simply by $V(n,k)$ $(0 \le k \le n)$, the set of linear isometric maps $\mathbb{R}^k \to \mathbb{R}$. Such a map is uniquely determined by the images of the vectors $\mathrm{ort}_1, \dots, \mathrm{ort}_k \in \mathbb{R}^k$, i.e, by an orthonormal k-frame in $\mathbb{R}^n$. The coordinates of the vectors of this frame form the matrix of the map, which has n rows and k columns. In this way, $V(n,k)$ can be interpreted as the set of orthonormal k-frames in $\mathbb{R}^n$, or as the set of the $n \times k$-matrices $\|v_{si}\|$ such that

$$\sum_{s=1}^{n} v_{si} v_{sj} = \delta_{ij} \quad (1 \le i \le j \le k). \tag{3.2.1.2}$$

We may regard a matrix $\|v_{si}\|$ as a point of $\mathbb{R}^{nk}$, if we index its entries v_{si} in dictionary order. Thus $V(n,k)$ becomes the subset of $\mathbb{R}^{nk}$ defined by the equations (3.2.1.2). An easy computation shows that the gradients of the left-hand sides of 3.2.1.2) do not vanish and are pairwise orthogonal on this subset. Hence $V(n,k)$ is an $[nk - k(k+1)/2]$-dimensional $\mathcal{C}^a$-submanifold without boundary of $\mathbb{R}^{nk}$ (see Remark 3.1.2.12). $V(n,k)$ is called the *Stiefel manifold*.

Clearly, $V(n,0)$ reduces to a point, $V(n,1)$ is just the sphere $\mathbb{S}^n$, and $V(n,2)$ is the submanifold of all vectors of unit length in $\operatorname{Tang} \mathbb{S}^{n-1}$.

Remark 3.2.1.3. The points of $V(n,n)$ are orthogonal transformations of $\mathbb{R}^n$, or orthogonal matrices of order n, and $V(n,n)$ is usually denoted by $\mathrm{O}(n)$. The composition of transformation (multiplication of matrices) induces a group structure on $\mathrm{O}(n)$. The subgroup of $\mathrm{O}(n)$ consisting of all matrices with determinant $+1$ is denoted by $\mathrm{SO}(n)$. The two sets $SO(n)$ and $\mathrm{O}(n) \setminus \mathrm{SO}(n)$ are open in $\mathrm{O}(n)$), and hence are $\mathcal{C}^a$-manifolds. They are actually $\mathcal{C}^a$-diffeomorphic: multiplication by an arbitrary matrix from $\mathrm{O}(n) \setminus \mathrm{SO}(n)$ establishes a diffeomorphism. Moreover, the manifold $\mathrm{SO}(n)$ is canonically $\mathcal{C}^a$-diffeomorphic to $V(n, n-1)$: a matrix from $V(n, n-1)$ is carried by this diffeomorphism into a matrix from $\mathrm{SO}(n)$ through the addition of a column; that is to say, we complete each orthonormal $(n-1)$-frame in $\mathbb{R}^n$ to a positive orthonormal n-frame.

We further note that $\mathrm{SO}(2) = V(2,1) = \mathbb{S}^1$, and the group structure on $\mathrm{SO}(2)$ agrees with the group structure on the circle $\mathbb{S}^1$, considered as the multiplicative group of complex numbers of modulus 1.

Remark 3.2.1.4. The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+q}$ induces a $\mathcal{C}^a$-embedding $V(n,k) \to V(n+q,k)$, which transforms each map $\varphi\colon \mathbb{R}^k \to \mathbb{R}^n$ into the composite map

$$\mathbb{R}^k \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{\text{incl}} \mathbb{R}^{n+q}.$$

There is also a canonical $\mathcal{C}^a$-embedding $V(n,k) \to V(n+q, k+q)$, which transforms $\varphi\colon \mathbb{R}^k \to \mathbb{R}^n$ into the map

$$\mathbb{R}^{k+q} = \mathbb{R}^k \times \mathbb{R}^q \xrightarrow{\varphi \times \mathrm{id}} \mathbb{R}^n \times \mathbb{R}^q = \mathbb{R}^{n+q}$$

Finally, the inclusion $\mathbb{R}^{k-q} \to \mathbb{R}^k$ induces a $\mathcal{C}^a$-submersion

$$V(n,k) \to V(n,k-q),$$

which transforms each map $\varphi\colon \mathbb{R}^k \to \mathbb{R}^n$ into the composite map

$$\mathbb{R}^{k-q} \xrightarrow{\text{incl}} \mathbb{R}^k \xrightarrow{\varphi} \mathbb{R}^n,$$

i.e, each frame $(v_1, \ldots, v_k)$ is taken into the frame $(v_1, \ldots, v_{k-q})$. Using Theorem 3.1.5.9, it is readily seen that this is indeed a submersion, because given any frame $(v_1^0, \ldots, v_k^0) \in V(n,k)$ we have an explicit construction of the neighbourhood V of the frame $(v_1^0, \ldots, v_{k-q}^0) \in V(n, k-q)$ and of the $\mathcal{C}^a$-map $g\colon V \to V(n,k)$ which are required by this Theorem. In fact, one can take as V the set of frames $((v_1, \ldots, v_{k-q}) \in V(n, k-q)$ such that the vectors

$$v_1, \ldots, v_{k-q}, v_{k-q+1}^0, \ldots, v_k^0$$

are linearly independent and then, for $(v_1, \ldots, v_{k-q}) \in V$ define $g((v_1, \ldots, v_{k-q})$ to be the frame obtained from

$$(v_1, \ldots, v_{k-q}, v_{k-q+1}^0, \ldots, v_k^0)$$

through standard orthogonalisation. Let us add that the pre-image of an arbitrary frame $(v_1^0, \ldots, v_{k-q}^0) \in V(n, k-q)$ under this submersion is the submanifold of $V(n,k)$ consisting of all the frames

$$(v_1^0, \ldots, v_{k-q}^0, v_{k-q+1}^0, \ldots, v_k^0),$$

where $(v_{k-q+1}^0, \ldots, v_k^0)$ is an orthonormal q-frame of the $(n-k+q)$-dimensional subspace of $\mathbb{R}^n$ which is orthogonal to the vectors $v_1^0, \ldots, v_{k-q}^0$; in particular, this submanifold is diffeomorphic to $V(n-k+q, q)$.

Remark 3.2.1.5. We see from equations (3.2.1.2) that the set $V(n,k)$ is bounded and closed in $\mathbb{R}^{nk}$. Therefore, $V(n,k)$ is a closed manifold.

The manifold $V(n, n-1) = \mathrm{SO}(n)$ is connected: each matrix of $\mathrm{SO}(n)$ can be expressed as $cu(\varphi_1, \ldots, \varphi_r)c^{-1}$, where

$$u(\varphi_1, \ldots, \varphi_r) = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 & & & & & & \\ \sin\varphi_1 & \cos\varphi_1 & & & & & & \\ & & \ddots & & & 0 & & \\ & & & \cos\varphi_r & -\sin\varphi_r & & & \\ & & & \sin\varphi_r & \cos\varphi_r & & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$

(with $\varphi_1, \ldots, \varphi_r \in \mathbb{R}$) and c is an orthogonal matrix. Therefore, each matrix can be joined to the identity matrix by the path

$$t \mapsto cu((1-t)\varphi_1, \ldots, (1-t)\varphi_r)c^1.$$

Since the manifolds $V(n,k)$ with $k < n-1$ are the images under continuous maps of $V(n,n-1)$ (see Remark 3.2.1.4), they are also connected. For $n > 0$, $V(n,n)$ has two connected components: $\mathrm{SO}(n)$ and $\mathrm{O}(n) \setminus \mathrm{SO}(n)$.

The Complex Case

Remark 3.2.1.6. Let $\mathbb{C}V(n,k)$, $0 \leq k \leq n$, be the set of linear isometric maps $\mathbb{C}^k \to \mathbb{C}^n$. In other words, $\mathbb{C}V(n,k)$ consists of the orthonormal k-frames in $\mathbb{C}^n$ or, equivalently, of the complex $n \times k$-matrices $\|v_{sj}\|$ such that

$$\sum_{s=1}^{n} v_{si}\overline{v}_{sj} = \delta_{ij} \quad (1 \leq i \leq j \leq k).$$

These equations show that $\mathbb{C}V(n,k)$ is a subset of $\mathbb{C}^{nk} = \mathbb{R}^{2nk}$. Now we separate their real and imaginary parts and obtain k^2 real equations such that the gradients of their left-hand sides do not vanish and are pairwise orthogonal on $\mathbb{C}V(n,k)$. Thus $\mathbb{C}V(n,k)$ is a $(2nk-k^2)$-dimensional $\mathcal{C}^a$-submanifold without boundary of $\mathbb{R}^{2nk}$, called the *complex Stiefel manifold*.

Warning: $\mathbb{C}V(n,k)$ is not a complex manifold in the sense of Remark 3.1.6.5. The present definition does not equip it with a complex structure; in fact, such a structure does not exist in general, since $\mathbb{C}V(n,k)$ is odd-dimensional for k odd.

Clearly, $\mathbb{C}V(n,0)$ reduces to a point, and $\mathbb{C}V(n,1)$ is just the sphere $\mathbb{S}^{2n-1}$. he points of the manifold $\mathbb{C}V(n,n)$ are unitary transformations of $\mathbb{C}^n$, and $\mathbb{C}V(n,n)$ is usually denoted by $\mathrm{U}(n)$. Like $\mathrm{O}(n)$, $\mathrm{U}(n)$ is a group under the composition operation $\circ$. The subgroup of $\mathrm{U}(n)$ consisting of all matrices with determinant 1 is denoted by $\mathrm{SU}(n)$ and is a $\mathcal{C}^a$-submanifold of $\mathrm{U}(n)$, canonically diffeomorphic to $\mathbb{C}V(n,n-1)$ (cf. Remark 3.2.1.3).

The manifold $\mathrm{U}(n)$ is canonically diffeomorphic to $\mathrm{SU}(n) \times \mathbb{S}^1$: this diffeomorphism takes each pair $(u,z) \in \mathrm{SU}(n) \times \mathbb{S}^1$ into the matrix obtained from u by multiplying its first row by z.

Warning: this diffeomorphism is *not* a group isomorphism between the direct product of groups $\mathrm{SU}(n) \times \mathbb{S}^1$ and $\mathrm{U}(n)$.

The $\mathcal{C}^a$-embeddings

$$\mathbb{C}V(n,k) \to \mathbb{C}V(n+q,k) \quad \text{and} \quad \mathbb{C}V(n,k) \to \mathbb{C}V(n+q,k+1),$$

and the $\mathcal{C}^a$-submersion

$$\mathbb{C}V(n,k) \to \mathbb{C}V(n,k-q)$$

are defined exactly as in the real case. Moreover, since each linear isometric map $\mathbb{C}^k \to \mathbb{C}^n$ may be regarded as a linear isometric map $\mathbb{R}^{2k} \to \mathbb{R}^{2n}$, there is a canonical $\mathcal{C}^a$-embedding $\mathbb{C}V(n,k) \to \mathbb{R}V(2n,2k)$.

The manifolds $\mathbb{C}V(n,k)$ are compact. Moreover, they are all connected. Indeed, $\mathbb{C}V(n,k) = \mathrm{U}(n)$ is connected, because every unitary matrix can be expressed as $cu(\varphi_1, \dots, \varphi_n)c^{-1}$, where

$$u(\varphi_1, \dots, \varphi_n) = \begin{bmatrix} \exp(i\varphi_1) & & 0 \\ & \ddots & \\ 0 & & \exp(i\varphi_n) \end{bmatrix} \qquad (\varphi_1, \dots, \varphi_n \in \mathbb{R}),$$

and c is a unitary matrix, and then joined to the identity matrix by the path

$$t \mapsto cu((1-t)\varphi_1, \dots, (1-t)\varphi_n).$$

The manifolds $\mathbb{C}V(n,k)$ with $k < n$ are connected as continuous images of $\mathbb{C}V(n,n)$.

The Quaternionic Case

Remark 3.2.1.7. The discussion in Remark 3.2.1.6 can be repeated almost word for word if one replaces the field of complex numbers by the skew field of quaternions.

(Recall that $\mathbb{H}^n$ is considered as a left vector space; see Definition 1.2.5.4; consequently, a linear map $\mathbb{H} \to \mathbb{H}^n$ is left-linear here, and the scalar product of the vectors $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ is defined as $\sum_{i=1}^n u_i \bar{v}_i$).

In this way, we obtain:

- the $\mathcal{C}^a$-manifold without boundary $\mathbb{H}V(n,k)$ $(0 \leq k \leq n)$ of dimension $4nk - (2k^2 - k)$, called the *quaternionic Stiefel manifold*;
- the $\mathbb{C}^a$-embeddings $\mathbb{H}V(n,k) \to \mathbb{H}V(n+q,k)$, $\mathbb{H}V(n,k) \to \mathbb{H}V(n+q,k+q)$, and $\mathbb{H}V(n,k) \to (V(2n,2k)$; and
- the $\mathcal{C}^a$-submersion $\mathbb{H}V(n,k) \to \mathbb{H}V(n,k-q)$.

Clearly, $\mathbb{H}V(n,0)$ reduces to a point, and $\mathbb{H}V(n,1)$ is just $\mathbb{S}^{4n-1}$. The manifold $\mathbb{H}V(n,n)$ is usually denoted by $\mathrm{Sp}(n)$ its points are the linear isometric transformations of $\mathbb{H}^n$, and the composition of transformations makes $\mathrm{Sp}(n)$ into a group.

All the manifolds $\mathbb{H}V(n,k)$ are compact and connected. Since the proof of connectedness along the lines in Remarks 3.2.1.5 or 3.2.1.6 requires the normal form of a matrix form $\mathrm{Sp}(n)$, which is less known than the normal forms for $\mathrm{SO}(n)$ and $\mathrm{U}(n)$, we remark that a different proof of connectedness is given in Chapter 5 (see Corollary 5.2.7.4).

Non-compact Stiefel Manifolds

Remark 3.2.1.8. Let $\mathbb{R}V'(n,k)$, or simply $V'(n,k)$, $0 \leq k \leq n$, denote the set of linear monomorphisms $\mathbb{R}^k \to \mathbb{R}^n$. Alternatively, one may describe $V'(n,k)$ as the set of non-degenerate k-frames in $\mathbb{R}^n$, or as the set of the real $n \times k$-matrices of rank k. Clearly, this set is open in the space $\mathbb{R}^{nk}$ of all real $n \times k$-matrices, and hence $V'(n,k)$ is an nk-dimensional $\mathcal{C}^a$-manifold containing $V(n,k)$ as a submanifold.

Let $T(k,\mathbb{R})$, or simply $T(k)$, be the set of real upper triangular matrices of order k with positive diagonal entries. Trivially, $T(k)$ is open in the space of all upper triangular matrices of order k, $\mathbb{R}^{k(k+1)/2}$, and is actually diffeomorphic to $\mathbb{R}^{k(k+1)/2}$. If we orthogonalise a given non-degenerate k-frame in $\mathbb{R}^n$ via the standard procedure, then the matrix corresponding to this frame takes the form ut, where $u \in V(n,k)$ and $t \in T(k)$. Moreover, it is obvious that this representation is unique and defines a diffeomorphism $V'(n,k) \to V(n,k) \times T(k)$, transforming $V(n,k)$ into the fibre $V(n,k) \times E$, where E is the identity matrix. Therefore, the manifold $V'(n,k)$ is $\mathcal{C}^a$-diffeomorphic to $V(n,k) \times \mathbb{R}^{k(k+1)/2}$, and $V(n,k)$ is its strong deformation retract. In particular, $V'(n,k)$ is connected for $k < n$, and $V'(n,n)$ has two components.

Again, $V'(n,0)$ reduces to a point, while $V'(n,1)$ coincides with $\mathbb{R}^n \setminus 0$. The manifold $V'(n,n)$ is usually denoted by $\mathrm{GL}(n,\mathbb{R})$ and its points are the non-degenerate linear transformations of $\mathbb{R}^n$ or, equivalently, the non-degenerate matrices of order n. The composition of transformations (multiplication of matrices) defines a group structure on $\mathrm{GL}(n,\mathbb{R})$. The subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of all matrices with positive determinant is denoted by $\mathrm{GL}_+(n,\mathbb{R})$. The sets $GL_+(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{R}) \setminus \mathrm{GL}_+(n,\mathbb{R})$ are open in $\mathrm{GL}(n,\mathbb{R})$ and $\mathcal{C}^a$-diffeomorphic to $\mathrm{SO}(n) \times \mathbb{R}^{n(n+1)/2}$; in fact, they are the components of the manifold $\mathrm{GL}(n,\mathbb{R})$.

Corresponding to each monomorphism $\varpi\colon \mathbb{R}^k \to \mathbb{R}^n$ we have the composite maps

$$\begin{aligned}
&\mathbb{R}^k \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{\text{incl}} \mathbb{R}^{n+q},\\
&\mathbb{R}^{k+q} \xrightarrow{\text{id}} \mathbb{R}^{k+q} \xrightarrow{\varphi\times\text{id}} \mathbb{R}^n \times \mathbb{R}^q \xrightarrow{\text{id}} \mathbb{R}^{n+q} \quad \text{and}\\
&\mathbb{R}^{k-q} \xrightarrow{\text{incl}} \mathbb{R}^k \xrightarrow{\varphi} \mathbb{R}^n,
\end{aligned}$$

and so we obtain the $\mathcal{C}^a$-embeddings

$$\begin{aligned}
&V'(n,k) \to V'(n+q,k),\\
&V'(n,k) \to V'(n+q,k+q),
\end{aligned}$$

and the $\mathcal{C}^a$-submersion

$$V'(n,k) \to V'(n,k-q),$$

respectively; cf. Remark 3.2.1.4.

Remark 3.2.1.9. We let $\mathbb{C}V'(n,k)$ $(0 \le k \le n)$ denote the set of linear monomorphisms $(\mathbb{C}^k \to \mathbb{C}^n$. Alternatively, $\mathbb{C}V'(n,k)$ is the set of non-degenerate k-frames in $\mathbb{C}^n$, or the set of complex $n \times k$-matrices of rank k. Again, it it clear that this is an open set in the space $\mathbb{C}^{nk}$ of all complex $n \times k$-matrices. Thus $\mathbb{C}V'(n,k)$ is a $2nk$-dimensional $\mathcal{C}^a$-manifold containing $\mathbb{C}V(n,k)$ as a submanifold.

Repeating what was said in Remark 3.2.1.8 (with obvious modifications), we obtain a $\mathcal{C}^a$-diffeomorphism $\mathbb{C}V'(n,k) \to \mathbb{C}V(n,k) \times T(k,\mathbb{C})$, where $T(k,\mathbb{C})$ is the manifold of complex upper triangular matrices of order k with positive diagonal elements. Clearly, $T(k,\mathbb{C})$ is $\mathcal{C}^a$-diffeomorphic $\mathbb{R}^{k^2}$, and hence $\mathbb{C}V'(n,k)$ is diffeomorphic to $\mathbb{C}V(n,k)\times\mathbb{R}^{k^2}$. In particular, the manifolds $\mathbb{C}V'(n,k)$ are all connected. $\mathbb{C}V'(n,0)$ reduces to a point, while $\mathbb{C}V'(n,1)$ is just $\mathbb{C}^n\setminus 0$. Usually the manifold $\mathbb{C}V'(n,n)$ is denoted by $\mathrm{GL}(n,\mathbb{C})$; it consists of all non-degenerate linear transformations of $\mathbb{C}^n$, and is a group with group operation $\circ$. There are also the natural $\mathcal{C}^a$-embeddings

$$\begin{aligned}\mathbb{C}V'(n,k)&\to\mathbb{C}V'(n+q,k),\\ \mathbb{C}V'(n,k)&\to\mathbb{C}V'(n+q,k+q),\end{aligned}$$

as well as the $\mathcal{C}^a$-submersion

$$\mathbb{C}V'(n,k)\to\mathbb{C}V'(n,k-q).$$

Remark 3.2.1.10. If we replace the commutative fields $\mathbb{R}$ and $\mathbb{C}$ by the skew field $\mathbb{H}$ in the previous definitions, we obtain:

- the $4nk$-dimensional $\mathcal{C}^a$-manifold $\mathbb{H}V'(n,k)$ of all linear monomorphisms $\mathbb{H}^k\to\mathbb{H}^n$;
- the $\mathcal{C}^a$-diffeomorphism $\mathbb{H}V'(n,k)\to\mathbb{H}V(n,k)\times\mathbb{R}^{2k^2-k}$;
- the $\mathcal{C}^a$-embeddings

$$\begin{aligned}\mathbb{H}V'(n,k)&\to\mathbb{H}V'(n+q,k)\quad\text{and}\\ \mathbb{H}V'(n,k)&\to\mathbb{H}V'(n+q,k+q)\end{aligned}$$

- and finally, the $\mathcal{C}^a$-submersion $\mathbb{H}V'(n,k)\to\mathbb{H}V'(n,k-q)$.

As before, $\mathbb{H}V'(n,0)$ reduces to a point, while $\mathbb{H}V'(n,1)$ is just $\mathbb{H}\setminus 0$. The manifold $\mathbb{H}V'(n,n)$ consists of all non-degenerate linear transformations of $\mathbb{H}^n$, is a group with group operation $\circ$, and is usually denoted by $\mathrm{GL}(n,\mathbb{H})$.

3.2.2 Grassmann Manifolds

Remark 3.2.2.1. Let $\mathbb{R}G(n,k)$, or simply $G(n,k)$, $0\le k\le n$, be the set of k-dimensional linear subspaces (or k-planes passing through 0) of $\mathbb{R}^n$. If γ is such a plane, we let U_γ denote the collection of planes in $G(n,k)$ whose projection on γ is non-degenerate. Let us fix an orthonormal basis $\mathbf{e}=\{e_1,\dots,e_k\}$ of γ and complete it to an orthonormal basis of $\mathbb{R}^n$, adding a frame $\varepsilon=\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$. For each $\gamma'\in U_\gamma$ there exists a unique frame $u_1,\dots,u_k$ in γ' which projects onto $\mathbf{e}$. Expressing the vectors $u_1,\dots,u_k$ in terms of the basis $e_1,\dots,e_k,\varepsilon_1,\dots,\varepsilon_{n-k}$, we obtain

$$u_i=e_i+\sum_{s=1}^{n-k}c_{is}\varepsilon_s\quad(i=1,\dots,k;\quad c_{is}\in\mathbb{R})$$

This construction yields a map $\varphi(e,\varepsilon)$ from the set U_γ into the space $\mathbb{R}^{k(n-k)}$ of real $k\times(n-k)$-matrices. Obviously, $\varphi(e,\varepsilon)$ is invertible, and the collection of maps $\varphi(e,\varepsilon))$ obtained from all the pairs e,ε is a $\mathcal{C}^a$-atlas of the set $G(n,k)$.

Since any two planes in $G(n,k)$ are contained in some set U_γ, the topology defined by the atlas $\{\varphi(e,\varepsilon)\}$ is Hausdorff. Moreover, this atlas contains finite subatlases (for example, the subatlas consisting of the charts $\varphi(e,\varepsilon)$ where e and ε are subframes of the standard frame $\mathrm{ort}_1, \dots, \mathrm{ort}_n$ of $\mathbb{R}^n$), and hence the above topology has a countable base. Thus, the atlas $\{\varphi(e,\varepsilon)\}$ transforms $G(n,k)$ into a $k(n-k)$-dimensional $\mathcal{C}^a$-manifold, called the *Grassmann manifold*.

Clearly, $G(n,0)$ and $G(n,n)$ both reduce to points, and $G(n,1)$, as a set and a topological space, is identical to $\mathbb{R}P^{n-1}$. So we see that the space $\mathbb{R}P^{n-1}$ has a $\mathbb{C}^a$-structure compatible with its topology, and hence $\mathbb{R}P^{n-1}$ is an $(n-1)$-dimensional $\mathbb{C}^a$-manifold. We remark that this $\mathbb{C}^a$-structure may be described directly and conveniently as follows: for $k=1$, the finite atlas of the manifold $G(n,k)$ given above consists of n charts

$$\varphi_1 \colon U_1 \to \mathbb{R}^{n-1}, \dots, \varphi_n \colon U_n \to \mathbb{R}^{n-1},$$

defined in homogeneous coordinates by the formulae

$$U_i = \{(x_1 : \cdots : x_n) | x_i \neq 0\},$$
$$\varphi_i((x_1 : \cdots : x_n)) = (x_1\ x_i, \dots x_{i-1}/x_u, x_{i+1}/x_i, \dots x_n/x_i).$$

It is also clear that relative to this $\mathcal{C}^a$-structure, the canonical homeomorphism $\mathbb{R}P^1 \to \mathbb{S}^1$ (see Remark 1.2.5.6) becomes a $\mathcal{C}^a$-diffeomorphism.

Remark 3.2.2.2. Obviously, we may modify the definition of the manifold $G(n,k)$ by replacing the non-oriented planes with oriented ones. More precisely, $G(n,k)$ is replaced by the set $G_+(n,k)$ of oriented k-dimensional planes (oriented k-planes, for short) of $\mathbb{R}^n$ passing through 0. One has to modify the set U_γ accordingly and take it to be the collection of all planes in $G_+(n,k)$ whose projections onto the plane $\gamma \in G_+(n,k)$ are non-degenerate and orientation preserving. The maps $\varphi(e,\varepsilon) \colon U \to \mathbb{R}^{n(n-k)}$ are defined as in Remark 3.2.2.1 and again they form a $\mathcal{C}^a$-atlas possessing finite subatlases. Since any pair of points of $G_+(n,k)$ is covered by a set of the form $U_\gamma \cup U_{\overline{\gamma}}$, where the plane $\overline{\gamma}$ differs from γ only by its orientation (obviously, $U_\gamma \cup U_{\overline{\gamma}} = \varnothing$) , the topology defined by the atlas $\{\varphi(e,\varepsilon)\}$ is Hausdorff. The $k(n-k)$-dimensional $\mathcal{C}^a$-manifold so obtained is termed the *upper Grassmann manifold*.

The manifolds $G_+(n,0)$ and $G_+(n,n)$ are canonically homeomorphic to $\mathbb{S}^0$, while $G_+(n,1)$ is canonically diffeomorphic to $\mathbb{S}^{n-1}$: under this diffeomorphism each point $x \in \mathbb{S}^{n-1}$ goes into the oriented line defined by the pair of points $0, x$.

Remark 3.2.2.3. If one associates to each plane $\gamma \in G(n,k)$ its orthogonal complement, one obtains a mapping of $G(n,k)$ onto $G(n,n-k)$ which is clearly a $\mathcal{C}^a$-diffeomorphism. A $\mathcal{C}^a$-diffeomorphism $G_+(n,k) \to G_+(n,n-k)$ is similarly defined, if the orientation given to the orthogonal plane $\gamma^\perp$ and the orientation of γ behave in accordance to the classical rule: let a basis of $\gamma^\perp$ which is compatible with this orientation be written to the right of a basis of γ which is compatible with its orientation; then the resulting basis of $\mathbb{R}^n$ should be compatible with the standard orientation of $\mathbb{R}^n$ (see Remark 3.1.3.10). In particular, $G(n,n-1) = \mathbb{R}P^{n-1}$ and $G_+(n,n-1) = \mathbb{S}^{n-1}$.

The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+q}$ induces obvious $\mathcal{C}^a$-embeddings $G(n,k) \to G(n+q,k)$ and $G_+(n,k) \to G_+(n+q,k)$. Furthermore, the formula $\gamma \mapsto \gamma \times \mathbb{R}^q$ defines $\mathcal{C}^a$-embeddings $G(n,k) \to G(n+q,n+k)$ and $G_+(n,k) \to G_+(n+q,n+k)$ (the orientation of the product $\gamma \times \mathbb{R}^q$ is defined by the orientation of its factors - see Remark 3.1.3.7). These embeddings and the previous diffeomorphisms form the following commutative diagrams:

$$\begin{tikzcd}
G(n,k) \arrow[r] \arrow[d] & G(n+q,k) \arrow[d] & G_+(n,k) \arrow[r] \arrow[d] & G_+(n+q,k) \arrow[d] \\
G(n,n-k) \arrow[r] & G(n+q,n-k+q) & G_+(n,n-k) \arrow[r] & G_+(n+q,n-k+q)
\end{tikzcd}$$

The map $G_+(n,k) \to G(n,k)$ which takes each oriented plane and "forgets" its orientation is clearly a $\mathcal{C}^a$-submersion such that the pre-image of each point of $G(n,k)$ consists of just two points. For $k = 1$ we recover the projection $\mathbb{S}^{n-1} \to \mathbb{R}P^{n-1}$.

If to each frame from $V(n,k)$ we associate the oriented plane that it spans, we obtain a $\mathcal{C}^a$-map $V(n,k) \to G_+(n,k)$. This is a submersion, a fact that can be readily checked with the aid of Theorem 3.1.5.9, if for a given frame $v^0 \in V(n,k)$ one explicitly indicates the neighbourhood V of the oriented plane γ^0 spanned by v^0 and the map $g\colon V \to V(n,k)$ which are needed in this theorem. One can take $V = U_{\gamma_0}$ (see Remark 3.2.2.2) and define $g(\gamma)$ for each $\gamma \in V$ to be the frame which is obtained from v^0 after projection on γ and standard orthogonalisation. The pre-image of an oriented plane $\gamma \in G_+(n,k)$ under this submersion is the set of all orthonormal positive k-frames of γ and, in particular, is diffeomorphic to $\mathrm{SO}(k)$.

The map of $V(n,k)$ onto $G(n,k)$ which takes each frame into the non-oriented plane that it spans is also a submersion. In fact, it is exactly the composition of the two previous submersions. The pre-image of a plane $\gamma \in G(n,k)$ consists of all the orthonormal frames of γ and, in particular, is diffeomorphic to $\mathrm{O}(k)$.

Finally, the maps $V'(n,k) \to G_+(n,k)$ and $V'(n,k) \to G(n,k)$, which transform each frame into the plane that it spans are both submersions. The pre-images of the points of $G_+(n,k)$ and $G(n,k)$ under these maps are diffeomorphic to $\mathrm{GL}_+(k,\mathbb{R})$ and $\mathrm{GL}(k,\mathbb{R})$, respectively.

Since the manifolds $G(n,k)$ and $G_+(n,k)$ are continuous images of $V(n,k)$, they are compact and, excepting $G_+(n,0)$ and $G_+(n,n)$, connected.

The family of manifolds $G(n,k)$ $(k \neq 0,n)$ contains both orientable and non-orientable manifolds. More precisely,

Theorem 3.2.2.4. *for $k \neq 0$, $G(n,k)$ is orientable if n is even and non-orientable if n is odd.*

Proof. To see this, use the atlas made of the charts $\varphi(e,\varepsilon)$, where e and ε are complementary subframes of the standard basis of $\mathbb{R}^n$, with the order of vectors they inherit from this basis (cf. Remark 3.2.2.1). Denote the indices of the vectors of e by

$$j_1(e), \ldots, j_k(e) \quad (1 \leq j_1(e) < \cdots < j_k(e) \leq n),$$

and say that two charts $\varphi(e, \varepsilon)$ and $\varphi(e', \varepsilon')$ of the atlas are *contiguous* if there exists ℓ such that $j_\ell(e') = j_\ell(e) \pm 1$ and $j_p(e') = j_p(e)$ for $p \neq \ell$. Obviously, for $k \neq 0, n$ we can always exhibit contiguous charts, and actually any two charts in the atlas may be connected by a finite chain of charts such that each two neighbouring charts in the chain are contiguous.

Suppose that c_{is} and c'_{is} are the coordinate functions of two contiguous charts, $\varphi(e, \varepsilon)$ and $\varphi(e', \varepsilon')$, such that $j_\ell(e') = j_\ell(e) + 1$. Then it is readily seen that the coordinate $c_{\ell m}$ with $m = j_\ell - \ell + 1$ does not vanish on the intersection of the supports of the two charts, but takes all the remaining real values on this intersection. A simple computation shows that

$$c'_{is} = \begin{cases} (c_{is}c_{\ell m} - c_{im}c_{\ell s})c_{\ell m}^{-1}, & \text{if} \quad i \neq \ell, s \neq m, \\ c_{\ell s}c_{\ell m}^{-1}, & \text{if} \quad i = \ell, s \neq m, \\ -c_{im}c_{\ell m}^{-1}, & \text{if} \quad i \neq \ell, s = m, \\ c_{\ell m}^{-1}, & \text{if} \quad i = \ell, s = m, \end{cases}$$

and thus the Jacobian is [see Remark 3.1.3.1]

$$J(\varphi(e, \varepsilon), \varphi(e', \varepsilon')) = (-1)^n c_{\ell m}^{-n}.$$

If n is odd, this Jacobian takes both positive and negative values; hence the manifold $G(n, k)$ is not orientable for such n. For n even, the formula

$$\varphi(e, \varepsilon) \mapsto (-1)^{k[j_1(e) + \cdots + j_k(e)]}$$

defines a map of the atlas under consideration into $\mathbb{S}^0$, which satisfies the compatibility condition in Remark 3.1.3.1: this condition obviously holds for contiguous charts, which in turn implies the compatibility for non-contiguous charts. Hence, for n even $G(n, k)$ is orientable. $\square$

Remark 3.2.2.5. The next constructions produce $\mathcal{C}^a$-embeddings of $G(n, k)$ and $G_+(n, k)$ in Euclidean spaces; see also Exercise 3.2.4.8.

Let us start with $G_+(n, k)$. For a matrix $v \in V'(n, k)$, let $M_{i_1 \cdots i_k}(v)$ be the minor constructed from the rows with indices $i_1, \ldots i_k$, and put $N_{i_1 \cdots i_k}(v) = M_{i_1 \cdots i_k}(v)/\mu(v)$, where $\mu(v)$ is the positive square root of the sum of the squares of all minors having maximal order. The functions $N_{i_1 \cdots i_k} \colon V'(n, k) \to \mathbb{R}$ are clearly analytic. Moreover, if two frames in $V'(n, k)$, v^1 and v^2, span the same k-plane, then $N_{i_1 \cdots i_k}(v^1) = N_{i_1 \cdots i_k}(v^2)$. Therefore, the map

$$N_{i_1 \cdots i_k} \colon V'(n, k) \to \mathbb{R}^{\binom{n}{k}}$$

with coordinate functions $N_{i_1 \cdots i_k}$ is the composition of the canonical submersion $V'(n, k) \to G_+(n, k)$, defined in Remark 3.2.2.3, with a $\mathcal{C}^a$-map

$$g_+ \colon G_+(n, k) \to \mathbb{R}^{\binom{n}{k}}$$

(see Theorem 3.1.5.10). We next show that g_+ is a $\mathcal{C}^a$-embedding.

Using Corollary 3.1.5.4, it suffices to verify that g_+ is an injective immersion. To demonstrate that g_+ is injective, one has to show that if $v^1, v^2 \in V'(n,k)$ and $N(v^1) = N(v^2)$, then the $n \times 2k$-matrix constructed by adjoining the matrices v^1 and v^2 has rank k. But this is plain, because any $(k+1)\times(k+1)$-minor of this $n \times 2k$-matrix, constructed from k columns of v^1 and one column of v^2 is equal to zero (expand the minor with respect to the column of v^2). To verify that g_+ is an immersion, it is enough to show that at each point $\gamma_0 \in G_+(n,k)$, the rank of the differential g_+ is $k(n-k)$. This in turn will be true if we can prove that at each point $v_0 \in V'(n,k)$, the differential $d_{v_0}N$ has rank $\geq k(n-k)$. Finally, to obtain this property of $d_{v_0}N$, we prove that at each point $v_0 \in V'(n,k)$, the differential $d_{v_0}M$ has rank $\geq k(n-k)+1$, where

$$M\colon V'(n,k) \to \mathbb{R}^{\binom{n}{k}}$$

is the map with coordinate functions $M_{i_1 \cdots i_k}$. So let A be any $k \times k$-submatrix of the matrix $v \in V'(n,k)$ such that A is non-degenerate for $v = v_0$, and let $v_{i_0 j_0}$ be an element of A such that its cofactor, call it α, does not vanish for $v = v_0$. Next isolate those minors $M_{i_1 \cdots i_k}(v)$ of the matrix v which have at least $k-1$ rows in common with A, and then form a submatrix of the Jacobi matrix of the map M as follows. Take the derivatives of the chosen minors with respect to those elements of the matrix v which do not appear in A and the one derivative with respect to $v_{i_0 j_0}$. We obtain a square matrix of order $k(n-k)+1$ which has, in a neighbourhood of the point v_0 (and for a suitable arrangement of the rows and columns) the form

$$\begin{bmatrix} (A^t)^{-1}\det A & 0 & 0 & \beta_1 \\ 0 & (A^t)^{-1}\det A & 0 & \\ & & \ddots & \vdots \\ 0 & 0 & (A^t)^{-1}\det A & \beta_{k(n-k)} \\ 0 & & 0 & \alpha \end{bmatrix}$$

where t indicates transposition. The determinant of this matrix is equal to $\alpha(\det A)^{(n-k)(k-1)}$ and therefore does not vanish.

Now let us turn to $G(n,k)$ and compose the embedding g_+ with the map

$$q\colon \mathbb{R}^{\binom{n}{k}} \to \mathbb{R}^{\binom{n}{k}(\binom{n}{k}+1)/2}$$

defined as

$$\begin{aligned} &q(x_1, \ldots, x_{\binom{n}{k}}) = \\ &(x_1^2, x_1x_2, \ldots x_1x_{\binom{n}{k}}, x_1^2, x_2x_3, \ldots, x_2x_{\binom{n}{k}}, \ldots, x_{\binom{n}{k}-1}x_{\binom{n}{k}}, x_{\binom{n}{k}}^2). \end{aligned}$$

Clearly, the restriction $q|_{\mathbb{R}^{\binom{n}{k}}\setminus 0}$ is a $\mathcal{C}^a$-immersion, and $q(x) = q(y)$ if and only if $x = \pm y$. Since the equality $g_+(\gamma') = g_+(\gamma)$ holds for planes $\gamma, \gamma' \in G_+(n,k)$

which coincide geometrically (but have opposite orientations), we see that $q\circ g_+$ is a $\mathbb{C}^a$-immersion of $G_+(n,k)$ into $\mathbb{R}^{\binom{n}{k}(\binom{n}{k}+1)/2}$. Moreover, $q\circ g_+(\gamma') = q\circ g_+(\gamma)$ if and only if γ and γ' coincide or differ only by their orientations. Therefore, the map $\mathrm{fact}(q\circ g_+)\colon G(n,k) \to \mathbb{R}^{\binom{n}{k}(\binom{n}{k}+1)/2}$ is well defined and is a $\mathcal{C}^a$-embedding.

Information 3.2.2.6. The coordinate functions $g^+_{i_1\cdots i_k}$ of the map $g_+\colon G_+(n,k) \to \mathbb{R}^{\binom{n}{k}}$ satisfy the relations

$$g^+_{i_1\cdots i_k} g^+_{j_1\cdots j_k} - \sum_{s=1}^{k} g^+_{i_1\cdots i_{k-1} j_s} g^+_{j_1\cdots j_{s-1} i_k j_{s+1}\cdots j_k} = 0.$$

Considering these relations as equations relative to the coordinates in $\mathbb{R}^{\binom{n}{k}}$, they define a subset of $\mathbb{R}^{\binom{n}{k}}$ which is exactly $g_+(G_+(n,k))$. See [9] for details.

The numbers $g^+_{i_1\cdots i_k}(\gamma)$ are known as the *Grassmann-Plücker coordinates* of the oriented plane γ.

The Complex and Quaternionic Cases

Remark 3.2.2.7. To obtain the complex version of the manifold $\mathbb{R}G(n,k)$, one has to take k-dimensional planes of $\mathbb{C}^n$ passing through 0 instead of k-dimensional planes of $\mathbb{R}^n$ passing through 0. The result is a complex manifold $\mathbb{C}G(n,k)$ of dimension $k(n-k)$, called the *complex Grassmann manifold*.

Obviously, $\mathbb{C}G(n,0))$ and $\mathbb{C}G(n,n)$ reduce to points, whereas $\mathbb{C}G(n,1))$ coincides, as a topological space, with $\mathbb{C}P^{n-1}$. Thus we equip the projective space $\mathbb{C}P^{n-1}$ with a complex structure compatible with its topology, which makes $\mathbb{C}P^{n-1}$ into an $(n-1)$-dimensional complex manifold. This structure may be given in homogeneous coordinates just as we described the $\mathcal{C}^a$-structure of $\mathbb{R}P^{n-1}$]. Relative to this structure the canonical homeomorphism $\mathbb{C}P^1 \to \mathbb{S}^2$ (see Remark 1.2.5.6) becomes a $\mathcal{C}^a$-diffeomorphism.

The complex analogues constructed in Remark 3.2.2.3, are defined in an obvious way:

Real	Complex	The analogue is a
$G(n,k) \to G(n,n-k)$	$\mathbb{C}G(n,k) \to \mathbb{C}G(n,n-k)$	biholomorphic map
$G(n,k) \to G(n+q,k)$	$\mathbb{C}G(n,k) \to \mathbb{C}G(n+q,k)$	holomorphic embedding
$G(n,k) \to G(n+q,k+q)$	$\mathbb{C}G(n,k) \to \mathbb{C}G(n+q,k+q)$	holomorphic embedding
$V(n,k) \to G(n,k)$	$\mathbb{C}V(n,k) \to \mathbb{C}G(n,k)$	$\mathcal{C}^a$-submersion
$V'(n,k) \to G(n,k)$	$\mathbb{C}V'(n,k) \to \mathbb{C}G(n,k)$	$\mathcal{C}^a$-submersion

We mention also the $\mathcal{C}^a$-embedding $\mathbb{C}G(n,k) \to \mathbb{C}G_+(2n,2k)$: when we transform $\mathbb{C}^n$ into $\mathbb{R}^{2n}$, every k-plane becomes an oriented $2k$-plane.

The manifold $\mathbb{C}G(n,k)$ is the image of $\mathbb{C}V(n,k)$ under a continuous map, and as such it is compact and connected. Moreover, it readily seen that $\mathbb{C}G(n,k)$ can be analytically embedded in $\mathbb{R}^{\binom{2n}{2k}}$: the composition of the embeddings

$$\mathbb{C}G(n,k) \to \mathbb{C}G_+(2n,2k) \to \mathbb{R}^{\binom{2n}{2k}}$$

is a $\mathcal{C}^a$-embedding.

Remark 3.2.2.8. Substituting quaternions for complex numbers in these definitions, we obtain a connected $4k(n-k)$-dimensional $\mathcal{C}^a$-manifold $\mathbb{H}G(n,k)$, called the *quaternionic Grassmann manifold*. For $k = 0, n$, $\mathbb{H}G(n,k)$ reduces to a point. For $k = 1, n-1$, $\mathbb{H}G(n,k)$ is topologically the quaternionic projective space $\mathbb{H}P^{n-1}$. Thus $\mathbb{H}P^{n-1}$ becomes a $\mathcal{C}^a$-manifold, and the canonical homeomorphism $\mathbb{H}P^{n-1} \to \mathbb{S}^4$ becomes a $\mathcal{C}^a$-diffeomorphism.

The quaternionic analogues of the maps described in Remarks 3.2.2.3 and 3.2.2.7 are $\mathcal{C}^a$-maps. In particular, there is a canonical $\mathcal{C}^a$-embedding $\mathbb{H}G(n,k) \to G_+(4n,4k)$. Composing it with the canonical embedding $G_+(4n,4k) \to \mathbb{R}^{\binom{4n}{4k}}$, we obtain a $\mathcal{C}^a$-embedding $\mathbb{H}G(n,k) \to \mathbb{R}^{\binom{4n}{4k}}$.

Remark 3.2.2.9. The maps

$$\begin{aligned} \mathbb{S}^{2n-1} &= \mathbb{C}V(n,1) \to \mathbb{C}G(n,1) = \mathbb{C}P^{n-1} \quad \text{and} \\ \mathbb{S}^{4n-1} &= \mathbb{H}V(n,1) \to \mathbb{H}G(n,1) = \mathbb{H}P^{n-1}, \end{aligned}$$

which are particular cases of the canonical maps in Remark 3.2.2.7 and coincide with the corresponding Hopf maps (see Remark 1.2.5.8). As a result, we see that these Hopf maps are $\mathcal{C}^a$-submersions. The Hopf map $\mathbb{S}^{15} \to \mathbb{S}^8$ is a $\mathcal{C}^a$-submersion too, a fact that follows directly from its definition.

Projective Cayley Plane

Remark 3.2.2.10. The projective Cayley plane is also equipped with a $\mathcal{C}^a$-structure which transforms $\mathbb{C}aP^2$ into a $\mathcal{C}^a$-manifold. We next describe this structure.

Identify $\mathbb{R}^{16}$ with $\mathbb{C}a^2$, and $\mathbb{R}^{17}$ with $\mathbb{C}a^2 \times \mathbb{R}$, and then define three maps $\mathbb{D}^{16} \to \mathbb{S}^{16}$ by the formulae

$$\begin{aligned} (y_1, y_2) &\mapsto (2(1-\rho^2)^{1/2} y_1, 2(1-\rho^2)^{1/2} y_2, 2\rho^2 - 1), \\ (y_1, y_2) &\mapsto (2\overline{y}_1 y_2, 2(1-\rho^2)^{1/2}\overline{y}_1, 1 - 2|y_1|^2), \\ (y_1, y_2) &\mapsto (2(1-\rho^2)^{1/2}\overline{y}_2, 2\overline{y}_2 y_1, 1 - 2|y_2|^2) \end{aligned}$$

where $y_1, y_2 \in \mathbb{C}a$ and $\rho = (|y_1|^2 + |y_2|^2)^{1/2}$ (the first map is just $\mathbb{DS}$, but we do not use this fact explicitly). These three maps yield a map

$$F \colon \mathbb{D}^{16} \to \mathbb{S}^{16} \times \mathbb{S}^{16} \times \mathbb{S}^{16},$$

and one can easily verify that

$$\operatorname{zer}(F) = \operatorname{zer}[\operatorname{proj} \colon \mathbb{D}^{16} \to \mathbb{C}aP^2].$$

Since the injective factor of F is a topological embedding of the quotient space $\mathbb{D}^{16}/\operatorname{zer}(F)$ into $\mathbb{S}^{16} \times \mathbb{S}^{16} \times \mathbb{S}^{16}$ (see Theorem 1.1.7.10), and since, according to the last equality, this quotient space coincides with $\mathbb{D}^{16}/\operatorname{zer}(\operatorname{proj}) = \mathbb{C}aP^2$, we can identify $\mathbb{C}aP^2$ with $F(\mathbb{D}^{16})$. To transform $\mathbb{C}aP^2$ into a $\mathcal{C}^a$-manifold, it suffices to show that $F(\mathbb{D}^{16})$ is a $\mathcal{C}^a$-submanifold of $\mathbb{S}^{16} \times \mathbb{S}^{16} \times \mathbb{S}^{16}$.

Define $f, g\colon \mathbb{S}^{16} \setminus \mathrm{ort}_{17} \to \mathbb{S}^{16}$ as

$$f(u_1, u_2, t) = (\frac{u_2\overline{u}_1}{1-t}, \overline{u}_1, \frac{|u_2|^2}{1-t} - t), \quad g(u_1, u_2, t) = (\overline{u}_2, \frac{u_1\overline{u}_2}{1-t}, \frac{|u_1|^2}{1-t} - t),$$
$$u_1, u_2 \in \mathbb{C}a, \quad t \in \mathbb{R},$$

and consider the three maps $h_1, h_2, h_3\colon \mathbb{S}^{16} \setminus \mathrm{ort}_{17} \to \mathbb{S}^{16} \times \mathbb{S}^{16} \times \mathbb{S}^{16}$ given by

$$h_1(x) = (x, f(x), g(x)), \quad h_2(x) = (g(x), x, f(x)), \quad h_3(x) = (f(x), g(x), x).$$

Since f and g are analytic, h_1, h_2 and h_3 are analytic embeddings. One can check directly that

$$F(\mathbb{D}^{16}) \cap \mathrm{proj}_i^{-1}(\mathbb{S}^{16} \setminus \mathrm{ort}_{17}) = h_i(\mathbb{S}^{16} \setminus \mathrm{ort}_{17}).$$

Therefore, the left-hand side intersections are $\mathcal{C}^a$-submanifolds of $\mathbb{S}^{16} \times \mathbb{S}^{16} \times \mathbb{S}^{16}$, and it remains to note that

$$\cup_i \mathrm{proj}_i^{-1}(\mathbb{S}^{16} \setminus \mathrm{ort}_{17}) = \mathbb{S}^{16} \times \mathbb{S}^{16} \times \mathbb{S}^{16} \setminus (\mathrm{ort}_{17}, \mathrm{ort}_{17}, \mathrm{ort}_{17}),$$

and that $(\mathrm{ort}_{17}, \mathrm{ort}_{17}, \mathrm{ort}_{17}) \notin F(\mathbb{D}^{16})$.

Non-compact Grassmann Manifolds

Remark 3.2.2.11. We denote by $\mathbb{R}G'(n,k)$, or simply by $G'(n,k)$ $(0 \leq k \leq n)$ the set of *all* k-dimensional planes of $\mathbb{R}^n$, i.e., the planes need not pass through 0. It is clear that given a plane $\gamma' \in G'(n,k)$, there is a unique $(k+1)$-plane γ in $\mathbb{R}^{n+1}$ that passes both through 0 and the k-plane which results by translating γ' by the vector ort_{n+1}. Moreover, the formula $\gamma \mapsto \gamma'$ defines an injective mapping $G'(n,k) \to G(n+1, k+1)$, which has an open image. So we may regard $G'(n,k)$ as an open subset of the manifold $G(n+1, k+1)$. In particular, $G'(n,k)$ is a $(k+1)(n-k)$-dimensional $\mathcal{C}^a$-manifold. We call it the *non-compact Grassmann manifold.* $G'(n,k)$ can be mapped naturally onto $G(n,k)$: for each plane $\gamma' \in G'(n,k)$, consider the parallel plane $\gamma \in G(n,k)$. It is a straightforward consequence of Theorem 3.1.5.9 that this map is a $\mathcal{C}^a$-submersion (one can take the entire $G(n,k)$ for the required neighbourhood V of γ in $G(n,k)$, and one can take the map which takes each plane from $G(n,k)$ into the parallel plane passing through an arbitrary, but fixed point of γ', for the required g). The pre-image of a plane $\gamma \in G(n,k)$ under this submersion is the set of all k-planes of $\mathbb{R}^n$ which are parallel to γ, and is canonically diffeomorphic to the orthogonal complement $\gamma^\perp$ of γ (a unique k-plane parallel to γ passes through each point of $\gamma^\perp$).

The oriented, complex, and quaternionic versions of these definitions are immediate.

3.2.3 Some Low-Dimensional Stiefel and Grassmann Manifolds

Theorem 3.2.3.1. SO(3) *is canonically* $\mathcal{C}^a$*-diffeomorphic to* $\mathbb{R}P^3$.

Proof. We let shi denote the map of $\mathbb{R}^3$ into $\mathbb{R}^3_1$ which takes (y_1, y_2, y_3) into $(0, y_1, y_2, y_3)$. The canonical $\mathcal{C}^a$-diffeomorphism $\mathbb{R}P^3 \to \mathrm{SO}(3)$ takes the line of $\mathbb{R}^3_1$ passing through the point $x \in \mathbb{R}^4 \setminus 0$ into the orthogonal transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by the quaternionic formula $\varphi(y) = \mathrm{shi}^{-1}(x\,\mathrm{shi}(y)x^{-1})$. The inverse diffeomorphism $\mathrm{SO}(3) \to \mathbb{R}^3$ takes each transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ into the line consisting of the quaternions of the form

$$q - \mathrm{shi}(\varphi(\mathrm{ort}_1))q\,\mathrm{ort}_2 - \mathrm{shi}(\varphi(\mathrm{ort}_2))q\,\mathrm{ort}_3 - \mathrm{shi}(\varphi(\mathrm{ort}_3))q\,\mathrm{ort}_4,$$

where q is an arbitrary quaternion, and ort_2, ort_3 , and ort_4 are regarded as quaternion units. It is routine to check that quaternions of this form describe precisely a line and that the constructed maps $\mathbb{R}P^3 \to \mathrm{SO}(3)$ and $\mathrm{SO}(3) \to \mathbb{R}^3$ are inverses of one another. □

Theorem 3.2.3.2. $\mathbb{R}V(4,2)$ *is canonically* $\mathcal{C}^a$*-diffeomorphic to* $\mathbb{S}^3 \times \mathbb{S}^2$.

Proof. The canonical $\mathcal{C}^a$-diffeomorphism $\mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{R}V(4,2)$ takes the pair $(x, y) \in \mathbb{S}^3 \times \mathbb{S}^2$ into the frame $\{x, x\,\mathrm{shi}(y)\}$. □

Theorem 3.2.3.3. $\mathrm{SO}(4)$ *is canonically* $\mathcal{C}^a$*-diffeomorphic to* $\mathbb{S}^3 \times \mathrm{SO}(3)$.

Proof. The canonical $\mathcal{C}^a$-diffeomorphism $\mathbb{S}^3 \times \mathrm{SO}(3) \to \mathrm{SO}(4)$ is defined by the quaternion formula $(x, \{y, z\}) \mapsto \{x, x\,\mathrm{shi}(y), x\,\mathrm{shi}(z)\}$ (here the points of the manifolds SO(3) and SO(4) are interpreted as frames). □

Theorem 3.2.3.4. $G_+(4,2)$ *is canonically* $\mathcal{C}^a$*-diffeomorphic to* $\mathbb{S}^2 \times \mathbb{S}^2$.

Proof. The canonical $\mathcal{C}^a$-diffeomorphism $G_+(4,2) \to \mathbb{S}^2 \times \mathbb{S}^2$ takes the oriented plane spanned by the frame $\{x, y\} \in V(4,2)$ into the pair $(\mathrm{shi}^{-1}(xy^{-1}), \mathrm{shi}^{-1}(x^{-1}y))$. The inverse diffeomorphism transforms each pair $(u, v) \in \mathbb{S}^2 \times \mathbb{S}^2$ into the two-dimensional plane consisting of quaternions of the form $\mathrm{shi}(u)q + q\,\mathrm{shi}(v)$, where q is an arbitrary quaternion. Again, it is routine to check that the pair $(\mathrm{shi}^{-1}(xy^{-1}), \mathrm{shi}^{-1}(x^{-1}y))$ is uniquely determined by the oriented plane spanned by the frame $\{x, y\}$, that the quaternions $\mathrm{shi}(u)q + q\,\mathrm{shi}(v)$ fill exactly a two-dimensional plane, and that the maps $G_+(4,2) \to \mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2 \to G_+(4,2)$ constructed above are inverses of one another. □

3.2.4 Exercises

Exercise 3.2.4.1. A homogeneous polynomial in $n+1$ variables and with real (complex) coefficients is *non-singular* if there are no points in $\mathbb{R}^{n+1} \setminus 0$ (respectively, in ${}^{n+1} \setminus 0$) where all its partial derivatives vanish. Show that the projection $\mathbb{R}^{n+1} \setminus 0 \to \mathbb{R}P^n$ (respectively, $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}P^n$) transforms the set of zeros different from 0 of such a polynomial into a submanifold of $\mathbb{R}P^n$ (respectively, $\mathbb{C}P^n$) .

Exercise 3.2.4.2. Let $p(x_1, x_2, x_3)$ be a non-singular homogeneous polynomial of degree k with real coefficients. Show that the submanifold of the projective plane $\mathbb{R}P^2$ defined by the equation $p(x_1, x_2, x_3) = 0$ has an orientable neighbourhood if and only if k is even.

Exercise 3.2.4.3. Let $p(x_1, x_2, x_3)$ be a non-singular homogeneous polynomial of degree 3 with real coefficients. Show that the submanifold of $\mathbb{R}P^2$ defined by the equation $p(x_1, x_2, x_3) = 0$ is homeomorphic to either $\mathbb{S}^1$ or $\mathbb{S}^1 \amalg \mathbb{S}^1$, and that both cases are realised.

Exercise 3.2.4.4. Show that the equation $x_1^2 + x_2^2 + x_3^2 = 0$ defines in $\mathbb{C}P^2$ a submanifold homeomorphic to $\mathbb{S}^2$.

Exercise 3.2.4.5. Show that the equation $x_1^3 + x_2^3 + x_3^3 = 0$ defines in $\mathbb{C}P^2$ a submanifold homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.

Exercise 3.2.4.6. Show that the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ defines in $\mathbb{C}P^3$ a submanifold homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$.

Exercise 3.2.4.7. Show that $\mathbb{R}G(n,k)$ $(\mathbb{C}G(n,k))$ admits a $\mathcal{C}^a$-embedding in $\mathbb{R}P^{\binom{n}{k}-1}$ (respectively, $\mathbb{C}P^{\binom{n}{k}-1}$).

Exercise 3.2.4.8. Show that the map $\mathbb{R}G(n,k) \to \mathbb{R}^{n^2}$ which takes each plane $\gamma \in \mathbb{R}G(n,k)$ into the matrix of the composite map

$$\mathbb{R}^n \xrightarrow{\text{proj}} \gamma \xrightarrow{\text{incl}} \mathbb{R}^n$$

(where proj is the orthogonal projection) is a $\mathcal{C}^a$-embedding. Show that the same is true for the map $\mathbb{C}G(n,k) \to \mathbb{C}^{n^2}$ which takes each plane $\gamma \in \mathbb{C}G(n,k)$ into the matrix of the composite map

$$\mathbb{C}^n \xrightarrow{\text{proj}} \gamma \xrightarrow{\text{incl}} \mathbb{C}^n$$

Exercise 3.2.4.9. Show that $\mathbb{R}V(8,k)$ is $\mathcal{C}^a$-diffeomorphic to $S^7 \times \mathbb{R}V(7,k-1)$ $(1 \leq k \leq 8)$. Show that $\mathbb{C}V(4,k)$ is $\mathcal{C}^a$-diffeomorphic to $S^7 \times \mathbb{C}V(3,k-1)$ $(1 \leq k \leq 4)$.

3.3 A DIGRESSION: THREE THEOREMS FROM CALCULUS

3.3.1 Polynomial Approximation of Functions

Remark 3.3.1.1. The purpose of this section is to state and prove three theorems from calculus, namely Theorem 3.3.1.7, 3.3.2.3, and 3.3.3.5. They differ in character and we grouped them together here because all three are needed in this chapter, and none of them is included in the traditional calculus course.

The main theorem of this subsection, Theorem 3.3.1.7, is a corollary of Lemma 3.3.1.4, whose proof, in turn, requires Lemma 3.3.1.2.

Lemma 3.3.1.2. *For any positive* $\delta < 1$

$$\lim_{k\to\infty} \frac{1}{a_k}\int_{-\delta}^{\delta}(1-t^2)^k dt = 1.$$

where

$$a_k = \int_{-1}^{1}(1-t^2)^k dt. \tag{3.3.1.3}$$

Proof. This is an easy consequence of the inequalities

$$0 < 1 - \frac{1}{a_k}\int_{-\delta}^{\delta}(1-t^2)^k dt < \frac{2(1-\delta)}{\delta}\left(\frac{1-\delta^2}{a-\delta^2/4}\right)^k$$

The left inequality is plain, while the right one follows from the estimates

$$a_k - \int_{-\delta}^{\delta}(1-t^2)^k dt = 2\int_{\delta}^{1}(1-t^2)^k dt < 2(1-\delta)(1-\delta^2)^k$$

and

$$a_k > \int_{-\delta/2}^{\delta/2}(1-t^2)^k dt > \delta(1-\delta^2/4)^k.$$

□

Lemma 3.3.1.4. *There exists a sequence of mappings*

$$\{p_k\colon \mathcal{C}(I^n,\mathbb{R}) \to \mathcal{C}(\mathbb{R}^n,\mathbb{R})\}_{k=1}^{\infty}$$

such that:

(i) $p_k(f)$ *is a polynomial for any* $f \in \mathcal{C}(I^n,\mathbb{R})$ *and any* k*;*

(ii) if f *equals* 0 *on* $\operatorname{Fr} I^n$*, then the sequence* $\{p_k|_{I^n}$ *converges uniformly to* f*;*

(iii) if f *equals* 0 *on* $\operatorname{Fr} I^n$ *and has continuous partial derivative* $D_i f$*, then* $p_k(D_i f) = D_i p_k(f)$*.*

Proof. For $f \in \mathcal{C}(l^n, \mathbb{R})$ we set

$$[p_k(f)](x_1, \dots, x_n) = \frac{1}{a_k^n} \int_{I^n} f(t_1, \dots, t_n) \prod_{j=1}^{n} [1-(t_j-x_j)^2]^k dt_1 \cdots dt_n \quad (3.3.1.5)$$

where a_k is defined by (3.3.1.3). This clearly defines a polynomial and therefore we must check only properties (ii) and (iii).

To check (ii), we extend the function f to $\mathbb{R}^n$, setting $f(x) = 0$ for $x \in \mathbb{R}^n \backslash I^n$, and denote by M the maximum of $|f(x)|$. Given an arbitrary $\varepsilon > 0$, we can find δ, $0 < \delta < 1$, such that

$$|f(x_1', \dots, x_n') - f(x_1, \dots, x_n)| < \varepsilon/2 \quad \text{for} \quad |x_1' - x_1| < \delta, \dots, |x_n' - x_n| < \delta.$$

Moreover, we can find a number K such that

$$1 - \left(\frac{1}{a^k} \int_{-\delta}^{\delta} (1-t^2)^k dt \right)^n < \frac{\varepsilon}{4M}$$

for all $k > K$ (see Lemma 3.3.1.2). We next show that $|[p_k(f)](x) - f(x)| < \varepsilon$ for $x \in I^n$ and $k > K$.

Write $[p_k(f)](x) - f(x)$ as

$$\frac{1}{a^k} \int_{[-1,1]^n} (f(t_1 + x_1, \dots, t_n + x_n) - f(x_1, \dots, x_n)) \prod_{j=1}^{n} (1-t_j^2)^k dt_1 \cdots dt_n$$

and then replace the integrand by its absolute value. Now divide the new integral into two integrals, one over the cube $[-\delta, \delta]^n$ and one over its complement $[-1,1]^n \setminus [-\delta, \delta]^n$. We obtain

$$|[p_k(f)](x) - f(x)| \leq$$
$$\frac{1}{a^k} \int_{[-\delta,\delta]^n} (f(t_1 + x_1, \dots, t_n + x_n) - f(x_1, \dots, x_n)) \prod_{j=1}^{n} (1-t_j^2)^k dt_1 \cdots dt_n +$$
$$\frac{1}{a^k} \int_{[-1,1]^n \setminus [-\delta,\delta]^n} (f(t_1 + x_1, \dots, t_n + x_n) - f(x_1, \dots, x_n)) \prod_{j=1}^{n} (1-t_j^2)^k dt_1 \cdots dt_n.$$

The first term is smaller than $\varepsilon/2$, since

$$|f(t_1 + x_1, \dots, t_n + x_n) - f(x_1, \dots, x_n)| < \varepsilon/2$$

for $(t_1, \dots, t_n) \in [-\delta, \delta]$ and

$$\frac{1}{a^k} \int_{[-1,1]^n} \prod_{j=1}^{n} (1-t_j^2)^k dt_1 \cdots dt_n = \left(\frac{1}{a^k} \int_{-\delta}^{\delta} (1-t^2)^k dt \right)^n < 1.$$

The second term is also smaller than $\varepsilon/2$, since

$$|f(t_1 + x_1, \dots, t_n + x_n) - f(x_1, \dots, x_n)| \leq 2M$$

$$
\begin{aligned}
&\frac{1}{a^k}\int_{[-1,1]^n\setminus[-\delta,\delta]^n}\prod_{j=1}^{n}(1-t_j^2)^k dt_1\cdots dt_n = \\
&\frac{1}{a^k}\int_{[-1,1]^n}\prod_{j=1}^{n}(1-t_j^2)^k dt_1\cdots dt_n - \frac{1}{a^k}\int_{[-\delta,\delta]^n}\prod_{j=1}^{n}(1-t_j^2)^k dt_1\cdots dt_n = \\
&1-\left(\frac{1}{a^k}\int_{-\delta}^{\delta}(1-t^2)^k dt\right)^n < \frac{\varepsilon}{4M}.
\end{aligned}
$$

Therefore, $|[p_k(f)](x) - f(x)| \leq \varepsilon$.

To check property (iii), replace f by $D_i f$ in definition (3.3.1.5) and then integrate by parts with respect to t_i in the right-hand side. We obtain

$$
\begin{aligned}
&[p_k(D_i f)](x_1,\ldots,x_n) = \\
&-\frac{1}{a^k}\int_{I^n} f(t_1,\ldots,t_n)\frac{\partial}{\partial t_i}\prod_{j=1}^{n}(1-(t_j-x_j)^2)^k dt_1\cdots dt_n.
\end{aligned}
\tag{3.3.1.6}
$$

Since

$$
\frac{\partial}{\partial t_i}\prod_{j=1}^{n}(1-(t_j-x_j)^2)^k = -\frac{\partial}{\partial x_i}\prod_{j=1}^{n}(1-(t_j-x_j)^2)^k,
$$

the right-hand side of (3.3.1.6) is equal to $[D_i p_k(f)](x_1,\ldots,x_n)$. □

Theorem 3.3.1.7. *Suppose X is a compact set in $\mathbb{R}^n$ and f is a real function defined and of class $\mathcal{C}$ in a neighbourhood of X. If $r \leq \infty$ then for any $\varepsilon > 0$ and any non-negative integer $s \leq r$, there is a polynomial $g\colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$
\max_{x\in X}|D_1^{s_1}\cdots D_1^{s_n}g(x) - D_1^{s_1}\cdots D_1^{s_n}f(x)| < \varepsilon
$$

for any collection $s_1,\ldots,s_n$ of non-negative integers with $s_1+\cdots+s_n \leq s$.

Proof. Clearly, one can assume that $X \subset \operatorname{int} I^n$. Denote by U the neighbourhood of X mentioned above, and let $\beta\colon \mathbb{R}^n \to \mathbb{R}$ be any $\mathcal{C}^r$-function equal to 0 on $\mathbb{D}^n$ and equal to 1 outside the concentric ball of radius 2. For $y \in (I^n \setminus U) \cup \operatorname{Fr} I^n$, we let $d(y)$ denote the ball of centre y and radius $\operatorname{Dist}(y,X)/4$. Cover $(I^n \setminus U) \cup \operatorname{Fr} I^n$ by a finite number of balls $\operatorname{int} d(y)$, say $\operatorname{int} d(y_1),\ldots,\operatorname{int} d(y_p)$, and define $h\colon I^n \to \mathbb{R}$ as

$$
h(y) = \begin{cases} f(y)\prod_{i=1}^{p}\beta(4(y-y_i)/\operatorname{Dist}(y_i,X)), & \text{if} \quad y \in U, \\ 0, & \text{if} \quad y \notin U. \end{cases}
$$

Then h is of class $\mathcal{C}^r$, agrees with f on X, and vanishes on a neighbourhood of $\operatorname{Fr} I^n$. This shows that one can take g to be $p_k(h)$, with p_k as in Lemma 3.3.1.4 and k large enough. □

3.3.2 Singular Values

Remark 3.3.2.1. In the present subsection we assume that we are given open subset U of $\mathbb{R}^n$ and a $\mathcal{C}^\infty$-map $f\colon U \to \mathbb{R}^q$ with $q \geq 1$. We let F denote the set of points in U where the rank of the Jacobi matrix of f is less than q. Our aim is to prove Theorem 3.3.2.3, which is needed in the next section (see Theorem 3.4.7.4).

We need two auxiliary notations: f_j for the j-th coordinate function of the map f ($j = 1, \ldots, n$) and F_S for the set of points in U where all partial derivatives of order $1, \ldots, s$ of the functions f vanish. Clearly, F_s is closed in U and $F \supset F_1 \supset F_2 \cdots$.

Lemma 3.3.2.2. *Suppose* $s > (n/q) - 1$ *and* C *is any compact part of the set* F_s*. The image* $f(C)$ *is nowhere dense.*

Proof. It is enough to show that for any n-dimensional cube $Q \subset U$, the set $f(C \cap Q)$ is nowhere dense. Indeed, one can cover C by a finite number of such cubes, and use the fact that a finite union of nowhere dense sets is nowhere dense.

Let a be the edge length of Q. Consider the standard partition of Q into m^n small cubes of edge length a/m (with m an arbitrary positive integer). Let Q' be a small cube in this partition which intersects C. Now apply Taylor's theorem to the functions f_j and use the fact that their partial derivatives of order $s + 1$ are bounded on Q to show that there is a constant b such that

$$\operatorname{dist}(f(x), f(y)) \leq b[\operatorname{dist}(x, y)]^{s+1}$$

for any $x \in F_s \cap Q$, $y \in Q$. Since Q' has diameter $a\sqrt{n}/m$, we see that $f(Q')$ is contained in a ball of radius $b(a\sqrt{n}/m)^{s+1}$. Therefore, $f(Q')$ is contained in a q-dimensional cube with edge length $2b(a\sqrt{n}/m)^{s+1}$, and $f(C \cap Q)$ - in a union of no more than m^n such cubes. The volume of each such cube is $[2b(a\sqrt{n}/m)^{s+1}]^q$, and so the sum of their volumes does not exceed

$$m^n[2b(a\sqrt{n}/m)^{s+1}]^q = cm^{n-q}(s + 1)$$

where c is independent of m. By hypothesis, this sum goes to 0 as $m \to \infty$. This shows that the set $f(C \cap Q)$ can have no interior points. Since the latter is a closed set, it is also nowhere dense. $\square$

Theorem 3.3.2.3. *The image* $f(C)$ *of any compact part of the set* F *is nowhere dense.*

Proof. We proceed inductively on n. For $n = 0$ there is nothing to prove; hence it is enough to show that the theorem holds for $n = k + 1$ if it holds for $n = k$.

We start with the special case $F_1 = \varnothing$. Since C is compact, it suffices to find, for each point $x \in C$, a neighbourhood V in U such that $f(C \cap V)$ is nowhere dense. Assume that $D_{k+1} f_q(x) \neq 0$ (we can always achieve this by re-indexing the coordinates in $\mathbb{R}^{k+1}$ and $\mathbb{R}^q$) and consider the map $g\colon U \to \mathbb{R}^{k+1}$ defined by the formula

$$y = (y_1, \ldots, y_{k+1}) \mapsto (y_1, \ldots, y_k, f_q(y)).$$

Its Jacobian at the point x does not vanish (being equal to $D_{k+1}f_q$); hence g is a $\mathcal{C}^\infty$-diffeomorphism of a neighbourhood W of x onto a neighbourhood of $g(x)$. Now one can take as V any neighbourhood of x with compact closure included in W. To see this, set h to be the composite map

$$g(W) \xrightarrow{(\operatorname{abr} g)^{-1}} W \xrightarrow{\operatorname{ab} f} \mathbb{R}^q.$$

It transforms each point from $g(W)$ into a point which has the same last coordinate. In particular, for any real number u, h can be compressed to a map

$$g(W) \cap [\mathbb{R}^k \times u] \mapsto \mathbb{R}^{q-1} \times u.$$

If we identify $g(W) \cap [\mathbb{R}^k \times u]$ with its orthogonal projection on $\mathbb{R}^k$ in the standard fashion, and $\mathbb{R}^{q-1} \times u$ - with its orthogonal projection on $\mathbb{R}^{q-1}$, we obtain a $\mathcal{C}$-map h_u of an open subset of $\mathbb{R}^k$ into $\mathbb{R}^{q-1}$. Clearly, the Jacobi matrix of h_u at the point $(y_1, \dots, y_k)$ is obtained from the Jacobi matrix of the map h at the point $(y_1, \dots, y_k, u)$ by deleting the last column and the last row, which has the form $0, \dots, 0, 1$. Therefore, the rank of the first matrix is less than $q - 1$ if and only if the rank of the second matrix is less than q, i.e, if $(y_1, \dots, y_k, u) \in g(F \cap W)$. Applying the induction hypothesis to h_u, we deduce that the intersection of $h(g(C \cap \operatorname{Cl} V))$ with each hyperplane $\mathbb{R}^{q-1} \times u$ is nowhere dense in $\mathbb{R}^{q-1} \times u$. But if this is the case, $h(g(C \cap \operatorname{Cl} V))$ has no interior points in $\mathbb{R}^q$ and we need only note that this set is closed and coincides with $f(C \cap \operatorname{Cl} V)$.

Now let us turn to a second special case: $C \subset F_s$ and $F_{s+1} = \varnothing$ (for some s). Again, it is enough to exhibit for each point $x \in C$ a neighbourhood V in U such that the set $f(C \cap V)$ is nowhere dense. Let φ be a derivative of order s of one of the functions f_j which satisfies the following condition: one of the derivatives $D_i\varphi$, say $D_{k+1}\varphi$, does not vanish at the point x. Consider the map $g \colon U \to \mathbb{R}^{q+1}$ defined as

$$y = (y_1, \dots, y_{k+1}) \mapsto (y_1, \dots, y_k, \varphi(y)).$$

Its Jacobian does not vanish at x (being equal to $D_k\varphi$); hence g yields a $\mathcal{C}^\infty$-diffeomorphism of a neighbourhood W of x onto a neighbourhood of $g(x)$. We show, with the aid of the composite map

$$g(W) \xrightarrow{(\operatorname{abr} g)^{-1}} W \xrightarrow{\operatorname{ab} f} \mathbb{R}^q \tag{3.3.2.4}$$

that one can take any neighbourhood of x with compact closure included in W for V. To do this, note that $g(C) \subset \mathbb{R}^k$ and restrict the map (3.3.2.4) to a map $h \colon g(W) \cap \mathbb{R}^k \to \mathbb{R}^q$. Clearly, all the derivatives of order $\leq s$ of the coordinate functions of h vanish on $g(C \cap W)$. Using the induction hypothesis, it is evident that h carries the compact parts of the set $g(C \cap W)$ into nowhere dense sets. Finally, we observe that $g(C \cap \operatorname{Cl} V)$ is a compact part of $g(C \cap W)$, and that $h(g(C \cap \operatorname{Cl} V))$ is just $f(C \cap \operatorname{Cl} V)$.

At last, we come to the general case. According to Lemma 3.3.2.2, there exists a number r such that the set $f(C \cap F_r)$ is nowhere dense. We shall prove by induction on r, i.e, assuming that $f(C \cap F_r)$ is nowhere dense, we show that, for $r = 1$, $f(C)$ is nowhere dense, and for $r > 1$, $f(C \cap F_{r-1})$ is nowhere dense.

Let G be an open non-empty subset of $\mathbb{R}^q$. Since $C \cap F_r$ is compact and $f(C \cap F_r)$ is nowhere dense, the set $C \cap F_r$ has a neighbourhood N in $\mathbb{R}^n$ such that $\operatorname{Cl} N$ is compact and $\operatorname{Cl} N \subset U$, $f(\operatorname{Cl} N) \not\supset G$. Next replace the map f by its restriction to $U \setminus F_r$ and C - by the set

$$C' = \begin{cases} C \setminus N, & \text{if} \quad r = 1 \\ (C \setminus F_r) \setminus N, & \text{if} \quad r > 1. \end{cases}$$

Now we are back to one of the cases covered by the first part of the proof (namely, in the first case for $r = 1$, and in the second one for $r > 1$). Therefore, we conclude that $f(C')$ does not cover $G \setminus f(\operatorname{Cl} N)$. Consequently, if $r = 1$ the set $f(C \cap F_r)$ does not cover G, while if $r > 1$ the set $f(C \cap F_r)$ does not cover G. This completes the proof, because $f(C)$ and $f(C \cap F_{r-1}$ are closed. □

Information 3.3.2.5. In Theorem 3.3.2.3, the condition that f be $\mathcal{C}^\infty$-smooth is unnecessarily strong: in fact, the proof uses only the fact that f is of class $\mathcal{C}^r$, with $r = 2 + \max(n - q, 0)$. A more precise analysis shows that this r can be decreased by 1 (see, for example, [21]), but no further (for $q = 1$, this is showed in [23], and the case $q > 1$ reduces easily to the case $q = 1$).

3.3.3 Non-degenerate Critical Points

Remark 3.3.3.1. Let f be a real $\mathcal{C}^2$-function defined on an open subset of $\mathbb{R}^n$. A critical point y of f is *non-degenerate* if the second differential of f at y (considered as a quadratic form) has rank n. The index of the second differential of f at y (i.e, the number of negative squares in the diagonal representation of this form) is called the *index of the point* y and is denoted by $\operatorname{ind}_f y$.

We remark that if φ is a $\mathcal{C}^2$-diffeomorphism of an open subset U of $\mathbb{R}^n$ onto another open subset of $\mathbb{R}^n$ and y is a non-degenerate critical point of $f\colon U \to \mathbb{R}$, then $\varphi(y)$ is a non-degenerate critical point of the function $f \circ \varphi : \varphi(U) \to \mathbb{R}$, and $\operatorname{ind}_{f\circ\varphi} \varphi(y) = \operatorname{ind}_f y$. Both conclusions remain true in the more general situation where φ is only of class $\mathcal{C}^1$ but the function $f \circ \varphi$ is of class $\mathcal{C}^2$.

Now consider the function $\mathbb{R}^n \to \mathbb{R}$ defined as

$$(x_1, \dots, x_n) \mapsto -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2 + c, \tag{3.3.3.2}$$

where c is a real number ($0 \le k \le n$). This function has a unique critical point, at 0, which clearly is non-degenerate and of index k. The main goal in the present subsection is to show that, in a suitably chosen system of coordinates, any sufficiently smooth function has the above form (3.3.3.2) in the vicinity of a non-degenerate critical point.

Lemma 3.3.3.3. *Let V be an open ball in $\mathbb{R}^n$ with centre 0, and let $f\colon V \to \mathbb{R}$ be a $\mathcal{C}^r$-function, $r \ge 1$, with $f(0) = 0$. There are $\mathcal{C}^{r-1}$-functions $f_1, \dots f_n\colon V \to \mathbb{R}$*

such that

$$f(x) = \sum_{i=1}^{n} x_i f_i(x) \tag{3.3.3.4}$$

for all points $x = (x_1, \ldots, x_n) \in V$.

Proof. To prove the lemma, it is enough to set

$$f_i(x) = \int_0^1 D_i f(tx) dt,$$

and then observe that (3.3.3.4) is an immediate consequence of the equality

$$\frac{\partial}{\partial t} f(tx) = \sum_{i=1}^{n} x_i D_i f(tx).$$

□

Theorem 3.3.3.5. *Suppose that* y *is a non-degenerate critical point of a* $\mathcal{C}^r$*-function* f *defined on an open subset of* $\mathbb{R}^n$. *If* $r \geq 3$, *then there exist a neighbourhood* U *of* y *and a diffeomorphism* φ *of* U *onto a neighbourhood* V *of* 0, *such that the restriction* $f|_U$ *coincides with the composite map* $U \xrightarrow{\varphi} V \xrightarrow{3.3.3.2} \mathbb{R}$, *where* $k = \operatorname{ind}_f y$ *and* $c = f(y)$.

Proof. Without loss of generality, we may assume that $y = 0$ and $f(y) = 0$. By Lemma 3.3.3.3,

$$f(x) = \sum_{i=1}^{n} x_i f_i(x)$$

in some neighbourhood of 0, where f_i are $\mathcal{C}^{r-1}$-functions. Differentiating, we obtain $D_i f(x) = \sum_{j=1}^{n} x_j D_i f_j(x)$, since $f_1(0) = \cdots = f_n(0)$. Again we apply Lemma 3.3.3.3 and write, in a neighbourhood V_0 of 0,

$$f_i(x) = \sum_{j=1}^{n} x_j f_{ij}(x) \quad (i = 1, \ldots, n)$$

with $\mathcal{C}^{r-2}$-functions f_{ij}. Therefore, for $x \in V_0$,

$$f(x) = \sum_{i,j=1}^{n} g_{ij}(x) x_i x_j,$$

where $g_{ij}(x) = (f_{ij}(x) + f_{ji}(x))/2$. Clearly, $g_{ij}(0) = D_i D_j f(0)/2$.

The subsequent constructions mimic the standard reduction of a quadratic form to canonical form through linear transformations. For $p = 0, \ldots, n$, we construct neighbourhoods V_p and W_p of the point 0 in $\mathbb{R}^n$, $\mathcal{C}^{r-2}$-diffeomorphisms $\varphi_p \colon W_p to V_p$, and also $\mathcal{C}^{r-2}$-functions $g_{ij}^p \colon V_p \to \mathbb{R}$, $i, j = p+1, \ldots, n$, such that:

(i) $W_p \subset V_{p-1}$;

(ii) $\varphi_p = 0$;

(iii) the composite map

$$V_p \xrightarrow{\varphi_p^{-1}} W_p \xrightarrow{\text{incl}} V_{p-1} \xrightarrow{\varphi_{p-1}^{-1}} \cdots \xrightarrow{\varphi_1^{-1}} W_1 \xrightarrow{\text{incl}} V_0 \xrightarrow{\operatorname{abr} f} \mathbb{R}$$

is represented by the formula

$$x \mapsto \pm x_1^2 \pm \cdots \pm x_p^2 + \sum_{ij=p+1}^{n} g_{ij}^p(x) x_i x_j; \tag{3.3.3.6}$$

(iv) $g_{ij} = g_{ji}$.

Then we will have finished, since one could take

$$\varphi_1^{-1}(\cdots(\varphi_n^{-1}(V_n))\cdots)$$

for V, define φ as the composition

$$V \xrightarrow{\operatorname{abr}\varphi_1} \varphi_1(V) \xrightarrow{\operatorname{abr}\varphi_2} \varphi_2(\varphi_1) \xrightarrow{\operatorname{abr}\varphi_2} \cdots \xrightarrow{\operatorname{abr}\varphi_n} V_n \xrightarrow{\operatorname{abr}\pi} \pi(V_n),$$

where π is a suitable permutation of the standard coordinates in $\mathbb{R}^n$, and set $U = \varphi^{-1}(V)$.

The neighbourhood V_0 is already given. We let $W_0 = V_0$, $\varphi_0 = \operatorname{id} V_0$, $g_{ij}^0 = g_{ij}$, and assume that we have constructed V_p, W_p, φ'_p, and g_{ij} satisfying (i) , (ii), (iii), and (iv) for $p \leq q$. It is clear that 0 is a non-degenerate critical point of the function (3.3.3.6) with $p = q$. Hence the matrix $G = \|g_{ij}^p(0)\|_{i,j=q+1}^k$ is non-degenerate and there exists a non-degenerate $(n-q) \times (n-q)$-matrix A such that the left upper element of the matrix A^tGA is not zero. Let ℓ denote the linear transformation of $\mathbb{R}^n$ having matrix

$$\begin{bmatrix} E & 0 \\ 0 & A \end{bmatrix}$$

where E is the $q \times q$-identity matrix. The composition of the diffeomorphism $\operatorname{abr}\ell\colon \ell^{-1}(V_q) \to V_q$ with the function (3.3.1.6) is given by

$$x \mapsto \pm x_1^2 \pm \cdots \pm x_q^2 + \sum_{i,j=q+1}^{n} h_{ij}(x) x_i x_j,$$

where $h_{ij} = h_{ji}$ and $h_{q+1,q+1} \neq 0$. Now consider the subset L of $\ell^{-1}(V_q)$ consisting of all the points x where $h_{q+1,q+1} \neq 0$ and has the same sign as $h_{q+1,q+1}(0)$, and then define $\psi\colon L \to \mathbb{R}^n$ as

$$\psi(x) = (x_1, \ldots, x_q, \xi\sqrt{|h_{q+1,q+1}(x)|}, x_{q+2}, \ldots, x_n),$$

where

$$\xi = x_{q+1} + \sum_{s>q+1} x_s \frac{h_{s,q+1}(x)}{h_{q+1,q+1}(x)}.$$

A simple computation shows that the Jacobian of ψ at the point 0 does not vanish. Therefore, the compression of ψ to a neighbourhood M of 0 and to its image $\psi(M)$ is a $\mathcal{C}^{r-2}$-diffeomorphism. It is now readily verified that the sets $V_{q+1} = \psi(M)$ and $W_{q+1} = \ell(M)$, the map $\varphi_{q+1} \colon W_{q+1} \to V_{q+1}$ defined by $\varphi_{q+1}(x) = \psi(\ell^{-1}(x))$, and the functions $g_{ij}^{q+1} \colon V_{q+1} \to \mathbb{R}$ defined as

$$g_{ij}^{q+1}(x) = h_{ij}(x) - \frac{h_{i,q+1}(x) h_{j,q+1}(x)}{h_{q+1,q+1}(x)}$$

enjoy the properties (i), (ii), (iii), and (iv). $\square$

3.4 EMBEDDINGS. IMMERSIONS. SMOOTHINGS. APPROXIMATIONS

3.4.1 Spaces of Smooth Maps

Remark 3.4.1.1. Let X and X' be $\mathcal{C}^{\geq r}$-manifolds ($0 \leq r \leq a$). We denote by $\mathcal{C}^r(X, X')$ the set of all $\mathcal{C}^r$-maps $X \to X'$. If $r \leq \infty$, we equip $\mathcal{C}^r(X, X')$ with the ***$\mathcal{C}^r$-topology*** which makes $\mathcal{C}^r(X, X')$ into a topological space, as follows. Given two arbitrary charts $\varphi \in \operatorname{Atl} X$ and $\varphi' \in \operatorname{Atl} X'$, a sequence of non-negative integers $r_1, \dots, r_n$ with $n = \dim X$ and $r_1 + \cdots + r_n \leq r$, a compact subset A of $\operatorname{im} \varphi$, and an open subset A' of $\mathbb{R}^{n'}$, where $n' = \dim X'$, consider the subset of $\mathcal{C}^r(X, X')$ consisting of all maps f such that

$$[D_1^{r_1} \cdots D_n^{r_n} \operatorname{loc}(\varphi, \varphi') f](A) \subset A'.$$

These subsets form a prebase of the $\mathcal{C}^r$-topology on $\mathcal{C}^r(X, X')$.

Clearly, $\mathcal{C}^0(X, X') \to \mathcal{C}(X, X')$, and the $\mathcal{C}^0$-topology is simply the compact-open topology (see Definition 1.2.7.1). Also, for $s < r$, the inclusion $\mathcal{C}^r(X, X') \to \mathcal{C}^s(X, X')$ is obviously continuous. Another direct consequence of the definition of the $\mathcal{C}^r$-topology is that all the spaces $\mathcal{C}^r(X, X')$ are regular. In addition, we note that the sets closed (open) in $\mathcal{C}^r(X, X')$ are exactly those sets closed (respectively, open) in all the $\mathcal{C}^r$-topologies with r finite.

For each pair of $\mathcal{C}^r$-maps $f \colon Y \to X$ and $f' \colon Y' \to X'$, there is a map $\mathcal{C}^r(X, X') \to \mathcal{C}^r(Y, Y')$ defined by the formula $g \mapsto f' \circ g \circ f$, and denoted by $\mathcal{C}^r(f, f')$.

$$\begin{tikzcd} X \arrow[r, "g", ""{name=U, below}] & X' \arrow[d, "f'"] \\ Y \arrow[u, "f"] \arrow[r, "f' \circ g \circ f"', ""{name=D, above}] & Y' \arrow[Rightarrow, from=U, to=D, "\mathcal{C}^r(f, f')"'] \end{tikzcd}$$

Obviously, $\mathcal{C}^0(f, f') = \mathcal{C}(f, f')$ (see Definition 1.2.7.1), $\mathcal{C}^r(f, f') = \operatorname{abr} \mathcal{C}^s(f, f')$ for $s < r$, and $\mathcal{C}^r(f, f')$ is continuous for all $r \leq \infty$.

We list some particular subsets of $\mathcal{C}^r(X, X')$ which are important in the sequel. These are the sets of all $\mathcal{C}^r$-embeddings, $\mathcal{C}^r$-immersions, $\mathcal{C}^r$-submersions, and $\mathcal{C}^r$-diffeomorphisms $X \to X'$, and they are denoted by $\operatorname{Emb}^r(X, X', \operatorname{Imm}^r(X, X')$, $\operatorname{Subm}^r(X, X')$, and $\operatorname{Diff}^r(X, X')$, respectively ($1 \leq r \leq a$). Moreover, we let $\mathcal{C}^r_\partial(X, X')$ denote the set of all $\mathcal{C}^r$-maps $f \colon X \to X'$ such that $f(\partial X) \subset \partial X'$ and $d_x f(\operatorname{Tang} X) \subset \operatorname{Tang}_{f(x)}(\partial X')$ for all $x \in \partial X$. Usually one writes $\operatorname{Diff}^r X$ instead of $\operatorname{Diff}^r(X, X)$.

For $1 \leq r \leq \infty$, the map $\mathcal{C}^r_\partial(X, X') \to \mathcal{C}^r(\partial X, \partial X')$, defined as $f \mapsto \operatorname{abr} f$ is continuous, and the map $\mathcal{C}^r(X, X') \to \mathcal{C}^{r-1}(\operatorname{Tang} X, TangX')$, defined as $f \mapsto df$ is a topological embedding.

Theorem 3.4.1.2. *If X is compact, then the set $\operatorname{Imm}^r(X, X')$ is open in $\mathcal{C}^r_\partial(X, X')$ ($1 \leq r \leq \infty$).*

Proof. We have to exhibit, for a given $\mathcal{C}^r_\partial$-immersion $f_0\colon X \to X'$, a neighbourhood of f_0 in $\mathcal{C}^r_\partial(X, X')$ consisting only of immersions. To do this, pick for each point $x \in X$ two charts, $\varphi_x \in \operatorname{Atl}_x X$ and $\varphi'_x \in \operatorname{Atl}_x X'$, such that $f_0(\operatorname{supp}\varphi) \subset \operatorname{supp}\varphi'_x$ and $\operatorname{loc}(\varphi_x, \varphi'_x)f$ equals one of the inclusions $\mathbb{R}^n \to \mathbb{R}^{n'}$, $\mathbb{R}^n_- \to \mathbb{R}^{n'}$, or $\mathbb{R}^n_- \to \mathbb{R}^{n'}_-$, (see Proposition 3.1.5.3 and Remark 3.1.5.1; here $n = \dim X$ and $n' = \dim X'$). Now cover X with a finite number of sets $U_x = \varphi_x^{-1}(\operatorname{int}\mathbb{D}^n)$, say $U_{x_1}, \dots, U_{x_s}$, and denote by $\mathcal{U}_i$ the subset of $\mathcal{C}^r(X, X')$ consisting of all the maps f such that $f(\operatorname{Cl} U_{x_i}) \subset \operatorname{supp}\varphi'_{x_i}$ and the upper $n \times n$-minor of the Jacobi matrix of the map $\operatorname{loc}(\varphi_{x_i}, \varphi'_{x_i})f$ has no zeros on $\mathbb{D}^n$. The intersection $\mathcal{U}_1 \cap \dots \cap \mathcal{U}_s$ is the desired neighbourhood of the map f_0. □

Theorem 3.4.1.3. *If X is compact and X' has no boundary, then the set $\operatorname{Subm}^r(X, X')$ is open in $\mathcal{C}^r(X, X')$ $(1 \le r \le \infty)$.*

Proof. We have to exhibit, for a given $\mathcal{C}^r$-submersion $f_0\colon X \to X'$, a neighbourhood of f_0 in $\mathcal{C}^r(X, X')$ consisting only of submersions. Again, for each point $x \in X$ we choose charts $\varphi_x \in \operatorname{Atl}_x X$ and $\varphi'_x \in \operatorname{Atl}'_x X$, such that $f_0(\operatorname{supp}\varphi_x) \subset \operatorname{supp}\varphi'_x$ and $\operatorname{loc}(\varphi_x, \varphi'_x)f$ equals one of the orthogonal projections $\mathbb{R}^n \to \mathbb{R}^{n'}$, $\mathbb{R}^n_- \to \mathbb{R}^{n'}$ (see Theorem 3.1.5.7). Now cover X with a finite number of sets $U_x = \varphi_x^{-1}(\operatorname{int}\mathbb{D}^n)$, say $U_{x_1}, \dots, U_{x_s}$, and denote by $\mathcal{U}_i$ the subset of $\mathcal{C}^r(X, X')$ consisting of all maps f such that $f(\operatorname{Cl}_{x_i}) \subset \operatorname{supp}\varphi_{x_i}$ and the left $n' \times n'$-minor of the Jacobi matrix of the map $\operatorname{loc}(\varphi, \varphi')f$ has no zeros on $\mathbb{D}^n$. The intersection $\mathcal{U}_1 \cap \dots \cap \mathcal{U}_s$ is the desired neighbourhood of the map f_0. □

Theorem 3.4.1.4. *If X is compact, then the set $\operatorname{Emb}^r(X, X')$ is open in $\mathcal{C}^r(X, X')$ $(1 \le r \le \infty)$.*

Proof. Given a $\mathcal{C}^r$-embedding $f_0\colon X \to X'$, Theorems 3.4.1.2 and Corollary 3.1.5.4 show that it is enough to produce a neighbourhood of f_0 in $\mathcal{C}^r(X, X')$ consisting only of injective maps. For each point $x \in X$, choose two charts, $\varphi_x \in \operatorname{Atl}_x X$ and $\varphi'_x \in \operatorname{Atl}_x X'$, such that $f_0(\operatorname{supp}\varphi_x) \subset \operatorname{supp}\varphi'_x$ and $\operatorname{loc}(\varphi_x, \varphi'_x)f$ coincides with one of the inclusions $\mathbb{R}^n_- \to \mathbb{R}^{n'}$, or $\mathbb{R}^n_- \to \mathbb{R}^{n'}_-$ (see Remark 3.1.5.1). Now cover X with a finite number of sets $U_x = \varphi_x^{-1}(\operatorname{int}\mathbb{D}^n)$, say $U_{x_1}, \dots, U_{x_s}$. Let $\mathcal{U}_i$ be the subset of $\mathcal{C}^r(X, X')$ consisting of all maps f such that $f(\operatorname{Cl} U_{x_i}) \subset \operatorname{supp}\varphi'_{x_i}$ and, if we symmetrise the upper $n \times n$-part of the Jacobi matrix of $\operatorname{loc}(\varphi_{x_i}, \varphi'_{xi})f$ and take all the principal minors, they are all positive on the ball $\mathbb{D}^n$. (The principal minors are the left-upper minors; the symmetrised matrix is half the sum of the matrix with its transpose.) Finally, denote by $\mathcal{U}$ that part of $\mathcal{C}^r(X, X')$ consisting of all maps such that the preimage of any point of $'$ lies in one of the sets U_{x_i}. Let us show that the intersection $\mathcal{V} = \mathcal{U}_1 \cap \dots \cap \mathcal{U}_s \cap \mathcal{U}$ is a neighbourhood of f_0 with the necessary property. It is clear that $f_0 \in \mathcal{V}$ and that all the sets $\mathcal{U}_i$ are open. Hence it suffices to verify that:

(i) $\mathcal{U}$ is open, and

(ii) the maps in $\mathcal{U}_i$ are injective.

To prove (i), note that $\mathcal{U}$ is the preimage of the set

$$\mathcal{W} = \mathcal{C}(X \times X, (X \times X) \setminus \cup_i (U_{x_i} \times U_{x_i}); X' \times X', (X' \times X') \setminus \operatorname{diag}(X'))$$

under the continuous mapping

$$\mathcal{C}^r(X, X') \to \mathcal{C}(X \times X, X' \times X'), \quad f \mapsto f \times f.$$

Since $(X \times X) \setminus \cup_i (U_{x_i} \times U_{x_i})$ is compact and $(X' \times X') \setminus \operatorname{diag}(X')$ is open in $X' \times X'$ (see Remark 1.2.2.4), $\mathcal{W}$ is open in $\mathcal{C}(X \times X, X' \times X')$ and $\mathcal{U}$ is open in $\mathcal{C}^r(X, X')$.

To prove (ii), given a map $f \in \mathcal{U}_i$ and arbitrary distinct points $y, z \in U_{x_i}$, let $s\colon I \to \mathbb{R}$ be the function which takes each point $t \in I$ into the inner product of the vectors $v = \varphi_{x_i}(z) - \varphi_{x_i}(y)$ and

$$[\operatorname{loc}(\varphi_{x_i}, \varphi'_{x_i}) f]((1-t)\varphi_{x_i}(y) + t\varphi_{x_i}(z)) - \operatorname{loc}(\varphi_{x_i}, \varphi'_{x_i}) f(y),$$

computed in $\mathbb{R}^{n'}$. Next denote by $J(u)$ the symmetrised upper $n \times n$-part of the Jacobi matrix of the map $\operatorname{loc}(\varphi_{x_i}, \varphi'_{x_i}) f$ at the point $u \in \operatorname{int} \mathbb{D}^n$, and by q_t - the bilinear form $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ having the matrix $J((1-t)\varphi_{x_i}(y) + t\varphi_{x_i}(z))$. The function s is smooth and its derivative at the point t is precisely $q_t(v, v)$, and is therefore positive (as a consequence of the definition of $\mathcal{U}_i$). Moreover, $s(0) = 0$, which implies that $s(1) > 0$, i.e., $f(y) \neq f(z)$. □

Corollary 3.4.1.5. *If X is compact, then the set of neat $\mathcal{C}^r$-embeddings $X \to X'$ is open in $\mathcal{C}^r_\partial(X, X')$ ($1 \le r \le \infty$).*

Proof. This is an immediate corollary of Theorem 3.4.1.4 since the set in question is just $\operatorname{Emb}^r(X, X') \cap \mathcal{C}^r_\partial(X, X')$. □

Corollary 3.4.1.6. *If X is compact, then the set $\operatorname{Diff}^r(X, X')$ is open in $\mathcal{C}^r_\partial(X, X')$ ($1 \le r \le \infty$).*

Proof. This is a corollary of Corollary 3.4.1.5 (see Remark 3.1.5.1). □

3.4.2 The Simplest Embedding Theorems

Theorem 3.4.2.1. *Every compact $\mathcal{C}^{\ge r}$-manifold, $1 \le r \le \infty$, admits a $\mathcal{C}^r$-embedding in a Euclidean space of sufficiently high dimension.*

Proof. Let n be the dimension of the given manifold X, and let $\alpha\colon \mathbb{R}^n \to I$ be a $\mathcal{C}^r$-function equal to 1 on $\mathbb{D}^n$, smaller than 1 outside $\mathbb{D}^n$, and equal to 0 outside the concentric ball of radius 2. For each point $x \in X$, fix a chart $\varphi_x \in \operatorname{Atl}_x X$ with $\operatorname{im} \varphi_x = \mathbb{R}^n$ or $\mathbb{R}^n_-$, and $\varphi_x = 0$. Define a map $j_x\colon X \to \mathbb{R} \times \mathbb{R}^n$ by the formula

$$j_x(y) = \begin{cases} (\alpha(\varphi_x(y)), \alpha(\varphi_x(y))\varphi_x(y), & \text{if } \ y \in \operatorname{supp} \varphi_x, \\ (0, 0), & \text{if } \ y \in X \setminus \operatorname{supp} \varphi_x. \end{cases}$$

Now cover X by a finite number of sets $U_x = \varphi_x^{-1}(\operatorname{int}\mathbb{D}^n)$, say $U_{x_1}, \dots, U_{x_s}$, and define

$$j\colon X \to (\mathbb{R}\times\mathbb{R}^n)\times\cdots\times(\mathbb{R}\times\mathbb{R}^n) = \mathbb{R}^{s(n+1)}, \quad y \mapsto (j_{x_1}(y), \dots, j_{x_s}(y)).$$

The map j is of class $\mathcal{C}^r$ and injective: if $y \in U_{x_i}$ and $y' \neq y$, then $j_{x_i}(y') \neq j_{x_i}(y)$.

Indeed, if $y' \notin U_{x_i}$, then $\alpha(\varphi_{x_i}(y')) < 1$, whereas $\alpha(\varphi_{x_i}(y)) = 1$; if $y' \in U_{x_i}$, then

$$\alpha(\varphi_{x_i}(y'))\varphi_{x_i}(y') = \varphi_{x_i}(y') \neq \varphi_{x_i}(y) = \alpha(\varphi_x(y))\varphi_x(y)$$

Moreover, j is an immersion, since j_{x_i} is an immersion on U_{x_i} (the second component $X \to \mathbb{R}^n$ of the map j agrees with φ_{x_i} on U_{x_i}). Therefore, j is a $\mathcal{C}^r$-embedding (see Corollary 3.1.5.4). $\square$

Supplement for the Case of Non-empty Boundary

Lemma 3.4.2.2. *On any compact $\mathcal{C}^{\geq r}$-manifold, $1 \leq r \leq \infty$, there is a (real) $\mathcal{C}^r$-function h, equal to 0 on ∂X, positive on $\operatorname{int} X$, and having no critical points on ∂X.*

Proof. Let $\alpha\colon \mathbb{R}^n \to I$, $n = \dim X$, be a $\mathcal{C}^r$-function equal to 1 on $\mathbb{D}^n$ and equal to 0 outside the concentric ball of radius 2. For each point $x \in \partial X$ fix a chart $\varphi_x \in \operatorname{Atl}_x X$ such that $\operatorname{im}\varphi_x = \mathbb{R}^n_-$ and $\varphi_x(x) = 0$, and define two functions $f_x, g_x\colon X \to \mathbb{R}$ through the formulae

$$(f_x(y), g_x(y)) = \begin{cases} (1-\alpha(\varphi_x(y)), \beta(\varphi_x)), & \text{if} \quad y \in \operatorname{supp}\varphi_x, \\ (1,0) & \text{if} \quad y \in X \setminus \operatorname{supp}\varphi_x, \end{cases}$$

Here $\beta\colon \mathbb{R}^n \to \mathbb{R}$ is given by $\beta(t_1, \dots, t_n) = t_1\alpha(t_1, \dots, t_n)$. Covering ∂X by a finite number of sets $U_x = \varphi_x^{-1}(\operatorname{int}\mathbb{D}^n)$, say $U_{x_1}, \dots, U_{x_s}$, and setting

$$h(y) = \prod_{i=1}^{s} f_{x_i}(y) + \sum_{i=1}^{s} g_{x_i}(y),$$

we obtain the needed function $h\colon X \to \mathbb{R}$. In fact, h vanishes identically on ∂X, since f is equal to 0 on U_{x_i} and all the functions g_{x_i} vanish identically on ∂X; h is positive on $\operatorname{int} X$, since all the functions f_{x_i}, g_{x_i} are non-negative and g_{x_i} is positive at all points of $\operatorname{int} X$, excepting the zeros of f_{x_i}. Finally, h has no critical points on ∂X, since $\sum g_{x_i}$ has no critical points on ∂X (the derivative with respect to the first coordinate of the local representative $\operatorname{loc}(\varphi_{x_k}, \operatorname{id}\mathbb{R})g_{x_i}$, i.e, of the composition $(\varphi_{x_i}|_{\operatorname{supp}\varphi_{x_k}}) \circ \varphi_{x_k}^{-1}$, is negative on $\mathbb{D}^n \cap \mathbb{R}^{n-1}_1$ for $k =$i and non-positive on $\mathbb{D}^n \cap \mathbb{R}^{n-1}_1$ for all k), while $\prod f_{x_i}$ vanishes identically on $\cup U_{x_i}$. $\square$

Theorem 3.4.2.3. *Every compact $\mathcal{C}^{\geq r}$-manifold, $1 \leq r \leq \infty$, admits a neat $\mathcal{C}^r$-embedding in a Euclidean space of sufficiently high dimension.*

Proof. The formula $x \mapsto (-h(x), j(x))$, where j is an arbitrary $\mathcal{C}^r$-embedding in $\mathbb{R}^n$ (see Theorem 3.4.2.1), and h is the function constructed in Lemma 3.4.2.2, defines a neat $\mathcal{C}^r$-embedding in $\mathbb{R}^{q+1}_- = \mathbb{R}^1_- \times \mathbb{R}^q$. $\square$

Information

Remark 3.4.2.4. The compactness assumption and the condition that $r \neq a$ may be eliminated from the formulations of 3.4.2.1 and 3.4.2.3. Any smooth manifold of class $\mathcal{C}^{\geq r}$, with $r \leq \infty$ or $r = a$, compact or not, can be $\mathcal{C}^r$-embedded in Euclidean space, and any smooth manifold of class $\mathcal{C}^{\geq r}$, with $r \leq \infty$ or $r = a$, compact or not, admits a neat $\mathcal{C}^{\geq r}$-embedding Euclidean space. For proofs see [22] and [8].

We should mention that the case $r = a$ in Theorems 3.4.2.1 and 3.4.2.3 is exceedingly difficult and this is the reason why we excluded it here. In the sequel we shall exclude it from other formulations too: cf., for example, Theorems 3.4.4.2, 3.4.5.3, 3.4.6.5, and 4.6.2.7.

3.4.3 Transversalisations and Tubes

Remark 3.4.3.1. In this subsection, we consider the image in Euclidean space of a smooth manifold under a differentiable embedding and study the structure of a neighbourhood of this image. The results are concentrated in Theorems 3.4.3.4, 3.4.3.5, and 3.4.3.7, and serve as the technical basis for the remaining part of the present section.

Remark 3.4.3.2. Let j be a differentiable embedding of the smooth, closed, n-dimensional manifold X in $\mathbb{R}^q$. A *transversalisation* of j is a continuous map $\tau\colon X \to G(q, q-n)$ such that, for each point $x \in X$, the plane $\tau(x)$ is *transverse* to the plane $d_x j(\operatorname{Tang}_x X)$ (i.e., the two planes intersect at only one point). A basic example is the *normal transversalisation* which associates to each point $x \in X$ the corresponding normal plane (i.e., the orthogonal complement to $d_x j(\operatorname{Tang}_x X)$ in $\mathbb{R}^q$); if j is of class $\mathcal{C}^r$, then its normal transversalisation is obviously of class $\mathcal{C}^{r-1}$ (cf. Remark 3.1.4.2).

Given an embedding $j\colon X \to \mathbb{R}^q$ and a transversalisation $\tau\colon X \to G(q, q-n)$ of j, one can construct the natural map $\widetilde{\tau}\colon X \to G'(q, q-n)$, which takes each point x into the plane $j(x) + \tau(x)$ (which is parallel to $\tau(x)$ and passes through $j(x)$). We denote the ball and the sphere with centre $j(x)$ and radius ρ in $j(x) + i(x)$ by $d_\tau(x, \rho)$ and $s_\tau(x, \rho)$, respectively. The unions $\cup_{x \in X} d_\tau(x, \rho)$ and $\cup_{x \in X} [d_\tau(x, \rho) \setminus s_\tau(x, \rho)]$ are called the *tube* (or the *tubular neighbourhood*) and the *open tube* (or the *open tubular neighbourhood*) of radius ρ of the transversalisation τ , and are denoted by $\operatorname{Tub}_\tau \rho$and $\operatorname{tub}_\tau \rho$, respectively.

A tube $\operatorname{Tub}_\tau \rho$ is said to be *neat* if there is a $\sigma > \rho$ such that:

(i) the open balls $d_\tau(x, \sigma) \setminus s_\tau(x, \sigma)$, $x \in X$, are pairwise disjoint and the open tube $\operatorname{tub}_\tau \sigma$ they form is a neighbourhood of $j(X)$ in $\mathbb{R}^q$;

(ii) the map of this neighbourhood onto X, which transforms all the points of $d_\tau(x, \sigma) \setminus s_\tau(x, \sigma)$ into x, is smooth.

The restrictions of the last map to $\operatorname{Tub}_\tau \rho$ or $\operatorname{tub}_\tau \rho$ (which obviously do not depend on the choice of σ) are called *projections* and are denoted by proj_τ.

If $\operatorname{Tub}_\tau \rho$ is neat, then all the tubes $\operatorname{Tub}_\tau \rho'$ with $\rho' < \rho$ are obviously neat.

Warning: it may happen that the normal transversalisation does not have a neat tube, or even a tube such that the balls $d_{tau}(x, \rho)$ are disjoint; see Exercise 3.4.11.4.

Remark 3.4.3.3. The following construction enables us to represent a neat tubular neighbourhood as the image of some ideal model of itself, and is required in the proofs of Theorems 3.4.3.4 and 3.4.3.5.

Let Tu_τ be that subset of the product $X \times \mathbb{R}^q$ consisting of all the pairs (x, t) with $t \in \tau(x)$. For $\rho > 0$, we let $\mathrm{Tu}_\tau\,\rho$ and $\mathrm{tu}_\tau\,\rho$ denote the pieces of Tu_τ such that $\mathrm{dist}(0, t) \leq \rho$ and $\mathrm{dist}(0, t) < \rho$, respectively. We also denote by $\mathrm{nat}\colon \mathrm{Tu}_\tau \to \mathbb{R}^q$ the map given by $\mathrm{nat}(x, t) = j(x) + t$. nat is obviously an isometry of each plane $x \times \tau(x)$ onto the corresponding plane $j(x) + \tau(x)$, and transforms $\mathrm{Tu}_\tau\,\rho$ $(\mathrm{tu}_\tau\,\rho)$ exactly into $Tub_\tau\rho$ (respectively, $\mathrm{tub}_\tau\,\rho$). It is also clear that nat is injective on $\mathrm{Tu}_\tau\,\rho(tu_\tau\rho)$ if and only if the balls $d_\tau(x, \rho)$ (respectively, the open balls $d_\tau(x, \rho) \setminus s_\tau(x, \rho)$) are pairwise disjoint. In this case nat transforms the restriction to $\mathrm{Tu}_\tau\,\rho$ of the projection $\mathrm{proj}_1\colon X \times \mathbb{R}^q \to X$ into the projection $\mathrm{proj}_\tau\colon \mathrm{Tub}_\tau\,\rho \to X$ (respectively, the restriction to $\mathrm{tu}_\tau\,\rho$ of proj_1 into $\mathrm{proj}_\tau\colon \mathrm{tub}_\tau \to X$).

We are interested only in smooth transversalisations τ. If j and τ are $\mathcal{C}^r$-maps with $r \geq 1$, then $X \times \mathbb{R}^q$ is a $\mathcal{C}^{\geq r}$-manifold, Tu_τ is a neat q-dimensional submanifold of $X \times \mathbb{R}^q$ (without boundary), and nat is a $\mathcal{C}^r$-map. Moreover, in this case $\mathrm{Tu}_\tau\,\rho$ is a compact q-dimensional submanifold of Tu_τ such that $\mathrm{int}(\mathrm{Tu}_\tau\,\rho) = \mathrm{tu}_\tau\,\rho$ and the restriction of the projection $\mathrm{proj}\,1\colon X \times \mathbb{R}^q \to X$ to each of the manifolds Tu_τ, $\mathrm{Tu}_\tau\,\rho$, and $\mathrm{tu}_\tau\,\rho$ is a $\mathcal{C}^r$-submersion.

Theorem 3.4.3.4. *Let the maps j and τ be of class $\mathcal{C}^r$, $r \geq 1$. If $\mathrm{Tub}_\tau\,\rho$ is a neat tube, then it is a $\mathcal{C}^r$-submanifold of $\mathbb{R}^q$ with $\mathrm{int}(\mathrm{Tub}_\tau\,\rho) = \mathrm{tub}_\tau\,\rho$, and $\mathrm{proj}_\tau\colon \mathrm{Tub}_\tau\,\rho \to X$ is a $\mathcal{C}^r$-submersion.*

Proof. Let $\sigma > \rho$ be such that the conditions (i) and (ii) in Remark 3.4.3.2 are satisfied. As Remark 3.4.3.3 shows, the map $\mathrm{abr\,nat}\colon \mathrm{tu}_\tau\,\sigma \to \mathrm{tub}_\tau\,\sigma$ is invertible and its inverse $\mathrm{abr\,nat}^{-1}\colon\ \colon \mathrm{tub}_\tau\,\sigma \to \mathrm{tu}_\tau\,\sigma$ is obviously given by $y \mapsto (\mathrm{proj}_\tau(y), y - j \circ \mathrm{proj}_\tau(y))$. This formula shows that $\mathrm{abr\,nat}^{-1}$ is smooth and thus a $\mathcal{C}^r$-diffeomorphism. Now it is evident that the properties of $\mathrm{Tub}_\tau\,\rho$ and the projection $\mathrm{proj}_\tau\colon \mathrm{Tub}_\tau\,\rho \to X$ which we have to verify are consequences of the properties established in Remark 3.4.3.3 for their models $\mathrm{Tu}_\tau\,\rho$ and $\mathrm{abr\,proj}_1\colon \mathrm{Tu}_\tau\,\rho \to X$. ☐

Theorem 3.4.3.5. *Every smooth transversalisation has a neat tube.*

Proof. Let τ be a smooth transversalisation of the embedding $j\colon X \to \mathbb{R}^q$. Since the planes $\tau(x)$ and $d_x j(\mathrm{Tang}_x X)$ are transverse, the differential $d_{x,t}\,\mathrm{nat}$ is nondegenerate when $t = 0$. Hence, nat defines a diffeomorphism of a neighbourhood U of $X \times 0$ in Tu_τ onto a neighbourhood of $j(X)$ (see Theorem 3.1.5.5). It is clear that if $\mathrm{Tu}_\tau\,\rho \subset U$, then $Tub_\tau\rho$ is a neat tube of the transversalisation τ. ☐

Lemma 3.4.3.6. *Suppose X and X' are closed $\mathcal{C}^{\geq r}$-manifolds, $1 \leq r \leq \infty$. If there exists a $\mathcal{C}^r$-embedding of X' in Euclidean space together with a $\mathcal{C}^r$-transversalisation, then the set $\mathcal{C}^r(X, X')$ is dense in $\mathcal{C}(X, X')$.*

Proof. Fix a $\mathcal{C}^r$-embedding $j\colon X \to \mathbb{R}^q$, and a $\mathcal{C}^r$-embedding $j'\colon X' \to \mathbb{R}^{q'}$ together with a $\mathcal{C}^r$-transversalisation τ^{prime}. It suffices to show that given an arbitrary continuous map $f\colon X \to X'$ and an arbitrary $\varepsilon > 0$, there is a $\mathcal{C}^r$-map $g\colon X \to X'$ such that

$$\max_{x \in X} \operatorname{dist}(j' \circ f(x), j' \circ g(x)) < \varepsilon$$

(see Theorem 1.2.7.3). We construct the neat tube $Tub_{\tau'}\rho'$ with $\rho' \leq \varepsilon/2$ and choose

$$\delta \overset{def}{=} \min(\varepsilon/2, \operatorname{Dist}(j'(X'), \mathbb{R}^{q'} \setminus \operatorname{tub}_{\tau'} \rho')).$$

Note that $\delta > 0$ (see Theorem 1.1.7.15). Then, according to Theorem 1.1.5.17, the composition

$$j(X) \xrightarrow{(\operatorname{abr} j)^{-1}} X \xrightarrow{f} X' \xrightarrow{j'} \mathbb{R}^{q'}$$

extends to a continuous map $f_1\colon \mathbb{R}^q \to \mathbb{R}^{q'}$. Now Theorem 3.3.1.7 yields a map $g_1\colon \mathbb{R}^q \to \mathbb{R}^{q'}$ with polynomial components and such that

$$\max_{x \in X} \operatorname{dist}(f_1 \circ j(x), g_1 \circ j(x)) < \delta.$$

This in turn shows that $g_1 \circ j(X) \subset \operatorname{tub}_{\tau'}, \rho'$, and it is clear that $g(x) = \operatorname{proj}_{\tau}(g_1 \circ j(x))$ defines the desired map $g\colon X \to X'$. ☐

Theorem 3.4.3.7. *Every $\mathcal{C}^r$-embedding of a closed $\mathcal{C}^{\geq r}$-manifold, $1 \leq r \leq a$, in Euclidean space admits a $\mathcal{C}^r$-transversalisation.*

Proof. If $r = a, \infty$, the normal transversalisation will suffice. If $1 \leq r < \infty$, the existence of the normal transversalisation shows that the set of all transversalisations of a given $\mathcal{C}^r$-embedding $j\colon X \to \mathbb{R}^q$ is not empty. Since the latter set is (trivially) open in $\mathcal{C}(X, G(q, q - \dim X))$, it is enough to show that $\mathcal{C}^r(X, G(q, q - \dim X))$ is dense in $\mathcal{C}(X, G(q, q - \dim X))$. But this is a consequence of Lemma 3.4.3.6, because $G(q, q - \dim X)$ can be analytically embedded in Euclidean space. ☐

3.4.4 Smoothing Maps in the Case of Closed Manifolds

Remark 3.4.4.1. Now we arrive at the main topic of the present section — approximating maps of one smooth manifold into another by maps which are more regular in a sense or another as, for example, maps of a higher differentiability class, or embeddings, or immersions.

In this subsection we consider only approximations which raise the differentiability class of maps without improving their other properties, and we restrict ourselves to the simplest case — that of the closed manifolds.

Theorem 3.4.4.2. *If $r \leq \infty$ then for any closed $\mathcal{C}^{\geq r}$-manifolds X and X' and any $s < r$, the set $\mathcal{C}^r(X, X')$ is dense in $\mathcal{C}^s(X, X')$. The same is true when $r = a$ provided that X and X' can be $\mathcal{C}^a$-embedded in Euclidean spaces.*

Proof. We have to show that given a map $f \in \mathcal{C}^s(X, X')$ and a neighbourhood $\mathcal{U}$ of f in $\mathcal{C}^s(X, X')$, there is a $\mathcal{C}^r$-map in $\mathcal{U}$. Fix $\mathcal{C}^r$-embeddings $j\colon X \to \mathbb{R}^q$ and $j'\colon X' \to \mathbb{R}^{q'}$, corresponding transversalisations τ and τ', and corresponding neat tubes $\operatorname{Tub}_\tau \rho$ and $\operatorname{Tub}_{\tau'} \rho'$. Now consider the mapping

$$\mathcal{C}^s(\operatorname{abr} j, \operatorname{proj}_{\tau'})\colon \mathcal{C}^s(\operatorname{Tub}_\tau(\rho/2, \operatorname{tub}_\tau, \rho') \to \mathcal{C}^s(X, X'),$$

where $\operatorname{abr} j = [\operatorname{abr} j\colon X \to \operatorname{Tub}_\tau(\rho/2)]$ (see Remark 3.4.1.1). Since it takes $\mathcal{C}^r$-maps into $\mathcal{C}^r$-maps, it suffices to prove that the preimage $\mathcal{V}$ of $\mathcal{U}$ under this mapping intersects $\mathcal{C}^r(\operatorname{Tub}_\tau(\rho/2), \operatorname{tub}_{\tau'}, \rho')$. But $\mathcal{V}$ is open and contains the restriction to $\operatorname{Tub}_\tau(\rho/2)$ of the composition

$$\operatorname{tub}_\tau \rho \xrightarrow{\operatorname{proj} \tau} X \xrightarrow{f} X' \xrightarrow{\operatorname{abr} j'} \operatorname{tub}_{\tau'} \rho'. \tag{3.4.4.3}$$

The image of this restriction is compact, and so it lies at a positive distance from $\mathbb{R}^{q'} \setminus \operatorname{tub}_{\tau'} \rho'$. Consequently, there exists $\varepsilon > 0$ such that if, at the points of $\operatorname{Tub}_\tau(\rho/2)$, the partial derivatives of order $0, 1, \ldots, s$ of the coordinate functions of a map $g \in \mathcal{C}^s(\operatorname{Tub}_\tau(\rho/2), \mathbb{R}^{q'})$ differ from the corresponding partial derivatives of the map (3.4.4.3) by less than ε, then

$$g(\operatorname{Tub}_\tau(\rho/2)) \subset \operatorname{tub}_{\tau'} \rho' \quad \text{and} \quad [\operatorname{abr} g\colon \operatorname{Tub}_\tau(\rho/2) \to \operatorname{tub}_{\tau'} \rho] \in \mathcal{V}.$$

Finally, apply Theorem 3.3.1.7 to the coordinate functions of (3.4.4.3) to deduce that there exists a map g with polynomial coordinate functions whose partial derivatives have the property above. Thus $\mathcal{V} \cap \mathcal{C}^r(\operatorname{Tub}_\tau(\rho/2), \operatorname{tub}_{\tau'} \rho') \neq \varnothing$. □

Remark 3.4.4.4. Comparing Theorem 3.4.4.2 with the Theorems 3.4.1.2, 3.4.1.3, 3.4.1.4, and Corollary 3.4.1.6, we see that for $r \leq \infty$ and $1 \leq s < r$, and for any given closed $\mathcal{C}^{\geq r}$-manifolds X and X', the following holds: $\operatorname{Imm}^r(X, X')$ is dense in $\operatorname{Imm}^s(X, X')$, $\operatorname{Subm}^r(X, X')$ is dense in $\operatorname{Subm}^s(X, X')$, $\operatorname{Emb}^r(X, X')$ is dense in $\operatorname{Emb}^s(X, X')$, and finally $\operatorname{Diff}^r(X, X')$ is dense in $\operatorname{Diff}^s(X, X')$. The same is true when $r = a$ provided that X and X' can be $\mathcal{C}^a$-embedded in Euclidean spaces.

Corollary 3.4.4.5. *If two closed $\mathcal{C}^{\geq r}$-manifolds, $1 \leq r \leq \infty$, are diffeomorphic, then they are $\mathcal{C}^r$-diffeomorphic. The same is true for $r = a$ provided that X and X' can be $\mathcal{C}^a$-embedded in Euclidean spaces.*

Information 3.4.4.6. Two closed homeomorphic $\mathcal{C}^a$-manifolds are not necessarily diffeomorphic. Historically, the first such examples where $\mathcal{C}^a$-manifolds which are homeomorphic, but not diffeomorphic to $\mathbb{S}^7$; see [15].

Supplement to to Theorem 2

Theorem 3.4.4.7. *If* $r \leq \infty$*, then* $\mathcal{C}^r(X, X')$ *is dense in* $\mathcal{C}^s(X, X')$*,* $s < r$*, in the following more general situation too:* X *is a closed* $\mathcal{C}^{\geq r}$*-manifold and* X' *is an open subset of a closed* $\mathcal{C}^{\geq r}$*-manifold* Y*. The same is true for* $r = a$ *if* X *and* Y *can be* $\mathcal{C}^a$*-embedded in Euclidean spaces.*

Proof. The proof reduces to observing that the mapping

$$\mathcal{C}^s(\mathrm{id}, \mathrm{incl})\colon \mathcal{C}^s(X, X')/to\mathcal{C}^s(X, Y)$$

is a topological embedding with open image which carries $\mathcal{C}^r(X, X')$ into the intersection of this image with $\mathcal{C}^r(X, Y)$. Since $\mathcal{C}^r(X, Y)$ is dense in $\mathcal{C}^s(X, Y)$, the above intersection is dense in this image, and $\mathcal{C}^r(X, X')$ is dense in $\mathcal{C}^s(X, X')$. □

Lemma 3.4.4.8. *Every pair of disjoint closed subsets of a closed* $\mathcal{C}^r$*-manifold with* $r \leq \infty$ *has a Urysohn function of class* $\mathcal{C}^r$*.*

Proof. Let $\varphi\colon X \to I$ be an arbitrary Urysohn function for the given pair of subsets A, B of X. According to Theorem 3.4.4.7 , there is a $\mathcal{C}^r$-function $\psi\colon X \to \mathbb{R}$ such that $\max_{x \in X} |\psi(x) - \varphi(x)| < 1/3$. If now $\lambda\colon \mathbb{R} \to I$ is a $\mathcal{C}^r$-function such that $\lambda(y) = 0$ for $y \leq 1/3$ and $\lambda(y) = 1$ for $y \geq 2/3$, then $\lambda \circ \psi$ is obviously a Urysohn $\mathcal{C}^r$-function for the pair A, B. □

Theorem 3.4.4.9. *Let* X *and* X' *be closed* $\mathcal{C}^{\geq r}$*-manifolds and let* A *be a closed subset of* X*. If* $0 \leq s < r \leq \infty$*, then that part of* $\mathcal{C}^s(X, X')$ *consisting of the* $\mathcal{C}^r$*-extensions of a given map* $\varphi\colon A \to X'$ *is dense in the part of* $\mathcal{C}^s(X, X')$ *consisting of the extensions of* φ *which are of class* $\mathcal{C}^r$ *in a neighbourhood of* A *(the neighbourhood depends upon the extension).*

Proof. Let $f \in \mathcal{C}^s(X, X')$ be an extension of φ which is of class $\mathcal{C}^r$ in a neighbourhood U of A. Given a neighbourhood $\mathcal{U}$ of f in $\mathcal{C}^s(X, X')$, we have to show that $\mathcal{U}$ contains a $\mathcal{C}^r$-extension of φ. Fix a $\mathcal{C}^r$-embedding $j'\colon X' \to \mathbb{R}^{q'}$, a $\mathcal{C}^r$-transversalisation τ' of j' and a neat tube $\mathrm{Tub}_{\tau'}\,\rho'$, and denote by $\mathcal{V}$ the piece of $\mathcal{C}^s(X, X')$ consisting of all the maps g such that

$$\max_{x \in X} \mathrm{dist}(j' \circ f(x), j' \circ g(x)) < \mathrm{Dist}(j'(X), \mathbb{R}^{q'} \setminus \mathrm{tub}_{\tau'}\,\rho').$$

It is obvious that $\mathcal{V}$ is open and that for any $g \in \mathcal{V}$, $x \in X$, the segment with endpoints $j' \circ f(x)$, $j' \circ g(x)$ is contained in $\mathrm{tub}_{\tau'}\,\rho'$. Next construct a Urysohn $\mathcal{C}^r$-function ψ for the pair A, $X \setminus U$ (see Lemma 3.4.4.8) and consider the map $\Phi\colon \mathcal{V} \to \mathcal{C}^s(X, X')$ which transforms g into the map

$$x \mapsto \mathrm{proj}_{\tau'}((1 - \psi(x))j' \circ f(x) + \psi(x)j' \circ g(x)).$$

One may check directly that Φ is continuous and $\Phi(f) = f$; hence the set $\Phi^{-1}(\mathcal{U})$ is open and non-empty. Now Theorem 3.4.4.2 shows that $\varphi^{-1}(\mathcal{U})$ contains a $\mathcal{C}^r$-map. Finally, we note that Φ takes $\mathcal{C}^r$-maps into $\mathcal{C}^r$-extensions of φ. □

Theorem 3.4.4.10. *Suppose that X and X' are closed $\mathcal{C}^{\geq r}$-manifolds and A is a submanifold of X which is itself closed as a manifold. Let $\varphi\colon A \to X'$ be a $\mathcal{C}^r$-map. If $0 \leq s < r \leq \infty$, then that part of $\mathcal{C}^s(X, X')$ consisting of the $\mathcal{C}^r$-extensions of φ is dense in the part of $\mathcal{C}^s(X, X')$ consisting of all the $\mathcal{C}^s$-extensions of φ.*

Proof. Given a $\mathcal{C}^s$-extension of φ and a neighbourhood $\mathcal{U}$ of this extension in $\mathcal{C}^s(X, X')$, we have to show that $\mathcal{U}$ contains a $\mathcal{C}^r$-extension of φ. Fix $\mathcal{C}^r$-embeddings $j\colon X \to \mathbb{R}^q$ and $j'\colon X' \to \mathbb{R}^{q'}$, a $\mathcal{C}^r$-transversalisation τ of the embedding $j|_A\colon A \to \mathbb{R}^q$ and a $\mathcal{C}^r$-transversalisation τ' of j', and corresponding neat tubes $\operatorname{Tub}_\tau \rho$ and $\operatorname{Tub}_{\tau'} \rho'$. Further, denote by $\mathcal{V}$ the piece of $\mathcal{C}^s(X, X')$ consisting of all the maps g such that

$$\max_{x \in X} \operatorname{dist}(j' \circ \varphi(x), j' \circ g(x)) < \operatorname{Dist}(j'(X'), \mathbb{R}^{q'} \setminus \operatorname{tub}_{\tau'} \rho').$$

Obviously, $\mathcal{V}$ is open and contains all the $\mathcal{C}^s$-extensions of φ to X. Now take any Urysohn $\mathcal{C}^r$-function ψ for the pair $X \setminus j^{-1]}(\operatorname{tub}_\tau \rho)$, A and consider the mapping $\Phi\colon \mathcal{V} \to \mathcal{C}^s(X, X')$ which transforms each map g into the map

$$x \mapsto \begin{cases} \operatorname{proj}_{\tau'} & (j \circ g(x) + \psi(x)[j' \circ \varphi \circ \operatorname{proj}_\tau(j(x)) - j' \circ g \circ \operatorname{proj}_\tau(j(x))]), \\ & \text{if} \quad j(x) \in \operatorname{Tub}_\tau \rho, \\ g(x), & \text{if} \quad j(x) \notin \operatorname{Tub}_\tau \rho. \end{cases}$$

It is clear that Φ is continuous and that $\Phi(g) = g$ whenever g extends φ. This implies that $\Phi^{-1}(\mathcal{U})$ is an open non-empty set which, according to Theorem 3.4.4.2, contains a $\mathcal{C}^r$-map. Finally, note that Φ takes $\mathcal{C}^r$-maps into $\mathcal{C}^r$-extensions of φ. □

3.4.5 Glueing Manifolds Smoothly

Remark 3.4.5.1. Our main task in this subsection is to make the necessary preparations for extending the basic approximation theorems given in the previous subsection, i.e., Theorem 3.4.4.2, Remark 3.4.4.4 and Corollary 3.4.4.5, in their non-analytic version, to include compact manifolds with boundary. The main tool used in the extension is that of smooth doubling of a compact manifold, an operation which transforms it into a closed manifold. However, we find it convenient to define and study a more general operation, which is useful for other purposes too - the smooth glueing of smooth compact manifolds. To begin with, we need to investigate the structure of a smooth compact manifold in the vicinity of its boundary.

Definition 3.4.5.2. A *collaring* of a compact $\mathcal{C}^r$-manifold X ($0 \leq r \leq a$) is a $\mathcal{C}^r$-embedding of the cylinder $\partial X \times I$ into X, which takes the point $(x, 0)$ into x, for each $x \in \partial X$. The image of $\partial X \times I$ under such an embedding is known as a *collar* (on X).

If X is a smooth manifold (i.e., $r \geq 1$), a collaring is a differentiable embedding and its image is a submanifold of codimension 0, whose boundary consists of ∂X and of a submanifold of $\operatorname{int} X$ diffeomorphic to ∂X.

Theorem 3.4.5.3. *If $1 \leq r \leq \infty$, every compact $\mathcal{C}^r$-manifold admits a collaring.*

Proof. Let X be the given manifold. Pick a neat $\mathcal{C}^r$-embedding $j\colon X \to \mathbb{R}^q$ (see Theorem 3.4.2.3), a $\mathcal{C}^r$-transversalisation τ of the composite embedding

$$X \xrightarrow{\operatorname{abr} j} \mathbb{R}^q_- \xrightarrow{\text{incl}} \mathbb{R}^q,$$

and a neat tube $\operatorname{Tub}_\tau \rho$. Consider the map $\varphi\colon j(\operatorname{tub}_\tau \rho) \to \partial X \times \mathbb{R}^1_-$, defined as $x \mapsto (\operatorname{proj}_\tau(j(x)), j_1(x))$, where j_1 is the first coordinate function of j. Since j is neat, the differential $d_x\varphi$ is non-degenerate at each point $x \in \partial X$, so that φ realises a diffeomorphism of a neighbourhood of ∂X onto a neighbourhood of $\partial X \times 0$ (see Theorem 3.1.5.5). Now let $\varepsilon > 0$ be small enough so that the product $\partial X \times [-\varepsilon, 0]$ is contained in the previous neighbourhood. Then the formula $(x,t) \mapsto \varphi^{-1}(x, -\varepsilon t)$ obviously defines a collaring of X. ☐

Information 3.4.5.4. The compact topological manifolds ($r = 0$) and the compact analytic manifolds ($r = a$) admit collarings too. The case $r = 0$ is considered in [4].

Glueing

Remark 3.4.5.5. Suppose that X and X' are compact n-dimensional $\mathcal{C}^r$-manifolds with $r \geq 1$, and let C and C' be submanifolds of ∂X and $\partial X'$, respectively, consisting of whole components of these boundaries. Assuming that C and C' are diffeomorphic, pick a $\mathcal{C}^r$-diffeomorphism $\varphi\colon C \to C'$ and attach X to X' by the composite map

$$C \xrightarrow{\varphi} C' \xrightarrow{\text{incl}} X'$$

(see Remark 1.2.4.8). The resulting space $Y = X' \cup_{\text{incl}\circ\varphi} X$ is obviously a compact, n-dimensional, topological manifold. However, if X and X' have collars then it turns out that Y has a natural $\mathcal{C}^r$-structure that makes it into a collared $\mathcal{C}^r$-manifold. The atlas that defines this $\mathcal{C}^r$-structure consists of the charts of $\operatorname{Atl}(X \setminus C)$ and $\operatorname{Atl}(X' \setminus C')$ (we regard X and X' as parts of Y), as well as the charts Ψ constructed from both the charts $\psi \in \operatorname{Atl} C$ and the collarings $k\colon \partial X \times I \to X$ and $k'\colon \partial X' \times I \to X'$ by the formulae

$$\operatorname{supp} \Psi = k(\operatorname{supp} \psi \times [0,1)) \cup k'(\operatorname{supp} \psi \times [0,1))$$

and

$$\left.\begin{aligned} \Psi(k(z,t)) &= (\psi(z), -t) \\ \Psi(k'(z,t)) &= (\psi(z), t) \end{aligned}\right\} \quad z \in C, \quad t \in [0,1)$$

($\operatorname{im} \psi \subset \mathbb{R}^{n-1}$ and $\operatorname{im} \Psi = \operatorname{im} \psi \times (-1,1) \subset \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$) . It is readily seen that these charts are all pairwise compatible. One constructs a collar on the resulting $\mathcal{C}^r$-manifold Y from those pieces of the collarings k and k' which are preserved under the above procedure. We say that the diffeomorphism φ *glues* X and X' into Y. It is clear that $\partial Y = (\partial X \setminus C) \cup (\partial X' \setminus C')$ and that X, X', and C are submanifolds of Y. The pieces of the collarings k and k' which are related to C and C' yield a *two-sided collaring of the manifold C in Y*, i.e., a $\mathcal{C}^r$-embedding $C \times [-1,1] \to Y$ such that $(z,0) \mapsto z$; its image is a *two-sided collar* of C in Y.

In the particular situation $X' = X$, $C' = C = \partial X$, and $\varphi = \mathrm{id}_{\partial X}$, we use the term *doubling* instead of glueing and denote Y by $\operatorname{dopp} X$. This definition agrees with Definition 3.1.1.10, i.e., $\mathcal{C}^0(\operatorname{dopp} X) = \operatorname{dopp}(\mathcal{C}^0 X)$.

Now suppose that X and X' are oriented manifolds and C and C' are equipped with the induced orientations (see Remark 3.1.3.4). If φ is orientation preserving, then Y is orientable and can be actually oriented in a canonical way. This canonical orientation is that which induces the original orientation on X and the orientation opposite to the original one on X'. In particular, $\operatorname{dopp} X$ is oriented for any oriented manifold X.

The simplest examples show that the $\mathcal{C}^r$-structure on Y depends not only upon the $\mathcal{C}^r$-structures of the manifolds X and X' and the diffeomorphism φ, but also upon the collarings k and k'. Our next objective is to demonstrate that for $r \neq a$ this last dependence is eliminated if we regard Y as distinct up to $\mathcal{C}^r$-diffeomorphisms.

Lemma 3.4.5.6. *Let X and X' be closed $\mathcal{C}^r$-manifolds, $1 \le r \le \infty$, and let $f\colon X \times [-1,1] \to X' \times [-1,1]$ be such that $f(X \times [-1,0]) \subset X' \times (-1,0]$ and $f(X \times [0,1]) \subset X' \times [0,1)$. Assume that $\operatorname{abr} f\colon X \times [-1,0] \to X' \times (-1,0]$ and $\operatorname{abr} f\colon X \times [0,1] \to X' \times [0,1)$ are differentiable embeddings of class $\mathcal{C}^r$, while $\operatorname{abr} f\colon X \times 0 \to X' \times 0$ is a diffeomorphism. Then there exists a $\mathcal{C}^r$-embedding $g\colon X \times [-1,1] \to X' \times [-1,1]$ such that*

$$\begin{aligned}
&g = f \quad \text{on} \quad (X \times [-1,-1/2]) \cup (X \times 0) \cup (X \times [1/2,1]),\\
&g(X \times [-1,0]) = f(X \times [-1,0]),\\
&g(X \times [0,1]) = f(X \times [0,1]).
\end{aligned}$$

Proof. In the proof that follows we let f_1 and f_2 denote the composite maps

$$\begin{tikzcd}
 & X' \\
X \times [-1,1] \arrow[ur, "f_1"] \arrow[r, "f"] \arrow[dr, "f_2"'] & X' \times [-1,1] \arrow[u, "\mathrm{proj}_1"'] \arrow[d, "\mathrm{proj}_2"] \\
 & {[-1,1]}
\end{tikzcd}$$

respectively.

For a start, assume that for some positive ε the map f_1 is constant on each set $x \times [-\varepsilon,\varepsilon]$, $x \in X$. In this case, fix positive numbers δ and η, such that $\delta \le \min(\varepsilon, 1/2)$ and for any $x \in X$ the derivative of the function $t \mapsto f_2(x,t)$ is not less than η on the intervals $[-\delta,0)$ and $(0,\delta]$.

Note that the existence of such δ and η results from the continuity of the functions $X \times [-1, 0] \to \mathbb{R}$ and $X \times [0, 1] \to \mathbb{R}$, given by the formula $(x, t) \mapsto \partial f_2(x, t)\partial\tau|_{\tau=t}$ (for $t = 0$ one takes the left derivative for the first function and the right derivative for the second one), and the positivity of both functions on $X \times 0$. Since X is compact, these functions are bounded from below by a positive constant on $X \times 0$, and hence on $X \times [-\delta, \delta]$ for some $\delta > 0$.

To proceed further, pick a $\mathcal{C}^r$-function $\alpha\colon [-\delta, \delta] \to I$ such that

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad |t| < \delta/4, \\ 1 & \text{for} \quad |t| \geq \delta/2. \end{cases}$$

It is not hard to verify that the formula

$$g(x, t) = \begin{cases} (f_1(x, t), (1 - \alpha(t))\eta t + \alpha(t) f_2(x, t)), & \text{if} \quad |t| \leq \delta, \\ f(x, t), & \text{if} \quad |t| \geq \delta, \end{cases}$$

defines a map $g\colon X \times [-1, 1] \to X' \times [-1, 1]$ with the desired properties.

In the general case, choose $0 < \varepsilon_1 \leq 1/2$ such that the map $\varphi_r\colon X \to X'$, defined as $\varphi_t(x) = f_1(x, t)$, is a diffeomorphism for all $|t| \leq \varepsilon_1$.

Note that the existence of such an ε_1 is a consequence of:

- the continuous dependence of φ_t on t in the $\mathcal{C}^r$-topology,
- the fact that φ_t is a diffeomorphism (recall that $\operatorname{abr} f\colon X \times 0 \to X' \times 0$ is a diffeomorphism), and
- the fact that $\mathrm{Diff}(X, X')$ is open in $\mathcal{C}^r(X, X')$ (see Corollary 3.4.1.6).

Let $\gamma\colon [-\varepsilon_1, \varepsilon_1] \to \mathbb{R}$ be a non-decreasing $\mathcal{C}^r$-function such that

$$\gamma(t) = \begin{cases} 0 & \text{for} \quad |t| \leq \varepsilon/4, \\ t & \text{for} \quad |t| \geq \varepsilon/2. \end{cases}$$

Now define $\widetilde{f}\colon X \times [-1, 1] \to X' \times [-1, 1]$ by

$$\widetilde{f}(x, t) = \begin{cases} f(\varphi_t^{-1} \circ \varphi_{\gamma(t)}(x), t), & \text{if} \quad |t| \leq \varepsilon_1, \\ f(x, t), & \text{if} \quad |t| \geq \varepsilon_1. \end{cases}$$

Then $\widetilde{f}$ satisfies all the conditions imposed to f in the statement of the lemma. Moreover, $\widetilde{f}$ satisfies the extra conditions under which the lemma has already been proved, namely that the composite map

$$X \times [-1, 1] \xrightarrow{\widetilde{f}} X' \times [-1, 1] \xrightarrow{\mathrm{proj}_1} X'$$

is constant on the sets $x \times [-\varepsilon, \varepsilon]$, with $\varepsilon = \varepsilon_1/4$. The map g corresponding to $\widetilde{f}$ via the above procedure has the needed properties because $\widetilde{f}$ agrees with f on $(X \times [-1, -1/2]) \cup (X \times 0) \cup (X \times [1/2, 1])$. $\square$

Theorem 3.4.5.7. *Suppose that* X_1, X_1', X_2, *and* X_2' *are collared* $\mathcal{C}^r$*-manifolds,* $1 \le r \le \infty$, C_1, C_1', C_2, *and* C_2' *are pieces of their boundaries consisting of whole components, and* $\varphi_1\colon C_1 \to C_1'$, *and* $\varphi_2\colon C_2 \to C_2'$ *are* $\mathcal{C}^r$*-diffeomorphisms. Assume that there are* $\mathcal{C}^r$*-diffeomorphisms* $F\colon X_1 \to X_2$ *and* $F'\colon X_1' \to X_2'$, *such that* $F(C_1) = C_2$, $F'(C_1') = C_2'$ *and the diagram*

$$\begin{tikzcd}
C_1 \arrow[r, "\varphi_1"] \arrow[d, "\operatorname{abr} F"'] & C_1' \arrow[d, "\operatorname{abr} F'"] \\
C_2 \arrow[r, "\varphi_2"'] & C_2'
\end{tikzcd}$$

is commutative. If Y_1 *is the result of glueing and* X_1 *and* X_1' *by* φ_1, *and* Y_2 *is the result of glueing and* X_2 *and* X_2' *by* φ_2, *then the manifolds* Y_1 *and* Y_2 *are* $\mathcal{C}^r$*-diffeomorphic. Moreover, there exists a* $\mathcal{C}^r$*-diffeomorphism* $G\colon Y_1 \to Y_2$ *such that* $G(X_1') = X_2'$, $G(C_1) = C_2$, *and* $[\operatorname{abr} G\colon C_1 \to C_2] = [\operatorname{abr} F\colon C_1 \to C_2]$.

Proof. Let $\ell_1\colon C_1 \times [-1,1] \to Y_1$ and $\ell_2\colon C_2 \times [-1,1] \to Y_2$ be two-sided collarings. Denote by $H\colon Y_1 \to Y_2$ the map defined by the formulae

$$H = \begin{cases} [\text{incl}\colon X_2 \to Y_2] \circ F & \text{on} \quad X_1 \\ [\text{incl}\colon X_2' \to Y_2] \circ F' & \text{on} \quad X_1' \end{cases}$$

and choose $\varepsilon > 0$ so that $H \circ \ell_1(C_1 \times [-\varepsilon, \varepsilon]) \subset \ell_2(C_2 \times [-1,1])$. Now apply Lemma 3.4.5.6 to the map

$$f\colon C_1 \times [-1,1] \to C_2 \times [-1,1], \quad (z,t) \mapsto \ell_2^{-1}(H \circ \ell_1(z, \varepsilon t)).$$

This lemma guarantees the existence of a $\mathcal{C}^r$-embedding $g\colon C_1 \times [-1,1] \to C_2 \times [-1,1]$ with

$$g = f \quad \text{on} \quad (C_1 \times [-1,-1/2]) \cup (C_1 \times 0) \cup (C_1 \times [1/2,1]),$$

and satisfies

$$g(C_1 \times [-1,0]) = f(C_1 \times [-1,0]), \quad g(C_1 \times [0,1]) = f(C_1 \times [0,1]).$$

Clearly,

$$G(y) = \begin{cases} H(y) & \text{if} \quad y \notin (C_1 \times [-\varepsilon, \varepsilon]), \\ \ell_2 \circ g(z, t/\varepsilon), & \text{if} \quad y = \ell_1(z,t) \quad \text{with} \quad z_1 \in C_1, \quad t \in [-\varepsilon, \varepsilon], \end{cases}$$

defines the required $\mathcal{C}^r$-diffeomorphism $G\colon Y_1 \to Y_2$. □

Cutting

Theorem 3.4.5.8. *Let* Y *be a* $\mathcal{C}^r$*-manifold,* $1 \le r \le \infty$, *and let* X *and* X' *be compact submanifolds of* Y *such that* $\dim X = \dim X' = \dim Y$ *and* $Y = X \cup X'$.

If $C = X \cap X'$ is a piece of both boundaries ∂X and $\partial X'$ consisting of whole components of ∂X and $\partial X'$, then there is a $\mathcal{C}^r$-embedding $\ell\colon C \times [-1,1] \to Y$ such that

$$\ell(z,0) = z \quad \textit{for any point} \quad z \in C,$$
$$\ell(C \times [-1,0)) \subset \operatorname{int} X, \quad \ell(C \times (0,1]) \subset \operatorname{int} X'.$$

Proof. Fix a $\mathcal{C}^r$-embedding $j\colon Y \to \mathbb{R}^q$, a $\mathcal{C}^r$-transversalisation τ of the embedding $j|_C\colon C \to \mathbb{R}^q$, and a neat tube $\operatorname{Tub}_\tau \rho$. Consider the map $\varphi\colon C \to \mathbb{S}^{q-1}$ which takes each point $z \in C$ into the unit vector tangent to $j(Y)$ at the point $j(z)$, contained in $\tau(z)$, and pointing towards $j(X')$. φ is continuous (in fact, of class $\mathcal{C}^{r-1}$), and so Theorem 3.4.4.2 yields a $\mathcal{C}^r$-map $\varphi_1\colon C \to \mathbb{S}^{q-1}$ such that the inner product $\langle\varphi(z), \varphi_1(z)\rangle$ is positive on C. Define

$$\psi\colon \operatorname{tub}_\tau \rho \to C \times \mathbb{R}, \quad z \mapsto (\operatorname{proj}_\tau(z), \langle z - j \circ \operatorname{proj}_\tau(z), (\varphi_1(z)\rangle).$$

Clearly, ψ is of class $\mathcal{C}^r$ and for $z \in C$ the differential

$$d_{j(z)}\psi\colon \operatorname{Tang}_{j(z)}(\operatorname{tub}_\tau \rho) \to \operatorname{Tang}_{(z,0)}(C \times \mathbb{R}) = (\operatorname{Tang}_z C) \oplus \mathbb{R}$$

induces an isomorphism of $\operatorname{Tang}_{j(z)} j(C)$ onto $\operatorname{Tang}_z C$ and carries the vector $\varphi(z)$ into $\langle\varphi(z), \varphi_1(z)\rangle \in \mathbb{R}$. Moreover, both $\operatorname{Tang}_{j(z)} j(C)$ and $\varphi(z)$ are contained in $\operatorname{Tang}_{j(z)} j(Y)$; hence $d_{j(z)}\psi$ takes $\operatorname{Tang}_{j(z)} j(Y)$ onto $\operatorname{Tang}_{(z,0)}(C \times \mathbb{R})$. Since $\dim Tang_{j(z)} = \dim Tang_{(z,0)}(C \times \mathbb{R})$, we see that $d_{j(z}\psi_{\operatorname{Tang}_{j(z)j(Y)}}$' i.e., the linear map $d_{j(z)}(\psi|_{j(Y)\cap\operatorname{tub}_\tau \rho})$, is an isomorphism. By Theorem 3.1.5.5, $\psi|_{j(Y)\cap\operatorname{tub}_\tau \rho}$, defines a diffeomorphism from a neighbourhood of $j(C)$ onto a neighbourhood of $C \times 0$ Accordingly, $C \times [-\varepsilon, \varepsilon]$ will lie in the previous neighbourhood provided that $\varepsilon > 0$ is small enough. Now it is plain that $\ell(z,t) = j^{-1}(\psi^{-1}(z, \varepsilon t))$ defines the desired embedding $\ell\colon C \times [-1,1] \to Y$. ☐

Corollary 3.4.5.9. *Let Y be a $\mathcal{C}^r$-manifold, $1 \le r \le \infty$, and let X and X' be compact submanifolds of Y such that $\dim X = \dim X' = \dim Y$ and $Y = X \cup X'$. If $X \cap X'$ is a piece of both boundaries ∂X and $\partial X'$, consisting of whole components of ∂X and $\partial X'$, then id_X and $\mathrm{id}_{X'}$ together define a $\mathcal{C}^r$-diffeomorphism of Y onto the manifold obtained from the appropriately collared manifolds X and X' glueing X and X' by $\mathrm{id}(X \cap X')$.*

The Simplest Application

Theorem 3.4.5.10. *Every smooth compact manifold is a CNRS.*

Proof. When the manifold is closed, this is a consequence of Theorems 3.4.2.1, 3.4.3.7, and 3.4.3.5, because the image of a smooth manifold under a differentiable embedding in Euclidean space is the retract of the interior of a neat tube corresponding to a smooth transversalisation of the given embedding. Theorem 1.3.6.4 enables us to reduce the case of manifolds with boundary to the closed case; namely, any compact smooth manifold has a smooth closed double (see Remark 3.4.5.5), and is obviously a retract of this double. ☐

3.4.6 Smoothing Maps in the Presence of a Boundary

Remark 3.4.6.1. The main results of this subsection are Thor ems 3.4.6.5, 3.4.6.10 and which generalise Theorem 3.4.4.2. Lemma 3.4.6.2 is necessary to the proof of Lemma 3.4.6.3, Lemma 3.4.6.3 - to the proof of Lemma 3.4.6.4, and Lemma 3.4.6.4 - to the proof of Theorem 3.4.6.5. Finally, Lemmas 3.4.6.7 and 3.4.6.8 are necessary to the proof of Theorem 3.4.6.10.

Lemma 3.4.6.2. *Let Y be a $\mathcal{C}^{\geq r}$-manifold with $r < \infty$, and let $f\colon Y \times \mathbb{R}^1_- \to \mathbb{R}$ be a $\mathcal{C}^r$-function. Then the function $F\colon Y \times \mathbb{R} \to \mathbb{R}$ defined by*

$$F(y,t) = \begin{cases} f(y,t), & \text{if} \quad t \leq 0, \\ \sum_{k=0} r(-1)^k \binom{r+1}{k+1} f(y,-kt), & \text{if} \quad t \geq 0 \end{cases}$$

is also of class $\mathcal{C}^r$.

Proof. All we must check is that the two expressions defining F, as well as their partial derivatives with respect to t and local coordinates on Y agree for $t = 0$. To see this, it suffices to note that the equality

$$\sum_{k=0}^{r} (-1)^{k+s} \binom{r+1}{k+1} k^s \mathbb{D}^s \varphi(0) = \mathbb{D}^s \varphi(0)$$

holds for $s \leq r$ and any $\mathcal{C}^s$-function $\varphi\colon \mathbb{R}^1_- \to \mathbb{R}$. Indeed, this last equality is equivalent to

$$\sum_{k=-1}^{s} (-1)^k \binom{r+1}{k+1} k^s = 0 \quad (s \leq r)$$

and this is valid if we interpret the sum as the "$(r+1)$-th difference" of the integral function $k \to k^s$ computed at $k = -1$. □

Lemma 3.4.6.3. *Let X be a collared $\mathcal{C}^{\geq r}$-manifold. If $1 \leq r < \infty$, then every function in $\mathcal{C}^r(X, \mathbb{R})$ extends to a function in $\mathcal{C}^r(\operatorname{dopp} X, \mathbb{R})$.*

Proof. Fix a two-sided collaring of the manifold X in $\operatorname{dopp} X$, $\ell\colon \partial X \times [-1,1] \to \operatorname{dopp} X$ (see Remark 3.4.5.5), and pick a $\mathcal{C}^r$-function $\alpha\colon \mathbb{R} \to I$ such that

$$\alpha(t) = \begin{cases} 1 & \text{for} \quad t \leq 0, \quad \text{and} \\ 0 & \text{for} \quad t \geq 1/r. \end{cases}$$

Let $\varphi \in \mathcal{C}^r(X, \mathbb{R})$. Consider the function $f\colon \partial X \to \mathbb{R}^1_-$ defined by the formula

$$f(y,t) = \begin{cases} \alpha(t)\varphi(\ell(y,t)), & \text{if} \quad t \leq 1/r, \\ 0, & \text{if} \quad t \geq 1/r. \end{cases}$$

Since f is $\mathcal{C}^r$, Lemma 3.4.6.2 processes it into a $\mathcal{C}^r$-function $F\colon \partial X \times j\mathbb{R} \to \mathbb{R}$ (take $Y = \partial X$). Now it is plain that

$$\psi(x) = \begin{cases} 0, & \text{if} \quad x \in \operatorname{dopp} X \setminus (X \cup \ell(\partial X \times [0, 1/r])), \\ \varphi(x), & \text{if} \quad x \in X, \\ F(\ell^{-1}(x)), & \text{if} \quad x \in \ell(\partial X \times [0, 1/r]), \end{cases}$$

defines a $\mathcal{C}^r$-function $\psi\colon \operatorname{dopp} X \to \mathbb{R}$ extending φ. □

Lemma 3.4.6.4. *Let X be a collared $\mathcal{C}^{\geq r}$-manifold, and let X' be a closed $\mathcal{C}^{\geq r}$-manifold. If $0 \leq r < \infty$, then every map in $\mathcal{C}^r(X, X')$ extends to a map in $\mathcal{C}^r(\operatorname{dopp} X, X')$.*

Proof. This is evident when $r = 0$. Suppose $r > 0$ and $f \in \mathcal{C}^r(X, X')$. Fix a $\mathcal{C}^r$-embedding $j'\colon X' \to \mathbb{R}^q$, a $\mathcal{C}^r$-transversalisation τ' of j', and a neat tube $\operatorname{Tub}_{\tau'} \rho'$. Lemma 3.4.6.3 ensures that the coordinate functions of $j' \circ f$ extend to $\mathcal{C}^r$-functions $\operatorname{dopp} X \to \mathbb{R}$, i.e., $j' \circ f$ extends to a $\mathcal{C}^r$-map $g\colon \operatorname{dopp} X \to \mathbb{R}^q$. Let U be the neighbourhood of ∂X in X consisting of the points x such that

$$\operatorname{dist}(j' \circ f(x), g(\operatorname{cop}(x))) < \operatorname{Dist}(j'(X'), \mathbb{R}^{q'} \setminus \operatorname{tub}_{\tau'} \rho').$$

Now construct a Urysohn $\mathcal{C}^r$-function $\varphi\colon \operatorname{dopp} X \to I$ for the pair $X, \operatorname{cop}(X \setminus U)$. It is clear that for any $x \in X$ the segment with endpoints $j' \circ f(x)$ and $g(\operatorname{cop}(x))$ lies in $\operatorname{tub}_{\tau'} \rho'$. Moreover, we see that the formulae

$$\begin{aligned} &h(x) = j' \circ f(x), && \text{if} \quad x \in X, \\ &h(\operatorname{cop}(x)) = j' \circ f(x), && \text{if} \quad x \in X \setminus U, \\ &h(\operatorname{cop}(x)) = (1 - \varphi(\operatorname{cop}(x)))g(\operatorname{cop}(x)) + \varphi(\operatorname{cop}(x))j' \circ f(x), && \text{if} \quad x \in \operatorname{Cl} U, \end{aligned}$$

define a $\mathcal{C}^r$-map $h\colon \operatorname{dopp} X \to \mathbb{R}^{q'}$ which extends $j' \circ f$ and satisfies $h(\operatorname{dopp} X) \subset \operatorname{tub}_{\tau'} \rho'$. Finally, the composite map

$$\operatorname{dopp} X \xrightarrow{\operatorname{abr} j} \operatorname{tub}_{\tau'} \rho' \xrightarrow{\operatorname{proj}_{\tau'}} X'$$

is the desired $\mathcal{C}^r$-extension of f to $\operatorname{dopp} X$. □

Theorem 3.4.6.5. *Let X and X' be compact $\mathcal{C}^{\geq r}$-manifolds with X' closed. If $0 \leq s < r \leq \infty$, then:*

(i) $\mathcal{C}^r(X, X'$ is dense in $\mathcal{C}^s(X, X')$;

(ii) given an arbitrary $\mathcal{C}^r$-map $\varphi\colon \partial X \to X'$, that part of $\mathcal{C}^r(X, X')$ consisting of the $\mathcal{C}^r$-extensions of φ is dense in the part of $\mathcal{C}^s(X, X')$ consisting of the $\mathcal{C}^s$-extensions of Φ.

Proof. The mapping

$$\mathcal{C}^s(\operatorname{incl}\colon X \to \operatorname{dopp} X, \operatorname{id} X')\colon \mathcal{C}^s(\operatorname{dopp} X, X') \to \mathcal{C}^s(X, X')$$

transforms $\mathcal{C}^r$-maps into $\mathcal{C}^s$-maps and, according to Lemma 3.4.6.4, its image is precisely $\mathcal{C}^s(X, X')$. Hence,
(i) is a consequence of the fact that $\mathcal{C}^r(\operatorname{dopp} X, X1)$ is dense in $\mathcal{C}^s(\operatorname{dopp} X, X')$ (see Theorem 3.4.4.2), while
(ii) follows from the fact that the set of all $\mathcal{C}^r$C-extensions of φ is dense in the part of $\mathcal{C}^s(\operatorname{dopp} X, X')$ consisting of the $\mathcal{C}^s$-extensions of φ (see Theorem 3.4.4.10). □

Corollary 3.4.6.6. *Let X and X' be closed $\mathcal{C}^{\geq r}$-manifolds, $1 \leq r \leq \infty$. If two maps in $\mathcal{C}^r(X, X')$ are homotopic, then they can be connected by a $\mathcal{C}^r$-homotopy $X \times I \to X'$.*

Lemma 3.4.6.7. *Let X and X' be compact $\mathcal{C}^{\geq r}$-manifolds, $1 \leq r \leq \infty$. Then the mapping $\mathcal{C}^r_\partial(X, X') \to \mathcal{C}^r(\partial X, \partial X')$, $f \mapsto \operatorname{abr} f$, is open.*

Proof. We have already seen in Remark 3.4.1.1 that this mapping is continuous. We presently show that it is open. Given a map $f \in \mathcal{C}^r_\partial(X, X')$, it is enough to find a neighbourhood $\mathcal{U}$ of the map $\operatorname{abr} f \colon \partial X \to \partial X'$ and a continuous mapping $\Phi \colon \mathcal{U} \to \mathcal{C}^r(\partial X, \partial X')$ such that $\Phi(\operatorname{abr} f) = f$ and $[\operatorname{abr}(\Phi(g)) \colon \partial X \to \partial X'] = g$ for all $g \in \mathcal{U}$ (see Theorem 1.1.4.5). Fix collarings $k \colon \partial X \times I \to X$ and $k' \colon \partial X' \times I \to X'$, a $\mathcal{C}^r$-embedding $j' \colon \partial X' \to \mathbb{R}^{q'}$, a C -transversalisation τ' of j', and a neat tube $\operatorname{Tub}_{\tau'} \rho'$. Now construct a $\mathcal{C}^r$-function $\alpha \colon I \to I$ such that

$$\alpha(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t \leq 1/3, \quad \text{and} \\ 0 & \text{for} \quad 2/3 \leq t \leq 1, \end{cases}$$

and choose $\varepsilon > 0$ with $f(k(\partial X \times [0, \varepsilon]) \subset k'(\partial X' \times I)$. Let f_1 and f_2 be the composite maps

$$\begin{tikzcd}
 & & & \partial X' \\
\partial X \times [0,\varepsilon] \arrow[r, "\operatorname{abr} k"] \arrow[urrr, bend left, "f_1"] \arrow[drrr, bend right, "f_2"'] & k(\partial X \times [0,\varepsilon]) \arrow[r, "\operatorname{abr} f"] & k'(\partial X' \times I) \arrow[r, "(\operatorname{abr} k')^{-1}"] & (\partial X' \times I) \arrow[u, "\mathrm{proj}_1"] \arrow[d, "\mathrm{proj}_2"'] \\
 & & & I
\end{tikzcd}$$

We define $\mathcal{U}$ as the set of all $g \in \mathcal{C}^r(\partial X, \partial X')$ such that

$$\max_{y \in \partial X} \operatorname{dist}(j' \circ f(y), j' \circ g(y)) < \operatorname{Dist}(j'(\partial X'), \mathbb{R}^{q'} \setminus \operatorname{tub}_{\tau'} \rho'),$$

and define Φ as

$$\begin{aligned}
&[\Phi(g)](x) = f(x), \quad \text{if} \quad x \in X \setminus k(\partial X \times [0, \varepsilon]), \\
&[\Phi(g)](k(y,t)) = k'(\operatorname{proj}_{\tau'}(j'(f_1(y,t)) + \alpha(t/\varepsilon)[j'(f(y)) - j'(g(y))]), f_2(y,t)), \\
&\text{if} \quad y \in \partial X, t \in [0, \varepsilon].
\end{aligned}$$

It is routine to check that $\mathcal{U}$ and Φ have the needed properties. ☐

Lemma 3.4.6.8. *Let X and X' be compact $\mathcal{C}^{\geq r}$-manifolds. If $1 \leq r \leq \infty$, then the set of all composite maps*

$$X \xrightarrow{f} X' \xrightarrow{\text{incl}} \operatorname{dopp} X'$$

with $f \in \mathcal{C}^r(X, X')$ is open in that part of $\mathcal{C}^r(X, \operatorname{dopp} X')$ consisting of the extensions of all maps $\partial X, \to \partial X'$.

Proof. Let $f_0 \in \mathcal{C}^r_\partial(X, X')$ and denote by g_0 the composite map

$$X \xrightarrow{f_0} X' \xrightarrow{\text{incl}} \operatorname{dopp} X'.$$

Fix a collaring of X, $k\colon \partial X \times I \to X$ and a two-sided collaring of $\partial X'$ in $\operatorname{dopp} X$, $\ell'\colon \partial X' \times [-1, 1] \to \operatorname{dopp} X'$. Further, take positive δ, η such that:

(i) $g_0(k(\partial X \times [0, \delta]) \subset \ell'(\partial X' \times (-1, 1))$;

(ii) for any $y \in \partial X$ the derivative of the function

$$\begin{aligned} &[0, \delta] \to [-1, 1] \\ &\quad t \mapsto [\operatorname{proj}_2\colon \partial X \times [-1, 1] \to [-1, 1]](\ell'^{-1}(g_0 \circ k(y, t)) \end{aligned} \tag{3.4.6.9}$$

is everywhere less than $-\eta$.

[The function $\partial X \times [0, 1] \to \mathbb{R}$, which carries each point (y, t) to the derivative of the function (3.4.6.9) at the point t, is continuous on $\partial X \times [0, 1]$ and negative on $\partial X \times 0$. Since ∂X is compact, this function is bounded above by a negative constant on $\partial X \times 0$, and hence on $\partial X \times [0, \delta]$, for some positive δ. This ensures the existence of δ and η as above.]

Clearly, the $\mathcal{C}^r$-maps $g\colon X \to \operatorname{dopp} X'$ which fulfil conditions (i) and (i) (writing g instead of g_0) and satisfy $g(X \setminus k(\partial X \times [0, \delta))) \subset \operatorname{int} X'$ form an open set in $\mathcal{C}^r(X, \operatorname{dopp} X')$. It remains to observe that the intersection of this set with that part of $\mathcal{C}^r(X, \operatorname{dopp} X')$ consisting of the extensions of all maps $\partial X \to \partial X'$ is a neighbourhood of g_0 in the set of all composite maps

$$X \xrightarrow{f} X' \xrightarrow{\text{incl}} \operatorname{dopp} X', \quad f \in \mathcal{C}^r_\partial(X, X').$$

□

Theorem 3.4.6.10. *Let X and X' be compact $\mathcal{C}^{\geq r}$-manifolds. If $1 \leq r \leq \infty$, then*

(i) $\mathcal{C}^r_\partial(X, X')$ is dense in $\mathcal{C}^s_\partial(X, X')$.

(ii) the part of $\mathcal{C}^s_\partial(X, X')$ consisting of all $\mathcal{C}^r_\partial$-extensions of a given map

$$\varphi \in \mathcal{C}^r(\partial X, \partial X')$$

is dense in the part of $\mathcal{C}^s_\partial(X, X')$ consisting of all $\mathcal{C}^s_\partial$-extensions of φ.

Proof. Lemma 3.4.6.7 and Theorem 3.4.4.2 (the latter applied to the manifolds ∂X and $\partial X'$) show that the set of all maps $g \in \mathcal{C}^s_\partial(X, X')$ such that

$$[\operatorname{abr} g\colon \partial X \to \partial X'] \in \mathcal{C}^r(\partial X, \partial X')$$

is dense in $\mathcal{C}^s_\partial(X, X')$. Therefore, (i) is a consequence of (ii).

To prove (ii), suppose that $f \in \mathcal{C}^s_\partial(X, X')$ is an extension of φ, and let $\mathcal{U}$ be a neighbourhood of f in $\mathcal{C}^s_\partial(X, X')$. We have to show that $\mathcal{U}$ contains a $\mathcal{C}^r$-extension of φ. Consider the composite map

$$X \xrightarrow{f} X' \xrightarrow{\text{incl}} \operatorname{dopp} X.$$

By virtue of Lemma 3.4.6.8, this map has a neighbourhood $\mathcal{V}$ in $\mathcal{C}^s(X, \operatorname{dopp} X')$ such that for any $g \in \mathcal{V}$ with $g(\partial X) \subset \partial X'$ one has

$$g(X) \subset X' \quad \text{and} \quad [\operatorname{abr} g \colon \partial X \to \partial X'] \in \mathcal{U}.$$

Thus, it suffices to find a $\mathcal{C}^r$-extension of φ in $\mathcal{V}$; but such an extension is provided by Theorem 3.4.6.5. □

Remark 3.4.6.11. Comparing Theorems 3.4.6.5 and 3.4.6.10 with Theorems 3.4.1.2 - 3.4.1.4 and Corollaries 3.4.1.5 - 3.4.1.5, we arrive at the following statements for $1 \leq s < r \leq \infty$. Given any compact $\mathcal{C}^{\geq r}$-manifold X and any closed $\mathcal{C}^{\geq r}$-manifold X', $\operatorname{Imm}^r(X, X')$ is dense in $\operatorname{Imm}^s(X, X')$, $\operatorname{Subm}^r(X, X')$ is dense in $\operatorname{Subm}^s(X, X')$, and $\operatorname{Emb}^r(X, X')$ is dense in $\operatorname{Emb}^s(X, X')$. For any compact $\mathcal{C}^{\geq r}$-manifolds X and X', the set of neat embeddings in $\mathcal{C}^r(X, X')$ is dense in the set of neat embeddings in $\mathcal{C}^s(X, X')$, and $\operatorname{Diff}^r(X, X')$ is dense in $\operatorname{Diff}^s(X, X')$.

Corollary 3.4.6.12. *Two compact $\mathcal{C}^{\geq r}$-manifolds, $1 \leq r \leq \infty$, which are diffeomorphic are $\mathcal{C}^r$-diffeomorphic.*

Information 3.4.6.13. As with Theorem 3.4.4.2, Theorems 3.4.6.5 and 3.4.6.10 remain valid for $r = a$ too (cf. Remark 3.4.2.4). We excluded this case in view of its difficulty.

Theorems 3.4.4.2, 3.4.6.5 and 3.4.6.10 (as well as their $\mathcal{C}^a$-variants) also hold for non-compact X and X'. However, this generalisation is of limited interest. For example, it does not suffice if one wants to eliminate the compactness assumption in Corollary 3.4.6.12 (which is actually possible). The appropriate extensions of Theorems 3.4.4.2, Theorems 3.4.6.5 and 3.4.6.10 to the non-compact case are related to topologies which are stronger than those defined in Remark 3.4.1.1, and require analytic tools stronger than Theorem 3.3.1.7.

3.4.7 General Position

Remark 3.4.7.1. The main result of this subsection is the final Theorem 3.4.7.7, which constitutes the basis of a large part of the material below. We emphasise that this theorem is formulated and proved only in the $\mathcal{C}^\infty$-case. In Subsection 3.4.9 we add a statement covering the case of $\mathcal{C}^r$-maps with r finite (see Theorem 3.4.9.10).

The technical part of the subsection is concentrated in Theorem 3.4.7.2, which establishes the fundamental topological property of the spaces $\mathcal{C}^r(X, X')$, and Theorem 3.4.7.4, which represents the only corollary of Theorem 3.3.2.3 that we need.

To simplify the formulations of Theorems 3.4.7.2 and 3.4.7.3, we give a special name to those topological spaces where the intersection of any countable collection of dense open sets is dense: we call them *Baire spaces*.

Technicalities

Theorem 3.4.7.2. *$\mathcal{C}^r(X, X')$ is a Baire space for any $\mathcal{C}^{\geq r}$ manifolds X and X' with $0 \leq r \leq \infty$.*

Proof. We have to show that given arbitrary open dense subsets $\mathcal{U}_1, \mathcal{U}_2, \dots$ of $\mathcal{C}^r(X, X')$, and an arbitrary open subset $\mathcal{W}$ of $\mathcal{C}^r(X, X')$, the intersection $W \cap (\cap_{i=1^\infty} \mathcal{U}_i)$ is not empty. Let $\{\varphi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ be atlases of the manifold X, indexed so that the set $K_i = \operatorname{Cl}\operatorname{supp}\psi_i$ is compact and contained in $\operatorname{supp}\varphi_i$ and $\psi_i = \operatorname{abr}\varphi_i$ for all i. Similarly, let $\{\varphi'_j\}_{j=1}^\infty$ and $\{\psi'_j\}_{j=1}^\infty$ be atlases of X', indexed so that the set $K'_j = \operatorname{Cl}\operatorname{supp}\psi'_j \subset \operatorname{supp}\varphi'_j$ and $\psi'_j = \operatorname{abr}\varphi'_j$ for all j. We construct a sequence of $\mathcal{C}^r$-maps $f_1 \colon X \to X', f_1 \colon X \to X', \dots$, a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of open subsets of $\mathcal{C}^r(X, X')$, and a sequence of positive integers $n(1), n(2), \dots$, such that:

(i) $f_i \in \mathcal{V}_i$;

(ii) if $i \geq 2$, then $\mathcal{V}_i \subset \mathcal{V}_{i-1}$;

(iii) $\operatorname{Cl}\mathcal{V}_i \subset \mathcal{W} \cap \mathcal{U}_i$;

(iv) $f_i(K_i) \subset \cup_{j=1}^{n(i)} \operatorname{supp}\psi'_j$;

(v) if $s \leq i$ and $t \leq n(s)$, then $f_i(K_s \cap f_s^{-1}(K'_t)) \subset \operatorname{supp}\varphi'_t$;

(vi) if $i \geq 2$, $s \leq i$, and $t \leq n(s)$, then the partial derivatives of order $\leq \min(r, i-2)$ of the coordinate functions of the local representatives $\operatorname{loc}(\varphi_s, \varphi'_t) f_i$ and $\operatorname{loc}(\varphi_s, \varphi'_t) f_{i-1}$ differ by less than 2^{-i} at the points of $\varphi_s(K_s \cap f_s^{-1}(K'_t)$.

Then we will have finished the proof. Indeed, (v) and (vi) imply that for any s, t such that $t \leq n(s)$ the sequence

$$\{\operatorname{loc}(\varphi_s, \varphi'_t) f_i|_{\varphi_s(\operatorname{supp}\psi_s \cap f_s^{-1}(\operatorname{supp}\psi'_t))}\}_{i=s}^\infty,$$

together with all its partial derivatives of order $\leq r$ converges uniformly on $\varphi_s(\operatorname{supp}\psi_s \cap f_s^{-1}(\operatorname{supp}\psi'_t))$ to a $\mathcal{C}^r$-map

$$g_{st} \colon \varphi_s(\operatorname{supp}\psi_s \cap f_s^{-1}(\operatorname{supp}\psi'_t)) \to \operatorname{im}\psi'_t.$$

Moreover, the composite maps

$$\begin{aligned} \operatorname{supp}\psi_s \cap f_s^{-1}(\operatorname{supp}\psi'_t) &\xrightarrow{\operatorname{abr}\varphi_s} \varphi_s(\operatorname{supp}\psi_s \cap f_s^{-1}(\operatorname{supp}\psi'_t)) \\ &\xrightarrow{g_{st}} \operatorname{im}\varphi'_t \xrightarrow{(\varphi'_t)^{-1}} \operatorname{supp}\varphi'_t \end{aligned}$$

together define a $\mathcal{C}^r$-map $g \colon X \to X'$ [(iv) shows that the sets $\operatorname{supp}\psi_s \cap f_s^{-1}(\operatorname{supp}\psi'_t)$ cover X and clearly the maps g_{st} are compatible on the intersections of these sets]. g is the limit (in $\mathcal{C}^r(X, X')$) of the sequence $f_1, f_2, \dots$ and (i)-(iii) show that $g \in \mathcal{W} \cap (\cap_{i=1}^\infty \mathcal{U}_i)$.

We proceed by induction. Take f_1 to be any element of $\mathcal{W} \cap \mathcal{U}_1$, and take $\mathcal{V}_1$ to be any neighbourhood of f_1 (in $\mathcal{C}^r(X, X')$) such that $\operatorname{Cl} \mathcal{V}_1 \subset \mathcal{W} \cap \mathcal{U}_1$ (recall that $\mathcal{C}^r(X, X')$ is a regular space - see Remark 3.4.1.1). Assume that for some $k \geq 2$, the maps $f_i \in \mathcal{C}^r(X, X')$, the open sets $\mathcal{V}_i \subset \mathcal{C}^r(X, X')$, and the positive integers $n(i)$ with $i < k$ have been defined and satisfy (i)-(vi). We let G denote the set of all $\mathcal{C}^r$-maps $g \colon X \to X'$ such that for $s \leq k-1$ and $t \leq n(s)$

$$g(K_s \cap f_s^{-1}(K'_t)) \subset \operatorname{supp} \varphi'_t,$$

and all partial derivative of order $\leq \min(r, k-2)$ of the coordinate functions of the local representatives $\operatorname{loc}(\varphi_s, \varphi'_t)g$ and $\operatorname{loc}(\varphi_s, \varphi'_t)f_{k-1}$ differ by less than 2^{-k} at the points of $\varphi_s(K_s \cap f_s^{-1}(K'_t))$. Obviously, G is open and $f_{k-1} \in G$. Therefore, $G \cap \mathcal{V}_{k-1} \neq \varnothing$ and, since $\mathcal{U}_k$ is dense, $G \cap (\mathcal{V}_{k-1} \cap \mathcal{U}_k) \neq \varnothing$. Choose $\mathcal{V}_k$ to be any non-empty open set with $\operatorname{Cl} \mathcal{V}_k \subset G \cap \mathcal{V}_{k-1} \cap \mathcal{U}_k$, f_k to be any element of $\mathcal{V}_k$ and $n(k)$ to be any positive integer such that $f_k(K_k) \subset \cup_{t=1}^{n(k)} \operatorname{supp} \psi'_t$. It is readily seen that the objects $\mathcal{V}_k$, f_k, and $n(k)$ satisfy conditions (i) - (vi) for $i = k$. □

Theorem 3.4.7.3. *Every topological manifold is a Baire space.*

Proof. This is a special case of Theorem 3.4.7.2: in fact, the topological manifold X may be regarded as the space $\mathcal{C}^0(\mathbb{D}^0, X)$. □

Theorem 3.4.7.4. *Let X and X' be manifolds of class $\mathcal{C}^\infty$ or $\mathcal{C}^a$. If $f \colon X \to X'$ is a $\mathcal{C}^\infty$-map and F is the set of all points $x \in X$ such that $d_x f(\operatorname{Tang}_x X) \neq \operatorname{Tang}_{f(x)} X'$, then $f(F)$ is the union of a countable family of nowhere dense sets.*

Proof. Let Φ and Φ' be arbitrary countable atlases of the manifolds X and X'. For each pair $(\varphi, \varphi') \in \Phi \times \Phi'$, write $\operatorname{supp}(\varphi \cap f^{-1}(\operatorname{supp} \varphi'))$ as the union of a sequence of compact sets $K_1(\varphi, \varphi'), K_2(\varphi, \varphi'), \ldots$, and let $C_i(\varphi, \varphi')$ be the set of all points $x \in K_i(\varphi, \varphi')$ such that the rank of the Jacobi matrix of $\operatorname{loc}(\varphi, \varphi')f$ at $\varphi(x)$ is less than $\dim X'$. Theorem 3.3.2.3 shows that the sets $f(C_i(\varphi, \varphi'))$ are all nowhere dense, and obviously $f(F) = \cup_i f(C_i(\varphi, \varphi'))$. □

The Basic Theorem

Remark 3.4.7.5. Let X_1, X_2, and X' be smooth manifolds, and let A_1 and A_2 be subsets of X_1 and X_2. Two smooth maps $f_1 \colon X_1 \to X'$ and $f_2 \colon X_2 \to X'$ are said to be *transverse* (*one to the other*) *on* A_1, A_2 if for any $x_1 \in A_1$, $x_2 \in A_2$ with $f_1(x_1) = f_1(x_1)$, the vector space $\operatorname{Tang}_{f(x_1)} X'$ is spanned by its subspaces $d_{x_1} f_1(\operatorname{Tang}_{x_1} X_1)$ and $d_{x_2} f_2(\operatorname{Tang}_{x_2} X_2)$, and the following holds:

- if $x_1 \in \partial X_1$, then $\operatorname{Tang}_{f(x_1)} X'$ is already spanned by its subspaces $d_{x_1} f_1(\operatorname{Tang}_{x_1} \partial X_1)$ and $d_{x_2} f_2(\operatorname{Tang}_{x_2} X_2)$;
- if $x_2 \in \partial X_2$, then $\operatorname{Tang}_{f(x_1)} X'$ is already spanned by its subspaces $d_{x_1} f_1(\operatorname{Tang}_{x_1} X_1)$ and $d_{x_2} f_2(\operatorname{Tang}_{x_2} \partial X_2)$;

- if $x_1 \in \partial X_1$, and $x_2 \in \partial X_2$ then $\operatorname{Tang}_{f(x_1)} X'$ is already spanned by its subspaces $d_{x_1} f_1(\operatorname{Tang}_{x_1} \partial X_1)$ and $d_{x_2} f_2(\operatorname{Tang}_{x_2} \partial X_2)$;

Two maps $f_1 \colon X_1 \to X'$ and $f_2 \colon X_2 \to X'$ which are transverse on X_1, X_2 are simply referred to as transverse.

Let us make three obvious remarks.

First if $\dim X_1 + \dim X_2 < \dim X'$, then the fact that f_1 and f_2 are transverse on A_1, A_2 implies that $f_1(A_1) \cap f_2(A_2) = \varnothing$.

Secondly if $f_!$ and f_2 are transverse on AA_1, A_2, then they are transverse on some neighbourhoods of A_1 and A_2.

Thirdly if X_1, X_2, and X' are of class $\mathcal{C}^r$ with $1 \leq r \leq \infty$, and A_1, A_2 are compact, then given $f_2 \in \mathcal{C}^r(X_2, X')$, the set of all $f_1 \in \mathcal{C}^r(X_1, X')$ such that $f_!$ and f_2 are transverse on A_1, A_2 is open in $\mathcal{C}^r(X_1, X')$.

Lemma 3.4.7.6. *Suppose X_1 and X_2 are $\mathcal{C}^\infty$- or $\mathcal{C}^a$-manifolds and $f_1 \colon X_1 \to \mathbb{R}^{q'}$ and $f_2 \colon X_2 \to \mathbb{R}^{q'}$ are $\mathcal{C}^\infty$-maps. Then there is a dense set V in $\mathbb{R}^{q'}$ such that for each vector $v \in V$, the map $X_1 \to \mathbb{R}^{q'}$ defined by $x_1 \mapsto f_1(x_1) + v$ is transverse to f_2.*

Proof. We may take V to be the set of all $v \in \mathbb{R}^{q'}$ which satisfy the following condition. Consider the four maps $\operatorname{int} X_1 \times \operatorname{int} X_2 \to \mathbb{R}^{q'}$, $\operatorname{int} X_1 \times \partial X_2 \to \mathbb{R}^{q'}$, $\partial X_1 \times \operatorname{int} X_2 \to \mathbb{R}^{q'}$, and $\partial X_1 \times \partial X_2 \to \mathbb{R}^{q'}$, given by $(x_1, x_2) \mapsto f_2(x_2) - f_1(x_1)$. Then given any of these maps, v is not the image of a point where the differential of that map has rank less than q'. The map $x_1 \mapsto f_1(x_1) + v$ with $v \in V$ is obviously transverse to f_2, and theorems 3.4.7.4 and 3.4.7.3 show that V is dense in $\mathbb{R}^{q'}$. □

Theorem 3.4.7.7. *Let X_1, X_2, and X' be $\mathcal{C}^\infty$- or $\mathcal{C}^a$-manifolds with $\partial X' = \varnothing$. If $f_2 \colon X_2 \to X'$ is a $\mathcal{C}^\infty$-map, then the subset of $\mathcal{C}^\infty(X_1, X')$ consisting of all maps transverse to f_2 is the intersection of a countable collection of dense open sets.*

Proof. For two sets $A_1 \subset X_1$ and $A_2 \subset X_2$, we let $\mathcal{F}(A_1, A_2)$ denote the set of all $\mathcal{C}^\infty$-maps $f_1 \colon X_1 \to X'$ such that $f_1$and f_2 are transverse on A_1, A_2. If A_1 and A_2 are compact, then $\mathcal{F}(A_1, A_2)$ is obviously open in $\mathcal{C}^\infty(X_1, X')$, and we shall presently show that if A_1 and A_2 are compact, then $\mathcal{F}(A_1, A_2)$ is dense in $\mathcal{C}^\infty(X_1, X')$. These two facts are enough: express X_1 as the union of a sequence of compact sets $K_{11}, K_{12}, \ldots$, express X_2 as the union of a sequence of compact sets $K_{21}, K_{22}, \ldots$, and then observe that the part in which we are interested (i.e, $\mathcal{F}(X_1, X_2)$) can be written as $\cap_{i,j} \mathcal{F}(K_{1i}, K_{2j})$.

Thus, suppose that A_1 and A_2 are compact, and let $\mathcal{U}$ be a neighbourhood (in $\mathcal{C}^\infty(X_1, X')$) of an arbitrarily given map $g_1 \colon X_1 \to X'$. We have to produce a map contained in both $\mathcal{F}(A_1, A_2)$ and $\mathcal{U}$. We do this in two steps: first, we compress $\mathcal{U}$ to a neighbourhood $\mathcal{V}$ of g_1 having a more special form, and then construct a map belonging to $\mathcal{F}(A_1, A_2)$ and $\mathcal{V}$.

Let $\dim X_1 = q_1$, $\dim X_2 = q_2$, and $\dim X' = q'$. To begin building the neighbourhood $\mathcal{V}$, fix for each point $x' \in f_2(A_2)$ a chart $\varphi'_{x'} \in \operatorname{Atl}_{x'} X'$ with $\operatorname{im} \varphi'_{x'} = \mathbb{R}^{q'}$ and $\varphi'_{x'}(x') = 0$. Further, fix for each point $x_2 \in A_2$ a chart $\varphi_{2x_2} \in \operatorname{Atl}_{x_2} X_2$ such that $\operatorname{im} \varphi_{2x_2} = \mathbb{R}^{q_2}$ or$\operatorname{im} \varphi_{2x_2} = \mathbb{R}^{q_2}_-$, $\varphi_{2x_2}(x_2) = 0$, $\operatorname{Cl} \operatorname{supp} \varphi_{2x_2}$ is compact, and $f_2(\operatorname{Cl} \operatorname{supp} \varphi_{2x_2}) \subset [\varphi'_{f_2(x_2)}]^{-1}(\operatorname{int} \mathbb{D}^{q'})$. Now cover A_2 by a finite number of sets $\varphi^{-1}_{2x_2}(\operatorname{int} \mathbb{D}^{q_2})$, say $\varphi^{-1}_{2x_{21}}(\operatorname{int} \mathbb{D}^{q_2}), \dots, \varphi^{-1}_{2x_{2\ell}}(\operatorname{int} \mathbb{D}^{q_2})$ and denote the chart $\varphi'_{f_2(x_{2j})}(x_2)$ simply by $\psi'_{j'}$, and the chart $\varphi_{2x_{2j}}$ - by ψ_{2j}. Continuing, for each point $x_1 \in A_1$ choose a chart $\varphi_{1x_1} \in \operatorname{Atl}_{x_1} X_1$ such that $\operatorname{im} \varphi_{1x_1} = \mathbb{R}^{q_2}$ or $\operatorname{im} \varphi_{1x_1} = \mathbb{R}^{q_2}_-$, $\varphi_{1x_1}(x_1) = 0$, $\operatorname{Cl} \operatorname{supp} \varphi_{1x_1}$ is compact, and $g_1(\operatorname{Cl} \operatorname{supp} \varphi_{1x_1})$ is contained in one of the sets $X' \setminus f_2(\operatorname{Cl} \operatorname{supp} \psi_{2j})$, $[\psi'_j]^{-1}(\operatorname{int} \mathbb{D}^{q'})$, for any given $j = 1, \dots, \ell$. Finally, cover A_1 by a finite number of the sets $\varphi^{-1}_{1x_1}(\operatorname{int} \mathbb{D}^{q_1})$, say $\varphi^{-1}_{1x_{11}}(\operatorname{int} \mathbb{D}^{q_1}), \dots, \varphi^{-1}_{1x_{1k}}(\operatorname{int} \mathbb{D}^{q_1})$, and denote the chart $\varphi_{1x_{1i}}$ by ψ_{1i}. At last, we may define $\mathcal{V}$ as the subset of $\mathcal{U}$ consisting of all maps $h_1 \colon X_1 \to X'$ such that:

$$
\begin{aligned}
&h_1(\operatorname{Cl} \operatorname{supp} \psi_{1i}) \subset X' \setminus f_2(\operatorname{Cl} \operatorname{supp} \psi_{2j}) \quad \text{if} \quad g_1(\operatorname{Cl} \operatorname{supp} \psi_{1i}) \subset X' \setminus f_2(\operatorname{Cl} \operatorname{supp} \psi_{2j}),\\
&h_1(\operatorname{Cl} \operatorname{supp} \psi_{1i}) \subset X' \setminus (\psi'_j)^{-1}(\operatorname{int} \mathbb{D}^{q'}) \quad \text{if} \quad g_1(\operatorname{Cl} \operatorname{supp} \psi_{1i}) \subset X' \setminus f_2(\operatorname{int} \mathbb{D}^{q'}).
\end{aligned}
$$

That $\mathcal{V}$ is a neighbourhood of g_1 in $\mathcal{C}^\infty(X_1, X')$ is plain.

Turning now to the final part of the construction, arrange the pairs (i, j), $i = 1, \dots, k$, $j = 1, \dots, \ell$, in a sequence $(i_1, j_1), \dots, (i_m, j_m)$, with $m = k \cdot \ell$. Next construct inductively maps $h_1^0, \dots, h_1^m \colon X_1 \to X'$ with the following properties:

(i) $h_1^s \in \mathcal{V}$;

(ii) if $r \le s$, then the maps h_1^r and f_2 are transverse on $\psi^{-1}_{1i_r}(\mathbb{D}^{q_1})$, $\psi^{-1}_{2j_r}(\mathbb{D}^{q_2})$.

Then h_1^m will belong to $\mathcal{F}(A_1, A_2) \cap \mathcal{V}$.

Put $h_1^0 = f_1$ and assume that the maps h satisfying (i) and (ii) are already defined for $s < t \le m$. If the maps h_1^{t-1} and f_2 are transverse on $\psi^{-1}_{1i_t}(\mathbb{D}^{q_1})$, $\psi^{-1}_{2j_t}(\mathbb{D}^{q_2})$, put $h_1^t = h_1^{t-1}$. Otherwise, $h_1^{t-1}(\psi_{1i_t}(\mathbb{D}^{q_1})) \subset (\psi_{1j_t})^{-1}(\mathbb{D}^{q'})$, and Lemma 3.4.7.6 guarantees the existence of a set V dense in $\mathbb{R}^{q'}$ such that the composite map

$$
\operatorname{im} \psi_{1i_t} \xrightarrow{\operatorname{loc}(\psi_{1i_t}, \psi'_{j_t}) h_1^{t-1}} \mathbb{R}^{q'} \xrightarrow{x \mapsto x+v} \mathbb{R}^{q'}
$$

is transverse to $\operatorname{loc}(\psi_{2j_t}, \psi_{j_t}) f_2 \colon \operatorname{im} \psi_{2j_t} \to \mathbb{R}^{q'}$, for each $v \in V$. Pick a $\mathcal{C}^\infty$-function $\alpha \colon \mathbb{R}^{q'} \to \mathbb{R}$, equal to 1 on $\mathbb{D}$ and equal to 0 outside the concentric ball $\mathbb{D}^{q'}$ of radius 2. Now for $v \in V$ define $g_v \colon X_1 \to X'$ by

$$
g_v(X_1) = \begin{cases}
h_1^{t-1}(x_1), \\
\qquad \text{if} \quad x_1 \in X_1 \setminus (h_1^{t-1})^{-1}[(\psi'_{j_t})^{-1}(2\mathbb{D}^{q'})], \\
(\psi'_{j_t})^{-1}((\psi'_{j_t})(h_1^{t-1}(x_1)) + \alpha((h_1^{t-1}(x_1)))v), \\
\qquad \text{if} \quad x_1 \in (h_1^{t-1})^{-1}(\operatorname{supp} \psi'_{j_t})
\end{cases}
$$

We easily see that $g_0 = h_1^{t-1}$, that g_V is $\mathcal{C}^\infty$, and that the map $\mathbb{R}^{q'} \to \mathcal{C}^\infty(X_1, X')$, $v \mapsto g_v$, is continuous. Consequently, there is an open set $U \subset \mathbb{R}^{q'}$, such that

$0 \in U$ and for $v \in U$ the map $g_v \in \mathcal{V}$, and g_v and f_2 are transverse on $\psi_{1i_r}^{-1}(\mathbb{D}^{q_1})$, $\psi_{2j_t}^{-1}(\mathbb{D}^{q_2})$ for $r < t$.

On the other hand, the definition of V shows that for $v \in V$, g_v f_2 are transverse on $\psi_{1i_r}^{-1}(\mathbb{D}^{q_1})$, $\psi_{2j_t}^{-1}(\mathbb{D}^{q_2})$, and hence one may take $h_1^t = g_v$ for any $v \in U \cap V$. ☐

3.4.8 Maps Transverse to a Submanifold

Remark 3.4.8.1. Theorem 3.4.7.7 is used mainly when X_2 is a submanifold of X' and f_2 is the corresponding inclusion. In such a situation we use a simpler terminology; namely, instead of saying that the map $f_1 \colon X_1 \to X'$ is transverse to the inclusion $\mathrm{incl} \colon X_2 \to X'$, we say that f_1 *is transverse to* X_2. Then Theorem 3.4.7.7 states that if X_1 and X' are of class $\mathcal{C}^\infty$ or $\mathcal{C}^a$ and $\partial X' = \varnothing$, then the set of all maps in $\mathcal{C}^\infty(X_1, X')$ which are transverse to X_2 is dense in $\mathcal{C}^\infty(X_1, X')$.

The more special case, when both X_1 and X_2 are submanifolds of a manifold X' and f_1 and f_2 are the corresponding inclusions, deserves particular attention. If the maps $\mathrm{incl} \colon X_1 \to X'$ and in: $\mathrm{incl} \colon X_2 \to X'$ are transverse, we say that the submanifolds X_1 and X_2 themselves are transverse. Comparing Theorems 3.4.7.7 and 3.4.1.4, we see that given arbitrary submanifolds X_1 and X_2 of a closed, $\mathcal{C}^\infty$ or $\mathcal{C}^a$-manifold X', every neighbourhood of the inclusion $\mathrm{incl} \colon X_1 \to X'$ in $\mathcal{C}^\infty(X_1, X')$ contains embeddings transverse to X_2. The last statement is frequently formulated in a more geometric and less formal fashion: *two submanifolds can be made transverse through an arbitrarily small displacement of either of them.*

Of course, the $\mathcal{C}^{<\infty}$-complement to Theorem 3.4.7.7 that was mentioned in Remark 3.4.7.1 (Theorem 3.4.9.10) applies to these special cases too. However, note that in order to bring two submanifolds into general position in this way one must displace both of them, and not only one.

Theorem 3.4.8.2. *Suppose X_1 and X' are $\mathcal{C}^r$-manifolds ($1 \le r \le a$) with $\dim X_1 = q_1$ and $\dim X' = q'$, and X_2 is a submanifold of X' with $\dim X_2 = q_2$. Let $f_1 \colon X_1 \to X'$ be a $\mathcal{C}^r$-map transverse to X_2 and such that $f_1(\partial X) \cap \partial X_2 \subset \partial X'$. Then $X_{12} = f_1^{-1}(X_2)$ is a $(q_1 + q_2 - q')$-dimensional $\mathcal{C}^r$-submanifold of X_1 with $\partial X_{12} = f_1^{-1}(\partial X_2)$. Moreover, this submanifold is neat whenever X_2 is neat. Finally, the linear map*

$$\operatorname{fact} d_{x_1} f_1 \colon \operatorname{Tang}_{x_1} X_1 / \operatorname{Tang}_{x_1} X_{12} \to \operatorname{Tang}_{f_1(x_1)} X' / \operatorname{Tang}_{f_1(x_1)} X_2 \quad (3.4.8.3)$$

is an isomorphism for any point $x_1 \in X_1$.

(It is understood that a submanifold of negative dimension is void.)

Proof. Note that the second assertion is a straightforward supplement to the first, while the third assertion becomes evident if one observes that both quotient spaces appearing in (3.4.8.3) have the same dimension. Therefore, we have to prove only the first assertion, and in order to do this, we follow the scheme in Remark 3.4.8.1.

To begin with, consider, given $x_1 \in X_{12}$, all possible positions of the point $f_1(x_1)$. The case $f_1(x_1) \in \operatorname{int} X_2 \cap \partial X'$ is excluded, because $\operatorname{int} X_2 \cap \partial X' = \varnothing$ (X_2 is a submanifold of X'). If $x_1 \in X_1$, the case $f_1(x_1) \in \partial X_2 \cap \partial X'$ is excluded since it would contradict the assumption that f_1 is transverse to X_2 (if $x_1 \in X_1$ and $f_1(x_1) \in \partial X_2 \cap \partial X'$, then the spaces $\operatorname{im} d_{x_1} f_1$ and $\operatorname{Tang}_{f_1(x_1} \partial X_2$ are both contained in $\operatorname{Tang}_{f_1(x_1)} \partial X'$, and so they cannot span $\operatorname{Tang}_{f_1(x_1)} X'$.) When $x_1 \in \partial X_1$, the relation $f_1(x_1) \in \partial X_2 \cap \operatorname{int} X'$ is impossible since $f_1(\partial X_1) \cap \partial X_2 \subset \partial X'$. Therefore, we are left with four possibilities:

(i) $x_1 \in \operatorname{int} X_1,\ f_1(x_1) \in \operatorname{int} X_2 \cap \operatorname{int} X'$;

(ii) $x_1 \in \operatorname{int} X_1,\ f_1(x_1) \in \partial X_2 \cap \operatorname{int} X'$;

(iii) $x_1 \in \partial X_1,\ f_1(x_1) \in \operatorname{int} X_2 \cap \operatorname{int} X'$;

(iv) $x_1 \in \partial X_1,\ f_1(x_1) \in \partial X_2 \cap \partial X'$.

Fix a chart $\varphi' \in \operatorname{Atl}_{f_1(x_1)} \mathcal{C}^r X'$ which transforms the triple

$$(\operatorname{supp} c\varphi', X_2 \cap \operatorname{supp} \varphi', f_1(x_1))$$

into one of the triples $(\mathbb{R}^{q'}, \mathbb{R}^{q_2}, 0)$, $(\mathbb{R}^{q'}, \mathbb{R}^{q_2}_-, 0)$, or $(\mathbb{R}^{q'}_-, \mathbb{R}^{q_2}_-, 0)$, with corresponding local coordinates $\varphi'_1, \dots, \varphi'_{q'}$. Define

$$\psi_1, \dots, \psi_q \colon f_1^{-1}(\operatorname{supp} \varphi') \to \mathbb{R}, \quad \psi_i(x) = \varphi'_i(f_1(x)).$$

In cases (i), (iii), and (iv) above, the intersection $X_{12} \cap f_1^{-1}(\operatorname{supp} \varphi')$ is defined by the equations $\psi_{q_2+1}(x) = 0, \dots, \psi_{q'}(x) = 0$, and in the case (ii) - by the same equations and the inequality $\psi_1(x) < 0$. Let us verify that for all cases (i)-(iv), the above equations and inequality satisfy the independence conditions displayed in Remark 3.1.2.12.

In case (i) , we have to show that $\psi_{q_2+1}, \dots, \psi_{q'}$ are independent at the point x_1. This follows from the equality

$$\dim(\cap_{j=q_2+1}^{q'} \ker(d_{x_1}\psi_j) = q' - q_2,$$

which in turn follows from the trivial inclusion

$$d_{x_1} f_1(\cap_{j=q_2+1}^{q'} \ker(d_{x_1}\psi_j)) \subset \operatorname{Tang}_{f_1(x_1)} X_2$$

and the equality

$$\operatorname{Tang}_{f_1(x_1)} X' = \operatorname{im} d_{x_1} f_1 + \operatorname{Tang}_{f_1(x_1)} X_2$$

(which is part of the definition of transversality).

In case (ii) , we have to show that $\psi_1, \psi_{q_2+1}, \dots, \psi_{q'}$ are independent at x_1. The proof is a repeat of the previous one, except that one must replace the equality

$$\operatorname{Tang}_{f_1(x_1)} X' = \operatorname{im} d_{x_1} f_1 + \operatorname{Tang}_{f_1(x_1)} X_2$$

by

$$\operatorname{Tang}_{f_1(x_1)} X' = \operatorname{im} d_{x_1} f_1 + \operatorname{Tang}_{f_1(x_1)} \partial X_2.$$

Finally, in cases (iii) and (iv), we have to produce a $\mathcal{C}^r$-function

$$\psi \colon f_1^{-1}(\operatorname{supp} \varphi') \to \mathbb{R}$$

which is zero on $\partial X_1 \cap f_1^{-1}(\operatorname{supp} \varphi')$, negative on $\operatorname{int} X_1 \cap f_1^{-1}(\operatorname{supp} \varphi')$, and is such that $\psi_1, \psi_{q_2+1}, \dots, \psi_{q'}$, are independent at x_1. The existence of such a function is equivalent i to the restrictions of $\psi_{q_2+1}, \dots, \psi_{q'}$ to $\partial X_1 \cap f_1^{-1}(\operatorname{supp} \varphi')$ being independent at x_1. The latter can be proved as in (i), employing the equality

$$\operatorname{Tang}_{f_1(x_1)} X' = \operatorname{im} d_{x_1} (f_1|_{\partial X_1}) + \operatorname{Tang}_{f_1(x_1)} X_2$$

rather than

$$\operatorname{Tang}_{f_1(x_1)} X' = \operatorname{im} d_{x_1} f_1 + \operatorname{Tang}_{f_1(x_1)} X_2.$$

□

Corollary 3.4.8.4. *Let X_1 and be X_2 transverse submanifolds of a smooth manifold X', and assume that X_1 is neat. Then $X_1 \cap X_2$ is a $(\dim X_1 + \dim X_2 - \dim X')$-dimensional submanifold of X', and is neat whenever X_2 is neat.*

The Simplest Applications

Theorem 3.4.8.5. *Let A be a closed subset of a closed $\mathcal{C}^r$-manifold X, and let U be a neighbourhood of A. If $1 \leq r \leq \infty$, then there is in U a compact submanifold B of codimension 0 such that $A \subset \operatorname{int} B$.*

Proof. Let $\varphi \colon X \to I$ be a Urysohn $\mathcal{C}^r$-function for the pair $A, X \backslash U$ (see Lemma 3.4.4.8), and suppose that $c \in (0,1)$ is not a critical value of φ (see Remark 3.4.8.11. Set $B = \varphi^{-1}([0,c])$. Then B is the preimage of the submanifold $(-\infty, c]$ of $\mathbb{R}$ under the composite $\mathcal{C}^r$-map $X \xrightarrow{\varphi} I \xrightarrow{\text{incl}} \mathbb{R}$. Since the latter is transverse to $(-\infty, c]$, B is a submanifold of codimension 0. It is immediate from the construction that B is closed as a subset, that $A \subset \operatorname{int} B$, and that $B \subset U$. □

Theorem 3.4.8.6. *Every CNRS is homeomorphic to a retract of a closed, orientable, $\mathcal{C}^\infty$-manifold.*

Proof. Let j be an embedding of the CNRS X in $\mathbb{S}^q$, with q large enough, and let U be a neighbourhood of $j(X)$ which retracts on $j(X)$. Theorem 3.4.8.5 provides a compact submanifold $B \subset U$ such that $B \supset j(X)$, and clearly $j(X)$ is a retract of the double of B. □

3.4.9 Raising the Smoothness Class of a Manifold

Remark 3.4.9.1. The main results of this subsection are Theorems 3.4.9.6 and 3.4.9.8. Lemmas 3.4.9.2-3.4.9.5 are needed for the proof of Theorem 3.4.9.6, while Lemma 3.4.9.7 in conjunction with Theorem 3.4.9.6 yield Theorem 3.4.9.8.

Lemma 3.4.9.2. *Let X and X' be $\mathcal{C}^{\geq r}$-manifolds, $1 \leq r \leq \infty$, X compact, and $\partial X' = \varnothing$. Suppose A' is a submanifold of X', $f\colon X \to X'$ is a $\mathcal{C}^r$-map transverse to A' such that $f(\partial X) \subset X' \setminus A'$, and $\rho\colon X \to f^{-1}(A')$ is both a retraction and a $\mathcal{C}^r$-submersion. Then there is a neighbourhood $\mathcal{U}$ of the map f in $\mathcal{C}^r(X, X')$ such that every $g \in \mathcal{U}$ satisfies:*

(i) g is transverse to A';

(ii) $g(\partial X) \subset X' \setminus A'$;

(iii) $\rho|_{g^{-1}}(A')\colon g^{-1}(A') \to f^{-1}(A')$ is a submersion,

Proof. We obtain $\mathcal{U}$ as the intersection of three open sets, $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$.

- $\mathcal{U}_1$ is the set of all maps in $\mathcal{C}^r(X, X')$ which are transverse to A'.
- $\mathcal{U}_2$ is the set of all $g \in \mathcal{C}^r(X, X')$ with $g(\partial X) \subset X' \setminus A'$.
- $\mathcal{U}_3$ is the set of all $g \in \mathcal{C}^r(X, X')$ such that the intersection of the subspaces $\ker d_x\rho$ and $(d_x g)^{-1}(\operatorname{Tang}_{g(x)} A')$ of $\operatorname{Tang}_x X$ reduces to 0 for all points $x \in g^{-1}(A')$.

We already know that $\mathcal{U}_1$ is open (see Remark 3.4.7.5). The openness of $\mathcal{U}_2$ is a consequence of the compactness of ∂X and the openness of $X' \setminus A'$. To prove that $\mathcal{U}_3$ is open, we shall describe it in a different way. Fix a $\mathcal{C}^r$-embedding $j\colon X \to \mathbb{R}^q$ and let $C \subset \operatorname{Tang} X$ be the subset of all vectors u such that $d\rho(u) = 0$ and $< dj(u), dj(u) >= 1$. Then clearly $\mathcal{U}_3$ is just the set of all $g \in \mathcal{C}^r(X, X')$ such that $dg(C) \subset \operatorname{Tang} X' \setminus \operatorname{Tang} A'$. The openness of this last set is a consequence of the following facts: C is compact, $\operatorname{Tang} X' \setminus \operatorname{Tang} A'$ is open in $(\operatorname{Tang} X')$, and the mapping $\mathcal{C}^r(X, X') \to \mathcal{C}^{r-1}(\operatorname{Tang} X, \operatorname{Tang} X')$, which takes each $g \in \mathcal{C}^r(X, X')$ into dg, is continuous (see Remark 3.4.1.1). Therefore, $\mathcal{U}$ is open, and we see at once that $f \in \mathcal{U}$ and that any map $g \in \mathcal{U}$ satisfies (i) and (ii). It is easily checked that (iii) also holds: indeed, $g \in \mathcal{U}_1$ implies that $g^{-1}(A')$ is a neat submanifold of X (see Theorem 3.4.8.2); since $g \in \mathcal{U}_2$, $g^{-1}(A') \subset \operatorname{int} X$, and as such it is closed as an independent manifold. Finally, $g \in \mathcal{U}_3$ implies that the differential $d_x(\rho|_{g^{-1}}(A')$ is non-degenerate at all points $x \in g^{-1}(A')$. □

Lemma 3.4.9.3. *Let X and X' be smooth closed manifolds of equal dimensions, and let $f\colon X \to X'$ be a submersion. If X' is connected and the preimage under f of one of its points reduces to a point, then f is a diffeomorphism.*

Proof. It suffices to show that f is invertible. Let $A' = f(X)$, and denote by B' the set of all $x' \in X$ such that $f(x')$ consists of more than one point. Being a submersion, the map f is open (see Corollary 3.1.5.8), so A' is an open set. Also, B' is open: if $f(x_1) = f(x_2) = x'$ and $x_1 \neq x_2$, then x_1 and x_2 have disjoint neighbourhoods, U_1 and U_2, in X such that $f|_{U_1}$ and $f|_{U_2}$ are differentiable embeddings, and $f(U_1) \cap f(U_2)$ is a neighbourhood of x' contained in B'. On the other hand, A' is closed because X is compact. Also, B' is closed, because if $U_1, \ldots, U_s$ are open sets coveringX and such that the restrictions $f|_{U_i}$ are differentiable embeddings, then

$$B' = \cap_{i=1}^{s} f(X \setminus U_i),$$

and the sets $f(X \setminus U_i)$ are closed. Finally, since X' is connected, $A' \neq \varnothing$, and $B' \neq X'$, we have $A' = X'$ and $B' = \varnothing$, i.e., f is invertible. ☐

Lemma 3.4.9.4. *Suppose X is a compact $\mathcal{C}^r$-submanifold of $\mathbb{R}^q$, $1 \leq r \leq \infty$, U is a neighbourhood of X in $\mathbb{R}^q$, and X' is an open subset of a closed $\mathcal{C}^a$-manifold admitting a $\mathcal{C}^a$-embedding in Euclidean space. If $f \in \mathcal{C}^r(U, X')$, then every neighbourhood in $\mathcal{C}^r(X, X')$ of the restriction $f|_X$ contains the restriction of some map belonging to $\mathcal{C}^a(U, X')$.*

Proof. For a start, suppose that X' is itself a closed $\mathcal{C}^a$-manifold. Denote by $\mathcal{U}$ the given neighbourhood of $f|_X$ in $\mathcal{C}^r(X, X')$, and fix a $\mathcal{C}^a$-embedding $j' \colon X' \to \mathbb{R}^q$, a $\mathcal{C}^a$-transversalisation τ' of j', and a neat tube $\operatorname{Tub}_{\tau'} \rho'$. Since

$$\mathcal{C}^r(\mathrm{id}, \mathrm{proj}_{\tau'}) \colon \mathcal{C}^r(X, \mathrm{tub}_{\tau'} \rho') \to \mathcal{C}^r(X, X')$$

is continuous, the preimage of $\mathcal{U}$ under this mapping is open; moreover, it is not empty because it contains the composite map

$$[\mathrm{abr}\, j' \colon X' \to \mathrm{tub}_{\tau'} \rho'] \circ f.$$

Therefore, Theorem 3.3.1.7 yields a map $g \colon U \to \mathrm{tub}_{\tau'} \rho'$ with polynomial components, such that $g|_X$ belongs to the above preimage of $\mathcal{U}$. Clearly, the composition $\mathrm{proj}_{\tau'} \circ g$ is the desired map belonging to $\mathcal{C}^a(U, X')$.

One can reduce the general case to the above situation: if X' is an open subset of the closed $\mathcal{C}^a$-manifold Y, then

$$\mathcal{C}^r(\mathrm{id}, \mathrm{incl}) \colon \mathcal{C}^r(X, X') \to \mathcal{C}^r(X, Y)$$

is a topological embedding with open image, and transforms $\mathcal{C}^a(X, X'$ into the intersection of this image with $\mathcal{C}^a(X, Y)$ (cf. Theorem 3.4.4.7). ☐

Lemma 3.4.9.5. *For each compact, q-dimensional $\mathcal{C}^r$-submanifold X of $\mathbb{R}^q$, $1 \leq r \leq a$, and each compact subset A of $\mathbb{R}^q$, the set of all $\mathcal{C}^r$-embeddings $f \colon X \to \mathbb{R}^q$ with $f(\operatorname{int} X) \supset A$ is open in $\mathbb{R}^q(X, \mathbb{R}^q)$.*

(In the present subsection we shall apply Lemma 3.4.9.5 only in the case where A is a point. However, we shall need it in full generality in the next subsection.)

Proof. For a start, suppose that A is the ball in $\mathbb{R}^q$ with centre c and radius ρ. Let $f\colon X \to \mathbb{R}^q$ be an embedding of class $\mathbb{R}^q$ such that $f(\operatorname{int} X) \supset A$, and denote by $\mathcal{U}$ the set of all $\mathbb{R}^q$-embeddings $g\colon X \to \mathbb{R}^q$ satisfying for any $x \in X$ the inequality

$$\operatorname{dist}(f(x), g(x)) < \min(\rho, \operatorname{Dist}(f(\operatorname{Fr} X), A)).$$

Since $\mathcal{U}$ is open in $\mathcal{C}^r(X, \mathbb{R}^q)$ (see Theorem 3.4.1.4) and $f \in \mathcal{U}$, it is enough to show that $g(\operatorname{int} X) \supset A$ for any $g \in \mathcal{U}$. But if $g \in \mathcal{U}$, then $g(\operatorname{Fr} X) \cap A = \varnothing$ and thus the two sets $g(X) \cap A$ and $g(\operatorname{int} X) \cap A$ are equal, while the first is closed in A and the second is open in A. Moreover, $g(\operatorname{int} X) \cap A \supset g(f^{-1}(c))$ (since $\operatorname{dist}(g(f^{-1}(c)), c) < \rho$), so that $g(\operatorname{int} X) \cap A \neq \varnothing$. Consequently, $g(\operatorname{int} X) \cap A = A$, i.e., $(\operatorname{int} X) \supset A$.

The more general situation where A is the union of a finite number of balls reduces to the case already considered. To prove the theorem in the most general case, it remains to observe that for any $\mathcal{C}^r$-embedding $f\colon X \to \mathbb{R}^q$ with $f(\operatorname{int} X) \supset A$, there is a finite number of balls whose union contains A and is contained in $f(\operatorname{int} X)$. ☐

Theorem 3.4.9.6. *Every closed $\mathcal{C}^r$-manifold X with $1 \leq r \leq \infty$ is $\mathcal{C}^r$-diffeomorphic to a $\mathcal{C}^a$-submanifold of Euclidean space.*

Proof. It is sufficient to consider a connected manifold X. Fix a $\mathcal{C}^r$-embedding $j\colon X \to \mathbb{R}^q$, a $\mathcal{C}^r$-transversalisation τ of j, and a neat tube $\operatorname{Tub}_\tau \rho$. Consider the map $f\colon \operatorname{Tub}_\tau \rho \to G'(q, n = \dim X)$ which takes each point $y \in \operatorname{Tub}_\tau \rho$ into the plane $y - j \circ \operatorname{proj}_\tau(y) + (\tau \circ \operatorname{proj}_\tau(y))^\perp$ (which passes through the point $y - j \circ \operatorname{proj}_\tau(y)$ and is orthogonal to $\tau \circ \operatorname{proj}_\tau(y)$). It is clear that f is transverse to $G(q,n)$ (even the maps $f|_{\operatorname{proj}_\tau^{-1}}$, $x \in X$, are transverse to $G(q,n)$) and that $f^{-1}(G(q,n)) = j(X)$. Pick some point $x_0 \in X$ and let $V \subset G'(q,n)$ be the set of all planes transverse to $\tau(c_0)$ (i.e., intersecting $\tau(x_0)$ at only one point) . Finally, let π be the map $V \to \tau(x_0)$ which takes each plane belonging to V into its intersection with $\tau(x_0)$. According to Lemma 3.4.9.2, $f|_{\operatorname{Tub}_\tau}(\rho/2)$ has in $\mathcal{C}^r(\operatorname{Tub}_\tau(\rho/2), G'(q,n))$ a neighbourhood $\mathcal{U}$ such that if $g \in \mathcal{U}$, then:

1. g is transverse to $G(q,n)$;
2. $g(\partial \operatorname{Tub}_\tau(\rho/2)) \subset G'(q,n) \setminus G(q,n)$;
3. $\operatorname{abr} \operatorname{proj}_\tau\colon g^{-1}(G'(q,n)) \to X$ is a submersion.

Moreover, as Lemma 3.4.9.5 shows, the set of all $\mathcal{C}^r$-embeddings

$$\varphi\colon d_\tau(x_0, \rho/2) \to \tau(x_0)$$

with $\varphi(\operatorname{int} d_\tau(x_0, \rho/2)) \ni 0$ is open in $\mathcal{C}^r(d_\tau(x_0, \rho/2)), \tau(x_0))$. Hence $f|_{\operatorname{Tub}_\tau(\rho/2)}$ has in $\mathcal{C}^r(\operatorname{Tub}_\tau(\rho/2), G'(q,n))$ a neighbourhood $\mathcal{V}$ such that for each $g \in \mathcal{V}$ one has $g(d_\tau(x_0, \rho/2)), \tau(x_0))$. Hence $f|_{\operatorname{Tub}_\tau(\rho/2)}$ has in $\mathcal{C}^r(d_\tau(x_0, \rho/2)) \subset V$ and $\pi \circ [\operatorname{abr} g\colon d_\tau(x_0, \rho/2) \to V]$ is a $\mathcal{C}^r$-embedding whose image contains 0. Since $G'(q,n)$ is an open subset of a closed $\mathcal{C}^a$-manifold (see Remark 3.2.2.11), Lemma 3.4.9.4 guarantees the existence of a $\mathcal{C}^a$-map $h\colon \operatorname{tub}_\tau \rho \to G'(q,n)$ whose restriction to $\operatorname{Tub}_\tau(\rho/2)$ belongs to $\mathcal{U} \cap \mathcal{V}$. Set $Y = [h|_{\operatorname{Tub}_\tau}(\rho/2)]^{-1}(G(q,n))$. By virtue of Theorem 3.4.8.2, Y is a $\mathcal{C}^a$-manifold, and we have only to verify that $\operatorname{proj}_\tau|_Y : Y \to X$ is a diffeomorphism. But this claim follows from Lemma 3.4.9.3. Indeed, since $h|_{\operatorname{Tub}_\tau(\rho/2)} \in \mathcal{U} \cap \mathcal{V}$, the conditions of this lemma are satisfied: the fact that $h|_{\operatorname{Tub}_\tau(\rho/2)} \in \mathcal{U}$ shows that $\operatorname{proj}_\tau|_Y$ is a submersion, while $h|_{\operatorname{Tub}_\tau(\rho/2)} \in \mathcal{V}$ implies that the preimage $(\operatorname{proj}_\tau|_Y)^{-1}(x_0)$ reduces to a point. □

Lemma 3.4.9.7. *Let X be a closed $\mathcal{C}^r$-manifold with $1 \le r \le \infty$, and let A and B be compact submanifolds of X such that $A \subset \operatorname{int} B$ and $\dim B = \dim X$. If B is endowed with the $\mathcal{C}^\infty$-structure which is the restriction of its $\mathcal{C}^r$-structure, then for any closed $\mathcal{C}^\infty$-manifold X', that part of the space $\mathcal{C}^r(X,X')$ consisting of all extensions of maps from $\mathcal{C}^\infty(A,X')$ is dense in $\mathcal{C}^r(X,X')$.*

Proof. Suppose $f \in \mathcal{C}^r(X,X')$ and $\mathcal{U}$ is a neighbourhood of f in $\mathcal{C}^r(X,X')$. We have to produce in $\mathcal{U}$ a map extending a map from $\mathcal{C}^\infty(A,X')$. Fix a $\mathcal{C}^\infty$-embedding $j'\colon X' \to \mathbb{R}^{q'}$, a $\mathcal{C}^\infty$-transversalisation τ' of j', and a neat tube $\operatorname{Tub}_{\tau'}, \rho'$. Let $\mathcal{V} \subset \mathcal{C}^r(B,X')$ be the set of all maps g such that

$$\max_{x\in B} \operatorname{dist}(j' \circ f(x), j'(g(x))) < \operatorname{Dist}(j'(X'), \mathbb{R}^{q'} \setminus \operatorname{tub}_{\tau'} \rho').$$

Clearly, $\mathcal{V}$ is open and the segment with endpoints $j' \circ f(x)$ and $j'(g(x))$ lies in $\operatorname{tub}_{\tau'} \rho'$, for any $g \in 1\mathcal{V}$ and $x \in B$. Now choose a neighbourhood U of ∂B in B such that $\operatorname{Cl} U \cap A = \varnothing$, construct a Urysohn $\mathcal{C}^\infty$-function $\varphi\colon B \to I$ for the pair $\operatorname{Cl} U$, A, and consider the mapping $\Phi\colon \mathcal{V} \to \mathcal{C}^r(X,X')$ which takes $g \in \mathcal{V}$ into the map

$$x \mapsto \begin{cases} f(x), & \text{if} \quad x \in X \setminus B, \\ \operatorname{proj}_{\tau'}((1-\varphi(x))j' \circ f(x) + \varphi(x)j'(g(x))), & \text{if} \quad x \in B. \end{cases}$$

Obviously, Φ is continuous, and $\Phi(f|_B) = f$. These properties of Φ show that the set $\Phi^{-1}(\mathcal{U})$ is open and non-empty. Applying Theorem 3.4.6.5, we deduce that $\Phi^{-1}(\mathcal{U})$ contains a $\mathcal{C}^\infty$-map. Finally, note that Φ transforms $\mathcal{C}^\infty$-maps into extensions of maps from $\mathcal{C}^\infty(A,X')$. □

Theorem 3.4.9.8. *Every compact $\mathcal{C}^r$-manifold with $1 \le r < \infty$ is $\mathcal{C}^r$-diffeomorphic to a $\mathcal{C}^\infty$-manifold. Moreover, if X is a compact $\mathcal{C}^r$-manifold with $1 \le r < \infty$, Y is a $\mathcal{C}^\infty$-manifold, and $\psi\colon Y \to \partial X$ is a $\mathcal{C}^r$-diffeomorphism, then there exists a $\mathcal{C}^\infty$-manifold X' together with a $\mathcal{C}^r$-diffeomorphism $\varphi\colon X \to X'$, such that the composite map*

$$Y \xrightarrow{\psi} \partial X \xrightarrow{\operatorname{abr} \varphi} \partial X'$$

is a $\mathcal{C}^\infty$-diffeomorphism.

Proof. Composing the $\mathcal{C}^r$-diffeomorphism $\psi \times \operatorname{id}\colon Y \times [-1,1] \to \partial X \times [-1,1]$ with an arbitrary two-sided $\mathcal{C}^r$-collaring $\partial X \times [-1,1] \to \operatorname{dopp} X$, we obtain a

$\mathcal{C}^r$-embedding $Y \times [-1,1] \to \operatorname{dopp} X$. Since $Y \times [-1,1]$ is a $\mathcal{C}^\infty$-manifold, the image B of this embedding inherits a $\mathcal{C}^\infty$-structure which is the restriction of the induced $\mathcal{C}^r$-structure, and obviously $\partial X \subset \operatorname{int} B$. Theorem 3.4.9.6 provides a closed $\mathcal{C}^a$-manifold Z together with a $\mathcal{C}^r$-diffeomorphism $\operatorname{dopp} X \to Z$. In addition, Lemma 3.4.9.7 and Corollary 3.4.1.6 imply that this $\mathcal{C}^r$-diffeomorphism may be taken of class $\mathcal{C}^\infty$ on ∂X. Now set X' to be the image of X under the diffeomorphism chosen in this way, and take φ to be the compression of this diffeomorphism to a diffeomorphism $X \to X'$. It is immediate that X' and φ have the desired properties. □

Information 3.4.9.9. Theorems 3.4.9.6 and 3.4.9.8 can be substantially strengthened: they are valid for non-compact manifolds too, and in Theorem 3.4.9.8 one may replace ∞ by a. However, one cannot eliminate the hypothesis that $r \geq 1$ from these theorems: there exist closed topological manifolds which are not homeomorphic to smooth manifolds. The first example of this kind appeared in [12].

Application: A Supplement to Theorem 3.4.7.7

Theorem 3.4.9.10. *Let X_1 and X_2 be compact $\mathcal{C}^{\geq r}$-manifolds with $1 \leq r \leq \infty$. If X' is closed, then the pairs (f_1, f_2) of transverse maps form a dense set in $\mathcal{C}^r(X_1, X') \times \mathcal{C}^r(X_2, X')$.*

Proof. By Theorem 3.4.9.8, there exist $\mathcal{C}^\infty$-manifolds Y_1, and Y_2, together with $\mathcal{C}^r$-diffeomorphisms $Y_1 \to X_1$, $Y_2 \to X_2$ and $Y' \to X'$. Hence it is enough to prove that the pairs of transverse maps are dense in $\mathcal{C}^r(Y_1, Y') \times \mathcal{C}^r(Y_2, Y')$. By Theorem 3.4.7.7, these pairs are dense in $\mathcal{C}^\infty(Y_1, Y') \times \mathcal{C}^\infty(Y_2, Y')$, and according to Theorem 3.4.6.5 this product is dense in $\mathcal{C}^r(Y_1, Y') \times \mathcal{C}^r(Y_2, Y')$. □

3.4.10 Approximation of Maps by Embeddings and Immersions

Remark 3.4.10.1. In this subsection we complete the programme outlined in Remark 3.4.4.1. The main results are Theorems thm:03-4-10-4 and thm:03-4-10-5. Lemma 3.4.10.3 is our basic tool, and this same lemma can be used to derive many other corollaries; see, in particular, Exercises 3.4.11.11, 3.4.11.12, and 3.4.11.13.

Auxiliary Manifolds

Remark 3.4.10.2. Suppose X is a closed n-dimensional $\mathcal{C}^{\geq r}$-manifolds with $1 \leq r \leq a$, and let $j\colon X \to \mathbb{R}^q$ be an embedding of class $\mathcal{C}^r$. Also, let m be a positive integer such that $0 < m \leq q$. We shall need two constructions.

The first construction We denote by Aux_1 or, more specifically by $\mathrm{Aux}_1(j; m)$, that subset of $\operatorname{Tang} X \times G(q, m)$ consisting of the pairs (u, γ) such that

$$< dj(u), dj(u) >= 1 \quad \text{and} \quad dj(u) \in \gamma.$$

Further, define

$$\mathrm{aux}_1\colon\ \mathrm{Aux}_1 \to G'(q,m),\quad (u,y)\mapsto j(\mathrm{proj}(u))+\gamma,\quad \mathrm{proj}=[\mathrm{proj}\colon\ \mathrm{Tang}\,X\to X].$$

Note that Aux_1 is a $[2n-1+(m-1)(q-m)]$-dimensional submanifold of $\mathrm{Tang}\,X\times G(q,m)$. Indeed, it is the preimage of the submanifold

$$\mathbb{S}^{q-1}\times G(q,m)\subset\mathbb{R}^q\times G'(q,m)$$

under the mapping

$$\mathrm{Tang}\,X\times G(q,m)\to\mathbb{R}^q\times G'(q,m),\quad (u,\gamma)\mapsto(dj(u),dj(u)+\gamma);$$

this mapping is transverse to $\mathbb{S}^{q-1}\times G(q,m)$.

aux_1 is of class $\mathcal{C}^{r-1}$ and its image consists of those m-planes of $\mathbb{R}^q$ which contain lines tangent to $j(X)$.

The second construction We denote by Aux_2 or, more specifically by $\mathrm{Aux}_2(j;m)$, the subset of $X\times X\times G(q,m)$ consisting of the triples (x,x',γ) such that $x\neq x'$ and $j(x')-j(x)\in\gamma$. Further, define

$$\mathrm{aux}_2\colon\ \mathrm{Aux}_2\to G'(q,m),\quad (x,x',\gamma)\mapsto j(x)+\gamma.$$

Note that Aux_2 is a $[2n+(m-1)(q-m)]$-dimensional submanifold of $X\times X\times G(q,m)$. Indeed, it is the preimage of the submanifold $G(q,m)$ of $G'(q,m)$ under the mapping

$$((X\times X)\setminus\mathrm{diag}\,X)\times G(q,m)\to G'(q,m),\quad (x,x',\gamma)\mapsto j(x')-j(x)+\gamma;$$

this mapping is transverse to $G(q,m)$.

aux_2 is of class $\mathcal{C}^r$ and its image consists of those m-planes of $\mathbb{R}^q$ which intersect $j(X)$ at more than one point.

The Basic Theorems

Lemma 3.4.10.3. *Let X be a closed n-dimensional $\mathcal{C}^{\geq r}$-manifold, $1\leq r\leq\infty$, and let $j\colon X\to\mathbb{R}^q$ be an embedding of class $\mathcal{C}^r$ together with a $\mathcal{C}^r$-transversalisation $\tau\colon X\to G'(q,q-n)$ and a neat tube $\mathrm{Tub}_\tau\,\rho$. Then there exist a neighbourhood $\mathcal{U}$ of the map $\widetilde{\tau}\colon X\to G'(q,q-n)$ (see Remark 3.4.3.2) in $\mathcal{C}^r(X,G'(q,q-n))$ and a continuous mapping $\Phi\colon\mathcal{U}\to\mathrm{Diff}^r\,X$ such that, for each map $g\in\mathcal{U}$:*

(i) $[j\circ\Phi(g)](x)\in g(x)$ *for all* $x\in X$*;*

(ii) the map

$$\tau_g\colon X\to G(q,q-n),\quad x\mapsto g(x)-[j\circ\Phi(g)](x)$$

is a transversalisation of the embedding $j\circ\Phi(g)\colon X\to\mathbb{R}^q$*;*

(iii) some neat tube of this transversalisation contains $\mathrm{Tub}_\tau(\rho/2)$.

Proof. Given $g \in \mathcal{C}^r(X, G'(q, q-n))$, let $h_g\colon \operatorname{Tub}_\tau \rho \to \mathbb{R}^q$ denote the map which carries each point $y \in \operatorname{Tub}_\tau \rho$ into its image under the orthogonal projection onto the plane $g(\operatorname{proj}(y))$. Obviously,

$$\mathcal{C}^r(X, G'(q, q-n)) \to \mathcal{C}^r(\operatorname{Tub}_\tau \rho, \mathbb{R}^q), \quad g \mapsto h_g$$

is a continuous mapping, and if $g = \widetilde{\tau}$, then $h_g = [\text{incl}\colon \operatorname{Tub}_\tau \rho \to \mathbb{R}^q]$. Consequently, $\widetilde{\tau}$ has a neighbourhood $\mathcal{V}$ in $\mathcal{C}^r(X, G'(q, q-n))$ such that, for any $g \in \mathcal{V}$, h_g is a $\mathcal{C}^r$-embedding and $h_g(\operatorname{tub}_\tau \rho) \supset \operatorname{Tub}_\tau(\rho/2)$ (see Lemma 3.4.9.5). For $g \in \mathcal{V}$, let i_g denote the composition

$$x \xrightarrow{\operatorname{abr} j} \operatorname{Tub}_\tau(\rho/2) \xrightarrow{(\operatorname{abr} h_g)^{-1}} h_g^{-1}(\operatorname{Tub}_\tau(\rho/2)) \xrightarrow{\text{incl}} \operatorname{Tub}_\tau \rho \xrightarrow{\operatorname{proj}_\tau} X$$

An obvious verification shows that $g \mapsto i_g$ is a continuous mapping $\mathcal{V} \to \mathcal{C}^r(X, X)$, and that $\widetilde{i_\tau} = \text{id}$. Therefore, $\widetilde{\tau}$ has a neighbourhood $\mathcal{U}$in $\mathcal{V}$ such that i_g is a diffeomorphism for all $g \in \mathcal{U}$. Set $\Phi(g) = i_g^{-1}$ for $g \in \mathcal{U}$. It is immediate that Φ is continuous and that $\mathcal{U}$ and Φ satisfy the conditions (i)-(iii). □

Theorem 3.4.10.4. *Suppose X is a compact n-dimensional $\mathcal{C}^{\geq r}$-manifold, $1 \leq r \leq \infty$, and X' is a closed n-dimensional $\mathcal{C}^{\geq r}$-manifold. Then for $n' \geq 2n$, $\operatorname{Imm}^r(X, X')$ is dense in $\mathcal{C}^r(X, X')$, and for $n' \geq 2n+1$, $\operatorname{Emb}^r(X, X')$ is dense in $\mathcal{C}^r(X, X')$.*

Proof. Without loss of generality, we shall prove these statements in the case $r = \infty$; when $r < \infty$, we simply apply Theorems 3.4.9.6 and 3.4.6.5 to reduce to the first case.

Let $f \in \mathcal{C}^\infty(X, X')$, and let $\mathcal{U}$ be a neighbourhood of f. We have to show that, for $n' \geq 2n$, $\mathcal{U}$ contains an immersion, and that for $n' \geq 2n+1$, $\mathcal{U}$ contains an embedding.

Fix $\mathcal{C}^\infty$-embeddings $j\colon X \to \mathbb{R}^q$ and $j'\colon X' \to \mathbb{R}^{q'}$, and define an embedding

$$J'\colon X' \to \mathbb{R}^{q+q'} = \mathbb{R}^q \times \mathbb{R}^{q'}, \quad x' \mapsto (j'(x'), 0).$$

Further, pick a $\mathcal{C}^\infty$-transversalisation τ' of J' and a neat tube $\operatorname{Tub}_{\tau'} \rho'$. Then $J_t(x) = (j'(f(x), tj(x))$ defines a $\mathcal{C}^\infty$-embedding $J_t\colon X \to \mathbb{R}^{q+q'}$, for any fixed $t > 0$. Pick ε small enough so that $J_\varepsilon(X) \subset \operatorname{Tub}_{\tau'},(\rho'/2)$ and denote J_ε simply by J. Applying Lemma 3.4.10.3 (to the manifold X', the embedding J', the transversalisation τ' and the tube $\operatorname{Tub}_{\tau'} \rho'$), we conclude that there are a neighbourhood $\mathcal{U}'$ of the map $\widetilde{\tau'}$ in $\mathcal{C}^\infty(X', G'(q'+q, q'+q-n))$ and a continuous mapping $\Phi'\colon \mathcal{U}' \to \operatorname{Diff}^\infty X'$ such that, for each $g' \in \mathcal{U}'$:

(i) $[J' \circ \Phi'(g')](x') \in g'(x')$ for all $x' \in X'$;

(ii) the map

$$\tau'_{g'}\colon X' \to G(q'+q, q'+q-n'), \quad (x') \mapsto g'(x') - [J' \circ \Phi'(g')](x')$$

is a transversalisation of the embedding $J' \circ \Phi'(g')$;

(iii) some neat tube of this transversalisation contains $\mathrm{Tub}_{\tau'}(\rho'/2)$.

Now consider the mapping

$$\Psi' \colon \mathcal{U}' \to C^\infty(X, X'), \quad [\Psi'(g')](x) = \mathrm{proj}_{\tau'_{g'}}(J(x)).$$

We see that Ψ' is continuous and $\Psi'(\tau') = f$. Hence, $(\Psi')^{-1}(\mathcal{U})$ is open in $\mathcal{C}^\infty(X', G'(q'+q, q'+q-n'))$ and non-empty. This fact together with Theorem 3.4.7.7 show that $(\Psi')^{-1}(\mathcal{U})$ contains a map h transverse to each of the maps

$$\begin{aligned}\mathrm{aux}_1 \colon\ & \mathrm{Aux}_1(J; q'+q-n') \to G'(q'+q, q'+q-n')\\ \mathrm{aux}_2 \colon\ & \mathrm{Aux}_2(J; q'+q-n') \to G'(q'+q, q'+q-n').\end{aligned}$$

We shall presently show that $\Psi'(h)$ is an immersion for $n' \geq 2n$, and a differentiable embedding for $n' \geq 2n+1$. This will complete the proof, since $\Psi'(h) \in \mathcal{U}$.

The inequality $n' \geq 2n$ is equivalent to

$$\dim X' + \dim \mathrm{Aux}_1(J; q'+q-n') < \dim G'(q'+q, q'+q-n');$$

hence for $n' \geq 2n$ the fact that aux_1 and h are transverse means that $h(X')$ does not intersect $\mathrm{im}\,\mathrm{aux}_1$, i.e., none of the planes $h(x'), x' \in X'$, contains a line tangent to $J(X)$. The latter, in turn, means that the differential $d_y(\mathrm{proj}_{\tau'_h}|_{J(x)})$ of the restriction $\mathrm{proj}_{\tau'_h}|_{J(x)}$ is a monomorphism for any point $y \in J(X)$. Thus $\mathrm{proj}_{\tau'_h}|_{J(x)}$ and also $\Psi'(h)$, are immersions.

Similarly, the inequality $n' \geq 2n+1$ is equivalent to

$$\dim X' + \dim \mathrm{Aux}_2(J; q'+q-n') < \dim G'(q'+q, q'+q-n');$$

hence for $n' \geq 2n+1$ the fact that aux_2 and h are transverse means that $h(X')$ does not intersect $\mathrm{im}\,\mathrm{aux}_2$, i.e., none of the planes $h(x'), x' \in X'$, intersects $J(X)$ at more than one point. This says that the restriction $\mathrm{proj}_{\tau'_h}|_{J(x)}$ is injective. We see at once that $\Psi'(h)$ is also an injective map, and since it is an immersion, $\Psi'(h)$ is a differentiable embedding (see Corollary 3.1.5.4). □

Embeddings and Immersions in Euclidean Spaces

Theorem 3.4.10.5. *Every compact n-dimensional $\mathcal{C}^{\geq r}$-manifold with $1 \leq r \leq \infty$ can be $\mathcal{C}^r$-immersed in $\mathbb{R}^{2n}$ and $\mathcal{C}^r$-embedded in $\mathbb{R}^{2n+1}$.*

Proof. This is a consequence of Theorem 3.4.10.4 (where we take X to be the given manifold, and set first $X' = \mathbb{S}^{2n}$, and then $X' = \mathbb{S}^{2n+1}$; we disregard the trivial case $n = 0$). □

Information 3.4.10.6. As a matter of fact, every n-dimensional $\mathcal{C}^{\geq r}$-manifold with $r > 1$ admits a $\mathcal{C}^r$-embedding in $\mathbb{R}^{2n}$ whenever $n \geq 1$, and a $\mathcal{C}^r$-immersion in $\mathbb{R}^{2n-1}$ whenever $n \geq 2$. If $n > 0$ and is not a power of 2, then every $\mathcal{C}^{\geq r}$-manifold with $r \geq 1$ admits a $\mathcal{C}^r$-embedding in $\mathbb{R}^{2n-1}$. However, for each

$n = 2^s$, $s \geq 0$, there are smooth, closed, n-dimensional manifolds which cannot be even topologically embedded in $\mathbb{R}^{2n-1}$ (such an example is $\mathbb{R}P^n$). Every n-dimensional $\mathcal{C}^r$-manifold, $r \geq 1$, without closed components, can be $\mathcal{C}^r$-embedded in $\mathbb{R}^{2n-1}$. Every orientable n-dimensional $\mathcal{C}^{\geq r}$-manifold with $r \geq 1$ and $n \neq 1, 4$ can be $\mathcal{C}^r$-embedded in $\mathbb{R}^{2n-1}$. [it is not known whether a smooth, closed, orientable, four-dimensional manifold admits a differentiable embedding in $\mathbb{R}^7$; a topological embedding always exists.]

More information, details, and references can be found in [20] and [18].

3.4.11 Exercises

Exercise 3.4.11.1. Show that for any $\mathcal{C}^{\geq r}$-manifolds X and X', with $0 \leq r \leq \infty$, the space $\mathcal{C}^r(X, X')$ has a countable base.

Exercise 3.4.11.2. Let X be a compact $\mathcal{C}^{\geq r}$-manifold, $1 \leq r \leq \infty$ and let X' be an arbitrary $\mathcal{C}^{\geq r}$-manifold. Show that the set $\operatorname{Subm}^r(X, X') \cap \mathcal{C}^r_\partial(X, X')$ is open in $\mathcal{C}^r_\partial(X, X')$.

Exercise 3.4.11.3. Show that every compact topological manifold is a CNRS. [cf. Theorem 3.4.5.10.]

Exercise 3.4.11.4. Suppose X is the smooth one-dimensional submanifold of $\mathbb{R}^2$, closed as an independent manifold, and containing the graph of the function $x \mapsto \sin(1/x)$ defined on the interval $(0, 1)$. Let τ be the normal transversalisation of the inclusion $X \to \mathbb{R}^2$. Show that for any ρ the segment $d_\tau((0, 0), \rho)$ intersects a segment $d_\tau(x, \rho)$ for some $x \neq (0, 0)$. [cf. Remark 3.4.3.2.]

Exercise 3.4.11.5. Suppose X is a compact $\mathcal{C}^r$-manifold with $1 \leq r \leq \infty$, and A is a submanifold of X. Show that for a suitable collaring of X, $A \cup \operatorname{cop}(A)$ is a submanifold of $\operatorname{dopp} X$.

Exercise 3.4.11.6. Let X and Y be compact $\mathcal{C}^r$-manifold with $1 \leq r \leq \infty$. Check that:

(i) the product $X \times Y$ has a $\mathcal{C}^r$-structure which induces the usual $\mathcal{C}^r$-structures on $interX \times Y$ and $X \times \operatorname{int} Y$;

(ii) the $\mathcal{C}^r$-manifold obtained by equipping $X \times Y$ with a $\mathcal{C}^r$-structure having these properties is unique up to a diffeomorphism.

Exercise 3.4.11.7. Let X_1, X_2, and X be compact $\mathcal{C}^\infty$-manifolds, and let $f_2 \in \mathcal{C}^\infty_\partial(X_2, X')$. Show that the subset of $\mathcal{C}^\infty_\partial(X_1, X')$ consisting of all maps transverse to f_2 is dense in $\mathcal{C}^\infty_\partial(X_1, X')$.

Exercise 3.4.11.8. Let X_1, X_2, X', and f_2 be as in exercise 3.4.11.7. Show that for any $\mathcal{C}^\infty$-map $\varphi\colon \partial X_1 \to \partial X'$ transverse to $\operatorname{abr} f_2\colon \partial X_2 \to \partial X'$, the subset of $\mathcal{C}^\infty - (X_1, X')$ consisting of all the extensions of φ which are transverse to f_2 is dense in the subspace of $\mathcal{C}^\infty_\partial(X_1, X')$ consisting of all the extensions of ϕ.

Exercise 3.4.11.9. Let X and X' be compact $\mathcal{C}^{\geq r}$-manifolds with $1 \leq r \leq \infty$, and let $\varphi\colon partialX \to \partial X'$ be a $\mathcal{C}^r$-immersion. If $\dim X' > 2 \dim X$, show that the subset of $\mathcal{C}^r_\partial(X, X')$ consisting of all $\mathcal{C}^r$-immersions that extend φ is dense in the subspace of $\mathcal{C}^r_\partial(X, X')$ consisting of all extensions of φ.

Exercise 3.4.11.10. Let X and X' be compact $\mathcal{C}^{\geq r}$-manifolds with $1 \leq r \leq \infty$, and let $\varphi \colon \partial X \to \partial X'$ be a $\mathcal{C}^r$-embedding. If $\dim X' \geq 2 \dim X + 1$, show that the subset of $\mathcal{C}^r_\partial(X, X')$ consisting of all $\mathcal{C}^r$-embeddings that extend φ is dense in the subspace of $\mathcal{C}^r_\partial(X, X')$ consisting of all extensions of φ. [In particular, every compact n-dimensional $\mathcal{C}^{\geq r}$-manifold admits a neat embedding in $\mathbb{D}^{2n+1}$.]

Exercise 3.4.11.11. Let X and X' be closed $\mathcal{C}^{\geq r}$-manifolds with $1 \leq r \leq \infty$. Show that the set of all $\mathcal{C}^r$-maps $f \colon X \to X'$ such that $\operatorname{Tang}_{f(x_1)} X' = \operatorname{im} d_{x_1} f + \operatorname{im} d_{x_2} f$ for any two distinct points $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, is dense in $\mathcal{C}^r(X, X')$.

Exercise 3.4.11.12. Let X and X' be closed $\mathcal{C}^{\geq r}$-manifolds with $1 \leq r \leq \infty$. If $\dim X' = 2 \dim X - 1$, show that the set of all $\mathcal{C}^r$-maps $f \colon X \to X'$ such that $\operatorname{rank} d_x f = \dim X$ for all but a finite number of points $x \in X$, where $\operatorname{rank} d_x f = \dim X - 1$, is dense in $\mathcal{C}^r(X, X')$.

Exercise 3.4.11.13. Let X and X'be closed $\mathcal{C}^{\geq r}$-manifolds with $1 \leq r \leq \infty$. If $2 \dim X' > 3 \dim X$, show that the set of all $\mathcal{C}^r$-maps $f \colon X \to X'$ such that the preimage of each point of X' under f contains at most two points, is dense in $\mathcal{C}^r(X, X')$.

3.5 THE SIMPLEST STRUCTURE THEOREMS

3.5.1 Morse Functions

Remark 3.5.1.1. The central result of this section is Theorem 3.5.2.10, whose main conclusion is that every compact n-dimensional $\mathcal{C}^\infty$-manifold can be obtained from an empty n-dimensional manifold through a finite number of fairly standard operations, namely, by attaching handles. The entire present subsection and that part of Subsection 3.5.2 preceding Theorem 3.5.2.10 are essentially devoted to the preparation of its formulation and proof. The remaining part of Subsection 3.5.2 contains corollaries of Theorem 3.5.2.10. In Subsection 3.5.3 this theorem is used to effectively classify the compact smooth two-dimensional manifolds.

It should come as no surprise that, in contrast to the previous sections, here we consider, in general, only the $\mathcal{C}^\infty$-case: the theorems concerning smoothing of diffeomorphisms and manifolds (i.e., Corollaries 3.4.4.5, 3.4.6.12, and Theorems 3.4.9.6, 3.4.9.8) show that we may replace the class $\mathcal{C}^\infty$ by any class $\mathcal{C}^r$, $1 \leq r < \infty$, without affecting the theory discussed here.

Cobordisms and Morse Functions

Definition 3.5.1.2. A compact $\mathcal{C}^\infty$-manifold X is called a *cobordism* if its boundary ∂X is the disjoint union of two parts, $\partial_0 X$ and $\partial 1X$, each consisting of whole components of ∂X. Those two parts are termed the *beginning* and the *end of the cobordism* X. Each of them may be empty; when both are empty, X is closed. In general, given a compact $\mathcal{C}^\infty$-manifold, one can transform it into a cobordism in 2^ℓ ways, where ℓ is the number of components of ∂X. Among these cobordisms, there is one without beginning ($\partial_0 X = \varnothing$, $\partial_1 X = \partial X$) and one without end ($\partial_0 X = \partial X$, $\partial_1 X = \varnothing$).

Two cobordisms, X and X', are said to be *diffeomorphic* if there is a diffeomorphism (and hence a $\mathcal{C}^\infty$-diffeomorphism) $f\colon X \to X'$ such that $f(\partial_0 X) = \partial_0 X'$ and $f(\partial_1 X) = \partial_1 X'$.

Suppose that X and X' are two cobordisms such that $\partial_1 X$ and $\partial_0 X'$ are diffeomorphic, and let $\varphi\colon \partial_1 X \to \partial_0 X'$ be a $\mathcal{C}^\infty$-diffeomorphism. Then one can form a manifold Y by glueing the somehow collared manifolds X and X' with the aid of φ. Now Y naturally becomes a cobordism if we set $\partial_0 Y = \partial_0 X$ and $\partial_1 Y = \partial_1 X'$. We say that *the cobordism Y is the result of glueing the cobordisms X and X' by φ.* If the cobordisms X and X' are oriented and φ is orientation reversing (here the orientations of $\partial_1 X$ and $\partial_0 X'$ are those induced by the orientations of X and X'; see Remark 3.1.3.4), then one can orient Y in such a manner that both embeddings, $X \to Y$ and $X' \to Y$, become orientation preserving. **Warning**: this definition of the orientation of the glued cobordism is not in accordance with the definition of the orientation of a glued manifold, given in Remark 3.4.5.5.

Two smooth closed manifolds, V_0 and V_1, are *cobordant* if there is a cobordism with the beginning and the end diffeomorphic to V_0 and V_1, respectively.

If, in addition, V_0 and V_1 are oriented, and there is an oriented cobordism X such that one of the diffeomorphisms $V_0 \to \partial_0 X$ and $V_1 \to \partial_1 X$ preserves orientation, whereas the other reverses it, then we say that V_0 and V_1 are *oriented cobordant.* Clearly, the cobordism and oriented cobordism relations are reflexive and symmetric, and since cobordisms can be glued, they are also transitive, i.e., they are genuine equivalence relations.

Definition 3.5.1.3. A critical point x of a $\mathcal{C}^2$-function $f\colon X \to \mathbb{R}$, where X is a $\mathcal{C}^{\geq 2}$-manifold is *non-degenerate* if for some chart $\varphi \in \operatorname{Atl}_x C^2 X$ (and hence for any such chart) $\varphi(x)$ is a non-degenerate critical point of the function $(f|_{\operatorname{supp}\varphi}) \circ \varphi^{-1}\colon \operatorname{im}\varphi \to \mathbb{R}$ (see Remark 3.3.3.1). The corresponding index is independent of the choice of the chart φ (see Remark 3.3.3.1), and is called the *index of the point x relative to f.*

Suppose X is a cobordism, and let $f\colon X \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function; f is a *Morse function* if the following holds:

- $\operatorname{im} f \subset I$;
- $f^{-1}(0) = \partial_0 X$, and $f^{-1}(1) = \partial_1 X$;
- all critical points of f lie in $\operatorname{int} X$ and are non-degenerate.

We say that a Morse function is *proper* if its values at distinct critical points are distinct.

The Local Structure of Morse Functions

Theorem 3.5.1.4. *Suppose that X is an n-dimensional cobordism and $f\colon X \to \mathbb{R}$ is a Morse function. Then for every point $x \in X$ there is a chart $\varphi \in \operatorname{Atl}_x X$ with $\varphi(x) = 0$, such that the restriction $f|_{\operatorname{supp}\varpi}$ coincides with the composite map*

$$\operatorname{supp}\varphi \xrightarrow{\varphi} \operatorname{im}\varphi \to \mathbb{R},$$

where the second arrow denotes one of the following functions:

$$(t_1, \dots, t_n) \mapsto \begin{cases} -t_1, & \textit{if} \quad x \in \partial_0 X; \\ 1 + t_1, & \textit{if} \quad x \in \partial_1 X; \\ f(x) + t_1, & \textit{if} \quad x \in \operatorname{int} X \quad \textit{is not a critical point of} \quad f; \end{cases}$$

Proof. To prove the first three cases we need only remark that the function $X \to \mathbb{R}$, defined by

$$y \mapsto \begin{cases} f(y) - x & \text{for} \quad x \in \operatorname{int} X \cup \partial_1 X \\ -f(y) & \text{for} \quad x \in \partial_0 X \end{cases}$$

can be completed, in a neighbourhood of x, to a system of coordinates (see Remark 3.1.2.12). For the fourth case, we refer to Theorem 3.3.3.5. □

Corollary 3.5.1.5. *A Morse function has only a finite number of critical points.*

An Existence Theorem

Lemma 3.5.1.6. *Given any $\mathcal{C}^\infty$-function $f\colon \mathbb{D}^n \to \mathbb{R}$, there exists an open dense subset A of $\mathbb{R}^n$ such that for $a \in A$ the function $\mathbb{D}^n \to \mathbb{R}$ defined by*

$$x \mapsto f(x) - < a, x > \tag{3.5.1.7}$$

has no degenerate critical points.

Proof. Consider the map $\operatorname{grad} f\colon \mathbb{D}^n \to \mathbb{R}$ with coordinate functions $D_1 f, \dots, D_n f$. One may take A to be $\mathbb{R}^n \setminus [\operatorname{grad} f](F)$, where $F = \{x \in \mathbb{D}^n | \operatorname{rank} d_x \operatorname{grad} f < n\}$. F is clearly open, and Theorem 3.4.7.4 implies that it is also dense in $\mathbb{R}$. Moreover, it is evident that if $x \in \mathbb{D}^n$ is a critical point of the function (3.5.1.7), then $[\operatorname{grad} f](x) = a$, and the matrix of the second-order partial derivatives of (3.5.1.7) at x is precisely the matrix of the differential $d_x \operatorname{grad} f$ relative to the standard coordinates in $\operatorname{Tang}_x \mathbb{D}^n$ and $\operatorname{Tang}_a \mathbb{R}^n$. Therefore, if $a \in A$, then this matrix of second-order partial derivatives is non-singular. ☐

Theorem 3.5.1.8. *On every cobordism there is a proper Morse function.*

Proof. First, let us show that if there exists some Morse function on the cobordism X, then there exists a proper Morse function on X. Let $x_1, \dots, x_m$ be the critical points of the Morse function $f\colon X \to \mathbb{R}$, and let $U_1, \dots, U_m$ be pairwise disjoint neighbourhoods of these points in $\operatorname{int} X$. Further, let $V_1, \dots, V_m$ be neighbourhoods of $x_1, \dots, x_m$ such that $\operatorname{Cl} V_1 \subset U_1, \dots, \operatorname{Cl} V_m \subset U_m$, and let $\varphi_1, \dots, \varphi_m$ be Urysohn $\mathcal{C}^\infty$-functions for the pairs $X \setminus U_1, \operatorname{Cl} V_1), \dots, (X \setminus U_m, \operatorname{Cl} V_m)$. (The existence of such Urysohn functions results from Lemma 3.4.4.8, applied here to the double of the manifold X.) Clearly, if the numbers $\varepsilon_1, \dots, \varepsilon_m$ are small enough, then the function

$$X \to \mathbb{R}, \quad x \mapsto f(x) + \varepsilon_1 \varphi_1(x) + \dots + \varepsilon_m \varphi_m(x)$$

is, together with f, a Morse function, and its only critical points are $x_1, \dots, x_m$. Moreover, its values at the points $x_1, \dots, x_m$ are $f(x_1) + \varepsilon_1, \dots f(x_m) + \varepsilon_m$, and so, by choosing suitable $\varepsilon_1, \dots, \varepsilon_m$ one can force these values to be distinct.

Now we show that there exists Morse functions on X. Fix a collaring $k\colon \partial X \times I \to X$ and a $\mathcal{C}^\infty$-function $\beta\colon I \to I$ which equals 1 on the segment $[0, 1/2]$ and vanishes on a neighbourhood of 1. Next define $g\colon X \to \mathbb{R}$ by the formulae

$$g(x) = \frac{1}{2}, \quad \text{for} \quad x \in X \setminus k(\partial X \times [0,1)),$$

$$g(k(z,t)) = \begin{cases} \frac{1}{2} + \frac{1}{2}(t-1)\beta(t), & \text{for} \quad z \in \partial_0 X, \quad t \in I, \\ \frac{1}{2} + \frac{1}{2}(1-t)\beta(t), & \text{for} \quad z \in \partial_1 X, \quad t \in I. \end{cases}$$

It is clear that g is $\mathcal{C}^\infty$, takes X into I, $\partial_0 X$ into 0, and $\partial_1 X$ into 1, and has no critical points in $k(\partial X \times [0, 1/2])$. Pick charts $\varphi_1, \dots, \varphi_s \in \operatorname{Atl} X$ such that $\operatorname{im} \varphi_1 = \cdots \operatorname{im} \varphi_s = \mathbb{R}^n$, $n = \dim X$, and the sets $\varphi_1^{-1}(\operatorname{int} \mathbb{D}^n), \dots, \varphi_s^{-1}(\operatorname{int} \mathbb{D}^n)$ cover $X \setminus k(\partial X \times [0, 1/2])$. Further, choose a $\mathcal{C}^\infty$-function $\alpha\colon \mathbb{R}^n \to I$ which equals 1 on $\mathbb{D}^n$ and 0 outside the concentric ball $2\mathbb{D}^n$ of radius 2. Using induction, we shall build functions $g_0, \dots, g_s \colon X \to \mathbb{R}$, such that g_i coincides with g in a neighbourhood of ∂X, maps $\operatorname{int} X$ into $\operatorname{int} I$, has no critical points in $k(\partial X \times [0, 1/2])$, and has no degenerate critical points in $\cup_{j=1}^{i} \varphi_j^{-1}(\mathbb{D}^n)$. Then g_s will be a Morse function on X.

Let $g_0 = g$ and assume that for some $k \geq 1$ functions enjoying the required properties are already constructed for $i < k$. For each point $a \in \mathbb{R}^n$, the formula

$$h_a(x) = \begin{cases} g_{k-1}(x), & \text{if} \quad x \in X \setminus \varphi_k^{-1}(2\mathbb{D}^n), \\ g_{k-1} - < a, \varphi_k > \alpha \circ \varphi_k(x), & \text{if} \quad x \in \operatorname{supp} \varphi_k, \end{cases}$$

defines a $\mathcal{C}^\infty$-function $h_a \colon X \to \mathbb{R}$ which agrees with g_{k-1} in a neighbourhood of ∂X, and for $a = 0$ coincides with g_{k-1} on all of X. Clearly, the point 0 has a neighbourhood U in $\mathbb{R}^n$ such that for $a \in U$ the function h_a has no critical points in $k(\partial X \times [0, 1/2])$, has no degenerate critical points in $\cup_{j=1}^{k-1} \varphi_j^{-1}(\mathbb{D}^n)$, and maps $\operatorname{int} X$ into $\operatorname{int} I$. Thus, by Lemma 3.5.1.6, we can find $a \in U$ such that h_a also has no degenerate critical points in $\varphi_k^{-1}(\mathbb{D}^n)$, and hence we can take $g_k = h_a$ for such a value of a. □

3.5.2 Cobordisms and Surgery

Remark 3.5.2.1. This subsection is devoted to two types of special operations on cobordisms, called *attaching of handles* and *spherical modifications*. The former were already mentioned in Remark 3.5.1.1. Spherical modifications are simpler that the attaching of handles, but their applications are more limited: we shall define them solely for the closed case, and starting with a closed $\mathcal{C}^\infty$-manifold they are capable of producing only manifolds cobordant with the given one.

Standard Cobordisms

Definition 3.5.2.2. We define *standard cobordisms* of two kinds:

- the standard trivial cobordisms, and
- the standard elementary cobordisms of index k.

On every standard cobordism there is a standard Morse function.

A standard trivial cobordism is constructed by taking an arbitrary closed $\mathcal{C}^\infty$-manifold V and simply forming the cylinder $V \times I$, with $\partial_0(V \times I) = V \times 0$ and $\partial_1(V \times I) = V \times 1$. The corresponding standard Morse function is defined as $(v, t) \mapsto t$ and has no critical points.

The standard elementary cobordism of index k is defined for an arbitrary closed $(n-1)$-dimensional $\mathcal{C}^\infty$-manifold V with $n \geq k$, and an arbitrary $\mathcal{C}^\infty$-embedding $\varphi \colon \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to V$, and is denoted by $\operatorname{El}(V, \varphi)$. Up to a canonical homeomorphism, $\operatorname{El}(V, \varphi)$ is simply $(V \times I) \cup_{\varphi_1} (\mathbb{D}^k \times \mathbb{D}^{n-k})$, where the map $\varphi_1 \colon \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to V \times I$ is defined by $\varphi_1(x, y) = (\varphi(x, y), 1)$. To define $\operatorname{El}(V, \varphi)$

as a $\mathcal{C}^\infty$-manifold, we let $E(n,k)$ denote the set

$$\{(t_1,\dots,t_n)\in\mathbb{R}^n|\sum_{k+1}^{n} t_i^2-\le 1, |\sum_{i=1}^{k} t_i^2-\sum_{k+1}^{n} t_i^2|\le\frac{1}{16}\}$$

and let el denote the homeomorphism of the intersection

$$E(n,k)\cap[\mathbb{R}^n\setminus(\mathbb{R}^k\times\operatorname{int}(\frac{1}{2}\mathbb{D}^{n-k}))]$$

onto the cylinder

$$\varphi(\mathbb{S}^k\times(\mathbb{D}^{n-k}\setminus\operatorname{int}(\frac{1}{2}\mathbb{D}^{n-k})))\times I$$

given by the formula

$$(t_1,\dots,t_n)\mapsto(\varphi((t_1,\dots,t_k)/(t_1^2,\dots,t_k^2)^{1/2},(t_{k+1},\dots,t_n)),$$
$$8(\frac{1}{16}-t_1^2-\cdots t_k^2+t_{k+1}^2+\cdots+t_n^2)).$$

As a topological space, $\mathrm{El}(V,\varphi)$ is the result of glueing the cylinder $[V\setminus\varphi(\mathbb{S}^k\times\operatorname{int}(\frac{1}{2}\mathbb{D}^{n-k}))]\times I$ and $E(n,k)$ by the homeomorphism el (see Fig. 3.1).

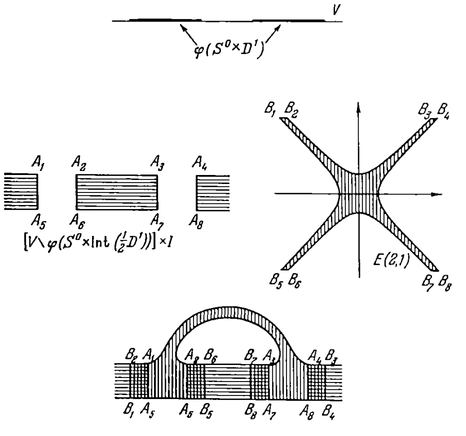

Figure 3.1: $(n=2,\ k=1)$

The $\mathcal{C}^\infty$-structure on $\mathrm{El}(V,\varphi)$ is fixed by an atlas consisting of the charts of arbitrary atlases of the $\mathcal{C}^\infty$-manifolds

$$[V\setminus\varphi(\mathbb{S}^{k-1}\times(\frac{1}{2}\mathbb{D}^{n-k}))]\times I \quad\text{and}\quad E(n,k)\cap(\mathbb{R}^k\times\operatorname{int}\mathbb{D}^{n-k})$$

(these manifolds are regarded as parts of the space $\mathrm{El}(V,\varphi)$). The beginning of $\partial_0 \mathrm{El}(V,\varphi)$ is formed by the two sets

$$[V \setminus \varphi(\mathbb{S}^{k-1} \times (\frac{1}{2}\mathbb{D}^{n-k}))] \times 0, \quad \text{and}$$
$$\{(t_1,\dots,t_n) \in E(n,k) | -t_1^2 - \dots - t_k^2 + t_{k+1}^2 \dots + t_n = -\frac{1}{16}\}.$$

Similarly, the sets

$$[V \setminus \varphi(\mathbb{S}^{k-1} \times (\frac{1}{2}\mathbb{D}^{n-k}))] \times 1, \quad \text{and}$$
$$\{(t_1,\dots,t_n) \in E(n,k) | -t_1^2 - \dots - t_k^2 + t_{k+1}^2 \dots + t_n = \frac{1}{16}\},$$

constitute the end $\partial_1 \mathrm{El}(V,\varphi)$. The standard Morse function on $\mathrm{El}(V,\varphi)$ is given by the two functions

$$[V \setminus \varphi(\mathbb{S}^{k-1} \times \mathrm{int}(\frac{1}{2}\mathbb{D}^{n-k}))] \times I \to \mathbb{R}, \quad E(n,k) \to \mathbb{R},$$

defined by the formulae

$$(v,t) \mapsto t, \quad (t_1,\dots,t_n) \mapsto 8(\frac{1}{16} - t_1^2 - \dots - t_k^2 + t_{k+1}^2 + \dots + t_n^2)$$

respectively. It has a unique critical point $0 \in E(n,k)$, of index k. We use the symbol mo to denote the standard Morse function.

Using Fig. 3.1 as a guide, it is readily seen that the manifold $\mathrm{El}(V,\varphi)$ is homeomorphic to $(V \times I) \cup_{\varphi_1} (\mathbb{D}^k \times \mathbb{D}^{n-k})$. The formulae describing this canonical homeomorphism are cumbersome, and we shall not burden the reader with their precise form.

The basic property shared by all standard cobordisms constructed from a given manifold V is that the beginning of each is canonically $\mathcal{C}^\infty$-diffeomorphic to V. For the standard trivial cobordism, this diffeomorphism is $v \mapsto (v,0)$. In the case of the cobordism $\mathrm{El}(V,\varphi)$, the diffeomorphism is defined as $v \mapsto (v,0)$ on the part

$$V \setminus \varphi(\mathbb{S}^{k-1} \times \mathrm{int}(\frac{1}{2}\mathbb{D}^{n-k})) \to [V \setminus \varphi(\mathbb{S}^{k-1} \times \mathrm{int}(\frac{1}{2}\mathbb{D}^{n-k}))] \times 0,$$

and as

$$\varphi(\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}) \to$$
$$\{(t_1,\dots t_n) \in E(n,k) | -t_1^2 - \dots - t_k^2 + t_{k+1}^2 + \dots + t_n^2 = \frac{1}{16}\},$$
$$t_k)(t_{k+1},\dots t_n)) \mapsto$$
$$(t_1\sqrt{\frac{1}{16} + t_{k+1}^2 + \dots + t_n^2}, \dots, t_1\sqrt{\frac{1}{16} + t_{k+1}^2 + \dots + t_n^2}, t_{k+1}, \dots, t_n)$$

Remark 3.5.2.3. The construction of the standard elementary cobordism the simplest when the index k is 0 or n.

If $k = 0$, then φ embeds the empty set into V, $E(n, k)$ is the ball $\frac{1}{4}\mathbb{D}^n$ and el is a homeomorphism of empty sets. In this case the similarity dilatation (with coefficient 4) of the ball $E(n, k)$ transforms $\mathrm{El}(V, \varphi)$ into the sum $(V \times I) \amalg \mathbb{D}^n$ with

$$\partial_0[(V \times I) \amalg \mathbb{D}^n] = \mathrm{incl}_1(V \times 0), \quad \partial_1[(V \times I) \amalg \mathbb{D}^n] = (V \times 1) \amalg \mathbb{S}^{n-1};$$

it also transforms the standard Morse function mo into the function

$$\begin{cases} V \times I \to \mathbb{R}, & (v, t) \mapsto t, \\ \mathbb{D}^n \to \mathbb{R}, & (t_1, \dots, t_n) \mapsto (1 + t_1^2 + \cdots + t_n^2)/2. \end{cases}$$

If $k = n$, then φ embeds $\mathbb{S}^{n-1}$ into V, the image of this embedding being one of the components of V (see Remark 3.1.5.1). Again, $E(n, k)$ is the ball $\frac{1}{4}\mathbb{D}^n$ and el is a homeomorphism of empty sets. Here the similarity dilatation of the ball $E(n, k)$ transforms $\mathrm{El}(V, \varphi)$ into the sum $([V \setminus \varphi(\mathbb{S}^{n-1})] \times I) \amalg \mathbb{D}^n$ with

$$\begin{aligned} \partial_0\{([V \setminus \varphi(\mathbb{S}^{n-1})] \times I) \amalg \mathbb{D}^n\} &= ([V \setminus \varphi(\mathbb{S}^{n-1})] \times 0) \amalg \mathbb{S}^{n-1}, \\ \partial_1\{([V \setminus \varphi(\mathbb{S}^{n-1})] \times I) \amalg \mathbb{D}^n\} &= \mathrm{incl}_1([V \setminus \varphi(\mathbb{S}^{n-1})] \times 1); \end{aligned}$$

while the standard Morse function mo is taken into the function

$$\begin{cases} [V \setminus \varphi(\mathbb{S}^{n-1})] \times I \to \mathbb{R}, & (v, t) \mapsto t, \\ \mathbb{D}^n \to \mathbb{R}, & (t_1, \dots, t_n) \mapsto (1 - t_1^2 + \cdots - t_n^2)/2. \end{cases}$$

We also remark that $k = 0$ and $k = n$ are the only values of the index for which the standard elementary cobordism can have a connected boundary: for $k = 0$ the boundary is connected if and only if the initial manifold V is empty, while for $k = n$ the boundary is connected if and only if φ is a diffeomorphism. We have seen that in both cases the cobordism is diffeomorphic to $\mathbb{D}^n$.

Trivial Cobordisms and Elementary Cobordisms

Definition 3.5.2.4. A cobordism is said to be *trivial* if it is diffeomorphic to a standard trivial cobordism. A cobordism is an *elementary cobordism of index* k if it is diffeomorphic to a standard elementary cobordism of index k.

From these definitions it follows that if X is a trivial cobordism, then the manifolds $\partial_0 X$ and $\partial_1 X$ are diffeomorphic, and that on a trivial cobordism there is a Morse function without critical points, whereas on an elementary cobordism of index k there is a Morse function with a single critical point of index k and no other critical points.

Lemma 3.5.2.5. *Suppose X and X' are two cobordisms such that the manifolds $\partial_1 X$ and $\partial_0 X'$ are diffeomorphic. If X' is trivial, the the cobordism obtained by glueing X and X' by an arbitrary $\mathcal{C}^\infty$-diffeomorophism $\partial_1 X \to \partial_0 X'$ (and using arbitrary collarings of X and X') is diffeomorphic to X.*

Proof. Corollary 3.4.5.9 shows that given a $\mathcal{C}^\infty$-diffeomorphism $\varphi\colon \partial_1 X \to \partial_0 X'$, it suffices to produce two $\mathcal{C}^\infty$-embeddings, $j\colon X \to X$ and $j'\colon X' \to X$, such that

$$j(X) \cup j'(X') = X, \quad j(X) \cap j'(X') = j(\partial_1 X) = j'(\partial_0 X'),$$

and the composite diffeomorphism

$$\partial_0 X' \xrightarrow{\operatorname{abr} j'} j'(\partial_0 X') \xrightarrow{\operatorname{abr} j^{-1}} \partial_1 X$$

coincides with φ^{-1}. In order to accomplish this, let us fix:

- a collaring $k\colon \partial X \times I \to X$;
- a $\mathcal{C}^\infty$-diffeomorphism $f\colon X' \to \partial_0 X' \times I$ such that $f(x') = (x', 0)$ for all $x' \in \partial X'$;
- an increasing $\mathcal{C}^\infty$-function $\alpha\colon I \to I$, such that

$$\alpha(t) = \begin{cases} 1/2 + t/3 & \text{for} \quad t \le 3/4, \\ t & \text{for} \quad t \ge 7/8. \end{cases}$$

Further, using the function

$$\beta\colon I \to I, \quad t \mapsto (1-t)/2,$$

set

$$\begin{cases} j(x) = x, & \text{if} \quad x \in X \setminus k(\partial_1 \times [0,1)), \\ j(k(z,t)) = k(z, \alpha(t)), & \text{if} \quad z \in \partial_1 X, \quad t \in I, \\ j'(x') = k \circ (\varphi^{-1} \times \beta) \circ f(x'), & \text{if} \quad x' \in X'. \end{cases}$$

We can verify directly that j and j' have the required properties. □

Lemma 3.5.2.6. *If on a cobordism there is a Morse function without critical point, then the cobordism is trivial,*

Proof. Let $f\colon X \to \mathbb{R}$ be a Morse function with no critical points. According to Corollary 3.1.5.8 (or, if one prefers, to Theorem 3.4.8.2), the preimage $f^{-1}(t)$ of any point $t \in (0,1)$ is a neat submanifold of X; moreover, $f^{-1}(t)$ is obviously closed as an independent manifold (the preimages $f^{-1}(0) = \partial_0 X$ and $f^{-1}(1) = \partial_1 X$ are also closed manifolds). By Theorems 3.4.5.3 and 3.4.5.8, the manifold $f^{-1}(t)$ has a neighbourhood U_t together with a $\mathcal{C}^\infty$-submersion $\pi_t\colon U_t \to f^{-1}(t)$ which is the identity map on $f^{-1}(t)$, for each fixed $t \in I$. Define, for $t \in I$, a $\mathcal{C}^\infty$-map

$$F_t\colon U_t \to f^{-1}(t) \times I, \quad x \mapsto (\pi_t(x), f(x)).$$

Obviously, the differential $d_t F_t$ is non-degenerate for $x \in f^{-1}(t)$, and F_t induces a diffeomorphism of $f^{-1}(t)$ onto $f^{-1}(t) \times t$. Consequently, F_t is a $\mathcal{C}^\infty$-embedding on a neighbourhood of $f^{-1}(t)$ (see 3.1.5.5), i.e., there is a neighbourhood Δ_t of the point t in I such that F_t induces a diffeomorphism of $F_t^{-1}(\Delta_t)$ onto

$f^{-1}(t) \times \Delta_t$. Let m be large enough so that if we divide I into m intervals of length $1/m$, then each such interval is contained in one of the sets Δ_t. Then all the cobordisms $f^{-1}([(i-1)/m, i/m])$, with

$$\partial_0 f^{-1}([(i-1)/m, i/m]) = f^{-1}((i-1)/m), \quad \partial_1 f^{-1}([(i-1)/m, i/m]) = f^{-1}(i/m),$$

are trivial, and now Lemma 3.5.2.5 shows that the entire cobordism X is trivial. □

Theorem 3.5.2.7. *If on a cobordism X there is a Morse function with a single critical point of index k and no other critical points, then X is an elementary cobordism of index k.*

Proof. The proof is quite long and we shall begin by constructing an auxiliary cobordism Y.

Fix a Morse function $f\colon X \to I$ with a single critical point of index k, say x, and no other critical points. By Theorem 3.5.1.4, there is a chart $\varphi \in \operatorname{Atl}_x X$, $\varphi(x) = 0$, such that $f|_{\operatorname{supp}\varphi}$ equals the composition of φ with the function $\operatorname{im}\varphi \to \mathbb{R}$ given by

$$(t_1, \dots, t_n) \mapsto f(x) - t_1^2 - \dots - t_k^2 + t_{k+1}^2 + \dots + t_n^2.$$

Obviously, $\operatorname{im}\varphi$ contains the subset A of $\mathbb{R}^n$ determined (in the standard coordinates) by the inequalities:

$$\begin{gathered} t_{k+1}^2 + \dots + t_n^2 \le 4\varepsilon^2, \\ -\frac{\varepsilon^2}{16} \le -t_1^2 - \dots - t_k^2 + t_{k+1}^2 + \dots + t_n^2 \le \frac{\varepsilon^2}{16}, \end{gathered}$$

for some $\varepsilon > 0$. Let

$$B = A \cap (\mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \operatorname{int}(\frac{\varepsilon}{2}\mathbb{D}^{n-k}))).$$

As a topological space, Y is the result of glueing the subset

$$f^{-1}([f(x) - \frac{\varepsilon}{16}, f(x) + \frac{\varepsilon}{16}]) \setminus \varphi^{-1}(A \setminus B)$$

of X and the product $\mathbb{S}^{k-1} \times 2\mathbb{D}^{n-k} \times I$ by the homeomorphism

$$\varphi^{-1}(B) \to \mathbb{S}^{k-1} \times [2\mathbb{D}^{n-k} \setminus \operatorname{int}(\frac{\varepsilon}{2}\mathbb{D}^{n-k})] \times I$$

given by

$$\varphi^{-1}(t_1, \dots, t_n) \mapsto \left(\frac{(t_1, \dots, t_k)}{\sqrt{(t_1^2 + \dots + t_k^2)}}, (t_{k+1}, \dots, t_n), \frac{8}{\varepsilon^2}(\frac{\varepsilon^2}{16} - t_1^2 - \dots - t_k^2 + t_{k+1}^2 + \dots + t_n^2) \right).$$

[Here we use the inequalities $-\frac{\varepsilon^2}{16} < f(x) < 1 - \frac{\varepsilon^2}{16}$, which are immediate consequences of the inclusion $A \subset \operatorname{im} \varphi$.]

The $\mathcal{C}^\infty$-structure of Y is fixed by the atlas consisting of arbitrary atlases of the $\mathcal{C}^\infty$-manifolds

$$f^{-1}([f(x) - \frac{\varepsilon^2}{16}, f(x) + \frac{\varepsilon^2}{16}]) \setminus \varphi^{-1}(A \setminus B)$$

and

$$\mathbb{S}^{k-1} \times \operatorname{int}(2\mathbb{D}^{n-k}) \times I$$

The beginning $\partial?0Y$ is the union of the two sets

$$f^{-1}(f(x) - \frac{\varepsilon^2}{16}) \setminus \varphi^{-1}(A \setminus B) f^{-1} \cap (f(x) - \frac{\varepsilon^2}{16})$$

and

$$\mathbb{S}^{k-1} \times \operatorname{int}(2\mathbb{D}^{n-k}) \times 0.$$

Similarly, the end $\partial_1 Y$ is the union of the two sets

$$f^{-1}(f(x) - \frac{\varepsilon^2}{16}) \setminus \varphi^{-1}(A \setminus B) f^{-1} \cap (f(x) - \frac{\varepsilon^2}{16})$$

and

$$\mathbb{S}^{k-1} \times \operatorname{int}(2\mathbb{D}^{n-k}) \times 1.$$

The functions

$$\begin{aligned} &f^{-1}([f(x) - \frac{\varepsilon^2}{16}, f(x) + \frac{\varepsilon^2}{16}]) \setminus \varphi^{-1}(A \setminus B) \to \mathbb{R} \\ &y \mapsto \frac{8}{\varepsilon^2}(\frac{\varepsilon^2}{16} + f(y) - f(x)), \end{aligned}$$

and

$$\begin{aligned} &\mathbb{S}^{k-1} \times \operatorname{int}(2\mathbb{D}^{n-k}) \times I \to \mathbb{R}, \\ &(u, v, t) \mapsto [u \in \mathbb{S}^{k-1}, v \in \operatorname{int}(2\mathbb{D}^{n-k}), t \in I], \end{aligned}$$

yield together a function $g \colon Y \to \mathbb{R}$. It is readily seen that g is a Morse function with no critical points. (In virtue of Lemma 3.5.2.6, this implies that Y is a trivial cobordism; however, in what follows this property of Y is not used directly.) Now consider the composite embedding

$$\mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \xrightarrow{(z_1,z_2) \mapsto (z_1,z_2,1/2)} \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \times I \xrightarrow{\text{incl}} \mathbb{S}^{k-1} \times \operatorname{int}(2\mathbb{D}^{n-k}) \times I \xrightarrow{\text{incl}} Y$$

and its compression $\psi \colon \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to g^{-1}(1/2)$. To complete the proof of the theorem we shall presently verify that the cobordism X is diffeomorphic to $\operatorname{El}(g^{-1}(1/2), \psi)$.

As a preliminary step, we find a number δ, $0 < \delta < 1/2$, and a $\mathcal{C}^\infty$-diffeomorphism

$$H\colon g^{-1}(\frac{1}{2}) \times [\frac{1}{2} - \delta, \frac{1}{2} + \delta] \to g^{-1}([\frac{1}{2} - \delta, \frac{1}{2} + \delta]),$$

such that

$$H((u, v, \frac{1}{2}), t) = (u, v, t)$$

for all $u \in \mathbb{S}^{k-1}$, $v \in \mathbb{D}^{n-k}$, and $t \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$.

By Theorem 3.4.5.8, one can find η, $0 < \eta < 1/2$, such that there is a $\mathcal{C}^\infty$-submersion $\pi\colon g^{-1}((\frac{1}{2} - \eta, \frac{1}{2} + \eta)) \to g^{-1}(\frac{1}{2})$ which is the identity map on $g^{-1}(\frac{1}{2})$. Fix a $\mathcal{C}^\infty$-function $\alpha\colon \mathbb{R}\, n-k \to \mathbb{R}$, equal to 1 on $\mathbb{D}^{n-k}$ and to 0 outside $2\mathbb{D}^{n-k}$, and define a new $\mathcal{C}^\infty$-submersion $\rho\colon g^{-1}((\frac{1}{2} - \eta, \frac{1}{2} + \eta)) \to g^{-1}(\frac{1}{2})$ by the formula

$$\rho(y) = \begin{cases} \pi(y), & \text{if } \ y \notin \mathbb{S}^{k-1} \times \operatorname{int}(2\mathbb{D}^{n-k}) \times I, \\ \pi(u, v, -1 + (t-1)(1 - \alpha(v))), & \text{if } \ y = (u, v, t), \\ & \text{where } \ u \in \mathbb{S}^{k-1}, v \in 2\mathbb{D}^{n-k}, t \in I. \end{cases}$$

Further, define

$$G\colon g^{-1}((\frac{1}{2} - \eta, \frac{1}{2} + \eta)) \to g^{-1}(\frac{1}{2}) \times (\frac{1}{2} - \eta, \frac{1}{2} + \eta)$$

by $G(y) = (\rho(y), g(y))$. Since G induces a diffeomorphism of $g^{-1}(\frac{1}{2})$ onto $g^{-1}(\frac{1}{2}) \times \frac{1}{2}$ and the differential d_yG is non-degenerate as long as $y \in g^{-1}(\frac{1}{2})$, there is δ, $0 < \delta < \eta$, such that the restriction of G to $g^{-1}([\frac{1}{2} - \delta, \frac{1}{2} + \delta])$ is a $\mathcal{C}^\infty$-embedding. Now it is clear that one can take for H the map

$$(\operatorname{abr} G)^{-1}\colon g^{-1}([\frac{1}{2}) \times [\frac{1}{2} - \delta, \frac{1}{2} + \delta] \to g^{-1}([\frac{1}{2} - \delta, \frac{1}{2} + \delta]).$$

Finally, to show that the cobordisms X and $\operatorname{El}(g^{-1}\frac{1}{2}, \psi)$ are diffeomorphic, cut each of them into three cobordisms:

- X into the cobordisms

  $f^{-1}([0, f(x) - \frac{\delta\varepsilon^2}{8}])$,
  $f^{-1}([f(x) - \frac{\delta\varepsilon^2}{8}, f(x) + \frac{\delta\varepsilon^2}{8}])$,
  $f^{-1}([f(x) + \frac{\delta\varepsilon^2}{8}, 1])$,

- $\operatorname{El}(g^{-1}(\frac{1}{2}), \psi)$ into the conformisms

  $\operatorname{mo}^{-1}([0, \frac{1}{2} - \delta])$,
  $\operatorname{mo}^{-1}([\frac{1}{2} - \delta, \frac{1}{2} + \delta])$,
  $\operatorname{mo}^{-1}([\frac{1}{2} + \delta, 1])$.

By Lemma 3.5.2.6, the first and the third cobordisms in each of these triples are trivial, and hence the second cobordism is diffeomorphic to the entire cobordism. Therefore, it is enough to exhibit a diffeomorphism

$$\operatorname{mo}^{-1}([\frac{1}{2}-\delta, \frac{1}{2}8+\delta] \to f^{-1}([f(x)-\frac{\delta\varepsilon^2}{8}, f(x)+\frac{\delta\varepsilon^2}{8}]).$$

To do this, consider the composite map

$$\begin{aligned}&(g^{-1}(\frac{1}{2}) \setminus \psi(\mathbb{S}^{k-1} \times \operatorname{int}\mathbb{D}^{n-k})) \times [\frac{1}{2}-\delta, \frac{1}{2}+\delta] \xrightarrow{\operatorname{abr} H}\\ &(Y \setminus (\mathbb{S}^{k-1} \times \operatorname{int}\mathbb{D}^{n-k}) \times I)) \cap g^{-1}([\frac{1}{2}-\delta, \frac{1}{2}+\delta]) \xrightarrow{\operatorname{id}}\\ &(X \setminus \varphi^{-1}(\mathbb{R}^k \times \operatorname{int}\mathbb{D}^{n-k})) \cap f^{-1}([f(x)-\frac{\delta\varepsilon^2}{8}, f(x)+\frac{\delta\varepsilon^2}{8}]),\end{aligned}$$

and the map

$$\begin{aligned}&E(n,k) \cap \operatorname{mo}^{-1}([\frac{1}{2}-\delta, \frac{1}{2}+\delta]) \to\\ &\varphi^{-1}(\operatorname{im}\varphi \cap (\mathbb{R}^k \times \varepsilon\mathbb{D}^{n-k})) \cap f^{-1}([f(x)-\frac{\delta\varepsilon^2}{8}, f(x)+\frac{\delta\varepsilon^2}{8}])\end{aligned}$$

given by the formula

$$(t_1, \dots, t_n) \mapsto \varphi^{-1}(\varepsilon t_1, \dots, \varepsilon t_{n)},$$

which together provide the desired diffeomorphism. □

Corollary 3.5.2.8. *Suppose that on a given n-dimensional cobordism X with connected boundary there exists a Morse function with a unique critical point. Then X is diffeomorphic to $\mathbb{D}^n$.*

Attaching Handless

Definition 3.5.2.9. Let X be an n-dimensional cobordism and let $\varphi\colon \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to \partial_1 X$ be a $\mathcal{C}^\infty$-embedding. The result of glueing X and the elementary cobordism $\operatorname{El}(\partial_1 X, \varphi)$ by the canonical diffeomorphism $\partial_1 X \to \partial_0 \operatorname{El}(\partial_1 X, \varphi)$ (see Definition 3.5.2.2) is said to be obtained from X by *attaching a handle of index k*.

By attaching a handle of index 0 to X we replace, up to a diffeomorphism, X by $X \amalg \mathbb{D}^n$; the new component of the boundary, i.e., $\operatorname{incl}_2(\mathbb{S}^{n-1})$, is added to $\partial_1 X$. To attach a handle of index n, we actually glue X and $\mathbb{D}^n$ by a diffeomorphism of one of the components of $\partial_1 X$ onto $\mathbb{S}^{n-1}$.

Theorem 3.5.2.10. *Every cobordism X can be obtained, up to a diffeomorphism, from the standard trivial cobordism $\partial_0 X \times I$, by attaching a finite number of handles. Moreover, given any proper Morse function $f\colon X \to \mathbb{R}$, one may choose these handles so that their number will not exceed the number of critical points of f.*

Proof. We prove this statement by induction on the number of critical points of f. If f has no critical points, Lemma 3.5.2.6 suffices. If f has $m \geq 1$ critical points, then there is $c \in (0,1)$ such that one of the critical values of f is greater than c, while the remaining ones are smaller than c. We cut X into two cobordisms: $f^{-1}([0,c])$ and $f^{-1}([c,1])$. On $f^{-1}([0,c])$ there is a proper Morse function with $m-1$ critical points, for example, $x \mapsto f(x)/c$. On $f^{-1}([c,1]))$ there is a Morse function with a unique critical point, for example, $x \mapsto (f(x)-c)/(1-c)$. Finally, note that by Theorem 3.5.2.7 the second cobordism is elementary. ☐

Theorem 3.5.2.11. *A closed n-dimensional $\mathcal{C}^\infty$-manifold X on which there is Morse function having only two critical points is homeomorphic to $\mathbb{S}^n$.*

Proof. We remark (leaving the trivial case $n=0$ aside) that every Morse function with only two critical points is proper (the two points are necessarily a maximum and a minimum). Thus Theorem 3.5.2.10 shows that X can be obtained from an empty manifold by attaching two handles. Obviously, the first handle has index 0, and the second index n, and hence X actually results from glueing two copies of $\mathbb{D}^n$ by a diffeomorphism of $\mathbb{S}^{n-1}$. ☐

A Homotopy Corollary

Lemma 3.5.2.12. *The cobordism* $\operatorname{El}(V,\varphi)$ *is homotopy equivalent to* $V \cup_f \mathbb{D}^k$*, where* $f\colon \mathbb{S}^{k-1} \to V$ *is given by* $f(y) = \varphi(y,0)$*. Moreover, there is a homotopy equivalence* $V \cup_f \mathbb{D}^k \to \operatorname{El}(V,\varphi)$ *which agrees on* V *with the inclusion* $V[= \partial_0 \operatorname{El}(V,\varphi)] \to \operatorname{El}(V,\varphi)$.

Proof. One can assemble such a homotopy equivalence from the above inclusion $V \to \operatorname{El}(V,\varphi)$ and the embedding $\mathbb{D}^k \to \operatorname{El}(V,\varphi)$ which takes each point $x \in \mathbb{D}^k$ into the point $x/4 \in E(n,k)$. To complete the proof, it is enough to remark that the constructed mapping $V \cup_f \mathbb{D}^k \to \operatorname{El}(V,\varphi)$ is a topological embedding whose image is a strong deformation retract of $\operatorname{El}(V,\varphi)$. ☐

Theorem 3.5.2.13. *Every compact n-dimensional smooth manifold is homotopy equivalent to a finite cellular space of dimension $\leq n$.*

Proof. The discussion in Definition 3.5.1.2 implies that one may assume that the given manifold is a cobordism with an empty beginning. Therefore, all we have to show is that if an n-dimensional cobordism is homotopy equivalent to a finite cellular space of dimension $\leq n$, then it retains this property after we attach to it an arbitrary handle; see Theorem 3.5.2.10. But from Lemma 3.5.2.12 it follows that attaching a handle of index k to a cobordism X has the same homotopy effect as attaching $\mathbb{D}^k$ to X by some embedding $f\colon \mathbb{S}^{k-1} \to X$. Now replace X by a finite cellular space Y of dimension $\leq n$ with the same homotopy type, replace the map f by its composition with a homotopy equivalence $X \to Y$, and subsequently replace this composition by a homotopic cellular map $g\colon \mathbb{S}^{k-1} \to Y$ (see Theorem 2.3.2.6). By Theorem 1.3.7.8, the cobordism which results by attaching a handle of index k to X is homotopy equivalent to the space $Y \cup_g \mathbb{D}^k$;

according to Remark 2.1.5.6, $Y \cup_g \mathbb{D}^k$ is a finite cellular space of dimension $\leq n$. ☐

Definition 3.5.2.14. Let V be a closed n-dimensional $\mathcal{C}^\infty$-manifold, and let $\varphi\colon \mathbb{S}^{k-1} \times \mathbb{D}^{n-k+1} \to V$ be a $\mathcal{C}^\infty$-embedding. Fix arbitrary collars on

$$Y \setminus \varphi(\mathbb{S}^{k-1} \times \operatorname{int} \mathbb{D}^{n-k+1}) \quad \text{and} \quad \mathbb{D}^k \times \mathbb{S}^{n-k},$$

and then glue these manifolds by the diffeomorphism

$$\operatorname{abr} \varphi\colon \mathbb{S}^{k-1} \times \mathbb{S}^{n-k} \to \varphi(\mathbb{S}^{k-1} \times \mathbb{S}^{n-k})$$

of the boundary of the second onto the boundary of the first. We say that the glued manifold is obtained from V by a *spherical modification along the embedding* φ[1]. The number k is the *index of the modification.*

Theorem 3.5.2.15. *If the cobordism X' is obtained from the cobordism X by attaching a handle using an embedding $\varphi\colon \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to \partial_1 X$, then $\partial_1 X'$ is obtained from by a spherical modification along the same embedding $\varphi\colon \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to \partial_1 X$.*

Proof. Since $\partial_1 X^{prime} = \partial_1 \operatorname{El}(\partial_1 X, \varphi)$, we actually claim that $\partial_1 \operatorname{El}(\partial_1 X, \varphi)$ is obtained from $\partial_1 X$ by a spherical modification along φ. Recall that $\operatorname{El}(\partial_1 X, \varphi)$ is the result of glueing the spaces $[\partial_1 X \setminus \varphi(\mathbb{S}^{k-1} \times \operatorname{int}(\frac{1}{2}\mathbb{D}^{n-k}))] \times I$ and $E(n,k)$ by el (see Definition 3.5.2.2). Obviously, $[\partial_1 X \setminus \varphi(\mathbb{S}^{k-1} \times \operatorname{int} \mathbb{D}^{n-k})] \times 1$ and $E(n,k) \cap \partial_1 \operatorname{El}(\partial_1 X, \varphi)$ are compact $(n-1)$-dimensional submanifolds of the manifold $\partial_1 \operatorname{El}(\partial_1 X, \varphi)$, which they cover, and they intersect along their common boundary. Consider the mappings

$$\operatorname{proj}_1\colon [\partial_1 X \setminus \varphi(\mathbb{S}^{k-1} \times \operatorname{int} \mathbb{D}^{n-k})] \times 1 \to \partial_1 X \ \ \varphi(\mathbb{S}^{k-1} \times \operatorname{int} \mathbb{D}^{n-k})$$

and

$$\psi\colon E(n,k) \cap \partial_1 \operatorname{El}(\partial_1 X, \varphi) \to \mathbb{D}^k \times \mathbb{S}^{n-k-1},$$
$$\psi(t_1, \dots, t_n) = \left(\frac{16}{15}(t_1, \dots, t_n), \frac{(t_{k+1}, \dots, t_n)}{\sqrt{(t_{k+1}^2, + \dots + t_n^2)}} \right).$$

It is clear that ψ is a $\mathcal{C}^\infty$-diffeomorphism such that $\operatorname{proj}(\psi^{-1}(z_1, z_2)) = \varphi(z_1, z2)$ for all $z_1 \in \mathbb{S}^{k-1}$ and $z_2 \in \mathbb{S}^{n-k-1}$. Now applying Corollary 3.4.5.9, we see that $\partial_1 \operatorname{El}(\partial_1 X, \varphi)$ is really obtained by glueing the manifolds

$$\partial_1 X \setminus \varphi(\mathbb{S}^{k-1} \times \operatorname{int} \mathbb{D}^{n-k}) \quad \text{and} \quad \mathbb{D}^k \times \mathbb{S}^{n-k-1}$$

by $\operatorname{abr} \varphi$. ☐

Corollary 3.5.2.16. *Given an arbitrary cobordism X, the manifold $\partial_1 X$ can be obtained from $\partial_0 X$ through a finite number of spherical modifications. In particular, the boundary of an arbitrary compact $\mathcal{C}^\infty$-manifold can be obtained from an empty manifold through a finite number of spherical modifications.*

---

[1] Translator's note: or by *doing a surgery on V using φ.*

3.5.3 Two-dimensional Manifolds

Remark 3.5.3.1. The theory presented in the two previous subsections represents, first of all, an attempt to somehow visualise, survey, and classify smooth compact manifolds of a given dimension. To apply this theory in complex situations, one needs to develop it further, but in the simplest cases Theorem 3.5.2.10 emerges as a sufficiently effective tool. For example, in the one-dimensional case every standard cobordism is a sum of segments, and hence Theorem 3.5.2.100 implies that every smooth compact one-dimensional manifold is diffeomorphic to the sum of a finite number of segments and circles.
Warning: although this differentiable classification coincides with the topological classification given in Theorem 3.1.1.18, it has an entirely different meaning. [We add that also in the non-compact case (which, due to its relative complexity, is not considered in our book in any systematic way), the differentiable classification of one-dimensional manifolds is identical with the topological one; see Exercise 3.5.4.1.]

The present subsection is devoted to the differentiable classification of the smooth, compact, two-dimensional manifolds, and again Theorem 3.5.2.10 is enough. We begin by drawing up a list of model manifolds (which play the same role as the segments and circles in the one-dimensional case). Next we show that every smooth, compact, connected, two dimensional manifold is diffeomorphic to one of the models. No two of the model manifolds are diffeomorphic or homeomorphic. However, we prove this fact only in Chapter 5 (see Remark **??**); here we merely prepare the geometric part of the proof, describing for the closed model manifolds canonical rigged cellular decomposition, and for the non-closed model manifolds - the bouquets of circles to which they are homotopy equivalent. As we did in the classification of one-dimensional manifolds, we shall not worry about the differentiability class (cf. Remark 3.5.1.1).

Model Surfaces

Remark 3.5.3.2. We begin with the elementary model surfaces: the *spheres with holes* and the *Möbius strips*.

A sphere with ℓ hole is $\mathbb{S}^2$ with the interiors of ℓ pairwise disjoint spherical caps (segments) removed; its boundary is a sum of ℓ circles and inherits a well-defined orientation. A sphere with one hole is diffeomorphic to $\mathbb{D}^2$, while a sphere with two holes is diffeomorphic to the cylinder $\mathbb{S}^1 \times \mathbb{D}^1$; we call the latter a handle (do not confuse with the "handles" in Subsection 3.5.2 !).

A Möbius strip is a submanifold of $\mathbb{R}^3$ produced by the motion of a segment of length 1 whose middle glides along a circle $\mathbb{S}^1$ in such a manner that the segment remains normal to the circle and turns uniformly through a total angle π (see Fig. 3.2). Every Möbiu strip is a non-orientable compact submanifold of $\mathbb{R}^3$ with boundary diffeomorphic to $\mathbb{S}^1$.

The list of all *model surfaces* is:

- the empty two-dimensional manifold,

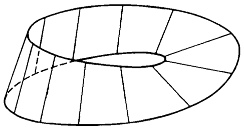

Figure 3.2:

- the *spheres with handles and holes*,
- the *spheres with cross-caps*.

A sphere with g handles and ℓ holes ($g \geq 0$, $\ell \geq 0$) is defined as the result of glueing a sphere with $2g + \ell$ holes and a sum of g handles by an orientation preserving diffeomorphism from the boundary of the sum of handles onto the union of $2g$ components of the sphere with holes. This object is a smooth, compact, orientable, two dimensional manifold, uniquely defined up to an orientation preserving diffeomorphism by the numbers g and ℓ, and whose boundary is diffeomorphic to a sum of ℓ circles. This manifold coincides with $\mathbb{S}^2$ if $g = 0$, $\ell = 0$; it is diffeomorphic to $\mathbb{D}^2$ if $g = 0$, $\ell = 1$, and to $\mathbb{S}^1 \times \mathbb{S}^1$ if $g = 1$, $\ell = 0$, and then it is known as a *torus*; for $g = 2$ and $\ell = 0$, it is called a *double torus* or a *pretzel*. For any g and ℓ, a sphere with g handles and ℓ holes can be differentiably embedded in $\mathbb{R}^3$; for $g = 3$ and $\ell = 2$, the standard embedding is depicted in Fig.3.3.

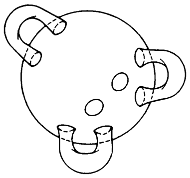

Figure 3.3:

A sphere with h cross-caps and ℓ holes ($h \geq 0$, $\ell \geq 0$) is defined as the result of glueing a sphere with $h + \ell$ holes and a sum of h Möbius strips by a diffeomorphism of the boundary of the sum of strips onto the union of h components of the sphere with holes. One obtains a smooth, compact, two-dimensional manifold, uniquely defined up to a diffeomorphism by the numbers h and ℓ, and whose boundary is diffeomorphic to a sum of ℓ circles. This manifold is orientable only for $h = 0$. For $h = 1$, $\ell = 0$, it is diffeomorphic to $\mathbb{R}P^2$, and for $h = 1$, $\ell = 1$, to a Möbius strip.

For $h = 2$, $\ell = 0$, this manifold is called the *Klein bottle* and is well known because of its immersion in $\mathbb{R}^3$, depicted in Fig. 3.4; for $h = 2$, $\ell = 1$, it is called a *disc with an inverted handle* and is depicted in Fig. 3.5, on the left

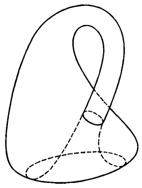

Figure 3.4:

(the drawing on the right in Fig. 3.5 demonstrates that by glueing a disc with an inverted handle and a usual disc by a diffeomorphism of their boundaries we actually get a Klein bottle).

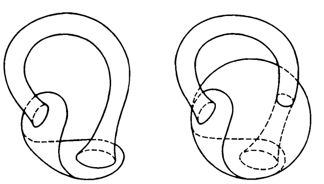

Figure 3.5:

We emphasise that in our list of model manifolds, the cross-caps do not meet with the handles, i.e., we excluded the case of a sphere with holes to which we attach both handles and Möbius strips. We show in Lemma 3.5.3.3 that for $h \geq 1$, every sphere with g handles, h cross-caps, and ℓ holes is diffeomorphic to a sphere with $2g + h$ cross-caps and ℓ holes.

Auxiliary Propositions

Lemma 3.5.3.3. *Let X_1 and X_2 be smooth, compact, two-dimensional manifolds, and let φ be a diffeomorphism of a component of $\partial_1 X$ onto a component of $\partial_2 X$. Denote by X the manifold obtained by glueing X_1 and X_2 by φ. Then :*

(i) if X_1 and X_2 are diffeomorphic to a sphere with g_1 handles and ℓ_1 holes, and to a sphere with g_2 handles and ℓ_2 holes, respectively, then X is diffeomorphic to a sphere with $g_1 + g_2$ handles and $\ell_1 + \ell_2$ holes;

(ii) if X_1 and X_2 are diffeomorphic to a sphere with h_1 cross-caps and ℓ_1 holes, and to a sphere with h_2 cross-caps and ℓ_2 holes, respectively, then X is diffeomorphic to a sphere with $h_1 + h_2$ cross-caps and $\ell_1 + \ell_2$ holes;

(iii) *if X_1 and X_2 are diffeomorphic to a sphere with g_1 handles and ℓ_1 holes, and to a sphere with $h_2 > 0$ cross-caps and ℓ_2 holes, respectively, then X is diffeomorphic to a sphere with $2g_1 + h_2$ cross-caps and $\ell_1 + \ell_2 - 2$ holes.*

Proof. The only case which needs proof is (iii), where we see immediately that X is diffeomorphic to a sphere with g_1 handles, h_2 cross-caps, and $\ell_1 + \ell_2 - 2$ holes.

To begin with, suppose that $g_1 = 1$, $h_2 = 1$, and $\ell_1 + \ell_2 - 2 = 1$. Then X is diffeomorphic to a Möbius strip with a handle attached; see the left drawing in Fig. 3.6.

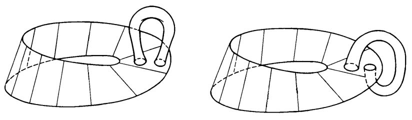

Figure 3.6:

The last manifold is obviously diffeomorphic to a Möbius strip with an inverted handle, as that in the right drawing in Fig. 3.6, and hence can be cut into a sphere with one cross-cap and a disc with an inverted handle. According to (ii), this implies that X is diffeomorphic to a sphere with three cross-caps and one hole.

In the general case, we use induction on g_1. If $g_1 = 0$, there is nothing to prove. If $g_1 > 0$, then X can be glued from a sphere with $g_1 - 1$ handles, cross-caps, and $\ell_1 + \ell_2 - 1$ holes and a sphere with one handle, one cross-cap, and one hole. But we already proved that this second manifold is diffeomorphic to a sphere with three cross-caps and one hole. Consequently, X is diffeomorphic to a sphere with $g_1 - 1$ handles, $h_2 + 2$ cross-caps, and $\ell_1 + \ell_2 - 2$ holes, and so diffeomorphic to a sphere with $2g_1 + h_2$ cross-caps and $\ell_1 + \ell_2 - 2$ holes. ☐

Lemma 3.5.3.4. *Let X be a smooth, compact, connected, two-dimensional manifold, and let X' be the result of attaching a handle to X by a diffeomorphism of the boundary of the handle onto the union of two components of ∂X. Then:*

(i) *if X is diffeomorphic to a sphere with g handles and ℓ holes, then X' is diffeomorphic to a sphere with $g + 1$ handles and $\ell - 2$ holes, or to a sphere with $2g + 2$ cross-caps and $\ell - 2$ holes;*

(ii) *if X is diffeomorphic to a sphere with h cross-caps and ℓ holes, then X' is diffeomorphic to a sphere with $h + 2$ cross-caps and $\ell - 2$ holes.*

Proof. This is a corollary of Lemma 3.5.3.3: indeed, in both cases one can cut X' into two manifolds, such that the first one differs from X by having one hole less, while the second is diffeomorphic to a sphere with one handle and one hole, or to a disc with an inverted handle. □

Theorem 3.5.3.5. *Let X_1 and X_2 be smooth, compact, two-dimensional manifolds, and let φ be a diffeomorphism of a non-empty union of whole components of $\partial_2 X_2$ onto a non-empty union of whole components of $\partial_1 X$. Denote by X the manifold glued from X_1 and X_2 by means of φ. If both X_1 and X_2 are diffeomorphic to model surfaces, then X is also diffeomorphic to one of the model surfaces.*

Proof. This is a consequence of Lemmas 3.5.3.3 and 3.5.3.4, because glueing by means of φ is equivalent to first glueing by the diffeomorphism of one of the components of $\partial_2 X_2$ onto the corresponding component of $\partial_1 X$, obtained by compressing φ, and subsequently attaching a number of handles equal to half the number of the components of $\partial_1 X$ and $\partial_2 X$ which remain to be identified. □

The Main Theorem

Theorem 3.5.3.6. *Every smooth, connected, compact, two-dimensional manifold is diffeomorphic to one of the model surfaces.*

Proof. Applying Theorems 2.10 and 5, all we need to show is that the components of the elementary two-dimensional cobordisms are diffeomorphic to model surfaces. And this is not hard to check directly by examining all possible cases, if we recall that every smooth, closed, one-dimensional manifold is diffeomorphic to a sum of circles. To spell it out,every elementary cobordism of index 0 constructed from a sum of m circles is diffeomorphic to a sum of m spheres with two holes and a sphere with one hole. Next, every elementary cobordism of index 2 constructed from a sum of m circles (and a differentiable embedding of a circle in this sum) is diffeomorphic to a sum of $m-1$ spheres with two holes and a sphere with one hole. And finally, every elementary cobordism of index 1 constructed from a sum of m circles and a differentiable embedding of $\mathbb{S}^0 \times \mathbb{D}^1$ in this sum is diffeomorphic to one of the following three manifolds:

- a sum of $m-2$ spheres with two holes and a sphere with three holes;
- a sum of $m-1$ spheres with two holes and a sphere with three holes;
- a sum of $m-1$ spheres with two holes and a sphere with one cross-cap and two holes.

For $m = 2$, one see the three cases in Fig. 3.7.

□

Information 3.5.3.7. Every compact, connected, two-dimensional topological manifold is homeomorphic to one of the model surfaces.

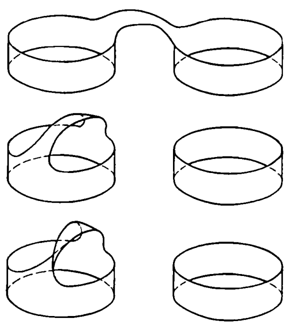

Figure 3.7:

Cellular Decompositions of the Closed Model Surfaces

Remark 3.5.3.8. The closed model surfaces possess standard rigged cellular decompositions, which generalize the canonical decompositions of the sphere $\mathbb{S}^2$, the complex projective space $\mathbb{C}P^2$, and the torus $\mathbb{S}^1 \times \mathbb{S}^1$ into two cells, three cells, and four cells, respectively. Each of these standard decompositions, except the no-cell decomposition of the empty model surface, contains only one 0-cell and only one 2-cell, while the number of 1-cells is $2g$ for a sphere with g handles, and h for a sphere with h cross-caps. Therefore, the 1-skeleton of a sphere with g handles is a bouquet of $2g$ circles, while the 1-skeleton of a sphere with h cross-caps is a bouquet of h circles. Moreover, the description of the entire rigged cellular decomposition reduces to the characterisation of the attaching map for the 2-cell, i.e., of a certain map of $\mathbb{S}^1$ into the aforementioned bouquet.

We disregard the values $g = 0, 1$ and $h = 0, 1$, already considered, and for the case of a sphere with g handles, we represent $\mathbb{S}^1$ as the contour of a regular polygon with first vertex ort_1 and $4g$ edges, arranged successively as

$$a_1, b_1, a'_1, b'_1, \dots, a_g, b_g, a'_g, b'_g$$

In the case of a sphere with h cross-caps, we represent $\mathbb{S}^1$ as the contour of a regular polygon with first vertex ort_1 and $2h$ edges, which are arranged successively as

$$c_1, c'_1, \dots, c_h, c'_h.$$

In both cases we form a quotient space of $\mathbb{S}^1$ by identifying each edge with the corresponding "primed" edge, as follows: a_1 is identified with a'_1, and b_1 with b'_1 through a reflection with respect to a line (relative to which these edges are symmetric), while c_1 is identified with c'_1 through a rotation of the polygon (around its centre). In either of cases the quotient space is a bouquet of circles:

in the first case the number of circles is $2g$, and in the second h. The projection of $\mathbb{S}^1$ onto this quotient space is the required attaching map.

Of course, we still have to convince the reader that the cellular spaces produced in this manner are homeomorphic to the model surfaces. To this end, let us divide our $4g$-gon into a g-gon and g pentagons, by drawing diagonals which cut out quadruplets a_i, b_i, a'_i, b'_i. Similarly, we divide our $2h$-gon into a h-gon and h triangles, by drawing diagonals which cut out pairs c_i, c'_i (see Fig. 3.8).

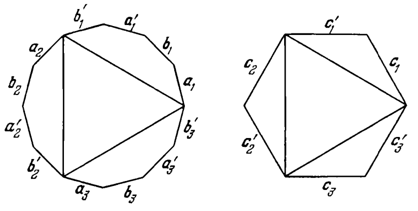

Figure 3.8: $g = 3$, $h = 3$

Identifying the edges and the way described above, all the vertices of the remaining g-gon become one and the same point, thus transforming the g-gon into a sphere with g circular apertures; at the same time, the edges of the pentagons are identified in such a manner that each pentagon becomes a torus with a circular aperture. Attaching these holed tori to the holed sphere in such a way as to restore all that was destroyed by the auxiliary (diagonal) cuts, we obtain, up to a homeomorphism, a sphere with g handles. Similarly, the prescribed identifications of the edges c_i, c'_i take all the vertices of the h-gon into one and the same point, thus transforming the h-gon into a sphere with h circular apertures; at the same time, each triangle becomes a Möbius strip, since two of its edges are identified. Attaching these strips to the holed sphere, we obtain, up to a homeomorphism,a sphere with h cross-caps. [**Warning**: the boundaries of the previous circular apertures (in the sphere) have a common point, and for $g = 2$ or $h = 2$, they even coincide.]

The Homotopy Structure of the Non-closed Model Surfaces

Theorem 3.5.3.9. *A sphere with g handles and ℓ holes is homotopy equivalent to a bouquet of $2g + \ell - 1$ circles. A sphere with h cross-caps and ℓ holes is homotopy equivalent to a bouquet of $h + \ell - 1$ circles.*

Proof. To prove these assertions, we first note that by attaching the 4g-gon to the bouquet of $2g$ circles as in Remark 3.5.3.8, we produce, up to a homeomorphism, a sphere with g handles and ℓ holes, provided we first remove the interiors of ℓ pairwise disjoint discs from the interior of the $4g$-gon. Now let us arrange these discs such that every line passing through the first vertex, ort_1, intersects no more than one of them. Denote by A the set consisting of:

(i) the contour of the $4g$-gon;

(ii) the $2\ell - 2$ segments tangent to $\ell - 1$ of the removed discs and passing through ort_1;

(iii) the outer arcs of the boundaries of these discs having as endpoints the tangency points.

Then A is obviously a strong deformation retract of the holed $4g$-gon. If we now project this $4g$-gon onto the sphere with g handles, the above strong deformation retraction is transformed into a strong deformation retraction of the holed sphere with g handles onto the image of A under the projection. Finally, note that this image is manifestly homeomorphic to a bouquet of $2g + \ell - 1$ circles.

The proof for a sphere with h cross-caps and ℓ holes is a verbatim repetition of the previous argument, with the $4g$-gon replaced by the $2h$-gon. □

3.5.4 Exercises

Exercise 3.5.4.1. Show that every smooth, connected, non-compact, one-dimensional manifold is homeomorphic to a line or a half-line (see Remark 3.5.3.1).

Exercise 3.5.4.2. Define a submanifold of $\mathbb{C}P^2$ in homogeneous coordinates by the equation $z_1^m + z_2^m + z_3^m = 0$, and show that it is diffeomorphic to a sphere with $(m-1)(m-2)/2$ handles (see Exercises 3.2.4.4 and 3.2.4.5).

Exercise 3.5.4.3. Show that the subset of $\mathbb{C}P^1 \times \mathbb{C}P^1$ consisting of the points $((z_1 : z_2), (w_1 : w_2))$ such that $z_1^q(w_1^p + w_2^p) = z_2^2(w_1^p - w_2^p)$ is a manifold diffeomorphic to a sphere with $(p-1)(q-1)$ handles.

Exercise 3.5.4.4. Show that every smooth, closed, connected, orientable, three-dimensional manifold can be obtained by glueing two copies of a handle-body by a diffeomorphism of its boundary. (A *handle-body* is a part of $\mathbb{R}^3$ bounded by a sphere with handles which is standardly embedded in $\mathbb{R}^3$) .

Exercise 3.5.4.5. Consider the manifold obtained by glueing two copies of the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ by a diffeomorphism of its boundary $\mathbb{S}^1 \times \mathbb{S}^1$, given by the formula $(z_1, z_2) \mapsto (z_1^a z_2^b, z_1^c z_2^d)$, where a, b, c, d are integers satisfying $ad - bc = \pm 1$. Show that this manifold is diffeomorphic to $\mathbb{S}^3$ for $a = 0$, to $\mathbb{S}^2 \times \mathbb{S}^1$ for $a = \pm 1$, and to $\mathbb{R}P^3$ for $a = \pm 2$.

Exercise 3.5.4.6. Show that on every connected closed $\mathcal{C}^\infty$-manifold there is a Morse function with a unique local minimum and a unique local maximum.

Exercise 3.5.4.7. Show that on every connected cobordism X with non-empty $\partial_0 X$ and $\partial_1 X$ there is a proper Morse function with no local maxima and minima lying in $\operatorname{int} X$.

Exercise 3.5.4.8. Show that on every cobordism there is a proper Morse function such that, for any of its critical points x_1 and x_2, of indices k_1 and k_2, $k_1 < k_2$ implies $f(x_1) < f(x_2)$.

Exercise 3.5.4.9. Suppose that on a cobordism X there is a Morse function with no critical points of index 1 and that the manifold $\partial_0 X$ is orientable. Show that the cobordism X is orientable and that every orientation of $\partial_0 X$ is induced by some orientation of X.

Chapter 4

BUNDLES

4.1 BUNDLES WITHOUT GROUP STRUCTURE

4.1.1 General Definitions

Definition 4.1.1.1. A *bundle* is a triple (T, p, B), where T and B are topological spaces and $p\colon T \to B$ is a continuous map. The spaces T and B are called the *total space* and the *base* of the bundle (T, p, B), respectively, and the map p is its *projection*. For a bundle ξ, we denote its total space, its base, and its projection by $\mathrm{tl}(\xi)$, $\mathrm{bs}(\xi)$, and $\mathrm{proj}(\xi)$, respectively: $\xi = (\mathrm{tl}(\xi), \mathrm{proj}(\xi), \mathrm{bs}(\xi))$.

The preimage $[proj(\xi)]^{-1}(b)$ of a point $b \in \mathrm{bs}(\xi)$ is called the *fibre of the bundle* ξ *over the point* b.

A *section* of the bundle ξ is a continuous map $s\colon \mathrm{bs}_\xi \to \mathrm{tl}(\xi)$ such that $\mathrm{proj}(\xi) \circ s = \mathrm{id}_{\mathrm{bs}(\xi)}$. Two sections of ξ are *homotopic* if they can be connected by a homotopy consisting only of sections, i.e., by a homotopy $h\colon \mathrm{bs}(\xi) \times I \to \mathrm{tl}(\xi)$ such that $\mathrm{proj}(\xi) \circ h$ equals $\mathrm{proj}_1\colon \mathrm{bs}(\xi) \times I \to \mathrm{bs}(\xi)$.

The *restriction of the bundle* ξ to a subspace $B \subset \mathrm{bs}(\xi)$ is the bundle

$$\xi|_B \overset{def}{=} ([\mathrm{proj}(\xi)]^{-1}(B), \mathrm{abr}\,\mathrm{proj}(\xi), B).$$

The *product of the bundles* ξ_1 and ξ_2 is the bundle

$$(\mathrm{tl}(\xi_1) \times \mathrm{tl}(\xi_2), \mathrm{proj}(\xi_1) \times \mathrm{proj}(\xi_2), \mathrm{bs}(\xi_1) \times \mathrm{bs}(\xi_2))$$

denoted $\xi 1 \times \xi_2$. The fibre of $\xi 1 \times \xi_2$ over a point $(b_1, b_2) \in \mathrm{bs}(\xi_1 \times \xi_2)$ is precisely the product of the fibres $[\mathrm{proj}(\xi_1)]^{-1}(b_1)$ and $[\mathrm{proj}(\xi_2)]^{-1}(b_2)$.

Definition 4.1.1.2. A *map of the bundle* ξ' *into the bundle* ξ is a pair of continuous maps $F\colon \mathrm{tl}(\xi') \to \mathrm{tl}(\xi)$, $f\colon \mathrm{bs}(\xi') \to \mathrm{bs}(\xi)$, such that the diagram

$$\begin{tikzcd}
\mathrm{tl}(\xi') \arrow[r, "F"] \arrow[d, "\mathrm{proj}(\xi')"'] & \mathrm{tl}(\xi) \arrow[d, "\mathrm{proj}(\xi)"] \\
\mathrm{bs}(\xi') \arrow[r, "f"'] & \mathrm{bs}(\xi)
\end{tikzcd} \tag{4.1.1.3}$$

is commutative. If $\Phi = (F, f)$ is such a pair, we write $\Phi : \xi' \to \xi$, $F = \mathrm{tl}(\Phi)$, $f = \mathrm{bs}(\Phi)$.

A map $\Phi\colon \xi' \to \xi$ is said to be an *isomorphism* if $\mathrm{tl}(\Phi)$ and $\mathrm{bs}(\Phi)$ are homeomorphisms, and an *equivalence* if, in addition, $\mathrm{bs}(\xi') = \mathrm{bs}(\xi)$ and $\mathrm{bs}(\Phi) = \mathrm{id}_{\mathrm{bs}(\xi)}$. If there is an isomorphism (equivalence) between ξ' and ξ, then the bundles ξ' and ξ are said to be *isomorphic* (respectively, *equivalent*).

A map $\Phi : \xi' \to \xi$ is called an *inclusion* if $\mathrm{tl}(\Phi)$ and $\mathrm{bs}(\Phi)$ are inclusions. For example, given any subset B of $\mathrm{bs}(\xi)$, the inclusions $\mathrm{incl}\colon [\mathrm{proj}(\xi)]^{-1}(B) \to \mathrm{tl}(\xi)$ and $\mathrm{incl}\colon B \to \mathrm{bs}(\xi)$, form the inclusion of the bundle $\xi|_B$ in ξ.

The commutativity of the diagram (4.1.1.3) implies that F is a *fibre preserving map* (or a *fibred map*), i.e., it takes each fibre of ξ' into a fibre of ξ. Obviously, if $\mathrm{proj}(\xi')(\mathrm{tl}(\xi')) = \mathrm{bs}(\xi')$, then given an arbitrary fibre preserving map $F\colon \mathrm{tl}(\xi') \to \mathrm{tl}(\xi)$ there is a unique map $f\colon \mathrm{bs}(\xi') \to \mathrm{bs}(\xi)$, which makes diagram (4.1.1.3) commutative. Moreover, if the map $\mathrm{proj}(\xi')$ is factorial, then the continuity of F implies the continuity of f. Therefore, we have

Theorem 4.1.1.4. *if ξ' is a bundle with factorial projection, then given any fibre preserving map $F\colon \mathrm{tl}(\xi') \to \mathrm{tl}(\xi)$, there is one and only one continuous map $\Phi : \xi' \to \xi$ such that $\mathrm{tl}(\Phi) = F$.*

Remark 4.1.1.5. Let f be a continuous map of a topological space B into the base of a bundle ξ. We may define a new bundle having base B, total space $\{(b, x) \in B \times \mathrm{tl}(\xi) | f(b) = [\mathrm{proj}(\xi)](x)\}$, and projection - the restriction of the projection $\mathrm{proj}_1\colon B \times \mathrm{tl}(\xi) \to B$ to the last space. This new bundle is called the *bundle induced from* ξ by f, and is denoted by $f^!\xi$.

It is clear that the restriction of the projection $\mathrm{proj}_2\colon B \times \mathrm{tl}(\xi) \to \mathrm{tl}(\xi)$ to $\mathrm{tl}(f^!(\xi))$ defines for each $b \in \mathrm{bs}((f^!(\xi))$ a homeomorphism of the fibre of $f^!(\xi)$ over the point b onto the fibre of ξ over the point $f(b) \in \mathrm{bs}(\xi)$, and determines, together with f, a map $f^!(\xi) \to \xi$. This map is called the *adjoint* of f and is denoted by $\mathrm{adj}\, f$.

The following observations also need no proofs or explanations. If f is a homeomorphism, then the adjoint map $\mathrm{adj}\, f\colon f^!(\xi) \to \xi$ is an isomorphism; if, in addition, $f = \mathrm{id}_{\mathrm{bs}(\xi)}$ then $\mathrm{adj}\, f$ is an equivalence. If f is an inclusion, then $\mathrm{adj}\, f\colon f^!(\xi) \to \xi$ establishes an equivalence between $f^!(\xi)$ and $\xi|_B$. Finally, given arbitrary continuous maps $f\colon B \to \mathrm{bs}(\xi)$ and $g\colon B' \to B$, the bundles $(f \circ g)^!(\xi)$ and $g^!(f^!(\xi))$ are canonically equivalent.

Definition 4.1.1.6. If $\Phi\colon \xi' \to \xi$ is a map of bundles, then the formula $x \mapsto ([\mathrm{proj}(\xi')](x), [(\mathrm{tl}(\Phi)](x))$ defines a continuous map $\mathrm{tl}(\xi') \to \mathrm{tl}([\mathrm{bs}(\Phi)]^!(\xi))$. This map defines, together with $\mathrm{id}_{\mathrm{bs}(\xi')}$, a map of ξ' into the bundle $[\mathrm{bs}(\Phi)]^!(\xi)$, which we denote by $\mathrm{corr}\,\Phi$; we say that $\mathrm{corr}\,\Phi$ *corrects* the map Φ.

Obviously, $\mathrm{adj}(\mathrm{bs}(\Phi)) \circ \mathrm{corr}\,\Phi = \Phi$.

4.1.2 Locally Trivial Bundles

Definition 4.1.2.1. The obvious example of a bundle having a given base B and fibres homeomorphic to a given space F is the *standard trivial bundle* (or the *product bundle*) $(B \times F, \mathrm{proj}_1, B)$. Its fibres are the fibres $b \times F$ of the product $B \times F$, and are obviously canonically homeomorphic to F.

Notice that there is a one-to-one correspondence between the continuous functions $B \to F$ and the sections $B \to B \times F$ of the standard trivial bundle $(B \times F, \mathrm{proj}_1, B)$: for each function $f\colon B \to F$ there is the corresponding section

$$s\colon B \to B \times F, \quad b \mapsto (b, f(b));$$

we say that f and s are *associated*.

Definition 4.1.2.2. A bundle ξ is *trivial* or, more specifically, *topologically trivial*, if it is equivalent to a standard trivial bundle. Any equivalence between a standard trivial bundle and ξ, is referred to as a *trivialisation* of ξ.

A bundle ξ is *locally trivial* or, more specifically, *topologically locally trivial*, if every point of bs_ξ has a neighbourhood U such that the bundle $\xi|_U$ is trivial.

Since the projection of a product of topological spaces onto one of its factors is an open map, the projection of a trivial bundle is open, and hence so is the projection of a locally trivial bundle.

It is immediate that the product of two trivial (locally trivial) bundles is a trivial (respectively, locally trivial) bundle. Furthermore, any bundle induced from a trivial (locally trivial) bundle is trivial (respectively, locally trivial). If $f\colon B \to \mathrm{bs}_\xi$ is constant, then $f^!\xi$ is a trivial bundle, for any ξ.

Remark 4.1.2.3. The fibres of a trivial bundle are, as those of the standard trivial bundle, homeomorphic to each other. However, in a trivial, but not standard trivial bundle, these homeomorphisms are not canonical any longer. If the base of a locally trivial bundle is connected, then its fibres are also mutually homeomorphic; indeed, the set of the points of the base having fibres homeomorphic to a given fibre is open, and the sets of this type form a partition of the base (see Theorem 1.3.3.5).

On the other hand, the example of the locally trivial bundle

$$((B \times F) \amalg (B' \times F'), \mathrm{proj}_1 \amalg \mathrm{proj}_1, B \amalg B')$$

where B, F, B', and F' are arbitrary topological spaces, demonstrates that in a locally trivial bundle the fibres over points situated in different components of the base are not necessarily homeomorphic. Moreover, we see that there are locally trivial bundles which are not trivial.

A non-trivial, locally trivial bundle may have a connected base; see Theorem 4.1.2.5 and Example 4.1.2.6.

Coverings

Remark 4.1.2.4. A locally trivial bundle is a covering in the broad sense if all its fibres are discrete spaces. In this case the total space and the projection are usually called a covering space and a covering projection, respectively.[1] Clearly

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[1] Translator's note: Frequently, the terms covering space and covering projection are themselves used to designate the whole covering.

, every point of a covering space has a neighbourhood such that the restriction of the projection to this neighbourhood is a homeomorphism onto its image in the base.

A covering in the broad sense is said to be a covering in the narrow sense or, simply, a covering, if both the covering and base spaces are connected and non-empty. According to Remark 4.1.2.33, all the fibres of a covering have the same cardinality, called the *number of sheets* (or the *multiplicity*) of the given covering.

Theorem 4.1.2.5. *A covering whose number of sheets is greater than one cannot be trivial.*

Proof. Indeed, the total space of a trivial bundle is homeomorphic to the product of the base and a fibre, and hence it cannot be connected when the fibre is discrete and has more than one point. ☐

Example 4.1.2.6. The bundle $(\mathbb{S}^1, \mathrm{hel}_m, \mathbb{S}^1)$, where

$$\mathrm{hel}_m \colon \mathbb{S}^1 \to \mathbb{S}^1, \quad z \mapsto z^m,$$

is an m-sheeted covering for any $m \neq 0$. The bundle $(\mathbb{R}^1, \mathrm{hel}, \mathbb{S}^1)$, where $\mathrm{hel}\colon \mathbb{R}^1 \to \mathbb{S}^1$ is defined as $\mathrm{hel}(x) = e^{2\pi i x}$, is a countably-sheeted covering.

If $k \neq 0, n$, then the bundle having total space $G_+(n, k)$, base $G(n, k)$, and projection equal to the submersion exhibited in Remark 3.2.2.3, is a two-sheeted covering. In particular, so is $(\mathbb{S}^n, \mathrm{proj}, \mathbb{R}P^n)$ for $n \geq 1$.

Finally, let us show that every sphere with h cross-caps admits as a two-sheeted covering space a sphere with $h - 1$ handles (see Subsection 3.5.3); here h is an arbitrary positive integer. For $h = 1$, we already encountered such a covering, namely $(\mathbb{S}^2, \mathrm{proj}, \mathbb{R}P^2)$. Generally, one may construct it starting with h copies of $(\mathbb{S}^2, \mathrm{proj}, \mathbb{R}P^2)$. To do this, restrict one copy of $(\mathbb{S}^2, \mathrm{proj}, \mathbb{R}P^2)$ to a covering over the projective plane with $h - 1$ holes, and restrict the remaining $h - 1$ copies to coverings over the projective plane with one hole (i.e., over the Möbius strip). Now glue the bases of these h restricted coverings into a sphere with h cross-caps by diffeomorphisms of the boundaries of the holes. The resulting glued space is the base of a new covering, whose total space is obtained by glueing the h total spaces of the above restricted coverings: one of these total spaces is a sphere with $h - 1$ pairs of antipodal holes, while the remaining $h - 1$ total spaces are spheres with two antipodal holes, i.e., cylinders over circles. (Each of these $h - 1$ cylinders has two possible covering attaching maps, and we may use either one of them.) Since "sealing" a pair of holes by a cylinder results in replacing this pair by a handle, what we actually obtain is a two-sheeted covering having a sphere with h cross-caps as the base and a sphere with $h - 1$ handles as the total space.

4.1.3 Serre Bundles

Definition 4.1.3.1. A bundle ξ is a *Serre bundle*[2] if it satisfies *Serre's condition*: for any positive integer r or $r = 0$, and every continuous maps $f\colon I^r \to \mathrm{bs}_\xi$ and $\widetilde{f}_0\colon I^r \to \mathrm{tl}(\xi)$, related by $\mathrm{proj}(\xi) \circ \widetilde{f}_0 = f|_{I^{r-1}}$, there is a continuous map $\widetilde{f}\colon I^r \to \mathrm{tl}(\xi)$ such that $\mathrm{proj}(\xi) \circ \widetilde{f} = f$ and $\widetilde{f}|_{I^{r-1}} = \widetilde{f}_0$. (We identify the cube I^{r-1} with that face of I^r whose points have the last coordinate equal to zero; see Remark 1.2.5.7.)

$$\begin{tikzcd}
I^{r-1} \arrow[r, "\widetilde{f}_0"] \arrow[d, hook] & \mathrm{tl}(\xi) \arrow[d, "\mathrm{proj}(\xi)"] \\
I^r \arrow[r, "f"'] \arrow[ur, dotted, "\widetilde{f}"] & \mathrm{bs}(\xi)
\end{tikzcd}$$

The requirement that $\mathrm{proj}(\xi)\circ f = f$ appearing in Serre's condition is fundamental in the theory of bundles, and is encountered also when f and $\widetilde{f}$ are defined on spaces other than cubes. If two maps, $\widetilde{f}\colon X \to \mathrm{tl}(\xi)$ and $f\colon X \to \mathrm{bs}(\xi)$, satisfy this last requirement, we say that $\widetilde{f}$ *covers* f (or that $\widetilde{f}$ is a *lifting* or *lift* of f; we also say that f can be lifted to $\mathrm{tl}(\xi)$; this terminology is valid for arbitrary ξ and X.

Obviously, the product of two Serre bundles and a bundle induced from a Serre bundle are again Serre bundles.

Example 4.1.3.2. Examples of bundles which do not satisfy Serre's condition are (I, p, I), where $p(x) = x/2$ or $p(x) = 4x(1-x)$. In the first case, take $r = 1$, $f = \mathrm{id}_I$, $\widetilde{f}_0(0) = 0$; in the second case, take $r = 2$, $f(x_1, x_2) = 4x_1(1-x_1)(1-x_2)$, $\widetilde{f}_0 = \mathrm{id}_I$. Then there is no continuous map such that $p \circ \widetilde{f} = f$ and $\widetilde{f}|_{I^{r-1}} = \widetilde{f}_0$.

We remark that the first bundle has both empty and non-empty fibres, while the second has a single connected fibre, the others being not connected. As we shall see later (see Theorem 5.4.3.6), such features of a bundle are not compatible with Serre's condition when the base is connected.

Serre's Condition is Local

Theorem 4.1.3.3. *If every point of the base* bs_ξ *of a bundle* ξ *has a neighbourhood* U *such that* $\xi|_U$ *is a Serre bundle, then* ξ *itself is a Serre bundle.*

Proof. Let $f\colon I^n \to \mathrm{bs}(\xi)$ and $\widetilde{f}_0\colon I^{n-1} \to \mathrm{tl}(\xi)$ be continuous maps satisfying $\mathrm{proj}(\xi) \circ \widetilde{f}_0 = f|_{I^{n-1}}$. Since $\mathrm{bs}(\xi)$ can be covered by open sets such that the restriction of ξ to each of these sets satisfies Serre's condition, Theorem 1.1.7.16 yields a positive integer N such that every cube of edge $1/N$ contained in I^n is taken by f into one of these open sets. Divide I^n into N^n cubes of edge $1/N$, arrange these cubes in dictionary order $Q_1, \dots, Q_{N^n}$, and set

$$W_i = I^{n-1} \cup (\cup_{j=1}^{i} Q_j), \quad Q'_i = Q_i \cap W_{i-1}.$$

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[2] Translator's note: Frequently called a Serre fibre space or a weak fibration.

It is clear that $I^{n-1} = W_0 \subset W_1 \subset \cdots \subset W_{N^n} = I^n$, and examining all possible cases, we see that each pair (Q_i, Q'_i) is homeomorphic to (I^n, I^{n-1}). Now assume that for some $i \leq N^n$ there is a map $\widetilde{f}_{i-1}\colon W_{i-1} \to \operatorname{tl}(\xi)$, such that

$$\operatorname{proj}(\xi) \circ f = f|_{W_{i-1}}, \quad \widetilde{f}_{i-1}|_{I^{n-1}} = \widetilde{f}_0.$$

Set $g_i = f|_{Q_i}$ and $\widetilde{g}_{0i} = \widetilde{f}_{i-1}|_{Q'_i}$. Since the restriction $\xi|_{f(Q_i)}$ satisfies Serre's condition, there is a continuous map $\widetilde{g}_i\colon \operatorname{tl}(\xi)$ such that

$$\operatorname{proj}(\xi) \circ \widetilde{g}_i = g_i \quad \text{and} \quad \widetilde{g}_i|_{Q'_i} = \widetilde{g}_{0i}.$$

Furthermore, because $\widetilde{f}_{i-1}$ and $\widetilde{g}_i$ agree on $W_{i-1} \cap Q_i = Q'_i$ together they form a continuous map $\widetilde{f}_i\colon W_{i-1} \cup Q_i \to \operatorname{tl}(\xi)$, and obviously $\operatorname{proj}(\xi) \circ \widetilde{f}_i = f|_{W_i}$ and $\widetilde{f}_i|_{I^{n-1}} = \widetilde{f}_0$. This shows that induction on i works, starting with $i = 0$, and the result is a map $\widetilde{f} = \widetilde{f}_{N^n}$ such that $\operatorname{proj}(\xi) \circ \widetilde{f} = f$ and $\widetilde{f}_i|_{I^{n-1}} = \widetilde{f}_0$. □

Serre's Condition and Local Triviality

Theorem 4.1.3.4. *Every locally trivial bundle is a Serre bundle.*

Proof. By Theorem 4.1.3.3, one need only consider the case $(B \times F, \operatorname{proj}_1 B)$, where B and F are arbitrary topological spaces. Let

$$f\colon I^n \to B \quad \text{and} \quad \widetilde{f}_0\colon I^{n-1} \to B \times F$$

be continuous maps such that $\operatorname{proj}_1 \circ \widetilde{f}_0 = f|_{I^{n-1}}$. Define $\widetilde{f}\colon I^n \to B \times f$ as

$$\widetilde{f}(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n), \operatorname{proj}_2 \circ f_0(x_1, \ldots, x_{n-1})).$$

Clearly, $\operatorname{proj}_1 \circ \widetilde{f} = f$ and $\widetilde{f}|_{I^{n-1}} = \widetilde{f}_0$. □

Example 4.1.3.5. The following example demonstrates that there are Serre bundles which are not locally trivial. Let T be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, and $(1,0)$, and let $p_1, p_2\colon T \to I$ be defined by $p_1(x_1, x_2) = x_1$, $p_2(x_1, x_2) = x_2$. The bundle (T, p_1, I) is not locally trivial; indeed, the fibres over the points 0 and 1 are not homeomorphic. However, (T, p_1, I) does satisfy Serre's condition: if $f\colon I^n \to I$ and $\widetilde{f}_0\colon I^{n-1} \to T$ are continuous, and $p_1 \circ \widetilde{f}_0 = f$ then the map $\widetilde{f}\colon I^n \to T$ defined by

$$\widetilde{f}(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n), \min(1 - f(x_1, \ldots, x_n), \operatorname{proj}_2 \circ f_0(x_1, \ldots, x_{n-1})))$$

is continuous, covers f, and equals $\widetilde{f}_0$ on I^{n-1}.

$$\begin{tikzcd}
I^{n-1} \arrow[r, "\widetilde{f}_0"] \arrow[d, hook] & T \arrow[d, "p_1"] \\
I^n \arrow[r, "f"'] \arrow[ur, dotted, "\widetilde{f}"] & I
\end{tikzcd}$$

This example shows also that in a Serre bundle with connected base there can be non-homeomorphic fibres. Actually, there are Serre bundles with connected base and in which some fibres are not even homotopy equivalent, being instead equivalent in a certain weaker sense (see Exercise 5.4.4.3 and Theorem 5.4.3.6) .

The Covering Homotopy Theorem

Theorem 4.1.3.6. *Suppose that ξ is a Serre bundle and (X, A) is a cellular pair. Then for every continuous map $\widetilde{f}\colon X\times \to \mathrm{tl}(\xi)$, every homotopy $F\colon X \times I \to \mathrm{bs}(\xi)$ of $\mathrm{proj}(\xi)\circ \widetilde{f}$, and every homotopy $G\colon A\times I \to \mathrm{tl}(\xi)$ of $\widetilde{f}|_A$ covering $F|_{A\times I}$, there is a homotopy of $\widetilde{f}$ which covers F and extends G.*

Proof. Assume that X is rigged, and that for some $r \geq 0$ there is a homotopy $\widetilde{F}_{r-1}\colon (A\cup \mathrm{skel}_{r-1} X)\times I \to \mathrm{tl}(\xi)$ of $\widetilde{f}|A_{A\cup\mathrm{skel}_{r-1} X}$ covering $F|_{(A\cup\mathrm{skel}_{r-1} X)\times I}$. If e is an r-ce11 from $X\setminus A$, then $\varphi_e(x, t) = F(\mathrm{char}(x), t)$ defines a continuous map $\varphi_e\colon \mathbb{D}^r \times I \to \mathrm{bs}(\xi)$, while the formula

$$\widetilde{\varphi}_{0,e}(x,t) = \begin{cases} \widetilde{f}(\mathrm{char}(x)), & \text{if} \quad t = 0, \\ \widetilde{F}_{r-1}(\mathrm{char}_e(x), t), & \text{if} \quad x \in \mathbb{S}^{r-1}, \end{cases}$$

defines a continuous map $\widetilde{\varphi}_{0,e}\colon (\mathbb{D}^r\times 0)\cup(\mathbb{S}^{r-1}\times I) \to \mathrm{tl}(\xi)$. Obviously, the pairs $(\mathbb{D}^r\times I, (\mathbb{D}^r \times 0)\cup(\mathbb{S}^{r^1}\times I))$ and (I^{r+1}, I^r) are homeomorphic, and

$$\mathrm{proj}(\xi)\circ\widetilde{\varphi}_{0,e} = \varphi_e|_{(\mathbb{D}^r\times 0)\cup(\mathbb{S}^{r-1}\times I)}.$$

Consequently, there is a continuous map $\widetilde{\varphi}_e\colon \mathbb{D}^r\times I \to \mathrm{tl}(\xi)$ covering ϕ_e and extending $\widetilde{\varphi}_{0,e}$. Since $\widetilde{\varphi}_e(x,t) = \widetilde{F}_{r-1}(\mathrm{char}_e(x), t)$ for all $x\in\mathbb{S}^{r-1}$, the maps $\widetilde{\varphi}_e$ corresponding to all r-cells from $X\setminus A$ together with $\widetilde{F}_{r-1}$ yield a continuous map $\widetilde{F}_r\colon (A\cup\mathrm{skel}_r X)\times I \to \mathrm{tl}(\xi)$, and it is evident that the following holds:

$$\mathrm{proj}(\xi)\circ\widetilde{F}_r = F|_{(A\cup\mathrm{skel}_r X)\times I}, \quad \widetilde{F}_r = F|_{(A\cup\mathrm{skel}_{r-1} X)\times I} = \widetilde{F}_{r-1}.$$

Hence, we may use induction on r, setting $\widetilde{F}_{-1} = G$, to produce a sequence

$$\{\widetilde{F}_r\colon (A\cup\mathrm{skel}_r X)\times I \to \mathrm{tl}(\xi)\}_{r=-1}^{\infty}$$

of homotopies which extend each other. These homotopies define a homotopy of $\widetilde{f}$ covering F and extending G. □

Theorem 4.1.3.7. *Let X be a cellular space and let $\widetilde{f}\colon X \to \mathrm{tl}(\xi)$ be continuous. If ξ is a Serre bundle, then every homotopy of $\mathrm{proj}(\xi)\circ\widetilde{f}$ is covered by a homotopy of $\widetilde{f}$.*

Proof. This is precisely Theorem 4.1.3.6 for the absolute case $A = \varnothing$, the proof is immediate. We note that for $X = I^n$, Theorem 4.1.3.7 reduces to Serre's condition for $r = n + 1$. □

The Case of Coverings

Proposition 4.1.3.8. *Suppose that ξ is a covering in the broad sense, X is a connected topological space, and $f, g\colon X \to \mathrm{tl}(\xi)$ are continuous maps. If*

$$\mathrm{proj}(\xi) \circ f = \mathrm{proj}(\xi) \circ g$$

and f equals g at some point, then $f = g$.

Proof. Since the set $\{x \in X | f(x) \neq g(x)\}$ is open and, by assumption, its complement is not empty, it suffices to show that this complement is also open. In other words, let us verify that if $f(x_0) = g(x_0)$, then x_0 has a neighbourhood U such that $f(x) = g(x)$ for all $x \in U$. Let V be a neighbourhood of $f(x_0)$ such that $\mathrm{proj}(\xi)|_V\colon V \to [\mathrm{proj}(\xi)](V)$ is a homeomorphism (see Remark 4.1.2.4), and take U to be any neighbourhood of x_0 with $f(U) \subset V$ and $g(U) \subset V$. Since $[\mathrm{proj}(\xi)](f(x)) = [\mathrm{proj}(\xi)](g(x))$ for all $x \in X$, we have $f(x) = g(x)$ for all $x \in U$. □

Theorem 4.1.3.9. *Suppose that ξ is a covering in the broad sense, X is a connected cellular space with a distinguished 0-cell x_0, and $f, g\colon X to\, \mathrm{tl}(\xi)$ are continuous. If the maps $\mathrm{proj}(\xi) \circ f$ and $\mathrm{proj}(\xi) \circ g$ are x_0-homotopic and $f(x_0) = g(x_0)$, then f and g are x_0-homotopic.*

Proof. By Theorem 4.1.3.6, any x_0-homotopy from $\mathrm{proj}(\xi) \circ f$ to $\mathrm{proj}(\xi) \circ g$ is covered by an x_0-homotopy from f to some map h. Since $h(x_0) = f(x_0) = g(x_0)$ and $\mathrm{proj}(\xi) \circ h = \mathrm{proj}(\xi) \circ g$, Proposition 4.1.3.8 yields $h = g$. □

4.1.4 Bundles With Map Spaces as Total Spaces.

Definition 4.1.4.1. We say that a bundle ξ satisfies the *strong Serre condition*[3] if for every topological space X, every continuous map $\widetilde{f}\colon X \to \mathrm{tl}(\xi)$, and every homotopy F of $\mathrm{proj}(\xi) \circ \widetilde{f}$ there is a homotopy of $\widetilde{f}$ which covers F.

$$\begin{tikzcd}
X \arrow[r, "\widetilde{f}"] \arrow[d, hook] & \mathrm{tl}(\xi) \arrow[d, "\mathrm{proj}(\xi)"] \\
X \times I \arrow[r, "F"'] \arrow[ur, dotted, "\widetilde{F}"] & \mathrm{bs}(\xi)
\end{tikzcd}$$

If we replace X by a cube of arbitrary dimension, then this becomes the simple Serre condition; moreover, when X is restricted to be an arbitrary cellular space, we obtain again a condition equivalent to the simple Serre condition; see Theorem 4.1.3.7.

Theorem 4.1.4.2. *Let (X, A) and Y be a Borsuk pair (= cofibration) and a topological space, respectively. If X is Hausdorff and locally compact, then the bundle*

$$(\mathcal{C}(X, Y), \mathcal{C}(\mathrm{incl}, \mathrm{id}), \mathcal{C}(A, Y))$$

---

[3] Translator's note: Such a bundle is frequently called a Hurewicz fibre space or a fibration.

satisfies the strong Serre condition.

Proof. Consider a topological space Z, a continuous map $\widetilde{f}\colon Z \to \mathcal{C}(X,Y)$, and a homotopy $F\colon Z \times I \to \mathcal{C}(A,Y)$ of $\mathcal{C}(\mathrm{incl},\mathrm{id}) \circ \widetilde{f}$. Since X is Hausdorff and locally compact, the maps $\widetilde{g}\colon Z \times X \to Y$ and $G\colon Z \times A \times I \to Y$ given by $\widetilde{g}(z,x) = (\widetilde{f})\wedge(z,x) = [\widetilde{f}(z)](x)$ and $G(z,x,t) = [F(z,t)](x)$ are continuous (see Theorem 1.2.7.6). It is clear that G is a homotopy of $\widetilde{g}|_{Z\times A}$. Now $(Z\times X, Z\times A)$ is a Borsuk pair (see Theorem 1.3.5.5 and Remark 1.3.5.3), and hence G extends to a homotopy $\widetilde{G}$ of $\widetilde{g}$.

$$\begin{tikzcd}
Z \times A \times 0 \arrow[rr, "\subset"] \arrow[dd, hook] \arrow[dr, "\widetilde{g}|_{Z\times A\times 0}"] & & Z \times X \times 0 \arrow[dd, hook] \arrow[dl, "G"'] \\
& Y & \\
Z \times A \times I \arrow[rr, "\subset"'] \arrow[ur, "\widetilde{g}"] & & Z \times X \times I \arrow[ul, dotted, "\widetilde{G}"]
\end{tikzcd}$$

Finally, the formula $[\widetilde{F}(z,t)](x) = \widetilde{G}(z,x,t)$ defines a homotopy $\widetilde{F}\colon Z \times I \to \mathcal{C}(X,Y)$ of $\widetilde{f}$ which covers F.

$$\begin{tikzcd}
Z \arrow[r, "\widetilde{f}"] \arrow[d, hook] & \mathcal{C}(X,Y) \arrow[d, "\mathcal{C}(\mathrm{incl},\mathrm{id})"] \\
Z \times I \arrow[r, "F"'] \arrow[ur, dotted, "\widetilde{F}"] & \mathcal{C}(A,Y)
\end{tikzcd}$$

□

Theorem 4.1.4.3. *In a bundle with connected base and satisfying the strong Serre condition, the fibres are pairwise homotopy equivalent.*

Proof. Let ξ be the given bundle, and let s be a path joining two given points of $\mathrm{bs}(\xi)$. Set $F_0 = [\mathrm{proj}(\xi)]^{-1}(s(0))$ and $\mathrm{F}_1 = [\mathrm{proj}(\xi)]^{-1}(s(1))$. Now consider two homotopies $J_0\colon F_0 \times I \to \mathrm{bs}(\xi)$ and $J_1\colon F_1 \times I \to \mathrm{bs}(\xi)$ of the composite maps

$$F_0 \xrightarrow{\mathrm{incl}} \mathrm{tl}(\xi) \xrightarrow{\mathrm{proj}(\xi)} \mathrm{bs}(\xi) \quad \text{and} \quad F_1 \xrightarrow{\mathrm{incl}} \mathrm{tl}(\xi) \xrightarrow{\mathrm{proj}(\xi)} \mathrm{bs}(\xi),$$

respectively, given by $J_0(x,t) = s(t)$ and $J_1(x,t) = s^{-1}(t)$. Since ξ satisfies the strong Serre condition, J_0 and J_1 are covered by two homotopies,

$$\widetilde{J}_0\colon F_0 \times I \to \mathrm{tl}(\xi) \quad \text{and} \quad \widetilde{J}_1\colon F_1 \times I \to \mathrm{tl}(\xi)$$

of the maps $\mathrm{incl}\colon F_0 \to \mathrm{tl}(\xi)$ and $\mathrm{incl}\colon F_1 \to \mathrm{tl}(\xi)$, respectively. Now we have

$$\widetilde{J}_0(F_0 \times 1) \subset F_1, \quad \widetilde{J}_1(F_1 \times 1) \subset F_0$$

and hence there are well-defined maps

$$f_0\colon F_0 \to F_1, \quad x \mapsto \widetilde{J}_0(x,1) \quad \text{and} \quad f_1\colon F_1 \to F_0, \quad x \mapsto \widetilde{J}_1(x,1).$$

We next show that $f_1 \circ f_0$ is homotopic to id_{F_0}, and since the construction is symmetric, $f_0 \circ f_1$ will be homotopic to id_{F_1}.

The formulae

$$j(x,t) = \begin{cases} \widetilde{J}_0(x,2t), & \text{if} \quad t \le 1/2, \\ \widetilde{J}_1(f_0(x),2t-1) & \text{if} \quad t \ge 1/2. \end{cases}$$

and

$$H(x,t,\tau) = s((1-\tau)(1-|1-2t|)$$

define a map $j\colon F_0 \times I \to \mathrm{tl}(\xi)$, and a homotopy $H\colon (F_0 \times I) \times I \to \mathrm{bs}(\xi)$ of $\mathrm{proj}(\xi) \circ j$. Again, using Serre's strong condition, H can be lifted to a homotopy $\widetilde{H}\colon (F_0 \times I) \times I \to \mathrm{tl}(\xi)$ of j. Since $(1-\tau)(1-|1-2t|) = 0$ for $\tau = 1$ or $t = 0, 1$, we see that

$$\widetilde{H}((F_0 \times (0 \cup 1)) \times I) \cup \widetilde{H}((F_0 \times I) \times 1) \subset F_0.$$

Therefore, the formula

$$K(x,t) = \begin{cases} \widetilde{H}((x,0),3t), & \text{if} \quad t \le 1/3, \\ \widetilde{H}((x,3t-1),1), & \text{if} \quad 1/3 \le t \le 1/3, \\ \widetilde{H}((x,1),3-3t), & \text{if} \quad t \ge 2/3. \end{cases}$$

defines a homotopy $K\colon F_0 \times I \to F_0$. Since

$$K(x,0) = \widetilde{H}((x,0),0) = j(x,0) = \widetilde{J}(x,0) = x \quad \text{and}$$
$$K(x,1) = \widetilde{H}((x,1),0) = j(x,1) = \widetilde{J}(f_0x,1) = f_1(f_0(x)),$$

it follows that K is a homotopy from id_{F_0} to $f_1 \circ f_0$. ☐

Theorem 4.1.4.4. *Given arbitrary points x_0, x_1, x'_0, x'_1 of a connected topological space X, the spaces $\mathcal{C}(I,0,1;X,x_0,x_1)$ and $\mathcal{C}(I,0,1;X,x'_0,x'_1)$ have the same homotopy type*

Proof. $\mathcal{C}(I,0,1;X,x_0,x_1)$ and $\mathcal{C}(I,0,1;X,x'_0,x'_1)$ are the fibres of the bundle $(\mathcal{C}(I,X),\mathcal{C}(\mathrm{incl},\mathrm{id}),\mathcal{C}(0\cup 1),X))$ over the points (x_0,x_1) and (x'_0,x'_1) of its base $\mathcal{C}(0\cup 1),X)) = X \times X$; hence by Theorems 4.1.4.2 and 4.1.4.3, they have the same homotopy type. ☐

The Adjoint Serre Bundle

Definition 4.1.4.5. Given an arbitrary bundle ξ, we let adj_ξ denote the bundle with the same base, total space

$$\{(x,s) \in \mathrm{tl}(\xi) \times \mathcal{C}(I,\mathrm{bs}(\xi))|s(0) = \mathrm{proj}(\xi)(x)\},$$

and projection $(x,s) \mapsto s(1)$. We call adj_ξ the *bundle adjoint to* ξ.

Notice that the total spaces $\mathrm{tl}(\mathrm{adj}\,\xi)$ and $\mathrm{tl}(\xi)$ have the same homotopy type: the formulae $x \mapsto (x, u_x)$ and $(x, s) \mapsto x$, where u_x is the constant path in $\mathrm{bs}(\xi)$ with $u_x(0) = [\mathrm{proj}(\xi)](x)$, define homotopy equivalences $\mathrm{tl}(\xi) \to \mathrm{tl}(\mathrm{adj}\,\xi)$ and $\mathrm{tl}(\mathrm{adj}\,\xi) \to \mathrm{tl}(\xi)$ which are inverses to one another. Indeed, the composition of the first map with the second one is $\mathrm{id}_{\mathrm{tl}(\xi)}$, while the composition of the second map with the first one is homotopic to $\mathrm{id}_{\mathrm{tl}_{\mathrm{adj}_\xi}}$ via the homotopy $((x, s), t) \mapsto (x, s_t)$, where s_t is the path in bs_ξ defined by $s_t(\tau) = s(t\tau)$.

Theorem 4.1.4.6. *The bundle* $\mathrm{adj}\,\xi$ *satisfies the strong Serre condition for any bundle* ξ*.*

Proof. Consider a topological space Z, a continuous map $\widetilde{f}\colon Z \to \mathrm{tl}(\mathrm{adj}\,\xi)$, and a homotopy $F\colon Z \times I \to \mathrm{bs}(\mathrm{adj}\,\xi)(= \mathrm{bs}(\xi))$ of $\mathrm{proj}(\mathrm{adj}\,\xi) \circ \widetilde{f}$. Denote by g_1 and g_2 the composite maps

$$\begin{tikzcd}
 & & & \mathrm{tl}(\xi) \\
Z \arrow[r, "\widetilde{f}"] & \mathrm{tl}(\mathrm{adj}\,\xi) \arrow[r, "\mathrm{incl}"] & \mathcal{C}(I, \mathrm{bs}(\xi)) \arrow[ur, "\mathrm{proj}_1"] \arrow[dr, "\mathrm{proj}_2"'] & \\
 & & & \mathcal{C}(I, \mathrm{bs}(\xi))
\end{tikzcd}$$

and define a homotopy $g\colon Z \times I \to (I, \mathrm{bs}(\xi))$ by

$$[g(z, t)](\tau) = \begin{cases} [g_2(z)](\tau(1 + t)), & \text{if} \quad \tau \le 1/(1 + t), \\ F(z, \tau(1 + t) - 1), & \text{if} \quad \tau \ge 1/(1 + t). \end{cases}$$

It is readily verified that the following diagramme commutes.

$$\begin{tikzcd}
Z \arrow[r, "\widetilde{f}"] \arrow[d, hook] & \mathrm{tl}(\mathrm{adj}\,\xi) \arrow[d, "\mathrm{proj}(\mathrm{adj}\,\xi)"] \\
Z \times I \arrow[r, "F"'] \arrow[ur, "\widetilde{F}"] & \mathrm{bs}(\mathrm{adj}\,\xi)(= \mathrm{bs}(\xi))
\end{tikzcd}$$

□

4.1.5 Exercises

Exercise 4.1.5.1. Show that for any $g \ge 1$ a sphere with g handles admits a sphere with $2g - 1$ handles as a covering space.

Exercise 4.1.5.2. Show that for any $h \ge 1$ a sphere with h cross-caps admits a sphere with $2h - 2$ cross-caps as a covering space.

Exercise 4.1.5.3. Show that the spaces $\mathcal{C}(\mathbb{S}^1, \mathrm{ort}_1; \mathbb{R}P^n, (1 : 0 : \cdots : 0))$ and $\mathcal{C}(\mathbb{S}^1, \mathrm{ort}_1; \mathbb{S}^n, \mathrm{ort}_1) \times \mathbb{S}^0$ are homeomorphic for any $n \ge 1$.

Exercise 4.1.5.4. Show that the bundle with total space $\mathcal{C}(I, 0; \mathbb{S}^n, \mathrm{ort}_1)$, base $\mathbb{S}^n$, and projection $s \mapsto s(1)$, is locally trivial (see Exercise 1.2.9.4).

4.2 A DIGRESSION: TOPOLOGICAL GROUPS AND TRANSFORMATION GROUPS

4.2.1 Topological Groups

Definition 4.2.1.1. A set G is a *topological group* or a *group space* if it is endowed with both a topology and a group structure such that the group operations, i.e, the maps $G \times G \to G$, $(g,h) \mapsto gh$, $G \to G$, and $g \mapsto g^{-1}$, are continuous. Obviously, the continuity of these two maps is equivalent to the continuity of the single map $G \times G \to G$, $(g,h) \mapsto g^{-1}h$.

By the definition of the (product) topology on $G \times G$, the continuity of the map $(g,h) \mapsto gh$ at the point (g_0,h_0) means that for every neighbourhood W of the point g_0h_0 one can find neighbourhoods U and V of g_0 and h_0 such that $UV \subset W$. Similarly, the continuity of the map $(g,h) \mapsto g^{-1}h$ means that for every neighbourhood W of the point g_0 and h_0 there are neighbourhoods U and V of g_0 and h_0 such that $U^{-1}V \subset W$.

Clearly, every group becomes a topological group if it is equipped with the discrete topology.

Remark 4.2.1.2. The continuity of the group operations implies that the left and right translations by group elements (i.e., the maps $G \to G$ given by $g \mapsto ag$ and $g \mapsto ga$), and the map $g \mapsto g^1$ are homeomorphisms of the space G. In particular, if $B \subset G$ is open or closed, then so are the sets B^{-1}, aB and Ba, for all $a \in G$.

We note also that if B is open and A is arbitrary, then AB and BA are open. Indeed, $AB = \cup_{a\in A} aB$ and $BA = \cup_{a\in A} Ba$.

Subgroups and Quotients

Remark 4.2.1.3. Any subset H of the topological group G which is a subgroup in the algebraic sense inherits both a group structure and a topology from G, and it is immediate that the group operations in H are continuous.

A subgroup of a topological group is *normal* if it is normal in the algebraic sense. As in ordinary group theory, normal subgroups are termed also *normal divisors* or *invariant subgroups*.

An example of a normal subgroup of a topological group G, which has no analogue in ordinary group theory, is the *component of the identity*, i.e., that component of the space G which contains the identity element, e_G, of G. This component is obviously a subgroup: if u and v are paths joining e_G with g and h, respectively, then the path $t \mapsto u(t)^{-1}v(t)$ joins e_G with $g^{-1}h$. Since the inner automorphisms of G are continuous and take e_G into itself, this subgroup is normal. It is also clear that the cosets of this subgroup in G are exactly the components of the space G, and the corresponding quotient group coincides, as a set, with $\operatorname{comp} G$.

Theorem 4.2.1.4. *Every open subgroup of a topological group is also closed.*

Proof. In fact, the complement of an open subgroup is a union of left cosets and by Remark 4.2.1.2 each of these cosets is open. Therefore, the complement is also open. □

Definition 4.2.1.5. The partition of a topological group G into the left cosets of a subgroup H is denoted by $\operatorname{zer}(G, H)$, and the corresponding quotient space is called the *space of left cosets of* H *in* G and is denoted by G/H. We shall not need here right coset spaces.

The basic topological property of the partition $\operatorname{zer}(G, H)$ and of the projection $G \to G/H$ is their openness. Indeed, the saturation of a set B relative to $\operatorname{zer}(G, H)$ equals BH, which is an open set whenever B is open (see Remark 4.2.1.2).

Theorem 4.2.1.6. *Given a topological group, the space of left cosets of a closed subgroup is regular. In particular, a topological group whose identity element is closed is regular.*

Proof. Since the cosets gH are closed (see Remark 4.2.1.2), G/H satisfies Axiom T_1. To show that G/H additionally satisfies Axiom T_3, it suffices to produce, given a coset g_0H and a neighbourhood U of g_0H which is saturated relative to $\operatorname{zer}(G, H)$, a saturated neighbourhood V of g_0H such that $\operatorname{Cl} V \subset U$. To see this, suppose that we have such a neighbourhood. Then for every point $\operatorname{proj}(g_0) \in G/H$ and every neighbourhood $\operatorname{proj}(U)$ of $\operatorname{proj}(g_0)$, there is a neighbourhood, $\operatorname{proj}(V)$, of $\operatorname{proj}(g_0)$, such that $\operatorname{Cl} \operatorname{proj}(V) \subset \operatorname{proj}(U)$. The last inclusion follows from the inclusion $\operatorname{proj}(V) \subset \operatorname{proj}(\operatorname{Cl} V)$ together with the fact that $\operatorname{proj}(\operatorname{Cl} V)$ is closed (which follows from the fact that $\operatorname{Cl} V$ is closed and saturated; $\operatorname{Cl} V$ is saturated because the partition $\operatorname{zer}(G, H)$ is open; see Remark 1.2.3.10).

Now to produce the desired neighbourhood V, note that $e_G^{-1} g_0 = g_0$ and so the points e_G and g_0 have neighbourhoods W and W_0 such that $W^{-1}W_0 \subset U$. Set $V = W_0H$. If $g \in \operatorname{Cl} V$, then Wg, being a neighbourhood of g, intersects V, i.e., there exist $w \in W$, $w_0 \in W_0$, and $h \in H$ such that $wg = w_0h$. We have $g = w^{-1}w_0h$, and thus $g = w^{-1}w_0h \in W^{-1}W_0H \subset UH = U$. Therefore, $\operatorname{Cl} V \subset U$. □

Definition 4.2.1.7. Let H be a normal subgroup of the topological group G. According to ordinary group theory, the set of cosets G/H is endowed with a group structure. Let us show that the map $G/H \times G/H \to G/H$, $(x, y) \mapsto x^{-1}y$, is continuous.

First, note that the composition $\psi \colon G \times G \to G/H$ of the map $G \times G \to G$, $(g, h) \mapsto g^{-1}h$, with the projection $G \to G/H$ is constant on the elements of the partition $\operatorname{zer}(G, H) \times \operatorname{zer}(G/H)$. Secondly, the map $(x, y) \mapsto x^{-1}y$ equals $\operatorname{fact} \psi \circ \alpha^{-1}$, where

$$\alpha^{-1} \colon G/H \times G/H \to (G \times G)/(\operatorname{zer}(G, H) \times \operatorname{zer}(G, H))$$

is the inverse of the injective factor, α, of the map

$$\operatorname{proj} \times \operatorname{proj} \colon G \times G \to G/H \times G/H,$$

and

$$\operatorname{fact}\psi\colon (G\times G)/(\operatorname{zer}(G,H)\to \operatorname{zer}(G,H))\to G/H.$$

Since the partition $\operatorname{zer}(G,H)$ is open, α is a homeomorphism (see Theorem 1.2.3.11), and this implies that the map $(x,y)\mapsto x^{-1}y$ is continuous.

We conclude that G/H is a topological group; G/H is called the *factor group* of the topological group G by H.

Homomorphisms

Remark 4.2.1.8. A map $f\colon G\to G'$, where G and G' are topological groups, is a *homomorphism* if it is an algebraic homomorphism as well as continuous.

As in ordinary group theory, the kernel $\ker f$ of f is defined as the preimage of the identity of G'. A homomorphism f is a *monomorphism* if it is injective, i.e., if its kernel $\ker f$ is the identity element of G, and an *epimorphism* if its image $\operatorname{im} f = f(G)$ is all of G'. An example of monomorphism (epimorphism) is the inclusion of a subgroup in a topological group (respectively, the projection of a topological group onto a factor group).

An invertible homomorphism whose inverse is also a homomorphism is called an *isomorphism*. In other words, an isomorphism of topological groups is a map which is both an algebraic homomorphism and a homeomorphism.

Theorem 4.2.1.9. *Let $f\colon G\to G'$ be a homomorphism of topological groups. Then*

(i) $\operatorname{im} f$ *is a subgroup of G', and the compression* $\operatorname{abr} f\colon G\to \operatorname{im} f$ *is an epimorphism;*

(ii) $\ker f$ *is a normal subgroup of G, and the injective factor of f,*

$$\operatorname{fact} f\colon G/\ker f\to G',$$

is a monomorphism.

Proof. In addition to recognising that this copies a well-known statement of ordinary group theory, one has to check that the maps $\operatorname{abr} f$ and $\operatorname{fact} f$ are continuous, which is trivial. ☐

Theorem 4.2.1.10. *An epimorphism $f\colon G\to G'$ is open if and only if its injective factor,* $\operatorname{fact} f\colon G/\ker f\to G'$*, is an isomorphism.*

Proof. The necessity of this condition is obvious. The openness of the projection $G\to G/\ker f$ shows that the condition is also sufficient. ☐

Theorem 4.2.1.11. *An epimorphism of a compact topological group onto a topological group with closed identity element is open.*

Proof. Let $f\colon G\to G'$ be the given epimorphism. Since G is compact, $G/\ker f$ is compact. Moreover, as the identity element of G' is closed, G' is Hausdorff (see Theorem 4.2.1.6). Finally, $\operatorname{fact} f\colon G/\ker f\to G'$ is invertible and continuous, and hence in our case a homeomorphism (see Theorem 1.1.7.10). Applying Theorem 4.2.1.10, f is open. ☐

Direct Products

Definition 4.2.1.12. Let G_1 and G_2 be topological groups. In ordinary group theory, the product $G_1 \times G_2$ is given a group structure, and in topology it is given a topology (see Remark 1.2.2.1), and it is clear that these two structures are compatible in the sense of Definition 4.2.1.1, i.e., the group operations in $G_1 \times G_2$ are continuous. The result is a topological group $G_1 \times G_2$, called the *direct product* of the topological groups G_1 and G_2.

This product operation is both commutative and associative: there are obvious canonical isomorphisms

$$G_1 \times G_2 \to G_2 \times G_1, \quad (G_1 \times G_2) \times G_3 \to G_1 \times (G_2 \times G_3).$$

We remark that the inclusions

$$\begin{aligned} \mathrm{incl}_1 \colon G_1 \to G_1 \times G_2, &\quad x_1 \mapsto (x_1, e_{G_2}) \\ \mathrm{incl}_2 \colon G_2 \to G_1 \times G_2, &\quad x_2 \mapsto (e_{G_1}, x_2) \end{aligned}$$

are monomorphisms (of topological groups), while the projections

$$\mathrm{proj}_1 \colon G_1 \times G_2 \to G_1, \quad \mathrm{proj}_2 \colon G_1 \times G_2 \to G_2$$

are open epimorphisms such that $\ker \mathrm{proj}_1 = \mathrm{incl}_2(G_2)$ and $\ker \mathrm{proj}_2 = \mathrm{incl}_1(G_1)$. The last observation together with Theorem 4.2.1.10 imply

$$\begin{aligned} \mathrm{fact}\,\mathrm{proj}_1 \colon (G_1 \times G_2)/\,\mathrm{incl}_2(G_2) &\xrightarrow{\simeq} G_1, \\ \mathrm{fact}\,\mathrm{proj}_2 \colon (G_1 \times G_2)/\,\mathrm{incl}_1(G_1) &\xrightarrow{\simeq} G_2. \end{aligned}$$

Definition 4.2.1.13. We say that the topological group G *decomposes into the direct product of its subgroups* G_1 and G_2 if the map $G_1 \times G_2$, $(g_1, g_2) \mapsto g_1 g_2$ is an isomorphism of topological groups. If this is the case, the groups G and G_1 and G_2 are usually identified via this isomorphism.

Recall that a similar definition exists in ordinary group theory, only there the isomorphism is simply an algebraic isomorphism. Moreover, in that theory, G decomposes into the direct product of its subgroups G_1 and G_2 if and only if G_1 and G_2 generate G, are normal subgroups of G, and $G_1 \cap G_2 = e_G$. Consequently, if these conditions are satisfied in our case, then $(g_1, g_2) \mapsto g_1 g_2$ is an algebraic isomorphism. This map is obviously continuous; however, there are obvious examples where the algebraic inverse isomorphism is not continuous. But the algebraic inverse isomorphism is continuous whenever the space G is compact and Hausdorff. Therefore, every compact Hausdorff topological group which decomposes algebraically into the direct product of two subgroups, decomposes also into the direct product of these subgroups in the sense of our topological definition.

The Simplest Examples

Remark 4.2.1.14. The real line $\mathbb{R}$ with addition as the group operation is a topological group, as is the space $\mathbb{R}^n$. Obviously, $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n factors; the product is understood as in Definition 4.2.1.12).

Remark 4.2.1.15. The punctured real line $\mathbb{R}^\star = \mathbb{R} \setminus 0$, with multiplication as group operation, is a topological group. Its subgroup $\mathbb{R}^\star_+$ consisting of the positive reals, is isomorphic to $\mathbb{R}$: an isomorphism $\mathbb{R} \to \mathbb{R}^\star_+$ is provided by the exponential function $x \mapsto a^x$, with arbitrary $a \neq 1$. Another evident subgroup of $\mathbb{R}^\star$ is $\mathbb{S}^0$, and obviously $\mathbb{R}^\star = \mathbb{S}^0 \times \mathbb{R}^\star_+$.

The punctured complex line $\mathbb{C}^\star = \mathbb{C} \setminus 0$ and the punctured quaternionic line $\mathbb{H}^\star = \mathbb{H} \setminus 0$ are also topological groups under multiplication. Here $\mathbb{S}^1$ is a subgroup of $\mathbb{C}^\star$, $\mathbb{S}^3$ is a subgroup of $\mathbb{H}^\star$, and $\mathbb{C}^\star = \mathbb{S}^1 \times \mathbb{R}^\star_+$, $\mathbb{H}^\star = \mathbb{S}^3 \times \mathbb{R}^\star_+$.

Remark 4.2.1.16. The map hel : $\mathbb{R} \to \mathbb{S}^1$ (see Example 4.1.2.6) is an open epimorphism. Its kernel is the subgroup of integers, $\mathbb{Z}$, of $\mathbb{R}$, and hence the factor group $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{S}^1$.

The map $\mathrm{hel} \times \cdots \times \mathrm{hel} \colon \mathbb{R}^n \to (\mathbb{S}^1)^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is also an open epimorphism. Its kernel is the integer lattice, $\mathbb{Z} \times \cdots \times \mathbb{Z} = \mathbb{Z}^n$, of $\mathbb{R}^n$, and hence the factor group $\mathbb{R}^n/\mathbb{Z}^n$ is isomorphic to $(\mathbb{S}^1)^n$.

Remark 4.2.1.17. The subgroup of $\mathbb{S}^3$ consisting of the real quaternions (i.e., of the quaternions (x_1, x_2, x_3, x_4) such that $x_1 = x_2 = x_3 = 0$) is simply $\mathbb{S}^0$. This is a normal subgroup, and the factor group $\mathbb{S}^3/\mathbb{S}^0$ is, as a topological space, the same as $\mathbb{R}P^3$.

The subgroup of $\mathbb{S}^3$ consisting of the complex quaternions (i.e., of the quaternions (x_1, x_2, x_3, x_4) such that $x_3 = x_4 = 0$) is simply $\mathbb{S}^1$. However, this is not a normal subgroup. The coset space $\mathbb{S}^3/\mathbb{S}^1$ is canonically homeomorphic to $\mathbb{S}^2$: this canonical homeomorphism is provided by the injective factor of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ (obviously, $\mathrm{zer}(\mathbb{S}^3, \mathbb{S}^1) = \mathrm{zer}(\mathbb{S}^3 \to \mathbb{S}^2)$).

4.2.2 Groups of Homeomorphism

Remark 4.2.2.1. By Remark 1.1.4.8, the homeomorphisms of a topological space are a subgroup of the group $\mathrm{Sym}\, X$ of all invertible transformations $X \to X$, i.e., they form a group under the $\circ$ (composition) operation. We denote the group of homeomorphisms of X by $\mathrm{Top}\, X$.

We may define two topologies on $\mathrm{Top}\, X$. The first one is induced by the inclusion $\mathrm{Top}\, X \subset \mathcal{C}(X, X)$ (see Definition 1.2.7.1), i.e., is defined by the prebase consisting of the sets $\mathrm{Nb}(K, O) = \mathcal{C}(X, K; X, O) \cap \mathrm{Top}\, X$, with K compact and $O0$ open. The second topology is defined by the prebase consisting of the sets U, U^{-1} where U is open in the first topology. Equivalently, the second topology is generated by the prebase consisting of the sets $\mathrm{Nb}(K, O)$, $[\mathrm{Nb}(K, O)]^{-1}$.

Lemma 4.2.2.2. *If X is a locally compact Hausdorff space, then the map* $\mathrm{Top}\, X \times \mathrm{Top}\, X \to \mathrm{Top}\, X$, $(g, h) \mapsto gh (= g \circ h)$, *is continuous in either of the above topologies.*

Proof. If $gh \in \mathrm{Nb}(K, O)$, then $h(K) \subset g^{-1}(O)$, and by Theorem 1.1.7.22, every point of $h(K)$ has a neighbourhood whose closure is compact and contained in $g^{-1}(O)$. Let O' denote the union of a finite collection of such neighbourhoods which cover $h(K)$. Clearly, $\mathrm{Cl}\, O'$ is compact, and $g \in \mathrm{Nb}(\mathrm{Cl}\, O', O)$, $h \in Nb(K, O')$, $\mathrm{Nb}(\mathrm{Cl}\, O', O)\, \mathrm{Nb}(K, O') \subset \mathrm{Nb}(K, O)$.

Now if $gh \in [\mathrm{Nb}(K, O)]^{-1}$, then $h^{-1}g^{-1} \in \mathrm{Nb}(K, O)$, and the above argument yields two sets, $U, V \subset \mathrm{Top}\, X$, open in the first topology, and such that $h^{-1} \in V$, $g^{-1} \in U$, $VU \subset \mathrm{Nb}(K, O)$. Clearly, $g \in U^{-1}$, $h \in V^{-1}$, and $U^{-1}V^{-1} \subset [\mathrm{Nb}(K, O)]^{-1}$. □

Theorem 4.2.2.3. *If X is a locally compact Hausdorff space, then* $\mathrm{Top}\, X$*, equipped with the second topology, is a topological group.*

Proof. This is a corollary of Lemma 4.2.2.2 and of the obvious continuity of the map $g \mapsto g^{-1}$ in the second topology. □

Theorem 4.2.2.4. *If X is a compact Hausdorff space, then the first and the second topologies on* $\mathrm{Top}\, X$ *are identical.*

Proof. This is an immediate corollary of the relation

$$[\mathrm{Nb}(K, O)]^{-1} = \mathrm{Nb}(X \setminus O, X \setminus K).$$

□

Lemma 4.2.2.5. *Let X be a locally compact, locally connected, Hausdorff space. Then for a prebase of the second topology on* $\mathrm{Top}\, X$ *it suffice to take the sets of the form* $\mathrm{Nb}(K, O)$*, where K is the closure of a connected open set and O is open.*

Proof. Given a compact K, an open O, and a homeomorphism $f \in \mathrm{Nb}(K, O)$, it suffices to produce connected open sets $U_1, \dots, U_s$ with compact closures, such that $f \in \cap_1^s \mathrm{Nb}(\mathrm{Cl}\, U_i, O) \subset \mathrm{Nb}(K, O)$. For each point $x \in K$, fix a connected neighbourhood of x, V_x with $\mathrm{Cl}\, V_x$ compact and $\mathrm{Cl}\, V_x \subset f^{-1}(O)$ (see Theorems 1.1.7.22 and 1.3.4.3). Now cover K by a finite number of the V_x's, say $V_{x_1}, \dots, V_{x_s}$. It is clear that the sets $U_i = V_{x_i}$ have the required properties. □

Theorem 4.2.2.6. *If X is a locally compact, locally connected, Hausdorff space, then the two topologies on* $\mathrm{Top}\, X$ *are identical.*

Proof. By Lemma 4.2.2.5, given an open connected U with compact $\mathrm{Cl}\, U$, an open O, and a homeomorphism $f \in \mathrm{Nb}(\mathrm{Cl}\, U, O)$, it is enough to find a subset $\mathcal{U} \subset \mathrm{Top}\, X$, open in the first topology, and such that $f \in \mathcal{U} \subset \mathrm{Nb}(\mathrm{Cl}\, U, O)$. Leaving aside the trivial case $U = \varnothing$, we fix a point $x_0 \in f(U)$ and find a set W with compact closure contained in O, such that $f(\mathrm{Cl}\, U) \subset W$, and then take an open set V satisfying $f(\mathrm{Cl}\, U) \subset V \subset \mathrm{Cl}\, V \subset W$. Now set

$$\mathcal{U} = \mathrm{Nb}(x_0, U) \cap \mathrm{Nb}(\mathrm{Cl}\, W \setminus U, f^{-1}(O) \setminus \mathrm{Cl}\, U).$$

The inclusion $f \in \mathcal{U}$ is trivial, and it remains is to show that $\mathcal{U}^{-1} \subset \mathrm{Nb}(\mathrm{Cl}\, U, O)$, i.e., that $g^{-1}(\mathrm{Cl}\, U) \subset O$ for any $g \in \mathcal{U}$. But

$$g \in \mathcal{U} \quad \Rightarrow \quad g(\mathrm{Cl}\, W \setminus U) \subset f^{-1}(O) \setminus \mathrm{Cl}\, U,$$

whence $U \subset g(V) \cup g(X \setminus \mathrm{Cl}\, W)$. Since $g(V)$ and $g(X \setminus \mathrm{Cl}\, W)$ are open and disjoint, and U is connected, we have only two possibilities: either $U \cap g(U) = \varnothing$ or $U \subset g(V)$. Since $g \in \mathcal{U}$, $g(x_0) \in U$; since $x_0 \in f(U) \subset V$, $g(x_0) \in g(V)$. Consequently, $U \cap g(V) \neq \varnothing$, and thus $U \subset g(V)$, i.e., $g^{-1}(U) \subset V$. Finally, $g^{-1}(\mathrm{Cl}\, U) \subset \mathrm{Cl}\, V \subset O$. □

Groups of Diffeomorphisms

Note: the following "7" is numbered as "8" in the original Russian text.

Remark 4.2.2.7. Let X be a $\mathcal{C}^{\geq r}$-manifold, $1 \leq r \leq \infty$. By Remark 3.1.2.9, the set of its $\mathcal{C}^r$-diffeomorphisms, $\mathrm{Diff}^r X$, is a group under the composition operation. By Remark 3.4.1.1, $\mathrm{Diff}^r X$ can be endowed with the $\mathcal{C}^r$-topology. We show that these two structures are compatible and conclude that $\mathrm{Diff}^r X$ is a *topological group.*

Obviously, the case $r = \infty$ reduces to $r < \infty$, and so we may assume from the beginning that r is finite. Consider the mapping

$$\underbrace{d \circ \cdots \circ d}_{r} \colon \mathrm{Diff}^r X \to \mathrm{Top}(\underbrace{\mathrm{Tang} \cdots \mathrm{Tang}}_{r} X).$$

This is clearly a group monomorphism. Moreover, by Remark 3.4.1.1, $d \circ \cdots \circ d$ is a topological embedding when the group $\mathrm{Top}(\mathrm{Tang} \cdots \mathrm{Tang}\, X)$ is equipped with the second topology. However, the first and the second topologies on $\mathrm{Top}(\mathrm{Tang} \cdots \mathrm{Tang}\, X)$ coincide (see Theorem 4.2.2.6), and hence the operations $\circ$ and $f \mapsto f^{-1}$ are continuous in these topologies. Consequently, both operations are continuous in $\mathrm{Diff}^r X$ also.

Let us add that the inclusion $\mathrm{Diff}^r X \to \mathrm{Top}\, X$ is a monomorphism of topological groups, and that the same is true for the inclusions $\mathrm{Diff}^r X \to \mathrm{Diff}^s X$ with $s < r$.

The Classical Groups

Note: the following "10" and "11" are numbered as "9" and "10" in the original Russian text.

Definition 4.2.2.8. The analytic manifolds $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, and also $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}_+(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{GL}(n, \mathbb{H})$, defined in Subsection 3.2.1 (see Remarks 3.1.1.2 and 3.2.1.10 - 3.2.1.10) and endowed there with group structures, are obviously topological groups. $\mathrm{O}(n)$ is called the *orthogonal group*, $\mathrm{SO}(n)$ - the *special orthogonal group*, $\mathrm{U}(n)$ - the *unitary group*, $\mathrm{SU}(n)$ - the *special unitary group*, and $\mathrm{Sp}(n)$ - the *symplectic group.* $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, and $\mathrm{GL}(n, \mathbb{H})$ are known as the *general linear groups.* It is immediate that $\mathrm{SO}(n)$ is the component of the identity of $\mathrm{O}(n)$, while $\mathrm{GL}_+(n, \mathbb{R})$ is the component of the identity of $\mathrm{GL}(n, \mathbb{R})$.

Remark 4.2.2.9. The topological group $\mathrm{GL}(n,\mathbb{R})$ is manifestly a subgroup of the topological group $\operatorname{Top}\mathbb{R}^n$ (in the sense of Remark 4.2.1.3). In the same sense, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, and $\mathrm{GL}_+(n,\mathbb{R})$ are subgroups of $\operatorname{Top}\mathbb{R}^n$.

Similarly, $\mathrm{GL}(n,\mathbb{C})$ and its subgroups $\mathrm{U}(n)$, $\mathrm{SU}(n)$, are subgroups of $\operatorname{Top}\mathbb{C}^n$, while $\mathrm{GL}(n,\mathbb{H})$ and its subgroup $\mathrm{Sp}(n)$ are subgroups of $\operatorname{Top}\mathbb{H}^n$.

We also note that the inclusions $\mathrm{U}(n) \subset \mathrm{SO}(2n)$, $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$, $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}_+(2n,\mathbb{R})$, and $\mathrm{GL}(n,\mathbb{H}) \subset \mathrm{GL}(2n,\mathbb{C})$ are inclusions of a subgroup into a group in the sense of Remark 4.2.1.3.

4.2.3 Actions

Definition 4.2.3.1. An *action of the group G on a set X* is a map $\mu\colon G\times X \to X$ with the following two properties:

(i) $\mu(e_G, x) = x$;

(ii) if $\mu(g_1, x) = x_1$ and $\mu(g_2, x_1) = x_2$, then $\mu(g_2 g_1, x) = x_2$.

The image of $\mu(g, x)$ under the given action is usually denoted by gx, and so one may write conditions (i), (ii) in the form:

- $e_G x = x$,
- $g_2(g_1 x) = (g_2 g_1)x$.

$$\begin{tikzcd}
G\times G\times X \arrow[r, "(\mathrm{id}_G,\mu)"] \arrow[d, "(\circ_G{,}\mathrm{id}_X)"'] & G\times X \arrow[d, "\mu"] \\
G\times X \arrow[r, "\mu"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
(g_2, g_1, x) \arrow[r, mapsto] \arrow[d, mapsto] & (g_2, g_1 x = x_1) \arrow[d, mapsto] \\
(g_2 g_1, x) \arrow[r, mapsto] & x_2 = (g_2 g_1)x = g_2(g_1 x) = g_2 x_1
\end{tikzcd}$$

Every element $g \in G$ defines a map $X \to X$, $x \mapsto gx$, called the *transformation induced by the element g*. We see from (i) and (ii) that

- this map is invertible (its inverse is the transformation induced by g^{-1}),
- the map $G \to \operatorname{Sym} X$ which takes each g into the corresponding transformation is a homomorphism.

We call it the *adjoint* homomorphism of the given action. Actually, this homomorphism uniquely determines the action, and it is clear that every homomorphism $h\colon G \to \operatorname{Sym} X$ is the adjoint homomorphism of a certain action, namely, of $(g, x) \mapsto (h(g))x$. Therefore, an action of a group G on X can be interpreted as a homomorphism $G \to \operatorname{Sym} X$.

We are mainly interested in the case where the adjoint homomorphism is a monomorphism. An action with this property is said to be *effective.* Generally, the kernel of the adjoint homomorphism will be referred to as the *non-effectiveness kernel* of the given action. If K is this kernel, then we can write the adjoint homomorphism as the composition of the projection $G \to G/K$ with the monomorphism $G/K \to \operatorname{Sym} X$. Moreover, the action itself may be expressed as the composition of the map

$$\operatorname{proj} \times \operatorname{id}_X \colon G \times X \to G/K \times X$$

with the effective action $G/K \times X \to X$. We call the action $G/K \times X \to X$ the *effective factor* of the action $G \times X \to X$.

The image of the set $G \times x$ under a given action $G \times X \to X$ is a subset of X called the *orbit* of the point x. Obviously, the orbits of two points are either identical or disjoint, and hence the orbits partition X. An action with only one orbit is said to be *transitive*. In general, we denote the space of orbits by X/G.

If $h \colon G_1 \to G$ is a group homomorphism, then by composing the mapping $h \times \operatorname{id}_X \colon G_1 \times X \to G \times X$ with an action $G \times X \to X$ of G on X, we obtain an action of G_1 on X. We say that this new action is *induced* by the initial action via the homomorphism h. The adjoint homomorphism of the induced action is simply the composition of h with the adjoint homomorphism of the initial action. If h is a monomorphism (epimorphism), then an effective (respectively, transitive) action induces an effective (respectively, a transitive) one.

If G_1 is a subgroup of G and h is the inclusion of G_1 in G, then we say that the induced action is obtained by *restricting* (or by *reducing*) *the group G to G_1*; we also say that the initial action is obtained by *extending* (or by *prolonging*) *the group G_1 to G*. The discussion above shows that when one restricts the group, an effective action remains effective. Also, when one extends the group, a transitive action remains transitive.

A subset X_1 of X is *invariant under the action $G \times X \to X$* if it is saturated with respect to the partition of X into orbits. If this is the case, then we have an action $G \times X_1 \to X_1$, and clearly this is effective whenever the initial action $G \times X \to X$ is effective.

Given two actions, $G_1 \times X_1 \to X_1$ and $G_2 \times X_2 \to X_2$, their *product* is the action defined by

$$(G_1 \times G_2) \times (X_1 \times X_2) \to X_1 \times X_2, \quad (g_1, g_2)(x_1, x_2) = (g_1 x_1, g_2 x_2).$$

A product of effective (transitive) actions is again effective (respectively, transitive).

Given two actions, $G \times X \to X$ and $G \times X' \to X'$, of the same group, a map $f \colon X \to X'$ is a *G-map* (or a *G-equivariant map*) if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$.

$$\begin{tikzcd}
G \times X \arrow[r] \arrow[d, "(\mathrm{id}_G, f)"'] & X \arrow[d, "f"] \\
G \times X' \arrow[r] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
(g, x) \arrow[r, mapsto] \arrow[d, mapsto] & gx \arrow[d, mapsto] \\
(g, f(x)) \arrow[r, mapsto] & g(fx) = f(gx)
\end{tikzcd}$$

We can describe the more general situation when we are given two actions, $G \times X \to X$ and $G' \times X' \to X'1$, of different groups, and a homomorphism

$\gamma\colon G \to G'$; then $f\colon X \to X'$ is called a *γ-map* if $f(gx) = \gamma(g)f(x)$ for all $x \in X$ and $g \in G$.

$$\begin{tikzcd}
G \times X \arrow[r] \arrow[d, "{(\gamma,f)}"'] & X \arrow[d, "f"] & (g,x) \arrow[r, mapsto] \arrow[d, mapsto] & gx \arrow[d, mapsto] \\
G' \times X' \arrow[r] & X' & (\gamma(g), f(x)) \arrow[r, mapsto] & \gamma(g)(f(x)) = f(gx)
\end{tikzcd}$$

Two actions, $G \times X \to X$ and $G \times X' \to X'$, are *equivalent* if there is an invertible G-map $X \to X'$.

Remark 4.2.3.2. The action defined in Definition 4.2.3.1 should actually be called a *left action*, to distinguish it from a *right action*, which is defined as a map $X \times G \to X$ with the following two properties:

(i) $\mu(x, e_G) = x$;

(ii) if $\mu(x, g_1) = x_1$ and $\mu(x_1, g_2) = x_2$, then $\mu(x, g_1g_2) = x_2$.

For a right action, we write xg instead of gx, and properties (i), (ii) in the form:

- $xe_G = x$,
- $(xg_1)g_2 = x(g_1g_2)$.

$$\begin{tikzcd}
X \times G \times G \arrow[r, "{(\mu,\mathrm{id}_G)}"] \arrow[d, "{(\mathrm{id}_X, \circ_G)}"'] & X \times G \arrow[d, "\mu"] & (x, g_1, g_2) \arrow[r, mapsto] \arrow[d, mapsto] & (xg_1 = x_1, g_2) \arrow[d, mapsto] \\
X \times G \arrow[r, "\mu"'] & X & (x, g_1g_2) \arrow[r, mapsto] & x_2 = x(g_1g_2) = (xg_1)g_2 = x_1g_2
\end{tikzcd}$$

Furthermore, the adjoint homomorphism becomes the adjoint anti-homomorphism, and the rest of the discussion in Definition 4.2.3.1 can be repeated word for word for a right action.

It is clear that the formula $xg = g^{-1}x$ transforms a left action into a right one, and that the formula $gx = xg^{-1}$ yields the inverse transformation. We say that the actions thus related are *conjugate*.

Henceforth, by *action* we shall mean a left action, unless we mention explicitly that we are dealing with a right action.

Remark 4.2.3.3. If the transformations induced by the elements of the group G, acting on X from the left (right), are elements of a subgroup H of $\operatorname{Sym} X$, then the action (respectively, right action) of G can be thought of as a homomorphism (respectively, anti-homomorphism) of G into H. In this case we say that G acts on X (respectively, acts from the right on X) *by transformations* from H.

The group H is not always indicated explicitly. For example, if X is a topological space and $H = \operatorname{Top} X$, then one simply says that G acts on X (acts on X from the right) by homeomorphisms. Similarly, if X is a group and H is a group of automorphisms of X, then one says that G acts on X (respectively, acts on X from the right) by automorphisms; in this case, the action itself will be referred to as a *group-action*.

Important special examples are the actions $G \times G \to G$ given by $(g, x) \mapsto gx$ or $(g, x) \mapsto gxg^{-1}$, and the right actions $G \times F \to G$ given by $(x, g) \mapsto xg$ and $(x, g) \mapsto g^{-1}xg$ (here all the products are taken in G). These are called, in order: the left *canonical action*, the *left inner action*, the *right canonical action*, and the *right inner action*. The canonical actions are effective and transitive, while the inner actions are group-actions.

Let us remark that by restricting the left canonical action $G \times G \to G$ to the action $G_1 \times G \to G$, where G_1 is a subgroup of G, the orbits become the (right) cosets of G_1. Thus, the two interpretations (the usual group-theoretic one, and that given in Definition 4.2.3.1) of the notation G/G_1 agree (if we denote the right coset space also by G/G_1).

Definition 4.2.3.4. The following generalisation of the left canonical action is already of general importance. Let G_1 be a subgroup of G. Since every left translation takes left cosets into left cosets, the map $(g, x) \mapsto gx$ induces a map $G \times G/G_1 \to G/G_1$, and this is clearly an action, called the *canonical action of the group G on G/G_1* . It is transitive, and its non-effectiveness kernel is the intersection of all subgroups of G which are conjugate to G_1. The projection $G \to G/G_1$ is a G-map with respect to the left canonical action of G on G and the canonical action of G on G/G_1.

It turns out that every transitive action of G is equivalent to the canonical action on some quotient G/G_1. Specifically, let $G \times X \to X$ be a transitive action, and let $x_1 \in X$ be an arbitrarily chosen point. Consider the map $f\colon G \to X$, $g \mapsto gx_1$. The preimage of x_1 under f is a subgroup G_1 of G, while the preimages under f of points $x \in X$, $x \neq x_1$, are left cosets of G_1. It is routine to check that the injective factor of f, $\operatorname{fact} f\colon G/G_1 \to X$, is a G-map.

The subgroup G_1 has a special name: it is known as the *isotropy* (or *stability*, or *stationary*) *subgroup of the action $G \times X \to X$* , or *of the group G, at the point x_1*. Obviously, the isotropy subgroup at the point gx_1 is gG_1g^{-1}, and so the isotropy subgroups of a transitive action of G constitute exactly one of its classes of conjugate subgroups.

Continuous Actions

Definition 4.2.3.5. A continuous action of a topological group G on a topological space X is a continuous map $G \times X \to X$ which is an action in the sense of Definition 4.2.3.1.

For a continuous action, the transformations induced by the elements of the group are manifestly homeomorphisms. Therefore, the adjoint homomorphism of a continuous action $G \times X \to X$ can be compressed to an algebraic isomorphism $G \to \operatorname{Top} X$. By Theorem 1.2.7.6, this last homomorphism is continuous in the first topology of the group $\operatorname{Top} X$ (see Remark 4.2.2.1). If, in addition, X is Hausdorff and locally compact, then the existence of a continuous (in the first topology on $\operatorname{Top} X$) compression $G \to \operatorname{Top} X$ of the adjoint homomorphism of the given action is equivalent to the continuity of the action. It is clear that an algebraic homomorphism $G \to \operatorname{Top} X$ which is continuous in one of the two topologies of $\operatorname{Top} X$ is continuous also in the other one, and we know that if X is locally compact and Hausdorff, then $\operatorname{Top} X$ with the second topology is a topological group (see Theorem 4.2.2.3). Therefore,

Theorem 4.2.3.6. *if X is a locally compact Hausdorff space, then a continuous action of G on X may be defined as a homomorphism $G \to \operatorname{Top} X$ of topological groups (see Remark 4.2.1.8).*

A discrete group which acts by homeomorphisms always acts continuously. Thus, we may regard the actions of non-topologised groups which act by homeomorphisms as continuous actions.

Definition 4.2.3.7. A *G-space* is a topological space endowed with a continuous action of the group G. A G-space is called *effective* if the action of G is effective. In the general case, by shifting to the effective factor of the action of G, the given G-space becomes an effective (G/K)-space, where K is the non-effectiveness kernel.

When the action $G \times X \to X$ is continuous, X/G is a topological space (a quotient space of X), known as an *orbit space*. Since the saturation of any subset $A \subset X$ with respect to the partition of X into orbits is the union of the sets gA, $g \in G$, this partition is always open. When the group G is finite, the partition into orbits is also closed. In particular, X/G is second countable together with X, and when G is finite, X/G is normal together with X; see Remark 1.2.3.10 and Theorem 1.2.3.9. Let us add that the partition into orbits is again closed whenever X is compact and Hausdorff and G is compact. Indeed, in this case the action $G \times X \to X$ is a closed map, and since it transforms every product $G \times A$ into the saturation of the set A, this saturation is closed whenever A is closed.

By restricting the topological group G to a subgroup G_1, we transform every G-space into a G_1-space. Any invariant subspace of a G-space is obviously a G-space; the G-spaces of this type are termed *subspaces* of the initial G-space. The product of two continuous actions is continuous, and hence the product of a G_1-space with a G_2-space is a $(G_1 \times G_2)$-space.

One can define the notions of G-map and γ-map for the case of continuous actions. To spell it out, a *G-map* (or a *G-equivariant map*) of a G-space into another G-space is any continuous map which is a G-map in the sense of Definition 4.2.3.1; similarly, a *γ-map of a G-space into a G'-space* is any continuous map which is also a γ-map in the sense of Definition 4.2.3.1 (here $\gamma\colon G \to G'$ is a homomorphism of topological groups). Two continuous actions of G are *equivalent* if the corresponding G-spaces are G-homeomorphic.

When G is a topological group, the special actions introduced in Remark 4.2.3.3, i.e., the left canonical and left inner actions, are continuous.

As in Remark 4.2.3.3, the notation G/G_1 can be interpreted in two ways (as the space of cosets of G_1 in G and as an orbit space), but again the two interpretations agree.

Theorem 4.2.3.8. *The canonical action of a group G on the space G/G_1 of cosets of a subgroup G_1 is also continuous.*

Proof. To see this, consider the composition $\psi\colon G \times G \to G/G_1$ of the map $G \times G \to G$, $(g, h) \mapsto gh$, with the projection $G \to G/G_1$. Clearly, ψ is constant on the elements of the partition $\operatorname{zer}(G, e_G) \times \operatorname{zer}(G, G_1))$. Furthermore, the action $G \times G/G_1 \to G/G_1$ in which we are interested is the composition of the map

$$G \times G/G_1 \to +(G \times G)/(\operatorname{zer}(G, e_G) \times \operatorname{zer}(G, G_1)),$$

given by the inverse of the injective factor of $\mathrm{id}_G \times \mathrm{proj}\colon G\times \to G \times G/G_1$, with

$$\operatorname{fact}\psi\colon (G \times G)/(\operatorname{zer}(G, e_G) \times \operatorname{zer}(G, G_1)) \to G/G_1.$$

Since the partitions $(\operatorname{zer}(G, e_G)$ and $\operatorname{zer}(G, G_1)$ are open, the above injective factor is a homeomorphism (see Theorem 1.2.3.11), which in turn implies the continuity of the action of G on G/G_1. ☐

Therefore,

Definition 4.2.3.9. G/G_1 becomes a G-space with respect to the canonical action $G \times G/G_1 \to G/G_1$, and is called a *homogeneous space*.

The equivalence between an arbitrary transitive action $G \times X \to X$ and the canonical action of G on G/G_1, where G_1 is an isotropy subgroup, is not complete in the case of continuous actions. More precisely, the map $f\colon G \to X$, $g \mapsto= gx_1$, and its injective factor, $\operatorname{fact} f\colon G/G_1 \to X$, are indeed continuous, but as the next example (provided by the translator) shows that $(\operatorname{fact} f)^{-1}$ is not necessarily continuous.

Example 4.2.3.10 ((An “irrational flow” on the torus)). Think of $\mathbb{S}^1$ as the set of complex numbers of modulus 1 and let α be irrational, i.e, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Take $G = \mathbb{R}$, $X = \mathbb{S}^1 \times \mathbb{S}^1$, and define

$$G \times X \to X, \quad (t, (\exp 2\pi x, \exp 2\pi y)) \mapsto (\exp 2\pi(x + t), \exp 2\pi(y + \alpha t)),$$

However, $(\operatorname{fact} f)^{-1}$ is continuous provided that G is compact and X is Hausdorff, i.e.,

Theorem 4.2.3.11. *every transitive continuous action of a compact topological group on a Hausdorff topological space with a distinguished point is canonically equivalent to the canonical action of the group on the space of cosets of the isotropy group at the distinguished point.*

Remark 4.2.3.12. A continuous action $G \times X \to X$ is *free* if for every point $x \in X$ the map $G \to X$ given by $g \mapsto gx$ is an embedding.

Every free action is clearly effective. Moreover, if the action $G \times X \to X$ is free, then by restricting G to one of its subgroups, or by restricting X to one of its G-subspaces, the action remains free. A product of free actions is free. The canonical action $G \times G \to G$ is free, while the canonical action $G \times G/G_1 \to G/G_1$ is not free unless $G = e_G$.

Given a free action $G \times X \to X$, consider the bundle $(X, \mathrm{proj}, X/G)$. If every point $x \in X$ has a neighbourhood U such that $gU \cap g'U = \varnothing$ for all $g, g' \in G$ with $g \neq g'$ (which happens, in particular, when G is finite and X is Hausdorff), then the image of U under the projection $\mathrm{proj}\colon X \to X/G$ is open, and the restriction of the bundle $(X, \mathrm{proj}, X/G)$ to $\mathrm{proj}(U)$ is a trivial bundle with discrete fibres. Therefore, in this case $(X, \mathrm{proj}, X/G)$ is a covering in the broad sense.

Definition 4.2.3.13. A continuous *right* action of a topological group G on a topological space X is a continuous map $X \times G \to X$ which is also a right action in the sense of Remark 4.2.3.2.

All definitions and facts discussed in Definition 4.2.3.5, Theorem 4.2.3.6, Definition 4.2.3.7, and Remark 4.2.3.12 can be adapted immediately to the case of right actions. In particular, a topological space endowed with a right continuous action is called a *right G-space*. We keep the simple term G-space *only* for *left* G-spaces.

Examples

Remark 4.2.3.14. Let X be a topological space. The identity homomorphism

$$\operatorname{Top} X \to \operatorname{Top} X$$

defines an effective action of $\operatorname{Top} X$ on X, and hence an effective action of any subgroup of $\operatorname{Top} X$ on X. If X is Hausdorff and locally compact, then $\operatorname{Top} X$ is a topological group and all these actions are continuous. In particular, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}_+(n,\mathbb{R})$, $\mathrm{O}(n)$, and $\mathrm{SO}(n)$ act effectively and continuously on $\mathbb{R}^n$, while $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ act the same manner on $\mathbb{C}^n$, and $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{Sp}(n)$ on - $\mathbb{H}^n$.

Given an arbitrary $\mathcal{C}^{\geq r}$-manifold X with $r \leq \infty$, the inclusion

$$\operatorname{Diff}^r X \to \operatorname{Top} X$$

defines an effective and continuous action of $\operatorname{Diff}^r X$ on X.

Remark 4.2.3.15. Since $\mathbb{S}^{n-1}$ is invariant under the action of $\mathrm{O}(n)$ on $\mathbb{R}^n$, $\mathrm{O}(n)$ and its subgroup $\mathrm{SO}(n)$ act continuously on $\mathbb{S}^{n-1}$ Similarly, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, being subgroups of $\mathrm{O}(2n)$, act continuously on $\mathbb{S}^{2n-1}$, while $\mathrm{Sp}(n)$, being a subgroup of $\mathrm{O}(4n)$, acts continuously on $\mathbb{S}^{4n-1}$. All these actions are effective, and, if we exclude the trivial cases $\mathrm{SO}(1) \times \mathbb{S}^0 \to \mathbb{S}^0$ and $\mathrm{SU}(1) \times \mathbb{S}^1 \to \mathbb{S}^1$, transitive.

The isotropy subgroups of the actions

$$\mathrm{O}(n) \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1} \quad \text{and} \quad \mathrm{SO}(n) \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$$

at ort_n are exactly $\mathrm{O}(n-1)$ and $\mathrm{SO}(n-1)$. Similarly, the isotropy subgroups of the actions

$$\mathrm{U}(n) \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1} \quad \text{and} \quad \mathrm{SU}(n) \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$$

at ort_{2n} are $\mathrm{U}(n-1)$ and $\mathrm{SU}(n-1)$, while the isotropy subgroup of the action $\mathrm{Sp}(n) \times \mathbb{S}^{4n-1} \to \mathbb{S}^{4n-1}$ at ort_{4n} is $\mathrm{Sp}(n-1)$. The corresponding homeomorphisms,

$$\mathrm{O}(n)/\mathrm{O}(n-1) \to \mathbb{S}^{n-1}, \quad \mathrm{SO}(n)/\mathrm{SO}(n-1) \to \mathbb{S}^{n-1}, \quad \mathrm{U}(n)/\mathrm{U}(n-1) \to \mathbb{S}^{2n-1},$$
$$\mathrm{SU}(n)/\mathrm{SU}(n-1) \to \mathbb{S}^{2n-1}, \quad \mathrm{Sp}(n)/\mathrm{Sp}(n-1) \to \mathbb{S}^{4n-1}$$

(see Theorem 4.2.3.8 and Definition 4.2.3.9) equal the injective factors of the submersions

$$V(n,n)[=\mathrm{O}(n)] \to V(n,1)[=\mathbb{S}^{n-1}], \quad V(n,n-1) \to V(n,1),$$
$$\mathbb{C}V(n,n) \to \mathbb{C}V(n,1), \quad \mathbb{C}V(n,n-1) \to \mathbb{C}V(n,1), \quad \mathbb{H}V(n,n) \to \mathbb{H}V(n,1),$$

defined in Subsection 3.2.1 (see Remarks 3.2.1.4, 3.2.1.6, and 3.2.1.7).

If we restrict $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$ ($n \geq 1$) to their subgroups which consists of scalar multiples of the identity matrix, and which are usually identified with $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$, respectively, we obtain continuous actions

$$\mathbb{S}^0 \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad \mathbb{S}^1 \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}, \quad \mathbb{S}^3 \times \mathbb{S}^{4n-1} \to \mathbb{S}^{4n-1}.$$

These are free actions, and the corresponding orbit spaces are

$$\mathbb{S}^{n-1}/\mathbb{S}^0 = \mathbb{R}P^{n-1}, \quad \mathbb{S}^{2n-1}/\mathbb{S}^1 = \mathbb{C}P^{n-1}, \quad \mathbb{S}^{4n-1}/\mathbb{S}^3 = \mathbb{H}P^{n-1}.$$

We remark also that $\mathbb{D}^n$, $\mathbb{D}^{2n}$, and $\mathbb{D}^{4n}$ are invariant under the actions of $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$ on $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$. Hence $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ act continuously on $\mathbb{D}^n$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ act continuously on $\mathbb{D}^{2n}$, and $\mathrm{Sp}(n)$ acts continuously on $\mathbb{D}^{4n}$. All these actions are effective.

Remark 4.2.3.16. The groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ ($\mathrm{O}(k)$ and $\mathrm{SO}(k)$) act continuously from the left (right) on the Stiefel manifolds $V(n,k)$: the left actions are defined by $(g,v) \mapsto g \circ v$ [$g \in \mathrm{O}(n)$ or $\mathrm{SO}(n)$, $v \in V(n,k)$; g and v are regarded as linear maps]; the right actions are given by $(v,g) \mapsto v \circ g$. Similarly, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ ($\mathrm{U}(k)$ and $\mathrm{SU}(k)$) act continuously from the left (respectively, from the right) on $\mathcal{C}V(n,k)$, and $\mathrm{Sp}(n)$ ($\mathrm{Sp}(k)$) acts continuously from the left (respectively, from the right) on $\mathbb{H}V(n,k)$.

For $k \neq 0$, all the left actions are effective, and the only intransitive ones are

$$\mathrm{SO}(n) \times V(n,n) \to V(n,n) \quad \text{and} \quad \mathrm{SU}(n) \times \mathbb{C}V(n,n) \to \mathbb{C}V(n,n), \quad n \geq 1.$$

The isotropy subgroups of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ at the point

$$[(x_1,\ldots,x_k) \mapsto (0,\ldots,0,x_1,\ldots,x_k)] \in V(n,k)$$

(the elements of $V(n,k)$ are considered as linear isometric maps $\mathbb{R}^k \to \mathbb{R}^n$) coincide with $\mathrm{O}(n-k)$ and $\mathrm{SO}(n-k)$. Similarly, the isotropy subgroups of $\mathrm{U}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$ at the points

$$[(x_1,\ldots,x_k) \mapsto (0,\ldots,0,x_1,\ldots,x_k)] \quad \text{of} \quad \mathbb{C}V(n,k) \quad \text{and} \quad \mathbb{H}V(n,k)$$

coincide with $\mathrm{U}(n-k)$, $\mathrm{SU}(n-k)$, and $\mathrm{Sp}(n-k)$, respectively. The corresponding homeomorphisms

$$\begin{aligned}&\mathrm{O}(n)/\,\mathrm{O}(n-k) \to V(n,k), \quad \mathrm{SO}(n)/\,\mathrm{SO}(n-k) \to V(n,k),\\ &\mathrm{U}(n)/\,\mathrm{U}(n-k) \to \mathbb{C}V(n,k), \quad \mathrm{SU}(n)/\,\mathrm{SU}(n-k) \to \mathbb{C}V(n,k),\\ &\mathrm{Sp}(n)/\,\mathrm{Sp}(n-k) \to \mathbb{H}V(n,k),\end{aligned}$$

are precisely the injective factors of the maps

$$\begin{aligned}&\mathrm{O}(n) \to V(n,k), \quad \mathrm{SO}(n) \to V(n,k), \quad \mathrm{U}(n) \to \mathbb{C}V(n,k),\\ &\mathrm{SU}(n) \to \mathbb{C}V(n,k), \quad \mathrm{Sp}(n) \to \mathbb{H}V(n,k),\end{aligned}$$

defined in Subsection 3.2.1 (see Remarks 3.2.1.4, 3.2.1.6, and 3.2.1.7). When $k = 1$, these actions reduce to those discussed in Remark 4.2.3.15.

All the right actions are free. The corresponding orbit spaces,

$$\begin{aligned}&V(n,k)/\,\mathrm{O}(k), \quad V(n,k)/\,\mathrm{SO}(k), \quad \mathbb{C}V(n,k)/\,\mathrm{U}(k), \quad \mathbb{C}V(n,k)/\,\mathrm{SU}(k),\\ &\mathbb{H}V(n,k)/\,\mathrm{Sp}(k),\end{aligned}$$

are canonically homeomorphic to the Grassmann manifolds $G(n,k)$, $G_+(n,k)$, $\mathbb{C}G(n,k)$, and $\mathbb{H}G(n,k)$, respectively; the corresponding canonical homeomorphisms are the injective factors of the maps

$$\begin{aligned}&V(n,k) \to G(n,k), \quad V(n,k) \to G_+(n,k), \quad \mathbb{C}V(n,k) \to \mathbb{C}G(n,k),\\ &\mathbb{H}V(n,k) \to \mathbb{H}G(n,k),\end{aligned}$$

defined in Subsection 3.2.2 (see Remarks 3.2.2.3, 3.2.2.7, and 3.2.2.8).

Remark 4.2.3.17. The same formulae, i.e., $(g,v) \mapsto g \circ v$, $(g,v) \mapsto v \circ g$, define left actions of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}_+(n,\mathbb{R})$ on $V'(n,k)$, of $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}V'(n,k)$, and of $\mathrm{GL}(n,\mathbb{H})$ on $\mathbb{H}V'(n,k)$, and right actions of $\mathrm{GL}(k,\mathbb{R})$ and $\mathrm{GL}_+(k,\mathbb{R})$ on $V'(n,k)$, of $\mathrm{GL}(k,\mathbb{C})$ on $\mathbb{C}V'(n,k)$ and of $\mathrm{GL}(k,\mathbb{H})$ on $\mathbb{H}V'(n,k)$.

All the left actions are effective and, excepting the action

$$\mathrm{GL}_+(n,\mathbb{R}) \times V'(n,n) \to V'(n,n),$$

transitive. The isotropy subgroups of $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, and $\mathrm{GL}(n,\mathbb{H})$ at the points

$$[(x_1,\dots,x_k) \mapsto (0,\dots,0,x_1,\dots,x_k)]$$

of $V'(n,k)$, $\mathbb{C}V'(n,k)$, and $\mathbb{H}V'(n,k)$ are

$$\mathrm{GL}(n-k,\mathbb{R}), \quad \mathrm{GL}(n-k,\mathbb{C}), \quad \mathrm{GL}(n-k,\mathbb{H}),$$

respectively. The corresponding homeomorphisms

$$\begin{aligned}&\mathrm{GL}(n,\mathbb{R})/\,\mathrm{GL}(n-k,\mathbb{R}) \to V'(n,k), \quad \mathrm{GL}(n,\mathbb{C})/\,\mathrm{GL}(n-k,\mathbb{C}) \to \mathbb{C}V'(n,k),\\ &\mathrm{GL}(n,\mathbb{H})/\,\mathrm{GL}(n-k,\mathbb{H}) \to \mathbb{H}V'(n,k)\end{aligned}$$

are the injective factors of the maps

$$\mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}V'(n,k), \quad \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}V'(n,k), \quad \mathrm{GL}(n,\mathbb{H}) \to \mathbb{H}V'(n,k),$$

defined in Subsection 3.2.1 (see Remarks 3.2.1.8, 3.2.1.9, and 3.2.1.10). The isotropy subgroup of $\mathrm{GL}_+(n,\mathbb{R})$ at the point

$$[(x_1,\dots,x_k) \mapsto (0,\dots,0,x_1,\dots,x_k)] \in V'(n,k)$$

is $\mathrm{GL}_+(n-k,\mathbb{R})$.

All the right actions are free. The corresponding orbit spaces,

$$V'(n,k)/\mathrm{GL}(k,\mathbb{R}), \quad V'(n,k)/\mathrm{GL}_+(k,\mathbb{R}), \quad \mathbb{C}V'(n,k)/\mathrm{GL}(k,\mathbb{C}),$$
$$\mathbb{H}V'(n,k)/\mathrm{GL}(k,\mathbb{H}),$$

are canonically homeomorphic to the Grassmann manifolds $G(n,k)$, $G_+(n,k)$, $\mathbb{C}G(n,k)$, and $\mathbb{H}G(n,k)$; the corresponding canonical homeomorphisms are the injective factors of the maps

$$V'(n,k) \to G(n,k), \quad V'(n,k) \to G_+(n,k), \quad \mathbb{C}V'(n,k) \to \mathbb{C}G(n,k),$$
$$\mathbb{H}V'(n,k) \to \mathbb{H}G(n,k),$$

defined in Subsection 3.2.2 (see Remarks 3.2.2.3, 3.2.2.7, and 3.2.2.8).

Remark 4.2.3.18. $\mathrm{GL}(n,\mathbb{R})$ and its subgroups $\mathrm{GL}_+(n,\mathbb{R})$, $\mathrm{O}(n)$, and $\mathrm{SO}(n)$ obviously act continuously from the left on the Grassmann manifolds $G(n,k)$, $G_+(n,k)$. Similarly, $\mathrm{GL}(n,\mathbb{C})$ and its subgroups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ act continuously from the left on $\mathbb{C}G(n,k)$, while $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{Sp}(n)$ act continuously from the left on $\mathbb{H}G(n,k)$. For k odd, the actions of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ on $G_+(n,k)$ are effective. The non-effectiveness kernels of the actions $\mathrm{GL}(n,\mathbb{R}) \times G_+(n,k) \to G_+(n,k)$ and $\mathrm{GL}_+(n,k) \times G_+(n,k) \to G_+(n,k)$ for k odd consist of scalar matrices with positive diagonal elements. If we exclude the trivial cases $k = 0$ and $k = n$, the non-effectiveness kernels of the remaining actions consists of all scalar matrices contained in the corresponding group. The only intransitive actions are

$$\mathrm{GL}(n,\mathbb{R}) \times G_+(n,0) \to G_+(n,0), \quad \mathrm{GL}_+(n,\mathbb{R}) \times G_+(n,0) \to G_+(n,0),$$
$$\mathrm{O}(n) \times G_+(n,0) \to G_+(n,0), \quad \mathrm{SO}(n) \times G_+(n,0) \to G_+(n,0),$$
$$\mathrm{GL}_+(n,\mathbb{R}) \times G_+(n,n) \to G_+(n,n), \quad \mathrm{SO}(n) \times G_+(n,n) \to G_+(n,n).$$

Take the + plane $x_1 = 0,\dots,x_{n-k} = 0$ (oriented in the case of $G_+(n,k)$) as a distinguished point in the manifolds $G(n,k)$, $G(n,k)$, $\mathbb{C}G(n,k)$, and $\mathbb{H}G(n,k)$. Then the isotropy subgroups of the actions

$$\mathrm{GL}(n,\mathbb{R}) \times G(n,k) \to G(n,k), \quad \mathrm{GL}(n,\mathbb{R}) \times G(n,k) \to G_+(n,k),$$
$$\mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}G(n,k) \to \mathbb{C}G(n,k), \quad \mathrm{GL}(n,\mathbb{H}) \times \mathbb{H}G(n,k) \to \mathbb{H}G(n,k)$$

at these distinguished points are the subgroups of all matrices of the form

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where A and B are non-singular matrices of order $n-k$ and k, respectively, and C is an arbitrary $(n-k)\times k$ matrix (and $B \in \mathrm{GL}_+(k,\mathbb{R})$ in the case of $G(n,k)$).

If we restrict the acting group to a subgroup, then the new isotropy subgroup is the intersection of the original isotropy subgroup with the new acting group. In particular, for the actions of $\mathrm{O}(n)$ on $G(n,k)$ and $G_+(n,k)$, the action of $\mathrm{SO}(n)$ on $G_+(n,k)$, the action of $\mathrm{U}(n)$ on $\mathbb{C}G(n,k)$, and the action of $\mathrm{Sp}(n)$ on $\mathbb{H}G(n,k)$, the corresponding isotropy subgroups are the images of the monomorphisms

$$\begin{aligned} &\mathrm{O}(n-k)\times\mathrm{O}(k)\to\mathrm{O}(n), \quad \mathrm{O}(n-k)\times\mathrm{SO}(k)\to\mathrm{O}(n), \\ &\mathrm{SO}(n-k)\times\mathrm{SO}(k)\to\mathrm{SO}(n), \quad \mathrm{U}(n-k)\times\mathrm{U}(k)\to\mathrm{U}(n), \\ &\mathrm{Sp}(n-k)\times\mathrm{Sp}(k)\to\mathrm{Sp}(n), \end{aligned}$$

all defined by the matrix formula

$$(A,B)\to\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

If we identify these product with their images, we obtain canonical homeomorphisms

$$\begin{aligned} &\mathrm{O}(n)/[\mathrm{O}(n-k)\times\mathrm{O}(k)]\to G(n,k), \quad \mathrm{O}(n)/[\mathrm{O}(n-k)\times\mathrm{SO}(k)]\to G_+(n,k), \\ &\mathrm{SO}(n)/[\mathrm{SO}(n-k)\times\mathrm{SO}(k)]\to G_+(n,k), \quad \mathrm{U}(n)/[\mathrm{U}(n-k)\times U(k)]\to G(n,k), \\ &\mathrm{Sp}(n)/[\mathrm{Sp}(n-k)\times\mathrm{Sp}(k)]\to\mathbb{H}G(n,k). \end{aligned}$$

Remark 4.2.3.19. Let m, $\ell_1,\dots,\ell_n$ be relatively prime positive integers. The complex-number formula

$$\begin{aligned} &(k,(z_1,\dots,z_n))\mapsto(z_1\exp(2\pi ik\ell_1/m),\dots,z_n\exp(2\pi ik\ell_n/m)), \\ &k\in\mathbb{Z}, \quad (z_1,\dots,z_n)\in\mathbb{S}^{2n-1} \end{aligned}$$

defines an action $\mathbb{Z}\times\mathbb{S}^{2n-1}\to\mathbb{S}^{2n-1}$ with non-effectiveness kernel $m\mathbb{Z}$, which becomes, by shifting to the effective factor, a free action of the group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. The orbit space $\mathbb{S}^{2n-1}/(\mathbb{Z}/m\mathbb{Z}$ is denoted by $L(m;\ell_1,\dots,\ell_n)$ and is called a *lens* (or a *lens space*).

There are also *infinite lenses* $L(m;\ell_1,\ell_2,\dots)$, with m, $\ell_q,\ell_2,\dots$ relatively prime positive integers. The lens $L(m;\ell_1,\ell_2,\dots)$ defined as the orbit space of the free action resulting from passing to the effective factor of the action

$$(k,(z_1,\dots,))\mapsto(z_1\exp(2\pi ik\ell_1/m),z_2\exp(2\pi ik\ell_2/m),\dots)$$

of on $\mathbb{S}^\infty$. An equivalent description:

$$\begin{aligned} &L(m;\ell_1,\ell_2,\dots)=\varinjlim_{\mathrm{incl}}(L(m;\ell_1,\dots,\ell_n)), \\ &\mathrm{incl}\colon L(m;\ell_1,\dots,\ell_n)\to L(m;\ell_1,\dots,\ell_{n+1}). \end{aligned}$$

The infinite lens $L(m; 1, 1, \dots)$ is denoted simply by $L(m)$.

According to Remark 4.2.3.12, the triples

$$(\mathbb{S}^{2n-1}, \mathrm{proj}, L(m; \ell_1, \dots, \ell_n)), \quad (\mathbb{S}^\infty, \mathrm{proj}, L(m; \ell_1, \ell_2, \dots))$$

are coverings.

Remark 4.2.3.20. The formula $(y, x) \mapsto yxy^{-1}$, where x and y are quaternions and y has norm 1, defines a continuous action $\mathbb{S}^3 \times \mathbb{R}^4 \to \mathbb{R}^4$. The space $\mathbb{R}^3_1$ of imaginary quaternions is invariant under this action, and hence $\mathbb{R}^3_1$, and also $\mathbb{R}^3$, are ^{3}-spaces. [We identify $\mathbb{R}^3_1$ with $\mathbb{R}^3$ via the map shi: $\mathbb{R}^3 \to \mathbb{R}^3_1$; see Theorem 3.2.3.1.] The non-effectiveness kernel of the action $\mathbb{S}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is obviously $\mathbb{S}^0$, and now it is clear that the effective action of the factor group $\mathbb{S}^3/\mathbb{S}^0 = \mathbb{R}P^3$ on $\mathbb{R}^3$ becomes the standard action of SO(3) on $\mathbb{R}^3$ (see Remark 4.2.3.15) under the canonical identification of the spaces $\mathbb{R}P^3$ and SO(3) (see Theorem 3.2.3.1).

Example 4.2.3.21. Let P be a convex regular polyhedron in $\mathbb{R}^3$ (a tetrahedron, cube, octahedron, dodecahedron, or icosahedron) with centre 0. Let GP be the subgroup of SO(3) consisting of those rotations which take P into itself, and let $\widetilde{GP}$ be the preimage of GP under the projection $\mathbb{S}^3 \to \mathrm{SO}(3)$ (see Remark 4.2.3.20). Obviously, GP and $\widetilde{GP}$ do not change if we replace P by the dual polyhedron, while they are transformed into conjugate subgroups of SO(3) and if we replace P by any convex regular polyhedron with the same number of faces and centre 0. Therefore, in SO(3) ($\mathbb{S}^3$) there are exactly three classes of conjugate subgroups GP (respectively, $\widetilde{GP}$). The groups in the first class are called *tetrahedral groups* (respectively, *binary tetrahedral groups*), those in the second class - *cube* or *octahedral groups* (respectively, *binary cube* or *octahedral groups*), and those in the third class - *dodecahedral* or *icosahedral groups* (respectively, *binary dodecahedral* and *icosahedral groups*).

To every rotation in GP we may associate the image of a marked oriented edge of the polyhedron P, and in this way define an invertible mapping of the group GP onto the set of oriented edges of P. Consequently, the order of the group GP is twice the number of edges of P, i.e., 12 when P is a tetrahedron, 24 when P is a cube or octahedron, and 60 when P is a dodecahedron or an icosahedron. The corresponding binary groups $\widetilde{GP}$ have order 24, 48, and 120.

The coset spaces $\mathrm{SO}(3)/GP$ and $\mathbb{S}^3/\widetilde{GP}$ are orbit spaces of the free actions induced by the left canonical actions of SO(3) and $\mathbb{S}^3$ under the inclusions $GP \to \mathrm{SO}(3)$ and $\widetilde{GP} \to \mathbb{S}^3$. Therefore, the triples $(\mathrm{SO}(3), \mathrm{proj}, \mathrm{SO}(3)/GP)$ and $(\mathbb{S}^3, \mathrm{proj}, \mathbb{S}^3/\widetilde{GP})$ are coverings (see Remark 4.2.3.12). Obviously, we can write $\mathrm{SO}(3)/GP = \mathbb{S}^3/\widetilde{GP}$.

4.2.4 Exercises

Exercise 4.2.4.1. Show that for any smooth manifold X the first and the second topologies on Top X coincide.

Exercise 4.2.4.2. Let X denote the subset of $\mathbb{R}$ consisting of the points 0 and 2^n, all $n \in \mathbb{Z}$. Show that the first and the second topologies on Top X are distinct.

Exercise 4.2.4.3. Show that the canonical diffeomorphism $\mathrm{SU}(2) \to \mathbb{S}^3$ (see Remark 3.2.1.6) is a group isomorphism.

Exercise 4.2.4.4. Show that the lenses $L(m; \ell_1, \dots, \ell_k)$ and $L(m; \ell_1', \dots, \ell_k')$ are homeomorphic whenever for each i the sum $\ell_i + \ell_i'$ or the difference $\ell_i - \ell_i'$ is a multiple of m.

Exercise 4.2.4.5. Show that the submanifold $\operatorname{Tang}_1 \mathbb{R}P^2$ of $\operatorname{Tang} \mathbb{R}P^2$ consisting of the unit tangent vectors (i.e., of the images under the map $d_{\mathrm{proj}} \colon \operatorname{Tang} \mathbb{S}^2 \to \operatorname{Tang} \mathbb{R}P$ of the unit tangent vectors) is homeomorphic to the lens $L(4; 1, 1)$.

Exercise 4.2.4.6. Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on the manifold $V(3, 2)$ of unit vectors tangent to $\mathbb{S}^2$, where the non-zero element of takes each vector v into $-v$. Show that the orbit space $V(3, 2)/(\mathbb{Z}/2\mathbb{Z})$ is homeomorphic to $(4; 1, 1)$.

Exercise 4.2.4.7. Consider the action $(\mathbb{Z}/2\mathbb{Z}) \times \operatorname{Tang}_1 \mathbb{R}P^2 \to \operatorname{Tang}_1 \mathbb{R}P^2$ (see Exercise 4.2.4.5), where the non-zero element of takes each vector v into $-v$. Show that the orbit space $\operatorname{Tang}_1 \mathbb{R}P^2/(\mathbb{Z}/2\mathbb{Z})$ is homeomorphic to the coset space $\mathbb{S}/H$, where H is the subgroup of $\mathbb{S}^3$ consisting of the quaternions $\pm \mathrm{ort}_1$, $\pm \mathrm{ort}_2$, $\pm \mathrm{ort}_3$, $\pm \mathrm{ort}_4$.

Exercise 4.2.4.8. Consider the action $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{C}P^2 \to \mathbb{C}P^2$, where the non-zero element of $\mathbb{Z}/2\mathbb{Z}$ takes each point $((z_1 : z_2 : z_3)$ into $((\overline{z}_1 : \overline{z}_2 : \overline{z}_3))$. Show that the orbit space $\mathbb{C}P^2/(\mathbb{Z}/2\mathbb{Z})$ is homeomorphic to $\mathbb{S}^4$.

Exercise 4.2.4.9. Consider the action $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$, where the non-zero element of $\mathbb{Z}/2\mathbb{Z}$ takes each point $((z_1 : z_2), (w_1 : w_2))$ into $((\overline{z}_1 : \overline{z}_2), (\overline{w}_1 : \overline{w}_2))$. Show that the orbit space $\mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}/2\mathbb{Z}$ is homeomorphic to $\mathbb{S}^4$.

Exercise 4.2.4.10. Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{S}^2 \times \mathbb{S}^2$, where the non-zero element of $\mathbb{Z}/2\mathbb{Z}$ takes each point (x, y) into (y, x). Show that the orbit space $\mathbb{S}^2 \times \mathbb{S}^2/\mathbb{Z}/2\mathbb{Z}$ is homomorphic to $\mathbb{C}P^2$.

4.3 BUNDLES WITH A GROUP STRUCTURE

4.3.1 Spaces With F-Structure

Remark 4.3.1.1. The bundles which we encounter most frequently have fibres that besides being merely topological spaces, carry some additional structure: for example, they may be vector, Euclidean, or Hermitian spaces. In the present section we shall introduce this concept of additional structure into the theory of bundles.

We begin by giving an exact description of the necessary type of structures and then fit them systematically into the basic definitions of the theory, given in §4.1 (see Subsections 4.1.1 and 4.1.2).

Definition 4.3.1.2. Let G be a topological group, and let F be an effective G-space. We say that the topological space W is endowed with an F-*structure* if there is given a non-empty set A of homeomorphism $F \to W$ such that, for an arbitrarily fixed homeomorphism α/inA, a given homeomorphism $\beta\colon F \to W$ belongs to A if and only if $\beta^{-1} \circ \alpha$ is the transformation induced by one of the elements of G. The homeomorphisms of A are called *marked*.

Every marked homeomorphism naturally carries the action of G from F to W. If G is commutative, then the resulting action $G \times W \to W$ does not depend upon the choice of the marked homeomorphism, and hence in this case the F-structure reduces to the action of G. If G is not commutative, then an F-structure does not define a canonical action of G on W.

We remark that F itself has a canonical F-structure, namely that whose marked homeomorphisms are the transformations induced by the elements of G.

In the simplest case when G is the trivial group, a space with an F-structure is simply a topological space canonically homeomorphic to F.

Example 4.3.1.3. If $G = \mathrm{GL}(n, \mathbb{R})$ and $F = \mathbb{R}^n$ with the usual action of this group, then a space with an F-structure is nothing else but an n-dimensional vector space, and fixing a marked homeomorphism is simply fixing a basis of the space.

Example 4.3.1.4. If G is one of the groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{O}(n)$, or $\mathrm{SO}(n)$, and F is $\mathbb{R}^n$ with the usual action of these groups, then a space with an F-structure is an oriented n-dimensional real vector space, ann-dimensional Euclidean space, or an oriented n-dimensional Euclidean space, respectively. When G is $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{U}(n)$, and F is $\mathbb{C}^n$ with the usual action of G, then a space with an F-structure is an n-dimensional complex vector space, or an n-dimensional Hermitian space, respectively.

Example 4.3.1.5. If $G = \mathrm{Diff}^r X$, where $X = F$ is a $\mathcal{C}^r$-manifold $(1 \leq r \leq a)$ and G acts as usual, then a space with an F-structure is a $\mathcal{C}^r$-manifold which is $\mathcal{C}^r$-diffeomorphic to X.

Example 4.3.1.6. If $G = \mathrm{Top}\, X$, where X is a locally compact Hausdorff space, and $F = X$ with the usual action of $\mathrm{Top}\, X$, then a space with an F-structure is simply a topological space homeomorphic to X.

Example 4.3.1.7. If G is the group of all simplicial auto-homeomorphisms of the unit simplex T^n, and $F = T^n$ with the standard action of this group, then a space with an F-structure is simply an n-dimensional topological simplex.

Remark 4.3.1.8. A homeomorphism $W \to W'$, where W and W' are spaces with F-structure, which takes the set of marked homeomorphisms of W into the set of marked homeomorphisms of W', is called an *isomorphism* or, more specifically, an *F-isomorphism.*

In each of the previous examples, the F-isomorphisms form a well-known class of maps: in the first and the fifth cases they are the linear isomorphism, in the second - the orientation preserving linear isomorphisms, in the third and sixth - the linear isometric isomorphisms, in the fourth - the orientation preserving linear isomorphisms, in the seventh - the $\mathcal{C}^r$-diffeomorphisms, in the eighth - the homeomorphisms, and in the ninth - the simplicial homeomorphisms.

Remark 4.3.1.9. Given a space W with an F-structure and a space W' with an F'-structure, the product $W \times W'$ is obviously a space with an $F \times F'$-structure (see Definition 4.2.3.7); the marked homeomorphisms $F \times F' \to W \times W'$ are those of the form $\alpha \times \alpha'$, where α and α' are marked homeomorphisms.

If the G_1-space F_1 is obtained from the G-space F by reducing the group G to $G_\prime$, then by returning to F from F_1 every space W_1 with an F_1-structure becomes a space W with an F-structure: topologically, W is the same as W_1, while the new marked homeomorphisms are defined as the compositions of the transformations induced by the elements of G with the old marked homeomorphisms. We say that W is obtained from W_1 by *extending* (or *prolonging*) the group G_1 to G.

4.3.2 Steenrod Bundles

Definition 4.3.2.1. Let G and F be a topological group and an effective G-space, respectively. A bundle ξ is a *weak F-bundle*, or a *$W - F$-bundle*, if each of its fibres is endowed with an F-structure. In this case, F and G are called the *standard fibre* and the *structure group of* ξ, respectively. The set of all marked homeomorphisms from F onto the fibres of ξ is denoted by $MH(\xi)$. The group G acts naturally from the right on $MH(\xi)$ by the rule:

$$[\alpha g](y) = \alpha(gy) \quad \alpha \in MH(\xi), \quad g \in G, \quad y \in F.$$

If ξ is a $W - F$-bundle and $f\colon B \to \mathrm{bs}(\xi)$ is continuous, then clearly the induced bundle $f^!(\xi)$ is a $W - F$-bundle: the F-structures on its fibres are defined via the homeomorphisms

$$\mathrm{abr}\,\mathrm{tl}(\mathrm{adj}\, f)\colon [\mathrm{proj}(f^!(\xi))]^{-1}(b) \to [\mathrm{proj}(\xi)]^{-1}(f(b)), \quad b \in B.$$

Given two $W - F$-bundles, ξ and η, a map f of ξ into η is called a *$W - F$-map* if the maps $\mathrm{abr}\,\mathrm{tl}(f)$ from the fibres of ξ into the fibres of η are isomorphisms (see Remark 4.3.1.8). A $W - F$-map which is an isomorphism (respectively, equivalence) in the pure topological sense, i.e., in the sense of Definition 4.1.1.2, is called a $W-F$-*isomorphism* (respectively, a $w-F$-*equivalence*). Two $W-F$-bundles which can be mapped into each other by a $W-F$-isomorphism ($W-F$-equivalence) are said to be $W-F$-*isomorphic* (respectively, $W-F$-*equivalent*).

To each $W-F$-map $f\colon \xi \to \eta$ corresponds the map $MH(\xi) \to MH(\eta)$, which takes each marked homeomorphism $\alpha\colon F \to [\mathrm{proj}(\xi)]^{-1}(b)$ into the composite homeomorphism

$$F \xrightarrow{\alpha} (\mathrm{proj}\,\xi)^{-1}(b) \xrightarrow{\mathrm{abr\,tl}(f)} [\mathrm{proj}(\eta)]^{-1}(\mathrm{bs}(f(b))).$$

Moreover, we see that $MH(f)$ is a G-map with respect to the natural right actions of G on $MH(\xi)$ and $MH(\eta)$.

The standard trivial bundle, $(B \times F, \mathrm{proj}_1, B)$, with B an arbitrary topological space, is obviously a $W-F$-bundle: the F-structures on its fibres are defined by the homeomorphisms $F \to b \times f$, $y \mapsto (b, y)$. As in Subsection 4.1.2, every $W-F$-bundle which is $W-F$-equivalent to a standard trivial $W-F$-bundle is called a $W-F$-*trivial* $W-F$-*bundle.*

Definition 4.3.2.2. A bundle ξ is a *strong* F-*bundle* or, simply, an F-*bundle* if it is a $W-F$-bundle and $MH(\xi)$ is endowed with a topology.

If ξ is an F-bundle and $f\colon B \to \mathrm{bs}(\xi)$ is continuous, then the induced bundle $f^!(\xi)$ is also an F-bundle: to introduce a topology on $MH(f^!(\xi))$, we use the injective mapping

$$MH(f^!(\xi)) \to B \times MH(\xi), \quad \alpha \mapsto ([\mathrm{proj}(f^!(\xi))](\alpha(F)), [MH(\mathrm{adj}\, f)](\alpha)).$$

A map $f\colon \xi \to \eta$, where ξ and η are F-bundles, is said to be an F-*map* if it is a $W-F$-map and$MH(f)$ is continuous. AnF-map f is an F-*isomorphism* (F-*equivalence*) if it is an isomorphism (respectively, equivalence) in the pure topological sense and $MH(f)$ is a homeomorphism.

The standard trivial bundle $(B \times F, \mathrm{proj}_1, B)$, with B an arbitrary topological space, is obviously an F-bundle: the F-structures of its fibres were already introduced in Definition 4.3.2.1, and one can introduce a topology on $MH((B \times F, \mathrm{proj}_1, B))$ by means of the invertible mapping $B \times G \to MH((B \times F, \mathrm{proj}_1, B))$, which takes each pair (b, g) into the homeomorphism $F \to b \times F, y \mapsto (b, gy)$. An F-bundle which is F-equivalent to a standard trivial bundle is called F-*trivial*, and every such equivalence is an F-trivialisation.

Definition 4.3.2.3. The F-bundle ξ, is *locally* F-*trivial* if every point of $\mathrm{bs}(\xi)$ has a neighbourhood U such that the restriction $\xi|_U$ is F-trivial. The locally F-trivial bundles are called *Steenrod* F-*bundles.*

Steenrod bundles play a major role in what follows, which accounts also for the importance of the F-bundles. The weak F-bundles are only auxiliary.

We remark that for Steenrod bundles the canonical right action of the structure group on the space of marked homeomorphisms is continuous and free. This is plainly true in the standard trivial case, to which the general case reduces.

Remark 4.3.2.4. If ξ is a Steenrod F-bundle and $f\colon B \to \mathrm{bs}(\xi)$ is continuous, then the induced bundle $f^!(\xi)$ is again a Steenrod F-bundle: by Definition 4.3.2.2, $f^!(\xi)$ is an F-bundle, and the obvious fact that $f^!(\xi)$ is F-trivial if ξ is so implies the local F-triviality of $f^!(\xi)$. Clearly, the map $\operatorname{adj} f\colon f^!(\xi) \to \xi$ is an F-map, the canonical equivalence $\mathrm{id}^!_{\mathrm{bs}(\xi)}(\xi) \to \xi$, and the canonical equivalences of the form $g^!(f^!(\xi)) \to (f \circ g)^!(\xi)$ (see Remark 4.1.1.5), are F-equivalences. Moreover, given any F-map h of ξ into another Steenrod F-bundle, the correcting map, $\operatorname{corr} h$, is an F-map. (Recall Definition 4.1.1.6.)

The product of a Steenrod F-bundle ξ with a Steenrod F'-bundle ξ' is a Steenrod $F \times F'$-bundle: the $F \times F'$-structures on its fibres is defined as in Remark 4.3.1.9; the topology on $MH(\xi \times \xi')$ is introduced by means of the invertible mapping

$$MH(\xi) \times MH(\xi') \to MH(\xi \times \xi'), \quad (\alpha, \alpha') \mapsto \alpha \times \alpha';$$

the local $F \times F'$-triviality of the resulting $F \times F'$-bundle follows from the fact that it is $F \times F'$-trivial whenever ξ is F-trivial and ξ' is F'-trivial.

If the G_1-space F_1 comes from the effective G-space F by reducing the group G to G_1, then by returning to F from F_1, every Steenrod F_1-bundle becomes a Steenrod F-bundle ξ:

- topologically, ξ is the same as ξ_1;
- the F-structures on the fibres of ξ are those described in Remark 4.3.1.9;
- further, to define a topology on $MH(\xi)$, consider the action

  $$G_1 \times (G \times MH(\xi_1)) \to G \times MH(\xi_1), \quad (g_1, (g, \alpha)) \mapsto (g_1 g, \alpha^{-1} g_1),$$

  where G_1 acts canonically from the right on $MH(x\text{ß}_1)$ (see Definition 4.3.2.1), and then use the invertible mapping $(G \times MH(\xi_1))/G_1 \to MH(\xi)$, which takes the orbit of the pair (g, α) into the homeomorphism $y \mapsto \alpha(gy)$, to transfer the topology of $(G \times MH(\xi_1))/G_1$ to $MH(\xi)$;
- finally, the local F-triviality of the resulting F-bundle is a consequence of its F-triviality in the case when ξ_1 is F_1-trivial.

This transformation of F_1-bundles into F-bundles is known as the *extension* (or *prolongation*) *of the structure group*. It takes F_1-maps into F-maps, and F_1-equivalences into F-equivalences. It is also clear that the extension of the structure group commutes with the induction operation; that is to say, if ξ is obtained from ξ_1 by extension of the structure group and $f\colon B \to \mathrm{bs}(\xi)$ is an arbitrary continuous map, then $f^!(\xi)$ is obtained from $f^!(\xi_1)$ by extension of the structure group.

Theorem 4.3.2.5. *Every Steenrod F-bundle with trivial structure group is F-trivial.*

Proof. Let ξ be a Steenrod bundle with standard fibre F and trivial structure group. Let Γ be an open cover of $\mathrm{bs}(\xi)$ such that the bundle $\xi|_U$ is F-trivial for any $U \in \Gamma$. Set $\eta = (\mathrm{bs}(\xi) \times F, \mathrm{proj}_1, \mathrm{bs}(\xi))$. Clearly, the F-trivialisation $\eta|_U \to \xi|_U$ is unique for any $U \in \Gamma$, and these F-trivialisations together yield an F-trivialisation $\eta \to \xi$. ☐

Theorems About F-maps

Theorem 4.3.2.6. *Suppose that ξ and ξ' are Steenrod bundles with standard fibre F, B is a topological space, and $p\colon B \to \mathrm{bs}(\xi)$ is a factorial map. If $\tau\colon \mathrm{tl}(\xi) \to \mathrm{tl}(\xi')$ and $\beta\colon \mathrm{bs}(\xi) \to \mathrm{bs}(\xi')$ are maps such that $(\tau \circ \mathrm{tl}(\mathrm{adj}\, p), \beta \circ p)$ is an F-map $p^!(\xi) \to \xi'$, then (τ, β) is an F-map $\xi \to \xi'$.*

Proof. We need only check the continuity of β, τ, and $MH(\tau, \beta)$. The continuity of β is an immediate consequence of the continuity of the composition $\beta \circ p$ and the fact that p is factorial (see Remark 1.2.3.4). As for τ and $MH(\tau, \beta)$, it is enough to verify their continuity when ξ and ξ' are standard trivial F-bundles. In this situation, τ is given by $\tau(b, y) = (\beta(b), \varphi(b)y)$, where φ is some map from $\mathrm{bs}(\xi)$ into the structure group G. Moreover, if we use the homeomorphisms

$$B \times G \to MH(p^!(\xi)), \quad \mathrm{bs}(\xi) \times G \to MH(\xi), \quad \mathrm{bs}(\xi') \times G \to MH(\xi')$$

(which define the topologies on $MH(p^!(\xi))$, $MH(\xi)$, and $MH(\xi')$, respectively; see Definition 4.3.2.2), then the maps

$$\begin{aligned} &MH(\tau \circ \mathrm{tl}(\mathrm{adj}\, p), \beta \circ p)\colon MH(p^!(\xi)) \to MH(\xi'), \\ &MH(\tau, \beta)\colon MH(\xi) \to MH(\xi') \end{aligned}$$

are transformed into the maps

$$\begin{aligned} B \times G \to \mathrm{bs}(\xi') \times G, &\quad (b, g) \mapsto (\beta \circ p(b), (\varphi \circ p(\neg))g), \\ \mathrm{bs}(\xi) \times G \to \mathrm{bs}(\xi') \times G, &\quad (b, g) \mapsto (\beta(b), \varphi(b)g), \end{aligned}$$

respectively. The first formula shows that $\varphi \circ p$ is continuous, and since p is factorial, φ is continuous. Finally, the continuity of φ implies the continuity of τ and $MH(\tau, \beta)$. ☐

Corollary 4.3.2.7. *Suppose that ξ and ξ' are Steenrod bundles with standard fibre F and $\beta\colon \mathrm{bs}(\xi) \to \mathrm{bs}(\xi')$ is continuous. If $\tau\colon \mathrm{tl}(\xi) \to \mathrm{tl}(\xi')$ is a map such that the restrictions $\tau|_{[\mathrm{proj}(\xi)]^{-1}(U)}$, $\beta|_u$ form an F-map $\xi|_U \to \xi'$, for each element U of some fundamental cover of* $\mathrm{bs}(\xi)$*, then (τ, β) is an F-map $\xi \to \xi'$.*

Proof. This is a corollary of Theorem 4.3.2.6: take p to be the map $\mathrm{proj}\colon \amalg_{U \in \Gamma} U \to \mathrm{bs}(\xi)$, where Γ is the given fundamental cover of $\mathrm{bs}(\xi)$. ☐

Theorem 4.3.2.8. *If the Steenrod F-bundles ξ and ξ' have the same base, then every F-map $f\colon \xi \to \xi'$ with $\mathrm{bs}(f) = \mathrm{id}_{\mathrm{bs}(\xi)}$ is an F-equivalence.*

Proof. All we need to prove is that $[\mathrm{tl}(f)]^{-1}$ and $[MH(f)]^{-1}$ are continuous, and it suffices to examine the case when $\xi' = \xi$ and ξ is a standard trivial F-bundle. Then $\mathrm{tl}(f), [\mathrm{tl}(f)]^{-1}\colon \mathrm{bs}(\xi) \times F \to \mathrm{bs}(\xi) \times F$ are given by

$$(b, y) \mapsto (b, \varphi(b)y), \quad (b, g) \mapsto (b, \psi(b)y) \quad [b \in \mathrm{bs}(\xi), y \in F].$$

where φ and ψ are some maps from $\mathrm{bs}(\xi)$ into the structure group G. Moreover, if we use the topologising homeomorphism $\mathrm{bs}(\xi) \times G \to MH(\xi)$, then $MH(f)$ and $[MH(f)]^{-1}$ become the maps $\mathrm{bs}(\xi) \times G \to \mathrm{bs}(\xi) \times G$ given by

$$(b, g) \mapsto (b, \varphi(b)g), \quad (b, g) \mapsto (b, \psi(b)g) \quad [b \in \mathrm{bs}(\xi), g \in G].$$

Obviously, $\psi(b) = [\varphi(b)]^{-1}$, and thus the continuity of $MH(f)$ first implies the continuity of φ and ψ, and then the continuity of and $[MH(f)]^{-1}$. ☐

Corollary 4.3.2.9. *The correcting map,* corr f, *is an* F*-equivalence for every* F*-map* f *between Steenrod* F*-bundles.*

Principal Bundles

Definition 4.3.2.10. A Steenrod bundle is called *principal* if its standard fibre is the structure group G which acts canonically from the left on itself (see Definition 4.2.3.7). We take the liberty to denote the last G-space simply by G and, accordingly, the principal bundles with structure group G will be referred to as *Steenrod G-bundles*.

A fundamental property of the principal bundles is that their spaces of marked homeomorphisms can be identified with their total spaces. More precisely, given a principal G-bundle ξ, the formula $\alpha \mapsto \alpha(e_G)$ defines a homeomorphism $MH(\xi) \to \mathrm{tl}(\xi)$. For a standard trivial bundle, this is evident, and the general case is readily reduced to the standard trivial one.

If we identify $MH(\xi)$ and $\mathrm{tl}(\xi)$ via the homeomorphism $\alpha \mapsto \alpha(e_G)$, then the natural right action of G on $MH(\xi)$ (see Definition 4.3.2.1) becomes the free right action of G on $\mathrm{tl}(\xi)$. This free, action, $\mathrm{tl})\xi) \times G \to \mathrm{tl}(\xi)$, can be also described directly: its orbits are exactly the fibres of ξ, and on each fibre the action is simply the right canonical action, transferred from G to the fibre by means of marked homeomorphisms.

Remark 4.3.2.11. This construction of the free right action of G on the total space of a principal bundle with structure group G can be partially reversed. Assume that the topological group G acts continuously and freely from the right on the topological space X, and consider the bundle $(X, \mathrm{proj}, X/G)$. Its fibres (orbits) carry natural G-structures: the marked homeomorphisms $G \to \mathrm{proj}^{-1}(b)$ $(b \in X/G)$ are given by $g \mapsto xg$, $x \in \mathrm{proj}^{-1}(b)$. Since to distinct points x correspond distinct homeomorphisms $g \mapsto xg$, we obtain also an invertible map of X onto the set of marked homeomorphisms, and thus we get a topology on the last set. Therefore, $(X, \mathrm{proj}, X/G)$ is a G-bundle.

To explain why we called this last construction a partial inversion of the original one, apply it now to the right action $\mathrm{tl}(\xi) \times G \to \mathrm{tl}(\xi)$ described in

Definition 4.3.2.10; the resulting bundle is exactly ξ . More precisely, the injective factor of the projection $\mathrm{proj}(\xi)$ maps $\mathrm{tl}(\xi)/G$ onto $\mathrm{bs}(\xi)$, and together with $\mathrm{id}_{\mathrm{tl}(\xi)}$ forms a G-isomorphism $(\mathrm{tl}(\xi), \mathrm{proj}, \mathrm{tl}(\xi)/G) \to \xi$.

Theorem 4.3.2.12. *If the G-bundle $(X, \mathrm{proj}, X/G)$, defined by a free right action of G , has a section, then it is G-trivial. In particular, every Steenrod G-bundle having a section is G-trivial.*

Proof. Indeed, if $s\colon X/G \to X$ is a section, then the map

$$f\colon ((X/G){\times}G, \mathrm{proj}_1, X/G) \to (X, \mathrm{proj}_1, X/G), \quad \text{given by} \quad \mathrm{tl}(f(b,g)) = s(b)g,$$

is a G-trivialisation of the bundle $(X, \mathrm{proj}, X/G)$. □

Corollary 4.3.2.13. *If the G-bundle $(X, \mathrm{proj}, X/G)$, defined by a free right action of G, is topologically trivial, then it is G-trivial. If $(X, \mathrm{proj}, X/G)$ is locally topologically trivial, then it is locally G-trivial, i.e., it is a Steenrod G-bundle.*

4.3.3 Associated Bundles

Remark 4.3.3.1. Let G be a topological group, and let F and F' be effective G-spaces. The construction below associates to each Steenrod F-bundle ξ a certain Steenrod F'-bundle having the same base.

The formula $g(\alpha, y) = (\alpha g, g^{-1}y)$, where $g \in G$, $\alpha \in MH(\xi)$, and $y \in F'$, defines a right action of G on $MH(\xi)xF'$ (here G acts canonically from the right on $MH(\xi)$; see Definition 4.3.2.3). Let ξ' denote the bundle with total space $(MH(\xi) \times F')/G$, base $\mathrm{bs}(\xi)$, and whose projection takes the orbit of a pair $(\alpha, y) \in MH(\xi) \times F'$ into the point $(\mathrm{proj}(\xi))(\alpha(F))$. The fibres of this bundle carry a natural F'-structure: the marked homeomorphisms $F' \to [(\mathrm{proj}(\xi')]^{-1}(b)$ are given by $y \mapsto \mathrm{proj}(\alpha, y)$, where $\alpha \in MH(\xi)$ is such that $\alpha(F) = [\mathrm{proj}(\xi)]^{-1}(b)$. Since distinct homeomorphisms α yield distinct homeomorphisms $y \mapsto \mathrm{proj}(\alpha, y)$, we obtain at the same time an invertible map $MH(\xi) \to MH(\xi')$, which we use to topologise $MH(\xi')$, and thus make from an F'-bundle. Finally, $\xi'|_U$ is $'$-trivial for each set U such that $\xi|_U$ is F-trivial. Consequently, ξ' is locally F'-trivial, i.e., it is a Steenrod F'-bundle. We say that is the *F'-bundle associated with* ξ and denote it by $\mathrm{assoc}(\xi, F')$.

Remark 4.3.3.2. We add four remarks to the above description of the assoc construction:

(i) The map $\mathrm{tl}(\xi) \to (MH(\xi) \times F)/G$ which takes each point $x \in \mathrm{tl}(\xi)$ into the orbit consisting of the pairs $(\alpha, y) \in MH(\xi) \times F$ with $\alpha(y) = x$, is obviously a homeomorphism; together with $\mathrm{id}_{\mathrm{bs}(\xi)}$, this map defines an F-equivalence $\xi \to \mathrm{assoc}(\xi, F)$. Therefore, the bundle $\mathrm{assoc}(\xi, F)$ is canonically F-equivalent to ξ.

(ii) The invertible map $MH(\xi) \to MH(\mathrm{assoc}(\xi, F'))$ that we used to topologise $MH(\mathrm{assoc}(\xi, F'))$, is a G-map with respect to the right canonical actions of G on $MH(\xi)$ and $MH(\operatorname{assoc}(\xi, F'))$. As a corollary, we may state that, given an arbitrary effective G-space F'', the product of the above invertible G-map with id''_F is a G-map

$$MH(\xi) \times F'' \to MH(\operatorname{assoc}(\xi, F')) \times F'',$$

where G acts fro the right on $MH(\xi) \times F''$ and $MH(\operatorname{assoc}(\xi, F')) \times F''$ by $g(\alpha, y) = (\alpha g, g^{-1}y)$. The resulting homeomorphism

$$(MH(\xi) \times F'')/G \to (MH(\operatorname{assoc}(\xi, F')) \times F'')/G,$$

together with $\mathrm{id}_{\mathrm{bs}(\xi)}$ define an F''-equivalence

$$\operatorname{assoc}(\xi, F'') \to \operatorname{assoc}(\operatorname{assoc}(\xi, F'), F'').$$

Therefore, the bundles $\operatorname{assoc}(\operatorname{assoc}(\xi, F'), F'')$ and $\operatorname{assoc}(\xi, F'')$ are canonically F''-equivalent.

(iii) The bundle $\operatorname{assoc}(\xi, G)$, i.e., the principal bundle associated with ξ is canonically G-isomorphic to the G-bundle $(MH(\xi), \mathrm{proj}, MH(\xi)/G)$ defined by the canonical right action of G on $MH(\xi)$ (see Remark 4.3.2.11). The canonical G-isomorphism $(MH(\xi), \mathrm{proj}, MH(\xi)/G) \to \operatorname{assoc}(\xi, G)$ is given by the homeomorphism $MH(\xi) \to (MH(\xi) \times G)/G$ which takes each $\alpha \in MH(\xi)$ into the orbit of (α, e_G).

(iv) If F' is a subspace of the G-space F'' (see Definition 4.2.3.7), then

$$(MH(\xi) \times F')/G \subset (MH(\xi) \times F''/G,$$

and the inclusion

$$(MH(\xi) \times F')/G \to (MH(\xi) \times F'')/G$$

together with $\mathrm{id}_{\mathrm{bs}(\xi)}$, yield an inclusion of the bundle $\operatorname{assoc}(\xi, F')$ into $\operatorname{assoc}(\xi, F'')$. Moreover, $MH(\operatorname{assoc}(\xi, F'))$ is exactly the set of maps

$$\operatorname{abr} \alpha \colon F' \to \alpha(F'), \quad \alpha \in MH(\operatorname{assoc}(\xi, F'')).$$

Behaviour With Respect to Maps

Definition 4.3.3.3. Let F and F' again be effective G-spaces, and suppose that ξ and η are Steenrod bundles with standard fibre F, and $f \colon \xi - to \eta$ is an arbitrary F-map. Define the map

$$\operatorname{assoc}(f, F') \colon \operatorname{assoc}(\xi, F') \to \operatorname{assoc}(\eta, F')$$

by the formulae

$$\begin{aligned}&\mathrm{bs}(\operatorname{assoc}(f, F')) = \mathrm{bs}(f) \quad \text{and}\\ &\mathrm{tl}(\operatorname{assoc}(f, F')) = [\operatorname{fact}(MH(f) \times \mathrm{id}_{F'}) \colon (MH(\xi) \times F')/G(MH(\eta) \times F'/G].\end{aligned}$$

Remark 4.3.3.4. It is clear that $\operatorname{assoc}(f, F')$ is an F'-map. Moreover, $\operatorname{assoc}(f, F')$ is an F'-isomorphism (F'-equivalence) whenever f is an F-isomorphism (respectively, F-equivalence). Next, consider the diagrammes

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & \operatorname{assoc}(\xi, F) \arrow[d, "\operatorname{assoc}(f,F)"] & \operatorname{assoc}(\xi, F'') \arrow[r] \arrow[d] & \operatorname{assoc}(\operatorname{assoc}(\xi, F'), F'') \arrow[d, "\operatorname{assoc}(\operatorname{assoc}(f,F'),F'')"] \\
\eta \arrow[r] & \operatorname{assoc}(\eta, F) & \operatorname{assoc}(\eta, F'') \arrow[r] & \operatorname{assoc}(\operatorname{assoc}(\eta, F'), F'')
\end{tikzcd}$$

and

$$\begin{tikzcd}
(MH(\xi), \operatorname{proj}, MH(\xi)/G) \arrow[r] \arrow[d, "{(MH(f), \operatorname{fact} MH(f)}"'] & \operatorname{assoc}(\xi, G) \arrow[d, "\operatorname{assoc}(f,G)"] \\
(MH(\eta), \operatorname{proj}, MH(\eta)/G) \arrow[r] & \operatorname{assoc}(\eta, G)
\end{tikzcd}$$

where F'' is any effective G-space, and the horizontal arrows denote successively the canonical F-equivalences described in Remark 4.3.3.2 (i) and 4.3.3.2 (ii), and the canonical F-isomorphisms from Remark 4.3.3.2 (iii). These diagrammes clearly commute.

Remark 4.3.3.5. The assoc and induction operations commute. Namely, the map

$$\operatorname{corr}[\operatorname{assoc}(\operatorname{adj} h, F')] \colon \operatorname{assoc}(h^!(\xi), F') \to h^!(\operatorname{assoc}(\xi, F'))$$

is an F'-equivalence, for any Steenrod F-bundle ξ and any continuous map $h \colon B \to \operatorname{bs}(\xi)$; see Corollary 4.3.2.9.

Furthermore, the assoc operation commutes with the extension of the structure group. That is to say, let F_1 and F_1' be the effective G_1-spaces obtained from F and F' by reducing the group G to a subgroup G_1. If the Steenrod F_1-bundle ξ_1 is taken into ξ by the extension of the group G_1 to G, then the map

$$\operatorname{fact}(\operatorname{incl} \times \operatorname{id}_{F'}) \colon (MH(\xi_1) \times F_1')/G_1 \to (MH(\xi) \times F')/G,$$

where $\operatorname{incl} = [\operatorname{incl} \colon MH(\xi_1) \to MH(\xi)]$, defines an F'-equivalence between the bundle obtained from $\operatorname{assoc}(\xi_1, F_1')$ by extending G_1 to G, and $\operatorname{assoc}(\xi, F')$.

Weakly Associated Bundles

Remark 4.3.3.6. The construction described in Remark 4.3.3.11 can be generalised to the situation where the action of G on F' is not effective: we need only shift, as a preliminary step, to the effective factor of this action, and thus transform F' into an effective G/K-space, $\underset{\sim}{F}'$, where K is the non-effectiveness kernel. Therefore, $\operatorname{assoc}(\xi, F')$ may be defined for any Steenrod F-bundle ξ, and any G-space F' and is, in the general case, a Steenrod bundle with structure group G/K and standard fibre $\underset{\sim}{F}'$. We say that $\operatorname{assoc}(\xi, F')$ is *weakly associated with* ξ.

The map $\operatorname{assoc}(f, F')$ defined in Definition 4.3.3.3 remains viable under this extension of the assoc construction, and becomes an $\underset{\sim}{F}'$-map. The properties of assoc discussed in Remarks 4.3.3.2, 4.3.3.4, and 4.3.3.5 must be modified in an obvious manner; for example, when F'' is not effective (but F' is effective), the canonical F''-equivalence

$$\operatorname{assoc}(\xi, F'') \to \operatorname{assoc}(\operatorname{assoc}(\xi, F'), F'')$$

becomes an $\underset{\sim}{F}'$-equivalence.

Sections Associated with F-Maps

Remark 4.3.3.7. Let ξ and ξ' be Steenrod bundles with structure group G and standard fibre F, and let $f\colon \operatorname{bs}(\xi) \to \operatorname{bs}(\xi^{prime)}$, be continuous. The construction below establishes a one-to-one correspondence between the F-maps $h\colon \xi \to \xi'$ with $\operatorname{bs}(h) = f$ and the sections of a specially constructed bundle, $\operatorname{Fibr}(\xi, \xi'; f)$.

Let $G^{\times}$ denote the group G endowed with the action of the group $G \times G$ given by $(g_1, g_2)g = g_1 g g_2^{-1}$ (generally speaking, this is not an effective action). Set

$$\operatorname{Fibr}(\xi, \xi'; f) = \operatorname{assoc}(\operatorname{diag}^!(\xi \times f^!(\xi')), G^{\times}),$$

where $\operatorname{diag} = [\operatorname{diag}\colon \operatorname{bs}(\xi) \to \operatorname{bs}(\xi) \times \operatorname{bs}(\xi)]$, and assoc is taken in the weak sense of Remark 4.3.3.6. It is clear that for every F-map h such that $\operatorname{bs}(h) = f$ and every point $b \in \operatorname{bs}(\xi)$, the composite homeomorphism

$$\begin{aligned} F \xrightarrow{\alpha} [\operatorname{proj}(\xi)]^{-1}(b) \xrightarrow{\operatorname{abr}\operatorname{tl}(h)} [\operatorname{proj}(\xi')]^{-1}(f(b)) \xrightarrow{[\operatorname{abr}\operatorname{tl}(\operatorname{adj} f)]^{-1}} \\ [\operatorname{proj} f^!(\xi')]^{-1}(\beta(F)) \xrightarrow{\beta^{-1}} F, \end{aligned}$$

where

$$\alpha \in MH(\xi), \quad \operatorname{proj}(\xi(\alpha(F))) = b, \qquad \beta \in MH(f^!(\xi')), \quad \operatorname{proj}(f^!(\xi'(\beta(F)))) = b,$$

is simply one of the transformations $F \to F$ induced by the elements of G. We denote the corresponding element by $g(\alpha, \beta)$, and note that

$$g(\alpha g_1, \beta g_2) = g_1 g(\alpha, \beta) g_2^{-1} \quad \forall g_1, g_2 \in G$$

(here G acts canonically from the right on $MH(\xi)$ and $MH(f^!(\xi'))$. This shows that the orbit of the pair

$$(\alpha \times \beta, g(\alpha, \beta)) \in MH(\xi \times f^!(\xi')) \times G^{\times}$$

under the right action of $G \times G$ on $MH(\xi \times f^!(\xi')) \times G^{\times}$, constructed as in Remark 4.3.3.1, does not depend upon the choice of α and β, provided h and b are fixed. When h is fixed, the map $\operatorname{bs}(\xi) \to (MH(\xi \times f^!(\xi')) \times G^{\times})/(G \times G)$ taking $b \in \operatorname{bs}(\xi)$ into this orbit is continuous and manifestly a section of the bundle $\operatorname{Fibr}(\xi, \xi'; f)$. We call it the *section associated with* h and denote it by $h^{\times}$. The correspondence $h \mapsto h^{\times}$ defines an invertible map from the set of all F-maps $h\colon \xi \to \xi'$ with $\operatorname{bs}(h) = f$ onto the set of sections of $\operatorname{Fibr}(\xi, \xi'; f)$:

the inverse map takes each section $s\colon \operatorname{bs}(\xi) \to \operatorname{tl}(\operatorname{Fibr}(\xi, \xi'; f))$ into the F-map $h\colon \xi \to \xi'$ given by

$$\operatorname{tl}(h(x)) = \operatorname{tl}(\operatorname{adj} f(\beta_x(g_x(\alpha^{-1}(x))))),$$

where $x \in \operatorname{tl}(\xi)$, $\alpha_x \in MH(\xi)$, $\beta_x \in MH(f^!(\xi'))$, $g_x \in G$, and $(\alpha_x \times \beta_x, g_x) \in s \circ \operatorname{proj}(\xi(x))$.

Let us remark that when $\operatorname{bs}(\xi) = \operatorname{bs})\xi)$ and $f = \operatorname{id}_{\operatorname{bs}(\xi)}$, an F-map $h\colon \xi \to \xi'$ with $\operatorname{bs}(h) = f$ is simply an F-equivalence (see Theorem 4.3.2.8). Therefore, when ξ and $\xi^{[\prime}]$ have the same base, the above construction yields a one-to-one correspondence between the F-equivalences $\xi \to \xi'$ and the sections of the bundle $\operatorname{Fibr}(\xi, \xi'; \operatorname{id}_{\operatorname{bs}(\xi)})$.

4.3.4 Ehresmann-Feldbau Bundles

Definition 4.3.4.1. An *Ehresmann-Feldbau bundle* is a W-F-bundle which is locally W-F-trivial; the last means that every point of the base has a neighbourhood such that the restriction of the bundle to this neighbourhood is W-F-trivial.

The theory of Ehresmann-Feldbau bundles is a variant of the theory of bundles with a group structure; it is simpler than the theory of Steenrod bundles (there are fewer structures), but also less pithy (there are no associated bundles). This relative poverty nearly deprives it of any independent value; however, the fact that it is equivalent, for a large class of standard fibres which includes the most important cases, to the theory of Steenrod bundles, makes it useful, as it enables us to simplify the latter.

The Case of Topologically Effective Actions

Definition 4.3.4.2. A continuous effective action $G \times X \to X$ is said to be *topologically effective* if given any topological space Y and any map $f\colon Y \to G$, the continuity of the composite map

$$Y \times X \xrightarrow{f \times \operatorname{id}} G \times X \to X \tag{4.3.4.3}$$

implies the continuity of f. In this case we also say that the G-space X is *topologically effective.*

Clearly, if we reduce the group, a topologically effective action remains so, and every G-space which has a topologically effective subspace is itself topologically effective.

The free actions are immediate examples of topologically effective actions; in particular, the left canonical action of a topological group (on itself) is always topologically effective. Also, the usual actions of $\operatorname{GL}(n, \mathbb{R})$ and of its subgroups on $\mathbb{R}^n$ are all topologically effective.

Theorem 4.3.4.4. *In order that a G-space be topologically effective, it is necessary that the map $c\colon G \to \mathcal{C}(X,X)$, which takes each $g \in G$ into the transformation induced by g, be a topological embedding; if X is locally compact and Hausdorff, then this condition is also sufficient.*

Proof. The necessity is plain (take $Y = c(G)$ and $f = [(\operatorname{abr} c)^{-1}\colon c(G) \to G]$). Now assume that the condition is satisfied. Then the continuity of $f\colon Y \to G$ is equivalent to the continuity of the composite map $Y \xrightarrow{f} G \xrightarrow{c} \mathcal{C}(X,X)$. By Theorem 1.2.7.6, for X locally compact and Hausdorff the last map is continuous because so is the map (4.3.4.3). In particular, for G compact, every effective, locally compact Hausdorff G-space is topologically effective (see Theorems 1.1.7.10 and 1.2.7.2). If X is Hausdorff, locally compact, and locally connected, then the usual action of the group $\operatorname{Top} X$ on X is topologically effective, and the same holds when X is Hausdorff and compact (see Lemma 4.2.2.5 and Theorem 4.2.2.6). ☐

Theorem 4.3.4.5. *If the standard fibre F is topologically effective, then every W-F-map (respectively, W-F-isomorphism, W-F-equivalence) between Steenrod bundles is an F-map (respectively, F-isomorphism, F-equivalence).*

Proof. Let ξ and ξ' be Steenrod F-bundles. We need only prove the continuity of the map $MH(f)$ corresponding to the given W-F-map $f\colon \xi \to \xi'$ and we may assume that ξ and ξ' are standard trivial bundles. In this case,

$$\operatorname{tl}(\xi) = \operatorname{bs}(\xi) \times F, \quad \operatorname{tl}(\xi') = \operatorname{bs}(\xi') \times F,$$

and $\operatorname{tl}(f)$ is given by $(b,y) \mapsto (\operatorname{bs}(f(b)), \varphi(b)y)$, where φ is some map of $\operatorname{bs}(\xi)$ into the structure group G. At the same time, if we use the canonical homeomorphisms

$$\operatorname{bs}(\xi) \times G \to MH(\xi), \quad \operatorname{bs}(\xi') \times G \to MH(\xi')$$

(see Definition 4.3.2.2), then $MH(f)$ becomes the map

$$\operatorname{bs}(\xi) \times G \to \operatorname{bs}(\xi') \times G, \quad (b,g) \mapsto (\operatorname{bs}(f(b), \varphi(b)g).$$

The continuity of $\operatorname{tl}(f)$ implies the continuity of the composition

$$\operatorname{proj}_2 \circ \operatorname{tl}(f)\colon \operatorname{bs}(\xi) \times F \to F,$$

which equals the composition

$$\operatorname{bs}(\xi) \times F \xrightarrow{\varphi \times \mathrm{id}} G \times F \to F,$$

where the last arrow denotes the action. Since this action is topologically effective, φ is continuous, and so is $MH(f)$. ☐

Theorem 4.3.4.6. *Given an Ehresmann-Feldbau bundle with a topologically effective standard fibre, there is a unique topology on the set of its marked homeomorphisms which transforms this bundle into a Steenrod bundle.*

Proof. The uniqueness of this topology is a consequence of Theorem 4.3.4.5. Let us prove its existence. Let ξ be an Ehresmann-Feldbau bundle with topologically effective fibre F. Cover $\operatorname{bs}(\xi)$ by open sets U such that $\xi|_U$ is W-F-trivial, and fix W-F-equivalences $h_U\colon (U \times F, \operatorname{proj}_1, U) \to \xi|_U$. Now topologise the sets $MH(\xi|_U)$ with the aid of the maps

$$MH(h_U)\colon MH((U \times F, \operatorname{proj}_1, U)) \to MH(\xi|_U).$$

We obtain a cover of $MH(\xi)$ by topological spaces $MH(\xi|_U)$, and Theorem 4.3.4.5 shows that these spaces induce the same topologies on their intersections, as required for the construction in Remark 1.2.4.3. The topology on $MH(\xi)$ produced by this construction transforms ξ into a Steenrod bundle. ☐

Locally Trivial Bundles as Ehresmann-Feldbau Bundles

Remark 4.3.4.7. If F is a locally compact Hausdorff space endowed with the usual action of the group $\operatorname{Top} F$, then an Ehresmann-Feldbau W-F-bundle is simply a locally trivial bundle with fibres homeomorphic to F. Therefore, any ordinary locally trivial bundle whose fibres are locally compact Hausdorff spaces homeomorphic one to another, may be regarded as an Ehresmann-Feldbau bundle, and as such it has an implicit group structure. If, in addition, the fibres are locally connected or compact, then such bundles can be also regarded as Steenrod bundles. We remark that the last assertion is also true for all coverings in the broad sense with connected bases.

4.3.5 Exercises

Exercise 4.3.5.1. Show that all the effective actions listed in Remarks 4.2.3.15, 4.2.3.16, 4.2.3.17, and 4.2.3.18 are topologically effective.

Exercise 4.3.5.2. Let X be an arbitrary $\mathcal{C}^r$-manifold ($r \geq 1$) of positive dimension. Show that the usual action of $\operatorname{Diff}^r X$ on X is topologically effective.

Exercise 4.3.5.3. Consider $\mathbb{R}$ as a $\mathbb{Z}$-space with the action $(n,t) \mapsto t+n$ ($n \in \mathbb{Z}$, $t \in \mathbb{R}$), and using the same formula, extend this action to an action of the additive group $\mathbb{R}$, equipped with the discrete topology. Show that this extension of the structure group takes $\operatorname{assoc}((\mathbb{R}, \operatorname{hel}, \mathbb{S}^1), \mathbb{R})$ into a bundle which is not trivial as a Steenrod bundle, but is trivial as an Ehresmann-Feldbau bundle. (Cf. Theorem 4.3.4.5.)

Exercise 4.3.5.4. Suppose that G is a connected topological group, F is an effective G-space, and ξ is a non-trivial Steenrod F-bundle with simply connected base. Denote G^δ, F^δ, and ξ^δ the group G equipped with the discrete topology, the space F, regarded as a G -space, and the bundle ξ, regarded as a W-F -bundle, respectively. Show that there is no topology on the set of marked homeomorphisms of ξ^δ which makes from ξ^δ a Steenrod F^δ-bundle. (Cf. Theorem 4.3.4.6.)

4.4 THE CLASSIFICATION OF STEENROD BUNDLES

4.4.1 Spaces With F-Structure

Remark 4.4.1.1. We now turn to the problem of classifying the Steenrod bundles with a given standard fibre F and a given cellular base B with respect to F-equivalence. Our main achievement in this section is to establish a canonical one-to-one correspondence between the classes of F-equivalent bundles over B and the homotopy classes of maps from B into a specially constructed space that depends only upon the structure group. This correspondence reduces the given classification problem to a problem in ordinary homotopy theory.

Lemmas About F-Trivial Bundles

Lemma 4.4.1.2. *Let ξ be a Steenrod bundle with standard fibre F, and let B_1 and B_2 be closed subspaces of* $\mathrm{bs}(\xi)$*, such that $B_1 \cup B_2 = \mathrm{bs}(\xi)$ and $B_1 \cap B2$ is a retract of B_2. If the restrictions $\xi|_{B_B1}$ and $\xi|_{B2}$ are F-trivial, then ξ is also F-trivial.*

Proof. Choose a retraction $\rho\colon B_2 \to B_1 \cap b_2$ and two F-trivialisations,

$$h_1\colon \eta_1 = (B_1 \times F, \mathrm{proj}_1, B_1) \to \xi|_{B_1}, \quad h_2\colon \eta_2 = (B_2 \times F, \mathrm{proj}_1, B_2) \to \xi|_{B_2}$$

and denote by f the composite F-equivalence

$$\begin{tikzcd}
\eta_2|_{B_1\cap B_2} \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & \eta_1|_{B_1\cap B_2} \arrow[d, "\mathrm{abr}\, h_1"] \\
\eta_2|_{B_1\cap B_2} & \xi|_{B_1\cap B_2} \arrow[l, "\mathrm{abr}\, h_2^{-1}"]
\end{tikzcd}$$

Obviously, $\mathrm{tl}(f)$ is given by

$$(b, y) \mapsto (b, (\varphi(b))y, \quad [b \in B_1 \cap B_2, y \in F],$$

where φ is some map of $B_1 \cap B_2$ into the structure group G. Moreover, if we use the canonical homeomorphism

$$(B_1 \cap B_2) \times G \to MH(\xi|_{B_1\cap B_2}),$$

then $MH(f)$ becomes the map

$$(B_1 \cap B_2) \times G \to (B_1 \cap B_2) \times G, \qquad (b, g) \mapsto (b, (\varphi(b))g) \quad [b \in B_1 \cap B_2, g \in G].$$

Therefore, the continuity of $MH(f)$ implies the continuity of φ, which in turn yields the continuity of the map

$$B_2 \times F \to B_2 \times F, \quad (b, y) \mapsto (b, (\varphi \circ \rho(b))y).$$

But this last map, together with id_{B_2}, form an F-equivalence $f'\colon \eta_2 \to \eta_2$, and the obvious equality

$$f = [\operatorname{abr} f'\colon \eta_2|_{B_1\cap B_2} \to \eta_2|_{B_1\cap B_2}]$$

shows that the composite F-maps

$$\eta_1 \xrightarrow{h_1} \xi|_{B_1} \xrightarrow{\text{incl}} \xi, \quad \eta_1 \xrightarrow{f'} \eta_2 \xrightarrow{h_2} \xi|_{B_2} \xrightarrow{\text{incl}} \xi$$

coincide on $B_1 \cap B_2$. By Corollary 4.3.2.7, from this it follows that these two maps yield an F-map $(B \times F, \mathrm{proj}_1, B) \to \xi$. Finally, by Theorem 4.3.2.8, the last map is an F-equivalence. □

Lemma 4.4.1.3. *Every Steenrod F-bundle with base I^n is F-trivial.*

Proof. Let η be an arbitrary Steenrod F-bundle with $\mathrm{bs}(\eta) = I^n$. Find a positive integer N such that η is F-trivial over any cube of edge $1/N$ contained in I^n. Now divide I^n, as usual, into N^n such cubes, arrange them in lexicographical order $Q_1, \dots, Q_{N^n}$, and set $W_j = \cup_{j=1}^{i} Q_j$. Induction shows that η is F-trivial over each of the sets $W_1, \dots, W_{N^n}$: to go from W_i to W_{i+1}, apply Lemma 4.4.1.2 to $\xi = \eta|_{W_{i+1}}$, $B_1 = W_i$, and $B_2 = Q_{i+1}$. We conclude that η is F-trivial over $W_{N^n} = I^n$. □

Lemma 4.4.1.4. *Let ξ_1 and ξ_2 be Steenrod bundles with common standard fibre F and common base B, and let A be a retract of B. If ξ_1 and ξ_2 are F-trivial, then for every F-equivalence $h\colon \xi_1|_A \to \xi_2|_A$ there is an F-equivalence $h'\colon \xi_1 \to \xi_2$ such that* $[\operatorname{abr} h'\colon \xi_1|_A \to \xi_2|_A] = h$.

Proof. It is enough to prove this assertion for the case where ξ_1 is the standard trivial F-bundle $(B \times F, \mathrm{proj}_1, B)$, and $\xi_2 = \xi_1$. Let $\rho\colon B \to A$ be a retraction. Obviously, $\mathrm{tl}(h)$ is given by

$$(a, y) \mapsto (a, (\varphi(a))y) \qquad a \in A, \quad y \in F,$$

where φ is some map of A into the structure group G. Moreover, via the canonical homeomorphism

$$A \times G \to MH(\xi_1|_A)[= MH(\xi_2|_A)], \quad MH(h)$$

becomes the map

$$A \times G \to A \times G, \qquad (a, g) \mapsto (a, (\varphi(a))g) \quad a \in A, g \in G.$$

Therefore, the continuity of $MH(h)$ implies the continuity of φ, which in turn implies the continuity of the map

$$B \times F \to B \times F, \quad (b, y) \mapsto (b, (\varphi \circ \rho p(b))y).$$

The latter and id_B yield an F-equivalence $h'\colon \xi_1 \to \xi_2$ which extends h. □

The Homotopy Invariance of the Induced Bundle

Theorem 4.4.1.5. *Let* ξ *be a Steenrod bundle with standard fibre* F*, and let* f_1 *and* f_2 *be continuous maps of a cellular space* X *into* $\operatorname{bs}(\xi)$*. If* f_1 *and* f_2 *are homotopic, then the bundles* $f_1^!(\xi)$ *and* $f_2^!(\xi)$ *are* F*-equivalent. Moreover, if* f_1 *and* f_2 *are* A*-homotopic, where* A *is a cellular subspace of* X*, then there is an* F*-equivalence* $f_1^!(\xi) \to f_2^!(\xi)$ *which is the identity on* $(f_1^!(\xi))|_A$*.*

Proof. Pick an A-homotopy, $H\colon X \times I \to \operatorname{bs}(\xi)$, from f_1 to f_2, and set $\xi = (f_1^!\xi) \times (I, \operatorname{id}_I, I)$, $\xi_2 = H^!\xi$. It is clear that $\xi_1|_{(X\times 0)\cup(A\times I)} = \xi_2|_{(X\times 0)\cup(A\times I)}$ and that the canonical homeomorphism $X \to X \times 1$ transforms $\xi_1|_{X\times 1}$ and $\xi_2|_{X\times 1}$ into $f_1^!\xi$ and $f_2^!\xi$, respectively. Therefore, it suffices to find an F-equivalence $\xi_1 \to \xi_2$ which is the identity over $(X \times 0) \cup (A \times I)$.

We produce such an F-equivalence by taking the limit of a sequence of F-equivalences, $h_i\colon \xi_1|_{C_i} \to \xi_1|_{C_i}$ where $C_i = (X \times 0) \cup (A \times I) \cup (\operatorname{skel}_i X \times I)$, such that each map h_i extends the preceding one. Take h_{-1} to be the identity map, and assume that the F-equivalence h_i is already constructed. To get h_{i+1}, suppose that X is rigged, and for each cell $e \in \operatorname{cell}_{i+1} X \setminus \operatorname{cell}_{i+1} A$ consider the bundles

$$[\operatorname{abr}(\operatorname{char}_e \times \operatorname{id}_I)]^!(\xi_1|_{C_i}) = [(\operatorname{char}_e \times \operatorname{id} I)^!\xi_1]|_{(\mathbb{D}^{i+1}\times 0)\cup(\mathbb{S}^i\times I)}$$
$$[\operatorname{abr}(\operatorname{char}_e \times \operatorname{id}_I)]^!(\xi_2|_{C_i}) = [(\operatorname{char}_e \times \operatorname{id} I)^!\xi_2]|_{(\mathbb{D}^{i+1}\times 0)\cup(\mathbb{S}^i\times I)}$$

where

$$\operatorname{abr}(\operatorname{char}_e \times \operatorname{id}_I) = [\operatorname{abr}(\operatorname{char}_e \times \operatorname{id}_I)\colon (\mathbb{D}^{i+1} \times 0) \cup (\mathbb{S}^i \times I) \to C_i].$$

Let g_e denote the F-equivalence of these bundles defined by h_i. By Lemma 4.4.1.3,

$$(\operatorname{char}_e \times \operatorname{id}_I)^!(\xi_1), \quad (\operatorname{char}_e \times \operatorname{id}_I)^!(\xi_2)$$

are F-trivial, and since $(\mathbb{D}^{i+1} \times 0) \cup (\mathbb{S}^i \times I)$ is a retract of $\mathbb{D}^{i+1} \times I$, Lemma 4.4.1.4 shows that g_e extends to an F-equivalence

$$\widetilde{g}_e\colon (\operatorname{char}_e \times \operatorname{id} I)^!\xi_1 \to (\operatorname{char}_e \times \operatorname{id} I)^!\xi_2$$

Further, note that the map $\operatorname{tl}\widetilde{g}_e$ is constant on the elements of the partition $\operatorname{zer}(\operatorname{tl}\operatorname{adj}(\operatorname{char}_e \times \operatorname{id}_I))$ and apply Theorem 4.3.2.6, with

$$B = \mathbb{D}^{i+1} \times I, \quad p = [\operatorname{abr}(\operatorname{char}_e \times \operatorname{id}_I)\colon \mathbb{D}^{i+1} \times I \to \operatorname{Cl} e \times I]$$

to conclude that the composite map

$$(\operatorname{char}_e \times \operatorname{id}_I)^!\xi_1 \xrightarrow{\widetilde{g}_e} (\operatorname{char}_e \times \operatorname{id}_I)^!\xi_2 \xrightarrow{\operatorname{abr}(\operatorname{char}_e \times \operatorname{id}_I)} \xi_2|_{\operatorname{Cl} e\times I}$$

defines an F-map $\xi_1|_{\operatorname{Cl} e\times I} \to \xi_2|_{\operatorname{Cl} e\times I}$. By Theorem 4.3.2.8, this is an F-equivalence, which we denote by h_e. Now note that for any cells $e_1, e_2 \in \operatorname{cell}_{i+1} X \setminus \operatorname{cell}_{i+1} A$, $\operatorname{tl} h_{e_1}$ and $\operatorname{tl} h_{e_2}$ agree over $(\operatorname{Cl} e_1 \times I) \cap (\operatorname{Cl} e_2 \times I)$, and that for each cell $e \in \operatorname{cell}_{i+1} X \setminus \operatorname{cell}_{i+1}$, $\operatorname{tl} h_e$and $\operatorname{tl} h_i$ agree over $(\operatorname{Cl}_e \times I) \cap C_i$.

Since the sets C_i and $\operatorname{Cl} e \times I$, $e \in \operatorname{cell}_{i+1} X \setminus \operatorname{cell}_{i+1} A$, constitute a fundamental cover of C_{i+1}, we use Corollary 4.3.2.7 to conclude that h_i and h_e, $e \in \operatorname{cell}_{i+1} X \setminus \operatorname{cell}_{i+1} A$, form together an F-map $\xi_1|_{C_{i+1}} \to \xi_2|_{C_{i+1}}$. We take this map for h_{i+1} and note that it obviously extends h_i; moreover, by Corollary 4.3.2.7, h_{i+1}^{-1} is also an F-map.

To check the rest, i.e., that the sequence $\{h_i \colon \xi_1|_{C_{i+1}} \to \xi_2|_{C_{i+1}}\}$ converges to an F-equivalence $\xi_1 \to \xi_2$, it is enough to remark that the sets constitute a fundamental cover of $X \times I$, and then apply Corollary 4.3.2.7 to the sequences $\{h_i\}$ and $\{h_i^{-1}\}$. □

The Sets Stnrd(B, F)

Remark 4.4.1.6. We let $\operatorname{Stnrd}(B, F)$ denote the set of F-equivalence classes of Steenrod F-bundles over B. Below we shall study the mappings of this set into itself, defined by the induced bundle construction, by the extension of the structure group, and by the associated bundle construction.

(**Transcriber′snote**: Here the original authours try to avoid functional treatment(s). This has the advantage that it does not force the reader to swallow a lot of category theory first, but in the end the reader may want to reconstruct everything in the framework of axiomatic homotopy theory.)

For any continuous map $f \colon B' \to B$, the rule $\xi \mapsto f^!\xi$

$$\begin{tikzcd}
\operatorname{tl}(f^!\xi) \arrow[r] \arrow[d] & \operatorname{tl}(\xi) \arrow[d] \\
B' \arrow[r, "f"'] & B
\end{tikzcd}$$

defines a mapping

$$f! \colon \operatorname{Stnrd}(B, F) \to \operatorname{Stnrd}(B', F).$$

If B' is a cellular space, Theorem 4.4.1.5 shows that $f^!$ depends only on the homotopy class of f. In particular, if B and B' are both cellular and f is a homotopy equivalence, then f ! is invertible.

The extension of the structure group, which transforms the G_1-space F into the effective G-space F, defines a mapping

$$\operatorname{ext} \colon \operatorname{Stnrd}(B, F_1) \to \operatorname{Stnrd}(B, F),$$

for any topological space B. This mapping is natural: that is to say, the diagramme

$$\begin{tikzcd}
\operatorname{Stnrd}(B, F_1) \arrow[r, "\operatorname{ext}"] \arrow[d, "f^!"'] & \operatorname{Stnrd}(B, F) \arrow[d, "f^!"] \\
\operatorname{Stnrd}(B', F_1) \arrow[r, "\operatorname{ext}"'] & \operatorname{Stnrd}(B', F)
\end{tikzcd}$$

commutes for any continuous map $f \colon B' \to B$; see Remark 4.3.2.4.

Given another effective G-space, F', the rule $\xi \mapsto \operatorname{assoc}(\xi, F')$ defines for any topological space B a mapping

$$\operatorname{assoc}\colon \operatorname{Stnrd}(B,F) \to \operatorname{Stnrd}(B,F').$$

This mapping is invertible [its inverse is $\operatorname{assoc}\colon \operatorname{Stnrd}(B,F') \to \operatorname{Stnrd}(B,F)$] and also natural, i.e., the diagramme

$$\begin{tikzcd}
\operatorname{Stnrd}(B,F) \arrow[r, "\operatorname{assoc}"] \arrow[d, "f^!"'] & \operatorname{Stnrd}(B,F') \arrow[d, "f^!"] \\
\operatorname{Stnrd}(B',F) \arrow[r, "\operatorname{assoc}"'] & \operatorname{Stnrd}(B',F')
\end{tikzcd}$$

commutes for any continuous map $f\colon B' \to B$; see Remark 4.3.3.5.

Moreover, the diagram

$$\begin{tikzcd}
\operatorname{Stnrd}(B,F_1) \arrow[r, "\operatorname{assoc}"] \arrow[d, "f^!"'] & \operatorname{Stnrd}(B,F_1') \arrow[d, "f^!"] \\
\operatorname{Stnrd}(B,F) \arrow[r, "\operatorname{assoc}"'] & \operatorname{Stnrd}(B,F')
\end{tikzcd} \tag{4.4.1.7}$$

commutes for any topological space B, any effective G_1-spaces F_1 and F_1', and any effective G-spaces, F and F', obtained from F_1 and F_1' by extension of the structure group; see Remarks 4.3.3.5 and 4.3.3.6.

4.4.2 Universal Bundles

Remark 4.4.2.1. Let F be an effective G-space. By Theorem 4.4.1.5, given any Steenrod F-bundle ξ, any cellular space B, and any continuous map $f\colon B \to \operatorname{bs}(\xi)$, we may consider the bundle $f^!\xi$. This defines a mapping $\pi(B, \operatorname{bs}\xi) \to \operatorname{Stnrd}(B,F)$, which we denote by $\operatorname{induz}(B,\xi)$: $\operatorname{induz}(B,\xi)$ (the homotopy class of f) = the F-equivalence class of $f^!\xi$.

The following diagramme is obviously commutative for any topological space C and any continuous map $g\colon C \to \operatorname{bs}\xi$

$$\begin{tikzcd}
\pi(B,C) \arrow[rr, "\operatorname{induz}(B{,}g^!\xi)"] \arrow[dr, "\pi(\operatorname{id}_B{,}g)"'] & & \operatorname{Stnrd}(B,F) \arrow[dl, "\operatorname{induz}(B{,}\xi)"] \\
& \pi(B,\operatorname{bs}\xi) &
\end{tikzcd}$$

Similarly, the diagramme

$$\begin{tikzcd}
\pi(B,\operatorname{bs}\xi) \arrow[rr, "\operatorname{induz}(B{,}\operatorname{assoc}(\xi{,}f')"] \arrow[dr, "\operatorname{induz}(B{,}\xi)"'] & & \operatorname{Stnrd}(B,F') \arrow[dl, "\operatorname{assoc}"] \\
& \operatorname{Stnrd}(B,F) &
\end{tikzcd}$$

commutes for any effective G-space F'.

Definition 4.4.2.2. A Steenrod F-bundle ζ is called *universal* if the map $\operatorname{induz}(B, \zeta)$ is invertible for any cellular space B. In other words, a Steenrod F-bundle ζ is universal if:

(i) given any Steenrod F-bundle ξ with cellular base, there is a continuous map $f\colon \operatorname{bs}\xi \to \operatorname{bs}\zeta$ such that the bundle $f^!\zeta$ is F-equivalent to ξ,

(ii) if for two arbitrary continuous maps, f_1 and f_2, of a cellular space into $\operatorname{bs}\zeta$ the bundles $f_1^!\zeta$ and $f_2^!\zeta$ are F-equivalent, then f_1 and f_2 are homotopic.

Conditions (i) and (ii) have an equivalent formulation respectively:

(i)') given any Steenrod F-bundle ξ with cellular base, there is an F-map $\xi \to \zeta$. Indeed, if $f\colon \operatorname{bs}\xi \to \operatorname{bs}\zeta$ is continuous and $g\colon \xi \to f^!\zeta$ is an F-equivalence, then $\operatorname{adj} f \circ g\colon \xi \to \zeta$ is an F-map. Conversely, if $h\colon \xi \to \zeta$ is an F-map, then $\operatorname{corr} h\colon \xi \to (\operatorname{bs}(h))^!\zeta$ is an F-equivalence (see Corollary 4.3.2.9).

(ii)') given any Steenrod F-bundle ξ with cellular base and any F-maps $h_0, h_1 : \xi \to \zeta$, $\operatorname{bs} h_0$ and $\operatorname{bs} h_1$ are homotopic. Indeed, suppose that ξ, h_0, h_1 have these properties; then both bundles, $(\operatorname{bs}(h_0))^!\zeta$ and $(\operatorname{bs}(h_1))^!\zeta$, are F-equivalent to ξ, and so, by (ii), $\operatorname{bs}(h_0))$ and $\operatorname{bs}(h_1)$ are homotopic. Conversely, if f_0 and f_1 are continuous maps of a cellular space into $\operatorname{bs}\xi$ and $h\colon f_0^!\zeta \to f_1!\zeta$ is an F-equivalence, then $f_0 = \operatorname{bs}\operatorname{adj} f_0$, $f_1 = \operatorname{bs}(\operatorname{adj} f_1 \circ h)$ and, by (ii'), f_0 and f_1 are homotopic.

We remark also that both (i') and (ii') (and hence (i) and (ii)) are consequences of the following condition: given an arbitrary F-bundle ξ with cellular base and an arbitrary subspace A of $\operatorname{bs}\xi$, every F-map $\xi|_A \to \zeta$ extends to an F-map $\xi \to \zeta$. To see that this condition implies (i'), it suffices to take $A = \varnothing$. To see that it implies (ii'), take the F-bundle $\xi \times (I, \mathrm{id}_I, I)$, the subspace

$$A = (\operatorname{bs}\xi \times 0) \cup (\operatorname{bs}\xi \times 1) \subset \operatorname{bs}(\xi \times (I, \mathrm{id}_I, I)) = \operatorname{bs}\xi \times I,$$

and take as the F-map that must be extended

$$g\colon \xi \times (I, \mathrm{id}_I, I)|_{(\operatorname{bs}\xi\times 0)\cup(\operatorname{bs}\xi\times 1)} \to \zeta,$$

with

$$\begin{aligned} \operatorname{bs} g(b,0) = \operatorname{bs} h_0(b), \operatorname{bs} g(b,1) = \operatorname{bs} h_1(b) \quad & [b \in \operatorname{bs}\xi], \\ \operatorname{tl} g(x,0) = \operatorname{tl} h_0(x), \operatorname{tl} g(x,1) = \operatorname{tl} h_1(x) \quad & [x \in \operatorname{tl}\xi]. \end{aligned}$$

Theorem 4.4.2.3. *Every bundle induced from a universal bundle by a homotopy equivalence is universal.*

Proof. Indeed, if ζ is a universal F-bundle and $f\colon B \to \operatorname{bs}\zeta$ is a homotopy equivalence, then the map $\operatorname{induz}(B, f^!\zeta)$ coincides with $\operatorname{induz}(B, \zeta) \circ \pi(f, \mathrm{id}_B)$ for any cellular space (see Remark 4.4.2.1), and hence it is invertible. □

Theorem 4.4.2.4. *If the F-bundles ζ and ζ' are universal and* $\mathrm{bs}(\zeta)$ *and* $\mathrm{bs}(\zeta')$ *are cellular spaces, then* $\mathrm{bs}(g)\colon\ \mathrm{bs}(\zeta) \to \mathrm{bs}(\zeta')$ *is a homotopy equivalence for any F-map $g\colon \zeta \to \zeta'$.*

Proof. Pick an F-map $g\colon \zeta' \to \zeta$. Since $g' \circ g\colon \zeta' \to \zeta'$ and $g' \circ g\colon \zeta \to \zeta$ are F-maps, condition in Definition 4.4.2.2 (ii') implies that the map

$$\mathrm{bs}(g') \circ \mathrm{bs}(g) = \mathrm{bs}(g' \circ g) \quad (\text{respectively}, \quad \mathrm{bs}(g) \circ \mathrm{bs}(g') = \mathrm{bs}(g \circ g'))$$

is homotopic to $\mathrm{id}_{(\mathrm{bs}(\zeta))}$ (respectively, to $\mathrm{id}_{(\mathrm{bs}(\zeta'))}$). □

Theorem 4.4.2.5. *A bundle associated with a universal bundle is itself universal.*

Proof. Indeed, let ζ be a universal F-bundle and let F' be another effective G-space. Then for any cellular space B, $\mathrm{induz}(B, \mathrm{assoc}(\zeta, F'))$ is precisely the composition of the maps $\mathrm{induz}(B, \zeta)$ and $\mathrm{assoc}\colon\ \mathrm{Stnrd}(B, F) \to \mathrm{Stnrd}(B, F')$ (see Remark 4.4.2.1), and hence is invertible. □

Classifying Spaces

Theorem 4.4.2.6. *As Theorem 4.4.2.5 shows, the base of a universal bundle with structure group G is simultaneously the base of all universal bundles with structure group G and all possible standard fibres, and so we may say that it does not depend upon the choice of the standard fibre. This base is called a classifying space of the group G.*

From Theorem 4.4.2.3 it follows that every space which has the same homotopy type as a classifying space of G is itself a classifying space of G. Moreover, it results from Theorem 4.4.2.4 that any two cellular classifying spaces of G have the same homotopy type.

k-Universal Bundles

Definition 4.4.2.7. A Steenrod F-bundle ζ is called *k-universal* if

$$\mathrm{induz}(B, \zeta)\colon \pi(B, \mathrm{bs}(\zeta)) \to \mathrm{Stnrd}(B, F)$$

is surjective for each cellular space B of dimension $\leq k$ and injective for each cellular space B of dimension $\leq k-1$. In other words, a Steenrod F-bundle ζ is k-universal if:

a) given any Steenrod F-bundle 5ξ with cellular base of dimension $\leq k$, there is a continuous map $f\colon\ \mathrm{bs}\,\xi \to \mathrm{bs}\,\zeta$ such that the bundle $f^!(\xi)$ is F-equivalent to ξ,

b) any two continuous maps, f_0 and f_1, from a cellular space of dimension $\leq k-1$ into $\mathrm{bs}\,\zeta$, such that the bundles $f_0^!(\zeta)$ and $f_1^!(\zeta)$ are F-equivalent, are homotopic.

Equivalent conditions are:

a') given any Steenrod F-bundle ξ with cellular base of dimension $\leq k$, there is an F-map $\xi \to \zeta$ and

b') for any Steenrod F-bundle ξ with cellular base of dimension $\leq k-1$ and any F-maps $g_0, g_1 \colon \xi \to \zeta$, $\mathrm{bs}(g_0)$ and $\mathrm{bs}(g_1)$ are homotopic.

In all these formulations k is a positive integer, and the universal bundles are sometimes termed ∞-*universal.* Every k-universal bundle is obviously ℓ-universal for $\ell \leq k$. Moreover, Theorems 2.3.2.6 and 2.3.2.7 show that the restriction of a k-universal bundle with cellular base to a subspace of the base which contains its ℓ-skeleton is ℓ-universal, for any $\ell \leq k$.

Theorems 4.4.2.3 and 4.4.2.5 have immediate analogues for k-universal bundles: every bundle induced from a k-universal bundle by a homotopy equivalence is k-universal; a bundle associated with a k-universal bundle is itself k-universal.

4.4.3 The Milnor Bundles

Definition 4.4.3.1. Below we shall construct for any topological group G a principal bundle with structure group G, called the *Milnor G-bundle* and denoted by $\mathrm{Mil}\,G$. In Remarks 4.4.3.2, 4.4.3.5 and Theorems 4.4.3.3, 4.4.3.5 we shall prove that $\mathrm{Mil}\,G$ is a universal G-bundle.

Let $TG(k)$ denote the join of k copies of the group G. Then $TG(k)$ embeds naturally in $TG(k+1)$ (as a base of the join $TG(k) \star G = TG(k+1)$) and so the $TG = \varinjlim_k TG(k)$ is meaningful. The right action $G \times G \to G$, $(g_1, g) \mapsto g^{-1}g_1$, extends to a free, continuous, right action of G on $TG(k)$. Since the inclusions $TG(k) \to TG(k+1)$ are G-maps with respect to this action, G acts also on TG. Therefore, a G-bundle $(TG, \mathrm{proj}, TG/G)$ results, and this is $\mathrm{Mil}\,G$.

If G_1 is a subgroup of G, then there exist the inclusions

$$\underbrace{\mathrm{incl} \star \cdots \star \mathrm{incl}}_{k} \colon TG_1(k) \to TG(k) \quad (k = 1, 2, \dots),$$

where $\mathrm{incl} = [\mathrm{incl} \colon G_1 \to G]$, and all of them are incl-maps. Moreover, they are compatible with the embeddings

$$TG(k) \to TG(k+1), \quad TG_1(k) \to TG_1(k+1),$$

and hence the limit map $TG_1 \to TG$ is meaningful (see Remark 1.2.4.4). This map is also an incl-map, and together with the map $TG_1/G_1 \to TG/G$ that it induces, it clearly yields an $(\mathrm{incl}, \mathrm{incl})$-map $\mathrm{Mil}\,G_1 \to \mathrm{Mil}\,G$. Therefore, to each inclusion $G_1 \to G$ corresponds an $(\mathrm{incl}, \mathrm{incl})$-map $\mathrm{Mil}\,G_1 \to \mathrm{Mil}\,G$.

$\mathrm{Mil}\,G$ **is Locally Trivial**

Remark 4.4.3.2. It is convenient to identify $TG(k)$ with that subspace of the product $\underbrace{\mathrm{cone}\,G \times \cdots \times \mathrm{cone}\,G}_{k}$ which consists of the points $\{\mathrm{proj}(g_i, t_i)\}_{i=1}^{k}$ such that $t_1 + \cdots + t_k = 1$, and is canonically homeomorphic to $TG(k)$ (see Remark 1.2.6.4; here $\mathrm{proj} = [\mathrm{proj}\colon G \times I \to \mathrm{cone}\, G]$). After this identification, the points of TG may be represented as sequences $\{\mathrm{proj}(g_i, t_i)\}_{i=1}^{\infty}$ such that $\sum t_i = 1$ and only a finite number of the t_i's are non-zero. Now the right action of G on TG (see Definition 4.4.3.1) is described as $\mathrm{proj}(g_i, t_i)g = \{\mathrm{proj}(g^{-1}g_i, t_i)\}$.

Theorem 4.4.3.3. $\mathrm{Mil}\, G$ *is locally* G*-trivial.*

Proof. Let U_s be the collection of sequences $\{\mathrm{proj}(g_i, t_i)\}$ with $t_s \neq 0$, and consider the sets $\mathrm{proj}\,\mathrm{Mil}\, G(U_1), \mathrm{proj}\,\mathrm{Mil}\, G(U_2), \dots$. These sets are open, cover TG/G, and over each of them the bundle $\mathrm{Mil}\, G$ is G-trivial: the G-trivialisation

$$\mathrm{proj}\,\mathrm{Mil}\, G(U_s) \times G \to (\mathrm{proj}\,\mathrm{Mil}\, G)^{-1}(\mathrm{proj}\,\mathrm{Mil}\, G(U_s)) \quad [= U_s]$$

takes each point (x, g) into the sequence $\{\mathrm{proj}(g_i, t_i)\}$ determined by the conditions: $\mathrm{proj}\,\mathrm{Mil}\, G(\{\mathrm{proj}(g_i, t_i)\}) = x$ and $g_s = g$. ☐

Mi G is Universal

Theorem 4.4.3.4. $\mathrm{Mil}\, G$ *is universal.*

Proof. According to Definition 4.4.2.2, it suffices to show that given any G-bundle ξ with cellular base and any subspace $A \subset \mathrm{bs}\,\xi$, every G-map $f\colon \xi|_A to\, \mathrm{Mil}\, G$ extends to a G-map $\xi \to \mathrm{Mil}\, G$.

We consider first the case $\mathrm{bs}\,\xi = \mathbb{D}^{r+1}$, $A = \mathbb{S}^r$, for some r. Then ξ is G-trivial (see Remark 4.4.1.3), and we may actually assume that ξ is the standard trivial G-bundle $(\mathbb{D}^{r+1} \times G, \mathrm{proj}_1, \mathbb{D}^{r+1})$. The desired extension $h\colon \xi \to \mathrm{Mil}\, G$ has an explicit description: let k be the smallest number s such that $TG(s) \supset \mathrm{tl}(f(\mathbb{S}^r \times e_G)$, and let φ_i denote the composite map

$$\mathbb{S}^r \times G \xrightarrow{\mathrm{abr}\,\mathrm{tl}(f)} TG(k) \xrightarrow{\mathrm{incl}} \underbrace{\mathrm{cone}\, G \times \cdots \times \mathrm{cone}\, G}_{k} \xrightarrow{\mathrm{proj}_1} \mathrm{cone}\, G$$

$(i = 1, \dots, k)$; further, define i$\psi\colon \mathbb{D}^{r+1} \times G \to TG(k+1)$ by

$$\psi(ty, g) = (t\varphi_1(y), \dots, t\varphi_k(y), \mathrm{proj}(g, 1-t)),$$

where $y \in \mathbb{S}^r$, $t \in I$, and $\mathrm{proj} = [\mathrm{proj}\colon G \times I \to \mathrm{cone}\, G]$. Now set $\mathrm{tl}(h) = [\mathrm{incl}\colon TG(k+1) \to TG] \circ \psi$.

The general case reduces to this special one. Indeed, assume that the space $\mathrm{bs}\,\xi$ is rigged and that a G-map $h\colon \xi|_{A \cup \mathrm{skel}_r \mathrm{bs}(\xi)} \to \mathrm{Mil}\, G$ extending f is already available. The above argument shows that for each cell

$$e \in \mathrm{cell}_{r+1} \mathrm{bs}(\xi) \setminus \mathrm{cell}_{r+1} A$$

the G-map

$$f_e = h_r \circ [\mathrm{abr}\,\mathrm{char}\colon \mathbb{S}^r \to A \cup \mathrm{skel}_r \mathrm{bs}(\xi)]\colon\ \mathrm{char}_e^!(\xi)|_{\mathbb{S}^r} \to \mathrm{Mil}\, G$$

extends to a G-map $g_e\colon \operatorname{char}^!_q(\xi) \to \operatorname{Mil} G$, and it is clear that $\operatorname{tl}(g_e)$ is constant on the elements of the partition $\operatorname{zer}(\operatorname{tl} \operatorname{adj} Char_e)$. Applying Theorem 4.3.2.6 (with $B = \mathbb{D}^{r+1}$ and $p = [\operatorname{abr} Char_e\colon \mathbb{D}^{r+1} \to \operatorname{Cl} e]$), we see that g_e defines a G-map $\xi|_{\operatorname{Cl} e} \to \operatorname{Mil} G$, which we denote by h_0. Further, note that for any cells

$$e_1, e_2 \in \operatorname{cell}_{r+1} \operatorname{bs}(\xi) \setminus \operatorname{cell}_{r+1} A,$$

$\operatorname{tl}(h_{e_1})$ and $\operatorname{tl}(h_{e_2})$ agree over $\operatorname{Cl} e_1 \cap \operatorname{Cl} e_2$, and that for any cell

$$e \in \operatorname{cell}_{r+1} \operatorname{bs}(\xi) \setminus \operatorname{cell}_{r+1} A,$$

$\operatorname{tl}(h_e)$ and $\operatorname{tl}(h_r)$ agree over $\operatorname{Cl} e \cap (A \cup \operatorname{skel}_r \operatorname{bs}(\xi))$. This implies that h_r and h_e, $e \in \operatorname{cell}_{r+1} \operatorname{bs}(\xi) \setminus \operatorname{cell}_{r+1} A$, yield together a G-map

$$h_{r+1}\colon \xi|_{A \cup \operatorname{skel}_{r+1} \operatorname{bs}(\xi)} \to \operatorname{Mil} G$$

extending h_r (see Corollary 4.3.2.7). Therefore, using induction, we can produce a sequence

$$\{h_s\colon \xi|_{A \cup \operatorname{skel}_s \operatorname{bs}(\xi)} \to \operatorname{Mil} G\}_{s=-1}^{\infty}$$

of G-maps with $h_{-1} = f$, such that h_s extends h_{s-1} for all $s \geq 0$. Since the sets $A \cup \operatorname{skel}_s \operatorname{bs}(\xi)$ constitute a fundamental cover of $\operatorname{bs}(\xi)$, using again Corollary 4.3.2.7 we conclude that the h_s's yield a G-map $\xi \to \operatorname{Mil} G$ extending f. □

A Promise

Remark 4.4.3.5. The base of the bundle $\operatorname{Mil} G$ is not a cellular space. However, we shall see in Chapter 5 that for any topological group G there are also universal G-bundles with cellular base; see Theorem 5.6.1.4. By Definition 4.4.2.7, this will imply the existence of k-universal G-bundles with cellular base of dimension $\leq k$, for any given topological group G and any positive integer k.

4.4.4 Reductions of the Structure Group

Definition 4.4.4.1. We say that the Steenrod F_1-bundle ξ_1 with structure group G_1 is obtained from the Steenrod F-bundle ξ with structure group G by *reducing the group G to G_1* if ξ is obtained from ξ_1 by extending the group G_1 to G.

While the extension of the structure group of a Steenrod bundle is a well-defined operation, the reduction of the structure group cannot be carried out for every Steenrod bundle, and even when it is possible, it may produce bundles which are not equivalent with respect to the reduced group. In other words, the mapping

$$\operatorname{ext}\colon \operatorname{Stnrd}(B, F) \to \operatorname{Stnrd}(B, F) \tag{4.4.4.2}$$

defined in Remark 4.4.1.6 may be both non-surjective and non-injective.

We remark that the set-theoretic properties of the mapping (4.4.4.2) are uniquely determined by the triple B, G, G_1, i.e., they are preserved when we replace F and F_1 by other effective G-spaces and their corresponding G_1-spaces, while keeping B, G, and G_1 the same; this is plain from diagramme (4.4.1.7).

Recall that given a cellular space B, $\operatorname{Stnrd}(B,F)$ can be interpreted as the set of homotopy classes of continuous maps from B into a classifying space of the structure group. Below we describe (4.4.4.2) in the same homotopy terms.

Definition 4.4.4.3. Let G, G_1, F, F_1 be as in Definition 4.4.4.1, and let ζ and ζ_1 be universal bundles with standard fibres F and F_1. A continuous map $\psi\colon \operatorname{bs}(\zeta_1) \to \operatorname{bs}\zeta$ is called *classifying* if $\psi^!(\zeta)$ is F-equivalent to the bundle obtained from ζ_1 by extending the structure group G_1 to G. By the definition of a universal bundle, such a map exists whenever $\operatorname{bs}(zeta_1)$ is a cellular space, and so it certainly exists when $\zeta_1 = \operatorname{Mil} G$ and $\zeta_1 = \operatorname{Mil} G_1$ (see Definition 4.4.3.1).

Theorem 4.4.4.4. *Our main claim is that the diagramme*

$$\begin{tikzcd}
\operatorname{Stnrd}(B,F_1) \arrow[r, "\operatorname{ext}"] & \operatorname{Stnrd}(B,F) \\
\pi(B,\operatorname{bs}(\zeta_1)) \arrow[u, "\operatorname{induz}(B,\zeta_1)"] \arrow[r, "\pi(\operatorname{id}_B,\psi)"'] & \pi(B,\operatorname{bs}(\zeta)) \arrow[u, "\operatorname{induz}(B,\zeta)"']
\end{tikzcd} \tag{4.4.4.5}$$

commutes for any classifying map ψ and any cellular space B.

Proof. The composition $\operatorname{induz}(B,C) \circ \pi(\operatorname{id}_B,\zeta)$ takes the homotopy class of $f_1\colon B \to \operatorname{bs}\zeta_1$ into the class of the bundle $(\psi \circ f_1)^!(\zeta)$, while the same homotopy class is taken by $\operatorname{ext}\circ\operatorname{induz}(B,\zeta_1)$ into the class of the bundle obtained from $f_1)^!\zeta_1$ extending the structure group G_1 to G. Since the extension of the structure group and the induction construction commute (see Remark 4.4.1.6), the last class contains $f_1^!(\psi^!\zeta)$, and it remains to observe that $f_1^!(\psi^!\zeta) = (\psi \circ f_1)^!\zeta$. □

Remark 4.4.4.6. The commutativity of the diagramme (4.4.4.5) and the invertibility of its vertical mappings imply that $\operatorname{induz}(B,\zeta_1)$ is an injective mapping from the set of homotopy classes of maps $g\colon B \to \operatorname{bs}(\zeta_1)$, such that $\psi \circ g$ is homotopic to a given map $f\colon B \to \operatorname{bs}(\zeta)$, onto the set of classes of F_1-equivalent F_1-bundles which are obtained from $f^!\zeta$ by reducing the structure group G to G_1. In particular, a Steenrod F-bundle ξ with cellular base admits the reduction of the group G to G_1 if and only if any continuous map $f\colon \operatorname{bs}\xi \to \operatorname{bs}\zeta$ such that $f^!(\zeta)$ is F-equivalent to ξ is homotopic to the composition of some continuous map $\operatorname{bs}(\xi) \to \operatorname{bs}(\zeta_1)$ with ψ.

4.4.5 Exercises

Exercise 4.4.5.1. Given a topological group G and a positive integer k, denote by $\operatorname{Mil}(G,k)$ the restriction of the bundle $\operatorname{Mil} G$ to $TG(k)/G$, i.e., the bundle $(TG(k), \operatorname{proj}, TG(k)/G)$. Show that $\operatorname{Mil}(G,k)$ is a $(k-1)$-universal G-bundle.

Exercise 4.4.5.2. Show that $\operatorname{Mil}(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $(\mathbb{S}^\infty, \operatorname{proj}, \mathbb{R}P^\infty)$, while $\operatorname{Mil}(\mathbb{Z}/2\mathbb{Z},k)$ (see Exercise 4.4.5.1) is isomorphic to $(\mathbb{S}^{k-1}, \operatorname{proj}, \mathbb{R}P^{k-1})$.

Exercise 4.4.5.3. Show that $\operatorname{Mil}\mathbb{S}^1$ is isomorphic to $(\mathbb{S}^\infty, \operatorname{proj}, \mathbb{C}P^\infty)$, while $\operatorname{Mil}(\mathbb{S}^1, k)$ is isomorphic to $(S^{2k-1}, \operatorname{proj}, \mathbb{C}P^{k-1})$.

Exercise 4.4.5.4. Let X be a compact n-dimensional $\mathcal{C}^r$-manifold, $1 \le r \le \infty$. Consider the right action of $\operatorname{Diff}^r X$ on $\operatorname{Emb}^r(X, \mathbb{R}^q)$, given by

$$(j, \varphi) \mapsto j \circ \varphi, \quad [j \in \operatorname{Emb}^r(X, \mathbb{R}^q), \varphi \in \operatorname{Diff}^r X],$$

and the limit right action of $\operatorname{Diff} X$ on

$$\lim(\operatorname{Emb}^r(X, \mathbb{R}^q), \operatorname{abr}\mathcal{C}^r(\operatorname{id} X, \operatorname{incl})\colon \operatorname{Emb}^r(X, \mathbb{R}^q) \to \operatorname{Emb}^r(X, \mathbb{R}^{q+1})).$$

Show that

$$(\lim \operatorname{Emb}^r(X, \mathbb{R}^q), \operatorname{proj}, [\lim \operatorname{Emb}^r(X, \mathbb{R}^q)] / \operatorname{Diff}^r X)$$

is a universal $\operatorname{Diff}^r X$-bundle, while

$$(\operatorname{Emb}^r(X, \mathbb{R}^q), \operatorname{proj}, \operatorname{Emb}^r(X, \mathbb{R}^q) / \operatorname{Diff}^r X)$$

is a $(q - 2n - 1)$-universal $\operatorname{Diff}^r X$-bundle, for any $q \ge 2n + 1$.

4.5 VECTOR BUNDLES

4.5.1 General Definitions

Remark 4.5.1.1. The main objective of this section is to study those Steenrod bundles whose standard fibre is either $\mathbb{R}^n$ with the usual action of one of the groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}_+(n,\mathbb{R})$, $\mathrm{O}(n)$, or $\mathrm{SO}(n)$, or $\mathbb{C}^n$ with the usual action of $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{U}(n)$.

Since all the standard fibres listed above are topologically effective, the corresponding bundles may be also regarded as Ehresmann–Feldbau bundles with the same standard fibres (see Subsection 4.3.4). We shall proceed in this way and ignore completely the topology on the set of marked homeomorphisms in the course of the entire section.

To simplify the discussion, we introduce a special notation for the above standard fibres: $\mathrm{GL}\,\mathbb{R}^n$, $\mathrm{GL}_+\,\mathbb{R}^n$, $\mathrm{O}\,\mathbb{R}^n$, $\mathrm{SO}\,\mathbb{R}^n$, and $\mathrm{GL}\,\mathbb{C}^n$, $\mathrm{U}\,\mathbb{C}^n$.

Standard Fibre $\mathrm{GL}\,\mathbb{R}^n$

Definition 4.5.1.2. A Steenrod bundle with standard fibre $\mathrm{GL}\,\mathbb{R}^n$ is called an *n-dimensional real vector bundle.*

Since a space with a $\mathrm{GL}\,\mathbb{R}^n$-structure is simply an n-dimensional real vector space (see Example 4.3.1.3), a W-$\mathrm{GL}\,\mathbb{R}^n$-bundle is simply a bundle whose fibres are n-dimensional real vector spaces. Moreover, a W-$\mathrm{GL}\,\mathbb{R}^n$-equivalence of W-$\mathrm{GL}\,\mathbb{R}^n$-bundles is an equivalence that is linear on fibres. Therefore, an n-dimensional real vector bundle is a bundle whose fibres are n-dimensional real vector spaces, and which is locally trivial in the natural vector sense: every point of the base has a neighbourhood U over which the given bundle is equivalent to $(U \times \mathbb{R}^n, \mathrm{proj}_1, U)$ via an equivalence which is linear on each fibre.

Theorem 4.5.1.3. *A bundle ξ whose fibres are n-dimensional real vector spaces is an n-dimensional real vector bundle (i.e., ξ is locally trivial in the previous vector sense) if and only if:*

1. *ξ is topologically locally trivial;*
2. *the partial vector operations in* $\mathrm{tl}(\xi)$*, i.e., the maps*

$$\begin{aligned}
&\mathbb{R} \times \mathrm{tl}(\xi) \to \mathrm{tl}(\xi), \quad (\lambda, x) \mapsto \lambda x,\\
&\{(x_1, x_2) \in \mathrm{tl}(\xi) \times \mathrm{tl}(\xi) \,|\, \mathrm{proj}\,\xi(x_1) = \mathrm{proj}\,\xi(x_2)\} \to \mathrm{tl}(\xi)),\\
&(x_1, x_2) \mapsto x_1 + x_2
\end{aligned}$$

are continuous.

Proof. The necessity of these conditions is obvious. Let us verify their sufficiency. Let $b_0 \in \mathrm{bs}(\xi)$. Fix an arbitrary basis, $v_1, \ldots, v_n$, of the vector space $(\mathrm{proj}(\xi))^{-1}(b_0)$, a neighbourhood U of b_0 such that $\xi|_U$ is topologically trivial, and a trivialisation $h\colon (U \times \mathbb{R}^n, \mathrm{proj}_1, U) \to \xi|_U$. Define a map,

$$h'\colon (U \times \mathbb{R}^n, \mathrm{proj}_1, U) \to \xi,$$

linear on fibres, by the formula

$$\operatorname{tl}(h'(b, \operatorname{ort}_i) = \operatorname{tl} h(b, \operatorname{proj}_2 \circ \operatorname{tl} h^{-1}(v_i)), \quad \operatorname{proj}_2 = [\operatorname{proj}_2 \colon U \times \mathbb{R}^n \to \mathbb{R}^n].$$

Now pick disjoint neighbourhoods, K and N, of the set

$$\operatorname{proj}_2 \circ \operatorname{tl}(h^{-1}) \circ \operatorname{tl} h'(b_0 \times \mathbb{S}^{n-1})$$

and of the point

$$\operatorname{proj}_2 \circ \operatorname{tl}(h^{-1}) \circ \operatorname{tl} h'(b_0, 0) \in \mathbb{R}^n,$$

respectively, and denote by V the neighbourhood of b_0 consisting of all $b \in U$ such that $\operatorname{tl} h'(b \times S^{n-1}) \subset \operatorname{tl} h(b \times K)$ and $\operatorname{tl} h'(b, 0) \in \operatorname{tl} h(b \times N)$. It is clear that

$$b \in V \Longrightarrow \operatorname{tl} h'(b, 0) \notin \operatorname{tl} h'(b \times \mathbb{S}^{n-1}),$$

and thus the map $\operatorname{abr} h' \colon (V \times \mathbb{R}^n, \operatorname{proj}_1, V) \to \xi|_V$ is non-degenerate on each fibre. Consequently, we can apply Theorem 4.3.2.8 to $\operatorname{abr} h'$, taking F to be $\mathbb{R}^n$, regarded as a $\operatorname{Top} \mathbb{R}^n$-space ($(V \times \mathbb{R}^n, \operatorname{proj}_1, V)$ and $\xi|_V$ are thought of as Steenrod F-bundles; see Remark 4.3.4.7). We conclude that $\operatorname{abr} h'$ is an equivalence in the topological sense, and since $\operatorname{abr} h'$ is also linear on fibres, the proof is complete. □

Standard Fibre $\mathrm{O}\,\mathbb{R}^n$

Definition 4.5.1.4. A Steenrod bundle with standard fibre $\mathrm{O}\,\mathbb{R}^n$ is called an *n-dimensional Euclidean bundle.*

Since a space with an $\mathrm{O}\,\mathbb{R}^n$-structure is an n-dimensional Euclidean space, a W-$\mathrm{O}\,\mathbb{R}^n$-bundle is simply a bundle whose fibres are n-dimensional Euclidean spaces. Moreover, it is clear that any W-$\mathrm{O}\,\mathbb{R}^n$-equivalence of W-$\mathrm{O}\,\mathbb{R}^n$-bundles is an equivalence which is an orthogonal map on each fibre. Therefore, an n-dimensional Euclidean bundle is a bundle whose fibres are n-dimensional Euclidean spaces and which is locally trivial in the natural Euclidean sense: every point of the base has a neighbourhood U over which the bundle is equivalent to $(U \times \mathbb{R}^n, \operatorname{proj}_1, U)$ via an equivalence which is an orthogonal map on each fibre.

Theorem 4.5.1.5. *A bundle ξ whose fibres are n-dimensional Euclidean spaces is an n-dimensional Euclidean bundle (i.e., ξ is locally trivial in the Euclidean sense) if and only if it satisfies*

(i) ξ is topologically locally trivial;

(ii) the partial vector operations in $\operatorname{tl}(\xi)$, i.e., the maps

$$\begin{aligned}
&\mathbb{R} \times \operatorname{tl}(\xi) \to \operatorname{tl}(\xi), \quad (\lambda, x) \mapsto \lambda x,\\
&\{(x_1, x_2) \in \operatorname{tl}(\xi) \times \operatorname{tl}(\xi) | \operatorname{proj} \xi(x_1) = \operatorname{proj} \xi(x_2)\} \to \operatorname{tl}(\xi)),\\
&(x_1, x_2) \mapsto x_1 + x_2
\end{aligned}$$

are continuous.

(iii) the function $\operatorname{tl}(\xi) \to \mathbb{R}$*, which takes each its length, is continuous.*

Note that (i) and (ii) are identical to 1. and 2. of Theorem 4.5.1.3.

Proof. These conditions are obviously necessary. Let us verify that they are also sufficient. By Theorem (4.5.1.3), (i) and (ii) imply that every point of $\operatorname{bs}(\xi)$ has a neighbourhood U together with a trivialisation linear on fibres, $h\colon (U \times \mathbb{R}^n, \operatorname{proj}_1, U) \to \xi|_U$. Let $v_1(b), \dots, v_n(b)$ be the basis of the vector space $(\operatorname{proj}(\xi))^{-1}(b)$, $b \in U$, resulting from the standard orthogonalisation of the basis $\operatorname{tl} h(b, \operatorname{ort}_1), \dots, \operatorname{tl} h(b, \operatorname{ort}_n)$. Now (iii) shows that the vectors $v_1(b), \dots, v_n(b)$ depend continuously on b, and it is clear that the map linear on fibres, $h'\colon (U \times \mathbb{R}^n, \operatorname{proj}_1, U) \to \xi|_U$ given by $\operatorname{tl} h'(b, \operatorname{ort}_i) = v_i(b)$ $(i = 1, \dots, n)$, is a trivialisation, orthogonal on each fibre, of the bundle $\xi|_U$. ☐

Remark 4.5.1.6. Since $\mathrm{O}(n) \subset \mathrm{GL}\,\mathbb{R}^n$, every n-dimensional Euclidean bundle ξ determines a unique n-dimensional real vector bundle ξ', through extension of the structure group. One may use Theorem 4.5.1.5 to interpret the reduction of the structure group transforming ξ' into ξ as enriching the bundle ξ' with an additional structure: namely, a Euclidean metric on each fibre, such that the corresponding length function $\operatorname{tl} \xi' \to \mathbb{R}$ is continuous. This additional structure is termed a *Euclidean metric* on ξ'.

Standard Fibres $\mathrm{GL}_+\,\mathbb{R}^n$ and $\mathrm{SO}\,\mathbb{R}^n$

Definition 4.5.1.7. A Steenrod bundle with standard fibre ${}_GL_+\mathbb{R}^n$ (respectively, $\mathrm{SO}\,\mathbb{R}^n$) is called an *$n$-dimensional oriented vector bundle* (respectively, an *n-dimensional oriented Euclidean bundle*).

Since a space with $\mathrm{GL}_+\,\mathbb{R}^n$-structure ($\mathrm{SO}\,\mathbb{R}^n$-structure) is simply an n-dimensional oriented vector space (respectively, an n-dimensional oriented Euclidean space), a W-$\mathrm{GL}_+\,\mathbb{R}^n$-bundle (a W-$\mathrm{SO}\,\mathbb{R}^n$-bundle) is simply a bundle whose fibres are n-dimensional oriented vector (respectively, Euclidean) spaces. It is also plain that a W-$\mathrm{GL}_+\,\mathbb{R}^n$-equivalence of W-$\mathrm{GL}_+\,\mathbb{R}^n$-bundles (a W-$\mathrm{SO}\,\mathbb{R}^n$-equivalence of W-$\mathrm{SO}\,\mathbb{R}^n$-bundles) is simply an equivalence which is orientation preserving and linear (respectively, orthogonal) on fibres. Consequently, an n-dimensional oriented vector bundle (Euclidean bundle) is a bundle whose fibres are n-dimensional oriented vector spaces (respectively, Euclidean spaces), and which is locally trivial in the following sense: every point of the base has a neighbourhood over which the bundle has a trivialisation that is orientation preserving and linear (respectively, orthogonal) on fibres.

Remark 4.5.1.8. To obtain a version of Theorem 4.5.1.3 which is suitable for the oriented case, note that the orientation existing on each fibre of an n-dimensional oriented vector bundle ξ maps the set of non-degenerate n-frames of the given fibre into $\mathbb{S}^0$. Furthermore, the set of all non-degenerate n-frames of the fibres of ξ is the total space of the associated bundle $\operatorname{assoc}(\xi, V'(n,n))$ [where $\mathrm{GL}_+(n, \mathbb{R})$ acts on $V'(n,n)$ as usual; see Theorem 2.3.1.3], and the orientations of the fibres combine to define a map $\operatorname{tl}\operatorname{assoc}(\xi, V'(n,n)) \to \mathbb{S}^0$. The “oriented” version of

Theorem 4.5.1.3 asserts that a bundleξ whose fibres are n-dimensional oriented real vector spaces is an n-dimensional oriented vector bundle if and only it satisfies conditions (1) and (2) of Theorem 4.5.1.3 and the condition
(iv): the function $\operatorname{tl}\operatorname{assoc}(\xi, V'(n,n)) \to \mathbb{S}^0$, defined by the orientations of the fibres of ξ, is continuous.

Theorem 4.5.1.5 must be modified in a similar fashion. Namely, given an n-dimensional oriented Euclidean bundle ξ, the orientation of each fibre of ξ maps the set of orthonormal n-frames of the given fibre into $\mathbb{S}^0$. Since the set of all orthonormal n-frames of all fibres of ξ equals the total space of the associated bundle $\operatorname{assoc}(\xi, V(n,n))$, we obtain a function $\operatorname{tl}\operatorname{assoc}(\xi, V(n,n)) \to \mathbb{S}^0$. The "oriented" version of Theorem 4.5.1.5 asserts that a bundle ξ whose fibres are n-dimensional oriented Euclidean spaces is an n-dimensional oriented Euclidean bundle if and only if it satisfies conditions (1) and (2) of Theorem 4.5.1.3, condition (iii) of Theorem 4.5.1.5, and condition
(v) : the function $\operatorname{tl}\operatorname{assoc}(\xi, V(n,n)) \to \mathbb{S}^0$, defined by the orientations of the fibres of ξ, is continuous.

Definition 4.5.1.9. Since $\mathrm{GL}_+(n,\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{R})$, every n-dimensional oriented real vector bundle ξ determines a unique n-dimensional real vector bundle ξ', obtained from ξ by extending the structure group. As it follows from the discussion in Remark 4.5.1.8, when we reduce the structure group and produce ξ from ξ', we are endowing the fibres of ξ'' with orientations which combine to define a continuous map

$$\operatorname{assoc}(\xi', V'(n,n) = \mathrm{GL}(n,\mathbb{R})) \to \mathbb{S}^0.$$

This additional structure is termed an *orientation of the bundle* ξ'.

Similarly, the inclusion $\mathrm{SO}(n) \subset \mathrm{O}(n)$ associates with every n-dimensional oriented Euclidean bundle ξ a unique n-dimensional Euclidean bundle ξ', obtained from ξ by extension of the structure group. Again, the reduction of the structure group which transforms ξ' into ξ is seen to provide ξ' with an orientation, i.e., as orienting its fibres in such a manner that the corresponding function

$$\operatorname{tl}\operatorname{assoc}(\xi', V(n,n) = \mathrm{O}(n)) \to \mathbb{S}^0$$

is continuous.

The real vector and Euclidean bundles possessing orientations are refereed to as *orientable*. Since every orientation may be replaced by the *opposite* one, as a result of multiplication by -1, every orientable bundle has at least two orientations.

Standard Fibres $\mathrm{GL}\,\mathbb{C}^n$ and $\mathrm{U}\,\mathbb{C}^n$

Definition 4.5.1.10. A Steenrod bundle with standard fibre $\mathrm{GL}\,\mathbb{C}^n$ ($\mathrm{U}\,\mathbb{C}^n$) is called an *n-dimensional complex vector* (respectively, *Hermitian*) bundle.

Our discussion of real vector bundles in Theorem 4.5.1.2 and Theorem 4.5.1.3 carries over, word-for-word, for complex vector bundles, and the same is true for the discussion in Definition 4.5.1.4, Theorem 4.5.1.5, and Remark 4.5.1.6 of Euclidean bundles and the Hermitian bundles. In particular, a W-GL $\mathbb{C}^n$-bundle (W-U $\mathbb{C}^n$-bundle) is simply a bundle whose fibres are n-dimensional complex vector spaces (respectively, n-dimensional Hermitian spaces); a W-GL $\mathbb{C}^n$-equivalence (W-U $\mathbb{C}^n$-equivalence) of W-GL $\mathbb{C}^n$-bundles(respectively, W-U $\mathbb{C}^n$-bundles) is simply an equivalence linear (respectively, Hermitian) on fibres; a W-GL $\mathbb{C}^n$-bundle (W-U $\mathbb{C}^n$-bundle) is locally trivial if and only if it is topologically locally trivial and, in addition, the vector operations (respectively, the vector operations and the length function) are continuous; the reduction of the structure group which turns a given complex vector bundle, ξ' into a Hermitian bundle ξ, may be interpreted as endowing ξ' with a Hermitian metric, i.e., as supplying a Hermitian metric (inner product) on each fibre of ξ', in such a manner that the resulting length function is continuous on $\operatorname{tl}(\xi')$.

Definition 4.5.1.11. Given an arbitrary n-dimensional complex vector bundle ξ, we may construct an n-dimensional complex vector bundle, $\operatorname{conj}\xi$, by replacing each marked homeomorphism α with the composition $\alpha \circ \operatorname{conj}$, where $\operatorname{conj}\colon \mathbb{C}^n \to \mathbb{C}^n$ is the usual complex conjugation; $\operatorname{conj}\xi$ is called the *bundle conjugate to* ξ.

This construction carries over to Hermitian bundles and produces again Hermitian bundles, i.e., to every n-dimensional Hermitian bundle ξ there corresponds the conjugate Hermitian bundle $\operatorname{conj}\xi$.

Definition 4.5.1.12. The extension of structure group defined by the inclusion $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$ turns n-dimensional complex vector bundles into $2n$-dimensional real vector bundles. Similarly, the extension defined by the inclusion $\mathrm{U}(n) \subset \mathrm{O}(2n)$ turns n-dimensional Hermitian bundles into $2n$-dimensional Euclidean bundles. In both cases we call the extension of the structure group *realification*, and we denote by $\mathbb{R}\xi$ the bundle obtained from ξ by realification.

We recall that the additional structure which turns a given $2n$-dimensional real vector space into an n-dimensional complex space may be described as a linear transformation whose square equals $-\operatorname{id}$ (a "multiplication by $-i$"). Similarly, the additional structure which turns a given $2n$-dimensional Euclidean space into an n-dimensional Hermitian space may be described as an orthogonal transformation whose square equals $-\operatorname{id}$. Accordingly, the additional structure which distinguishes between the n-dimensional complex bundle ξ and $\mathbb{R}\xi$ may be regarded as a $\mathrm{GL}\,\mathbb{R}^{2n}$-equivalence, $I\colon \mathbb{R}\xi \to \mathbb{R}\xi$, such that $I^2 = -(\operatorname{id}_{\mathbb{R}\xi})$ (the minus sign is defined fibre-wise),and that which distinguishes between the n-dimensional Hermitian bundle ξ and $\mathbb{R}\xi$ — as an $\mathrm{O}\,\mathbb{R}^{2n}$ -equivalence,$I\colon \mathbb{R}\xi \to \mathbb{R}\xi$, such that $I^2 = -(\operatorname{id}_{\mathbb{R}\xi})$. Moreover, we may think of the reduction of structure group which turns $\mathbb{R}\xi$ into ξ as endowing $\mathbb{R}\xi$, with one of the last two equivalences.

Obviously, $\mathbb{R}\operatorname{conj}\xi = \mathbb{R}\xi$ for any complex vector or Hermitian bundle ξ, and the shift $\xi \mapsto \operatorname{conj}\xi$ may be described in the previous language as the shift $I \mapsto -I$.

Remark 4.5.1.13. Since we have not only $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$ and $\mathrm{U}(n) \subset \mathrm{O}(2n)$, but also $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}_+(2n,\mathbb{R})$ and $\mathrm{U}(n) \subset \mathrm{SO}(2n)$, the realification of a given n-dimensional complex vector or Hermitian bundle may be effected in two steps: one may first extend $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{GL}_+(2n, \mathbb{R})$ ($\mathrm{U}(n)$ to $\mathrm{SO}(2n)$), and then extend $\mathrm{GL}_+(2n, \mathbb{R})$ to $\mathrm{GL}(2n, \mathbb{R})$ (respectively, $\mathrm{SO}(2n)$ to $\mathrm{O}(2n)$). Therefore, every complex vector or Hermitian bundle ξ provides $\mathbb{R}\xi$ with a canonical orientation.

Maps

Definition 4.5.1.14. A map of a real or complex vector bundle into another is said to be *linear* if it is linear on fibres. Those linear maps which are nondegenerate (injective) on fibres are called *linear monomorphisms*. A map between Euclidean bundles which is both isometric and linear on fibres is called an *orthogonal monomorphism*. A map between Hermitian bundles which is both isometric and linear on fibres is called a *unitary monomorphism*.

Note that a linear monomorphism between n-dimensional vector bundles is nothing else but a $\mathrm{GL}\,\mathbb{R}^n$-map in the real case and a $\mathrm{GL}\,\mathbb{C}^n$-map in the complex case. Similarly, an orthogonal (unitary) monomorphism between two n-dimensional Euclidean (respectively, Hermitian) bundles is simply an $\mathrm{O}\,\mathbb{R}^n$-map (respectively, a $\mathrm{U}\,\mathbb{C}^n$-map)) .

Vector Fields

Definition 4.5.1.15. Sections of vector bundles are called *vector fields*. This term is applied equally to the real and complex vector bundles, to the Euclidean and Hermitian bundles, and to the oriented bundles of both types.

A sequence of k vector fields is termed a *field of k-frames*. A point of the base where the corresponding frame is degenerate is a *singularity* of the given field. A field of k-frames with no singularities in an n-dimensional vector bundle may be regarded as a section of the associated bundle with fibre $\mathbb{R}V'(n, k)$ or $\mathbb{C}V'(n, k)$. Similarly, a field of orthonormal k-frames with no singularities in an n-dimensional Euclidean or Hermitian bundle may be thought of as a section of the associated bundle with fibre $\mathbb{R}V(n, k)$ or $\mathbb{C}V(n, k)$ [in all cases, the structure group acts as usual].

For every vector bundle there is the *zero* vector field, which takes each point of the base into the zero vector of the fibre over the given point. As we shall see, there are vector bundles having no sections without zeros. An n-dimensional real (complex) vector bundle ξ admits a field of n-frames with no singularities if and only if ξ is $\mathrm{GL}\,\mathbb{R}^n$-trivial (respectively, $\mathrm{GL}\,\mathbb{C}^n$-trivial); conversely, any such field yields a $\mathrm{GL}\,\mathbb{R}^n$-trivialisation (respectively, a $\mathrm{GL}\,\mathbb{C}^n$-trivialisation) of ξ. Similarly, for an n-dimensional Euclidean (Hermitian) bundle ξ, giving a field of orthonormal n-frames is equivalent to giving an $\mathrm{O}\,\mathbb{R}^n$-trivialisation (respectively, a $\mathrm{U}\,\mathbb{C}^n$-trivialisation) of ξ. Moreover, given an n-dimensional oriented real vector (Euclidean) bundle ξ, a field of n-frames without singularities (respectively, a field of orthonormal n-frames) such that the orientations of the fibres are positive on the frames of the field, yields a $\mathrm{GL}_+\,\mathbb{R}^n$-trivialisation (respectively, an $\mathrm{SO}\,\mathbb{R}^n$-trivialisation) of ξ.

4.5.2 Constructions

Remark 4.5.2.1. In this subsection we discuss a number of constructions which shall be applied afterwards to vector, Euclidean, and Hermitian bundles, and which were not covered by the general theory of Steenrod bundles from §4.3.

Subbundles

Definition 4.5.2.2. Let ξ be an n-dimensional real vector bundle, and let g be a section of the bundle $\operatorname{assoc}(\xi, \mathbb{R}G(n,k))$ weakly associated with ξ [here $\mathrm{GL}(n,\mathbb{R})$ acts on $\mathbb{R}G(n,k)$ in the usual way], i.e., a continuous function which takes each point $b \in \operatorname{bs}\xi$ into a k-dimensional subspace of the fibre $(\operatorname{proj}(\xi))^{-1}(b)$. Denote by T the union of all these k-dimensional subspaces, and let $\xi|_g$ be the bundle $(T, \operatorname{proj}(\xi)|_T, \operatorname{bs}(\xi))$. Since the fibres of $\xi|_g$ are the subspaces singled out by g, $\xi|_g$ is a $\mathrm{GL}\,\mathbb{R}^k$-bundle, and it is clear that $\xi|_g$ is locally $\mathrm{GL}\,\mathbb{R}^k$-trivial.

In fact, let ξ be the standard trivial bundle $(B \times \mathbb{R}^n, \operatorname{proj}_1, B)$. Given $b_0 \in B$, choose a linear isomorphism $\ell\colon \mathbb{R}^k \to g(b_0)$. Then for a sufficiently small neighbourhood U of b_0, the restriction of $\xi|_g$ to U admits even a canonical $\mathrm{GL}\,\mathbb{R}^n$-trivialisation,

$$h\colon (U \times \mathbb{R}^k, \operatorname{proj}_1, U) \to (\xi|_g)|_U, \quad \operatorname{tl}(h(b,v)) = \operatorname{proj}_b(b, \ell(v));$$

here proj_b denotes the projection of the fibre $b \times \mathbb{R}^n$ onto its subspace $g(b)$.

Therefore, $\xi|_g$ is a k-dimensional real vector bundle. We call it the *subbundle of ξ associated with g*.

Remark 4.5.2.3. The subbundles of Euclidean, complex vector, or Hermitian bundles are defined similarly. In the Euclidean case, $\mathbb{R}G(n,k)$ is regarded as an $\mathrm{O}(n)$-space, g remains a section of $\operatorname{assoc}(\xi, \mathbb{R}G(n,k))$, and the resulting subbundle $\xi|_g$, turns out to be Euclidean (its local $\mathrm{O}\,\mathbb{R}^k$-triviality is a consequence of its local triviality linear on fibres, established in Definition 4.5.2.2, and of the continuity of the length; see Theorem 4.5.1.5). The discussion in Definition 4.5.2.2 carries over, word-for word, to the complex case: all we have to do is to replace $\mathbb{R}$ by $\mathbb{C}$ [in particular, the $\mathrm{GL}(n,\mathbb{R})$-space $\mathbb{R}G(n,k)$ must be replaced by the $\mathrm{GL}(n,\mathbb{C})$-space $\mathbb{C}G(n,k)$]. The resulting subbundle, $\xi|_g$, is a complex vector bundle. Finally, in the Hermitian case, $\mathbb{C}G(n,k)$ is considered as a $\mathrm{U}(n)$-space and the sub-bundle $\xi|_g$ is Hermitian ($\xi|_g$ is locally $U\mathbb{C}^k$ -trivial because it is locally trivial, linearly on fibres, and the length function is continuous; see Definition 4.5.1.10).

Definition 4.5.2.4. It is clear that the inclusion $\xi|_g \to \xi$ is a linear monomorphism in both vector cases, an orthogonal monomorphism in the Euclidean case, and a unitary monomorphism in the Hermitian case. Conversely, to every linear, orthogonal, or unitary monomorphism, $f\colon \xi_1 \to \xi$, with $\operatorname{bs}(\xi_1) = \operatorname{bs}(\xi)$ and $\operatorname{bs}(f) = \mathrm{id}$, there corresponds a subbundle of ξ, namely the subbundle associated with the section $b \mapsto \operatorname{tl} f((\operatorname{proj}(\xi_1))^{-1}(b))$ of $\operatorname{assoc}(\xi, \mathbb{R}G(n, \dim\xi_1))$ or $\operatorname{assoc}(\xi, \mathbb{C}G(n, \dim\xi_1))$. This subbundle is the *image* of the monomorphism f, denoted $\operatorname{im} f$. By Theorem 4.3.2.8, $\operatorname{abr} f\colon \xi_1 \to \operatorname{im} f$ is a $\mathrm{GL}\,\mathbb{R}^n_{-}$, $\mathrm{GL}\,\mathbb{C}^n$-, $\mathrm{O}\,\mathbb{R}^n$-, or $\mathrm{U}\,\mathbb{C}^n$-equivalence, depending on whether we are in the real vector, complex vector, Euclidean, or Hermitian case.

We note that the correcting map, $\operatorname{corr} f\colon \xi_1 \to (\operatorname{bs} f)^!(\xi)$, which is a linear, orthogonal, or unitary monomorphism together with f (see Remark 4.3.2.4), always satisfies $\operatorname{bs}\operatorname{corr} f = \mathrm{id}$. Therefore, $\operatorname{im}\operatorname{corr} f$ is meaningful for any linear, orthogonal, or unitary monomorphism f.

Orthogonal Complements and Quotient Bundles

Definition 4.5.2.5. If ξ is an n-dimensional Euclidean (Hermitian) bundle, then to each section g of $\operatorname{assoc}(\xi, \mathbb{R}G(n,k))$ (respectively, of $\operatorname{assoc}(\xi, \mathbb{C}G(n,k))$) there corresponds the section $g^\perp$ orthogonal to g: $g^\perp$ is the section of $\operatorname{assoc}(\xi, \mathbb{R}G(n, n-k))$ (respectively, of $\operatorname{assoc}(\xi, \mathbb{C}G(n, n-k))$) which takes each $b \in \operatorname{bs}(\xi)$ into the orthogonal complement of the subspace $g(b)$ in the fibre $(\operatorname{proj}(\xi))^{-1}(b)$. Therefore, to every k-dimensional subbundle $\eta = \xi|_g$ of the n-dimensional Euclidean or Hermitian bundle ξ there corresponds an $(n-k)$-dimensional subbundle, $\xi_{g^\perp}$, called the *orthogonal complement* of the subbundle η, and denoted $\eta^\perp$.

Definition 4.5.2.6. A subbundle of a vector bundle has no orthogonal complement, but a corresponding *quotient bundle* is well defined. Namely, let η be a k-dimensional subbundle of the n-dimensional real or complex vector bundle ξ. Consider the bundle $(T, \operatorname{fact}\operatorname{proj}(\xi), \operatorname{bs}(\xi))$, where T is the quotient space of $\operatorname{tl}(\xi)$ by its partition into the sets $x + (\operatorname{proj}(\xi))^{-1}(b)$, with $x \in (\operatorname{proj}(\xi))^{-1}(b)$. The fibres of this bundle are the quotient spaces $(\operatorname{proj}(\xi))^{-1}(b)/(\operatorname{proj}(\eta))^{-1}(b)$, and so it is a $\mathrm{GL}\,\mathbb{R}^{n^k}$- or $\mathrm{GL}\,\mathbb{C}^{n^k}$–bundle. $(T, \operatorname{fact}\operatorname{proj}(\xi), \operatorname{bs}(\xi))$ is called the *quotient bundle* of ξ by η, denoted by ξ/η.

This construction and the previous one are related: indeed, if we apply the quotient bundle construction to a Euclidean or Hermitian bundle, we obtain the result of the orthogonal complement construction. More precisely, in the Euclidean case each quotient $(\operatorname{proj}(\xi))^{-1}(b)/(\operatorname{proj}(\xi))^{-1}(b)$ is an $(n-k)$-dimensional Euclidean space; hence, ξ/η is an $\mathrm{O}\,\mathbb{R}^{n-k}$-bundle, and the map $h\colon \eta \to \xi/\eta$, given by $\operatorname{tl} h(x) = \operatorname{proj}(x)$, where $\operatorname{proj} = [\operatorname{proj}\colon \operatorname{tl}(\xi) \to \operatorname{tl}(\xi/\eta)]$, is an $\mathrm{O}\,\mathbb{R}^n$-equivalence. Similarly, in the Hermitian case, $(\operatorname{proj}(\xi))^{-1}(b)/(\operatorname{proj}(\xi))^{-1}(b)$ are $(n-k)$-dimensional Hermitian spaces, and hence ξ/η is a $\mathrm{U}\,\mathbb{C}^{n-k}$-bundle, and the same h is a $\mathrm{U}\,\mathbb{C}^{n-k}$-equivalence. Therefore, the quotient bundle of a Euclidean (Hermitian) bundle ξ by a subbundle η is a Euclidean (respectively, Hermitian) bundle of dimension $\dim\xi - \dim\eta$.

From this it is readily seen that the quotient of a vector bundle ξ by a subbundle η is a vector bundle of dimension $\dim\xi - \dim\eta$. All we have to check is that ξ/η is locally trivial linearly on fibres whenever ξ is a standard trivial bundle. But this is immediate from the previous discussion if we note that such a ξ may be assumed to be Euclidean in the real case, and Hermitian in the complex case .

Remark 4.5.2.7. For real vector or Euclidean bundles, the orientation of each of the three spaces, $(\operatorname{proj}(\xi))^{-1}(b)$, $(\operatorname{proj}(\eta))^{-1}(b)$, $(\operatorname{proj}(\xi))^{-1}(b)/(\operatorname{proj}(\eta))^{-1}(b)$, is uniquely determined by the orientations of the other two (see Remark 3.1.3.10). Consequently, the orientability of two out of the three bundles, ξ, η, ξ/η implies the orientability of the third, and given orientations of two of them canonically determine the orientation of the third.

Sums

Definition 4.5.2.8. The vector bundle ξ is said to *decompose into the (direct) sum* of its subbundles ξ_1 and ξ_2 if each fibre $(\mathrm{proj}(\xi))^{-1}(b)$ is the direct sum of its subspaces $(\mathrm{proj}(\xi)_1)^{-1}(b)$ and $(\mathrm{proj}(\xi_2))^{-1}(b)$. A Euclidean or Hermitian bundle ξ *decomposes into the (orthogonal) sum* of its subbundles ξ_1 and ξ_2 if each fibre $(\mathrm{proj}(\xi))^{-1}(b)$ is the orthogonal sum of $(\mathrm{proj}(\xi)_1)^{-1}(b)$ and $(\mathrm{proj}(\xi_2))^{-1}(b)$.

In the Euclidean and Hermitian cases, every subbundle ξ_1 of the given bundle ξ splits ξ into the sum of its subbundles ξ_1 and $\xi_1^\perp$. We show in Subsection 4.5.4 that given any vector bundle ξ with cellular base and any subbundle ξ_1 of ξ, there is a subbundle ξ_2 of ξ such that ξ decomposes into the direct sum of ξ_1 and ξ_2; see Remark 4.5.4.7.

If the vector bundle ξ decomposes into the sum of its subbundles ξ_1 and ξ_2, then the quotient bundle ξ/ξ_1 is canonically $\mathrm{GL}\,\mathbb{R}^{\dim \xi_2}$- or $\mathrm{GL}\,\mathbb{C}^{\dim \xi_2}$-equivalent to ξ_2: this canonical equivalence $h\colon \xi_2 \to \xi/\xi_1$ is given by

$$\mathrm{tl}\, h(x) = \mathrm{proj}(x), \quad \mathrm{proj} = [\mathrm{proj}\colon\ \mathrm{tl}(\xi) \to \mathrm{tl}(\xi/\xi_1)].$$

Such an equivalence exists also in the Euclidean and Hermitian cases, when O or U replaces GL (in fact, we have already established this in Definition **??**). We now introduce a construction which reverses the above process and, in particular, allows us to recover a bundle which decomposes into a sum of subbundles from its summands.

Definition 4.5.2.9. Let ξ_1 and ξ_2 be real vector bundles of dimensions n_1 and n_2 and with common base. Then $\xi_1 \times \xi_2$ is a Steenrod bundle with base $\mathrm{bs}(\xi_1) \times \mathrm{bs}(\xi_2)$ and structure group $\mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{GL}(n_2, \mathbb{R})$. The bundle $\mathrm{diag}^!(\xi_1 \times \xi_2)$ where $\mathrm{diag}\colon \mathrm{bs}(\xi_1) \to \mathrm{bs}(\xi_1) \times \mathrm{bs}(\xi_1)$, has the same structure group (and base $\mathrm{bs}(\xi_1)$). Extending this group to $\mathrm{GL}(n_1 + n_2, \mathbb{R})$, we turn $\mathrm{diag}^!(\xi_1 \times \xi_2)$ into an $(n_1 + n_2)$-dimensional real vector bundle, called the (*direct*) *sum of the bundles* ξ_1 *and* ξ_2, and denoted $\xi_1 \oplus \xi_2$.

In the Euclidean, complex vector, and Hermitian cases, the definition and notation of a sum of two bundles are the same. In the Euclidean case, $\mathrm{O}(n_1) \times \mathrm{O}(n_2)$ is extended to $\mathrm{O}(n_1 + n_2)$, in the complex vector case, $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C})$ is extended to $\mathrm{GL}(n_1 + n_2, \mathbb{C})$, and in the Hermitian case, $\mathrm{U}(n_1) \times \mathrm{U}(n_2)$ is extended to $\mathrm{U}(n_1 + n_2)$; moreover, a sum of Euclidean, complex vector, or Hermitian bundles is again a Euclidean, complex vector, or Hermitian bundle, respectively.

In all cases,

$$\mathrm{bs}(\xi_1 \oplus \xi_2) = \mathrm{bs}(\xi_1)(= \mathrm{bs}(\xi_2)),$$
$$(\mathrm{proj}(\xi_1 \oplus \xi_2))^{-1}(b) = (\mathrm{proj}(\xi_1))^{-1}(b) \oplus (\mathrm{proj}(\xi_2))^{-1}(b);$$

the last sum is orthogonal in the Euclidean and Hermitian situations. These equalities define linear, orthogonal, or unitary monomorphisms, $\xi_1 \to \xi_1 \oplus \xi_2$ and $\xi_2 \to \xi_1 \oplus \xi_2$ which act as the identity on the base. These monomorphisms identify ξ_1 and ξ_2 with subbundles of $\xi_1 \oplus \xi_2$ and split $\xi_1 \oplus \xi_2$ into the sum of these subbundles.

Note that the same identifications allow us to take the quotients $(\xi_1 \oplus \xi_2)/\xi_1$ and $\xi_1 \oplus \xi_2/\xi_2$ and using the canonical equivalences defined in Definition 4.5.2.8, we may actually write $(\xi_1 \oplus \xi_2)/\xi_1 = \xi_2$ and $(\xi_1 \oplus \xi_2)/\xi_2 = \xi_1$. In particular, in the real vector and Euclidean cases, the orientability of two out of the three bundles, ξ_1 ξ_2, and $\xi_1 \oplus \xi_2$ implies the orientability of the third, and the orientations of any two of them define canonically an orientation of the third.

Let us add that $\xi \oplus \xi$ is always orientable and has a canonical orientation, for any real vector or Euclidean bundle ξ. This canonical orientation is determined on each fibre by arbitrary orientations of its summands, provided that we take identical orientations for both summands.

Definition 4.5.2.10. The sum of a real vector or Euclidean bundle ξ and the one-dimensional trivial $\mathrm{GL}\,\mathbb{R}^1$- or $\mathrm{O}\,\mathbb{R}^1$-bundle $(\mathrm{bs}(\xi) \times \mathbb{R}, \mathrm{proj}_1, \mathrm{bs}(\xi))$ is called the *suspension of* ξ, and is denoted by $\mathrm{sus}\,\xi$. Similarly, for any complex vector or Hermitian bundle ξ, the suspension of ξ, $\mathrm{sus}\,\xi$, is the sum of ξ and the one-dimensional standard trivial $\mathrm{GL}\,\mathbb{C}^1$- or $\mathrm{U}\,\mathbb{C}^1$-bundle $(\mathrm{bs}(\xi) \times (\mathbb{C}, \mathrm{proj}_1, \mathrm{bs}(\xi))$.

Two real vector bundles, ξ_1 and ξ_2 with $\mathrm{bs}(\xi_1) = \mathrm{bs}(\xi_2)$ are said to be *stably equivalent* if there exist k_1 and k_2 such that $\dim \xi_1 + k_1 = \dim \xi_2 + k_2$ and the bundles $\mathrm{sus}^{k_1} \xi_1$ and $\mathrm{sus}^{k_2} \xi_2$ are $\mathrm{GL}\,\mathbb{R}^{\dim \xi_1 + k_1}$-equivalent. Stable equivalence of Euclidean, complex vector, and Hermitian bundles is similarly defined (replacing $\mathrm{GL}\,\mathbb{R}^{\dim \xi_1 + k_1}$ by $\mathrm{O}\,\mathbb{R}^{\dim \xi_1 + k_1}$, $\mathrm{GL}\,\mathbb{C}^{\dim \xi_1 + k_1}$, and $\mathrm{U}\,\mathbb{C}^{\dim \xi_1 + k_1}$, respectively). A bundle which is stably equivalent to a standard trivial bundle is called *stably trivial.*

Finally, we note that for real vector or Euclidean bundles, the orientability of one of the bundles ξ and $\mathrm{sus}\,\xi$ implies the orientability of the other, and any orientation of one of them canonically defines an orientation of the other; see Definition 4.5.2.9.

Complexification

Definition 4.5.2.11. Given an n-dimensional real vector bundle ξ, consider the map $I\colon \xi \oplus \xi \to \xi \oplus \xi$, given by $\mathrm{tl}\, I(x, y) = (-y, x)$ [x and y sit in the same fibre of ξ]. Obviously, I is a $\mathrm{GL}\,\mathbb{R}^{2n}$-equivalence satisfying $I^2 = -\,\mathrm{id}$. Therefore, I turns $\xi \oplus \xi$ into an n-dimensional complex vector bundle (see Definition 4.5.1.12), called the *complexification of* ξ, and denoted $\mathbb{C}\xi$.

If ξ is an n-dimensional Euclidean bundle, the same construction turns $\xi \oplus \xi$ into an n-dimensional Hermitian bundle, which is also called the *complexification of* ξ and is denoted $\mathbb{C}\xi$.

The operation $\xi \to \mathbb{C}\xi$ is called *complexification* in both cases. Note that $\mathbb{R}\mathbb{C}\xi = \xi \oplus \xi$ in both cases.

Remark 4.5.2.12. Each of the bundles $\mathbb{R}\mathbb{C}\xi$ and $\xi \oplus \xi$, appearing in the last equality, carries a canonical orientation; see Remark 4.5.1.13 and Definition 4.5.2.9. It may be shown that these orientations coincide when $n \equiv 0, 1 \mod 4$, and are opposite when $n \equiv 2, 3 \mod 4$.

Indeed, pick an arbitrary fibre $(\operatorname{proj}(\xi))^{-1}(b)$ of ξ and an arbitrary basis (respectively, orthonormal basis) $v_1, \ldots, v_n$ in $(\operatorname{proj}(\xi))^{-1}(b)$. The canonical orientation of the fibre $(\operatorname{proj}(\xi))^{-1}(b) \times (\operatorname{proj}(\xi))^{-1}(b)$ of $\mathbb{R}\mathbb{C}$ takes the value $+1$ on the basis $(v_1, 0), (0, v_1), \ldots, (v_n, 0), (0, v_n))$ of this fibre, and the same holds for the canonical orientation of the fibre $(\operatorname{proj}(\xi))^{-1}(b) \times (\operatorname{proj}(\xi))^{-1}(b)$ of $\xi \oplus \xi$ and its basis $(v_1, 0), (0, v_1), \ldots, (v_n, 0), (0, v_n))$. It remains to note that, in order to shift from one basis to the other, we have to perform $n(n-1)/2$ permutations of adjacent vectors, and that this number is even for $n \equiv 0, 1 \mod 4$, and odd for $n \equiv 2, 3 \mod 4$.

Theorem 4.5.2.13. *The map*

$$\operatorname{conj}\colon \mathbb{C}\xi \to \operatorname{conj}\mathbb{C}\xi, \quad \textit{given by} \quad \operatorname{tl}\operatorname{conj}(x, y) = (x, -y),$$

is a $\operatorname{GL}\mathbb{C}^n$*-equivalence (*$\operatorname{U}\mathbb{C}^n$*-equivalence) for any* n*-dimensional real vector (respectively, Euclidean) bundle* ξ*.*

Proof. Indeed, the equivalences $I_1, I_2\colon \xi \oplus \xi \to \xi \oplus \xi$, which turn $\xi \oplus \xi$ into $\mathbb{C}\xi$ and $\operatorname{conj}\mathbb{C}\xi$, are given by

$$\operatorname{tl} I_1(x, y) = (-y, x), \quad I_2 = -I_1,$$

and hence $I_2 \circ \operatorname{conj} = \operatorname{conj} \circ I_1$. □

Theorem 4.5.2.14. *The map*

$$K\colon \xi \oplus \operatorname{conj}\xi \to \mathbb{C}\mathbb{R}\xi, \quad \textit{given by} \quad \operatorname{tl} K(x, y) = (\frac{1}{2})(x + y), \frac{1}{2}(\operatorname{tl} I(y) - \operatorname{tl} I(x))),$$

where I *is the equivalence which turns* $\mathbb{R}\xi$ *into* ξ*, is a* $\operatorname{GL}\mathbb{C}^n$*-equivalence (*$\operatorname{U}\mathbb{C}^n$*-equivalence) for any* n*-dimensional complex vector (respectively, Hermitian) bundle.*

Proof. The equivalences $I_1, I_2\colon \mathbb{R}\xi \oplus \mathbb{R}\xi \to \mathbb{R}\xi \oplus \mathbb{R}\xi$, which turn $\mathbb{R}\xi \oplus \mathbb{R}\xi$ into $\xi \oplus \operatorname{conj}\xi$, and $\mathbb{C}\mathbb{R}\xi$, are given by

$$\operatorname{tl} I_1(x, y) = (\operatorname{tl} I(x), -\operatorname{tl} I(y)), \quad \operatorname{tl} I_2(x, y) = (-y, x),$$

and hence $I_2 \circ K = K \circ I_1$. □

4.5.3 The Classical Universal Vector Bundles

Remark 4.5.3.1. The main value of the construction in Subsection 4.4.3 is that it establishes the existence of a universal G-bundle for a completely arbitrary topological group G. However, for the groups $\operatorname{GL}(n, \mathbb{R})$, $\operatorname{GL}_+(n, \mathbb{R})$, $\operatorname{O}(n)$, $\operatorname{SO}(n)$, $\operatorname{GL}(n, \mathbb{C})$, and $\operatorname{U}(n)$, there are more convenient, classical constructions, which are described in the present subsection.

The Grassmann Spaces

Definition 4.5.3.2. Set

$$\begin{aligned} G(\infty, n) &= \lim(G(m,n), \mathrm{incl}\colon G(m,n) \to G(m+1,n)), \\ G_+(\infty, n) &= \lim(G_+(m,n), \mathrm{incl}\colon G_+(m,n) \to G_+(m+1,n)), \\ \mathbb{C}G(\infty, n) &= \lim(\mathbb{C}G(m,n), \mathrm{incl}\, \mathbb{C}G(m,n) \to \mathbb{C}G(m+1,n)). \end{aligned}$$

$G(\infty, n)$ consists of all n-dimensional planes of $\mathbb{R}^\infty$ passing through 0, and is called the *n-th (real) Grassmann space*. Similarly, the *n-th upper Grassmann space*, $G_+(\infty, n)$, consists of all oriented n-dimensional planes in $\mathbb{R}^\infty$ passing through 0. Finally, the *n-th complex Grassmann space*, $\mathbb{C}G(\infty, n)$, consists of all n-dimensional planes in $\mathbb{C}$ passing through 0.

The canonical maps:

$$\begin{aligned} &G(m,n) \to G(m,n), \quad \mathbb{C}G(m,n) \to G(2m,2n), \quad \mathbb{C}G(m,n) \to G_+(2m,2n), \\ &G(m,n) \to G(m+q,n+q), \quad G_+(m,n) \to G_+(m+q,n+q), \\ &\mathbb{C}G(m,n) \to \mathbb{C}G(m+q,n+q) \end{aligned}$$

(see Remarks 3.2.2.3 and 3.2.2.7) define for any n the following maps:

$$\begin{aligned} &G_+(\infty,n) \to G(\infty,n), \quad \mathbb{C}G(\infty,n) \to G(\infty,2n), \quad \mathbb{C}G(\infty,n) \to G_+(\infty,2n), \\ &G(\infty,n) \to G(\infty,n+q), \quad G_+(\infty,n \to G_+(\infty,n+q), \\ &\mathbb{C}G(\infty,n) \to \mathbb{C}G(\infty,n+q). \end{aligned}$$

The first of these maps (as the canonical map $G_+(m,n) \to G(m,n)$ with $m < \infty$) is a two-sheeted covering projection; the second is the composition of the first and the third; and all these maps, except the first, are embeddings.

Remark 4.5.3.3. The Grassmann spaces possess natural cellular decompositions which we shall presently describe.

First consider $G(\infty, n)$. Let Ω_n denote the set of all sequences of integers, $\omega = \{\omega(1), \ldots, \omega(n)\}$, with $0 \le \omega(1) \le \cdots \le \omega(n)$, and let us agree to add the term $\omega(0) = 0$ to each sequence $\omega \in \Omega_n$. Further, let $e(\omega)$ denote the subset of $G(\infty, n)$ consisting of those n-planes γ in $\mathbb{R}^\infty$ (passing through 0) such that

$$\dim(\gamma \cap \mathbb{R}^m) = \max\{s | \omega(s) + s \le m\},$$

for all m. We show that the sets $e(\omega)$, $\omega \in \Omega_n$, yield a cellular decomposition of $G(\infty, n)$, with $\dim e(\omega) = d(w)$, where $d(\omega) = \omega(1) + \cdots + \omega(n)$.

We have to produce a characteristic map, $\mathrm{char}_{e(\omega)}\colon \mathbb{D}^{d(\omega)} \to G(\infty, n)$. Fix ω and, for points $u, v \in \mathbb{S}^{\omega(n)+n-1}$ such that $u + v \neq 0$, denote by $r(u,v) \in \mathrm{SO}(\omega(n)+n)$ the orthogonal transformation which takes u into v and keeps fixed all the vectors of $\mathbb{R}^{\omega(n)+n}$ which are orthogonal to u and v. Further, let H_i denote the $\omega(i)$-dimensional hemisphere consisting of all points $(x_1, \ldots x_{\omega(i)+i}) \in \mathbb{S}^{\omega(i)-i-1}$ such that $x_{\omega(j)+j} = 0$ for $j = 1, \ldots, i+1$, and $x_{\omega(i)+i}) \ge 0$. Consider the map $\varphi\colon H_1 \times \cdots \times H_n \to G(\infty, n)$ which takes each sequence $(u_1, \ldots, u_n)$ into the plane spanned by the n-frame: (which is obviously orthonormal)

$$\begin{aligned} u_1,&[r(\mathrm{ort}_{\omega(1)+1}, u_1)](u_2), \ldots, \\ &[r(\mathrm{ort}_{\omega(1)+1}, u_1) \circ \cdots \circ r(\mathrm{ort}_{\omega(n-1)]n-1}, u_{n-1})](u_n). \end{aligned} \tag{4.5.3.4}$$

Then φ is continuous and maps $\mathrm{int}(H_1 \times \cdots \times H_n)$ into $e(\omega)$, and $\partial(H_1 \times \cdots \times H_n)$ into a union of sets $e(\omega')$ with $d(\omega')) < d(\omega)$. Moreover, its compression $\mathrm{abr}\,\varphi\colon \mathrm{int}(H_1 \times \cdots \times H_n) \to e(\omega)$ is a homeomorphism: its inverse takes each plane $\gamma \in e(\omega)$ into the sequence $(u_1, \ldots, u_n)$, with

$$\begin{aligned} &u_1 = v_1, u_2 = [r(\mathrm{ort}_{\omega(1)+1}), u_1]^{-1}(v_2), \ldots, \\ &u_n = [r(\mathrm{ort}_{\omega(1)+1}, u_1) \circ \cdots \circ [r(\mathrm{ort}_{\omega(n-1)+n-1}, u_{n-1})]^{-1}(v_n), \end{aligned}$$

where $v_1, \ldots, v_n$ is an orthogonal basis of γ, selected in such a way that v_i sits in the hemisphere $x_{\omega(i)+i} > 0$ of $\mathbb{S}^{\omega(i)+i-1}$ (the vectors $v_1, \ldots, v_n$ are uniquely determined by these requirements). Therefore, one may take $\mathrm{char}_{e(\omega)}$ to be the composition

$$\mathbb{D}^{d(\omega)} \to \mathbb{D}^{\omega(1)} \times \cdots \times \mathbb{D}^{\omega(n)} \to H_1 \times \cdots \times H_n \xrightarrow{\varphi} G(\infty, n),$$

where the left map is the inverse of the homeomorphism indicated in Remark 1.2.6.9, and the middle map is the product of the homeomorphisms $\mathbb{D}^{\omega(i)} H_i$ given by

$$(x_1, \ldots, x_{\omega(i)}) \mapsto$$

$$(x_1, \ldots, x_{\omega(i)}, 0, x_{\omega(i)+1}, \ldots, X_{\omega(2)}, 0, \ldots, 0, x_{\omega(i-1)+1}, \ldots, x_{\omega(i)}, (1 - \sum_1^{\omega(i)} x_j^2)^{1/2}).$$

The cellular decomposition of $G_+(\infty, n)$ has twice as many cells as that of $G(\infty, n)$. Namely, over each cell $e(\omega)$ sit two cells, $e_+(\omega)$ and $e_-(\omega)$, of $G_+(\infty, n)$, which are homeomorphically mapped onto $e(\omega)$ by the projection $G_+(\infty, n) \to G(\infty, n)$: $e_+(\omega)$ is made up of planes oriented in such a way that the orientation is positive on the basis $v_1, \ldots, v_n$ described above, while $e_-(\omega)$ is made up of the same planes, but with the opposite orientations. The characteristic maps for $e_+(\omega)$ and $e_-(\omega)$ are constructed in the same manner as $\mathrm{char}_{e(\omega)}$, but now the plane spanned by the frame (4.5.3.4) is oriented [its orientation is positive (negative) on (4.5.3.4) for $e_+(\omega)$ (respectively, for $e_-(\omega)$)].

The cellular decomposition of $\mathbb{C}G(\infty, n)$ is given by the cells $\mathbb{C}e(\omega)$, $\omega \in \Omega_n$, which are defined precisely as the $e(\omega)$'s if one replaces $\mathbb{R}^\infty$ and $\mathbb{R}^m$ by $\mathbb{C}^\infty$ and $\mathbb{C}^m$; $\dim \mathbb{C}e(\omega) = 2d(\omega)$, and $\mathrm{char}_{\mathbb{C}e(\omega)}$ is the exact complex-Hermitian analogue of the map $\mathrm{char}_{e(\omega)}$.

Remark 4.5.3.5. The above cellular decompositions contain only a finite number of cells of a given dimension, and hence satisfy property (**C**). Since each of the manifolds $G(m, n)$, $G_+(m, n)$, and $\mathbb{C}G(m, n)$ is covered by a finite number of cells, and since these manifolds constitute fundamental covers of the spaces

$G(\infty, n)$, $G_+(\infty, n)$, and $\mathbb{C}G(\infty n)$, our cellular decompositions satisfy also property (**W**). Finally, from Theorem 1.2.4.6 it follows that $G(\infty, n)$, $G_+(\infty, n)$, and $\mathbb{C}G(\infty n)$ are normal. Therefore, the above cellular decompositions turn $G(\infty, n)$, $G_+(\infty, n)$, and $\mathbb{C}G(\infty n)$ into cellular spaces.

It is clear that $G(m, n)$, $G_+(m, n)$, and $\mathbb{C}G(m, n)$ are subspaces of $G(\infty, n)$, $G_+(\infty, n)$, and $\mathbb{C}G(\infty n)$ in the cellular sense, and so they are cellular spaces too.

For $n = 1$, $G(m, n))$, $\mathbb{C}G(m, n)$, $G(\infty, n)$, and $\mathbb{C}G(\infty, n)$ are simply $\mathbb{R}P^{m-1}$, $\mathbb{C}P^{m-1}$, $\mathbb{R}P^\infty$, and $\mathbb{C}P^\infty$, respectively, and the above cellular decompositions are identical with those described in Remarks 2.1.3.4 and 2.1.3.5.

Remark 4.5.3.6. Note that the canonical embeddings

$$G(\infty, n) \to G(\infty, n+q), \quad G_+(\infty, n) \to G_+(\infty, n+q), \quad \mathbb{C}G(\infty, n) \to \mathbb{C}G(\infty, n+q)$$

(see Definition 4.5.3.2) are cellular. The image of the first, the second and the third respectively contain

$$\operatorname{skel}_n G(\infty, n+q), \quad \operatorname{skel}_n G_+(\infty, n+q), \quad \operatorname{skel}_{2n+1} \mathbb{C}G(\infty, n+q).$$

The inclusions

$$G(m, n) \supset \operatorname{skel}_{m-n} G(\infty, n), \quad G_+(m, n) \supset \operatorname{skel}_{m-n} G_+(\infty, n),$$
$$\mathbb{C}G(m, n) \supset \operatorname{skel}_{2m-2n+1} \mathbb{C}G(\infty, n)$$

are equally evident.

The Grassmann Bundles

Definition 4.5.3.7. Let $0 \le n \le m \le \infty$ and $n < \infty$. We let $T(m, n)$, $T_+(m, n)$, and $\mathbb{C}T(m, n)$, denote those subsets of the respective products $G(m, n) \times \mathbb{R}^m$, $G_+(m, n) \times \mathbb{R}^m$, and $\mathbb{C}G(m, n) \times \mathbb{C}^m$, consisting of all pairs (γ, x) such that $x \in \gamma$. Consider the bundles $(T(m, n), \operatorname{proj}, G(m, n))$, $(T(m, n), \operatorname{proj}, G_+(m, n))$, and $(\mathbb{C}T(m, n), \operatorname{proj}, \mathbb{C}G(m, n))$, where $\operatorname{proj}(\gamma, x) = \gamma$. The fibres of the first (second; third) bundle are Euclidean spaces (respectively, oriented Euclidean spaces; Hermitian spaces). Moreover, the first (second; third) bundle is locally W-O $\mathbb{R}^n$-trivial (respectively, locally W-SO $\mathbb{R}^n$-trivial; locally W-U $\mathbb{C}^n$-trivial), and hence it is a Euclidean (respectively, oriented Euclidean; Hermitian) bundle. We denote these bundles by $\operatorname{Grass}(m, \mathrm{O}(n))$, $\operatorname{Grass}(m, \mathrm{SO}(n))$, and $Grass(m, \mathrm{U}(n))$. In addition, by extending the respective structure groups, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, and $\mathrm{U}(n)$, to $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}_+(n, \mathbb{R})$, and $\mathrm{GL}(n, \mathbb{C})$, we obtain bundles denoted by $\operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$, $\operatorname{Grass}(m, \mathrm{GL}_+(n, \mathbb{R}))$, and $\operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{C}))$. The bundles of these six series are called *Grassmann bundles.* For $m = \infty$, we use the simpler notations $\operatorname{Grass} \mathrm{O}(n)$, $\operatorname{Grass} \mathrm{SO}(n)$, $\operatorname{Grass} \mathrm{U}(n)$, $\operatorname{Grass} \mathrm{GL}(n, \mathbb{R})$, $\operatorname{Grass} \mathrm{GL}_+(n, \mathbb{R})$, and $\operatorname{Grass} \mathrm{GL}(n, \mathbb{C})$.

Note that for $m < \infty$, $\operatorname{Grass}(m, \mathrm{O}(n))$ is nothing else but the subbundle of the standard trivial bundle $(G(m, n) \times \mathbb{R}^m, \operatorname{proj}, G(m, n))$ (viewed as a Euclidean bundle) associated with the diagonal section, $\gamma \mapsto (\gamma, \gamma)$ of the bundle

$(G(m,n) \times G(m,n), \mathrm{proj}_1, G(m,n))$. The same holds, with obvious modifications, for the remaining five series.

Theorem 4.5.3.8. *The bundles* $\operatorname{Grass} G$ *with* $G = \mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ *are universal.*

Proof. The proofs for the different groups G differ only in some obvious details, and are all very similar to the proof of Theorem 4.4.3.4. We shall treat here only the group $\mathrm{GL}(n,\mathbb{R})$. According to Definition 4.4.2.2, it suffices to show that given any n-dimensional real vector bundle ξ with cellular base and any subspace $A \subset \mathrm{bs})\xi)$, every $\mathrm{GL}\,\mathbb{R}^n$-map $g\colon \xi|_A \to \operatorname{Grass}\mathrm{GL}(n,\mathbb{R})$ extends to a $\mathrm{GL}\,\mathbb{R}^n$-map $\xi \to \operatorname{Grass}\mathrm{GL}(n,\mathbb{R})$.

Assume first that $\mathrm{bs}(\xi) = \mathbb{D}^{r+1}$ (for some r) and $A = \mathbb{S}^r$. In this case ξ is $\mathrm{GL}\,\mathbb{R}^n$-trivial, and so we may actually assume that ξ is the standard trivial bundle $(\mathbb{D}^{r+1} \times \mathbb{R}^n, \mathrm{proj}_1, \mathbb{D}^{r+1})$. The desired extension of g, $f\colon \xi \to \operatorname{Grass}\mathrm{GL}(n,\mathbb{R})$, can be described explicitly: let g_1 be the composite map

$$\mathbb{S}^r \times \mathbb{R}^n \xrightarrow{\mathrm{tl}(g)} \mathrm{tl}(\operatorname{Grass}\mathrm{GL}(n,\mathbb{R})) \xrightarrow{\mathrm{incl}} G(\infty,n) \times \mathbb{R}^\infty \xrightarrow{\mathrm{proj}_2} \mathbb{R}^\infty$$

and define $f_1\colon \mathbb{D}^{r+1} \times \mathbb{R}^n \to \mathbb{R}^\infty$ by

$$f_1(ty,(x_1,\dots,x_n)) = tg_1(y,(x_1,\dots,x_n)) + (1-t^2)^{1/2}(\underbrace{0,\dots,0}_{m}, x_1,\dots,x_n,0,\dots),$$

where $y \in \mathbb{S}^r$, $t \in I$, and m is the smallest number s such that $\mathbb{R}^s \supset g_1(\mathbb{S}^r \times \mathbb{S}^{n-1})$; finally, set $\mathrm{tl}(f(y,x)) = (f_1(y \times \mathbb{R}^n), f_1(y,x))$.

The general case reduces to this special situation. Assume that the cellular space $\mathrm{bs}(\xi)$ is rigged and that we already have a $\mathrm{GL}\,\mathbb{R}^n$-map

$$f_r\colon \xi|_{A \cup \mathrm{skel}_r \mathrm{bs}(\xi)} \to \operatorname{Grass}\mathrm{GL}(n,\mathbb{R})$$

which extends g. The above argument shows that for every cell $e \in \mathrm{cell}_{r+1}\mathrm{bs}(\xi) \setminus \mathrm{cell}_{r+1} A$ the $\mathrm{GL}\,\mathbb{R}^n$-map

$$g_e = f_r \circ \mathrm{adj}[\mathrm{abr\,char}_e\colon \mathbb{S}^r \to A \cup \mathrm{skel}_r \mathrm{bs}(\xi)]\colon\ \mathrm{char}_e^!\,\xi|_{\mathbb{S}^r} \to \operatorname{Grass}\mathrm{GL}(n,\mathbb{R})$$

extends to a $\mathrm{GL}\,\mathbb{R}^n$-map $h_e\colon \mathrm{char}_e^!\,\xi \to \operatorname{Grass}\mathrm{GL}(n,\mathbb{R})$, and it is clear that $\mathrm{tl}(h_e)$ is constant on the elements of the partition $\mathrm{zer}(\mathrm{tl\,adj\,char}_e)$. Applying Theorem 4.3.2.6 (with $B = \mathbb{D}^{r+1}$ and $p = [\mathrm{abr}\, \mathit{Char}_e\colon \mathbb{D}^{r+1} \to \mathrm{Cl}(e)]$), we conclude that h_e defines a $\mathrm{GL}\,\mathbb{R}^n$-map $\xi|_{\mathrm{Cl}(e)} \to \operatorname{Grass}\mathrm{GL}(n,\mathbb{R})$, which we denote by f_e.

Now note that for any two cells,

$$e_1, e_2 \in \mathrm{cell}_{r+1}\mathrm{bs}(\xi) \setminus \mathrm{cell}_{r+1} A,$$

$\mathrm{tl}(f_e)$ and $\mathrm{tl}(f_r)$ agree over $\mathrm{Cl}(e_1) \cap \mathrm{Cl}(e_2)$, and that for any cell

$$e \in \mathrm{cell}_{r+1}\mathrm{bs}(\xi) \setminus \mathrm{cell}_{r+1} A,$$

$\operatorname{tl}(f_e)$ and $\operatorname{tl}(f_r)$ agree over $\operatorname{Cl}(e) \cap (A \cup \operatorname{skel}_r \operatorname{bs}(\xi))$. From this compatibility it follows that the maps f_r and f_e, $e \in \operatorname{cell}_{r+1} \operatorname{bs}(\xi) \setminus \operatorname{cell}_{r+1} A$, combine to define a $\operatorname{GL}\mathbb{R}^n$-map

$$f_{r+1} \colon \xi|_{A \cup \operatorname{skel}_{r+1}} \operatorname{bs}(\xi) \to \operatorname{Grass}\operatorname{GL}(n, \mathbb{R})$$

which extends $_r$; see Corollary 4.3.2.7. Therefore, using induction, we can produce a sequence of $\operatorname{GL}\mathbb{R}^n$-maps,

$$\{f_s \colon \xi|_{A \cup \operatorname{skel}_s} \operatorname{bs}(\xi) \to \operatorname{Grass}\operatorname{GL}(n, \mathbb{R})_{s=-1}^{\infty}\}$$

such that \$$f_{-1} = g$ and f_s extends f_{s-1}, $s \geq 0$. Finally, the maps f_s define a $\operatorname{GL}\mathbb{R}^n$-map $\xi \to \operatorname{Grass}\operatorname{GL}(n, \mathbb{R})$ extending g. □

Corollary 4.5.3.9. *The bundles*

$$\operatorname{Grass}(m, \operatorname{GL}(n, \mathbb{R})), \operatorname{Grass}_+(m, \operatorname{GL}(n, \mathbb{R})), \operatorname{Grass}(m, \operatorname{O}(n)), \operatorname{Grass}(m, \operatorname{SO}(n))$$

are $(m - n)$*-universal. The bundles*

$$\operatorname{Grass}(m, \operatorname{GL}(n, \mathbb{C})), \operatorname{Grass}(m, \operatorname{U}(n))$$

are $(2m - 2n + 1)$*-universal*

Proof. This is a corollary of Theorem 4.5.3.8 (see Remark 4.5.3.6 and Definition 4.4.2.7). □

Associated Principal Bundles

Definition 4.5.3.10. When $m < \infty$, the total spaces of the principal bundles associated with the Grassmann bundles

$$\begin{aligned}&\operatorname{Grass}(m, \operatorname{GL}(n, \mathbb{R})), \quad \operatorname{Grass}(m, \operatorname{GL}_+(n, \mathbb{R})), \quad \operatorname{Grass}(m, \operatorname{GL}(n, \mathbb{C})),\\ &\operatorname{Grass}(m, \operatorname{O}(n)), \quad \operatorname{Grass}(m, \operatorname{SO}(n)), \quad \operatorname{Grass}(m, \operatorname{U}(n)),\end{aligned}$$

are obviously $V'(m, n)$, $V'(m, n)$, $\mathbb{C}V'(m, n)$, and $V(m, n)$, $V(m, n)$, $\mathbb{C}V(m, n)$. The corresponding projections are the maps described in Remarks 3.2.2.3 and 3.2.2.7:

$$\begin{aligned}&V'(m, n) \to G(m, n), \quad V'(m, n) \to G_+(m, n), \quad \mathbb{C}V'(m, n) \to \mathbb{C}G(m, n),\\ &V(m, n) \to G(m, n), \quad V(m, n) \to G_+(m, n), \quad \mathbb{C}V(m, n) \to \mathbb{C}G(m, n).\end{aligned} \tag{4.5.3.11}$$

The same is true for $m = \infty$, if $V'(\infty, n)$, $\mathbb{C}V'(\infty, n)$, $V(\infty, n)$, and $\mathbb{C}V(m, n)$ are understood as $\varinjlim(V'(m, n), \operatorname{incl})$, $\varinjlim(\mathbb{C}V'(m, n), \operatorname{incl})$, $\varinjlim(V(m, n), \operatorname{incl})$, and $\varinjlim(\mathbb{C}V(m, n), \operatorname{incl})$, and the projections (4.5.3.11) with $m = \infty$ as the limits of the projections (4.5.3.11), $m < \infty$. $V'(\infty, n)$, $\mathbb{C}V'(\infty, n)$, $V(\infty, n)$, and $\mathbb{C}V(m, n)$ are are called *Stiefel spaces.*

It is clear that for $m < \infty 0$ the canonical right actions of the structure groups on the above total spaces (see Definition 4.3.2.10) are exactly the right actions described in Remarks 4.2.3.16 and 4.2.3.17, while for $m = \infty$ they are the colimits (inductive limits) of the latter.

The Bundles $\operatorname{assoc}(\operatorname{Grass}\mathrm{O}(1),\mathrm{O}(1))$ **and** $\operatorname{assoc}(\operatorname{Grass}\mathrm{U}(1),\mathrm{U}(1))$

Theorem 4.5.3.12. *The principal bundle associated with* $\operatorname{Grass}\mathrm{O}(1)$ *is* $\mathrm{O}(1)$-*isomorphic to* $\operatorname{Mil}\mathrm{O}(1)$. *The principal bundle associated with* $\operatorname{Grass}\mathrm{U}(1)$ *is* $\mathrm{U}(1)$-*isomorphic to* $\operatorname{Mil}\mathrm{U}(1)$.

Proof. It suffices to find an $\mathrm{O}(1)$-homeomorphism

$$\operatorname{tl}(\operatorname{Mil}\mathrm{O}(1)) \to \operatorname{tl}(\operatorname{assoc}(\operatorname{Grass}\mathrm{O}(1),\mathrm{O}(1))),$$

when we regard $\operatorname{tl}(\operatorname{Mil}\mathrm{O}(1))$ and

$$\operatorname{tl}(\operatorname{assoc}(\operatorname{Grass}\mathrm{O}(1),\mathrm{O}(1))) = V(\infty,1)[=\mathbb{S}^\infty]$$

as right $\mathrm{O}(1)$-spaces ; similarly, viewing $\operatorname{tl}(\operatorname{Mil}\mathrm{U}(1))$ and

$$\operatorname{tl}(\operatorname{assoc}(\operatorname{Grass}\mathrm{U}(1),\mathrm{U}(1))) = \mathbb{C}V(\infty,1)[=\mathbb{S}^\infty]$$

as right $\mathrm{U}(1)$-spaces, we need only exhibit a $\mathrm{U}(1)$-homeomorphism

$$\operatorname{tl}(\operatorname{Mil}\mathrm{U}(1)) \to \operatorname{tl}(\operatorname{assoc}(\operatorname{Grass}\mathrm{U}(1),\mathrm{U}(1)));$$

see Corollary 4.3.2.9 and Definition 4.3.2.10. In both cases such a homeomorphism is given by the formula

$$\{\operatorname{proj}(g_i,t_i)\}_{i=1}^\infty \to \{g_i\sqrt{t_i}\}_{i=1}^\infty.$$

The meaning of the left-hand side was explained in Remark 4.4.3.2, while in the right-hand side the elements g_i of $\mathrm{O}(1)$ or $\mathrm{U}(1)$ are thought of as numbers (the following inclusions are used: $V(\infty,1)\subset\mathbb{R}^\infty$, $\mathbb{C}V(\infty,1)\subset\mathbb{C}^\infty$, $\mathrm{O}(1)=\mathbb{S}^0\subset\mathbb{R}$, and $\mathrm{U}(1)=\mathbb{S}^1\subset\mathbb{C}$. ☐

4.5.4 The Most Important Reductions of the Structure Group

Remark 4.5.4.1. The use of Grassmann bundles enables us to apply the scheme presented in Subsection 4.4.4 to the problems raised in Subsection 4.5.1 concerning reductions of the structure group. This is the subject of the present subsection.

Recall that the reductions corresponding to the inclusions

$$\mathrm{O}(n)\subset\mathrm{GL}(n,\mathbb{R}),\quad \mathrm{SO}(n)\subset\mathrm{GL}_+(n,\mathbb{R}),\quad \mathrm{U}(n)\subset\mathrm{GL}(n,\mathbb{C}), \tag{4.5.4.2}$$

are equivalent to the introduction of a Euclidean or Hermitian metric, while the reductions resulting from the inclusions

$$\mathrm{GL}_+(n,\mathbb{R})\subset\mathrm{GL}(n,\mathbb{R}),\quad \mathrm{SO}(n)\subset\mathrm{O}(n), \tag{4.5.4.3}$$

are equivalent to the introduction of an orientation. Finally, the reductions resulting from the inclusions

$$\mathrm{GL}(n,\mathbb{C})\subset\mathrm{GL}(2n,\mathbb{R}),\quad \mathrm{U}(n)\subset\mathrm{O}(2n), \tag{4.5.4.4}$$

mean the introduction of a complex structure.

For each of the inclusions (4.5.4.2), (4.5.4.3), and (4.5.4.4), we shall exhibit a canonical classifying map, and then list the most obvious consequences of these constructions. Moreover, we shall carry out the same programme for the inclusions

$$\begin{aligned}&\mathrm{GL}(n-s,\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{R}), \quad \mathrm{GL}_+(n-s,\mathbb{R}) \subset \mathrm{GL}_+(n,\mathbb{R}),\\ &\mathrm{GL}(n-s,\mathbb{C}) \subset \mathrm{GL}(n,\mathbb{C}),\end{aligned} \tag{4.5.4.5}$$

and

$$\mathrm{O}(n-s) \subset \mathrm{O}(n), \quad \mathrm{SO}(n-s) \subset \mathrm{SO}(n), \quad \mathrm{U}(n-s) \subset \mathrm{U}(n). \tag{4.5.4.6}$$

The reductions of the structure group corresponding to the six inclusions (4.5.4.5) and (4.5.4.6) may be interpreted as the representation of the given n-dimensional bundle as the s-fold suspension of an $(n-s)$-dimensional bundle.

Remark 4.5.4.7. The outlined programme is simple to carry out for inclusions (4.5.4.2). Indeed, the bundles $\mathrm{Grass}\,\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{Grass}\,\mathrm{O}(n)$ have the same base, $G(\infty,n)$, and the same is true for $\mathrm{Grass}\,\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{Grass}\,\mathrm{SO}(n)$, with the base $G_+(\infty,n)$, and for $\mathrm{Grass}\,GL(n,\mathbb{C})$ and $\mathrm{Grass}\,\mathrm{U}(n)$, with the base $\mathbb{C}G(\infty,n)$. It is obvious that in all three cases the identity map of the base is classifying. Therefore, the mappings

$$\begin{aligned}&\mathrm{ext}\colon\ \mathrm{Stnrd}(B,\mathrm{O}\,\mathbb{R}^n) \to \mathrm{Stnrd}(B,\mathrm{GL}\,\mathbb{R}^n),\\ &\mathrm{ext}:\mathrm{Stnrd}(B,\mathrm{SO}\,\mathbb{R}^n) \to \mathrm{Stnrd}(B,\mathrm{GL}_+\,\mathbb{R}^n),\\ &\mathrm{ext}\colon\ \mathrm{Stnrd}(B,\mathrm{U}\,\mathbb{C}n) \to \mathrm{Stnrd}(B,\mathrm{GL}\,\mathbb{C}^n)\end{aligned}$$

are invertible for any cellular space B; see Definition 4.4.4.3. In particular, every real (complex) vector bundle with cellular base admits a Euclidean (respectively, Hermitian) metric.

As a corollary, we obtain the theorem already formulated in Definition 4.5.2.8: given any vector bundle ξ with cellular base and any subbundle ξ_1, there exists a subbundle ξ_2 of ξ such that ξ decomposes into the sum of ξ_1 and ξ_2.

Remark 4.5.4.8. Similarly, the projection $G_+(\infty,n) \to G(\infty,n)$ is classifying for both inclusions (4.5.4.3), while the inclusion $\mathbb{C}G(\infty,n) \to G(\infty,2n)$ is classifying for both inclusions (4.5.4.4). However, a study of the homotopy properties of these classifying maps is already a quite difficult task. We shall return to the first of them in §5.6, armed with more sophisticated tools.

Remark 4.5.4.9. For the inclusions (4.5.4.5) and (4.5.4.6) there are also obvious classifying maps:

for both left inclusions, - the canonical embedding $G(\infty,n-s) \to G(\infty,n)$,

for both middle inclusions - the canonical embedding $G_+(\infty,n-s) \to G(\infty,n)$,

for both last inclusions - the canonical embedding $\mathbb{C}G(\infty,n-s) \to \mathbb{C}G(\infty,n)$

(see Definition 4.5.3.2). Identifying $G(\infty, n-s)$, $G_+(\infty, n-s)$, and $\mathbb{C}G(\infty, n-s)$ with their images under these embeddings, and using Remark 4.5.3.6, we can write:

$$G(\infty, n-s) \supset \operatorname{skel}_{n-s} G(\infty, n), \quad G_+(\infty, n-s) \supset \operatorname{skel}_{n-s} G(\infty, n)$$
$$\mathbb{C}G(\infty, n-s) \supset \operatorname{skel}_{2n-2s+1} \mathbb{C}G(\infty, n).$$

From the first inclusion it follows that the pair $(G(\infty, n), G(\infty, n-s))$ is $(n-s)$-connected (see Theorem 2.3.2.4), which in turn implies (by Theorems 2.3.2.6 and 2.3.2.7) that the map

$$\pi(\mathrm{id}, \mathrm{incl})\colon \pi(B, G(\infty, n-s)) \to \pi(B, G(\infty, n))$$

is invertible for any cellular space B with $\dim B \le n-s$, and surjective for any cellular space B with $\dim B = n-s$. Consequently,

$$\mathrm{ext}\colon \operatorname{Stnrd}(B, \mathrm{GL}\,\mathbb{R}^{n-s}) \to \operatorname{Stnrd}(B, \mathrm{GL}\,\mathbb{R}^{n})$$
$$\mathrm{ext}\colon \operatorname{Stnrd}(B, \mathrm{O}\,\mathbb{R}^{n-s}) \to \operatorname{Stnrd}(B, \mathrm{O}\,\mathbb{R}^{n})$$

are invertible for any cellular B with $\dim B \le n-s$, and surjective for any cellular B with $\dim B = n-s$. In exactly the same manner the inclusion $G_+(\infty, n-s) \supset \operatorname{skel}_{n-s} G_+(\infty, n)$ leads to the invertibility (surjectivity) of the mappings

$$\mathrm{ext}\colon \operatorname{Stnrd}(B, \mathrm{GL}_+\,\mathbb{R}^{n-s}) \to \operatorname{Stnrd}(B, \mathrm{GL}_+\,\mathbb{R}^{n})$$
$$\mathrm{ext}\colon \operatorname{Stnrd}(B, \mathrm{SO}\,\mathbb{R}^{n-s}) \to \operatorname{Stnrd}(B, \mathrm{SO}\,\mathbb{R}^{n})$$

for any cellular B with $\dim B \le n-s$ (respectively, $\dim B = n-s$), while the inclusion $\mathbb{C}G(\infty, n-s) \supset \operatorname{skel}_{2n-2s+1} \mathbb{C}G(\infty, n)$ implies the invertibility (surjectivity) of the mappings

$$\mathrm{ext}\colon \operatorname{Stnrd}(B, \mathrm{GL}\,\mathbb{C}^{n-s}) \to \operatorname{Stnrd}(B, \mathrm{GL}\,\mathbb{C}^{n})$$
$$\mathrm{ext}\colon \operatorname{Stnrd}(B, \mathrm{U}\,\mathbb{C}^{n-s}) \to \operatorname{Stnrd}(B, \mathrm{U}\,\mathbb{C}^{n})$$

for any cellular B with $\dim B \le 2(n-s)$ (respectively, $\dim B = 2n-2s+1$). Therefore, every n-dimensional real (complex) vector bundle with cellular base of dimension $\le n-s$ (respectively, $\le 2n-2s+1$) is $\mathrm{GL}\,\mathbb{R}^n$-equivalent (respectively, $\mathrm{GL}\,\mathbb{C}^n$-equivalent) to the s-fold suspension of an $(n-s)$-dimensional bundle; furthermore, if given two $(n-s)$-dimensional real (complex) vector bundles with cellular base of dimension $< n-s$ (respectively, $< 2n-2s+1$) their s-fold suspensions are $\mathrm{GL}\,\mathbb{R}^n$-equivalent (respectively, $\mathrm{GL}\,\mathbb{C}^n$-equivalent), then the bundles themselves are $\mathrm{GL}\,\mathbb{R}^{n-s}$-equivalent (respectively, $\mathrm{GL}\,\mathbb{C}^{n-s}$S-equivalent).

4.5.5 Exercises

Exercise 4.5.5.1. Let ξ be an n-dimensional real vector bundle. Show $\operatorname{assoc}(\xi, \mathbb{R}^n \setminus 0)$ is equivalent (in the sense of Definition 4.1.1.2) to the bundle with total space

$\{xG \in \mathrm{tl}(\xi) | x \neq 0\}$ and whose projection is the restriction of $\mathrm{proj}(\xi)$ to this subspace of $\mathrm{tl}(\xi)$.

Let ξ be an n-dimensional Euclidean bundle. Show that $\mathrm{assoc}(\xi, \mathbb{D}^n)$ and $\mathrm{assoc}(\xi, \mathbb{S}^{n-1})$ are equivalent to the bundles whose total spaces are the subspaces of $\mathrm{tl}(\xi)$ consisting of the vectors of length ≤ 1 and $= 1$, respectively, and whose projections are the appropriate restrictions of $\mathrm{proj}\,\xi$.

Exercise 4.5.5.2. Let ξ be an n-dimensional real (complex) vector bundle. Show that $\mathrm{assoc}(\xi, V'(n,k))$ (respectively, $\mathrm{assoc}(\xi, \mathbb{C}V'(n,k))$) is equivalent with the bundle with total space

$$\{(x_1, \dots, x_k) \in \underbrace{\mathrm{tl}(\xi) \times \cdots \times \mathrm{tl}(\xi)}_{k} \,|\, \mathrm{proj}\,\xi(x_1)) = \cdots = \mathrm{proj}(\xi(x_k)),$$

$$x_1, \dots, x_k \quad \text{linearly independent},$$

and whose projection is the restriction of the composite map

$$\mathrm{tl}(\xi) \times \cdots \times \mathrm{tl}(\xi) \xrightarrow{\mathrm{proj}_1} \mathrm{tl}(\xi) \xrightarrow{\mathrm{proj}(\xi)} \mathrm{bs}(\xi).$$

Let ξ be an n-dimensional Euclidean (Hermitian) bundle. Show that $\mathrm{assoc}(\xi, V(n,k))$ (respectively, $\mathrm{assoc}(\xi, \mathbb{C}V(n,k))$) is equivalent with the bundle with total space

$$\{(x_1, \dots, x_k) \in \underbrace{\mathrm{tl}(\xi) \times \cdots \times \mathrm{tl}(\xi)}_{k} \,|\, \mathrm{proj}\,\xi(x_1)) = \cdots = \mathrm{proj}(\xi(x_k)),$$

$$x_1, \dots, x_k \quad \text{is an orthonormal frame},$$

and whose projection is the restriction of the composite map

$$\mathrm{tl}(\xi) \times \cdots \times \mathrm{tl}(\xi) \xrightarrow{\mathrm{proj}_1} \mathrm{tl}(\xi) \xrightarrow{\mathrm{proj}(\xi)} \mathrm{bs}(\xi).$$

Exercise 4.5.5.3. Consider the spaces $\mathcal{T}$ and $\mathcal{S}$ introduced in Exercise 1.2.9.5. Now $GL_+(1, \mathbb{R})$ acts on $\mathcal{T} \setminus 0$ from the right by $(\{x_i\}, t) \mapsto \{tx_i\}$. Clearly $(\mathcal{T} \setminus 0)/\mathrm{GL}_+(1, \mathbb{R}) = \mathcal{S}$. Show that the $\mathrm{GL}_+(1, \mathbb{R})$-bundle defined by this action is locally trivial, but not trivial. Show that the associated oriented one-dimensional real vector bundle does not admit a Euclidean metric.

4.6 SMOOTH BUNDLES

4.6.1 Fundamental Concepts

Definition 4.6.1.1. Let $1 \leq r \leq a$. A bundle ξ is called a *bundle of class* $\mathcal{C}^r$, or a $\mathcal{C}$-bundle if $\operatorname{tl}(\xi)$ and $\operatorname{bs}(\xi)$ are $\mathcal{C}^r$-manifolds, and for each point $b_0 \in \operatorname{bs}(\xi)$ there are a neighbourhood U of b_0, a $\mathcal{C}^r$-manifold F with $\partial F = \varnothing$ if $\partial U \neq \varnothing$, and a $\mathcal{C}^r$-diffeomorphism $h \colon U \times F \to (\operatorname{proj}(\xi))^{-1}(U)$, such that $(\operatorname{proj}(\xi))^{-1}(h(b, x)) = b$ for all $b \in U$ and $x \in F$.

$$\begin{tikzcd}
U \times F \arrow[r, "h"] \arrow[dr, "\operatorname{proj}_1"'] & (\operatorname{proj}(\xi))^{-1}(U) \arrow[r, phantom, "\subset"] \arrow[d, "\operatorname{proj}(\xi)"] & \operatorname{tl}(\xi) \arrow[d, "\operatorname{proj}(\xi)"] \\
b_0 \arrow[r, phantom, "\in"] & U \arrow[r, phantom, "\subset"] & \operatorname{bs}(\xi)
\end{tikzcd}$$

The $\mathcal{C}^s$-bundles with $s \geq r$ will be referred to as bundles of class $\geq r$, or $\mathcal{C}^{\geq r}$-bundles. The $\mathcal{C}^{\geq 1}$-bundles are called *smooth.*

If ξ is a $\mathcal{C}^r$-bundle, then $\operatorname{proj}(\xi)$ is obviously a $\mathcal{C}^r$-submersion. In particular, the fibres of a smooth bundle are neat submanifolds of the total manifold $\operatorname{tl}(\xi)$ (see Corollary 3.1.5.8). Moreover, the fibres over points belonging to the same component of the base of a $\mathcal{C}^r$-bundle are pairwise $\mathcal{C}^r$-diffeomorphic. If $\operatorname{bs}(\xi)$ is connected and $\partial \operatorname{bs}(\xi) \neq \varnothing$, then the fibres have no boundary, and

$$\partial \operatorname{tl}(\xi) = (\operatorname{proj}(\xi))^{-1}(\partial \operatorname{bs}(\xi)),$$

whereas if $\partial \operatorname{bs}(\xi) = \varnothing$, then

$$\partial \operatorname{tl}(\xi) = \cup_{b \in \operatorname{bs})\xi)} \partial[(\operatorname{proj}(\xi))^{-1}(b)];$$

in the first situation, the restriction

$$(\partial \operatorname{tl}(\xi), \operatorname{abr} \operatorname{proj}(\xi), \partial[(\operatorname{proj}(\xi))^{-1}(b)]$$

of the bundle ξ to $\partial \operatorname{bs}(\xi)$ is a $\mathcal{C}^r$-bundle, whereas in the second

$$(\partial \operatorname{tl}(\xi), \operatorname{abr} \operatorname{proj}(\xi), \operatorname{bs}(\xi))$$

is a $\mathcal{C}^r$-bundle.

Given two $\mathcal{C}^r$-bundles, ξ_1 and ξ_2 such that $\partial \operatorname{bs}(\xi) = \varnothing$ and $\partial \operatorname{tl} = \varnothing$, the product $\xi_1 \times \xi_2$ is a $\mathcal{C}^r$-bundle.

The restriction of a $\mathcal{C}^r$-bundle to a neat submanifold of its base is clearly a $\mathcal{C}^r$-bundle.

Suppose that ξ is a $\mathcal{C}^r$-bundle, B is a $\mathcal{C}^r$-manifold, and $f \colon B \to \operatorname{bs}(\xi)$ is a $\mathcal{C}^r$-map such that the fibre $(\operatorname{proj}(\xi))^{-1}(f(b))$ has no boundary for all $b \in \partial B$. Then $f^!(\xi)$ is a $\mathcal{C}^r$-bundle, and we say that the bundle $f^!(\xi)$ is *neatly induced.* For example, given a $\mathcal{C}^r$-bundle ξ, $\operatorname{incl}^!(\xi)$ is always neatly induced when incl is either the inclusion of a neat submanifold $\operatorname{incl} \operatorname{bs}(\xi)$, or the inclusion $\partial \operatorname{bs}(\xi) \to \operatorname{bs}(\xi)$; obviously, $\operatorname{incl}^!(\xi)$ coincides, as a $\mathcal{C}^r$-bundle, with the corresponding restriction of ξ.

Definition 4.6.1.2. Let $0 \leq s \leq r$. A map φ from one $\mathcal{C}^{\geq r}$-bundle into another is said to be a $\mathcal{C}^s$-*map* (a $\mathcal{C}^s$-isomorphism) if $\mathrm{tl}(\xi)$ and $\mathrm{bs}(\xi)$ are $\mathcal{C}^s$-maps (respectively, $\mathcal{C}^s$-diffeomorphisms for $s \geq 1$, and homeomorphisms for $s = 0$). A $\mathcal{C}^s$-isomorphism which is also an equivalence is called a $\mathcal{C}^s$-equivalence.

A $\mathcal{C}^{\geq r}$-bundle Â£ is said to be $\mathcal{C}^s$-trivial if it is $\mathcal{C}^s$-equivalent to a standard trivial bundle $(\mathrm{bs}(\xi) \times F, \mathrm{proj}_1, \mathrm{bs}(\xi))$, where F is a $\mathcal{C}^r$-manifold (such that $\partial F = \varnothing$ if $\partial\,\mathrm{bs}(\xi) \neq \varnothing$). Every $\mathcal{C}^r$-bundle is obviously locally $\mathcal{C}^r$-trivial, meaning that each point of $\mathrm{bs}(\xi)$ has a neighbourhood U such that $\xi|_U$ is $\mathcal{C}^r$-trivial; in particular, every smooth bundle is topologically locally trivial.

If $f^!(\xi)$ is neatly induced from the $\mathcal{C}^{\geq r}$-bundle ξ by a $\mathcal{C}^r$ -map f, then $\mathrm{adj}\, f\colon f^!(\xi) \to \xi$ s a $\mathcal{C}^r$-map. Furthermore, if $\varphi\colon \xi' \to \xi$ is a $\mathcal{C}^r$-map, where ξ and ξ' are $\mathcal{C}^r$-bundles, and $(\mathrm{bs}(\varphi))^!(\xi)$ is neatly induced, then $\mathrm{corr}\,\varphi\colon \xi' \to (\mathrm{bs}(\varphi))^!(\xi)$ is also a $\mathcal{C}^r$-map.

Smooth Bundles and Submersions

Theorem 4.6.1.3. *Let $r \leq \infty$ and let $f\colon X \to Y$ be a $\mathcal{C}^r$-submersion, where X and Y are $\mathcal{C}^r$-manifolds, X is compact, and $f^{-1}(\partial Y) = \partial X$. Then (X, f, Y) is a $\mathcal{C}^r$-bundle. The same holds true when $r = a$, provided that X admits a $\mathcal{C}^a$-embedding in Euclidean space.*

(See Exercise 4.6.6.1 for a supplement to this theorem.)

Proof. It suffices to examine the case $f(X) = Y$: indeed, in the general case the set $f(X)$ is both open (see Corollary 3.1.5.8) and closed (see Theorem 1.1.7.9), and hence is a union of whole components of Y. We show that for each point $y_0 \in Y$ there are a neighbourhood U of y_0, a closed $\mathcal{C}^r$-manifold F, and a $\mathcal{C}^r$-diffeomorphism $h\colon U \times f \to f^{-1}(U)$, such that $f(h(y, x)) = y$ for all $y \in U$ and $x \in F$.

From Corollary 3.1.5.8 (or, if it is more convenient, from Theorem 3.4.8.2), it follows that $f^{-1}(y_0)$ is a neat submanifold of X for $y_0 \in \mathrm{int}\, Y$, and a neat submanifold of ∂X for $y_0 \in \partial Y$, and in both cases $f^{-1}(y_0)$ is closed as an independent manifold. Set $F = f^{-1}(y_0)$, and pick a $\mathcal{C}^r$-embedding $j\colon X \to \mathbb{R}^q$, a $\mathcal{C}^r$-transversalisation τ of the embedding $j|_F\colon F \to \mathbb{R}^q$ and a neat tube $\mathrm{Tub}_\tau\, \rho$. Now consider the map

$$\varphi\colon j^{-1}(\mathrm{tub}_\tau\, \rho) \to Y \times F, \quad \varphi(x) = (f(x), \mathrm{proj}_\tau(j(x))).$$

The following properties of φ are immediate:

- φ is of class $\mathcal{C}^r$;
- $\varphi(\partial(j^{-1}(\mathrm{tub}_\tau\, \rho))) \subset \partial(Y \times f)$;
- φ is injective on F;
- the differential $d_x\varphi$ is non-degenerate for all $x \in F$.

Since F is compact, we conclude that φ defines a diffeomorphism of a neighbourhood of F onto a neighbourhood of $\varphi(F) = y_0 \times F$ (see Theorem 3.1.5.5). Using once more the compactness of F, we see that the last neighbourhood contains a set of the form $V \times F$, where V is a neighbourhood of y_0 (in Y). Let U be a smaller neighbourhood of y_0, such that $j(f^{-1}(U)) \subset \operatorname{tub}_\tau \rho$. Then $\varphi^{-1}(U \times F) = f^{-1}(U)$, and we can finally set $h = (\operatorname{abr} \varphi)^{-1} \colon U \times F \to f^{-1}(U)$. $\square$

Example 4.6.1.4. The bundles

$(V(n,k), \operatorname{proj}, G(n,k))$, $(V(n,k), \operatorname{proj}, G_+(n,k))$, $(\mathbb{C}V(n,k), \operatorname{proj}, \mathbb{C}G(n,k))$, $(\mathbb{H}V(n,k), \operatorname{proj}, \mathbb{H}G(n,k))$,

whose projections are the submersions defined in Subsection 3.2.2, are principal $\mathcal{C}^a$-bundles with structure groups $\mathrm{O}(k)$, $\mathrm{SO}(k)$, $\mathrm{U}(k)$, and $\mathrm{Sp}(k)$, respectively. Similarly,

$$(V(n,k), \operatorname{proj}, V(n,k-q)), \quad (\mathbb{C}V(n,k), \operatorname{proj}, \mathbb{C}G(n,k-q)),$$
$$(\mathbb{H}V(n,k), \operatorname{proj}, \mathbb{H}G(n,k-q)),$$

whose projections are the submersions defined in Subsection 3.2.1, are Steenrod $\mathcal{C}^a$-bundles with structure group

$$\mathrm{O}(n-k+q), \quad \mathrm{U}(n-k+q), \quad \mathrm{Sp}(n-k+q),$$

and standard fibres

$$V(n-k+q,q), \quad \mathbb{C}V(n-k+q,q), \quad \mathbb{H}V(n-k+q,q)$$

(on which the above groups act canonically; see Remark 4.2.3.16). The coverings

$$(\mathbb{R}, \operatorname{hel}, \mathbb{S}^1), \quad (\mathbb{S}^1, \operatorname{hel}_m, \mathbb{S}^1), \quad (G_+(n,k), \operatorname{proj}, G(n,k)),$$

defined in Example 4.1.2.6, are also principal $\mathcal{C}^a$-bundles.

Among the previously listed principal $\mathcal{C}^a$-bundles we find

$$(\mathbb{S}^3, \operatorname{proj}, \mathbb{S}^2), \quad (\mathbb{S}^7, \operatorname{proj}, \mathbb{S}^4),$$

whose projections are the Hopf submersions (see Remark 3.2.2.9); they are called the ***Hopf bundles***. The Hopf submersion $\mathbb{S}^{15} \to \mathbb{S}^8$ defines a $\mathcal{C}^a$-bundle, which is also known as a Hopf bundle; its fibres are diffeomorphic to $\mathbb{S}^7$ (but this bundle is not endowed with any special structure group).

The Smooth Bundles as Steenrod Bundles

Remark 4.6.1.5. Let F be a $\mathcal{C}^r$-manifold with $r \geq 1$. According to Remark 4.2.3.14, F is an effective $\operatorname{Diff}^r F$-space, and thus every $\mathcal{C}^r$-bundle ξ whose fibres are $\mathcal{C}^r$-diffeomorphic to F is a W-F-bundle (see Example 4.3.1.5 and Definition 4.3.2.1). Moreover, ξ is clearly locally W-F-trivial, i.e., it is a Ehresmann–Feldbau bundle. However, the procedure that enabled us in Subsection 4.3.4 to turn Ehresmann-Feldbau bundles into Steenrod bundles does not work here: as we already had the occasion to note (see Exercise 4.3.5.2), when $\dim F > 0$, the natural action of $\operatorname{Diff}^r F$ on F is not topologically effective. Nevertheless, the set $MH(\xi)$ carries a natural topology, transferred from $\mathcal{C}^r(F, \operatorname{tl}(\xi))$ with the aid of the injective mapping $MH(\xi) \to \mathcal{C}^r(F, \operatorname{tl}(\xi))$ which takes each diffeomorphism $\alpha \in MH(\xi)$ into the map $[\text{incl}\colon \alpha(F) \to \operatorname{tl}(\xi)] \circ \alpha$. It is clear that with this topology on $MH(\xi)$, ξ becomes a Steenrod F-bundle.

It is instructive to compare the implicit group structures described above for smooth bundles with the implicit group structures of locally trivial bundles (see Remark 4.3.4.7). Here we merely mention two difficulties encountered in the differential situation. Firstly, not every Steenrod F-bundle can be smoothed - the fact that its base need not be a manifold is already an obstruction. Secondly, simple examples show that there are F-maps of $\mathcal{C}^r$-bundles with fibres diffeomorphic to F, which are not $\mathcal{C}^r$-maps.

4.6.2 Smoothings and Approximations

Remark 4.6.2.1. This subsection is similar in character with §3.4: here we generalise some of the results obtained there for smooth manifolds to smooth bundles. Although part of these results are indeed rather important, some problems are not touched upon at all. For the sake of brevity, we shall consider below only the closed case; the reader can find some additional information concerning the (more general) compact case in Subsection 4.6.6 (see Exercises 4.6.6.2-4.6.6.5).

We shall need two notations, for $0 \leq s \leq r$: if ξ is a $\mathcal{C}^r$-bundle, we let $\operatorname{Sect}^s \xi$ denote the set of all $\mathcal{C}^s$-sections of ξ; and if ξ and ξ' are $\mathcal{C}^r$-bundles, we let $\mathcal{C}^s(\xi, \xi')$ denote the space of all $\mathcal{C}^s$-maps $\xi \to \xi'$. If $s \neq a$, both sets carry natural topologies: $\operatorname{Sect}^s \xi$ is a subspace of $\mathcal{C}^s(\operatorname{bs}(\xi), \operatorname{tl}(\xi))$, while $\mathcal{C}^s(\xi, \xi')$ is a subspace of the product $\mathcal{C}^s(\operatorname{tl}(\xi), \operatorname{tl}(\xi')) \times \mathcal{C}^s(\operatorname{bs}(\xi), \operatorname{bs}(\xi'))$.

ξ-Transversalisations and Tubes

Our immediate task is to adapt the definitions and theorems of Subsection 3.4.3 for use in the more general setting of this subsection.

Definition 4.6.2.2. We start with the transversalisations. Let ξ be a smooth bundle such that $\operatorname{tl}(\xi)$ is closed, and let $j\colon \operatorname{tl}(\xi) \to \mathbb{R}^q$ be a differentiable embedding. A continuous map $t\colon \operatorname{tl}(\xi) \to G(q, q-n)$, where $n = \dim \operatorname{tl}(\xi) - \dim \operatorname{bs}(\xi)$, is called a *$\xi$-transversalisation* of the embedding j if the restriction $\tau|_{(\operatorname{proj}(\xi))^{-1}(y)}$ is a transversalisation of the embedding $j|_{(\operatorname{proj}(\xi))^{-1}(y)}$ for all points $y \in \operatorname{bs}(\xi)$. A fundamental example is the *normal ξ-transversalisation*, which takes each point $x \in \operatorname{tl}(\xi)$ into the orthogonal complement of the plane $d_x j(\operatorname{Tang}[(\operatorname{proj}(\xi)]^{-1}(\operatorname{proj}(\xi)(x))))$ in $\mathbb{R}^q$. Our ξ-version of Theorem 3.4.3.7 asserts that if ξ, and j are of class $\mathcal{C}^r$ $(1 \leq r \leq a)$, then j admits a ξ-transversalisation of class $\mathcal{C}^r$. The proof is an obvious modification of that of Theorem 3.4.3.7.

Now we move on to tubes. Let τ be an arbitrary ξ-transversalisation of the embedding j. We define the tube $\operatorname{Tub}_\tau \rho$ and the open tube $\operatorname{tub}_\tau \rho$ as the following subsets of $\mathrm{bs}(\xi) \times \mathbb{R}^q$:

$$\begin{aligned}\operatorname{Tub}_\tau \rho &= \cup_{x\in \mathrm{tl}(\xi)}((\mathrm{proj}(\xi))(x) \times d_\tau(x,\rho))\\ \operatorname{tub}_\tau \rho &= \cup_{x\in \mathrm{tl}(\xi)}((\mathrm{proj}(\xi))(x) \times (d_\tau(x,\rho) \setminus s_\tau(x,\rho))),\end{aligned}$$

where $d_\tau(x,\rho)$ and $s_\tau(x,\rho)$ are the ball, and respectively the sphere, with centre $j(x)$ and radius ρ in the plane $j(x)+\tau(x)$. Equivalently,

$$\begin{aligned}\operatorname{Tub}_\tau \rho &= \cup_{y\in \mathrm{bs}(\xi)}(y \times \operatorname{Tub}_{\tau|_{(\mathrm{proj}(\xi))^{-1}(y)}} \rho)\\ \operatorname{tub}_\tau \rho &= \cup_{y\in \mathrm{bs}(\xi)}(y \times \operatorname{tub}_{\tau|_{(\mathrm{proj}(\xi))^{-1}(y)}} \rho).\end{aligned}$$

The tube $\operatorname{Tub}_\tau \rho$ is *neat* if there is a $\sigma > \rho$ such that:

- the sets $(\mathrm{proj}(\xi))(x) \times (d_\tau(x,\sigma) \setminus s_\tau(x,\sigma))$ are pairwise disjoint,
- $\operatorname{tub}_\tau \sigma$ is open in $\mathrm{bs}(\xi) \times \mathbb{R}^q$,
- the map $\operatorname{tub}_\tau \sigma \to \mathrm{tl}(\xi)$, which takes $(\mathrm{proj}(\xi))(x) \times (d_\tau(x,\sigma) \setminus s_\tau(x,\sigma))$ into x, is smooth.

The restrictions of this last map to $\operatorname{Tub}_\tau \rho$ and ${}_tub_\tau\rho$ are called *projections* and are denoted by proj_τ (they clearly do not depend upon the choice of σ). If ξ, j, and τ are of class $\mathcal{C}^r$, $r \geq 1$, then the following are true:

- there exists a neat tube;
- every neat tube $\operatorname{Tub}_\tau \rho$ is a submanifold of $\mathrm{bs}(\xi) \times \mathbb{R}^q$, with $\operatorname{int}\operatorname{Tub}_\tau \rho = \operatorname{tub}_\tau \rho$;
- $\mathrm{proj}_\tau \colon \operatorname{Tub}_\tau \rho \to \mathrm{tl}(\xi)$ is a $\mathcal{C}^r$-submersion.

Again, the proof is an obvious modification of the proofs of Theorems 3.4.3.4 and 3.4.3.5. We must also modify appropriately the construction in Remark 3.4.3.3: now the model Tu_τ is defined as the subset

$$\{(x,t) \in \mathrm{tl}(\xi) \times \mathbb{R}^q | t \in \tau(x)\} \subset \mathrm{tl}(\xi) \times \mathbb{R}^q,$$

while $\mathrm{nat}\colon \mathrm{Tu}_\tau \to \mathrm{bs}(\xi) \times \mathbb{R}^q$ is given by $\mathrm{nat}(x,t) = ([\mathrm{proj}_\tau(\xi)](x), j(x)+t)$.

The Basic Theorems

Theorem 4.6.2.3. *Let $\leq \infty$ and let ξ and ξ' be $\mathcal{C}^r$-bundles with closed total spaces $\mathrm{tl}(\xi)$ and $\mathrm{tl}(\xi')$, and closed bases $\mathrm{bs}(\xi)$ and $\mathrm{bs}(\xi')$. Then $\mathcal{C}^r(\xi,\xi')$ is dense in $\mathcal{C}^s(\xi,\xi')$ for any $s < r$. The same holds $r = a$, provided that $\mathrm{tl}(\xi)$, $\mathrm{tl}(\xi')$, $\mathrm{bs}(\xi)$ and $\mathrm{bs}(\xi')$ admit $\mathcal{C}^a$-embeddings in Euclidean spaces.*

Proof. Pick a $\mathcal{C}^r$-embedding $j'\colon \mathrm{tl}(\xi') \to \mathbb{R}^{q'}$, a ξ'-transversalisation τ' of j' of class $\mathcal{C}^r$, and a neat tube $\operatorname{Tub}_{\tau'} \rho'$. Let $\mathcal{U}$ denote the subset of

$$\mathcal{C}^s(\mathrm{tl}(\xi), \mathrm{tl}(\xi') \times \mathcal{C}^s(\mathrm{bs}(\xi), \mathrm{bs}(\xi'))$$

consisting of all the pairs

$$(F\colon \operatorname{tl}(\xi) \to \operatorname{tl}(\xi'), f\colon \operatorname{bs}(\xi) \to \operatorname{bs}(\xi'))$$

such that $(f((\operatorname{proj}(\xi))(x)), j'(F(x))) \in \operatorname{tub}_{\tau'} \rho'$ for all $x \in \operatorname{tl})\xi)$. Then $\mathcal{U}$ is open and contains $\mathcal{C}^s(\xi, \xi')$. For $(F, f) \in \mathcal{U}$, define $\Phi(F, f)\colon \operatorname{tl}(\xi) \to \operatorname{tl}(\xi')$ by

$$x \mapsto \operatorname{proj}_{\tau'}(f((\operatorname{proj}(\xi))(x)), j'(F(x))).$$

Obviously, $(\varphi(F, f), f) \in \mathcal{C}^s(\xi, \xi')$ and the map

$$\Psi\colon \mathcal{U} \to \mathcal{C}^s(\xi, \xi'), \Psi(F, f) = (\Phi(F, f), f),$$

is a retraction which takes

$$\mathcal{U} \cap (\mathcal{C}^r(\operatorname{tl}(\xi), \operatorname{tl}(\xi')) \times \mathcal{C}^r(\operatorname{bs}(\xi), \operatorname{bs}(\xi')))$$

into $\mathcal{C}^r(\xi, \xi')$. Since

$$\mathcal{C}^r(\operatorname{tl}(\xi), \operatorname{tl}(\xi')) \times \mathcal{C}^r(\operatorname{bs}(\xi), \operatorname{bs}(\xi'))$$

is dense in,

$$\mathcal{C}^s(\operatorname{tl}(\xi), \operatorname{tl}(\xi')) \times \mathcal{C}^s(\operatorname{bs}(\xi), \operatorname{bs}(\xi'))$$

(see Theorem 3.4.4.2), the existence of such a retraction implies that $\mathcal{C}^s((\xi, \xi')$ is dense in $\mathcal{C}^s((\xi, \xi')$. □

Theorem 4.6.2.4. *Let $s < r \leq \infty$, let ξ and ξ' be arbitrary $\mathcal{C}^{\geq r}$-bundles such that $\operatorname{tl}(\xi)$, $\operatorname{tl}(\xi')$, $\operatorname{bs}(\xi)$, $\operatorname{bs}(\xi')$ are closed manifolds, and let $f : \operatorname{bs}(\xi) \to \operatorname{bs}(\xi')$ be a $\mathcal{C}^r$-map. Then the set*

$$\{\Phi \in \mathcal{C}^r(\xi, \xi') | \operatorname{bs}(\Phi) = f\}$$

is dense in

$$\{\Phi \in \mathcal{C}^s(\xi, \xi') | \operatorname{bs}(\Phi) = f\}.$$

The same holds true for $r = a$, provided that $\operatorname{tl}(\xi)$, $\operatorname{tl}(\xi\prime)$ admit $\mathcal{C}^a$-embeddings in Euclidean spaces.

Proof. Let $\mathcal{F}$ denote the subspace of $\mathcal{C}^s(\operatorname{bs}(\xi), \operatorname{bs}(\xi'))$ consisting of the maps $\operatorname{tl}(\Phi)$ such that $\Phi \in \mathcal{C}^s(\xi, \xi')$ and $\operatorname{bs}(\Phi) = f$. We show that $\mathcal{F} \cap \mathcal{C}^r(\xi, \xi')$ is dense in F

Pick a $\mathcal{C}^r$-embedding $j'\colon \operatorname{tl}(\xi' \to \mathbb{R}^q$, a ξ'-transversalisation τ' of j' of class $\mathcal{C}^r$, and a neat tube $\operatorname{Tub}_{\tau'} \rho$. Consider the subset $\mathcal{U} \subset \mathcal{C}^s(\operatorname{tl}(\xi), \operatorname{tl}(\xi'))$ consisting of all maps $F\colon \operatorname{tl}(\xi) \to \operatorname{tl}(\xi')$ such that $f((\operatorname{proj}(\xi))(x), j'(F(x))) \in \operatorname{tub}_{\tau'} \rho$ for all $x \in \operatorname{tl}(\xi)$. It is clear that $\mathcal{U}$ is open and contains $\mathcal{F}$. Moreover, the mapping $\mathcal{U} \to \mathcal{F}$, transforming each $F \in \mathcal{U}$ into the map

$$x \mapsto \operatorname{proj}_{\tau'}(f((\operatorname{proj}(\xi))(x)), j'(F(x))).$$

is a retraction which takes

$$\mathcal{U} \cap (\mathcal{C}^r(\operatorname{tl}(\xi), \operatorname{tl}(\xi')) \times \mathcal{C}^r(\operatorname{bs}(\xi), \operatorname{bs}(\xi')))$$

into $\mathcal{F} \cap \mathcal{C}^r(\xi, \xi')$. Since $\mathcal{C}^r(\operatorname{tl}(\xi), \operatorname{tl}(\xi'))$ is dense in $\mathcal{C}^s(\operatorname{tl}(\xi), \operatorname{tl}(\xi'))$, the existence of such a retraction shows that $\mathcal{F} \cap \mathcal{C}^s((\xi, \xi')$ is dense in $\mathcal{F}$, as claimed. □

Theorem 4.6.2.5. *Let $r \leq \infty$, and let ξ and ξ' be arbitrary $\mathcal{C}^{\geq r}$-bundles such that the manifolds* $\mathrm{tl}(\xi)$, $\mathrm{tl}(\xi')$, $\mathrm{bs}(\xi)$ *and* $\mathrm{bs}(\xi')$ *are closed. If $0 < s < r$, then the set of all $\mathcal{C}^r$-isomorphisms ($\mathcal{C}^r$-equivalences) $\xi \to \xi'$ is dense in the subspace of all $\mathcal{C}^s$-isomorphisms (respectively, $\mathcal{C}^s$-equivalences) of $\mathcal{C}^s(\xi, \xi')$. The same holds true for $r = a$, provided that* $\mathrm{tl}(\xi)$, $\mathrm{tl}(\xi')$, $\mathrm{bs}(\xi)$, $\mathrm{bs}(\xi')$ *(respectively,* $\mathrm{tl}(\xi)$ *and* $\mathrm{tl}(\xi')$*) admit $\mathcal{C}^a$-embeddings in Euclidean spaces.*

Proof. This is a consequence of Theorems 4.6.2.3, 4.6.2.4, and Corollary 3.4.1.6. □

Corollary 4.6.2.6. *If two $\mathcal{C}^{\geq r}$-bundles with closed total manifolds and bases are $\mathcal{C}^1$-isomorphic and $r < \infty$, then they are $\mathcal{C}^r$-isomorphic. The same holds true for $r = a$, provided that the total manifolds and the bases admit $\mathcal{C}^a$-embeddings in Euclidean spaces.*

If two $\mathcal{C}^{\geq r}$-bundles with closed total manifolds are $\mathcal{C}^1$-equivalent and $r \leq \infty$ then they are $\mathcal{C}^r$-equivalent. The same holds true for $r = a$, provided that the total manifolds and the bases admit $\mathcal{C}^a$-embeddings in Euclidean spaces.

Theorem 4.6.2.7. *If $r \leq \infty$, then given any $\mathcal{C}^r$-bundle ξ with closed* $\mathrm{tl}(\xi)$ *and* $\mathrm{bs}(\xi)$*, the space* $\mathrm{Sect}^r(\xi)$ *is dense in* $\mathrm{Sect}^s(\xi)$*, for any $s < r$. The same holds true for $r = a$, provided that* $\mathrm{tl}(\xi)$ *and* $\mathrm{bs}(\xi)$ *admit $\mathcal{C}^a$-embeddings in Euclidean spaces.*

Proof. This is a result of Theorem 4.6.2.4, applied to the bundles ξ and $(\mathrm{bs}(\xi), \mathrm{id}_{\mathrm{bs}(\xi)}, \mathrm{bs}(\xi))$ and to the map $\mathrm{id}_{\mathrm{bs}(\xi)}$. □

Theorem 4.6.2.8. *Every $\mathcal{C}^{\geq r}$-bundle ξ such that* $\mathrm{tl}(\xi)$ *and* $\mathrm{bs}(\xi)$ *are closed manifolds is $\mathcal{C}^r$-isomorphic to a $\mathcal{C}^a$-bundle η with the property that* $\mathrm{tl}(\eta)$ *and* $\mathrm{bs}(\eta)$ *can be $\mathcal{C}^a$-embedded in Euclidean spaces.*

Proof. By Theorem 3.4.9.6, there exist $\mathcal{C}^a$-manifolds, T and B, admitting $\mathcal{C}^a$-embeddings in Euclidean spaces, together with $\mathcal{C}^r$-diffeomorphisms

$$F\colon T \to \mathrm{tl}(\xi), \quad f\colon B \to \mathrm{bs}(\xi).$$

Pick a $\mathcal{C}^r$-embedding $j\colon\ \mathrm{tl}(\xi) \to \mathbb{R}^q$ a ξ-transversalisation τ of j of class $\mathcal{C}^r$, and a neat tube $\mathrm{Tub}_\tau\,\rho$. Let $\mathcal{U}$ denote the subset of $\mathcal{C}^r(T, B)$ consisting of all the submersions $p\colon T \to B$ such that the image of the composite map

$$T \xrightarrow{\mathrm{diag}} T \times T \xrightarrow{(f\circ p)\times(j\circ F)} \mathrm{bs}(\xi) \times \mathbb{R}^q$$

is contained in $\mathrm{tub}_\tau\,\rho$. Obviously:

- $\mathcal{U}$ is open;
- $f^{-1} \circ \mathrm{proj}(\xi) \circ F \in \mathcal{U}$;
- the mapping $\Phi\colon \mathcal{U} \to \mathcal{C}^r(T, \mathrm{tl}(\xi))$, which takes each $g \in \mathcal{U}$ into the map $x \mapsto \mathrm{proj}_\tau(f \circ g(x), j \circ F(x))$, is continuous and $\Phi(f^{-1} \circ \mathrm{proj}(\xi) \circ F) = F$.

Since F is a $\mathcal{C}^r$-diffeomorphism, there is a neighbourhood $\mathcal{V}$ of $f^{-1} \circ \operatorname{proj}(\xi) \circ F$ in $\mathcal{C}^r(T, B)$ with the property that $\mathcal{V} \subset \mathcal{U}$ and $\Phi(g)$ is a $\mathcal{C}^r$-diffeomorphism for all $g \in \mathcal{V}$. Now pick some $\mathcal{C}^a$-map h and set $\eta = (T, h, B)$. By Theorem 4.6.1.3, η is a $\mathcal{C}^a$-bundle, and it is plain that $\Phi(h)$ and f yield a $\mathcal{C}^r$-isomorphism $\eta \to \xi$. $\square$

4.6.3 Smooth Vector Bundles

Remark 4.6.3.1. Usually, when we encounter a bundle which is smooth, it carries some additional structures, most frequently a group structure of Steenrod type. In such cases, smoothness plays the same role as does the topology in the theory of Steenrod bundles discussed earlier (§§4.3, 4.4), and it is natural to try developing this analogy into a weighty theory of smooth Steenrod bundles.

Unfortunately, such a program is beyond the scope of our book. Therefore, we shall restrict ourselves to the basic facts concerning smooth *vector* bundles, which are of main interest to this study, and can be derived in a less cumbersome manner than the general theory.

We remark that because the fibres of a vector bundle of positive dimension are not compact, it is not possible to deduce the smoothing and approximation theorems below (Theorems 4.6.3.8 - 4.6.3.12) from the results of the previous subsection without resorting to additional devices. However, we prefer to give simple, straightforward proofs of these theorems, so that this subsection becomes independent of the previous one.

Fundamental Concepts

Definition 4.6.3.2. ξ is an n-dimensional real vector $\mathcal{C}^r$-bundle $(1 \leq r \leq a)$ if it is both an n-dimensional real vector bundle and a $\mathcal{C}^r$-bundle, and these two structures are compatible, meaning that the restriction of ξ over a small enough neighbourhood of an arbitrary point of $\operatorname{bs}(\xi)$ is $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-trivial (i.e., is $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-equivalent to a standard trivial bundle, where, of course, a $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-equivalence is just a $\operatorname{GL}\mathbb{R}^n$-equivalence which is simultaneously a $\mathcal{C}^r$-equivalence). The Euclidean, complex vector, and Hermitian $\mathcal{C}^r$-bundles are similarly defined.

Products of vector, Euclidean, or Hermitian $\mathcal{C}^{\geq r}$-bundles, as well as bundles induced (in particular, obtained by reducing) such bundles, are again $\mathcal{C}^{\geq r}$-bundles of the same kind, provided the conditions imposed by the corresponding definitions from Subsection 4.6.1 (see Definition 4.6.1.2) are fulfilled.

We add that the statements and proofs of Theorem 4.3.2.8 and Corollary 4.3.2.9 carry over, with obvious modifications, to vector, Euclidean, and Hermitian $\mathcal{C}^{\geq r}$-bundles. Here we formulate only the $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-version of Corollary 4.3.2.9: let $f\colon \xi \to \eta$ be a $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-map, where ξ and η are $\mathcal{C}^{\geq r}$-$\operatorname{GL}\mathbb{R}^n$-bundles, and suppose that the bundle $(\operatorname{bs}(f))^!\eta$ is neatly induced (see Definition 4.6.1.1.); then $\operatorname{corr} f$ is a $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-equivalence.

Remark 4.6.3.3. The explicit descriptions of the vector, Euclidean, and Hermitian bundles, given in Subsection 4.5.1, have obvious $\mathcal{C}^r$-analogues ($1 \leq r \leq a$). The $\mathcal{C}^r$-analogue of Theorem 4.5.1.3 asserts that a $\mathcal{C}^r$-bundle whose fibres are n-dimensional real vector spaces is an n-dimensional real vector $\mathcal{C}^r$-bundle if and only if the partial vector operations indicated in Theorem 4.5.1.3 are $\mathcal{C}^r$-maps. Similarly, the Theorem $\mathcal{C}^r$-analogue of Theorem 4.5.1.5 asserts that a $\mathcal{C}^r$-bundle whose fibres are n-dimensional Euclidean is an n-dimensional Euclidean $\mathcal{C}^r$-bundle if and only if the partial vector operations and the metric (equivalently, the square of the length of vectors, considered as a function on the total space) are of class $\mathcal{C}^r$. In particular, in order to turn a real vector $\mathcal{C}^r$-bundle into a Euclidean $\mathcal{C}^r$-bundle, one has to equip it with a Euclidean $\mathcal{C}^r$-metric. The corresponding complex formulations (i.e., the $\mathcal{C}^r$-analogues of the theorems in Definition 4.5.1.10) are obtained by replacing Euclidean bundles and Euclidean metrics by Hermitian bundles and Hermitian metrics.

Smooth Grassmann Bundles

Remark 4.6.3.4. The Grassmann bundles defined in Definition 4.5.3.7 provide (for $m < \infty$) fundamental examples of smooth vector, Euclidean, and Hermitian bundles. Namely, if $0 \leq n \leq m < \infty$, then $\operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$, $\operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{C}))$, $\operatorname{Grass}(m, \mathrm{O}(n))$, and $\operatorname{Grass}(m, \mathrm{U}(n))$ are obviously real vector, complex vector, Euclidean, and Hermitian $\mathcal{C}^a$-bundles, respectively, all of them n-dimensional. The third (fourth) bundle differs from the first (respectively, second) by having a Euclidean $\mathcal{C}^a$-metric (respectively, a Hermitian $\mathcal{C}^r$-metric).

Theorem 4.6.3.5 below may be thought of as a weakened $\mathcal{C}^r$-analogue (for $r \neq a$) of that part of Corollary 4.5.3.9 concerning $\operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$ and $\operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{C}))$.

Theorem 4.6.3.5. *Let ξ be an n-dimensional real (complex) vector $\mathcal{C}^r$-bundle with compact base. If $r \neq a$, then there are a number m and a $\mathcal{C}^r$-map $f\colon \operatorname{bs}(\xi) \to G(m, n)$ (respectively, $f\colon \operatorname{bs}(\xi) \to \mathbb{C}G(m, n)$) such that ξ is $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{R}^n$-equivalent to the bundle $f^! \operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$ (respectively, $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{C}$-equivalent to the bundle $f^! \operatorname{Grass}(m, \mathrm{GL}(n, \mathbb{C}))$).*

Proof. Since the proofs of the real and complex cases differ only in some obvious details, we shall prove only the former. Choose, for every point $b \in \operatorname{bs}(\xi)$, a chart $\psi_b \in \operatorname{Atl}_b \operatorname{bs}(\xi)$ such that

$$\psi_b(\operatorname{supp} \psi_b, b) = (\mathbb{R}q, 0) \quad \text{or} \quad (\mathbb{R}^q_-, 0) \quad [q = \dim \operatorname{bs}(\xi)]$$

and $\xi|_{\operatorname{supp} \psi_b}$ is $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{R}^n$-trivial, and then fix a $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{R}^n$-trivialisation,

$$\tau_b\colon (\operatorname{supp} \psi_b \times \mathbb{R}^n, \operatorname{proj}_1, \operatorname{supp} \psi_b) \to \xi|_{\operatorname{supp} \psi_b}.$$

Now cover $\operatorname{bs}(\xi)$ by a finite number of sets $\psi_b^{-1}(\mathbb{D}^q)$, say $\psi_{b_1}^{-1}(\mathbb{D}^q), \ldots, \psi_{b_s}^{-1}(\mathbb{D}^q)$ and pick a $\mathcal{C}^r$-function $\alpha\colon \mathbb{R}^q \to \mathbb{R}$ which equals 1 on $\mathbb{D}^q$ and 0 outside $2\mathbb{D}^q$.

Finally, define $H_1, \ldots, H_s\colon \operatorname{tl}(\xi) \to \mathbb{R}$ by

$$H_i(x) = \begin{cases} \alpha(\psi_{b_i}((\operatorname{proj}(\xi))(x)))\operatorname{proj}_2 \circ \operatorname{tl}\tau_{b_i}^{-1}(x), & \text{if } \ x \in (\operatorname{proj}(\xi))^{-1}(\operatorname{supp}\psi b_i), \\ 0, & \text{if } \ x \notin (\operatorname{proj}(\xi))^{-1}(\operatorname{supp}\psi b_i), \end{cases}$$

where $\operatorname{proj}_2 = [\operatorname{proj}_2\colon \operatorname{supp}\psi_{b_i} \times \mathbb{R}^n \to \mathbb{R}^n]$. Define

$$H\colon \operatorname{tl}(\xi) \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{sn}, \quad H(x) = (H_1(x), \ldots, H_s(x)).$$

Clearly, H is a $\mathcal{C}^r$-map and its restrictions $H|_{(\operatorname{proj}(\xi))^{-1}(b)}$ are linear monomorphisms for all $b \in \operatorname{bs}(\xi)$. Set $m = sn$ and

$$f\colon \operatorname{bs}(\xi) \to G(m,n), \quad f(b) = H((\operatorname{proj}(\xi))^{-1}(b).$$

To verify that the bundles ξ and $f^!\operatorname{Grass}(m, \operatorname{GL}(n,\mathbb{R}))$ are $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-equivalent, it is enough to produce a $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-map $\varphi\colon \xi \to \operatorname{Grass}(m, \operatorname{GL}(n,\mathbb{R}))$ with $\operatorname{bs}(\varphi) = f$. Such a φ is defined by

$$\operatorname{tl}(\varphi)\colon \operatorname{tl}(\xi) \to \operatorname{tl}(\operatorname{Grass}(m, \operatorname{GL}(n,\mathbb{R}))), \quad \operatorname{tl}(\varphi)(x) = (f(\operatorname{proj}(\xi))(x), H(x))$$

(recall that $\operatorname{tl}\operatorname{Grass}(m, \operatorname{GL}(n,\mathbb{R})) = \{(\gamma, y) \in G(m,n) \times \mathbb{R}^n | y \in \gamma\}$). ☐

An Application

Theorem 4.6.3.6. *If $1 \le r \le \infty$, then every real (complex) vector $\mathcal{C}^r$-bundle with compact base has a Euclidean (respectively, Hermitian) $\mathcal{C}^r$-metric.*

Proof. Since $\operatorname{Grass}(m, \operatorname{GL}(n,\mathbb{R}))$ $(\operatorname{Grass}(m, \operatorname{GL}(n,\mathbb{C})))$ has a Euclidean (respectively, Hermitian) $\mathcal{C}^a$-metric, Theorem 4.6.3.6 is a consequence of Theorem 4.6.3.5. ☐

Smoothings and Approximations

Remark 4.6.3.7. Given two real or complex vector $\mathcal{C}^{\ge r}$-bundles, ξ and ξ', and a $\mathcal{C}^s$-map $f\colon \operatorname{bs}(\xi') \to \operatorname{bs}(\xi)$ with $r \ge s \ge 0$, we let $L^s(\xi, \xi'; f)$ denote, in Theorems 4.6.3.8 and 4.6.3.10 below, the set of all linear $\mathcal{C}^s$-maps $\varphi\colon \xi' \to \xi$ such that $\operatorname{bs}(\varphi) = f$. If $s \ne a$, then $L^s(\xi, \xi'; f)$ inherits a natural topology as a subset of $\mathcal{C}^s(\xi', \xi)$.

When $\dim\xi = \dim\xi' = n$, $\operatorname{bs}\xi = \operatorname{bs}\xi'$, and $f = \operatorname{id}$, $L^s(\xi, \xi'; f)$ contains the set of all $\mathcal{C}^r$-$\operatorname{GL}\mathbb{R}^n$-equivalences $\xi' \to \xi$ in the real case, and the set of all $\mathcal{C}^s$-$\operatorname{GL}\mathbb{C}^n$-equivalences $\xi' \to \xi$ in the complex case. In both cases this subset is open for any $s \ne a$.

Notice that among the spaces $L^s(\xi, \xi'; f)$ we find $\operatorname{Sect}^s(\xi)$, $0 \le s \le r$ (see Remark 4.6.2.1). More precisely, $\operatorname{Sect}^s(\xi)$ is canonically homeomorphic to $L^s(\xi, \xi'; f)$, where ξ' is the standard trivial bundle $(\operatorname{bs}(\xi) \times \mathbb{R}, \operatorname{proj}_1, \operatorname{bs}(\xi))$, and $f = \operatorname{id}_{\operatorname{bs}(\xi)}$; the canonical homeomorphism $L^s(\xi, \xi'; f) \to \operatorname{Sect}^s(\xi)$ takes each map $\varphi\colon \xi' \to \xi$ into the section $b \mapsto (\operatorname{tl}(\varphi))(b, 1)$.

Theorem 4.6.3.8. *Let* ξ *and* ξ' *be real or complex vector* $\mathcal{C}^{\geq r}$*-bundles with compact bases. If* $0 \leq s < r \leq \infty$*, then* $L^r(\xi, \xi'; f)$ *is dense in* $L^s(\xi, \xi'; f)$ *for any* $\mathcal{C}^r$*-map* $f\colon \operatorname{bs}(\xi') \to \operatorname{bs}(\xi)$*.*

Proof. Since the proofs of the real and complex cases differ only in some obvious details, we shall again prove only the former. By Theorem 4.6.3.5, we may assume that

$$\xi = g^! \operatorname{Grass}(m, \operatorname{GL}(n, \mathbb{R})), \quad g'^! \operatorname{Grass}(m', \operatorname{GL}(n', \mathbb{R}),$$

where

$$g\colon \operatorname{bs}(\xi) \to G(m, n), \quad g'\colon \operatorname{bs}(\xi') \to G(m', n')$$

are some $\mathcal{C}^r$-maps. Then we can identify $\operatorname{tl}(\xi)$ with the $\mathcal{C}^r$-submanifold

$$\{(b, y) \in \operatorname{bs}(\xi) \times \mathbb{R}^m | y = g(b)\} \subset \operatorname{bs}(\xi) \times \mathbb{R}^n$$

and, similarly, $\operatorname{tl}(\xi')$ with the $\mathcal{C}^r$-submanifold

$$\{(b', y') \in \operatorname{bs}(\xi') \times \mathbb{R}^{m'} | y' = g(b')\} \subset \operatorname{bs}(\xi') \times \mathbb{R}^{n'}.$$

The orthogonal projections of $\mathbb{R}^m$ onto its subspaces $g(b)$ with $b \in \operatorname{bs}(\xi)$ combine to define a $\mathcal{C}^r$-map $p\colon \operatorname{bs}(\xi) \times \mathbb{R}^m \to \operatorname{tl}(\xi)$, and a $\mathcal{C}^r$-map $p\colon \operatorname{bs}(\xi') \times \mathbb{R}^{m'} \to \operatorname{tl}(\xi')$ is similarly defined.

Let A be the Euclidean space of all linear maps $\mathbb{R}^m \to \mathbb{R}^{m'}$ (which is the same as the space of all real $(m \times m')$-matrices; cf. Remarks 3.2.1.1 or 3.2.1.8), and consider the mappings

$$\Phi\colon L^s(\xi, \xi'; f) \to \mathcal{C}^s(\operatorname{bs}(\xi'), A), \quad \Psi\colon \mathcal{C}^s(\operatorname{bs}(\xi'), A) \to> L^s(\xi, \xi'; f),$$

given by

$$\begin{aligned}&\{[\Phi(\varphi)](b')\}(y') = [\operatorname{proj}_2\colon \operatorname{bs}(\xi) \times \mathbb{R}^m \to \mathbb{R}^m](\operatorname{tl}(\varphi) \circ p'(b', y'))\\&[\operatorname{tl}(\psi(h))](b', y') = p(f(b'), [h(b')](y'))\end{aligned}$$

(where $\varphi \in L^s(\xi, \xi'; f)$, $b' \in \operatorname{bs}(\xi')$, $y' \in \mathbb{R}^{m'}$, and $h \in \mathcal{C}^s(\operatorname{bs}(\xi'), A)$). Clearly, ψ is continuous (and so is φ) and takes $\mathcal{C}^r$-maps into $\mathcal{C}^r$-maps. Moreover, $\psi \circ \varphi = \operatorname{id}_{L^s(\xi, \xi'; f)}$, and hence ψ is surjective. Since $\mathcal{C}^r(\operatorname{bs} \xi', A)$ is dense in $\mathcal{C}^s(\operatorname{bs} \xi', A)$ (see Theorem 3.4.6.5), we conclude that $L^r(\xi, \xi'; f)$ is dense in $L^s(\xi, \xi'; f)$. [Explanation: Theorem 3.4.6.5 is applied after we have completed the space A to a sphere by adding a point; cf. Theorem 3.4.4.2 and 3.4.4.7.] □

Theorem 4.6.3.9. *Let* ξ *be a real or complex vector* $\mathcal{C}^{\geq r}$*-bundle with compact base. If* $0 \leq s < r \leq \infty$*, then* $\operatorname{Sect}^r(\xi)$ *is dense in* $\operatorname{Sect}^s(\xi)$*.*

Proof. This is a consequence of Theorem 4.6.3.8: $\operatorname{Sect}^r(\xi) = L^r(\xi, \xi'; \operatorname{id}_{\operatorname{bs}(\xi)})$ and $\operatorname{Sect}^s(\xi) = L^s(\xi, \xi'; \operatorname{id}_{\operatorname{bs}(\xi)})$, where $\xi' = (\operatorname{bs}(\xi) \times \mathbb{R}, \operatorname{proj}_1, \operatorname{bs}(\xi))$; see Remark 4.6.3.7. □

Theorem 4.6.3.10. *Let ξ and $\xi^{[\prime]}$ be n-dimensional real (complex) vector $\mathcal{C}^{\geq r}$-bundles with common compact base. If $0 \leq s < r \leq \infty$, then the set of $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{R}^n$-equivalences (respectively, $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{C}^n$-equivalences) $\xi' \to \xi$ is dense in the subset of $L^s(\xi, \xi'; \mathrm{id})$ consisting of all $\mathcal{C}^s$-$\mathrm{GL}\,\mathbb{R}^n$-equivalences (respectively, $\mathcal{C}^s$-$\mathrm{GL}\,\mathbb{C}^n$-equivalences) .*

Proof. Since the last subset is open in $L^s(\xi, \xi'; \mathrm{id})$, Theorem 4.6.3.10 is a consequence of Theorem 4.6.3.8. □

Corollary 4.6.3.11. *If two n-dimensional real (complex) vector$\mathcal{C}^{\geq r}$-bundles with compact base are $\mathrm{GL}\,\mathbb{R}^n$-equivalent (respectively, $\mathrm{GL}\,\mathbb{C}^n$–equivalent) and $r \neq a$, then they are $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{R}^n$-equivalent (respectively, $\mathcal{C}^r$-$\mathrm{GL}\,\mathbb{C}^n$-equivalent) .*

Theorem 4.6.3.12. *If the base of the n-dimensional real (complex) vector bundle ξ is a compact $\mathcal{C}^{\geq r}$-manifold with $1 \leq r \leq \infty$, then ξ is $\mathrm{GL}\,\mathbb{R}^n$-equivalent ($\mathrm{GL}\,\mathbb{C}\mathbb{R}^n$-equivalent) to a real (complex) vector $\mathcal{C}^{\geq r}$-bundle. If the base of the n-dimensional real (complex) vector $\mathcal{C}^s$-bundle ξ is a compact $\mathcal{C}^{\geq r}$-manifold, where $1 \leq r \leq \infty$, then ξ is $\mathcal{C}^s$-$\mathrm{GL}\,\mathbb{R}^n$-equivalent ($\mathcal{C}^s$-$\mathrm{GL}\,\mathbb{C}^n$-equivalent) to a real (complex) vector $\mathcal{C}^{\geq r}$-bundle.*

Proof. We shall prove again only the real case. Let ξ be an n-dimensional real vector bundle with $\mathrm{bs}(\xi)$ a compact $\mathcal{C}^{\geq r}$-manifold. By Corollary 4.5.3.9 and Theorem 3.5.2.13, ξ is $\mathrm{GL}\,\mathbb{R}^n$-equivalent to $f^! \,\mathrm{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$, where m is large enough and f is some continuous map $\mathrm{bs}(\xi) \to G(m, n)$. If ξ is an n-dimensional real vector $\mathcal{C}^s$-bundle such that $\mathrm{bs}(\xi)$ is a compact $\mathcal{C}^{\geq r}$-manifold, then by Theorem 4.6.3.5 ξ is $\mathcal{C}^s$-$\mathrm{GL}\,\mathbb{R}^n$-equivalent to $f^! \,\mathrm{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$, where m is large enough and f is some $\mathcal{C}^s$-map $\mathrm{bs})\xi) \to G(m, n)$. In both cases f is homotopic to a $\mathcal{C}^{\geq r}$-map $g \colon \mathrm{bs}(\xi) \to G(m, n)$ (see Theorems 3.4.6.5, 1.3.6.6, and 3.4.5.10), so that ξ is $\mathrm{GL}\,\mathbb{R}^n$-equivalent to $g^! \,\mathrm{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$ (see Theorem 4.4.1.5). This completes the proof of the first claim; as for the second, we need only add that, by Corollary 4.6.3.11, ξ is $\mathcal{C}^s$-$\mathrm{GL}\,\mathbb{R}^n$-equivalent to $g^! \,\mathrm{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$. □

Constructions

Remark 4.6.3.13. We conclude this subsection with a short review of the constructions described in §4.5.

By definition, a $\mathcal{C}^s$-subbundle of a (real or complex) vector $\mathcal{C}^r$-bundle ξ is a subbundle of ξ in the sense of Definition 4.5.2.2 or Remark 4.5.2.3, whose total space is a $\mathcal{C}^s$-submanifold of $\mathrm{tl}(\xi)$. A $\mathcal{C}^s$-subbundle is clearly a vector $\mathcal{C}^s$-bundle. The $\mathcal{C}^s$-subbundles of Euclidean or Hermitian $\mathcal{C}^r$-bundles are similarly defined. The $\mathcal{C}^r$-bundles of $\mathcal{C}^r$-bundles will be simply referred to as subbundles.

According to Definition 4.5.2.5, every subbundle η of a Euclidean or Hermitian bundle ξ has an orthogonal complement $\eta^\perp$, and it is clear that:

- $\eta^\perp$ is a $\mathcal{C}^s$-subbundle of ξ together with η;
- the canonical equivalence $\eta^\perp \to \xi/\eta$ (see Definition **??**) turns ξ/η into a Euclidean or Hermitian $\mathcal{C}^s$-bundle (and thus becomes a $\mathcal{C}^s$-equivalence).

We see that in the (real or complex) vector $\mathcal{C}^s$-case, ξ/η becomes a vector $\mathcal{C}^s$-bundle by introducing on ξ a (Euclidean or Hermitian) $\mathcal{C}^s$-metric. Recall, however, that we have established the existence of such a metric only under the assumptions that the base is compact and $s \neq a$ (see Theorem 4.6.3.6).

Let ξ_1 and ξ_2 be real vector $\mathcal{C}^r$-bundles with a common boundary-less base. Then the construction of $\xi_1 \oplus \xi_2$ (see Definition 4.5.2.9) shows that this sum is again a real vector $\mathcal{C}^r$-bundle. The difficulty occurring when the base has a boundary (i.e., the fact that the product $\xi_1 \times \xi_2$ is not defined as a $\mathcal{C}^s$-bundle)can be circumvented with the aid of the formulae

$$\mathrm{tl}(\xi_1 \oplus \xi_2) = \mathrm{tl}((\mathrm{proj}(\xi_1))^! \xi_2), \quad \mathrm{proj}(\xi_1 \oplus \xi_2) = (\mathrm{proj}(\xi_2)) \circ \mathrm{proj}((\mathrm{proj}(\xi_1))^! \xi_2).$$

If the conditions in Definition 4.5.2.9 are satisfied, then these formulas are equivalent to the definition of and this remains valid under our present circumstances, provided that the base has no boundary; the same formulas are now taken as the definition of the sum when the boundary is present. One can repeat the argument for complex vector, Euclidean, and Hermitian $\mathcal{C}^r$-bundles. In particular, we can define the suspension (see Definition 4.5.1.10) of a $\mathcal{C}^r$-bundle.

The $\mathcal{C}^r$-variants of the other constructions described in §4.5 and their mutual relations are already evident. In particular, the conjugate of a complex vector (Hermitian) $\mathcal{C}^r$-bundle is a complex (respectively, Hermitian) $\mathcal{C}^r$-bundle; the realification (see Definition 4.5.1.12) of a complex vector (Hermitian) $\mathcal{C}^r$-bundle is a real vector (respectively, Euclidean) $\mathcal{C}^r$-bundle; the complexification (see Definition 4.5.2.11) of a real vector (Euclidean) $\mathcal{C}^r$-bundle is a complex vector (respectively, Hermitian) $\mathcal{C}^r$-bundle; and in the $\mathcal{C}^r$-versions of Theorems 4.5.2.13 and 4.5.2.13, the equivalences conj and K become $\mathcal{C}^r$-equivalences.

4.6.4 Tangent and Normal Bundles

Remark 4.6.4.1. The basic notions of tangent and normal bundles have actually already been introduced and used in Chapter 3. However, only now,that we have acquired the idea of a smooth vector bundle, can we present the full-fledged definitions of tangent and normal bundles and give them the general, correct treatment that they deserve.

Tangent Bundles

Definition 4.6.4.2. Recall that in Chapter 3 we defined, for an arbitrarily given $\mathcal{C}^r$-manifold with $r \geq 1$, the real vector spaces $\mathrm{Tang}_x X$ ($x \in X$), the $\mathcal{C}^{r-1}$-manifold $\mathrm{Tang}\, X$, and the projection $\mathrm{proj}\colon \mathrm{Tang}\, X \to X$ (see Remarks 3.1.4.1 and 3.1.4.2). Comparing these objects with the general definitions given in Definitions 4.5.1.2 and 4.6.3.2, we readily see that $(\mathrm{Tang}\, X, \mathrm{proj}, X)$ is a real vector bundle of dimension $\dim X$ and, for $r \geq 2$, a real vector $\mathcal{C}^{r-1}$-bundle of dimension $\dim X$, called the *tangent bundle of the manifold* X, and is denoted by $\mathrm{tang}\, X$.

Similarly, confronting the definition of the differential $df\colon \mathrm{Tang}\, X \to \mathrm{Tang}\, Y$ of a $\mathcal{C}^{r-1}$-map $f\colon X \to Y$ (see Remark 3.1.4.5) with the general definitions given in Definitions 4.5.1.14 and 4.6.3.2, we conclude that (df, f) is a linear $\mathcal{C}^{r-1}$-map $\operatorname{tang} X \to \operatorname{tang} Y$. If f is a $\mathcal{C}^{r-1}$-diffeomorphism then (df, f) is a linear $\mathcal{C}^{r-1}$-isomorphism.

Remark 4.6.4.3. The notion of vector field has been defined twice: once for smooth manifolds (see Definition 3.1.4.7), and once for vector, Euclidean, and Hermitian bundles (see Definition 4.5.1.15). Now it is plain that the second definition generalises the first one: a vector field on a smooth manifold X is simply a vector field in its tangent bundle $\operatorname{Tang} X$.

In particular, the parallelisability ($\mathcal{C}^s$-parallelisability) of an n-dimensional smooth manifold X is equivalent to the $\operatorname{GL}\mathbb{R}^n$-triviality (respectively, $\mathcal{C}^s$-$\operatorname{GL}\mathbb{R}$-triviality) of the bundle $\operatorname{tang} X$. Comparing this with Corollary 4.6.3.11, we see that a parallelisable compact $\mathcal{C}^r$-manifold with $r \leq \infty$ is $\mathcal{C}^{r-1}$ -parallelisable.

Definition 4.6.4.4. A smooth manifold is *stably parallelisable* if its tangent bundle is stably trivial. The discussion in Remark 4.5.4.9 and Theorem 3.5.2.13 show that if a compact manifold is stably parallelisable, then the stabilisation occurs already at the first step, i.e., the bundle $\operatorname{sus}\operatorname{tang} X$ is $\operatorname{GL}\mathbb{R}^{n+1}$-trivial for any stably parallelisable n-dimensional manifold X.

Remark 4.6.4.5. Recall that, given a point x of the smooth manifold X, each chart $\varphi \in \operatorname{Atl}_x X$ defines a φ-basis for the tangent space $\operatorname{Tang} X$, and the matrix of the transformation from the φ)-basis to the ψ-basis is just the Jacobi matrix of the map $\operatorname{loc}(\varphi, \psi)\operatorname{id}$, computed at $\varphi(x)$. From this it follows that the values of any orientation of X (assumed on the charts of $\operatorname{Catl} X$) are correctly transferred to the bases of the spaces $\operatorname{Tang}_x X$, and in this manner an orientation is defined on the bundle $\operatorname{tang} X$. This procedure is clearly reversible, and hence there is a one-to-one correspondence between the orientations of the smooth manifold X and the orientations of its tangent bundle $\operatorname{tang} X$. In particular, X is orientable if and only if $\operatorname{tang} X$ is orientable, and $\operatorname{tang} X$ is a $\operatorname{GL}_+\mathbb{R}^n$-bundle for any oriented smooth n-dimensional manifold X.

We already know that every parallelisable manifold is orientable (see Theorem 3.1.4.8 and Remark 4.6.4.3). Now we can add that every stably parallelisable manifold is also orientable.

Definition 4.6.4.6. One attractive feature of tangent bundles is that they can naturally be induced from Grassmann bundles. Namely, if $j\colon X \to \mathbb{R}^q$ is a $\mathcal{C}^r$-immersion (for example, a $\mathcal{C}^r$-embedding), then dj maps each tangent space $\operatorname{Tang}_x X$ onto an n-dimensional plane of passing through 0. Thus a map $t\colon X \to G(q, n)$ is defined, and it is clear that $\operatorname{tang} X$ is nothing else but $t^! \operatorname{Grass}(q, \operatorname{GL}(n, \mathbb{R}))$ (up to a correcting $\mathcal{C}^{r-1}$-$\operatorname{GL}\mathbb{R}^n$-equivalence) . [In fact, this obvious observation is older that the theory of bundles and was one of the factors which stimulated its creation.] If X is orientable, then one can generalise this observation and replace $G(q, n)$, the standard fibre $\operatorname{GL}\mathbb{R}^n$, and the bundle $\operatorname{Grass}(q, \operatorname{GL}(n, \mathbb{R}))$, by $G(q, n)$, $GL_+\mathbb{R}^n$, and $\operatorname{Grass}(q, \operatorname{GL}_+(n, \mathbb{R}))$, respectively. In all cases t is known as a *tangential map.* If j is an embedding and no orientation is involved, then t coincides with the composition of the normal transversalisation $X \to G(q, q-n)$ of j with the canonical diffeomorphism $G(q, q-n) \to G(q, n)$ (see Remark 3.2.2.3).

Definition 4.6.4.7. A smooth manifold whose tangent bundle is equipped with a Euclidean metric is called a *Riemannian manifold.* In this case the metric is usually termed a *Riemannian metric.* If X is of class $\mathcal{C}^r$, then the metric can be at most of class $\mathcal{C}^{r-1}$. In fact, Theorem 4.6.3.6 shows that there is a Riemannian $\mathcal{C}^{r-1}$-metric on every compact $\mathcal{C}^r$-manifold with $1 \leq r \leq \infty$. Incidentally, the same result may be extracted from Theorem 3.4.2.1. It is customary to look upon the tangent bundle of a Riemannian manifold as a Euclidean bundle. If the metric is of class $\mathcal{C}^s$, $s \geq 1$, then this bundle is a Euclidean $\mathcal{C}^s$-bundle.

Normal Bundles

Definition 4.6.4.8. The initial data involved in the concept of *normal bundle* are two smooth manifolds, X and X', and an immersion $j\colon X \to X'$. The most important case is that of an embedding j. In the definitions below $n = \dim X$ and $n' = \dim X'$.

Let us examine first the simplest case: X is a submanifold of the Riemannian manifold X', and j is the inclusion $X \to X'$. One may naturally view $\operatorname{tang} X$ as a subbundle of $\operatorname{tang} X'|_X$. By Definition 4.5.2.5, has in $\operatorname{tang} X'|_X$ an orthogonal complement, $(\operatorname{tang} X)^\perp$. This is an $(n'-n)$-dimensional Euclidean bundle with base X, called the *normal bundle* of X, and denoted $\operatorname{norm} X$. If X and X' are of class $\mathcal{C}^{\geq r}$ with $r \geq 2$, and the Riemannian metric is of class $\mathcal{C}^{r-1}$, then $\operatorname{norm} X$ is a Euclidean $\mathcal{C}^{\geq r-1}$-bundle; see Remark 4.6.3.13. In all cases, the sum $\operatorname{tang} X \oplus \operatorname{norm} X$ is canonically $\mathrm{O}\,\mathbb{R}^{n'}$-equivalent to $\operatorname{tang} X'|_X$; when X and X' are of class $\mathcal{C}^{\geq r}$ with $r \geq 2$, this equivalence is of class $\mathcal{C}^{r-1}$.

To define the normal bundle for an arbitrary immersion $j\colon X \to X'$ and without recourse to a Riemannian metric, we have to replace the restriction $\operatorname{tang} X'|_X$ by the induced bundle $j^! \operatorname{tang} X'$ and then pass to a quotient instead of taking an orthogonal complement. More exactly, the normal bundle, $\operatorname{norm} j$, of the immersion $j\colon X \to X'$, with X and X' smooth manifolds, is defined as

$$\operatorname{norm} j = j^! \operatorname{tang} X' / \operatorname{im} \operatorname{corr}(dj, j).$$

This formula defines an $(n'-n)$-dimensional real vector bundle over X. The sum $\operatorname{tang} X \oplus \operatorname{norm} X$ is canonically $\mathrm{GL}\,\mathbb{R}^n$-equivalent to the induced bundle $j^! \operatorname{tang} X'$. Unfortunately, there is less to say about the differentiability class of $\operatorname{norm} j$ and of the canonical equivalence $j^! \operatorname{tang} X' \to \operatorname{tang} X \oplus \operatorname{norm} j$: if X' is either compact or diffeomorphic to an open subset of a compact manifold, and j is $\mathcal{C}^r$ with $2 \leq r \leq r$, then $\operatorname{norm} j$ is a real vector $\mathcal{C}^{\geq r-1}$-bundle, while the above canonical equivalence is $\mathcal{C}^{r-1}$. These limitations are obviously due to the fact that we have had no theorems which guarantee the existence of a Riemannian metric in the non-compact and analytic situations.

If j is an inclusion, we may simply write $\operatorname{norm} X$ instead of $\operatorname{norm} j$.

Notice that, as the definition of the normal bundle $\operatorname{norm} j$ shows, the orientability of two of the three bundles $j^! \operatorname{tang} X'$, $\operatorname{tang} X$, and $\operatorname{norm} j$ implies the orientability of the third, and given orientations on two of them canonically define the orientation of the third; see Remark 4.5.2.7. Comparing this with the discussion in Remark 4.6.4.5, we see that if X' is orientable, then the orientability of $\operatorname{norm} j$ is equivalent to the orientability of the manifold X; moreover, if X' is oriented, then the orientations of X' and of $\operatorname{norm} j$ canonically determine each other.

Remark 4.6.4.9. The introduction of tangent and normal bundles allows us to better formulate and essentially complete the main result of Subsection 3.4.8, i.e., Theorem 3.4.8.2.

We may sharpen the formulation of the third part of this theorem, which asserts that the linear homomorphisms $\operatorname{fact} d_{x_1} f_1$ are actually isomorphisms. Now we can say that the maps $\operatorname{fact} d_{x_1} f_1$ combine with $\operatorname{abr} f_1 \colon X_{12} \to X_2$ to define a linear isomorphism from the normal bundle of the manifold X_{12} (taken in X_1) onto the normal bundle of the manifold X_2 (taken in X'). When X' is compact and $2 \le r \le \infty$, this is a linear $\mathcal{C}^{r-1}$-isomorphism $\operatorname{norm} X_{12} \to \operatorname{norm} X_2$.

To complete Theorem 3.4.8.2, we consider orientations. Namely, suppose that X_1, X', and X_2 are orientable (oriented). Then, as an immediate consequence of Remark 4.6.4.5 and Definition 4.6.4.8, the manifold, X_{12} is orientable (respectively, canonically oriented). In particular, under the assumptions of Corollary 3.4.8.4, the orientability of X', X_1, and X_2 implies the orientability of $X_1 \cap X_2$, and given orientations of X', X_1, and X_2 canonically orient $X_1 \cap X_2$.

Theorem 4.6.4.10. *An n-dimensional smooth compact manifold is stably parallelisable if and only if it admits a differentiable embedding in some $\mathbb{R}^q$ having a $\operatorname{GL}\mathbb{R}^{q-n}$-trivial normal bundle.*

Proof. Let $j \colon X \to \mathbb{R}^q$ be an embedding enjoying the above property. Then $\operatorname{tang} X \oplus \operatorname{norm} j$ is $\operatorname{GL}\mathbb{R}$-equivalent to $j^! \operatorname{tang} \mathbb{R}^q$, and hence the condition is sufficient. To prove its necessity, consider an arbitrary differentiable embedding $j \colon X \to \mathbb{R}^q$ and the composite embedding $X \xrightarrow{j} \mathbb{R}^q \xrightarrow{\text{incl}} \mathbb{R}^{q+n+k}$, where k is large enough so that the suspension $\operatorname{sus} \operatorname{tang} X$ is $\operatorname{GL}\mathbb{R}$-trivial (actually, it suffices to take $k = 1$; see Definition 4.6.4.4). The normal bundle of this composite embedding is just $\operatorname{sus}^{n+k} \operatorname{norm} j$, and it is $\operatorname{GL}\mathbb{R}^{q+k}$-trivial, being $\operatorname{GL}\mathbb{R}^{q+k}$-equivalent to the bundle

$$\operatorname{norm} j \oplus \operatorname{sus} \operatorname{tang} X = \operatorname{sus}^k(\operatorname{norm} j \oplus \operatorname{tang} X) = \operatorname{sus}^k(j^! \operatorname{tang} \mathbb{R}^q).$$

□

The Complex Case

Definition 4.6.4.11. The basic definitions of this subsection, i.e., those the tangent bundle, $\operatorname{tang} X$, Riemannian metric, and normal bundles, $\operatorname{norm} X$ and *normj*, carry over,word-for-word, to complex manifolds. The bundles become complex bundles, the Riemannian metric is replaced by a Hermitian one, while

j is assumed to be a holomorphic immersion (i.e., to be locally a holomorphic embedding). If $f\colon X \to X'$ is holomorphic, then (df, f) is a linear map $\operatorname{tang} X \to \operatorname{tang} X'$. The definition and properties of tangential maps are preserved, but they become less universal (see Theorem 3.1.6.10). Finally, the realification of a complex manifold (see Remark 3.1.6.9) leads to the realification of its tangent bundle (see Definition 4.5.1.12 and Remark 4.5.1.13), and turns Hermitian metrics into Riemannian ones.

It is impossible not to notice the eclectic character of these definitions. The reason for this inconsistency is that, whereas the notion of differentiable structure, which lies at the heart of the theory of smooth vector bundles, is specifically a real notion even when we pass to complex vector bundles, in the complex case the tangent and normal bundles carry an additional, complex-differentiable structure - the so-called *holomorphic structure*. Unfortunately, the theory of holomorphic vector bundles is beyond the scope of this book.

4.6.5 Degree

Definition 4.6.5.1. In this subsection we shall apply some of the simplest results of differential topology to homotopy theory. Namely, given any oriented, compact, smooth manifold X, any oriented, compact, connected, smooth manifold Y with $\dim Y = \dim X$, and any continuous map $f\colon (X, \partial X) \to (Y, \partial Y)$, we define an integer which depends only upon the homotopy class of f. This number is called the *degree* of the map f and is denoted by $\deg f$.

Although the degree $\deg f$ is a global characteristic of f, and is actually defined for maps which are merely continuous, we shall approach this notion by infinitesimal methods: we start by assuming that

$$f \in \mathcal{C}(X, \partial X; Y, \partial Y) \cap \mathcal{C}^1(X, Y)$$

and choose a point $y \in \operatorname{int} Y$ such that f is transverse to y. Consider $f^{-1}(y)$. It consists of a finite number of points, each of them having a neighbourhood which is mapped diffeomorphically by f onto a neighbourhood of y (see Theorems 3.4.8.2 and 3.1.5.5), and each of these diffeomorphisms is either orientation preserving or orientation reversing, where the neighbourhoods are oriented in agreement with the orientations of X and Y (see Remark 4.6.4.9). The *degree of the map f at the point y*, denoted $\deg_y f$, is the number of the points of $f(y)$ where the orientation is preserved, minus the number of points of $f^{-1}(y)$ where the orientation is reversed. A popular shorter version of this definition is: $\deg_y f$ is the *algebraic number of the preimages of the point y*.

In this definition of $\deg_y f$, the assumption that $y \in \operatorname{int} Y$ is essential. However, one may repeat the definition for a boundary point y, provided that

$$f \in \mathcal{C}^1_\partial(X, Y) \cap \mathcal{C}(X, \operatorname{int} X; Y, \operatorname{int} Y)$$

and $\operatorname{abr} f\colon \partial X \to \partial Y$ is transverse to y. The degree $\deg_y f$ thus defined is obviously the same as $\deg_y[\operatorname{abr} f\colon f^{-1}(Z) \to Z]$, where Z is the component of ∂Y containing y.

We add that if the degree is defined for a point y, i.e., either

- $y \in \operatorname{int} Y$, $f \in \mathcal{C}(X, \partial X; Y, \partial Y) \cap \mathcal{C}^1(X, Y)$, and f is transverse to y, or
- $y \in \partial Y$, $f \in \mathcal{C}^1_\partial(X, Y) \cap \mathcal{C}(X, \operatorname{int} X; Y, \operatorname{int} Y)$, and $\operatorname{abr} f \colon \partial X \to \partial Y$ is transverse to y,

then it is defined for any point y' in a neighbourhood of y (in Y), and $\deg_{y'} f = \deg_y f$.

Lemma 4.6.5.2. *Let X and Y be oriented, compact, $\mathcal{C}^\infty$-manifolds, with* $\dim X = \dim Y$ *and Y connected, and let*

$$g, h \in \mathcal{C}^\infty_\partial(X, Y) \cap \mathcal{C}(X, \operatorname{int} X; Y, \operatorname{int} Y).$$

Further, let $y, z \in Y$ be such that $\deg_y g$ *and* $\deg_z h$ *are defined (see Definition 4.6.5.1). If the maps*

$$\operatorname{rel} g, \operatorname{rel} h \colon (X, \partial X) \to (Y, \partial Y)$$

are homotopic, then $\deg_z h = \deg_y g$.

Proof. We disregard the trivial case $\dim Y = 0$ and assume for a start that $h = g$. Using Definition 4.6.5.1, we find neighbourhoods U and V of y and z, such that

$$\deg_{y'} g = \deg_y g, \forall y' \in U, \quad \deg_{z'} h = \deg_z h, \forall z' \in V.$$

It is clear that one can join y to z by a path which is a $\mathcal{C}^\infty$-embedding $I \to Y$. By Theorems 3.4.1.4 and 3.4.7.7, there is a path $s \colon I \to \operatorname{int} Y$ such that $s(0) \in U$, $s(1) \in V$, and s is a $\mathcal{C}^\infty$-embedding transverse to g. The preimage $g^{-1}(s(I))$ is an oriented, compact, one-dimensional $\mathcal{C}^\infty$-submanifold of X, and

$$\partial g^{-1}(s(I)) = g^{-1}(s(0)) \cup g^{-1}(s(1))$$

(see Theorem 3.4.8.2 and Remark 4.6.4.9). Obviously,

$$\deg_{s(0)} g = \deg_{s(0)} \operatorname{abr} g, \quad \deg_{s(1)} g = \deg_{s(1)} \operatorname{abr} g,$$

where $\operatorname{abr} g = [\operatorname{abr} g \colon g^{-1}(s(I)) \to s(I)]$. But the contribution of a point from $g^{-1}(s(1))$ to $\deg_{s(1)} \operatorname{abr} g$ equals the value of the orientation which this point inherits a s a component of $\partial g^{-1}(s(I))$. Similarly, the contribution of a point from $g^{-1}(s(0))$ to $\deg_{s(0)} \operatorname{abr} g$ is opposite to the value of the orientation which this point inherits as a component of $\partial g^{-1}(s(I))$. Consequently, $\deg_{s(1)} g - \deg_{s(0)} g$ is the number of points of $\partial g^{-1}(s(I))$ which inherit the orientation $+1$ from $g^{-1}(s(I))$, minus the number of points of $\partial g(s(I))$ which inherit the orientation -1 from $g^{-1}(s(I))$. However, this difference must be 0, because every component of $g^{-1}(s(I))$ is diffeomorphic to either $\mathbb{S}^1$ or $\mathbb{D}^1$ (see Remark 3.5.3.1).

Now drop the assumption that $h = g$, and suppose that Y is closed. Then X is also closed, and from the fact that g and h are homotopic it follows that there exists a $\mathcal{C}^\infty$-homotopy F from g to h (see Corollary 3.4.6.6). The map $\Phi : X \times I \to Y \times I$, $\Phi(x,t) = (F(x,t),t)$, obviously belongs to the intersection

$$\mathcal{C}^\infty_\partial(X \times I, Y \times I) \cap \mathcal{C}(X \times I, \mathrm{int}(X \times I); Y \times I, \mathrm{int}(Y \times I)).$$

Furthermore, if we identify $X \times 0$ and $X \times 1$ with X, and $Y \times 0$ and $Y \times 1$ with Y, then $\mathrm{abr}\,\Phi\colon X \times 0 \to Y \times 0$ and $\mathrm{abr}\,\Phi\colon X \times 1 \to Y \times 1$ become g and h, and we can write $\deg_{(y,0)} \Phi = \deg_y g$ and $\deg_{(z,1)} \Phi = \deg_z h$. By the argument above, $\deg_{(z,1)} \Phi = \deg_{(y,0)} \Phi$ and thus $\deg_z h = \deg_y g$.

Finally, if $\partial Y \neq \varnothing$, we find a component Z of ∂Y and points $y', z' \in Z$ such that $\mathrm{abr}\, g\colon \partial X \to \partial Y$ is transverse to y', while $\mathrm{abr}\, h\colon \partial X \to \partial Y$ is transverse to z'. We already know that

$$\deg_y g = \deg_{y'} g, \quad \deg_z h = \deg_{z'} h,$$

while, according to Definition 4.6.5.1,

$$\begin{aligned} \deg_{y'} g &= \deg_{y'}[\mathrm{abr}\, g\colon g^{-1}(Z) \to Z] \\ \deg_{z'} h &= \deg_{z'}[\mathrm{abr}\, h\colon h^{-1}(Z) \to Z]. \end{aligned}$$

But

$$\deg_{y'}[\mathrm{abr}\, g\colon g^{-1}(Z) \to Z] = \deg_{z'}[\mathrm{abr}\, h\colon h^{-1}(Z) \to Z]. \tag{4.6.5.3}$$

Indeed, since $\mathrm{rel}\, g$ and $\mathrm{rel}\, h$ are homotopic, $h^{-1}(Z) = g^{-1}(Z)$, and thus (4.6.5.3) follows from that part of the lemma which we have already proved. □

Lemma 4.6.5.4. *For any compact $\mathcal{C}^\infty$-manifolds X and Y, the set*

$$\mathcal{C}^\infty_\partial(X, Y) \cap \mathcal{C}(X, \mathrm{int}\, X; Y, \mathrm{int}\, Y)$$

is dense in $\mathcal{C}(X, \partial X; Y, \partial Y)$.

Proof. Construct the doubles, $\mathrm{dopp}\, X$ and $\mathrm{dopp}\, Y$, together with two-sided collarings

$$k\colon \partial X \times \mathbb{D}^1 \to \mathrm{dopp}\, X, \quad \ell\colon \partial Y \times \mathbb{D}^1 \to \mathrm{dopp}\, Y$$

Pick a $\mathcal{C}^\infty$-embedding $j\colon \mathrm{dopp}\, Y \to \mathbb{R}^q$, a $\mathcal{C}^\infty$-transversalisation τ of $j|_{\partial Y}$, and a neat tube $\mathrm{Tub}_\tau\, \rho$. Then it suffices, given a map $f \in \mathcal{C}(X, \partial X; Y, \partial Y)$ and $\varepsilon > 0$, to find

$$\mathcal{C}^\infty_\partial(X, Y) \cap \mathcal{C}(X, \mathrm{int}\, X; Y, \mathrm{int}\, Y)$$

such that $\mathrm{dist}(j(f(x)), j(g(x))) < \varepsilon$ for all $x \in X$. To produce such a g, we shall construct successively three auxiliary maps, $h_1, h_2, h_3\colon \mathrm{dopp}\, X \to \mathrm{dopp}\, Y$.

The map h_1 is very simply defined by

$$h_1(x) = f(x), \quad h_1(\mathrm{cop}(x)) = \mathrm{cop}(f(x)) \quad [x \in X].$$

To construct h_2, fix δ, $0 < \delta < 1$, such that:

(i) $\operatorname{dist}(j \circ \ell(z,t), j \circ \ell(z,t')) < e/4$ for $|t - t'| < \delta$ where $z \in \partial Y,\ t, t' \in \mathbb{D}^1$;

(ii) $\operatorname{dist}(j(h_1(k(z,t))), j(h_1(k(z,t')))) < \varepsilon/4$ for $|t - t'| < \delta$ where $z \in \partial Y,\ t, t' \in \mathbb{D}^1$;

(iii) for any $z \in \partial Y$, the ball with centre $j(z)$ and radius δ lies in $\operatorname{Tub}_\tau \rho$, while its image under the map $j \circ \operatorname{proj}_\tau \colon \operatorname{Tub}_\tau \rho \to \mathbb{R}^q$ lies in the ball with centre $j(z)$ and radius $\varepsilon/4$.

[The existence of such a δ is a consequence of the continuity of j, k, ℓ, h_1, and proj_τ.] Further, pick a $\mathcal{C}^\infty$-map $\varpi \colon \partial X \to \partial Y$ such that

$$\operatorname{dist}(j(\varphi(z)), j \circ f(z)) < \delta \quad \forall z \in \partial X$$

(Theorem 3.4.4.2 guarantees that such a map φ exists), and define, for each $t \in I$, the map

$$\varphi_t \colon \partial X \to \partial Y, \quad \varphi_t(x) = \operatorname{proj}_\tau(tj(\varphi(x)) + (1-t)j \circ f(x)).$$

Next, pick a $\mathcal{C}^\infty$-map $\alpha \colon \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad |t| \le 1/3, \\ 1 & \text{for} \quad |t| \ge 2/3, \end{cases}$$

and define $k_1 \colon X \to X$ and $\ell_1 \colon Y \to Y$ by

$$\begin{cases} k_1(k(z,t)) = k(z, (1-\delta)t + \delta), & \text{if} \quad z \in \partial X,\ t \in I, \\ k_1(x) = x, & \text{if} \quad x \in X \setminus k(\partial X \times I), \end{cases}$$

and

$$\begin{cases} \ell_1(\ell(z,t)) = \ell(z, (1-\delta)t + \delta), & \text{if} \quad z \in \partial Y,\ t \in I, \\ \ell_1(y) = y, & \text{if} \quad y \in Y \setminus \ell(\partial Y \times I),. \end{cases}$$

It is clear that k_1 and ℓ_1 are topological embeddings. Now define h_2 by

$$\begin{cases} h_2(k(z,t)) = \ell(\varphi_{\alpha(\delta t)}(z), t), & \text{if} \quad z \in \partial X,\ |t| < \delta, \\ h_2(x) = \ell_1(f(k_1^{-1}(x))), & \text{if} \quad x \in X \setminus k(\partial X \times [0,\delta)), \\ h_2(\operatorname{cop}(x)) = \operatorname{cop}(\ell_1(f(k_1^{-1}(x))) & \text{if} \quad x \in X \setminus k(\partial X \times [0,\delta)). \end{cases}$$

Then the following facts are evident:

- $h_2 \in \mathcal{C}(\operatorname{dopp} X, \operatorname{int} X, \partial X : \operatorname{dopp} Y, \operatorname{int} Y, \partial Y)$;
- $\operatorname{dist}(j \circ h_1(x), j \circ h_2(x)) < \varepsilon/2$ for all $xin \operatorname{dopp} X$;
- the restriction of h_2 to $k(\partial X \times [-\delta/3, \delta/3])$ is of class $\mathcal{C}^\infty$;
- if $z \in \partial X$, then $\operatorname{im} d_z(h_2|_{k(partialX \times [-\delta/3,\delta/3])}) \not\subset \operatorname{Tang}_{h_2(z)} \partial Y$.

Finally, take h_3 to be any $\mathcal{C}^\infty$-map $\operatorname{dopp} X \to \operatorname{dopp} Y$ which equals h_2 on $k(\partial X \times [-\delta/6, \delta/6]$ and enjoys the properties:

$$\begin{aligned} &h_3(X \setminus k(\partial X \times [0, \delta/6))) \subset \operatorname{int} Y; \\ &h_3(\operatorname{cop}(X \setminus k(\partial X \times [0, \delta/6))) \subset \operatorname{cop}(\operatorname{int} Y); \\ &\operatorname{dist}(j \circ h_3(x), j \circ h_2(x)) < \varepsilon/2 \quad \forall x \in X. \end{aligned}$$

[Theorem 3.4.4.9 guarantees that such an $_3$ exists.]

Clearly, $h_3(X) \subset Y$. Now define the desired map g by $g = [\operatorname{abr} h_3 \colon X \to Y]$, and check directly that it has all the necessary properties. ☐

Definition 4.6.5.5. Let X and Y be (as in Definition 4.6.5.1) oriented, compact, manifolds with $\dim X = \dim Y$, and let $f \in \mathcal{C}(X, \partial X; Y, \partial Y)$. Restrict the differentiable structures of the manifolds X and Y to $\mathcal{C}^\infty$-structures (see Theorem3.4.9.8), and find a map $g \in \mathcal{C}^\infty_\partial(X, Y) \cap \mathcal{C}(X, \partial X; Y, \partial Y)$ which is close enough to f in the $\mathcal{C}^0$-topology, and such that the maps $f, g \colon X \to Y$, as well as the maps $\operatorname{abr} f, \operatorname{abr} g \colon \partial X \to \partial Y$ are homotopic (see Lemma 4.6.5.4, Theorems 3.4.5.10, and 1.3.6.6).

Now compute $\deg_y g$ at some point $y \in \operatorname{int} Y$ such that g is transverse to y. By Lemma 4.6.5.2, $\deg_y g$ does not depend upon the choice of g or y, while Remark 3.4.6.11 and Corollary 3.4.1.6 show that $deg_y g$ does not depend upon the modality of restricting the differentiable structures of X and Y to $\mathcal{C}^\infty$-structures (according to the aforementioned theorems, the $\mathcal{C}^\infty$-manifolds resulting from the restriction of the differentiable structure of a given compact, smooth manifold, effected in two distinct ways, are $\mathcal{C}^\infty$-diffeomorphic via a diffeomorphism which can be as $\mathcal{C}^0$-close to the identity diffeomorphism as we choose). We call $\deg_y g$ the *degree of the map* f, denoted $\deg f$.

The main properties of the degree are immediate consequences of its definition and of Lemmas 4.6.5.2 and 4.6.5.4. We list here some of them:

1. if $f, f' \colon (X, \partial X) \to (Y, \partial Y)$ are homotopic, then $\deg f = \deg f'$ (by Lemma 4.6.5.2);
2. the degree of the composite map $(X, \partial X) \xrightarrow{f} (Y, \partial Y) \xrightarrow{g} (Z, \partial Z)$ is $\deg f \cdot \deg g$ (by the definition of deg);
3. the degree of the identity map is 1 (trivial);
4. if $f \colon : (X, \partial X) \to (Y, \partial Y)$ is a homotopy equivalence, then $\deg f = 1$ (proof: if g is a homotopy inverse of f, then $\deg f \cdot \deg g = \deg(g \circ f) = \deg \operatorname{id}_{(X, \partial X) = 1})$;
5. if $f \colon (X, \partial X) \to (Y, \partial Y)$ is such that $f(X) \neq Y$, then $\deg f = 0$ (indeed, one can approximate f as closely as desired by a map from
$$\mathcal{C}^\infty_\partial(X, Y) \cap \mathcal{C}(X, \operatorname{int} X; Y, \operatorname{int} Y)$$
enjoying the same properties);

6. if Z is any component of Y, then

$$\deg[f\colon (X,\partial X) \to (Y,\partial Y)] = \deg[\operatorname{abr} f\colon f^{-1}(Z) \to Z]$$

(a result of the discussion in Definition 4.6.5.1) and, in particular, $\deg f = 0$ whenever X is closed but Y is not.

As examples, consider the maps

$$f\colon (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$$

and $\operatorname{abr} f\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, defined by an orthogonal $(n \times n)$-matrix V $(n \geq 2)$. Obviously, $\deg f = \deg \operatorname{abr} f = \det V$, i.e.,

$$\deg f = \deg \operatorname{abr} f = \begin{cases} 1 & \text{if} \quad V \in \mathrm{SO}(n), \\ -1 & \text{if} \quad V \in \mathrm{O}(n) \setminus \mathrm{SO}(n). \end{cases}$$

Thus, the degree of the antipodal map $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, $x \mapsto -x$, equals 1 if n is even and -1 if n is odd.

The Non-oriented Case

Definition 4.6.5.6. The discussion in Definition 4.6.5.1, Lemmas 4.6.5.2, and 4.6.5.4 can be carried over to non-oriented manifolds if we replace integers by integers modulo 2. This enables us to define $\deg f \in \mathbb{Z}/2\mathbb{Z}$ for any continuous map $f\colon (X,\partial X) \to (Y,\partial Y)$, where X and Y are smooth, compact manifolds, and Y is connected (no orientability needed). All the properties of the integral degree listed in Definition 4.6.5.5 are preserved. For the case of oriented manifolds, when both degrees (the integral and mod 2) are defined, we continue to use the same notation for both, because misunderstandings are usually eliminated by the context.

Applications

Theorem 4.6.5.7. *Smooth closed manifolds of positive dimension are not contractible.*

Proof. This is plain if the given manifold is not connected. In the connected case, the degree of the identity map of a closed manifold is 1, whereas the degree of any map which takes the whole manifold into one of its points is zero (here we use the -degree defined in Definition 4.6.5.6). ☐

Theorem 4.6.5.8. *If $n \neq m$, then $\mathbb{S}^n$ and $\mathbb{S}^m$ are not homotopy equivalent.*

Proof. Indeed, if $m < n$, then every continuous map $\mathbb{S}^m \to \mathbb{S}^n$ is homotopic to a constant map (see Corollary 2.3.2.5 and Theorem 2.3.1.6), whereas $\mathrm{id}\colon \mathbb{S}^n \to \mathbb{S}^n$ is not homotopic to a constant map (see Theorem 4.6.5.7). ☐

Theorem 4.6.5.9. *The boundary of a non-empty, compact, smooth manifold is not a retract of the manifold.*

Proof. It suffices to assume that the given manifold X is compact, smooth, connected, and with $\partial X \neq \varnothing$. Let $\rho\colon X \to \partial X$ be a retraction, and let Z be any component of ∂X. Consider the composite map $X \xrightarrow{\rho} X \xrightarrow{\text{incl}} X$. Since its image is not all of X, its degree is 0 (see Definitions 4.6.5.5 and 4.6.5.6). On the other hand, this degree equals the degree of $\operatorname{abr}(\text{incl} \circ \rho)\colon Z \to Z$, which is 1, the last map being $\operatorname{id} Z$. ☐

Theorem 4.6.5.10. *Every continuous map $\mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.*

Proof. Suppose that $f\colon \mathbb{D}^n \to \mathbb{D}^n$ is continuous and has no fixed points. Then the map $\mathbb{D}^n \to \mathbb{S}^{n-1}$ taking each point $x \in \mathbb{D}^n$ into its projection on $\mathbb{S}^{n-1}$ from the point $f(x)$ is a retraction, and hence $\mathbb{S}^{n-1}$ is a retract of $\mathbb{D}^n$: contradiction (see Theorem 4.6.5.9). ☐

Theorem 4.6.5.11. *If an m-dimensional locally Euclidean space is homeomorphic to an n-dimensional locally Euclidean space, then $n = m$. (Cf. Remark 3.1.1.4).*

Proof. Every point of $\mathbb{R}^q$ can be covered (in $\mathbb{R}^q$) by a Euclidean q-simplex. Therefore, every point of a q-dimensional locally Euclidean space lies in the interior of a finitely-triangulated subset, and its link in this subset is homeomorphic to $\mathbb{S}^{q-1}$. By Theorem 2.2.6.4, this link is a homotopy invariant, and Theorem 4.6.5.8 shows that the spheres $\mathbb{S}^n$ and $\mathbb{S}^m$ cannot have the same homotopy type unless $m = n$. ☐

Remark 4.6.5.12. Theorem 4.6.5.11 clarifies not only the definition of a locally Euclidean space, but also that of a cellular space. Namely, it shows that the dimension of a cell is uniquely determined by this cell.

Therefore, the dimension function which we introduced into the definition of the cellular decomposition as an additional element of its structure, is actually redundant, being completely determined by the decomposition itself.

Theorem 4.6.5.13. *The boundary of the half space $\mathbb{R}^n_-$ is $\mathbb{R}^{n+1}_1$.*
(Cf. Remark 3.1.1.4.)

Proof. It suffices to show that the point 0 has in $\mathbb{R}^n_-$ no neighbourhood homeomorphic to $\mathbb{R}^n$; see Remark 3.1.1.4.

Assume that such a neighbourhood exists. Then 0 is an interior point of a finitely-triangulated subset of this neighbourhood, where its link is homeomorphic to $\mathbb{S}^{n-1}$ (cf. the proof of Theorem 4.6.5.11). On the other hand, 0 is an interior point of a finitely-triangulated subset, where its link is homeomorphic to $\mathbb{D}^{n-1}$: take any Euclidean n-simplex which lies in $\mathbb{R}^n_-$ and contains 0 in the interior of one of its $(n-1)$-faces. Since $\mathbb{S}^{n-1}$ is not contractible, whereas $\mathbb{D}^{n-1}$ is, we contradict Theorem 2.2.6.4. ☐

4.6.6 Exercises

Exercise 4.6.6.1. Let $r \leq \infty$, and let $f\colon X \to Y$ be a $\mathcal{C}^r$-submersion, where X and Y are $\mathcal{C}^r$-manifolds, X compact and Y closed. Show that (X, f, Y) is a $\mathcal{C}^r$-bundle. (Combined with Theorem 4.6.1.3, this result shows that for $r \leq \infty$, (X, f, Y) is a $\mathcal{C}^r$-bundle whenever X is a compact $\mathcal{C}^r$-manifold and Y is a $\mathcal{C}^r$-manifold, while $f\colon X \to Y$ is a $\mathcal{C}^r$-submersion.)

Exercise 4.6.6.2. Let $1 \leq r \leq \infty$, and let ξ be a $\mathcal{C}^r$-bundle with closed base. Show that there is a collaring $k\colon \partial\,\mathrm{tl}(\xi) \times I \to \mathrm{tl}(\xi)$, such that $k(z \times I) \subset (\mathrm{proj}(\xi))^{-1}(\mathrm{proj}(\xi)(z))$ for every point $z \in \partial\,\mathrm{tl}(\xi)$.

Exercise 4.6.6.3. Let $1 \leq r \leq \infty$, and let ξ be a $\mathcal{C}^r$-bundle with

$$\partial\,\mathrm{tl}(\xi) = (\mathrm{proj}(\xi))^{-1}(\partial\,\mathrm{bs}(\xi)).$$

Show that there are collarings

$$k\colon \partial\,\mathrm{bs}(\xi) \times I \to \mathrm{bs}(\xi), \quad \ell\colon \partial\,\mathrm{tl}(\xi) \times I \to \mathrm{tl}(\xi)$$

such that the diagram

$$\begin{tikzcd}
\partial\,\mathrm{tl}(\xi) \times I \arrow[r, "\ell"] \arrow[d, "\mathrm{abr}\,\mathrm{proj}(\xi)\times\mathrm{id}_I"'] & \mathrm{tl}(\xi) \arrow[d, "\mathrm{proj}(\xi)"] \\
\partial\,\mathrm{bs}(\xi) \times I \arrow[r, "k"'] & \mathrm{bs}(\xi)
\end{tikzcd}$$

commutes.

Exercise 4.6.6.4. Show that if $r \leq \infty$, then for every $\mathcal{C}^{\geq r}$-bundle ξ with compact $\mathrm{bs}(\xi)$ and $\mathrm{tl}(\xi)$, $\mathrm{Sect}^r(\xi)$ is dense in $\mathrm{Sect}^s(\xi)$ for any $s < r$. (This generalises Theorem 4.6.2.7 for $r \neq a$.)

Exercise 4.6.6.5. Show that every $\mathcal{C}^{\geq r}$-bundle ξ with compact $\mathrm{bs}(\xi)$ and $\mathrm{tl}(\xi)$ is $\mathcal{C}^r$-isomorphic to a $\mathcal{C}^\infty$-bundle (cf. Theorem 4.6.2.8).

Exercise 4.6.6.6. Show that $\mathrm{sus}\,\mathrm{tang}\,\mathbb{R}P^n$ is $\mathcal{C}^a$-$\mathrm{GL}\,\mathbb{R}^{n+1}$-equivalent to the sum of $n+1$ copies of $\mathrm{Grass}(n+1, \mathrm{GL}(1, \mathbb{R}))$, while $\mathrm{sus}\,\mathrm{tang}\,\mathbb{C}P^n$ is $\mathcal{C}^a$-$\mathrm{GL}\,\mathbb{C}^{n+1}$-equivalent to the sum of $n+1$ copies of $\mathrm{Grass}(n+1, \mathrm{GL}(1, \mathbb{C}))$.

Exercise 4.6.6.7. Show that the normal bundle of the $\mathcal{C}^a$-embedding $G(m, n) \to G(m{+}1, n)$, described in Remark 3.2.2.3, is $\mathcal{C}^a$-$\mathrm{GL}\,\mathbb{R}^n$-equivalent to $\mathrm{Grass}(m, \mathrm{GL}(n, \mathbb{R}))$, while the normal bundle of the $\mathcal{C}^a$-embedding $\mathbb{C}G(m, n) \to \mathbb{C}G(m+1, n)$, described in Remark 3.2.2.7, is $\mathcal{C}^a$-$\mathrm{GL}\,\mathbb{C}^n$-equivalent to $\mathrm{Grass}(m, \mathrm{GL}(n, \mathbb{C}))$.

Exercise 4.6.6.8. Let $p_1, \ldots, p_{n+1}$ be homogeneous complex polynomials of degree m in $+1$ variables, whose only common zero is the point 0. Show that the map $\mathbb{C}P^n \to \mathbb{C}P^n$ given by

$$(z_1 : \cdots : z_{n+1}) \mapsto (p_1(z_1 \ldots, z_{n+1}) : \cdots : p_{n+1}(z_1 \ldots, z_{n+1}))$$

has degree m^n.

Exercise 4.6.6.9. Show that for $n \geq 1$ every continuous map $\mathbb{S}^n \to \mathbb{S}^n$ whose degree is not $(-1)^{n+1}$ has a fixed point.

Exercise 4.6.6.10. Show that for $n \geq 1$ every continuous map $\mathbb{S}^n \to \mathbb{S}^n$ having odd degree transforms some pair of antipodal points into another such pair.

Exercise 4.6.6.11. Show that for odd $n > 1$ the degree of any map $\mathbb{S}^n \to \mathbb{R}P^n$ is even.

Exercise 4.6.6.12. Let f be a simplicial map of the standard 2-simplex onto the standard 1-simplex. Show that the simplicial mapping cylinder, $\operatorname{Scyl} f$, is not homeomorphic to $\operatorname{Cyl} f$.

Chapter 5

HOMOTOPY GROUPS

5.1 THE GENERAL THEORY

5.1.1 Absolute Homotopy Groups

Definition 5.1.1.1. Let (X, x_0) be a pointed space, and let $r \geq 0$ be an integer. To simplify the notation, let us agree to write $\operatorname{Sph}_r(X, x_0)$ for the set $\mathcal{C}(I, \operatorname{Fr} I; X, x_0)$ of all continuous maps $(I, \operatorname{Fr} I) \to (X, x_0)$ and denote the set of homotopy classes of such maps (i.e., $\pi(I^r, \operatorname{Fr} I^r; X, x_0)$ by $\pi_r(X, x_0)$. The elements of $\operatorname{Sph}_r(X, x_0)$ will be referred to as *r-dimensional spheroids* (or simply *r-spheroids*) of the space X with origin x_0.

For $r > 0$ and two arbitrary spheroids $\varphi, \psi \in \operatorname{Sph}_r(X, x_0)$, we define their product, $\varphi\psi$, as the spheroid in $\operatorname{Sph}_r(X, x_0$ given by

$$\varphi\psi(t_1, t_2, \ldots, t_r) = \begin{cases} \varphi(2t_1, t_2, \ldots, t_r), & \text{if} \quad 0 \leq t_1 \leq 1/2, \\ \psi(2t_1 - 1, t_2, \ldots, t_r), & \text{if} \quad 1/2 \leq t_1 \leq 1. \end{cases} \tag{5.1.1.2}$$

For $r > 0$ and $\varphi \in \operatorname{Sph}_r(X, x_0)$, the spheroid φ^{-1}, called the *inverse* of φ, is defined by $\varphi^{-1}(t_1, t_2, \ldots, t_r) = \varphi(1 - t_1, t_2, \ldots, t_r)$. Obviously, if $\varphi, \varphi_1, \psi, \psi_1 \in \operatorname{Sph}_r(X, x_0)$ are such that φ_1 is homotopic to φ and ψ_1 is homotopic to ψ, then the spheroids $\varphi_1\psi_1$ and $\varphi\psi$ are homotopic. Therefore, (5.1.1.2) defines a multiplication on $\pi_r(X, x_0)$. It turns out that

Theorem 5.1.1.3. *this multiplication is associative, that the homotopy class of the constant spheroid* const *(which takes I^r into x_0) is a two-sided identity element and that the homotopy classes of the spheroids φ and φ^{-1} are inverses of one another.*

Proof. The associativity of the multiplication means that the products $(\varphi\psi)\chi$ and $\varphi(\psi\chi)$ are homotopic for any spheroids $\varphi, \psi, \chi \in \operatorname{Sph}_r(X, x_0)$. Indeed, the formula

$$((t_1, t_2, \dots, t_r), t) \mapsto \begin{cases} \varphi\left(\frac{4t_1}{11+t}, t_2, \dots, t_r\right), & \text{if} \quad 0 \leq t_1 \leq \frac{1+t}{4}, \\ \psi(4t_1 - t - 1, t_2, \dots, t_r), & \text{if} \quad \frac{1+t}{4} \leq t_1 \leq \frac{2+t}{4}, \\ \chi\left(\frac{4t_1-t-2}{2-t}, t_2, \dots, t_r\right), & \text{if} \quad \frac{2+t}{4} \leq t_1 \leq 1, \end{cases} \tag{5.1.1.4}$$

defines a homotopy $I^r \times I \to X$ from $(\varphi\psi)\chi$ to $\varphi(\psi\chi)$-

To prove the second claim, we have to show that the products $\varphi(\text{const})$ and $(\text{const})\varphi$ are both homotopic to φ, for any $\varphi \in \operatorname{Sph}_r(X, x_0)$. The formulae

$$((t_1, t_2, \dots, t_r), t) \mapsto \begin{cases} \varphi\left(\frac{2t_1}{1+t}, t_2, \dots, t_r\right), & \text{if} \quad 0 \leq t_1 \leq \frac{1+t}{2}, \\ x_0, & \text{if} \quad \frac{1+t}{2} \leq t_1 \leq 1, \end{cases} \tag{5.1.1.5}$$

and

$$((t_1, t_2, \dots, t_r), t) \mapsto \begin{cases} x_0, & \text{if} \quad 0 \leq t_1 \leq \frac{1+t}{2}, \\ \varphi\left(\frac{2t_1-1+t}{1+t}, t_2, \dots, t_r\right), & \text{if} \quad \frac{1+t}{2} \leq t_1 \leq 1, \end{cases} \tag{5.1.1.6}$$

define homotopies from $\varphi(\text{const})$ and $(\text{const})\varphi$ to φ.

Finally, the third claim is that the products $\varphi\varphi^{-1}$ and $\varphi^{-1}\varphi$ are both homotopic to const: indeed, a homotopy from $\varphi\varphi^{-1}$ to const is given by

$$((t_1, t_2, \dots, t_r), t) \mapsto \begin{cases} \varphi(2t_1, t_2, \dots, t_r), & \text{if} \quad 0 \leq t_1 \leq \frac{1-t}{2}, \\ \varphi(1 - t_1, t_2, \dots, t_r), & \text{if} \quad \frac{1-t}{2} \leq t_1 \leq \frac{1+t}{2}, \\ \varphi(2 - 2t_1, t_2, \dots, t_r), & \text{if} \quad \frac{1+t}{2} \leq t_1 \leq 1. \end{cases} \tag{5.1.1.7}$$

□

The set $\pi_r(X, x_0)$, $r > 0$, with this group structure is called the *r-th homotopy group of the space X at the point x_0*.

If $r > 0$, then each r-spheroid maps I^r into the component X_0 of X containing x_0. Consequently, for $r > 0$ the groups $\pi_r(X, x_0)$ an $\pi_r(X_0, x_0)$ are isomorphic.

By Theorem 2.3.4.3, for a countable cellular space X all the sets $\pi_r(X, x_0)$ are countable.

The Case $r = 0$

Definition 5.1.1.8. Since I^r is a point and $\operatorname{Fr} I^r = \varnothing$, $\operatorname{Sph}_0(X, x_0)$ and $\pi_0(X, x_0)$ can be identified with X and with the set $\operatorname{comp} X$ of components of X, respectively. $\pi_0(X, x_0)$ has no natural group structure. However, it does have a distinguished element which, in analogy with the higher-dimensional case, will be referred to as an identity: this is the homotopy class of the 0-spheroid const, i.e., the component of X containing x_0.

To be able to use the same language for the cases $r > 0$ and $r = 0$, we shall call $\pi_0(X, x_0)$ the *0-th homotopy group of X at x_0*, and we shall apply the group-theoretic terminology to sets with distinguished elements and their maps. In particular, by a direct product we understand the usual product, a homomorphism is a map preserving distinguished elements, the kernel of a homomorphism is the preimage of the distinguished element, and an isomorphism is an invertible homomorphism.

The Case $r = 1$

Remark 5.1.1.9. One-dimensional spheroids are nothing else but closed paths, and the multiplication, inversion, and homotopy of spheroids, as defined in Theorem 5.1.1.3, coincide with the multiplication, inversion, and homotopy of paths, as defined in Definition 1.3.2.1 and Remark 1.3.2.3. The 1-st homotopy group is alternatively known as the *fundamental group*. It was defined some decades before the higher homotopy groups were introduced, and we shall see below that it holds a special position amongst the homotopy groups.

Remark 5.1.1.10. When $r = 1$ the homotopies (5.1.1.4)-(5.1.1.7) are defined not only for loops φ, ψ, χ with common origin: the only condition that the paths φ, ψ, χ must satisfy is that the products involved be meaningful. As in the case of loops, the homotopy class of a product is uniquely determined by the homotopy classes of its factors, provided that the origin of the paths in the second class coincides with the end of the paths in the first class. This multiplication is associative; the class of the constant path is a left identity element for the class of paths with the same origin, and a right identity element for the class of paths with the same end; and the classes of the paths s and s^{-1} are inverses of one another, i.e., their product, taken in any order, is homotopic with the corresponding identity element.

The Case $r > 1$

Theorem 5.1.1.11. *For $r > 1$ the group $\pi_r(X, x_0)$ is Abelian.*

Proof. We have to verify that the products $\varphi\psi$ and $\psi\varphi$ are homotopic for any spheroids $\varphi\psi \in \mathrm{Sph}_r(X, x_0)$, $r > 1$. Consider the following three homotopies $I^r \times I \to X$:

$$((t_1, t_2, t_3, \dots, t_r), t) \mapsto$$
$$\begin{cases} \varphi(2t_1, (1+t)t_2, t_3, \dots, t_r), & \text{if } \ 0 \le t_1 \le \frac{1}{2}, \quad 0 \le t_2 \le \frac{1}{1+t}, \\ \psi(2t_1 - 1, (1+t)t_2 - t, t_3, \dots, t_r), & \text{if } \ \frac{1}{2} \le t_1 \le 1, \quad \frac{t}{1+t} \le t_2 \le 1, \\ x_0, & \text{otherwise;} \end{cases}$$

$$((t_1, t_2, t_3, \dots, t_r), t) \mapsto$$
$$\begin{cases} \varphi(2t_1 - t, 2t_2, t_3, \dots, t_r), & \text{if } \ \frac{t}{2} \le t_1 \le \frac{1+t}{2}, \quad 0 \le t_2 \le \frac{1}{2}, \\ \psi(2t_1 + t - 1, 2t_2 - 1, t_3, \dots, t_r), & \text{if } \ \frac{1-t}{2} \le t_1 \le \frac{2-t}{2}, \quad \frac{1}{2} \le t_2 \le 1, \\ x_0, & \text{otherwise;} \end{cases}$$

and

$$((t_1, t_2, t_3, \dots, t_r), t) \mapsto$$
$$\begin{cases} \varphi(2t_1 - 1, (2-t)t_2, t_3, \dots, t_r), & \text{if } \tfrac{1}{2} \le t_1 \le 1, \quad 0 \le t_2 \le \frac{1}{2-t}, \\ \psi(2t_1, (2-t)t_2 + t - 1, t_3, \dots, t_r), & \text{if } 0 \le t_1 \le \tfrac{1}{2}, \quad \frac{1-t}{2-t} \le t_2 \le 1, \\ x_0, & \text{otherwise;} \end{cases}$$

Their successive product is a homotopy from $\varpi\psi$ to $\psi\varphi$. [These homotopies are pictured in Fig. 5.1, where the shaded regions are mapped into x_0.] □

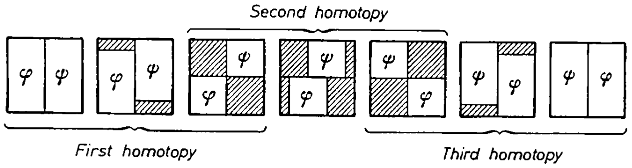

Figure 5.1: $(r = 2)$

Behaviour Under Continuous Maps

Definition 5.1.1.12. Let $f\colon (X, x_0) \to (X', x'_0)$ be a continuous map of pointed spaces. Then to each spheroid $\varphi\colon (I^r, \operatorname{Fr} I^r) \to (X, x_0)$ there corresponds the spheroid $f \circ \varphi\colon (I^r, \operatorname{Fr} I^r) \to (X', x'_0)$. This defines a map

$$f_\# = f_{\#r}\colon \operatorname{Sph}_r(X, x_0) \to \operatorname{Sph}_r(X', x'_0),$$

and clearly $f_\#$ takes homotopic spheroids into homotopic ones, and takes the constant spheroid into the constant one. Moreover, $f_{\#r}(\varphi\psi) = f_{\#r}(\varphi) f_{\#r}(\psi)$ for $r > 0$. Therefore, $f_{\#r}$ defines a homomorphism $\pi_r(X, x_0) \to \pi_r(X', x'_0))$ for each $r > 0$, called the *homomorphism induced by the map* f, and denoted f_* or, more specifically, f_{*r}.

Theorem 5.1.1.13. *For any two continuous maps,* $f\colon (X, x_0) \to (Y, y_0)$ *and* $g\colon (Y, y_0) \to (Z, z_0)$*, and any* $r > 0$*,*

$$(g \circ f)_{*r} = g_{*r} \circ f_{*r}.$$

If $f = \mathrm{id}_{(X,x_0)}$, *then* $f_{*r} = \mathrm{id}_{\pi_r(X,x_0)}$.

Proof. $(g \circ f)_{\#r} = g_{\#r} \circ f_{\#r}$ and $\mathrm{id}_{(X,x_0)_{\#r}} = \mathrm{id}\,\mathrm{Sph}_r(X,x_0)$. □

Theorem 5.1.1.14. *If the continuous maps* $f, f' \colon (X,x_0) \to (Y,y_0)$ *are homotopic, then* $f_{*r} = f'_{*r}$ *for all* r. *If* $f \colon (X,x_0) \to (Y,y_0)$ *is a homotopy equivalence, then* f_{*r} *is an isomorphism for all* r.

Proof. The spheroids $f \circ \varphi$ and $f^{\varphi} \circ \varphi$ are homotopic for any $\varphi \in \mathrm{Sph}_r(X,x_0)$, which proves the first assertion. The second assertion follows from the equalities $g_{*r} \circ f_{*r} = (g \circ f)_{*r} = \mathrm{id}$ and $f_{*r} \circ = (f \circ g)_{*r} = \mathrm{id}$, where g is any homotopy inverse of f (see Theorem 5.1.1.13). □

A Multiplication Theorem

Theorem 5.1.1.15. *Let* (X,x_0) *and* (Y,y_0) *be arbitrary pointed spaces. Then for any* $r \geq 0$ *the homotopy group* $\pi_r(X \times Y, (x_0,y_0))$ *is canonically isomorphic to the direct product* $\pi_r(X,x_0) \times \pi_r(Y,y_0)$. *The canonical isomorphism* $\pi_r(X \times Y, (x_0,y_0)) \to \pi_r(X,x_0) \times \pi_r(Y,y_0)$ *is given by* $\alpha \mapsto (\mathrm{proj}_{1*}(\alpha), \mathrm{proj}_{2*}(\alpha))$. *If* (X',x'_0) *and* (Y',y'_0) *is another pair of pointed spaces and* $f \colon (X,x_0) \to (X',x'_0)$ *and* $g \colon (Y,y_0) \to (Y',y'_0)$ *are continuous, then the diagramme*

$$\begin{tikzcd}
\pi_r(X \times Y, (x_0,y_0)) \arrow[r] \arrow[d, "(f\times g)_*"'] & \pi_r(X,x_0) \times \pi_r(Y,y_0) \arrow[d, "f_* \times g_*"] \\
\pi_r(X' \times Y', (x'_0,y'_0)) \arrow[r] & \pi_r(X',x'_0) \times \pi_r(Y',y'_0)
\end{tikzcd}$$

commutes (the horizontal maps are the canonical homomorphisms).

Proof. The proof is immediate. □

5.1.2 A Digression: Local Systems

Definition 5.1.2.1. We say that on the topological space X there is given a *local system of groups* if for each point $x \in X$ there is a group G_x, and for each path $s \colon I \to X$ there is a homomorphism $T_s \colon G_{s(0)} \to G_{s(1)}$, such that three conditions are satisfied:

(i) if $s_1(0) = s(1)$, then $T_{ss_1} = T_{s_1} \circ T_s$;

(ii) if s is a constant path, then T_s is the identical automorphism of $G_{s(0)}$;

(iii) if s and s_1 are homotopic paths, then $T_s = T_{s_1}$.

Condition (iii) shows that we may write $T_\sigma$instead of T_s, where σ is the homotopy class of the path s. Moreover, from (i)-(iii) it follows that all the homomorphisms T_s (T_σ) are actually isomorphisms, and that $T_s^{-1} = T_{s^{-1}}$ (respectively, $T_\sigma^{-1} = T_{\sigma^{-1}}$): indeed, using Remark 5.1.1.10, the paths ss^{-1} and $s^{-1}s$ are homotopic to a constant path, and hence $T_{s^{-1}} \circ T_s = T_{ss^{-1}} = \mathrm{id}_{G_{s(0)}}$ and $T_s \circ T_{s^{-1}} = T_{ss^{-1}} = \mathrm{id}\,G_{s(1)}$.

The isomorphism T_s is called the *translation along* s.

Remark 5.1.2.2. In particular, if s is a loop with origin x or, equivalently, if σ is an element of the fundamental group $\pi_1(X, x)$, then $T_s = T_\sigma$ is an automorphism of G_x. Comparing this with conditions (i) and (ii) in Definition 5.1.2.1, we see that the rule $\sigma \mapsto T_\sigma$ defines a right group-action of $\pi_1(X, x)$ on G_x.

To each path $s\colon I \to X$ there corresponds a natural isomorphism $t_s\colon \pi_1(X, s(0)) \to \pi_1(X, s(1))$, given by $t_s\omega = \sigma^{-1}\omega\sigma$, where σ is the homotopy class of s; sometimes we denote t_s by t_σ. One can check directly that $T_s\colon G_{s(0)} \to G_{s(1)}$ is a t_σ map (see Definition 4.2.3.1).

Definition 5.1.2.3. Let

$$(X, \{G_x\}, \{T_s\}), \quad (X', \{G'_{x'}\}, \{T'_{s'}\})$$

be local systems of groups, given on two spaces, X and X', and let $f : X \to X'$ be continuous. Let us assume further that for each point $x \in X$ we are given a homomorphism $h\colon G_x \to G'_{f(x)}$. We say that the homomorphisms h_x and the map f form a *homomorphism of the first local system into the second* if $h_{s(1)} \circ T_s = T'_{f\circ s} \circ h_{s(0)}$ for any path $s\colon I \to X$. A homomorphism $(f, \{h_x\})$ is an *isomorphism* if f is a homeomorphism and all h_x are isomorphisms; $(f, \{h_x\})$ is an *equivalence* if it is an isomorphism and, in addition, $X' = X$ and $f = \mathrm{id}_X$.

If $(X', \{G'_{x'}\}, \{T'_{s'}\})$ is a local system of groups and $f\colon X \to X'$ is continuous, then the induced local system $(X, \{G_x\}, \{T_s\})$ arises on X: set $G_x = G'_{f(x)}$ and $T_s = T'_{f\circ s}$. Obviously, $(f, \{\mathrm{id}_{G_x}\})$ is a homomorphism of the induced local system into the original one.

Theorem 5.1.2.4. *Let X be a connected space with base point x_0. Two local systems of groups, $(X, \{G_x\}, \{T_s\})$ and $(X', \{G'_{x'}\}, \{T'_{s'}\})$, are equivalent if and only if the two corresponding actions of $\pi_1(X, x_0)$ on G_{x_0} and $G_{x'_0}$ are isomorphic, i.e., if and only if there is a group isomorphism $G_{x_0} \to G_{x'_0}$ which is also a $\pi_1(X, x_0)$-map.*

Proof. That the actions of $\pi_1(X, x_0)$ on G_{x_0} and $G_{x'_0}$ arising from equivalent local systems are isomorphic is obvious. To prove the converse, fix a $\pi_1(X, x_0)$-isomorphism $h\colon G_{x_0} \to G_{x'_0}$ and choose, for each $x \in X$, some path s_x with origin x_0 and end x. It is readily verified that $(\mathrm{id}, \{h_x\})$, where $h_x = T'_{s_x} \circ h \circ T^{-1}_{S_x}$, is an equivalence. □

Remark 5.1.2.5. A local system of groups on a topological space X is said to be *simple* if it is equivalent to a *canonical simple local system* $(X, \{G_x\}, \{T_s\})$, where all the G_x are equal to some fixed group G, and all the homomorphisms T_s are the identical automorphism of G.

By Theorem 5.1.2.4, a local system of groups on a connected topological space X with base point x_0 is simple if and only if the induced action of $\pi_1(X, x_0)$ on G is the identical action. In particular, a local system is simple whenever $\pi_1(X, x_0)$ is trivial or the groups G_x are all isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Remark 5.1.2.6. It is readily seen that the discussion above may be extended from local systems of groups to local systems of other algebraic objects, such as vector spaces or rings.

In the present section we shall encounter, in addition to local systems of groups, local systems of sets with an identity (a distinguished element).

5.1.3 Local Systems of Homotopy Groups of Topological Spaces

Definition 5.1.3.1. Two spheroids, $\varphi_0 \in \operatorname{Sph}_r(X, x_0)$ and $\varphi_1 \in \operatorname{Sph}_r(X, x_1)$, are said to be *freely homotopic* if the maps $\operatorname{abrs}\varphi_0, \operatorname{abrs}\varphi_1 \colon I^r \to X$ can be connected by a homotopy consisting only of spheroids. More precisely φ_0 and φ_1 are freely homotopic if there is a continuous map $h\colon I^r \times I \to X$, constant on each set $\operatorname{Fr} I^r \times t$ $(t \in I)$, and such that $h(y, 0) = \varphi_0(y), h(y, 1) = \varphi_1(y)$ for all $y \in I^r$.

An essential element of such a homotopy is the path described by the origin of the spheroid, i.e., $t \mapsto h(\operatorname{Fr} I^r \times t)$. We say that h is a *free homotopy connecting the spheroids* φ_0 *and* φ_1 *along this path.*

Theorem 5.1.3.2. *Every spheroid with origin* x_0 *admits a free homotopy along any path with origin* x_0*. Free homotopies of homotopic spheroids along homotopic paths produce homotopic spheroids.*

Proof. Let $\varphi \in \operatorname{Sph}_r(X, x_0)$, and let s be a path with $s(0) = x_0$. To exhibit a free homotopy of φ along s, it is enough to extend somehow the homotopy $(y, t) \mapsto s(t)$ $[y \in \operatorname{Fr} I^r, t \in I]$ of the constant map $\varphi|_{\operatorname{Fr} I^r}$ to a homotopy of $\varphi\colon l^r \to X$. (That such an extension exists follows from Theorem 2.3.1.3.)

To prove the second claim, let φ_0 and φ_0' be homotopic spheroids, and let $h, h'\colon I^r \times I \to X$ be free homotopies of φ_0 and φ_0' along the homotopic paths s and s'. To show that the spheroids φ_1 and φ_1' are homo topic, where

$$\varphi_1(T_1, \dots, t_r) = h((t_1, \dots, t_r), t), \quad \varphi_1'(T_1, \dots, t_r) = h'((t_1, \dots, t_r), t)$$

pick some homotopies, $: I^r \times I \to X$, from φ_0 to φ_0', and $g\colon I \times I \to X$, from s to s', and define a subset K of the cube $l^{r+2} = I^r \times I \times I$ and a map $H\colon K \to X$, as follows:

$$K = \operatorname{Fr}_{\mathbb{R}^{r+2}} I^{r+2} \setminus [(\operatorname{int}_{\mathbb{R}^{r+1}} I^{r+1}) \times 1]$$

$$H(y, u, v) = \begin{cases} f(y, u), & \text{if} \quad v = 0, \\ g(u, v), & \text{if} \quad y \in \operatorname{Fr}_{\mathbb{R}^r} I^r, \\ h(y, v), & \text{if} \quad u = 0, \\ h'(y, v), & \text{if} \quad u = 1 \end{cases}$$

where $y \in I^r, u \in I, v \in I$.

There exists a homeomorphism $k\colon I^r \times I \to K$ such that $k(y, u) = (y, u, 1)$ for all $(y, u) \in \operatorname{Fr}(I^r \times I)$; for example, take the inverse of the homeomorphism $k_1\colon K \to I^r \times I$,

$$k_1(y, u, v) = (y_0, \frac{1}{2}) + \frac{1}{2}(1 + v)(y - y_0, u - \frac{1}{2}),$$

where $y = (\mathrm{ort}_1 + \cdots + \mathrm{ort}_r)/2$. Now it is clear that $H \circ k \colon I^r \times I \to X$ is a homotopy from φ_1 to φ_1'. ☐

Definition 5.1.3.3. According to the previous theorem, the free homotopies along a path $s \colon I \to X$ define a map $T_s \colon \pi_r(X, s(0)) \to \pi_r(X, s(1))$ for any $r \geq 0$. The same theorem demonstrates that the maps T_s fulfil property 5.1.2.1 (iii), and it is obvious that they enjoy also the properties 5.1.2.1 (i), 5.1.2.1 (ii), and are homomorphisms. The resulting local system, $(X, \{(X, x)\}, \{T_s\})$, is a local system of groups for any $r \geq 1$, and a local system of sets with distinguished elements for $r = 0$; $(X, \{(X, x)\}, \{T_s\})$ is called the *local system of the r-th homotopy groups of X*. In particular, $\pi_1(X, x)$ acts naturally from the right on $\pi_r(X, x)$, for any $x \in X$ and $r \geq 1$.

Theorem 5.1.3.4. *If $r = 1$, then the isomorphism T_s acts by the rule $T_s = \sigma^{-1}\omega\sigma$, where σ is the homotopy class of the path s (i.e., T_s coincides with the homomorphism t_s from Remark 5.1.2.2). In particular, the right action of $\pi_1(X, x)$ on $\pi_1(X, x)$ is the inner right action.*

Proof. Let w be any loop in the class ω, and define a path $s_t \colon I \to X$, $t \in I$, by $s_t(y) = s(ty)$. Consider the loop $w_t = (s_t^{-1} w) s_t$. Since s_0 is the constant path, belongs to ω, and the formula $(y, t) \mapsto w_t(y)$ defines a free homotopy $I \times I \to X$ from w_0 to the loop $w_1 = (s^{-1} w) s$ along s; w_1 belongs to the class $\sigma^{-1}\omega\sigma$. ☐

Definition 5.1.3.5. As Definition 5.1.3.3 shows, for arbitrary fixed r all the homotopy groups $\pi_r(X, x_0)$ of a connected topological space X are isomorphic. For $r = 1$, this was already a corollary of Remark 5.1.2.2.

A space X is r-simple if it is connected and the local system of its r-th homotopy groups is simple.
(*Remark by the transcriber*: For the definition of “simplicity”, see Remark 5.1.2.5.)

In this case, the groups $\pi_r(X, x)$ are not only isomorphic, but are manifestly canonically isomorphic, and hence they may be identified with a unique group, $\pi_r(X)$, referred to as the *r-th homotopy group of X without base point*. The elements of $\pi_r(X)$ are classes of freely homotopic spheroids. A space is *simple* if it is r-simple for all r.

If X is not r-simple, then one cannot use the isomorphisms T_s to identify the groups $\pi_r(X, x)$ with different x. In this situation one can speak of the group $\pi_r(X)$ of X only as an abstract group.

Obviously, the local system of the 0-th homotopy groups of a topological space is always simple, and for connected spaces, it becomes a local system of sets, each reduced to one point.

According to Theorem 5.1.3.4, a space is 1-simple if and only if it is connected and its fundamental group is Abelian (see Remark 5.1.2.5).

Remark 5.1.3.6. As Definition 5.1.1.12 shows, every continuous map $f \colon X \to X'$ induces a a homomorphism $f = (f_*)_X \colon \pi_r(X, x) \to \pi_r(X', f(x))$, for any $x \in X$. If h is a free homotopy from the spheroid φ_0 to the spheroid φ_1 along the path s, then $f \circ h$ is a free homotopy from $f \circ \varphi_0$ to $f \circ \varphi_1$ along the path $f \circ s$, and so $(f_*)_{s(1)} \circ T_s = T_{f\circ s} \circ (f_*)_{s(0)}$. Thus, given any $r \geq 0$, the induced homomorphisms $(f_{*r})_X$ combine with f to define a homomorphism of the local system of the r-th homotopy groups of X into the local system of the r-th homotopy groups of X'.

Three special cases deserve to be mentioned:

(a) X is r-simple;

(b) X' is r-simple;

(c) X and X' are both r-simple.

In case (a), the homomorphisms $(f_{*r})_X$ take the same group, $\pi_r(X)$, into $\pi(X', f(x))$, $x \in X$, and for any path $s\colon I \to X$ the diagramme

$$\begin{tikzcd}
 & \pi_r(X) \arrow[dl, "(f_*)_{s(0)}"'] \arrow[dr, "(f_*)_{s(1)}"] & \\
\pi_r(X', f\circ s(0)) \arrow[rr, "T_{f\circ s}"'] & & \pi_r(X', f\circ s(1))
\end{tikzcd}$$

commutes. In case (b) , the homomorphisms $(f_{*r})_X$ map the groups $\pi_r(X)$, $x \in X$, into the same group, $\pi_r(X')$, and for any path $s\colon I \to X$ the diagramme

$$\begin{tikzcd}
\pi_r(X, s(0)) \arrow[rr, "T_s"] \arrow[dr, "(f_*)_{s(0)}"'] & & \pi_r(X, s(1)) \arrow[dl, "(f_*)_{s(1)}"] \\
 & \pi_r(X') &
\end{tikzcd}$$

commutes. Finally, in case (c), the local systems of the r-th homotopy groups of X and X' reduce to two groups, $\pi_r(X)$ and $\pi_r(X')$, and the homomorphisms $(f_{*r})_X$ become one homomorphism $f_{*r}\colon \pi_r(X) \to \pi_r(X')$.

Theorem 5.1.3.7. *If $f\colon X \to X'$ is a homotopy equivalence, then all induced homomorphisms $(f_{*r})_(X)\colon pi_r(X, x) \to \pi_r(X', f(x))$ are isomorphisms.*

Proof. Let $f'\colon X' \to X$ be a homotopy inverse of f. If $H\colon X \times I \to X$ is a homotopy from $f' \circ f$ to id_X, then given any spheroid $\varphi \in \mathrm{Sph}_r(X, x)$, the map $I^r \times I \to I$, $(y, t) \mapsto H(\varphi(y), t)$, is a free homotopy from φ to the spheroid $f' \circ f \circ \mathrm{Sph}_r(X, f' \circ f(x))$, along the path s, $s(t) = H(x, t)$. Therefore, the homomorphism

$$(f'_*)_{f(x)} \circ (f_*)_x = ((f' \circ f)_*)_x \colon \pi_r(X, x) \to \pi_r(X, f' \circ f(x))$$

is simply the translation T_s; In particular, it is an isomorphism, implying that $(f'_*)_{f(x)}$ is an epimorphism. On the other hand, $(f_*)_{f'\circ f(x)} \circ (f'_*)_{f(x)}$ is also an isomorphism, and so $(f'_*)_{f(x)}$ a monomorphism. We conclude that $(f'_*)_{f(x)}$ an isomorphism, and hence so is $(f_*)_x = [(f'_*)_{f(x)}]^{-1} \circ T_s$. ☐

Theorem 5.1.3.8. *Let* (X, x_0) *be a pointed topological space, and let* $0 \leq k < \infty$. *The homotopy groups* $\pi_r(X, x_0)$ *are trivial for all* $r \leq k$ *if and only if* X *is* k*-connected. The homotopy groups* $\pi_r(X, x_0)$ *are all trivial if and only if* X *is* ∞*-connected*

Proof. If X is k-connected, then $\pi_r(X, x_0)$ is trivial for all $r \leq k$: to see this, compare the definition of $\pi_r(X, x_0)$ and that of k-connectedness (see Definition 1.3.3.7; in Theorem 1.3.3.6 one can replace $(\mathbb{D}^{r+1}, \mathbb{S}^r)$ by the homeomorphic pair $(I^{r+1}, \operatorname{Fr} I^{r+1})$). The same two definitions prove the converse statement, since the triviality of $\pi_r(X, x_0)$ for all $r \leq k$ implies the triviality of $\pi_r(X, x)$ for all $r \leq k$ and any $x \in X$ (see Definitions 5.1.1.8 and 5.1.3.5). ☐

5.1.4 Relative Homotopy Groups

Definition 5.1.4.1. Set $J^{r-1} = F_{\mathbb{R}^r} I^r \setminus \operatorname{int}_{\mathbb{R}^{r-1}} I^{r-1}$. Given any topological pair (X, A) with base point $x_0 \in A$ and any positive integer r, we let $\operatorname{Sph}_r(X, A, x_0)$ denote the set $\mathcal{C}(I^r, \operatorname{Fr} I^r, J^{r-1}; X, A, x_0)$ of all continuous maps $(I^r, \operatorname{Fr} I^r, J^{r-1}) \to (X, A, x_0)$. The elements of $\operatorname{Sph}_r(X, A, x_0)$ are called *r-dimensional spheroids* (or *r-spheroids*) *with* origin x_0 of the pair (X, A). The set $\pi_r(I^r, \operatorname{Fr} I^r, J^{r-1}; X, A, x_0)$ of homotopy classes of such spheroids is simply denoted by $\pi_r(X, A, x_0)$.

Notice that every spheroid $\varphi \in \operatorname{Sph}_r(X, A, x_0)$, such that $\varphi(I^r) \subset A$, is homotopic to the constant spheroid. In fact, there is even a standard homotopy $I^r \times I \to I$ from φ to the constant spheroid:

$$((t_1, \ldots, t_{r-1}, t_r), t) \mapsto (t_1, \ldots, t_{r-1}, (1-t)t_r + t).$$

A 1-spheroid with origin x_0 of the pair (X, A) is simply a path with origin in A and *end* x_0. *Warning*: a homotopy of such a spheroid is stationary at the point 1, but if A is not reduced to x_0, it is not necessarily stationary at the point 0.

When $r > 2$, formula (5.1.1.2) defines a multiplication on $\operatorname{Sph}_r(X, A, x_0)$, and this induces a multiplication on $\pi_r(X, A, x_0)$, which turns $\pi_r(X, A, x_0)$ into a group. The identity element of $\pi_r(X, A, x_0)$ is the homotopy class of the constant spheroid, while the class of the spheroid φ^{-1}, with

$$\varphi^{-1}(t_1, t_2, \ldots, t_r) = \varphi(1 - t_1, t_2, \ldots, t_r),$$

is the inverse of the class of φ. The proof of these assertions is entirely analogous to that given in the case of absolute homotopy groups.

For $r \geq 2$, $\pi_r(X, A, x_0)$ is called the *r-th homotopy group of the pair* (X, A) *at the point* x_0. The first homotopy group of (X, A) at x_0 is defined to be the set $\pi_1(X, A, x_0)$ with an identity (a distinguished element), namely the homotopy class of the constant 1-spheroid.

If $A = x_0$, then $\pi_r(X, A, x_0)$ equals $\pi_r(X, x_0)$ (i.e., $\pi_r(X, x_0, x_0)$ and $\pi_r(X, x_0)$ coincide as groups for $r \geq 2$, and as sets with distinguished elements for $r = 1$).

For $r > 1$, $\pi_r(X, A, x_0)$ is canonically isomorphic to $\pi_r(X_0, A_0, x_0)$, where X_0 and A_0 are the components of X and A which contain x_0.

For $r > 2$, $\pi_r(X, A, x_0)$ is Abelian; the proof entails obvious modifications of the proof of Theorem 5.1.1.11.

Definition 5.1.4.2. Every continuous map $f\colon (X, A, x_0) \to (X', A', x'_0)$ yields the *induced homomorphism* $f_*\colon \pi_r(X, A, x_0) \to \pi_r(X', A', x'_0)$, $r \geq 1$, defined as in the absolute case. If $A = x_0$ and $A' = x'_0$, we recover the absolute induced homomorphism, $f_*\colon \pi_r(X, x_0) \to \pi_r(X', x'_0)$. As with the absolute case, $(g \circ f)_* = g_* \circ f_*$ and $\mathrm{id}_* = \mathrm{id}$. If f and f' are homotopic, then $f_* = f'_*$. If f is a homotopy equivalence, then f_* is an isomorphism.

The Boundary Homomorphism

Definition 5.1.4.3. Given a spheroid $\varphi \in \mathrm{Sph}_r(X, A, x_0)$, its compression, $\mathrm{abr}\,\varphi\colon (I^{r-1}, \mathrm{Fr}\, I^{r-1}) \to (A, x_0)$, is a spheroid belonging to $\mathrm{Sph}_{r-1}(A, x_0)$, called the *boundary* of φ, and denoted $\partial\varphi$. The resulting map, $\partial\colon\ \mathrm{Sph}_r(X, A, x_0) \to \mathrm{Sph}_{r-1}(A, x_0)$, takes homotopic spheroids into homotopic ones, takes the sum of two spheroids into the sum of their boundaries, and takes the constant spheroid into the constant one. Therefore, it defines, for every $r \geq 1$, a homomorphism $\partial\colon \pi_r(X, A, x_0) \to \pi_{r-1}(A, x_0)$, called the *boundary homomorphism.*

Given any continuous map, $f\colon (X, A, x_0) \to (X', A', x'_0)$, the diagramme

$$\begin{tikzcd}
\pi_r(X, A, x_0) \arrow[r, "\partial"] \arrow[d, "f_*"'] & \pi_{r-1}(A, x_0) \arrow[d, "(\mathrm{abr}\, f)_*"] \\
\pi_r(X', A', x'_0) \arrow[r, "\partial"'] & \pi_{r-1}(A', x'_0)
\end{tikzcd}$$

commutes for any $r \geq 1$. Indeed, we already know that the similar diagram with Sph instead of π, and with $f_\#$ and $(\mathrm{abr}\, f)_\#$ instead of f_* and $(\mathrm{abr}\, f)_*$, commutes.

Local Systems of Homotopy Groups of a Topological Pair

Definition 5.1.4.4. Two spheroids, $\varphi_0 \in \mathrm{Sph}_r(X, A, x_0)$ and $\varphi_1 \in \mathrm{Sph}_r(X, A, x_1)$, are said to be *freely homotopic* if there is a homotopy from φ_0 to φ_1 consisting only of spheroids, where φ_0 to φ_1 are viewed as maps $(I^r, \mathrm{Fr}\, I - r) \to (X, A)$. In other words, φ_0 to φ_1 are freely homotopic if there is a map $h\colon I^r \times I \to I$ such that $h(\mathrm{Fr}\, I^r \times I) \subset A$, h is constant on every set $J^{r-1} \times t$ $(t \in I)$, and $h(y, 0) = \varphi_0(y)$, $h(y, 1) = \varphi_1(y)$ for all $y \in I^r$. We say that h is a *free homotopy from φ_0 to φ_1, along the path $t \mapsto h(J^{r-1} \times t)$.*

Theorem 5.1.4.5. *Every spheroid with origin x_0 of (X, A) admits a free homotopy along any given path, $s\colon I \to A$, with $s(0) = x_0$. Moreover, free homotopies of homotopic spheroids along homotopic paths of A produce homotopic spheroids.*

Proof. The proof differs from that of Theorem 5.1.3.2 in two details:

- in the first part of the proof, we have to start with $\operatorname{abr}\varphi\colon J^{r-1} \to A$ (instead of $\varphi|_{\operatorname{Fr} I^r}$) and extend its homotopy, $(y,t) \mapsto s(t)$, initially to a homotopy of $\operatorname{abr}\varphi\colon \operatorname{Fr} I^r \to A$, and then to a homotopy of $\varphi\colon I^r \to X$;
- in the second part of the proof, we must replace $g\colon I\times I \to X$ by $g\colon I\times I \to A$.

□

This theorem shows that given a path $s\colon I \to A$, the free homotopies along s define a map $T_s\colon \pi_r(X,A,s(0)) \to \pi_r(X,A,s(1))$ (for each $r \geq 1$). As in the absolute case, T_s are homomorphisms and enjoy properties 5.1.2.1 (i)-(iii). Therefore, a local system, $(A, \{\pi_r(X,A,x)\}, \{T_s\})$, arises on A, which is a local system of groups for $r \geq 2$, and a local system of sets with distinguished elements for $r = 1$. This is the *local system of the r-th homotopy groups of the pair* (X,A). In particular, for any $x \in A$ and $r \geq 1$, $\pi_1(A,x_0)$ acts naturally from the right on $\pi_r(X,A,x)$; this is a group-action for $r > 1$, and it fixes the distinguished element for $r = 1$.

From the existence of this local system it follows that, for any $r \geq 1$, the r-th homotopy groups $\pi_r(X,A,x)$, $x \in A$, are all isomorphic whenever A is connected.

A pair (X,A) with A connected is said to be *r-simple* if the local system of its r-th homotopy groups is simple. In this case all the homotopy groups $\pi_r(X,A,x)$, $x \in A$, can be identified with a single group, the *r-th homotopy group of the pair (X,A) without base point*, $\pi_r(X,A)$; the elements of $\pi_r(X,A)$ are classes of freely homotopic spheroids. A pair is *simple* if it is r-simple for any $r \geq 1$. For example, every pointed space is a simple pair.

Remark 5.1.4.6. Given any pair (X,A) and any path $s\colon I \to A$, the diagramme

$$\begin{tikzcd}
\pi_r(X,A,s(0)) \arrow[r, "\partial"] \arrow[d, "T_s"'] & \pi_{r-1}(A,s(0)) \arrow[d, "T_s"] \\
\pi_r(X,A,s(1)) \arrow[r, "\partial"'] & \pi_{r-1}(A,s(1))
\end{tikzcd}$$

obviously commutes. Therefore, the boundary homomorphisms, $\partial = \partial_x(X,A,x) \to \pi_{r-1}(A,x)$, combine with id_A to define a homomorphism of the local system of the r-th homotopy groups of the pair (X,A) into the local system of the $(r-1)$-th homotopy groups of the space A.

Further, given any continuous map, $f\colon (X,A) \to (X',A')$, the homomorphisms

$$f_* = (f_*)_x\colon \pi_r(X,A,x) \to \pi_r(X',A',f(x)),$$

combined with f to define a homomorphism of the local system of the r-th homotopy groups of the pair (X,A) into the corresponding local system of (X',A'). As in Remark 5.1.3.6, we mention three special cases:

(a) (X,A) is r-simple;

(b) (X', A') is r-simple;

(c) both (X, A) and (X', A') are r-simple.

In cases (a) and (b), any path $s\colon I \to A$ yields commutative diagrammes similar to those in Remark 5.1.3.6; in case (c), the local systems of the r-th homotopy groups of the pairs (X, A) and (X', A') reduce to single groups, $\pi_r(X, A)$ and $\pi_r(X', A')$, while the homomorphisms reduce to a single homomorphism, $f_*\colon \pi_r(X, A) \to \pi_r(X', A')$.

Theorem 5.1.4.7. *If f is a homotopy equivalence, then all the homomorphisms $(f_*)_x$ are isomorphisms.*

Proof. The proof is similar to that given in the absolute case (see Theorem 5.1.3.7). ☐

Theorem 5.1.4.8. *Let (X, A) be a pair with base point $x_0 \in A$. If X, A are connected, then the triviality of all the homotopy groups $\pi_r(X, A, x_0)$ is equivalent to the ∞-connectedness of (X, A); the triviality of the homotopy groups $\pi_r(X, A, x_0)$ for $1 \leq r \leq k$ is equivalent to the k-connectedness of (X, A).*

Proof. The proof is a repetition of the proof of Theorem 5.1.3.8, with obvious modifications (instead of referring to Definition 1.3.3.7, we refer to Remark 1.3.3.9). ☐

The Group $\pi_2(X, A, x_0)$

Theorem 5.1.4.9. *If $\alpha, \beta \in \pi_2(X, A, x_0)$, then $\alpha^{-1}\beta\alpha = T_{\partial\alpha}\beta$.*

Proof. (For an alternative proof, see Subsection 5.1.10). We have to check that, given two arbitrary spheroids, $\varphi, \psi \in \mathrm{Sph}_r(X, A, x_0)$, there is a free homotopy from *psi* to $\varphi^{-1}\psi\varphi$ along the loop $\partial\varphi$. We shall exhibit such a homotopy as a family of maps, $\chi_t\colon I^2 \to X$ $(t \in I)$, constructed as follows.

Set $f(t) = \frac{1}{16} - \frac{1}{8}|t - \frac{1}{2}|$, and divide I^2 into eight parts, as shown in Fig. 5.2: the points $A_1(t)$, $A_2(t)$, $A_3(t)$, $A_4(t)$ have abscissae $f(t)$, $t/4$, $1-(t/2)$, $1-f(t)$, respectively, while the points $B_1(t)$, $B_2(t)$, $B_3(t)$, $B_4(t)$ lie above these points at the height $1 - f(t)$. Further, let $\alpha_j\colon Q_j(t) \to I^2$, $j = 1, 2, 3$, be the affine maps defined by the conditions

$$\begin{aligned} &\alpha_1(A_1(t)) = (t, 0), \quad \alpha_1(A_2(t)) = (0, 0), \quad \alpha_1(B_1(t)) = (t, 1), \\ &\alpha_2(A_2(t)) = (0, 0), \quad \alpha_2(A_3(t)) = (1, 0), \quad \alpha_2(B_2(t)) = (0, 1), \\ &\alpha_3(A_3(t)) = (0, 0), \quad \alpha_3(A_4(t)) = (t, 0), \quad \alpha_3(B_3(t)) = (0, 1). \end{aligned}$$

Now set

$$\chi_t|_{Q_1(t)} = \varphi \circ \alpha_1, \quad \chi_t|_{Q_2(t)} = \psi \circ \alpha_2, \quad \chi_t|_{Q_3(t)} = \varphi \circ \alpha_3$$

and consider the resulting continuous map $Q_1(t) \cup Q_2(t) \cup Q_3(t) \to X$. Extend it firstly to a map $\cup_1^7 Q_i(t) \to X$ which is constant on the horizontals in $Q_4(t) \cup$

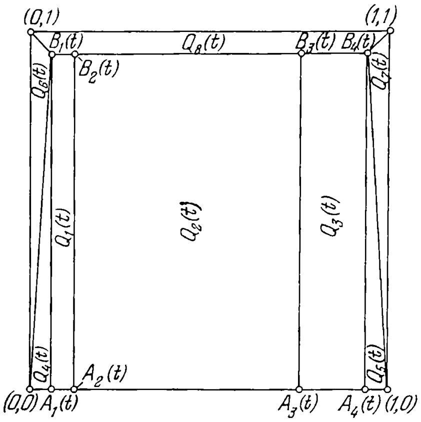

Figure 5.2:

$Q_5(t)$ and on the verticals in $Q_6(t) \cup Q_7(t)$, and then to a map $\cup_1^8 Q_i(t) \to X$ which is constant on the horizontals in $Q_{89}(t)$. (The latter is possible, since the already extended map$\cup_1^7 Q_i(t) \to X$ is constant on the segment $[B_1(t), B_4(t)]$, and assumes the same values at those points of the segments $[B_1(t), (0,1)]$ and $[B_4(t), (1,1)]$ which lie at the same height.) The continuity of the map $I^2 \times I \to X$ defined by the family $\chi_t \colon I^2 \to X$ follows from its continuity on each of the eight polyhedrons $\cup_{t \in I}(Q_i(t) \times t)$, $1 \leq i \leq 8$. $\square$

The Action of the Group $\pi_1(X, x_0)$ on $\pi_1(X, A, x_0)$

Definition 5.1.4.10. Given a spheroid $w \in \mathrm{Sph}_r(X, A, x_0)$ and a loop $s \in \mathrm{Sph}_1(X, x_0)$, the product ws is well defined, and obviously $ws \in \mathrm{Sph}_1(X, A, x_0)$. Moreover, the homotopy class of ws is uniquely determined by the homotopy classes of w and s, and hence we may define the product $\omega\sigma$ for any $\omega \in \pi_1(X, A, x_0)$ and $\sigma \in \pi_1(X, x_0)$. Using again the homotopies described in Theorem 5.1.1.3, we see that $\omega(\sigma\sigma') = (\omega)\sigma)\sigma'$ and $\omega e_{\pi_1(X, x_0)} = \omega$ for any $\omega \in \pi_1(X, A, x_0)$ and $\sigma, \sigma' \in \pi_1(X, x_0)$. That is to say, the rule $(\omega, \sigma) \mapsto \omega\sigma$ defines a right action Of $\pi_1(X, x_0)$ on $\pi_1(X, A, x_0)$.

If $A = x_0$, then we clearly recover the canonical right action of the group $\pi_1(X, x_0)$ (on itself) . It is readily seen that the translation

$$T_s \colon (X, A, s(0)) \to (X, A, s(1))$$

is a $[T_{\mathrm{incl} \circ s} \colon \pi_1(X, s(0)) \to t\pi_1(X, s(1))]$-map for any path $s \colon I \to A$, i.e., $T_s(\omega\sigma) = T_s(\omega) T_{\mathrm{incl} \circ s}(\sigma)$ for all $\sigma \in \pi_1(X, s(0))$ and $\omega \in \pi_1(X, A, s(0))$. Furthermore, given any continuous map $f \colon (X, A, x_0) \to (X', A', x_0')$, the homomorphism

$$f_* \colon \pi_1(X, A, x_0) \to \pi_1(X', A', x_0')$$

is a $f_*\colon \pi_1(X, x_0) \to \pi_1(X', x'_0)$-map, i.e.,

$$f_*(\omega\sigma) = f_*(\omega) f_*(\sigma) \quad \forall \omega \in \pi_1(X, A, x_0), \forall \sigma \in \pi_1(X, x_0).$$

In particular, if we consider the map $\mathrm{rel} = [\mathrm{incl}\colon (X, x_0, x_0) \to (X, A, x_0)]$, then

$$(\mathrm{rel}_* \, \omega)\sigma = \mathrm{rel}_*(\omega\sigma)$$

for all $\omega, \sigma \in \pi 1(X, x_0)$.

Theorem 5.1.4.11. *For any* $\omega \in \pi_1(X, A, x_0)$ *and* $\sigma in \pi_1(A, x_0)$ *we have*

$$T_\sigma \omega = \omega(\mathrm{incl}_* \, \sigma),$$

where $\mathrm{incl} = [\mathrm{incl}\colon (A, x_0) \to (X, x_0)]$.

Proof. Let w and s be spheroids in the classes ω and σ. Consider, for each fixed $t \in I$, the path $s_t\colon I \to A$ given by $s_t(y) = s(ty)$, and set $w_t = w(\mathrm{incl} \circ s_t)$. Since s_0 is a constant path, w and w_0 are homotopic. On the other hand, the formula $(y, t) \mapsto w_t(y)$ defines a free homotopy $I \times I \to I$ from w_0 to $w_1 = w(\mathrm{incl} \circ s)$, along s. □

5.1.5 A Digression: Sequences of Groups and Homomorphisms, and π-Sequences

Definition 5.1.5.1. A *sequence of groups and homomorphism* is a finite or infinite (on one or both sides) sequence of groups such that for each two adjacent groups, G_i and G_{i+1}, there is given a homomorphism $G_i \to G_{i+1}$.

A *homomorphism* of a sequence of groups and homomorphisms, into another such sequence,

$$\{G_i, h_i\colon G_i \to G_{i+1}\} \quad \text{into} \quad \{G'_i, h'_i\colon G'_i \to G'_{i+1}\}$$

is a sequence of homomorphisms $\{H_i\colon G_i \to G_{i+1}\}$ such that the diagram

$$\begin{tikzcd}
{} \arrow[r, dash, dotted] & G_{i-1} \arrow[r, "h_{i-1}"] \arrow[d, "H_{i-1}"] & G_i \arrow[r, "h_i"] \arrow[d, "H_i"] & G_{i+1} \arrow[r, dash, dotted] \arrow[d, "H_{i+1}"] & {} \\
{} \arrow[r, dash, dotted] & G'_{i-1} \arrow[r, "h'_{i-1}"'] & G'_i \arrow[r, "h'_i"'] & G'_{i+1} \arrow[r, dash, dotted] & {}
\end{tikzcd}$$

commutes. A homomorphism $\{H_i\}$ such that each $H_i\colon G_i \to G'_i$ is and isomorphism is called an *isomorphism*.

Definition 5.1.5.2. A sequence of groups and homomorphisms, $\{G_i, h_i\}$, is *exact* if for each group G_i, excepting the initial and final ones, the kernel $\ker h_i$ of the homomorphism h_i equals the image $\operatorname{im} h_{i-1}$ of the homomorphism h_{i-1}.

The following three properties are common to all exact sequences $\{G_i, h_i\}$.

(i) If G_i is an inner (i.e., neither initial, nor final) term of the given sequence, then h_{i-1} is trivial if and only if h_i is a monomorphism, while h_i is trivial if and only if h_{i-1} is an epimorphism; h_{i-1} and h_i are both trivial if and only if the group G_i is trivial.

(ii) If G_i and G_{i+1} are inner terms, then h_{i-1} and h_{i+1} are simultaneously trivial if and only if h_i is an isomorphism.

(iii) In particular, the triviality of G_{i-1} and G_{i+1} implies the triviality of G_i, while the triviality of G_{i-1} and G_{i+2} implies that h_i is an isomorphism.

Definition 5.1.5.3. An exact sequence of the form

$$1 \to F \xrightarrow{f} G \xrightarrow{g} H \to 1$$

is called *short* (here 1 denotes the trivial group). An example is

$$1 \to F \xrightarrow{\text{incl}} G \xrightarrow{\text{proj}} G/F \to 1,$$

where F is a normal subgroup of G. This example has a universal character: every short exact sequence,

$$1 \to F \xrightarrow{f} G \xrightarrow{g} H \to 1$$

is canonically isomorphic to a sequence of this type, namely, to

$$1 \to \operatorname{im} f \xrightarrow{\text{incl}} G \xrightarrow{\text{proj}} G/\operatorname{im} f \to 1;$$

the canonical isomorphism is obviously

$$\{\mathrm{id}_1, \operatorname{abr} f\colon F \to \operatorname{im} f, \mathrm{id}_G, h \mapsto \operatorname{proj}(g^{-1}(h)), \mathrm{id}_1\}.$$

Splitting

Definition 5.1.5.4. Let $\{G_i, h_i\}$ be a sequence of groups and homomorphisms. We say that this sequence is *split from the right at the term* G_α *by the homomorphism* $\zeta\colon G_{\alpha+1} \to G_\alpha$ if $h_\alpha \circ \zeta = \mathrm{id}_{G_{\alpha+1}}$. Such a splitting is said to be *normal* if $\operatorname{im}\zeta$ is a normal subgroup of G_α.

Similarly, $\{G_i, h_i\}$ is *split from the left at the term* G_α *by the homomorphism* $\zeta\colon G_\alpha \to G_{\alpha-1}$ if $\zeta \circ h_{\alpha-1} = \mathrm{id}_{G_{\alpha-1}}$. We sometimes say simply that the given sequence is *split*, or that it *splits* (at right or at left) at G_α.

Lemma 5.1.5.5. *Let A and B be groups, and let $u\colon A \to B$ and $v\colon B \to A$ be homomorphisms. If $\operatorname{im} v$ is a normal subgroup of A and $u \circ v = \mathrm{id}_B$, then $A = \ker u \times \operatorname{im} v$.*

Proof. Every element $a \in A$ can be represented as $[a(v \circ u(a))^{-1}](v \circ u(a))$, and obviously $a(v \circ u(a))^{-1} \in \ker u$ and $v \circ u(a) \in \operatorname{im} v$. If $a \in \ker u \cap \operatorname{im} v$, then $u(a) = e_B$, and there is $b \in B$ with $v(b) = a$. Thus, $b = u \circ v(b) = u(a) = e_B$ and $a = v(b) = e_A$. □

Theorem 5.1.5.6. *If the exact sequence* $\{G_i, h_i\}$ *splits normally from the right at* G_α *and splits from the right at* $G_{\alpha-3}$, *then it also splits from the left at* G_α , *and* $G_\alpha \overset{\sim}{=} G_{\alpha-1} \times G_{\alpha+1}$. *More precisely, under these hypotheses,* $h_{\alpha-1}$*is a monomorphism,* h_α *is an epimorphism, and* G_α *decomposes into the direct product of* $\operatorname{im} h_{\alpha-1}$ *and a subgroup which is mapped isomorphically onto* $G_{\alpha+1}$ *by* h_α. *Moreover, every homomorphism* $\zeta\colon G_{\alpha+1} \to G_\alpha$ *which splits the sequence from the right is a monomorphism, and if* ζ *is also normally splitting, then for a direct complement of* $\operatorname{im} h_{\alpha-1}$ *one may take* $\operatorname{im}\zeta$.

Proof. The equality $G_\alpha = \operatorname{im} h_{\alpha-1} \times \operatorname{im}\zeta$ is a consequence of Lemma 5.1.5.5 and of the exactness of the given sequence. From $h_\alpha \circ \zeta = \mathrm{id}_{G_{\alpha+1}}$, it follows that h_α is an epimorphism and that ζ is a monomorphism. Since $\{G_i, h_i\}$ splits from the right at $G_{\alpha-3}$, $h_{\alpha-3}$ is an epimorphism, and now the exactness of $\{G_i, h_i\}$ implies that $h_{\alpha-2}$ is trivial, while $h_{\alpha-1}$ is a monomorphism. Finally, as a homomorphism splitting the given sequence at from the left one can take the homomorphism which is the inverse of on an equals the identity on $\operatorname{im}\zeta$. ☐

Theorem 5.1.5.7. *If the exact sequence* $\{G_i, h_i\}$ *splits from the left at* G_α *and* $G_{\alpha+3}$, *then it splits normally from the right at* G_α, *and*

$$G_\alpha \overset{\sim}{=} G_{\alpha-1} \times G_{\alpha+1}.$$

More precisely, under these hypotheses, $h_{\alpha-1}$ *is a monomorphism,* h_α *is an epimorphism, and* G_α *decomposes into the direct product of* $\operatorname{im} h_{\alpha-1}$ *and a subgroup which is mapped isomorphically onto* $G_{\alpha+1}$ *by* h_α. *Moreover, every homomorphism* $\zeta\colon\ : G_\alpha \to G_{\alpha-1}$ *which splits the sequence from the left is an epimorphism, and for a direct complement of* $\operatorname{im} h_{\alpha-1}$ *one can take* $\ker\zeta$.

Proof. The equality $G_\alpha = \operatorname{im} h_{\alpha-1} \times \operatorname{im}\zeta$ is a consequence of Lemma 5.1.5.5. From $\zeta \circ h_{\alpha-1} = \mathrm{id}_{G_{\alpha-1}}$ it follows that $h_{\alpha-1}$ is a monomorphism and that ζ is an epimorphism. Since $\{G_i, h_i\}$ splits from the left at $G_{\alpha+3}$ and is exact, $h_\alpha 3$ is an epimorphism. As a homomorphism splitting the given sequence normally from the right at G_α, one can take the composition

$$G_{\alpha+1} = G_\alpha/\ker h_\alpha = G_\alpha/\operatorname{im} h_{\alpha-1} = (\operatorname{im} h_{\alpha-1} \times \ker\zeta)/\operatorname{im} h_{\alpha-1} = \ker\zeta \xrightarrow{\text{incl}} G_\alpha.$$

☐

Theorem 5.1.5.8. *An exact sequence*

$$1 \to F \xrightarrow{f} G \xrightarrow{g} H \to 1$$

splits at G *from the left if and only if it splits at* G *normally from the right, and this happens if and only if the subgroup* $\operatorname{im} f = \ker g$ *of* G *has a direct complement.*

Proof. This is a corollary of Lemma 5.1.5.5. ☐

Five Lemma

Theorem 5.1.5.9. *If*

$$\begin{tikzcd}
G_1 \arrow[r, "h_1"] \arrow[d, "\varphi_1"] & G_2 \arrow[r, "h_2"] \arrow[d, "\varphi_2"] & G_3 \arrow[r, "h_3"] \arrow[d, "\varphi_3"] & G_4 \arrow[r, "h_1"] \arrow[d, "\varphi_4"] & G_5 \arrow[d, "\varphi_5"] \\
G'_1 \arrow[r, "h'_1"'] & G'_2 \arrow[r, "h'_2"'] & G'_3 \arrow[r, "h'_3"'] & G'_4 \arrow[r, "h'_4"'] & G'_5
\end{tikzcd}$$

is a homomorphism of exact sequences, and if φ_1 is an epimorphism, φ_2, φ_4 are isomorphisms, and φ_5 is a monomorphism, then φ_3 is an isomorphism.

Proof. Let us show first that φ_3 is a monomorphism. If $a \in \ker \varphi_3$, then $\varphi_4 \circ h_3(a) = h'_3 \circ \varphi_3(a) = e_{G'_4}$, and so $h_3(a) = e_{G_4}$, i.e., $a \in \ker h_3 = \operatorname{im} h_2$. Let $a = h_2(b)$, $b \in G_2$. Since

$$h'_2 \circ \varphi_2(b) = \varphi_3 \circ h_2(b) = \varphi_3(a) = e_{G'_3}$$

there is $c \in G'_1$ such that $h'_1(c) = \varphi_2(b)$. Therefore, there is $d \in G_1$ such that $h'_1 \circ \varphi_1(d) = \varphi_2(b)$. On the other hand, $h'_1 \circ \varphi_1(d) = \varphi_2 \circ h_1(d)$, and hence $\varphi_2(b) = \varphi_2 \circ h_1(d)$. Consequently, $b = h_1(d)$ and $a = h_2 \circ h_1(d) = e_{G_3}$.

Now let us verify that φ_3 is an epimorphism. Let $a \in G'_3$. Then

$$\varphi_5 \circ h_4 \circ \varphi_4^{-1} \circ h'_3(a) = h'_4 \circ h'_3(a) = e_{G'_5},$$

and so $h_4 \circ \varphi_4^{-1} \circ h_3(a) = e_{G'_5}$, i.e., $\varphi_4^{-1} \circ h'_3(a) \in \ker h_4 = \operatorname{im} h_3$. Let

$$\varphi_4^{-1} \circ h'_3(a) = h_3(b), \quad b \in G_3.$$

Since

$$h'_3(a(\varphi_3(b))^{-1}) = (\varphi_4 \circ \varphi_4^{-1} \circ h'_3(a))(h'_3 \circ \varphi_3(b))^{-1} = \varphi_4(\varphi_4^{-1} \circ h'_3(a))(h'_3(b))^{-1}) = e_{G'_4},$$

there is $c \in G'_2$ such that $h'_2(c) = a(\varphi_3(b))^{-1}$, and hence there is $d \in G_2$ such that $h'_2 \circ \varphi_2(d) = a(\varphi_3(b))^{-1}$. On the other hand, $h'_2 \circ \varphi_2(d) = \varphi_3 \circ h_2(d)$, and hence $a(\varphi_3(b))^{-1} = \varphi_3 \circ h_2(d)$. Consequently, $a = \varphi_3(h_2(d)b)$. □

π-Sequences

Remark 5.1.5.10. In the next subsections we shall handle the so-called *homotopy sequences*. These are rather cumbersome entities which are similar to sequences of groups and homomorphisms, but possess additional properties and structures. Such sequences are encountered in various geometric situations, but they are all algebraically related. The rest of the present subsection is devoted to a preliminary, purely algebraic description and study of these sequences.

Definition 5.1.5.11. Consider a left-infinite sequence

$$\to \Pi_7 \xrightarrow{\rho_6} \Pi_6 \xrightarrow{\rho_5} \Pi_5 \xrightarrow{\rho_4} \Pi_4 \xrightarrow{\rho_3} \Pi_3 \xrightarrow{\rho_2} \Pi_2 \xrightarrow{\rho_1} \Pi_1 \xrightarrow{\rho_0} \Pi_0, \qquad (5.1.5.12)$$

where:

Π_0, Π_1, Π_2 are sets with an identity (distinguished element),

Π_3, Π_4, Π_5 are groups,

Π_6, Π_7, $\ldots$ are Abelian groups,

ρ_0, ρ_1, ρ_2 are homomorphisms in the sense of Definition 5.1.1.8,

ρ_3, ρ_4, $\ldots$ are group homomorphism.

Then (5.1.5.12) is called a *π-sequence* if there are given

right group-actions of Π_3 on the groups Π_{3k} with $k \geq 2$,

right group-actions of Π_4 on the groups Π_{3k+1} with $k \geq 2$,

right group-actions of Π_4 on the groups Π_{3k-1} with $k \geq 2$,

right group-actions of Π_3 on the set Π_2,

such that

(i) ρ_{3k} is a ρ_3-homomorphism for all $k \geq 2$;

(ii) ρ_{3k+1} a Π_4-homomorphism for all $k \geq 2$;

(iii) ρ_{3k-1} is a Π_4-homomorphism for all $k \geq 2$, with respect to the right group-action of Π_4 on Π_{3k} induced by the given action of on Π_3 via ρ_3;

(iv) ρ_4 is a Π_4-homomorphism with respect to the right inner action of Π_4;

(v) the transformation of the group Π_2 induced by the image $\rho_4(\alpha) \in \Pi_4$ of an arbitrary element $\alpha \in \Pi_4$ is the inner automorphism $\beta \mapsto \alpha^{-1}\beta\alpha$;

(vi) the transformation of the set Π_2 induced by an arbitrary element $\alpha \in \Pi_3$ coincides on $\rho_2(\Pi_3)$ with the transformation $\rho_2(\omega)\sigma = \rho_2(\omega\sigma)$.

A *homomorphism* of the π-sequence $\{\Pi_i, \rho_i\}_{i=0}^{\infty}$ into the π-sequence $\{\Pi'_i, \rho'_i\}_{i=0}^{\infty}$ is a sequence of homomorphisms $h_i\colon \Pi_i \to \Pi'_i$ such that

$\rho i \circ h_{i+1} = h_i \circ \rho_i$ for all $i \geq 0$;

h'_{3k}, h'_{3k+1} and h'_{3k-1} $(k \geq 2)$ are h_3-, h_4- and h_5-homomorphisms, respectively;

$h_2(\omega)h_3(\sigma) = h_2(\omega\sigma)$ for all $\omega \in \Pi_2$ and $\sigma \in \Pi_3$.

An *isomorphism* is a homomorphism such that all h_i's are isomorphisms.

Remark 5.1.5.13. Among conditions (i)-(vi) above, two refer to ρ_4, namely (iv) and (v). From (iv) it follows that if Π_4 acts identically on Π_5, then $\operatorname{im}\rho_4$ is contained in the centre of the group Π_4. From (v) it follows that if Π_4 acts identically on Π_5, then Π_5 is Abelian, and that the converse is true provided ρ_4 is an epimorphism. In general, (v) implies that $\ker\rho_4$ is contained in the centre of Π_5.

Definition 5.1.5.14. The π-sequence (5.1.5.12) is *exact* if $\ker \rho_i = \operatorname{im} \rho_{i+1}$ for all $i \geq 0$ and, in addition, the preimages of the elements of Π_1 under ρ_1 are nothing but the orbits of the action of Π_3 on π_2.

The π-sequence (5.1.5.12) is *exact* and $i \geq 0$ is arbitrary, then obviously the homomorphism ρ_i is trivial if and only if ρ_{i+1} is an epimorphism, while $\ker \rho_i$ is trivial if and only if ρ_{i+1} is trivial. In general, when $i \geq 3$, $\ker \rho_i$ is trivial if and only ρ_i is injective, because ρ_i is a group homomorphism. Further, the triviality of $\ker \rho_2$ means that ρ_2 is injective: if $\rho_2(\alpha) = \rho_2(\beta)$, then

$$\rho_2(\alpha\beta^{-1}) = \rho_2(\alpha)\beta^{-1} = \rho_2(\beta)\beta^{-1} = \rho_2(\beta\beta^{-1}) = \rho_2(e_{P i_3})$$

[see condition (vi) in Definition 5.1.5.11], and hence $\alpha = \beta$. The triviality of $\ker \rho_0$ does not imply the injectivity of ρ_0, and this is also valid for $\ker \rho_1$ and ρ_1. However, in the case of an exact π-sequence (5.1.5.12), the injectivity of ρ_1 is guaranteed if the group is trivial, or if it acts identically on Π_2.

The above discussion makes clear that,

in the case of an exact π-sequence (5.1.5.12) and for $i \geq 1$, the triviality of ρ_i and ρ_{i+2} is equivalent to the invertibility of ρ_{i+1},

the triviality of Π_i and Π_{i+2} implies the triviality of Π_{i-1},

the triviality of Π_{i-1} and Π_{i+2} implies the invertibility of ρ_i (cf. Definition 5.1.5.2).

Theorem 5.1.5.15. *Let (5.1.5.12) be an exact π-sequence. If the action of Π_4 on Π_2 induced by the action of Π_3 via ρ_3 is identical, then $\operatorname{im} \rho_3$ is a normal subgroup of Π_3. The converse is true provided ρ_2 is an epimorphism.*

Proof. Assume that Π_4 acts identically on Π_2. If $\alpha \in \Pi_4$, $\beta \in \Pi_3$, then $\rho_2(\beta)\rho_3(\alpha) = \rho_2(\beta)$, and hence

$$\rho_2(\beta\rho_3(\alpha)\beta^{-1}) = \rho_2(\beta\rho_3(\alpha))\beta^{-1} = [\rho_2(\beta)\rho_3(\alpha)]\beta^{-1} = \rho_2(\beta)\beta^{-1} = \rho_2(\beta\beta^{-1}) = e_{\Pi_2}$$

[see condition (vi) in Definition vi]. Therefore, $\beta\rho_3(\alpha)\beta^{-1} \in \ker \rho_2 = \operatorname{im} \rho_3$, and hence $\operatorname{im} \rho_3$ is a normal subgroup of Π_3.

Now assume that ρ_2 is an epimorphism and that $\operatorname{im} \rho_3$ is a normal subgroup of Π_3. If $\alpha \in \Pi_4$ and $\gamma \in \Pi_2$, then there is $\beta \in \Pi_3$ such that $= \rho_2 = \gamma$, and since $\beta\rho_3(\alpha)\beta^{-1} \in \operatorname{im} \rho_3 = \ker \rho_2$, we have

$$\begin{aligned}\gamma\rho_3(\alpha) &= \rho_2(\beta)\rho_3(\alpha) = \rho_2(\beta\rho_3(\alpha)) = \rho_2(\beta\rho_3(\alpha)\beta^{-1}\beta) = e_{\Pi_2}\beta \\ &= \rho_2(e_{\Pi_3})\beta = \rho_2(\beta) = \gamma\end{aligned}$$

[see again (vi) in Definition vi]. Consequently, Π_4 acts identically on Π_2. □

Remark 5.1.5.16. Every sequence of *Abelian* groups and *group* homomorphisms of the form (5.1.5.12) can be viewed as a π-sequence, where the action of Π_3 on Π_{3k}, $k \geq 2$, and the actions of Π_4 on Π_{3k+1} and Π_{3k-1}, $k \ \geq 2$, are identical, while the action of Π_3 on Π_2 is given by $\omega\sigma = \omega\rho_2(\sigma)$ $[\omega \in \Pi_2,\, \sigma \in \Pi_3]$. Then it is readily seen that a sequence which is exact in the sense of Definition 5.1.5.2 is also exact as a π-sequence, and that the homomorphisms of sequences in the sense of Definition 5.1.5.1 are also homomorphisms in the sense of Definition 5.1.5.11.

Splitting of π–Sequences

Definition 5.1.5.17. We say that the π-sequence (5.1.5.12) is split from the right at the term Π_α by the homomorphism $\zeta\colon \Pi_{\alpha-1} \to \Pi_\alpha$ if $\rho_{\alpha-1} \circ \zeta = \mathrm{id}_{\Pi_{\alpha-1}}$. This splitting is normal if

$\alpha = 0, 1, 2$ or

$\alpha \geq 3$ and $\operatorname{im}\zeta$ is a normal subgroup of Π_α.

We say that the π-sequence (5.1.5.12) is split from the left at the term Π_α by the homomorphism $\zeta\colon \Pi_\alpha \to \Pi_{\alpha+1}$ if $\zeta \circ \rho_\alpha = \mathrm{id}_{\Pi_{\alpha+1}}$. ($\zeta$ is a group homomorphism when this makes sense, and a homomorphism of sets with identity elements otherwise.)

Theorem 5.1.5.18. *If $\alpha \geq 5$, then any right splitting of the π-sequence (5.1.5.12) at Π_α is normal. If (5.1.5.12) is exact and Π_4 acts identically on Π_5, then any right splitting at Π_4 is normal.*

Proof. Since for $\alpha > 5$ the groups Π_α are Abelian, we need consider a only Π_4 and Π_5. Suppose that the homomorphism $\zeta\colon \Pi_4 \to \Pi_5$ splits (5.1.5.12) from the right at Π_5. If $\alpha, \beta \in \Pi_5$, then

$$\begin{aligned}\beta\alpha\beta^{-1} &= [\beta\zeta(\rho_4(\beta^{-1}))][\zeta(\rho_4(\beta))\alpha\beta^{-1}]\\ &= [\zeta(\rho_4(\beta))\alpha\beta^{-1}][\beta\zeta(\rho_4(\beta^{-1}))] = \zeta(\rho_4(\beta))\alpha\zeta(\rho_4(\beta^{-1}));\end{aligned}$$

permuting the factors is permissible because $\beta\zeta(\rho_4(\beta^{-1})) \in \ker\rho_4$ (as shown by the equality $\rho_4 \circ \zeta = \mathrm{id}_{\Pi_4}$), and hence $\beta\zeta(\rho_4(\beta^{-1}))$ belongs to the centre of Π_5 (see Remark 5.1.5.13). This representation of $\beta\alpha\beta^{-1}$ shows that if $\alpha \in \operatorname{im}\zeta$, then $\beta\alpha\beta^{-1} \in \operatorname{im}\zeta$ for all $\beta \in \Pi_5$.

Now suppose that $\zeta\colon \Pi_3 \to \Pi_4$ splits the exact π-sequence (5.1.5.12) from the right at Pi_4, and that Π_4 acts identically on Π_5. If $\alpha, \beta \in \Pi_4$, then

$$\begin{aligned}\beta\alpha\beta^{-1} &= [\beta\zeta(\rho_3(\beta^{-1}))][\zeta(\rho_3(\beta))\alpha\beta^{-1}]\\ &= [\zeta(\rho_3(\beta))\alpha\beta^{-1}][\beta\zeta(\rho_3(\beta^{-1}))] = \zeta(\rho_3(\beta))\alpha\zeta(\rho_3(\beta^{-1}));\end{aligned}$$

permuting the factors is permissible because $\beta\zeta(\rho_3(\beta^{-1})) \in \ker\rho_4$ (as shown by the equality $\rho_3 \circ \zeta = \mathrm{id}_{\Pi_3}$), and hence $\beta\zeta(\rho_3(\beta^{-1}))$ belongs to the centre of Π_4. (see Remark 5.1.5.13). This representation of $\beta\alpha\beta^{-1}$ shows that if $\alpha \in \operatorname{im}\zeta$, then $\beta\alpha\beta^{-1} \in \operatorname{im}\zeta$ for all $\beta \in \Pi_4$. □

Remark 5.1.5.19. Let the π-sequence (5.1.5.12) be exact, normally split from right at Π_α and split from the right at $\Pi_{\alpha+3}$. If $\alpha \geq 4$, then, according to Theorem 5.1.5.6, sequence (5.1.5.12) also splits from the left at Π_α, and Π_α decomposes into the product of a subgroup canonically isomorphic to $\Pi_{\alpha+1}$ and a subgroup isomorphic to Π_α. When $\alpha = 1, 2, 3$, (5.1.5.12) also splits from the left at Π_α (to see this, repeat the arguments of Theorem 5.1.5.6), but obvious examples demonstrate that the above isomorphism $\Pi_\alpha \overset{\sim}{=} \Pi_{\alpha+1} \times \Pi_{\alpha-1}$ is not necessarily valid.

Now let the π-sequence (5.1.5.12 be exact and split from the left at Π_α and $\Pi_{\alpha-3}$. If $\alpha \geq 6$, then, according to Theorem 5.1.5.7, (5.1.5.12) is normally split from the right at Π_α, and Π_α decomposes into the product of a subgroup canonically isomorphic to $\Pi_\alpha + 1$ and a subgroup isomorphic to $\Pi_{\alpha-1}$. A word-for-word repetition of the arguments in Theorem 5.1.5.7 shows that this holds also for $\alpha = 4, 5$. If $\alpha = 3$, all we can say is that (5.1.5.12) splits from the right at Π_3.

In what follows, we shall often encounter π-sequences which are exact, and split at every third term. The discussion above shows that if the π-sequence (5.1.5.12) is exact and normally split from the right (split from the left) at every term Π_{i_0+3k} with $i_0 + 3k \geq 1$, then it also splits from the left at these terms (respectively, it also splits from the right at Π_{i_0+3k} with $i_0 + 3k \geq 3$, and splits normally from the right at Π_{i_0+3k} for $i_0 + 3k \geq 4$).

The π-Variant of the Five Lemma

Theorem 5.1.5.20. *Let* $\{\Pi_i, \rho_i\}_{i=0}^{\infty}$ *and* $\{\Pi'_i, \rho'_i\}_{i=0}^{\infty}$ *be exact* π*-sequences, and let* $\{h_i \colon \Pi_i \to \Pi'_i\}_{i=0}^{\infty}$ *be a homomorphism of the first sequence into the second. If* $h_{\alpha-1}$ *and* $h_{\alpha+1}$ *are isomorphisms and* $\ker h_{\alpha-2} = e_{\Pi_{\alpha-2}}$, $\operatorname{im} h_{\alpha+2} = \Pi'_{\alpha+2}$, *then* $\ker h_\alpha = e_{\Pi_\alpha}$ *and* $\operatorname{im} h_\alpha = \Pi'_\alpha$ *(and hence* h_α *is a group isomorphism for all* $\alpha \geq 3$*).*

Proof. If $\alpha \geq 5$, then this is contained in theorem 5.1.5.9. The proof for $\alpha = 2, 3, 4$ is similar. □

5.1.6 The Homotopy Sequence of a Pair

Definition 5.1.6.1. Let (X, A) be a topological pair with base point $x_0 \in A$. According to Subsections subsect:05-1-1 and 5.1.4, the homotopy groups $\pi_r(X, x_0)$ and
$pi_r(A, x_0)$ are defined for any $r \geq 0$, whereas the homotopy groups $\pi_r(X, A, x_0)$ are defined for any $r \geq 1$. Moreover, by Definition 5.1.4.3, there are the homomorphisms $\partial \colon \pi_r(X, A, x_0) \to \pi_{r-1}(A, x_0)$. To these we add the homomorphisms $\operatorname{incl}_* \colon \pi_r(A, x_0) \to \pi_r(X, x_0)$ and $\operatorname{rel}_* \colon \pi_r(X, x_0) \to \pi_r(X, A, x_0)$, induced by the inclusions $\operatorname{incl} \colon A \to X$ and $\operatorname{rel} \colon (X, x_0, x_0) \to (X, A, x_0)$. These three series of homotopy groups and three series of homomorphisms can be

assembled into the left-infinite sequence

$$\begin{aligned}\cdots \xrightarrow{\partial} \pi_2(A,x_0) \xrightarrow{\mathrm{incl}_*} \pi_2(X,x_0) \xrightarrow{\mathrm{rel}_*} \pi_2(X,A,x_0) \xrightarrow{\partial} \pi_1(A,x_0) \xrightarrow{\mathrm{incl}_*} \\ \pi_1(X,x_0) \xrightarrow{\mathrm{rel}_*} \pi_1(X,A,x_0) \xrightarrow{\partial} \pi_0(A,x_0) \xrightarrow{\mathrm{incl}_*} \pi_0(X,x_0)\end{aligned} \tag{5.1.6.2}$$

Here, all the terms, except for the last six, are Abelian groups, all the terms, except for the last three, are groups, the last three terms are sets with an identity, all the maps, except for the last three, are group homomorphisms, and the last three maps are homomorphisms of sets with identity. By Definitions 5.1.3.3 and 5.1.4.4, $\pi_1(X,x_0)$ acts from the right on $\pi_r(X,x_0)$, while $\pi_1(A,x_0)$ acts from the right on $\pi_r(A,x_0)$ and $\pi_r(X,A,x_0)$, and all these are group-actions. Furthermore, the homomorphisms incl_*, rel_*, and ∂ are compatible with these actions, as required in Definition 5.1.5.11 (see Remarks 5.1.3.6, 5.1.4.6, and Theorem 5.1.3.4), and $\alpha^{-1}\beta\alpha = T_{\partial\alpha}\beta$ for all $\alpha,\beta \in \pi_2(X,A,x_0)$ (see Theorem 5.1.4.9). From Definition 5.1.4.10 and Theorem 5.1.4.11 it follows that $\pi_1(X,x_0)$ acts from the right on $\pi_1(X,A,x_0)$ in such a manner that $\mathrm{rel}_*(\omega)\sigma = \mathrm{rel}_*(\omega\sigma)$ for all $\omega,\sigma \in \pi_1(X,x_0)$. Therefore, (5.1.6.2) is a π-sequence, called the *homotopy sequence of the pair* (X,A) *with base point* x_0.

Theorem 5.1.6.3. *Sequence (5.1.6.2) is exact.*

Proof. The proof is a routine, direct verification of the six inclusions $\operatorname{im}\mathrm{incl}_* \subset \ker\mathrm{rel}_*$, $\ker\mathrm{rel}_* \subset \operatorname{im}\mathrm{incl}_*$, $\operatorname{im}\mathrm{rel}_* \subset \ker\partial$, $\ker\partial \subset \operatorname{im}\mathrm{rel}_*$, $\operatorname{im}\partial \subset \ker\mathrm{incl}_*$, and $\ker\mathrm{incl}_* \subset \operatorname{im}\partial$, plus a justification of the fact that, given $\alpha,\beta \in \pi_r(X,A,x_0)$, there is a $\alpha \in \pi_1(X,x_0)$ such that $\alpha\sigma = \beta$ if and only if $\partial\alpha = \partial\beta$. □

Theorem 5.1.6.4. *Given any path* $s\colon I \to A$, *the vertical isomorphisms*

$$\begin{tikzcd}
{} \arrow[r] & \pi_r(A,s(0)) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "T_s"] & \pi_r(X,s(0)) \arrow[r, "\mathrm{rel}_*"] \arrow[d, "T_{\mathrm{incl}\circ s}"] & \pi_r(X,A,s(0)) \arrow[r, "\partial"] \arrow[d, "T_s"] & \pi_{r-1}(A,s(0)) \arrow[r] \arrow[d, "T_s"] & {} \\
{} \arrow[r] & \pi_r(A,s(1)) \arrow[r, "\mathrm{incl}_*"'] & \pi_r(X,s(1)) \arrow[r, "\mathrm{rel}_*"'] & \pi_r(X,A,s(1)) \arrow[r, "\partial"'] & \pi_{r-1}(A,s(0)) \arrow[r] & {}
\end{tikzcd}$$

define an isomorphism of the first sequence onto the second.

Proof. The commutativity of the first square has been established in Remark 5.1.3.6, while the commutativity of the second and the third in Remark 5.1.4.6. From the fact that the local systems $(A,\{\pi_r(A,x)\},\{T_s\})$, $(X,\{\pi_r(,x)\},\{T_s\})$, and $(A,\{\pi_r(X,A,x)\},\{T_s\})$ satisfy property in Definition 5.1.2.1 (i), and from the equality $T_s(\omega\sigma) = T_s(\omega)T_{\mathrm{incl}\circ s}(\sigma)$ $[\sigma \in \pi_1(X,s(0)),\ \omega \in \pi_1(X,A,s(0))]$, deduced in Definition 5.1.4.10, it follows that the vertical homomorphisms are compatible with the actions of the fundamental groups. □

Theorem 5.1.6.5. *Given a continuous map $f\colon (X, A, x_0) \to (X', A', x_0')$, the vertical homomorphisms*

$$\begin{tikzcd}
{} \arrow[r] & \pi_r(A, x_(0)) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "(\mathrm{abr}\, f)_*"] & \pi_r(X, x_(0)) \arrow[r, "\mathrm{rel}_*"] \arrow[d, "f_*"] & \pi_r(X, A, x_(0)) \arrow[r, "\partial"] \arrow[d, "f_*"] & \pi_{r-1}(A, x_(0)) \arrow[r] \arrow[d, "(\mathrm{abr}\, f)_*"] & {} \\
{} \arrow[r] & \pi_r(A', x_(0)') \arrow[r, "\mathrm{incl}_*"'] & \pi_r(X', x_(0)') \arrow[r, "\mathrm{rel}_*"'] & \pi_r(X', A', x_(0)') \arrow[r, "\partial"'] & \pi_{r-1}(A', x_(0)') \arrow[r] & {}
\end{tikzcd} \tag{5.1.6.6}$$

induced by f yield a homomorphism of the first sequence into the second.

Proof. The commutativity of the first two squares follows from Theorem 5.1.1.13 and Definition 5.1.4.2. The commutativity of the third square has been established in Definition 5.1.4.3, and the compatibility of the vertical homomorphisms with the actions of the fundamental groups - in Remarks 5.1.3.6, 5.1.4.6, and Definition 5.1.4.10. ☐

The Most Important Special Cases

Remark 5.1.6.7. If X is ∞-connected, then all the homomorphisms

$$\partial\colon \pi_r(X, A, x_0) \to \pi_{r-1}(A, x_0)$$

are isomorphisms. If X is k-connected and $k < \infty$, then

$$\partial\colon \pi_r(X, A, x_0) \to \pi_{r-1}(A, x_0)$$

is an isomorphism for all $r \leq k$, while

$$\partial\colon \pi_{k+1}(X, A, x_0) \to \pi_k(A, x_0)$$

is an epimorphism. The converse of both statements is true provided that X is connected.

If A is ∞-connected, then all the homomorphisms

$$\mathrm{rel}_*\colon (X, x_0) \to \pi_r(X, A, x_0)$$

are isomorphisms. If A is k-connected, and $k < \infty$ then

$$\mathrm{rel}_*\colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(X, A, x_0)$$

is an isomorphism for all $r \leq k$, while

$$\mathrm{rel}_*\colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(X, A, x_0)$$

is an epimorphism. If one of the spaces X, A is connected, then again, the converse of both statements is true.

If the pair (X, A) is ∞-connected, then all the homomorphisms

$$\mathrm{incl}_*\colon \pi_r(A, x_0) \to \pi_r(X, x_0)$$

are isomorphisms. If (X, A) is k-connected and $k < \infty$, then

$$\mathrm{incl}_*\colon \pi_r(A, x_0) \to \pi_r(X, x_0)$$

is an isomorphism for all $r < k$, while

$$\mathrm{incl}_*\colon \pi_k(A, x_0) \to \pi_k(X, x_0)$$

is an epimorphism. The converse is true in both cases (with no supplementary conditions). In particular, if incl: $A \to X$ is a homotopy equivalence, then the pair (X, A) is ∞-connected (see Theorem 5.1.3.7 and cf. Remark 1.3.3.9).

Theorem 5.1.6.8. *If A is a retract of X, then the sequence (5.1.6.2) splits from the left at $\pi_r(X, x_0)$, and any retraction $\rho\colon X \to A$ induces splitting homomorphisms $\rho_*\colon \pi_r(X, x_0) \to \pi_r(A, x_0)$.*

Proof. Since $\rho \circ \mathrm{incl} = \mathrm{id}_A$, $\rho_* \circ \mathrm{incl}_* = \mathrm{id}_{\pi_r(A,x_0)}$. □

Theorem 5.1.6.9. *Suppose that (X, x_0) can be contracted to (A, x_0), i.e., id_X is x_0-homotopic to some map $h\colon X \to X$ such that $h(X) \subset A$. Then the sequence (5.1.6.2) splits from the right at $\pi_r(A, x_0)$, and as splitting homomorphisms one may take $(\mathrm{abr}\, h)_*\colon \pi_r(X, x_00 \to \pi_r(A, x_0)$.*

Proof. The composition $\mathrm{incl} \circ \mathrm{abr}\, h$ is x_0-homotopic to id_X , and hence

$$\mathrm{incl}_* \circ (\mathrm{abr}\, h)_* = \mathrm{id}_{\pi_r(X,x_0)}$$

□

Corollary 5.1.6.10. *Suppose that (A, x_0) is contractible in (X, x_0), i.e., the inclusion $A \hookrightarrow X$ is x_0-homotopic to the constant map. Then the sequence (5.1.6.2) splits from the right at $\pi_r(X, A, x_0)$. As splitting homomorphisms $\pi_r(A, x_0) \to \pi_{r+1}(X, A, x_0)$ one may take those induced by the maps, given by*

$\gamma_r\colon \mathrm{Sph}_r(A, x_0) \to \mathrm{Sph}_{r+1}(X, A, x_0)$, $\quad [\gamma_r(\varphi)](t_1, \ldots, t_r) = h(\varphi(t_1, \ldots, t_r), t_{r+1})$,
$[\varphi \in \mathrm{Sph}_r(A, x_0)]$,

where $h\colon A \times I \to X$ is any homotopy from incl: $A \hookrightarrow X$ *to the constant map.*

Proof. This is a corollary of the obvious equality $\partial \circ \gamma_r = \mathrm{id}_{\mathrm{Sph}_r(A,x_0)}$. □

Remark 5.1.6.11. The following remarks are concerned not with the homotopy sequences of pairs themselves, but with the homomorphism (5.1.6.6) induced by a map $f\colon (X, A, x_0) \to (X', A', x_0')$ between pairs with base point.

The π-variant of the Five Lemma (see Theorem 5.1.5.20) shows that:

- if

$$(\mathrm{abr}\, f)_*\colon \pi_r(A, x_0) \to \pi_r(A', x_0'), \quad \forall r \geq 0,$$
$$f_*\colon \pi_r(X, A, x_0) \to \pi_r(X', A', x_0'), \quad \forall r \geq 1,$$

  are isomorphisms, then so are $f_*\colon \pi_r(X, x_0) \to \pi_r(X', x_0')$, for all $r \geq 1$;

- if

$$f_*\colon \pi_r(X,x_0)\to\pi_r(X',x_0'),\quad \forall r\geq 1,$$
$$f_*\colon \pi_r(X,A,x_0)\to\pi_r(X',A',x_0'),\quad \forall r\geq 1,$$

  are isomorphisms, then so are $(\operatorname{abr} f)_*\colon \pi_r(A,x_0)\to\pi_r(A',x_0'),\quad \forall r\geq 1$;

- if

$$f_*\colon \pi_r(X,x_0)\to\pi_r(X',x_0'),\quad \forall r\geq 0,$$
$$(\operatorname{abr} f)_*\colon \pi_r(A,x_0)\to\pi_r(A',x_0'),\quad \forall r\geq 0,$$

  are isomorphisms, then
  - so are $f_*\colon \pi_r(X,A,x_0)\to\pi_r(X',A',x_0')$, for all $r\geq 2$,
  - while $f_*\colon \pi_r(X,A,x_0)\to->\pi_r(X',A',x_0')$ is an epimorphism with trivial kernel.

In the last case, $f_*\colon \pi_1(X,A,x_0)\to\pi_1(X',A',x_0')$ is not necessarily injective; see Example 5.3.8.9. However, this map is certainly injective (and hence, an isomorphism) if we assume, in addition, that all the homomorphisms

$$(\operatorname{abr} f)_*\colon \pi_r(A,x)\to\pi_r(A',f(x)),\quad x\in A,$$

are epimorphic. To see this, let ω_1,ω_2 with $f_*(\omega_1)=f_*(\omega_2)$, and let w_1 and w_2 be spheroids in the classes ω_1 and ω_2. Then there is a path $s'\colon I\to A'$ such that $s'(0)=f\circ w_1(1)$, $s'(1)=f\circ w_2(1)$, and the loop

$$((f\circ w_1)([\text{incl}\colon A'\to X']\circ s'))(f\circ w_2)^{-1} \tag{5.1.6.12}$$

is homotopic to the constant loop. Since $(\operatorname{abr} f)_*\colon \pi_1(A,x_0)\to\pi_1(A',x_0')$ is an isomorphism and $(\operatorname{abr} f)_*\colon \pi_1(A,w_1)\to\pi_1(A',f(w_1'))$ is an epimorphism, there is a path $s\colon I\to A$ such that $s(0)=w_1(1)$, $s(1)=w_2(1)$, and the path $\operatorname{abr} f\circ s\colon I\to A'$ is homotopic to s'. For such a choice of s, f takes the loop

$$(w_1([\text{incl}\colon A\to X]\circ s))w_2^{-1} \tag{5.1.6.13}$$

into a loop homotopic to (5.1.6.12), and therefore homotopic to the constant loop. Finally, from the fact that $f_*\colon \pi_1(X,x_0)\to\pi_1(X',x_0')$ is an isomorphism, it follows that (5.1.6.13) itself is homotopic to the constant loop, i.e., the spheroids w_1 and w_2 are homotopic, and $\omega_1=\omega_2$.

The Homotopy Sequence of a Triple

Definition 5.1.6.14. Let (X,A,B) be a topological triple with base point $x_0\in B$. According to Subsection 5.1.4, when $r\geq 1$ the homotopy groups

$$\pi_r(X,A,x_0),\quad \pi_r(X,B,x_0),\quad \pi_r(A,B,x_0)$$

and the homomorphisms

$$\mathrm{incl}_*\colon \pi_r(A,B,x_0)\to\pi_r(X,B,x_0),\quad \mathrm{rel}_*\colon \pi_r(X,B,x_0)\to\pi_r(X,A,x_0),$$

induced by the inclusions incl: $(A,B)\to(X,B)$ and rel: $(X,B)\to(X,A)$, are well defined. If $r\geq 2$, we define an additional homomorphism,

$$\partial\colon (X,A,x_0)\to\pi_{r-1}(A,B,x_0),$$

as the composition of the boundary homomorphism $\pi_r(X,A,x_0)\to\pi_{r-1}(A,x_0)$ and the homomorphism $\pi_{r-1}(A,x_0)\to\pi_{r-1}(A,B,x_0)$, induced by the inclusion $(A,x_0,x_0)\to(A,B,x_0)$.

Now we may assemble these three series of groups and three series of homomorphisms into a left-infinite sequence

$$\begin{aligned}\cdots\xrightarrow{\partial}\pi_2(A,B,x_0)\xrightarrow{\mathrm{incl}_*}\pi_2(X,B,x_0)\xrightarrow{\mathrm{rel}_*}\pi_2(X,A,x_0)\\ \xrightarrow{\partial}\pi_1(A,B,x_0)\xrightarrow{\mathrm{incl}_*}\pi_1(X,B,x_0)\xrightarrow{\mathrm{rel}_*}\pi_1(X,A,x_0)\end{aligned}\tag{5.1.6.15}$$

As was (5.1.6.2), (5.1.6.15) is a π-sequence:

the right group-actions of $\pi_2(X,A,x_0)$ on the groups $\pi_r(X,A,x_0)$, and the right group-actions of $\pi_2(X,B,x_0)$ on the groups $\pi_r(A,B,x_0)$ and $\pi_r(X,B,x_0)$, are induced by the actions of $\pi_1(A,x_0)$ and $\pi_1(B,x_0)$ via the homomorphisms $\partial\colon pi_2(X,A,x_0)\to\pi_1(A,x_0)$ and $\partial\colon pi_2(X,B,x_0)\to\pi_1(B,x_0)$;

similarly, the right action of $\pi_2(X,A,x_0)$ on $\pi_1(A,B,x_0)$ is induced by the action of $\pi_1(A,x_0)$ via the homomorphism $\partial\colon\pi_2(X,A,x_0)\to\pi_1(A,x_0)$;

finally, Definition 5.1.4.4, Remark 5.1.4.6, Theorem 5.1.4.9, and Definition 5.1.4.10 show that the conditions imposed by Definition 5.1.5.11 are satisfied.

π-Sequence (5.1.6.15) is called the *homotopy sequence of the triple* (X,A,B) *with base point* x_0.

Sequence (5.1.6.15) is exact; cf. Theorem 5.1.6.3.

Given any path $s\colon I\to B$, the translations

$$\begin{aligned}&\pi_r(X,A,s(0))\to\pi_r(X,A,s(1)),\quad \pi_r(X,B,s(0))\to\pi_r(X,B,s(1)),\\ &\pi_r(A,B,s(0))\to\pi_r(A,B,s(1))\end{aligned}$$

define an isomorphism of the homotopy sequence of the triple (X,A,B) with base point $s(0)$ into the homotopy sequence of the triple (X,A,B) with base point $s(1)$; cf. Theorem 5.1.6.4.

Given any continuous map f from a triple (X,A,B) with base point x_0 into a triple (X',A',B') with base point x'_0 $(x\in B, x'_0\in B')$, the homomorphisms

$$\begin{aligned}&f_*\colon\pi_r(X,A,x_0)\to\pi_r(X',A',x'_0),\quad f_*\colon\pi_r(X,B,x_0)\to\pi_r(X',B',x'_0),\\ &(\mathrm{abr}\,f)_*\colon\pi_r(A,B,x_0)\to\pi_r(A',B',x'_0),\end{aligned}$$

constitute a homomorphism from the homotopy sequence of the first triple into the homotopy sequence of the second triple; cf. Theorem 5.1.6.5.

5.1.7 The Local System of Homotopy Groups of the Fibres of a Serre Bundle

Definition 5.1.7.1. Suppose that ξ is a Serre bundle, F_0 and F_1 are fibres of ξ, and $x_0 \in F_0$, $x_1 \in F_1$. Two spheroids, $\varphi_0 \in \operatorname{Sph}_r(F_0, x_0)$ and $\varphi_1 \in \operatorname{Sph}_r(F_1, x_0)$, are said to be *fibre homotopic* if the spheroids

$$[\text{incl}\colon F_0 \to \operatorname{tl}(\xi)]\circ\varphi_0 \in \operatorname{Sph}_r(\operatorname{tl}(\xi), x_0), \quad [\text{incl}\colon F_1 \to \operatorname{tl}(\xi)]\circ\varphi_1 \in \operatorname{Sph}_r(\operatorname{tl}(\xi), x_1)$$

can be connected by a free homotopy consisting of spheroids of $\operatorname{tl}(\xi)$ which take I^r into fibres of ξ. In other words, φ_0 and φ_1 are fibre homotopic if there is a map $h\colon I^r \times I \to \operatorname{tl}(\xi)$ such that: h is constant on each set $\operatorname{Fr} I^r \times t$, $t \in I$, $h(y, 0) = \varphi_0$, $h(y, 1) = \varphi_1$, $y \in I^r$, and the map $\operatorname{proj}(\xi) \circ h$ is constant on each set $I^r \times t$, $t \in I$. We say that h is a *fibre homotopy from φ_0 to φ_1 along the path* $t \mapsto h(\operatorname{Fr} I^r \times t)$.

Theorem 5.1.7.2. *Given any spheroid φ with origin of the fibre F_0, there is a fibre homotopy of φ along any path with origin x_0 in $\operatorname{tl}(\xi)$. Fibre homotopies of homotopic spheroids along homotopic paths of $\operatorname{tl}(\xi)$, always lead to homotopic spheroids. Fibre homotopies of freely homotopic spheroids along paths which cover homotopic paths of $\operatorname{bs}(\xi)$ lead to freely homotopic spheroids.*

Proof. Let $s\colon I \to \operatorname{tl}(\xi)$ be a path with $s(0) = x_0$, and let $\varphi \in \operatorname{Sph}_r(F_0, x_0)$. Define homotopies

$$\begin{aligned} &H\colon I^r \times I \to \operatorname{bs}(\xi), \quad H(y,t) = \operatorname{proj}(\xi) \circ s(t), \\ &G\colon I^r \times I \to \operatorname{tl}(\xi), \quad G(y,t) = s(t). \end{aligned}$$

Since

$$H(y,0) = \operatorname{proj}(\xi)(\varphi(y)),\ \forall y \in I^r, \quad G(y,0) = \varphi(y)\ \forall y \in \operatorname{Fr} I^r,$$

there is a homotopy, $\widetilde{H}\colon I^r \times I \to \operatorname{tl}(\xi)$, which covers H and satisfies

$$\widetilde{H}(y,t) = G(y,t),\ \forall y \in \operatorname{Fr} I^r, \quad \widetilde{H}(y,0) = \varphi(y),\ \forall y \in I^r$$

(see Theorem 4.1.3.6). $\widetilde{H}$ is manifestly a fibre homotopy of φ along s.

To prove the second assertion, it suffices to show that two spheroids, $\varphi, \psi \in \operatorname{Sph}_r(F_0, x_0)$, which are fibre homotopic along a loop $s\colon I \to \operatorname{tl}(\xi)$ homotopic with the constant loop, are homotopic in the usual sense. Choose a fibre homotopy $\varphi : I \times I \to \operatorname{tl}(\xi)$, from φ to ψ along s, and a homotopy $h\colon I \times I \to \operatorname{tl}(\xi)$, from s to the constant loop. Now define

$$\begin{aligned} &\widetilde{f}\colon I^{r+1} \to \operatorname{tl}(\xi), \quad (t_1, \ldots, t_{r+1}) \mapsto \varphi((t_1, \ldots, t_r), t_{r+1}), \\ &H\colon I^{r+1} \times I \to \operatorname{bs}(\xi), \quad ((t_1, \ldots, t_r), t) \mapsto \operatorname{proj}(\xi) \circ h(t_{r+1}, t), \\ &G\colon \operatorname{Fr} I^{r+1} \times I \to \operatorname{tl}(\xi), \\ &((t_1, \ldots, t_{r+1}), t) \mapsto \begin{cases} \varphi(t_1, \ldots, t_{r+1}), t), & \text{if} \quad t_{r+1} = 0, \\ \psi(t_1, \ldots, t_{r+1}), t), & \text{if} \quad t_{r+1} = 0, \\ h(t_{r+1}, t), & \text{if} (t_1, \ldots, t_{r+1}) \in \operatorname{Fr} I^r. \end{cases} \end{aligned}$$

Since

$$\begin{aligned}&H(y,0)=\operatorname{proj}(\xi)(\widetilde{f}(y)), \quad y\in I^{r+1},\\&G(y,0)=\widetilde{f}(y), \quad y\in \operatorname{Fr} I^{r+1},\end{aligned}$$

there is a homotopy $\widetilde{H}\colon I^{r+1}\times I\to \operatorname{tl}(\xi)$, which covers H and satisfies

$$\begin{aligned}&\widetilde{H}(y,t)=G(y,t), \quad y\in \operatorname{Fr} I^{r+1}\\&\widetilde{H}(y,0)=\widetilde{f}(y), \quad y\in I^{r+1}.\end{aligned}$$

Now let $\psi((t_1,\dots,t_r),t)=\widetilde{H}((t_1,\dots,t_r),t),1)$ and note that $\Psi\colon I^r\times ItimesI\to F_0$ is a (usual) homotopy from φ to ψ.

Let us prove the last part of the theorem. Suppose that the spheroids $\varphi_0\in \operatorname{Sph}_r(F_0,x_0)$ and $\varphi_1\in \operatorname{Sph}_r(F_1,x_1)$ are fibre homotopic along the path $u\colon I\to \operatorname{tl}(\xi)$, and that the same holds for the spheroids $\psi_0\in \operatorname{Sph}_r(F_0,x_0)$ and $\psi_1\in \operatorname{Sph}_r(F_1,x_1)$ and the path $v\colon I\to \operatorname{tl}(\xi)$. Further, suppose that the paths $\operatorname{proj}(\xi)\circ u, \operatorname{proj}(\xi)\circ v\colon I\to \operatorname{bs}(\xi)$, are homotopic, and that φ_0 and ψ_0 are freely homotopic along a path $w\colon I\to F_0$. The last means that there is a fibre homotopy from φ_0 to ψ_0 along the path $w_0=[\text{incl}\colon F_0\to \operatorname{tl}(\xi)]\circ w$. It is clear that the loop $\operatorname{proj}(\xi)\circ(u^{-1}(w_0v))\colon I\to \operatorname{bs}(\xi)$ is homotopic to the constant loop, which in turn implies that the path $u^{-1}(w_0v)$ is homotopic to some path $w_1\colon I\to \operatorname{tl}(\xi)$ with $w_1(I)\subset F_1$ (see Theorem 4.1.3.6). By the first part of the theorem, there exists a fibre homotopy of φ_1 along w_1, and now the second part guarantees that this homotopy yields a spheroid which is homotopic to ψ_1 [φ_1 is fibre homotopic to ψ_1 along the path $u^{-1}(w_0v)$]. Consequently, φ_1 is fibre homotopic to ψ_1 along a path in F_1, i.e., the spheroids φ_1 and ψ_1 are freely homotopic. □

Definition 5.1.7.3. By Theorem 5.1.7.2, the fibre homotopies along a given path $s\colon I\to \operatorname{tl}(\xi)$ define (for any $r\geq 0$) a map

$$\begin{aligned}&T_s\colon \pi_r(F_0,s(0))\to \pi_r(F_1,s(1)),\\&\qquad F_0=(\operatorname{proj}(\xi))^{-1}(\operatorname{proj}(\xi)(s(0))),\ F_1=(\operatorname{proj}(\xi))^{-1}(\operatorname{proj}(\xi)(s(1))).\end{aligned}$$

The maps T_s are obviously homomorphisms and satisfy conditions (i)-(iii) in Definition 5.1.2.1. Therefore, a local system

$$(\operatorname{tl}(\xi),\{\pi_r(\operatorname{proj}(\xi))^{-1}(\operatorname{proj}(\xi)(x),x)\},\{T_s\})$$

arises on $\operatorname{tl}(\xi)$. This is a local system of groups for $r\geq 1$, and a local system of sets with identity for $r=0$. It is called the *upper local system of the r-th homotopy groups of the fibres of ξ*. In particular, given any $x\in \operatorname{tl}(\xi)$ and $r\geq 1$, there is a natural right group-action of $\pi_1(\operatorname{tl}(\xi),x)$ on $\pi_r(\operatorname{proj}(\xi))^{-1}(\operatorname{proj}(\xi)(x),x)$.

Clearly, by restricting this local system to any fibre of ξ we obtain the local system of the r-th homotopy groups of the given fibre. Moreover, the homomorphisms

$$\text{incl}_*\colon \pi_r(\operatorname{proj}(\xi))^{-1}(\operatorname{proj}(\xi)(x),x)\to \pi_r(\operatorname{tl}(\xi),x)$$

combine with $\mathrm{id}_{\mathrm{tl}(\xi)}$ to define a homomorphism of the upper local system into the local system of the r-th homotopy groups of $\mathrm{tl}(\xi)$.

Definition 5.1.7.4. Suppose now that the fibres of ξ are r-simple. Then, given any point $bin\,\mathrm{bs}(\xi)$, all the homotopy groups

$$\pi_r((\mathrm{proj}(\xi))^{-1}(b), x), \quad x \in (\mathrm{proj}(\xi))^{-1}(b),$$

may be identified with a unique group, $\pi_r((\mathrm{proj}(\xi))^{-1}(b)$ (see Definition 5.1.3.5). In this case, for any path $s\colon I \to \mathrm{bs}(\xi)$, we can define

$$T_s\colon \pi_r((\mathrm{proj}(\xi))^{-1}(s(0))) \to \pi_r((\mathrm{proj}(\xi))^{-1}(s(1)))$$

to be the translation

$$\widetilde{T}_s\colon \pi_r((\mathrm{proj}(\xi))^{-1}(s(0)), \widetilde{s}(0)) \to \pi_r((\mathrm{proj}(\xi))^{-1}(s(1)), \widetilde{s}(1))$$

along some path $\widetilde{s}\colon I \to \mathrm{tl})\xi)$, which covers s; from Theorem 5.1.7.2 it follows that T_s does not depend upon the choice of s. The maps T_s are obviously homomorphisms and satisfy properties (i)-(iii) in Definition 5.1.2.1. Therefore, a local system

$$(\mathrm{bs}(\xi), \{\pi_r((\mathrm{proj}(\xi))^{-1}(b))\}, \{T_s\})$$

arises on $\mathrm{bs}(\xi)$, which consists of groups (sets with identity) for any $r \geq 1$ (respectively, for $r = 0$). This is called the *lower local system of the r-th homotopy groups of the fibres of ξ*. In particular, for any $r \geq 1$ and any $b \in \mathrm{bs}(\xi)$, there is a natural right group-action of $\pi_1(\mathrm{bs}(\xi), b)$ on $\pi_r((\mathrm{proj}(\xi))^{-1}(b))$.

It is readily seen that the lower local system induces the upper local system, defined in Definition 5.1.7.3 on $\mathrm{tl}(\xi)$, via the projection $\mathrm{proj}(\xi)$.

Remark 5.1.7.5. Let $\varphi\colon \xi \to \xi_1$ be a map of Serre bundles. Then $\mathrm{tl}(\varphi)$ and the homomorphisms

$$\begin{aligned}\mathrm{abr}(\mathrm{tl}(\varphi))_*\colon \pi_r((\mathrm{proj}(\xi))^{-1}(\mathrm{proj}(\xi)(x), x) \to \\ \pi_r((\mathrm{proj}(\xi))^{-1}(\mathrm{proj}(\xi_1)(\mathrm{tl}(\varphi)(x)), \mathrm{tl}(\varphi)(x)) \quad [x \in \mathrm{tl}(\xi)]\end{aligned}$$

combine to define a homomorphism of the upper local system of the r-th homotopy groups of the fibres of ξ into the similar system for ξ_1. Furthermore, if the fibres of ξ and ξ_1 are r-simple, then $\mathrm{bs}(\varphi)$ and

$$\pi_r((\mathrm{proj}(\xi))^{-1}(b)) \to pi_r((\mathrm{proj}(\xi))^{-1}(\mathrm{bs}(\varphi)(b))) \quad [b \in \ \mathrm{bs}(\xi)]$$

combine to define a homomorphism of the lower local system of the r-th homotopy groups of the fibres of ξ into the similar system for ξ_1.

5.1.8 The Homotopy Sequence of a Serre Bundle

Lemma 5.1.8.1. *Let ξ be a Serre bundle with a base point $x_0 \in \mathrm{tl}(\xi)$, and let B be a subset of $\mathrm{bs}(\xi)$ with $b_0 = \mathrm{proj}(\xi)(x_0) \in B$. Then*

$$(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi), (\mathrm{proj}(\xi))^{-1}(B), x_0) \to \pi_r(\mathrm{bs}(\xi), B, x_0)$$

and, in particular,

$$(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi), (\mathrm{proj}(\xi))^{-1}(b_0), b_0) \to \pi_r(\mathrm{bs}(\xi), b_0)$$

are isomorphisms for any $r \geq 1$.

Proof. $(\mathrm{proj}(\xi))_*$ *is epimorphic.*
Let $\varphi \in \mathrm{Sph}_r(\mathrm{bs}(\xi), B, b_0)$. Define

$$\begin{aligned}
&\widetilde{f}\colon I^{r-1} \to \mathrm{tl}(\xi), \quad \widetilde{f}(I^{r-1}) = x_0\\
&H\colon I^{r-1} \times I \to \mathrm{bs}(\xi), \quad H((t_1, \dots, t_{r-1}), t) = \varphi(t_1, \dots, 1-t),\\
&G\colon \mathrm{Fr}\, I^{r-1} \times I \to \mathrm{tl}(\xi), \quad G(\mathrm{Fr}\, I^{r-1} \times I) = x_0.
\end{aligned}$$

Since $H(y,0) = (\mathrm{proj}(\xi))(\widetilde{f}(y))$ for $y \in I^{r-1}$, and $G(y,0) = \widetilde{f}(y)$ for $y \in \mathrm{Fr}\, I^{r-1}$, there is a homotopy $\widetilde{H}\colon I^{r-1} \times I \to \mathrm{tl}(\xi)$, which covers H and equals G on $\mathrm{Fr}\, I^{r-1} \times I$ (see Theorem 4.1.3.6). Now the formula

$$\psi(t_1, \dots, t_r) = \widetilde{H}((t_1, \dots, t_{r-1}), 1 - t_r)$$

defines a spheroid

$$\psi \in \mathrm{Sph}_r(\mathrm{tl}(\xi), (\mathrm{proj}(\xi))^{-1}(B), x_0)$$

such that $(\mathrm{proj}(\xi))_\#(\psi) = \varphi$.
$(\mathrm{proj}(\xi))_*$ *is monomorphic.*
Let

$$\psi \in \mathrm{Sph}_r(\mathrm{tl}(\xi), (\mathrm{proj}(\xi))^{-1}(B), x_0),$$

and suppose that the spheroid

$$(\mathrm{proj}(\xi))_\#(\psi) \in \mathrm{Sph}_r(\mathrm{bs}(\xi), B, b_0)$$

is homotopic to the constant spheroid. Choose a homotopy $\varphi\colon I^r \times I \to \mathrm{bs}(\xi)$ from $(\mathrm{proj}(\xi))_\#(\psi)$ to the constant spheroid, and define

$$\begin{aligned}
&\widetilde{f}\colon I^r \to \mathrm{tl}(\xi), \quad \widetilde{f}(I^r) = x_0,\\
&H\colon I^r \times I \to \mathrm{bs}(\xi), \quad H((t_1, \dots., t_r), t) = \varphi((t_1, \dots., t_{r-1}, 1-t), t_r),\\
&G\colon \mathrm{Fr}\, I^r \times I \to \mathrm{tl}(\xi),\\
&G((t_1, \dots., t_r), t) = \begin{cases} \psi(t_1, \dots., t_{r-1}, 1-t), & \text{if } t_r = 0,\\ x_0, & \text{if } (t_1, \dots., t_r) \in \mathrm{Fr}\, I^r,\ t_r \neq 0. \end{cases}
\end{aligned}$$

Since $H(y,0) = (\mathrm{proj}(\xi))(\widetilde{f}(y))$ for $y \in I^r$, and $G(y,0) = \widetilde{f}(y)$ for $y \in \mathrm{Fr}\, I^r$, there exists a homotopy $\widetilde{H}\colon I^r \times I \to \mathrm{tl}(\xi)$ which covers H and equals G on $\mathrm{Fr}\, I^r \times I$. Now it is plain that

$$\psi(t_1, \dots, t_r, t) = \widetilde{H}((t_1, \dots, t_{r-1}, t), 1 - t_r)$$

defines a homotopy $\psi\colon I^r \times I \to \mathrm{tl}(\xi)$ from ψ to the constant spheroid. □

The Action of $\pi_1(\mathrm{bs}(\xi), b_0)$ on $\mathrm{comp}(F_0)$

Remark 5.1.8.2. Let ξ be a Serre bundle with base point $b_0 \in \mathrm{bs}(\xi)$. Set $F_0 = (\mathrm{proj}(\xi))^{-1}(b_0)$ and define a right action of the group $\pi_1((\xi), b_0)$ on $\mathrm{comp}(F_0)$ as follows: for $C \in \mathrm{comp}(F_0)$ and $\sigma \in \pi_1(\mathrm{bs}(\xi), b_0)$, $C\sigma$ is the component of F_0 which contains the origins of those paths which end in C and cover loops in the class σ. That this action is well defined follows from Lemma 5.1.8.1: a path which ends in C and covers a loop in the class σ can be regarded as a spheroid of the pair $(\mathrm{tl}(\xi), F_0)$ with origin in C, and which is carried into a loop in class σ by $(\mathrm{proj}(\xi))_\#$. If $s_1 \in \mathrm{Sph}_1(\mathrm{tl}(\xi), F_0, x_1)$ and $s_2 \in \mathrm{Sph}_1(\mathrm{tl}(\xi), F_0, x_2)$ are two such spheroids, and w is a path in C with $w(0) = x_1$ and $w(1) = x_2$, then the loops $(\mathrm{proj}(\xi))_\#(s_1 w)$ and $(\mathrm{proj}(\xi))_\#(s_2)$ are homotopic. Now Lemma 5.1.8.1 implies that the spheroids $s_1 w$, $s_2 \in \mathrm{Sph}_r(\mathrm{tl}(\xi), F_0, x_2)$ are homotopic, which, in turn, implies that the components of F_0 containing $s_1(0)$ and $s_2(0)$ coincide. It is readily seen that this is indeed a right action.

This action is compatible with the action of the fundamental group of $\mathrm{tl}(\xi)$ on the homotopy groups of the fibres of ξ (see Definition 5.1.7.3), namely $C(\mathrm{proj}(\xi))_*(\sigma) = T_\sigma C$ for all $C \in \mathrm{comp}(F_0) = \pi_0(F_0, x_0)$, $\sigma \in \pi_1(\mathrm{tl}(\xi), x_0)$, and $x_0 \in F_0$. Moreover, if $f\colon \xi \to \xi'$ is a map of Serre bundles, then

$$\mathrm{fact}\,\mathrm{abr}\,\mathrm{tl}(f)\colon\ \mathrm{comp}(F_0) \to \mathrm{comp}(\mathrm{proj}(\xi'))^{-1}(\mathrm{bs}(f)(b_0))),$$

where

$$\mathrm{abr}\,\mathrm{tl}(f) = [\mathrm{abr}\,\mathrm{tl}(f)\colon F_0 \to \mathrm{proj}(\xi')^{-1}(\mathrm{bs}(f)(b_0))],$$

is a $[(\mathrm{bs}(f))_*\colon \pi_1(\mathrm{bs}(\xi), b_0) \to \pi_1(\mathrm{bs}(\xi'), \mathrm{bs}(f)(b_0))]$-map.

Theorem 5.1.8.3. *If $C \in \mathrm{comp}(F_0)$ and $x_0 \in C$, then the isotropy subgroup of $\pi_1(\mathrm{bs}(\xi), b_0)$ at x_0 (see Definition 4.2.3.4) equals the image of the homomorphism $(\mathrm{proj}(\xi))_*\colon \pi_1(\mathrm{tl}(\xi), x_0) \to \pi_1(\mathrm{bs}(\xi), b_0)$.*

Proof. In fact, the equality $C\sigma = C$ means that there is a path $s\colon I \to \mathrm{tl}(\xi)$ such that $s(0), s(1) \in C$ and s covers a loop in the class σ. This, in turn, guarantees the existence of a loop with origin x_0 which covers a loop in the class σ. □

Construction of the Sequence

Definition 5.1.8.4. Let ξ be a Serre bundle with base point $x_0 \in \mathrm{tl}(\xi)$. Let $b_0 = (\mathrm{proj}(\xi))(x_0)$, $F_0 = (\mathrm{proj}(\xi))^{-1}(b_0)$, and apply Lemma 5.1.8.1 to transform the homotopy sequence of the pair $(\mathrm{tl}(\xi), F_0)$ with base point x_0 into a new sequence. Namely, for each $r \geq 1$, we replace

- the homotopy group $\pi_r(\mathrm{tl}(\xi), F_0, x_0)$ by $\pi_r(\mathrm{bs}(\xi), b_0)$,
- the homomorphism $\mathrm{rel}_*\colon \pi_r(\mathrm{tl}(\xi), x_0) \to \pi_r(\mathrm{tl}(\xi), F_0, x_0)$ - by its composition with the isomorphism $(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi), F_0, x_0) \to \pi_r(\mathrm{bs}(\xi), b_0)$,
- the homomorphism $\partial\pi_r(\mathrm{tl}(\xi), F_0, x_0) \to \pi_{r-1}(\mathrm{tl}(\xi), x_0)$ - by the composition $\Delta = \partial \circ (\mathrm{proj}(\xi))^{-1}\colon \pi_r(\mathrm{tl}(\xi), F_0, x_0) \to (\mathrm{bs}(\xi), b_0, b_0)$

Since the composition of the inclusion $\mathrm{rel}\colon (\mathrm{tl}(\xi), x_0, x_0) \to (\mathrm{tl}(\xi), b_0, b_0)$ with the projection $\mathrm{proj}(\xi)\colon (\mathrm{tl}(\xi), F_0, x_0) \to (\mathrm{bs}(\xi), b_0, b_0)$ is simply

$$\mathrm{proj}(\xi)\colon (\mathrm{tl}(\xi), x_0, x_0) \to (\mathrm{bs}(\xi), b_0)$$

we see that $[(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi), F_0, x_0) \to \pi_r(\mathrm{bs}(\xi), b_0)] \circ \mathrm{rel}_*$ is nothing else but

$$(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi), x_0) \to \pi_r(\mathrm{bs}(\xi), b_0).$$

Finally, if we attach the homotopy group $\pi_0(\mathrm{bs}(\xi), b_0)$ to the right of the resulting sequence by means of the homomorphism

$$(\mathrm{proj}(\xi))_*\colon \pi_0(\mathrm{tl}(\xi), x_0) \to \pi_0(\mathrm{bs}(\xi), b_0)$$

we obtain the sequence

$$\begin{aligned}
\to \pi_2(F_0, x_0) &\xrightarrow{\mathrm{incl}_*} \pi_2(\mathrm{tl}(\xi), x_0) \xrightarrow{\mathrm{proj}(\xi)} \pi_2(\mathrm{bs}(\xi), b_0) \xrightarrow{\Delta} \\
\pi_1(F_0, x_0) &\xrightarrow{\mathrm{incl}_*} \pi_1(\mathrm{tl}(\xi), x_0) \xrightarrow{\mathrm{proj}(\xi)} \pi_1(\mathrm{bs}(\xi), b_0) \xrightarrow{\Delta} \\
\pi_0(F_0, x_0) &\xrightarrow{\mathrm{incl}_*} \pi_0(\mathrm{tl}(\xi), x_0) \xrightarrow{\mathrm{proj}(\xi)} \pi_0(\mathrm{bs}(\xi), b_0).
\end{aligned} \tag{5.1.8.5}$$

By Definitions 5.1.3.3, 5.1.7.3, and Remark 5.1.8.2, there are right group-actions of $\pi_1(\mathrm{tl}(\xi), x_0)$ on $\pi_r(\mathrm{tl}(\xi), x_0)$ and $\pi_r(F_0, x_0)$, and also a right action of $\pi_1(\mathrm{bs}(\xi), b_0)$ on the set $\pi_0(F_0, x_0)$. The homomorphisms incl_*, $(\mathrm{proj}(\xi))_*$, and Δ are compatible with these actions, as required by Definition 5.1.5.11 (see Remarks 5.1.3.6, 5.1.4.6, Definition 5.1.7.3, and Remark 5.1.8.2). Therefore, (5.1.8.5) is a π-sequence, called the *homotopy sequence of the bundle* ξ *with base point* x_0.

Theorem 5.1.8.6. *Sequence (5.1.8.5) is exact.*

Proof. This is a corollary of the exactness of the homotopy sequence of the pair $(\mathrm{tl}(\xi), F_0)$ and of two additional and evident facts:

- the kernel of $(\mathrm{proj}(\xi))_*\colon \pi_0(\mathrm{tl}(\xi), x_0) \to \pi_0(\mathrm{bs}(\xi), b_0)$ equals the image of $\mathrm{incl}_*\, \pi_0(F_0, x_0) \to \pi_0(\mathrm{tl}(\xi), x_0)$;
- $\alpha, \beta \in \pi_0(F_0, x_0)$, there is $\sigma \in \pi_1(\mathrm{bs}(\xi), b_0)$ such that $\beta = \alpha \circ \sigma$ if and only if $\mathrm{incl}_*(\alpha) = \mathrm{incl}_*(\beta)$.

□

Theorem 5.1.8.7. *Given a map* $f\colon \xi \to \xi'$, *of Serre bundles, the vertical homomorphisms*

$$\begin{tikzcd}
{} \arrow[r] & \pi_r(F_0, x_0) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "(\mathrm{abr}\,\mathrm{tl}(f))_*"'] & \pi_r(\mathrm{tl}(\xi), x_0) \arrow[r, "(\mathrm{proj}(\xi))_*"] \arrow[d, "((\mathrm{tl}(f))_*"'] & \pi_r(\mathrm{bs}(\xi), b_0) \arrow[r, "\Delta"] \arrow[d, "((\mathrm{bs}(f))_*"'] & \pi_{r-1}(F_0, x_0) \arrow[r] \arrow[d, "(\mathrm{abr}\,\mathrm{tl}(f))_*"'] & {} \\
{} \arrow[r] & \pi_r(F'_0, x_0) \arrow[r, "\mathrm{incl}_*"'] & \pi_r(\mathrm{tl}(\xi'), x'_0) \arrow[r, "(\mathrm{proj}(\xi'))_*"'] & \pi_r(\mathrm{bs}(\xi'), b'_0) \arrow[r, "\Delta"'] & \pi_{r-1}(F'_0, x'_0) \arrow[r] & {}
\end{tikzcd}$$

where $x'_0 = \mathrm{tl}(f)(x_0)$, $b'_0 = \mathrm{bs}(f)(b_0)$, *and* $F'_0 = (\mathrm{proj}(\xi'))^{-1}(b'_0)$, *constitute a homomorphism of the first homotopy sequence into the second.*

Proof. The commutativity of the first two squares follows from Theorem 5.1.1.13, while the commutativity of the third follows from Definitions 5.1.4.2 and 5.1.4.2. The compatibility of the vertical homomorphisms with the actions of the fundamental groups was established in Remarks5.1.3.6, 5.1.4.6, 5.1.7.5, and 5.1.8.2. □

The Most Important Special Cases

Remark 5.1.8.8. If $\mathrm{tl}(\xi)$ is ∞-connected, then all the homomorphisms

$$\Delta\colon \pi_r(\mathrm{bs}(\xi), b_0) \to \pi_{r-1}(F_0, x_0)$$

are isomorphisms. If $\mathrm{tl}(\xi)$ is k-connected and $k < \infty$, then

$$\Delta\colon \pi_r(\mathrm{bs}(\xi), b_0) \to \pi_{r-1}(F_0, x_0)$$

is an isomorphism for all $r \leq k$, while

$$\Delta\colon \pi_{k+1}(\mathrm{bs}(\xi), b_0) \to \pi_k(F_0, x_0)$$

is an epimorphism. If $\mathrm{tl}(\xi)$ is connected, then the converse is true in both cases.

If $\mathrm{bs}(\xi)$ is ∞-connected, then all the homomorphisms

$$\mathrm{incl}_*\colon \pi_r(F_0, x_0) \to \pi_r(\mathrm{tl}(\xi), x_0)$$

are isomorphisms. If $\mathrm{bs}(\xi)$ is k-connected and $k < \infty$,

$$\mathrm{incl}_*\colon \pi_r(F_0, x_0) \to \pi_r(\mathrm{tl}(\xi), x_0)$$

is an isomorphism for all $r \leq k$, while

$$\mathrm{incl}_*\colon \pi_{k+1}(F_0, x_0) \to \pi_{k+1}$$

is an epimorphism. If $\mathrm{bs}(\xi)$ is connected, then the converse is true in both cases.

If F_0 is ∞-connected, then all the homomorphisms

$$(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi), x_0) \to \pi_r(\mathrm{bs}(\xi), b_0)$$

are isomorphisms. If F_0 is k-connected and $k < \infty$, then

$$(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi), x_0) \to \pi_r(\mathrm{bs}(\xi), b_0)$$

is an isomorphism for all $r \leq k$, while

$$(\mathrm{proj}(\xi))_*\colon \pi_{k+1}(\mathrm{tl}(\xi), x_0) \to \pi_{k+1}(\mathrm{bs}(\xi), b_0)$$

is an epimorphism. The converse is true in both cases (with no supplementary conditions).

Theorem 5.1.8.9. *If the bundle ξ has a section s such that $s(b_0) = x_0$, then sequence (5.1.8.2) splits from the right at the terms $\pi_r(\mathrm{tl}(\xi), x_0)$, and $s_*\colon \pi_r(\mathrm{bs}(\xi), b_0) \to \pi_r(\mathrm{tl}(\xi), x_0)$ are splitting homomorphisms for any such section, $s\colon (\mathrm{bs}(\xi), b_0) \to (\mathrm{tl}(\xi), x_0)$.*

Proof. Since $(\mathrm{proj}(\xi)) \circ s = \mathrm{id}_{\mathrm{bs}(\xi)}$, $(\mathrm{proj}(\xi))_* \circ s_* = \mathrm{id}\, \pi_r(\mathrm{bs}(\xi), b_0)$. ☐

Theorem 5.1.8.10. *If F_0 is a retract of $\mathrm{tl}(\xi)$, then sequence (5.1.8.5) splits from the left at the terms $\pi_r(\mathrm{tl}(\xi), x_0)$, and any retraction $\rho\colon \mathrm{tl}(\xi) \to F_0$ induces splitting homomorphisms $\rho_*\colon \pi_r(\mathrm{tl}(\xi), x_0) \to \pi_r(F_0, x_0)$.*

Proof. Since $\rho \circ \mathrm{incl} = \mathrm{id}_{F_0}$, $\rho_* \circ \mathrm{incl}_* = \mathrm{id}_{\pi_r(F_0,x_0)}$. ☐

Theorem 5.1.8.11. *If the inclusion $\mathrm{incl}\colon F_0 \to \mathrm{tl}(\xi)$ is x_0-homotopic to the constant map, then sequence (5.1.8.5) splits from the right at the terms $\pi_r(\mathrm{bs}(\xi), b_0)$. Moreover, given any x_0-homotopy $h\colon F_0 \times I \to \mathrm{tl}(\xi)$ from incl to the constant map, consider the maps*

$$\gamma_r\colon \mathrm{Sph}_r(F_0, x_0) \to \mathrm{Sph}_{r+1}(\mathrm{bs}(\xi), b_0)$$

given by

$$[\gamma_r(\varphi)](t_1, \ldots, t_{r+1}) = \mathrm{proj}(\xi) \circ h(\varphi(t_1, \ldots, t_r), t_{r+1}), \quad \varphi \in \mathrm{Sph}_r(F_0, x_0).$$

Then the homomorphisms $\pi_r(F_0, x_0) \to \pi_{r+1}(\mathrm{bs}(\xi), b_0)$ induced by γ_r split the sequence.

Proof. Given an arbitrary spheroid $\mathrm{Sph}_r(F_0, x_0)$, it suffices to find a spheroid $\psi \in \mathrm{Sph}_{r+1}(\mathrm{tl}(\xi), F_0, b_0)$ such that $\partial\psi = \varphi$ and $(\mathrm{proj}(\xi))_\# = \gamma_r(\varphi)$. We can set, for example, $\psi(t_1, \ldots, t_{r+1}) = h(\varphi(t_1, \ldots, t_r), t_{r+1})$. ☐

Theorem 5.1.8.12. *If $\mathrm{proj}(\xi)$ is x_0-homotopic to the constant map, then sequence (5.1.8.5) splits from the left at $\pi_r(F_0, x_0)$. Moreover, given any x_0-homotopy $h\colon \mathrm{tl}(\xi) \times I \to \mathrm{bs}(\xi)$, from $\mathrm{proj}(\xi)$ to the constant map, consider the maps*

$$\begin{aligned}\gamma_r\colon\ & \mathrm{Sph}_r(F_0, x_0) \to \mathrm{Sph}_{r+1}(\mathrm{bs}(\xi), b_0),\\ & [\gamma_r(\varphi)](t_1, \ldots, t_{r+1}) = h(\varphi(t_1, \ldots, t_r)), t_{r+1}), \quad \varphi \in \mathrm{Sph}_r(F_0, x_0).\end{aligned}$$

Then the homomorphisms $\pi_r(F_0, x_0) \to \pi_{r+1}(\mathrm{bs}(\xi), b_0)$ Induced by γ_r split the sequence.

Proof. Given an arbitrary spheroid $\psi \in \mathrm{Sph}_{r+1}(\mathrm{tl}(\xi), F_0, s_0)$, it suffices to show that the spheroids $\gamma_r \circ \partial(\varphi)$ and $(\mathrm{proj}(\xi))_\#(\psi)$, which belong to $\mathrm{Sph}_{r+1}(\mathrm{bs}(\xi), b_0)$, are homotopic. Clearly the formula

$$((t_1, \ldots, t_r), t) \mapsto h(\psi(t_1, \ldots, t_r, tt_{r+1}), (1-t)t_{r+1})$$

defines such a homotopy. ☐

Corollary 5.1.8.13. *If ξ is a covering in the broad sense, then*

$$(\mathrm{proj}(\xi))_*\colon \pi_r(\mathrm{tl}(\xi),x_0)\to(\mathrm{bs}(\xi),b_0)$$

is an isomorphism for $r\geq 2$, and a monomorphism for $r=1$. If ξ is a covering (in the narrow sense) , then, in addition, the map

$$\mathrm{fact}\,\Delta\colon \pi_1(\mathrm{bs}(\xi),b_0)/\operatorname{im}(\mathrm{proj}(\xi))_*\to F_0$$

induced by $\Delta\colon \pi_1(\mathrm{bs}(\xi),b_0)\to\pi_0(F_0,X_0)=F_0$ is invertible.

Proof. This is a corollary of the exactness of the homotopy sequence of the bundle and of the fact that $\pi_r(F_0x_0)=0$ for all $r>0$ and $\pi_0(\mathrm{tl}(\xi),x_0)=0$ whenever ξ is a covering in the narrow sense. ☐

Remark 5.1.8.14. Let ξ and ξ' be Serre bundles with base points $x_0\in\mathrm{tl}(\xi)$ and $x'_0\in\mathrm{tl}(\xi')$, and let $f\colon\xi\to\xi'$, with $\mathrm{tl}(f)(x_0)=x'_0$. Set

$$\begin{aligned}b_0=(\mathrm{proj}(\xi))(x_0),\quad & b'_0=(\mathrm{proj}(\xi'))(x'_0),\\ F_0=(\mathrm{proj}(\xi))^{-1}(b_0),\quad & F'_0=(\mathrm{proj}(\xi'))^{-1}(b'_0).\end{aligned}$$

From Theorems 5.1.8.7 and 5.1.5.20, we derive the following conclusions.

- If
$$\begin{aligned}&(\mathrm{bs}(f))_*\colon \pi_r(\mathrm{bs}(\xi),b_0)\to\pi_r(\mathrm{bs}(\xi'),b'_0),\quad r\geq 1,\\ &(\mathrm{abr}\,\mathrm{tl}(f))_*\colon \pi_r(F_0,x_0)\to\pi_r(F'_0,x'_0),\quad r\geq 0,\end{aligned}$$
are isomorphisms, then so are
$$(\mathrm{tl}(f))_*\colon \pi_r(\mathrm{tl}(\xi),x_0)\to\pi_r(\mathrm{tl}(\xi'),x'_0),\quad r\geq 1.$$
- If
$$\begin{aligned}&(\mathrm{tl}(f))_*\colon \pi_r(\mathrm{tl}(\xi),x_0)\to\pi_r(\mathrm{tl}(\xi'),x'_0),\quad r\geq 0,\\ &(\mathrm{abr}\,\mathrm{tl}(f))_*\colon \pi_r(F_0,x_0)\to\pi_r(F'_0,x'_0),\quad r\geq 0,\end{aligned}$$
are isomorphisms, then so are
$$(\mathrm{bs}(f))_*\colon \pi_r(\mathrm{bs}(\xi),b_0)\to\pi_r(\mathrm{bs}(\xi'),b'_0),\quad r\geq 1.$$
- If
$$\begin{aligned}&(\mathrm{bs}(f))_*\colon \pi_r(\mathrm{bs}(\xi),b_0)\to\pi_r(\mathrm{bs}(\xi'),b'_0),\quad r\geq 0,\\ &(\mathrm{tl}(f))_*\colon \pi_r(\mathrm{tl}(\xi),x_0)\to\pi_r(\mathrm{tl}(\xi'),x'_0),\quad r\geq 0,\end{aligned}$$
are isomorphisms, then so are
$$(\mathrm{abr}\,\mathrm{tl}(f))_*\colon \pi_r(F_0,x_0)\to\pi_r(F'_0,x'_0),\quad r\geq 1,$$
while
$$(\mathrm{abr}\,\mathrm{tl}(f))_*\colon \pi_0(F_0,x_0)\to\pi_0(F'_0,x'_0)$$
is an epimorphism with trivial kernel.

We remark that in the last case,

$$(\operatorname{abr}\operatorname{tl}(f))_*\colon \pi_0(F_0, x_0) \to \pi_0(F_0', x_0')$$

is also an isomorphism if we make the additional assumption that all the homomorphisms

$$(\operatorname{tl}(f))_*\colon \pi_1(\operatorname{tl}(\xi), x_0) \to \pi_1(\operatorname{tl}(\xi'), x_0'), \quad x \in F_0$$

are epimorphisms. Indeed, let $x_1, x_2 \in F_0$ be such that $\operatorname{tl}(f)(x_1)$ and $\operatorname{tl}(f)(x_2)$ lie in the same component of the fibre F_0', and let $s'\colon I \to F_0'$ be a path with $s'(0) = \operatorname{tl}(f)(x_1)$, $s'(1) = \operatorname{tl}(f)(x_2)$. Since $(\operatorname{tl}(f))_*\colon \pi_0(\operatorname{tl}(\xi), x_0) \to \pi_0(\operatorname{tl}(\xi'), x_0')$ is an isomorphism, and $(\operatorname{tl}(f))_*\colon \pi_1(\operatorname{tl}(\xi), x_1) \to \pi_1(\operatorname{tl}(\xi'), (\operatorname{tl}(f))f(x_1))$ is an epimorphism, there is a path $s\colon I \to \operatorname{tl}(\xi)$ such that $s(0) = x_1$, $s(1) = x_2$, and $(\operatorname{tl}(f)) \circ s\colon I \to \operatorname{tl}(\xi')$ is homotopic to the path $[\operatorname{incl}\colon F_0' \to \operatorname{tl}(\xi')] \circ s'$. Then the loop $(\operatorname{proj}(\xi')) \circ \operatorname{tl}(f) \circ s$ is homotopic to the constant loop, and from the fact that $(\operatorname{bs}(f))_*\colon \pi_1(\operatorname{bs}(\xi), b_0) \to \pi_1(\operatorname{bs}(\xi'), b_0')$ is an isomorphism it follows that $\operatorname{proj}(\xi) \circ s$ is also homotopic to the constant loop. Now, applying Theorem 4.1.3.6 to the map s, an arbitrary homotopy from $\operatorname{proj}(\xi) \circ s$ to the constant loop, and the constant homotopy of the map $s|_{\operatorname{Fr} I}$, we obtain a homotopy from s to a path $u\colon I \to \operatorname{tl}(\xi)$ such that $u(I) \subset F_0$, $u(0) = x_1$, and $u(1) = x_2$.

5.1.9 The Influence of Other Structures Upon Homotopy Groups

Remark 5.1.9.1. In this subsection we discuss the most elementary properties of homotopy groups which are due to the presence of an additional, group-like structure, compatible with the topology of the space under consideration. The most important such property we shall consider is simplicity.

The Case of Topological Groups

Theorem 5.1.9.2. *If X is a topological group and $s\colon I \to X$ is an arbitrary path, then the translation $\pi_r(X, s(0)) \to \pi_r(X, s(1))$ coincides, for any $r \geq 0$, with the isomorphism induced by the left group translation by the element $[s(1)][s(0)]^{-1}$.*

Proof. In fact, there is even a canonical free homotopy from any spheroid $\varphi \in \operatorname{Sph}_r(X, s(0))$ to $[s(1)][s(0)]^{-1}\varphi$ along s, given by

$$((t_1, \dots, t_r), t) \mapsto [s(t)][s(0)]^{-1}\varphi(t_1, \dots, t_r).$$

□

Corollary 5.1.9.3. *The components of a topological group are simple spaces. In particular, the fundamental groups of these components are Abelian.*

Remark 5.1.9.4. If X is a topological group, then, besides the multiplication on the sets $\operatorname{Sph}_r(X, e = e_X)$ defined in Definition 5.1.1.1, there is another one, resulting from the group operation on X [the product of two spheroids $\varphi, \psi \in \mathrm{Sph}_r(X,e)$ is given by $y \mapsto \varphi(y)\psi(y)$]. Moreover, the second product makes also sense for $r = 0$, when the first product is not even defined. Obviously, this new multiplication turns $\mathrm{Sph}_r(X,e)$, $r \geq 0$, into a group; the spheroids homotopic to the constant one form a normal subgroup, and the resulting quotient group equals, as a set, $\pi_r(X,e)$. When $r = 0$, $\pi_0(X,e)$ equals the quotient group X/X_0, where X_0 is the component of e. When $r \geq 1$, the new group structure on $\pi_r(X,e)$ coincides with the original one; in fact, given $\varphi, \psi \in \mathrm{Sph}_r(X,e)$, the formula

$$\begin{aligned}&((t_1,\dots,t_r),t) \mapsto\\ &\varphi(\min(1, \frac{2t_1}{1+t}), t_2,\dots,t_r)\psi(\max(0, \frac{2t_1+t-1}{1+t}), t_2,\dots,t_r)\end{aligned}$$

defines a homotopy $I^r \times I \to X$ from the original product of φ and ψ to the new one.

Let us add that the existing multiplication on X also turns the set

$$U_{x\in X}\, \mathrm{Sph}_r(X,x)$$

of *all* spheroids of X into a group. The spheroids homotopic to the constant ones form a normal subgroup, and the resulting quotient group is canonically isomorphic to $\pi_r(X,e)$, for any $r \geq 1$.

Remark 5.1.9.5. Every inner automorphism of the topological group X induces automorphisms of the groups $\pi_r(X,e)$, and thus the inner right action of X defines a right group-action of X on each group $\pi_r(X,e)$. The transformations induced by the elements of the subgroup X_0 (see Remark 5.1.9.4) are all identical: if $w\colon I \to X$ is a path with $s(0) = e$ and $s(1) = x$, and $\varphi \in \mathrm{Sph}_r(X,e)$, then the formula $(y,t) \mapsto [w(t)]^{-1}\varphi(y)w(t)$defines a homotopy $I \times I \to X$ from φ to the spheroid $y \mapsto x^{-1}\varphi(y)x$. Therefore, there are natural right group-actions of $\pi_0(X,e) = X/X_0$ on the groups $\pi_r(X,e)$.

The Case of Homogeneous Spaces

Theorem 5.1.9.6. *Let G be a topological group, and let H be a connected subgroup of G. If $(G, \mathrm{proj}, X = G/H)$ is a Serre bundle, then given any path $s\colon I \to X$, the translation $\pi_r(X,s(0)) \to \pi_r(X,s(1))$ coincides, for any $r \geq 0$, with the isomorphism induced by any transformation of X (under the canonical action) which is given by an element of G which takes $s(0)$ into $s(1)$.*

Proof. Let $g \in G$ be any element such that $gs(0) = s(1)$, and let $\widetilde{s}\colon I \to G$ be any path covering s. Since $\widetilde{s}(1)$ and $g\widetilde{s}(0)$ lie in the same coset of H, and since the cosets of a connected group are connected, there is a path w in the coset containing $\widetilde{s}(1)$ and $g\widetilde{s}(0)$ such that $w(0) = \widetilde{s}(1)$ and $w(1) = g\widetilde{s}(0)$. Now given any $\varphi \in \mathrm{Sph}_r(X,s(0))$, the formula

$$((t_1,\dots,t_r),t) \mapsto [s_1(t)][s_1(0)]^{-1}\varphi(t_1,\dots,t_r)$$

where $s_1 = \widetilde{s}w$, defines a free homotopy $l^r \times I \to X$ from φ to $[s_1(1)][s(0)]^{-1}\varphi$, along a path homotopic to s (namely, the product of s and the constant path). □

Corollary 5.1.9.7. *If H is a connected subgroup of the topological group G and $(G, \mathrm{proj}, G/H)$ is a Serre bundle, then the components of G/H are simple spaces. In particular, the fundamental groups of the components of G/H are Abelian.*

Remark 5.1.9.8. Henceforth, $(G, \mathrm{proj}, G/H)$ will automatically be a Serre bundle (as required in Theorem 5.1.9.6 and Corollary 5.1.9.7): G and its quotient space G/H will always be closed smooth manifolds, while the projection $G \to G/H$ will be a submersion and, by Theorems 4.6.1.3 and 4.1.3.4, these properties imply that $(G, \mathrm{proj}, G/H)$ is a Serre bundle.

The Case of H-Spaces

Definition 5.1.9.9. A pointed topological space (X, e) is called an H-space if there exists a continuous map $\mu\colon X \times X \to X$ such that $\mu(e, e) = e$, and the maps $X \to X$, given by $x \mapsto \mu(e, x)$ and $x \mapsto \mu(x, e)$, are e-homotopic to id_X. The map μ is called a *multiplication*, and e is called an *identity* (or a *homotopy identity*). Usually, one writes xy instead of $\mu(x, y)$.

An H-space X is *homotopy associative* if the maps $X \times X \times X \to X$, given by $(x_1, x_2, x_3) \mapsto (x_1x_2)x_3$ and $(x_1, x_2, x_3) \mapsto x_1(x_2x_3)$, are homotopic, and *homotopy commutative* (or *Abelian*) if the maps $X \times X \to X$, given by $(x_1, x_2) \mapsto x_1x_2$ and $(x_1, x_2) \mapsto x_2x_1$, are homotopic.

Given a H-space X, a continuous map $\nu\colon X \to X$ is said to be a *homotopy inverse* if the maps $X \to X$, given by $x \mapsto x\nu(x)$ and$x \mapsto \nu(x)x$, are homotopic to the constant map which takes X into e. Usually, one writes x^{-1} instead of $\nu(x)$.

The primary examples of H-spaces are topological groups. Every topological group (viewed as a H-space) is homotopy associative and has a homotopy inverse, and every Abelian topological group is homotopy Abelian.

Remark 5.1.9.10. The spaces of spheroids provide another important class of H-spaces. For a given pointed topological space (X, x_0), the sets $\mathrm{Sph}_r(X, x_0)$ with $r \geq 1$ become H-spaces if we equip them with the topology they inherit as subsets of $\mathcal{C}(I^r, X)$ and with the usual multiplication, and take the constant spheroid as the identity. In fact, the map $\varphi \mapsto (\mathrm{const})\varphi$ is homotopic to $\mathrm{id}_{\mathrm{Sph}_r(X,x_0)}$ via the const-homotopy $\mathrm{Sph}_r(X, x_0) \times I \to \mathrm{Sph}_r(X, x_0)$, given by $(\varphi, t) \mapsto \varphi_t$, where φ_t is the spheroid whose value at the point $(t_1, \dots, t_r) \in I^r$ is given by the right-hand side of formula (5.1.1.5); further, the map $\varphi/to\varphi(\mathrm{const})$ is homotopic to $\mathrm{id}_{\mathrm{Sph}_r(X,x_0)}$ via the const-homotopy which is similarly defined by formula (5.1.1.6). The H-spaces $\mathrm{Sph}_r(X, x_0)$ are homotopy associative and have a homotopy inverse for $r \geq 1$, and are homotopy Abelian for $r \geq 2$; the formulae in Subsection 5.1.1 again provide us with the necessary homotopies.

Similarly, given any topological pair (X, A) with base point x_0, the sets $\mathrm{Sph}_r(X, A, x_0)$ with $r \geq 2$ are homotopy associative H-spaces possessing a homotopy inverse. If $r \geq 3$, these H-spaces are homotopy Abelian.

Theorem 5.1.9.11. *Every connected H-space is simple and, in particular, has an Abelian fundamental group.*

Proof. Suppose X is a connected H-space with identity e, $\varphi \in \mathrm{Sph}_r(X, e)$ with arbitrary r, and $s \in \mathrm{Sph}_1(X, e)$ is a loop. Then, the formula $(y, t) \mapsto s(t)\varphi(y)$ defines a free homotopy $I^r \times I \to X$ from the spheroid $y \mapsto e\varphi(y)$ (which is homotopic to φ) to the same spheroid $y \mapsto e\varphi(y)$, along the loop $t \mapsto s(t)e$ (which is homotopic to s) . Thus, $\pi_1(X, e)$ acts identically on the groups $\pi_r(X, e)$. ☐

Remark 5.1.9.12. It is clear that the second description of the homotopy groups $\pi_r(X, e)$, given in Remark 5.1.9.4 for the case of topological groups, carries over to H-spaces X with identity e. Generally speaking, the multiplication that each set $\mathrm{Sph}_r(X, e)$ inherits from X does not turn this set into a group. However, this multiplication does induce the usual group structure on $\pi_r(X, e)$, $r \geq 1$. Moreover, if X is homotopy associative and has a homotopy inverse, then the set $\pi_0(X, e)$ is a group.

The Local System of Homotopy Groups of the Total Space of a Principal Bundle

Lemma 5.1.9.13. *Let ξ be a principal bundle with structure group G, and let $u, v \colon I \to \mathrm{tl}(\xi)$ be paths satisfying $\mathrm{proj}(\xi) \circ u = \mathrm{proj}(\xi) \circ v$. If $g_0, g_1 \in G$ are such that $u(0)g_0 = v_0$ and $u(1)g_1 = v(1)$ [here G acts canonically from the right on $\mathrm{tl}(\xi)$; see Definition 4.3.2.10], then the diagramme*

$$\begin{tikzcd}
\pi_r(\mathrm{tl}(\xi), u(0)) \arrow[r, "T_u"] \arrow[d] & \pi_r(\mathrm{tl}(\xi), u(1)) \arrow[d] \\
\pi_r(\mathrm{tl}(\xi), v(0)) \arrow[r, "T_v"'] & \pi_r(\mathrm{tl}(\xi), v(1))
\end{tikzcd}$$

where the vertical isomorphisms are induced by the transformations $x \mapsto xg_0$ and $x \mapsto xg_1$, commutes.

Proof. Recall that the canonical right action $\mathrm{tl}(\xi) \times G \to \mathrm{tl}(\xi)$ is free and its orbits coincide with the fibres of ξ. Since $u(t)$ and $v(t)$ lie in the same fibre, for each $t \in I$ there is a unique $g(t) \in G$ such that $u(t)g(t) = v(t)$. Therefore, if $h \colon I^r \times I \to \mathrm{tl}(\xi)$ is a free homotopy from $\varphi_0 \in \mathrm{Sph}_r(\mathrm{tl}(\xi), u(0))$ to $\varphi_1 \in \mathrm{Sph}_r(\mathrm{tl}(\xi), u(1))$ along u, then $(y, t) \mapsto h(y, t)g(t)$ yields a free homotopy from the spheroid $y \mapsto \varphi_0 g_0$ to the spheroid $y \mapsto \varphi_1 g_1$ along v. ☐

Definition 5.1.9.14. Suppose that ξ is a principal bundle, $b \in \mathrm{bs}(\xi)$, and $r \geq 0$. The right canonical action of the structure group of ξ on $\mathrm{tl}(\xi)$ induces isomorphisms between the homotopy groups $\pi_r(\mathrm{tl}(\xi), x)$ with $x \in (\mathrm{proj}(\xi))^{-1}(b)$, and so we may identify these groups. We shall call the resulting group (which actually is a group for $r > 0$ and a set with identity for $r = 0$) the *r-th homotopy group of the space* $\mathrm{tl}(\xi)$ *over* b, and we shall write $\pi_r(\mathrm{tl}(\xi), b)$.

Given a path $s\colon I \to \mathrm{bs}(\xi)$, we define $T_s\colon (\mathrm{tl}(\xi), s(0)) \to \pi_r(\mathrm{tl}(\xi), s(1))$ as the translation $T_s\colon (\mathrm{tl}(\xi), \widetilde{s}(0)) \to \pi_r(\mathrm{tl}(\xi), \widetilde{s}(1))$ along any path $\widetilde{s}$ which covers s. That this is well defined follows from Lemma 5.1.9.13. Obviously, T_s are homomorphisms and satisfy conditions 5.1.2.1 (i)-(iii). Therefore, we have produced a local system on $\mathrm{bs}(\xi)$, $(\mathrm{bs}(\xi), \{\pi_r(\mathrm{tl}(\xi), b)\}, \{T_s\})$, which we call the *lower local system of the r-th homotopy groups of the total space of ξ*.

It is clear that the local system on $\mathrm{tl}(\xi)$ induced by this system via $\mathrm{proj}(\xi)$ is nothing but the usual local system of the r-th homotopy groups of $\mathrm{tl}(\xi)$.

Given a monomorphism $\varphi\colon\ : G \to G'$ and $r \geq 0$, every φ-map of the Steenrod G-bundle ξ into the Steenrod G'-bundle ξ' induces a homomorphism of the lower local system of the r-th homotopy groups of $\mathrm{tl}(\xi)$ into the corresponding system of $\mathrm{tl}(\xi')$.

The Homotopy Sequence of a Principal Bundle

Definition 5.1.9.15. Let ξ be a principal G-bundle with base point $x_0 \in \mathrm{tl}(\xi)$. If in sequence (5.1.8.5) we replace (F_0, x_0) by the Pair $(G, e = e_G)$ [which is canonically homeomorphic to (F_0, x_0) via $g \mapsto x_0 g$] and attach 1 to the right of the resulting sequence, we obtain

$$\begin{aligned} &\to \pi_2(G,e) \xrightarrow{\mathrm{incl}_*} \pi_2(\mathrm{tl}(\xi), x_0) \xrightarrow{(\mathrm{proj}(\xi))_*} \pi_2(\mathrm{bs}(\xi), b_0) \\ &\xrightarrow{\Delta} \pi_1(G,e) \xrightarrow{\mathrm{incl}_*} \pi_1(\mathrm{tl}(\xi), x_0) \xrightarrow{(\mathrm{proj}(\xi))_*} \pi_1(\mathrm{bs}(\xi), b_0) \\ &\xrightarrow{\Delta} \pi_0(G,e) \xrightarrow{\mathrm{incl}_*} \pi_0(\mathrm{tl}(\xi), x_0) \xrightarrow{(\mathrm{proj}(\xi))_*} \pi_0(\mathrm{bs}(\xi), b_0) \to 1 \\ &\quad (b_0 = (\mathrm{proj}(\xi))(x_0). \end{aligned} \tag{5.1.9.16}$$

Recall that $\pi_0(G, e)$ is a group (see Remark 5.1.9.4) and that $\pi_1(G, e)$ is Abelian (see Corollary 5.1.9.3). It is immediate that $\Delta\colon \pi_1(\mathrm{bs}(\xi), b_0) \to \pi_0(G, e)$ is a group homomorphism. Moreover, $\pi_0(G, e)$ acts from the right on $\pi_r(G, e)$ (see Remark 5.1.9.5), while $\pi_1(\mathrm{bs}(\xi), b_0)$ acts similarly on both $\pi_r(\mathrm{bs}(\xi), b_0$ (see Definition 5.1.3.3) and $\pi_r(\mathrm{tl}(\xi), x_0) = \pi_r(\mathrm{tl}(\xi), b_0)$ (see Definition 5.1.9.14). The canonical right action $\mathrm{tl}(\xi) \times G \to \mathrm{tl}(\xi)$ induces a right action of G on $\pi_0(\mathrm{tl}(\xi), x_0) = \mathrm{comp}(\mathrm{tl}(\xi))$, and thus a right action of $\pi_0(G, e)$ on $\pi_0(\mathrm{tl}(\xi), x_0)$. [We regard $\pi_0(G, e)$ as the quotient group of G by the component of e; see Remark 5.1.9.4. The action of this component on $\pi_0(\mathrm{tl}(\xi), x_0)$ is identical.] Finally, it is clear that the homomorphisms incl_*, proj_*, and Δ are compatible with the above actions, as required in Definition 5.1.5.11. Consequently, (5.1.9.16) is a π-sequence, called the *homotopy sequence of the G-bundle ξ with base point x_0*.

Obviously, $(\mathrm{proj}(\xi))_*\colon \pi_0(\mathrm{tl}(\xi), x_0) \to \pi_0(\mathrm{bs}(\xi), b_0)$ is an epimorphism, and the partition of $\pi_0(\mathrm{tl}(\xi), x_0)$ into the orbits of $\pi_0(G, e)$ is exactly $\mathrm{zer}((\mathrm{proj}(\xi))_*)$. Therefore, sequence (5.1.9.16) is exact.

Given a monomorphism $\varphi\colon G' \to G$, every φ-map f of the principal G'-bundle ξ' with base point $x_0' \in \mathrm{tl}(\xi')$ into the principal G-bundle ξ with base point $x_0 \in \mathrm{tl}(\xi)$, such that $\mathrm{tl}\, f(x_0') = x_0$, induces a homomorphism of the homotopy sequence of ξ' into the homotopy sequence of ξ.

5.1.10 Alternative Descriptions of the Homotopy Groups

Definition 5.1.10.1. The spheroid $\mathbb{DS} \circ I\mathbb{D} \in \mathrm{Sph}_r(\mathbb{S}^r, \mathrm{ort}_1)$ (see Definition 1.2.8.9) is called the *fundamental spheroid of the sphere* $\mathbb{S}^r$, denoted $I\mathbb{S}$, and we let sph_r denote the element of $\pi_r(\mathbb{S}^r, \mathrm{ort}_1)$ that it defines. The spheroid $I\mathbb{D} \in \mathrm{Sph}_r(\mathbb{D}^r, \mathbb{S}^{r-1}, \mathrm{ort}_1)$ is called the *fundamental spheroid of the ball* $\mathbb{D}^r$, and we let kug_r denote the element of $\pi_r(\mathbb{D}^r, \mathbb{S}^{r-1}, \mathrm{ort}_1)$ that it defines.

Obviously, $\partial(I\mathbb{D}) = I\mathbb{S}$, whence $\partial(\mathrm{kug}_r) = \mathrm{sph}_{r-1}$.

Remark 5.1.10.2. We let $\mathrm{Sph}_r^{\bigcirc}(X, x_0)$ denote the set of all continuous maps from the pointed space $(\mathbb{S}^r, \mathrm{ort}_1)$ into the pointed space (X, x_0), and define

$$I\mathbb{S}^{\#}\colon \mathrm{Sph}_r^{\bigcirc}(X, x_0) \to \mathrm{Sph}_r(X, x_0), \quad \varphi \mapsto \varphi \circ I\mathbb{S}.$$

Clearly, this map is invertible, and Theorem 1.3.7.6 implies that two maps, $\varphi, \psi \in \mathrm{Sph}_r^{\bigcirc}(X, x_0)$, are homotopic if and only if the spheroids $I\mathbb{S}^{\#}(\varphi)$, $I\mathbb{S}^{\#}(\psi)$ are homotopic. Consequently, replacing our "cubic" spheroids and their homotopies by the "spheric" spheroids from $\mathrm{Sph}_r^{\bigcirc}(X, x_0)$ and their homotopies, we are led to an equivalent description of the set $\pi_r(X, x_0)$.

It is readily seen that the identity spheroid $\mathrm{id}_{\mathbb{S}^r}$ belongs to the class sph_r, and that the element of $\pi_r(X, x_0)$ given by a spheroid

$$f\colon (\mathbb{S}^r, \mathrm{ort}_1) \to (X, x_0)$$

equals $f_*(\mathrm{sph}_r)$.

If $r \geq 1$, then $I\mathbb{S}^{\#}$ transfers the multiplication in $\mathrm{Sph}_r(X, x_0)$ to $\mathrm{Sph}_r^{\bigcirc}(X, x_0)$. The resulting multiplication in $\mathrm{Sph}_r^{\bigcirc}(X, x_0)$ may also be described directly: the product

$$\varphi\psi\colon (\mathbb{S}^r, \mathrm{ort}_1) = (\mathbb{S}^1, \mathrm{ort}_1) \otimes \cdots \otimes (\mathbb{S}^1, \mathrm{ort}_1) \to (X, x_0)$$

of the spheroids

$$\varphi, \psi\colon (\mathbb{S}^r, \mathrm{ort}_1) = (\mathbb{S}^1, \mathrm{ort}_1) \otimes \cdots \otimes (\mathbb{S}^1, \mathrm{ort}_1) \to (X, x_0)$$

is given by

$$\varphi\psi(y_1, y_2, \ldots, y_r) = \begin{cases} \varphi(y_1^2, y_2, \ldots, y_r), & \text{if } \Im y_1 \geq 0, \\ \pi(y_1^2, y_2, \ldots, y_r), & \text{if } \Im y_1 \leq 0, \end{cases} \tag{5.1.10.3}$$

where $y_1, y_2, \ldots, y_r$ are complex numbers of modulus 1, and $\Im$ denotes the imaginary part. The multiplication that this operation induces on $\pi_r(X, x_0)$ coincides with the existing one. One can use (5.1.10.3) to study directly the homotopy properties of the multiplication in $\mathrm{Sph}_r^{\bigcirc}(X, x_0)$ and get an independent description of the homotopy groups $\pi_r(X, x_0)$ in the language of spheric spheroids.

It is particularly simple to describe in this language the spheroid

$$\varphi^{-1} = (I\mathbb{S}^{\#})^{-1}([I\mathbb{S}^{\#}(\varphi)]^{-1})$$

that is, the inverse of the spheroid

$$\varphi \in \mathrm{Sph}_r^{\bigcirc}(X, x_0)\colon \varphi^{-1}(x_1, x_2, x_3, \dots, x_{r+1}) = \varphi(x_1, -x_2, x_3, \dots x_{r+1})$$

Definition 5.1.10.4. Let $\mathrm{Sph}_r \bigcirc(X, A, x_0)$ be the set of all continuous maps $(\mathbb{D}^r, \mathbb{S}^{r-1}, \mathrm{ort}_1) \to (X, A, x_0)$, and define

$$I\mathbb{D}^{\#}\colon\ \mathrm{Sph}_r^{\bigcirc}(X, A, x_0) \to \mathrm{Sph}_r(X, A, x_0), \quad \varphi \mapsto \varphi \circ I\mathbb{D}.$$

This map is invertible, and Theorem 1.3.7.6 implies that two maps,

$$\varphi, \psi \in \mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$$

are homotopic if and only if the spheroids $I\mathbb{D}^{\#}(\varphi)$ and $I\mathbb{D}^{\#}(\psi)$ are homotopic. If $r \geq 2$, then $I\mathbb{D}^{\#}$ transfers the multiplication from $\mathrm{Sph}_r(X, A, x_0)$ to $\mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$. The resulting multiplication in $\mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$ may also be described directly: given two maps,

$$\varphi, \psi\colon (\mathbb{D}^r, \mathrm{ort}_1) = (\mathbb{S}^1, \mathrm{ort}_1) \otimes \dots \otimes (\mathbb{S}^1, \mathrm{ort}_1) \otimes (I, 1) \to (X, x_0),$$

formula (5.1.10.3), where $y_1, \dots, y_{r-1}$ are complex numbers of modulus 1 and $y_r \in I$, defines a map

$$\varphi\psi\colon (\mathbb{D}^r, \mathrm{ort}_1) \to (X, x_0),$$

and this map belongs to $\mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$ whenever $\varphi, \psi \in \mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$. As in the absolute case, $\varphi^{-1} = (I\mathbb{D}^{\#})([I\mathbb{D}^{\#}(\varphi)]^{-1})$, that is, the inverse of the spheroid $\varphi \in \mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$, is given by

$$\varphi^{-1}(x_1, x_2, x_3, \dots, x_r) = \varphi(x_1, -x_2, x_3, \dots, x_r).$$

Therefore, by replacing the cubic spheroids by spheric ones, we get an adequate description of the homotopy groups $\pi_r(X, A, x_0)$.

Obviously, the identity spheroid $\mathrm{id}_{\mathbb{D}^r}$ belongs to the class kug_r, and the element of $\pi_r(X, A, x_0)$ given by a spheroid $f\colon (\mathbb{D}^r, \mathbb{S}^{r-1}, \mathrm{ort}_1) \to (X, A, x_0)$ equals $f_*(\mathrm{kug}_r)$.

Unlike the cubic sets $\mathrm{Sph}_r(X, x_0, x_0)$ and $\mathrm{Sph}_r(X, x_0)$, the sets $\mathrm{Sph}_r^{\bigcirc}(X, x_0, x_0)$ and $\mathrm{Sph}_r^{\bigcirc}(X, x_0)$ are not identical, being merely related through the canonical invertible map

$$(I\mathbb{D}^{\#})^{-1} \circ (I\mathbb{S}^{\#})\colon\ \mathrm{Sph}_r^{\bigcirc}(X, x_0) \to \mathrm{Sph}_r^{\bigcirc}(X, x_0, x_0);$$

this map can be described more directly as $\varphi \mapsto \varphi \circ \mathbb{DS}$.

The *boundary* $\partial\varphi$ of a spheric spheroid $\varphi \in \mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$ is the element $\partial\varphi \in \mathrm{Sph}_{r-1}(A, x_0)$ given by

$$\partial\varphi = [\mathrm{abr}\,\varphi\colon (\mathbb{S}^{r-1}, \mathrm{ort}_1) \to (A, x_0)].$$

Clearly, $I\mathbb{S}^{\#} \circ \partial = \partial \circ I\mathbb{D}^{\#}$, which demonstrates that this definition of the boundary leads to the same boundary homomorphism $\partial\colon \pi_r(X, A, x_0) \to \pi_{r-1}(A, x_0)$ as the cubic theory does.

Remark 5.1.10.5. For each continuous map,

$$f\colon (X, x_0) \to (X', x'_0) \quad \text{or} \quad f\colon (X, A, x_0) \to (X', A', x'_0)$$

we have the map

$$f_\#\colon \mathrm{Sph}_r^{\bigcirc}(X, x_0) \to \mathrm{Sph}_r^{\bigcirc}(X', x'_0) \quad \text{or} \quad f_\#\colon \mathrm{Sph}_r^{\bigcirc}(X, A, x_0) \to \mathrm{Sph}_r^{\bigcirc}(X', A', x'_0)$$

given by $f_\#(\varphi) = f \circ \varphi$. Trivially, $I\mathbb{S}^\# \circ f_\# = f_\# \circ I\mathbb{S}^\#$ and $I\mathbb{D}^\# \circ f_\# = f_\# \circ I\mathbb{D}^\#$, so that these maps $f_\#$ lead to the same induced homomorphisms

$$f_*\colon \pi_r(X, x_0) \to \pi_r(X', x'_0) \quad \text{and} \quad f_*\colon \pi_r(X, A, x_0) \to \pi_r(X', A', x'_0)$$

as the cubic theory does.

A free homotopy from a spheroid $\varphi_0 \in \mathrm{Sph}_r^{\bigcirc}(X, x_0)$ to a spheroid $\psi_0 \in \mathrm{Sph}_r^{\bigcirc}(X, x_1)$ along a path $s\colon I \to X$ is defined as a usual homotopy $\mathbb{S}^r \times I \to X$ from φ_0 to φ_1 which takes (ort_1, t) into $s(t)$ for any $t \in I$. If h is such a homotopy, then $h \circ (I\mathbb{S} \times \mathrm{id}_I)$ is a free homotopy from $I\mathbb{S}^\#(\varphi_0) \in \mathrm{Sph}_r(X, x_0)$ to $I\mathbb{S}^\#(\varphi_1) \in \mathrm{Sph}_r(X, x_1)$ along s. Therefore, the spheric free homotopies along s yield the same isomorphism $T_s\colon \pi_r(X, x_0) \to \pi_r(X, x_1)$ as do the cubic ones. It is readily seen that all this carries over to the relative case.

An Alternative Proof of Theorem 5.1.4.9

Remark 5.1.10.6. Using spheric spheroids and the homotopy sequence of a pair, one can give a second proof of Theorem 5.1.4.9, which is less direct but, in return, more transparent.

Consider first a model situation: $X = (\mathbb{D}^2, \mathrm{ort}_1) \vee (\mathbb{D}^2, \mathrm{ort}_1)$, $A = (\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^1, \mathrm{ort}_1)$, x_0 is the centre of both bouquets, $\alpha = \mathrm{Imm}_{1*}(\mathrm{kug}_2)$, and $\beta = \mathrm{Imm}_{2*}(\mathrm{kug}_2)$. Since $\partial(\alpha^{-1}\beta\alpha) = (\partial\alpha)^{-1}(\partial\beta)(\partial\alpha) = T_{\partial\alpha}\partial\beta$ (see Theorem 5.1.3.4) and $T_{\partial\alpha}\partial\beta = \partial T_{\partial\alpha}\beta$ (see Remark 5.1.4.6), we have $\partial(\alpha^{-1}\beta\alpha) = \partial T_{\partial\alpha}\beta$. Moreover, since X is contractible, β is an isomorphism (see Remark 5.1.6.7), and hence $\alpha^{-1}\beta\alpha = T_{\partial\alpha}\beta$.

In the general case, pick two arbitrary spheroids $4\varphi, \psi \in \mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$ in the classes α, β and define

$$f\colon ((\mathbb{D}^2, \mathrm{ort}_1) \vee (\mathbb{D}^2, \mathrm{ort}_1), (\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^1, \mathrm{ort}_1) \to (X, A)$$

by

$$f(\mathrm{Imm}_1(x)) = \varphi(x), \quad f(\mathrm{Imm}_2(x)) = \psi(x), \quad x \in \mathbb{D}^2.$$

Since $\varphi = f \circ \mathrm{Imm}_1$, $\psi = f \circ \mathrm{Imm}_2$, we have

$$\alpha = (f \circ \mathrm{Imm}_1)_*(\mathrm{kug}_2), \quad \beta = (f \circ \mathrm{Imm}_2)_*(\mathrm{kug}_2),$$

and hence

$$\begin{aligned} \alpha^{-1}\beta\alpha &= f_*([\mathrm{Imm}_{1*}(\mathrm{kug}_2)]^{-1}[\mathrm{Imm}_{2*}(\mathrm{kug}_2)][\mathrm{Imm}_{1*}(\mathrm{kug}_2)] \\ &= f_*(T_{\partial \circ \mathrm{Imm}_{1*}(\mathrm{kug}_2)}\, \mathrm{Imm}_{2*}(\mathrm{kug}_2)) = T_{\partial\alpha}\beta. \end{aligned}$$

Spheroids in Spheroid Spaces

Remark 5.1.10.7. Let (X, x_0) be a pointed space. With every spheroid $\varphi \in \mathrm{Sph}_{r+s}(X, x_0)$ we may associate an r-spheroid of the space $\mathrm{Sph}_s(X, x_0)$ [see Remark 5.1.9.10] with base point the constant spheroid, by means of each of the formulae

$$[\psi(t_1, \dots, t_r)](u_1, \dots, u_s) = \varphi(t_1, \dots, t_r, u_1, \dots, u_s),$$
$$[\psi(t_1, \dots, t_r)](u_1, \dots, u_s) = \varphi(u_1, \dots, u_s, t_1, \dots, t_r).$$

This leads to two maps,

$$\mathrm{Cub}, \mathrm{Buc}\colon \mathrm{Sph}_{r+s}(X, x_0) \to \mathrm{Sph}_r(\mathrm{Sph}_s(X, x_0), \mathrm{const}).$$

These maps are invertible and both they and their inverses take homotopic spheroids into homotopic ones. If $r > 0$, then the multiplication in $\mathrm{Sph}_{r+s}(X, x_0)$ is transferred by Cub into the usual multiplication in $\mathrm{Sph}_r(\mathrm{Sph}_s(X, x_0), \mathrm{const})$ [that is, the multiplication of spheroids]. If $s > 0$, then the multiplication in $\mathrm{Sph}_{r+s}(X, x_0)$ is transferred by Buc into the multiplication in $\mathrm{Sph}_r(\mathrm{Sph}_s(X, x_0), \mathrm{const})$ which arises from the fact that $\mathrm{Sph}_s(X, x_0)$ is an H-space with identity element const (see Theorem 5.1.9.11 and Remark 5.1.9.10). Therefore, when $r + s > 0$, the maps Cub and Buc define group isomorphisms,

$$\mathrm{cub}, \mathrm{buc}\colon \pi_{r+s}(X, x_0) \to \pi_r(\mathrm{Sph}_s(X, x_0), \mathrm{const})$$

[the group structure of$\pi_0(\mathrm{Sph}_s(X, x_0), \mathrm{const})$, $s > 0$, was explained in Remark 5.1.9.12]. Therefore, one can identify $\pi_r(X, x_0)$ with any of the groups $(\mathrm{Sph}_{r-q}(X, x_0), \mathrm{const})$ with $q \leq r$.

Finally, note that the isomorphism $\mathrm{cub}\colon \pi_r(X, x_0) \to \pi_{r-1}(\mathrm{Sph}_1(X, x_0), \mathrm{const})$ appears also as the isomorphism $\Delta\colon (X, x_0) \to \pi_{r-1}(\mathrm{Sph}_1(X, x_0), \mathrm{const})$ in the homotopy sequence of the Serre bundle

$$\xi = (\mathcal{C}(I, 0; X, x_0), \mathrm{abr}\, \mathcal{C}([\mathrm{incl}\colon \operatorname{Fr} I \to I], \mathrm{id}), X = \mathcal{C}(\operatorname{Fr} I, 0; X, x_0)),$$

whose fibre over the point x_0 is $\mathrm{Sph}_1(X, x_0)$. That ξ is a Serre bundle follows from Theorem 4.1.4.2.

5.1.11 Additional Theorems

Theorem 5.1.11.1. *Let $d_1, \dots, d_m$ be pairwise disjoint balls in $\mathbb{R}^r$ such that $d_1, \dots, d_m \subset \mathbb{D}^r$, and let $g \in \mathrm{Sph}_r^{\bigcirc}(X, A, x_0)$ be a spheroid with $g(C) \subset A$, where $C = \mathbb{D}^r \setminus \cup_{i=1}^m \operatorname{int} d_i$. Let $\tau_i\colon \mathbb{D}^r \to \mathbb{D}^r$ denote the map $\tau_i(y) =$ (centre of d_i) $+$ (radius of d_i)y. Suppose further that the segments joining the points $\tau_1(\mathrm{ort}_1), \dots, \tau_m(\mathrm{ort}_1)$ to ort_1 are contained in C. Then, for $r > 2$,*

$$\gamma = (T_{s_1}\gamma_1)(T_{s_2}\gamma_2) \cdots (T_{s_m}\gamma_m), \tag{5.1.11.2}$$

where $\gamma \in \pi_r(X, A, x_0)$ *and* $\gamma_i \in \pi_r(X, A, g \circ \tau_i(\mathrm{ort}_1))$ *are the elements represented by the spheroids* g *and* $g \circ \tau_i \in \mathrm{Sph}^{\bigcirc}_r(X, A, g \circ \tau_i(\mathrm{ort}_1))$, *and* s_i *is the path in* A *given by*

$$s_i(t) = g((1-t)\tau_i(\mathrm{ort}_1) + t\,\mathrm{ort}_1), \quad i = 1, \dots, m.$$

The same conclusion holds true for $r = 2$, *provided* $d_1, \dots, d_m$ *are indexed naturally, i.e., each of the* 2*-frames* $(\tau_i(\mathrm{ort}_1) - \mathrm{ort}_1, \tau_{i+1}(\mathrm{ort}_1) - \mathrm{ort}_1)$ *defines the natural orientation of* $\mathbb{R}^2$.

Proof. The proof is quite involved and we proceed by induction on m. We make two preliminary remarks, denoting by ℓ_i the (rectilinear) path in C given by $\ell_i(t) = (1-t)\tau_i(\mathrm{ort}_1) + t\,\mathrm{ort}_1$.

1. For given $d_1, \dots, d_m$, it suffices to prove the theorem when

   $$(X, A, x_0) = (\mathbb{D}^r, C, \mathrm{ort}_1), \quad g = \mathrm{rel}\,\mathrm{id}_{\mathbb{D}^r}, \quad s_i = \ell_i.$$

   Indeed, $g_*\colon (\mathbb{D}^r, C, \mathrm{ort}_1) \to \pi_r(X, A, x_0)$ takes the class of the spheroid $\mathrm{rel}\,\mathrm{id}_{\mathbb{D}^r}$ into γ, while it takes the class of the spheroid τ_i translated along ℓ_i, into $T_{s_i}\gamma_i$.

2. For a given m, it suffices to prove the theorem for a standard choice of $d_1, \dots, d_m$, namely, for the balls of radius $1/2m$ centred at

   $$\frac{m-1}{m}\mathrm{ort}_2, \frac{m-3}{m}\mathrm{ort}_2, \dots, \frac{3-m}{m}\mathrm{ort}_2, \frac{1-m}{m}\mathrm{ort}_2\,.$$

   To see this, consider, along with these standard balls and the corresponding C, τ_i, ℓ_i, arbitrary balls $d'_1, \dots, d'_m$ satisfying the conditions of the theorem, with the corresponding C', τ'_i, ℓ'_i. Clearly, there exists a continuous map $h\colon \mathbb{D}^r \to \mathbb{D}^r$, which is $\mathbb{S}^r$-homotopic to $\mathrm{id}_{\mathbb{D}^r}$ and satisfies $h(C) \subset C'$, $h \circ \tau_i = \tau'_i$, and $h \circ \ell_i = \ell i'$. Then

   $$\mathrm{rel}\,h_*\colon \pi_r(\mathbb{D}^r, C, \mathrm{ort}_1) \to \pi_r(\mathbb{D}^r, C', \mathrm{ort}_1)$$

   takes the class of the spheroid $\mathrm{id}_{\mathbb{D}^r}$ into the class of $\mathrm{id}_{\mathbb{D}^r}$, while taking the class of the spheroid τ_i translated along ℓ_i, into the class of the spheroid τ'_i translated along ℓ'_i, $i = 1, \dots, m$.

Now back to our induction. The cases $m = 0$ and $m = 1$ are trivial; consider $m = 2$. By our remarks, we may assume that $(X, A, x_0) = (\mathbb{D}^r, C, \mathrm{ort}_1)$, $g = \mathrm{id}$, and d_1, d_2 are the standard balls (with radius $1/4$ and centres $\mathrm{ort}_2/2$ and $-\mathrm{ort}_2/2$). Let ρ_φ denote the rotation of $\mathbb{D}^r$ by an angle φ around the subspace given by the equations $x_1 = x_2 = 0$. Further, consider the homotopies $\mathbb{D}^r \times I \to \mathbb{D}^r$ given by

$$(y, t) \mapsto [(1+t)\rho_{\pi t/2}(y) - 2\,\mathrm{ort}_2]/4,$$
$$(y, t) \mapsto [(1+t)\rho_{-\pi t/2}(y) + 2\,\mathrm{ort}_2]/4,$$

($y \in \mathbb{D}^r$, $t \in I$). These can be viewed as free homotopies of spheroids of the pair $\mathbb{D}^r, C)$; as such, they connect τ_1 and τ_2 with two spheroids with origin 0, σ_1 and σ_2 along two paths, which we call u_1 and u_2. Obviously, $u_1^{-1}\ell_1$ and $u_2^{-1}\ell_2$ are both homotopic to the rectilinear path ℓ which joins 0 to ort_1. Consequently, the product of the classes obtained by translating the classes of τ_1 and τ_2 to ort_1 along ℓ_1 and ℓ_2 is the same as the product of the classes obtained by translating the classes of σ_1 and σ_2 to ort_1 along ℓ, that is, the class of the product of the spheroids σ_1 and σ_2 translated to ort_1 along ℓ. It remains to observe that there is a free homotopy from the product $\sigma_1\sigma_2$ to $\mathrm{id}_{\mathbb{D}^r}$ along ℓ, for example, a rectilinear homotopy.

Finally, let $m \geq 3$. As in the case $m = 2$, we shall assume that

$$(X, A, x_0) = (\mathbb{D}^r, C, \mathrm{ort}_1), \quad g = \mathrm{id},$$

and $d_1, \dots, d_m$ are the standard balls with radius $1/2m$ and centres

$$\frac{m-1}{m}\,\mathrm{ort}_2, \frac{m-3}{m}\,\mathrm{ort}_2, \dots, \frac{3-m}{m}\,\mathrm{ort}_2, \frac{1-m}{m}\,\mathrm{ort}_2\,.$$

Let d be the ball of radius $3/2m$ centred at $\frac{2-m}{m}\,\mathrm{ort}_2$ (note that $d_{m-1}, d_m \subset d$), and let $\tau\colon \mathbb{D}^r \to \mathbb{D}^r$ be defined by $\tau(y) = (\text{centre of } d) + (\text{radius of } d)y$. Further, let ℓ, u, and v be the rectilinear paths joining $\tau(\mathrm{ort}_1)$ to ort_1, $\tau(\mathrm{ort}_1)$ to $\tau_m(\mathrm{ort}_1)$, and $\tau_m(\mathrm{ort}_1)$ to $\tau(\mathrm{ort}_1)$, respectively. We let $\delta \in \pi_r(\mathbb{D}^r, C, \tau(\mathrm{ort}_1))$ denote the class of τ. Since the products $u\ell$, $v\ell$ are clearly homotopic to the paths ℓ_{m-1}, ℓ_m,

$$T_{\ell_1}\gamma_1 \cdots T_{\ell_m}\gamma_m = T_{\ell_1}\gamma_1 \cdots T_{\ell_{m-2}}\gamma_{m-2} T_\ell(T_u\gamma_{m-1}T_v\gamma_m). \tag{5.1.11.3}$$

Now apply the theorem, first for the case of two factors, and then for the case of $m-1$ factors, to conclude that

$$\begin{aligned} T_u\gamma_{m-1}T_v\gamma_m &= \delta \\ T_{\ell_1}\gamma_1 \cdots T_{\ell_{m-2}}\gamma_{m-2}\delta &= \gamma. \end{aligned} \tag{5.1.11.4}$$

(In the first case, the theorem is applied to $(X, A, x_0) = (\mathbb{D}^r, C, \tau(\mathrm{ort}_1))$, $g = \tau$, and the balls $\tau^{-1}(d_{m-1})$, $\tau^{-1}(d_m)$, while in the second case we take $(X, A, x_0) = (\mathbb{D}^r, C, \mathrm{ort}_1)$, $g = \mathrm{id}$, and the balls $d_1, \dots, d_{m-2}, d$). At last, (5.1.11.3), and (5.1.11.4) yield (5.1.11.2). □

Theorem 5.1.11.5. *Let $X = \lim(X_k, \varphi_k)$, where X_k are T_1-spaces, and let $x \in X, x_0 \in X_0, x_1 \in X_1, \dots$ be points such that $\mathrm{Imm}_k(x_k) = x$. If for some r all the homomorphisms $(\varphi_K)_{*r}\colon \pi_r(X_k, x_k) \to \pi_r(X_{k+1}, x_{k+1})$ are isomorphism, then so are $(\mathrm{Imm}_k)_{*r}\colon \pi_r(X_k, x_k) \to \pi_r(X, x)$ (with the same r).*

Proof. Notice that, according to Theorem 1.2.4.5, every spheroid $I^r \to X$ may be expressed as the composition of a spheroid $I^r \to X_\ell$ with the embedding Imm_ℓ, for ℓ large enough; similarly, every homotopy $I^r \times I \to X$ is the composition of some homotopy $I^r \times I \to X_\ell$ with Imm_{ell}, for ℓ large enough. Now the fact that $(\mathrm{Imm}_k)_{*r}$ are epimorphisms and monomorphisms is seen to be a consequence of the analogous properties of the compositions

$$(\varphi_{\ell-1})_{*r} \circ (\varphi_{\ell-2})_{*r} \circ \cdots \circ (\varphi_k)_{*r} \colon \pi_r(X_k, x_k) \to \pi_r(X_\ell, x_\ell).$$

□

5.1.12 Exercises

Exercise 5.1.12.1. Let (X, x_0) be a pointed space and suppose that there is given a right group-action of $\pi_1(X, x_0)$ on a group G. Show that there exists a local system of groups, $(X, \{G_x\}, \{T_s\})$, with $G_{x_0} = G$, which determines the given action.

Exercise 5.1.12.2. Let (X, A) be a cellular pair with base point $x_0 \in A$, and suppose that X is countable. Show that all the groups $\pi_r(X, A, x_0)$ are countable.

Exercise 5.1.12.3. Let (X, A) be a cellular pair with base point $x_0 \in A$, and suppose that the groups ${}_pi_r(X, x_0)$ and $\pi_r(A, x_0)$ are finitely generated (for all $r \geq 1$). Show that if X is simply connected, then the groups $\pi_r(X, A, x_0)$ with $r \geq 2$ are also finitely generated.

Exercise 5.1.12.4. Let ξ be a Serre bundle, and let E be a subspace of $\mathrm{tl}(\xi)$ such that $(E, (\mathrm{proj}(\xi))|_E, \mathrm{bs}(\xi))$ is also a Serre bundle. Show that for any point $x \in E$ and any $r \geq 1$

$$\mathrm{incl}_* \colon \pi_r((\mathrm{proj}(\xi))^{-1}((\mathrm{proj}(\xi))(x)), (\mathrm{proj}(\xi))^{-1}((\mathrm{proj}(\xi))(x) \cap E, x) \to \pi_r(\mathrm{tl}(\xi), E, x)$$

is an isomorphism.

Exercise 5.1.12.5. Show that if the base of a covering is k-simple, then its total space is also k-simple.

Exercise 5.1.12.6. Let $r > 0$ and $s > 0$. Show that the homomorphisms

$$\mathrm{cub}, \mathrm{buc} \colon \pi_{r+s}(X, x_0) \to \pi_r(\mathrm{Sph}_s(X, x_0), \mathrm{const})$$

differ only by the constant factor $(-1)^{rs}$.

5.2 THE HOMOTOPY GROUPS OF SPHERES AND OF CLASSICAL MANIFOLDS

5.2.1 Suspension in the Homotopy Groups of Spheres

Definition 5.2.1.1. The *suspension of a spheroid* $\varphi \in \mathrm{Sph}_r(X, x_0)$ is the spheroid $\operatorname{sus}\varphi \in \mathrm{Sph}_{r+1}(\operatorname{sus}(X, x_0), \mathrm{bp})$, given by

$$\operatorname{sus}\varphi(t_1, \dots, t_{r+1}) = \operatorname{proj}(\varphi(t_1, \dots, t_r), t_{r+1}), \quad \operatorname{proj} = [\operatorname{proj}\colon X \times I \to \operatorname{sus}(X, x_0)].$$

Obviously, suspensions of homotopic spheroids are homotopic, the suspension of the product of two spheroids of positive dimensions equals the product of their suspensions, and the suspension of the constant spheroid is again the constant spheroid. Consequently, the mapping $\varphi \mapsto \operatorname{sus}\varphi$ yields a homomorphism $pi_r(X, x_0) \to \pi_{r+1}(\operatorname{sus}(X, x_0), \mathrm{bp})$, for any $r \geq 0$. This homomorphism is also called *suspension* and is denoted by sus.

Recall that we have already defined the suspension of a continuous map on two occasions: in Definition 1.2.6.2, for maps of topological spaces, and in Definition 1.2.8.5, for maps of pointed topological spaces. The present, third definition, is more special; it concerns maps from the pair $(I, \operatorname{Fr} I^r)$ into pointed spaces, and has no intersection with the previous ones. At the same time, it is compatible with the second definition, in the sense that we may obtain the third definition from the latter by shifting from spheric spheroids to cubic ones. More precisely, the spheroids in $\mathrm{Sph}^{\bigcirc}_r(X, x_0)$, being maps between pointed spaces, have suspensions in the sense of Definition 1.2.8.5, and the diagramme

$$\begin{tikzcd}
\mathrm{Sph}^{\bigcirc}_r(X, x_0) \arrow[r, "\operatorname{sus}"] \arrow[d, "I\mathbb{S}^{\#}"'] & \mathrm{Sph}^{\bigcirc}_{r+1}(\operatorname{sus}(X, x_0), \mathrm{bp}) \arrow[d, "I\mathbb{S}^{\#}"] \\
\mathrm{Sph}_r(X, x_0) \arrow[r, "\operatorname{sus}"'] & \mathrm{Sph}_{r+1}(\operatorname{sus}(X, x_0), \mathrm{bp})
\end{tikzcd}$$

commutes.

Let us add two important, yet obvious remarks. Firstly, if $f\colon (X, x_0) \to (Y, y_0)$ is continuous, then the diagramme

$$\begin{tikzcd}
\pi_r(X, x_0) \arrow[r, "\operatorname{sus}"] \arrow[d, "f_*"'] & \pi_{r+1}(\operatorname{sus}(X, x_0), \mathrm{bp}) \arrow[d, "(\operatorname{sus} f)_*"] \\
\pi_r(Y, y_0) \arrow[r, "\operatorname{sus}"'] & \pi_{r+1}(\operatorname{sus}(Y, y_0), \mathrm{bp})
\end{tikzcd}$$

(where $\operatorname{sus} f$ is understood as in Definition 1.2.8.5) commutes for any $r \geq 0$. Secondly, since $(\operatorname{sus}(\mathbb{S}^n, \mathrm{ort}_1), \mathrm{bp}) = (\mathbb{S}^{n+1}, \mathrm{ort}_1)$, our construction, when applied to spheres, yields a homomorphism $\pi_r(\mathbb{S}^n, \mathrm{ort}_1) \to \pi_{r+1}(\mathbb{S}^{n+1}, \mathrm{ort}_1)$. The main theorem of this subsection, Theorem 5.2.1.4, is devoted precisely to this homomorphism.

Remark 5.2.1.2. Alternatively, one can describe

$$\mathrm{sus}\colon \pi_r(X, x_0) \to \pi_{r+1}(\mathrm{sus}(X, x_0), \mathrm{bp})$$

by means of the map

$$\mathrm{lp}\colon (X, x_0) \to \mathrm{Sph}_1(\mathrm{sus}(X, x_0), \mathrm{bp}), \quad [\mathrm{lp}(x)](t) = (x, t).$$

Namely, every spheroid $\varphi \in \mathrm{Sph}_r(X, x_0)$ is taken into $\mathrm{sus}\,\varphi$ by the composition

$$\mathrm{Sph}_r(X, x_0) \xrightarrow{\mathrm{lp}_\#} \mathrm{Sph}_r(\mathrm{Sph}_1(\mathrm{sus}(X, x_0), \mathrm{bp}), \mathrm{const}) \xrightarrow{\mathrm{Cub}} \mathrm{Sph}_{r+1}(\mathrm{sus}(X, x_0), \mathrm{bp})$$

(see Remark 5.1.10.7), and hence the homomorphism

$$\mathrm{sus}\colon \pi_r(X, x_0) \to \pi_{r+1}(\mathrm{sus}(X, x_0), \mathrm{bp})$$

may be defined as $\mathrm{sus} = \mathrm{cub} \circ \mathrm{lp}_*$ (to check this facts is routine).

A new description of the homomorphism sus emerges if we view $\mathrm{sus}(X, x_0)$ as the quotient space of the cone $\mathrm{cone}(X, x_0)$ by its base (which is identified with X). Indeed, given any $\varphi \in \mathrm{Sph}_r(X, x_0)$, consider the spheroid in $\mathrm{Sph}_{r+1}(\mathrm{cone}(X, x_0), X, x_0)$ defined as

$$(t_1, \dots, t_{r+1}) \mapsto (\varphi(t_1, \dots, t_r), t_{r+1}).$$

The latter is taken into $\mathrm{sus}\,\varphi$ by the map

$$\begin{aligned} \mathrm{proj}_\#\colon\ & \mathrm{Sph}_{r+1}(\mathrm{cone}(X, x_0), X, x_0) \to \mathrm{Sph}_{r+1}(\mathrm{sus}(X, x_0), \mathrm{bp}), \\ & \mathrm{proj} = [\mathrm{proj}\colon \mathrm{cone}(X, x_0) \to \mathrm{sus}(X, x_0], \end{aligned}$$

and is transformed back into φ by the map

$$\partial\colon \mathrm{Sph}_{r+1}(\mathrm{cone}(X, x_0), X, x_0) \to \mathrm{Sph}_r(X, x_0).$$

Consequently, $\mathrm{sus}\colon \pi_r(X, x_0) \to \pi_{r+1}(\mathrm{sus}(X, x_0), \mathrm{bp})$ is nothing but the composition

$$\pi_r(X, x_0) \xrightarrow{\partial^{-1}} \pi_{r+1}(\mathrm{cone}(X, x_0), X, x_0) \xrightarrow{\mathrm{proj}_*} \pi_{r+1}(\mathrm{sus}(X, x_0), \mathrm{bp})$$

($\partial\colon \pi_{r+1}(\mathrm{cone}(X, x_0), X, x_0) \to \pi_r(X, x_0)$ is invertible because the cone is contractible; see Remark 5.1.6.7).

If, for example, $(X, x_0) = (\mathbb{S}^n, \mathrm{ort}_1)$, then $\mathrm{cone}(X, x_0) = \mathbb{D}^{n+1}$, $\mathrm{sus}(X, x_0) = \mathbb{S}^{n+1}$ and $\mathrm{proj} = \mathbb{DS}$.

The Suspension Theorem

Lemma 5.2.1.3. *Let K and L be closed disjoint subsets of I^m, and assume that K is covered by a finite number of k-dimensional planes, that L is covered by a finite number of ℓ-dimensional planes, and that $K \cap \mathrm{Fr}\, I^m \subset I^{m-1} \times [0, 1/2)$ and $L \cap \mathrm{Fr}\, l^m \subset I^{m-1} \times (1/2, 1]$. If $k + \ell \le m - 2$, then there is a $\mathrm{Fr}\, I^m$ -homotopy, $F\colon I^m \times I \to I^m$, such that:*

(i) the maps $I^m \to I^m$ which form F are homeomorphisms, and their inverses also form a homotopy;

(ii) F connects id_{I^m} with a map $I^m \to I^m$ which takes K into $I^{m-1} \times [0, 1/2]$ and L into $I^{m-1} \times [1/2, 1]$.

Proof. Suppose first that $L \subset I^{m-1} \times [1/2, 1]$. Since the lines which intersect both K and L constitute a set which can be covered by a finite number of planes of dimension $k + \ell + 1 \leq m - 1$, there is a point $a \in \operatorname{int} I^{m-1} \times (0, 1/2)$ which does not lie on any such line. Let us project K and L on $\operatorname{Fr} I^m$ from a. Their images are closed disjoint subsets of $\operatorname{Fr} I^m$, and so there is a Urysohn function, $\alpha\colon \operatorname{Fr} I^m \to I$, for these images. Choose $\varepsilon > 0$ such that the similarity transformation with centre a and coefficient ε pulls I^m into $I^m \times [0, 1/2]$, while the similarity transformation with the same centre and coefficient $1/(1 - \varepsilon)$ does not take $K \cap (I^m \times [1/2, 1])$ out of I^m. Now let $\varphi_x\colon I \to I^m$ denote the rectilinear path which joins a to a given point $x \in \operatorname{Fr} I^m$, and let $\psi_t\colon I \to I$ denote the homeomorphism which takes linearly $[0, (1 + t)/2]$ onto $[0, (1 - t)/2]$ and $[(1 + t)/2, 1]$ onto $[(1 - t)/2, 1]$. The formula

$$F(\varphi_x(u), t) = \varphi_x \circ \psi_{t(1-\varepsilon)\alpha(x)}(u),$$

where $x \in \operatorname{Fr} I^m$ and $t, u \in I$, defines the desired homotopy.

To reduce everything to this special case, we produce, in the general case, a $\operatorname{Fr} I^m$-homotopy $F\colon I^m \times I \to I^m$ which satisfies property (i) and connects id_{I^m} with a map which takes L into $I^m \times [1/2, 1]$. We can define such a homotopy F by

$$F((x, u), t) = (x, \psi_{-t\delta}(u)), \quad [x \in I^{m-1}, t, u \in I],$$

where $\delta \in (0, 1)$ is any number such that $L \subset I^{m-1} \times [(1 - \delta)/2, 1]$. □

Theorem 5.2.1.4. sus: $\pi_r(\mathbb{S}^n, \mathrm{ort}_1) \to \pi_{r+1}(\mathbb{S}^{n+1}, \mathrm{ort}_1)$ *is an isomorphism for* $r \leq 2n - 2$ *and an epimorphism for* $r = 2n - 1$.

Proof. **a**) To see that sus is epimorphic for $r \leq 2n - 1$, we have to verify that, given any spheroid $\varphi \in \mathrm{Sph}_{r+1}(\mathbb{S}^{n+1}, \mathrm{ort}_1)$, there is a spheroid $\psi \in \mathrm{Sph}_r(\mathbb{S}^n, \mathrm{ort}_1)$ such that $\operatorname{sus} \psi$ is homotopic to φ.

The proof is quite simple when $\varphi(I^r \times [0, 1/2])$ is contained in the upper half $\{x_{n+2} \geq 0\}$ of $\mathbb{S}^{n+1}$ while $\varphi(I^r \times [1/2, 1])$ is contained in the lower half $\{x_{n+2} \leq 0\}$ of $\mathbb{S}^{n+1}$ (here $x_1, \ldots, x_{n+2}$ are the standard coordinates in $\mathbb{R}^{n+2}$). In this case, $\varphi(I^r \times (1/2))$ lies in the intersection of these two hemispheres, i.e., $\varphi(I^r \times (1/2)) \subset \mathbb{S}^n$, and the required *psi* is given by

$$\psi(t_1, \ldots, t_r) = \varphi(t_1, \ldots, t_r, 1/2).$$

The formula

$$((t_1, \ldots, t_r), t) \mapsto \begin{cases} \varphi(t_1, \ldots, t_r, \frac{t_{r+1}}{1-t}) & \text{if } \ 0 \leq t_{r+1} \leq \frac{1-t}{2}, \\ \mathrm{proj}(\varphi(t_1, \ldots, t_r, \frac{1}{2}), t_{r+1}) & \text{if } \ \frac{1-t}{2} \leq t_{r+1} \leq \frac{1+t}{2}, \\ \varphi(t_1, \ldots, t_r, \frac{t_{r+1}-t}{1-t}) & \text{if } \ \frac{1+t}{2} \leq t_{r+1} \leq 1, \end{cases}$$

where $\mathrm{proj} = [\mathrm{proj}\colon \mathbb{S}^n \times I \to \mathrm{sus}(\mathbb{S}^n, \mathrm{ort}_1)]$, defines a homotopy from φ to $\mathrm{sus}\,\psi$.

Consider now the more general case where $\varphi^{-1}(\mathrm{ort}_{n+2}) \subset I^r \times [0, 1/2)$ and $\varphi^{-1}(-\mathrm{ort}_{n+2}) \subset I^r \times (1/2, 1]$. This case is readily reduced to the previous one. Indeed, pick $\varepsilon > 0$ such that the last coordinate of $\varphi(y)$ is $\le 1 - \varepsilon$ ($\ge -(1-\varepsilon)$) for all $y \in I^r \times [1/2, 1]$ (respectively, for all $y \in I^r \times [0, 1/2]$). Now define $h\colon (\mathbb{S}^{n+1}, \mathrm{ort}_1) \to (\mathbb{S}^{n+1}, \mathrm{ort}_1)$ by

$$h(x_1, \dots, x_{n+2}) = \begin{cases} (x_1, \dots, x_{n+1}, (x_{n+2}^2 - (1-\varepsilon^2)^{1/2}))/(1-(1-\varepsilon^2))^{1/2}, \\ \qquad \text{if} \quad |x_{n+2}| \ge 1 - \varepsilon, \\ (x_1, \dots, x_{n+1}, 0)/(x_1^2 + \dots + x_{n+1}^2)^{1/2}, \\ \qquad \text{if} \quad |x_{n+2}| \le 1 - \varepsilon, \end{cases}$$

[This map stretches the polar caps of $\mathbb{S}^{n+1}$, defined by the inequalities $x_{n+2} \ge 1 - \varepsilon$ and $x_{n+2} \le -(1-\varepsilon)$ over the upper and lower hemispheres, respectively, and contracts the equatorial belt $-(1-\varepsilon) < x_{n+2} < 1 - \varepsilon$ to the equator $\mathbb{S}^n$.] It is clear that h is ort_1-homotopic to $\mathrm{id}_{\mathbb{S}^{n+1}}$ [such a homotopy moves each point $x \in \mathbb{S}^{n+1}$ uniformly towards $h(x)$ along the shortest arc of the great circle passing through x and $h(x)$]. Moreover, $h \circ \varphi$ takes $I^r \times [0, 1/2]$ into the upper hemisphere, and takes $I^r \times [1/2, 1]$ into the lower hemisphere.

Finally, in the most general case, we triangulate $\mathbb{S}^{n+1}$ in such a way that ort_1 becomes a vertex, while ort_{n+2} and ort_{n+2} become interior points of $(n+1)$-simplices, and then consider a rectilinear triangulation of I^n which ensures that φ has simplicial approximations. Let φ_1 be such an approximation. Then the sets $K = \varphi_1^{-1}(\mathrm{ort}_{n+2})$ and $L = \varphi_1^{-1}(-\mathrm{ort}_{n+2})$ satisfy the conditions of Lemma 5.2.1.3, with $m = r + 1$ and $k = \ell = r - n$ [here the intersections $K \cap \mathrm{Fr}\, I^{r+1}$ and $L \cap \mathrm{Fr}\, I^{r+1}$ are actually empty]. Let $G\colon I^{r+1} \times r \to I^{r+1}$ be the homotopy made up of the inverses of the homeomorphisms which form the homotopy F provided by Lemma 5.2.1.3. The spheroid φ_1 is obviously homotopic to φ, and we need only remark that

$$\varphi_1 \circ \mathrm{rel}\, G\colon (I^{r+1} \times I, \mathrm{Fr}\, I^{r+1} \times I) \to (\mathbb{S}^{n+1}, \mathrm{ort}_1)$$

is a homotopy from φ_1 to a spheroid φ_2 such that $\varphi_2^{-1}(\mathrm{ort}_{n+2}) \subset I^r \times [0, 1/2]$ and $\varphi_2^{-1}(-\mathrm{ort}_{n+2}) \subset I^r \times [1/2, 1]$.

b) To see that sus is a monomorphism for $r \le 2n+2$, we must show that every spheroid $\psi\colon (I^r, \mathrm{Fr}\, I^r) \to (\mathbb{S}^n, \mathrm{ort}_1)$ whose suspension $\mathrm{sus}\,\psi\colon (I^{r+1}, \mathrm{Fr}\, I^{r+1}) \to (\mathbb{S}^{n+1}, \mathrm{ort}_1)$ is homotopic to the constant spheroid is itself homotopic to the constant spheroid.

Let $\varphi : (I^{r+1} \times I, \mathrm{Fr}\, I^{r+1} \times I) \to (\mathbb{S}^{n+1}, \mathrm{ort}_1)$ be a homotopy from $\mathrm{sus}\,\psi$ to const. Consider a triangulation of $\mathbb{S}^{n+1}$ with the properties: the equator $\mathbb{S}^n$ is a simplicial subspace, ort_1 is a vertex, and ort_{n+2}, $-\mathrm{ort}_{n+2}$ are interior points of $(n+1)$-simplices. Further, triangulate rectilinearly $I^{r+1} \times I = I^r \times I \times I$ so that $I^r \times (1/2) \times 0$ is a simplicial subspace and φ admits simplicial approximations. If φ_1 is such an approximation, then $\varphi_1(I^r \times (1/2) \times 0) \subset \mathbb{S}^n$, and the formula $\psi(y) = (y, 1/2, 0)$ defines a spheroid $\psi_1\colon (I^r, \mathrm{Fr}\, I^r) \to (\mathbb{S}^n, \mathrm{ort}_1)$ which is homotopic to ψ. Consider the map

$$\mathrm{perm}\colon I^{r+2} \to I^{r+2}, \quad (t_1, \dots, t_{r+1}, t_{r+2}) \mapsto (t_1, \dots, t_r, t_{r+2}, t_{r+1})$$

Clearly, $K = \mathrm{perm}(\psi_1^{-1}(\mathrm{ort}_{n+2})$ and $L = \mathrm{perm}(\psi_1^{-1}(-\mathrm{ort}_{n+2}))$ satisfy the conditions of Lemma 5.2.1.3, with $m = r + 2$ and $k = \ell = r - n + 1$. Thus, let $G\colon I^{r+2} \times I \to I^{r+2}$ be the homotopy made up of the inverses of the homeomorphisms which form the homotopy F provided by this lemma, and let $\rho\colon \mathbb{S}^{n+1} \setminus (\mathrm{ort}_{n+2} \cup (-\mathrm{ort}_{n+2})) \to \mathbb{S}^n$ be a retraction. One may verify directly that

$$(y,t) \mapsto \rho(\varphi_1 \circ \mathrm{perm} \circ G(y,t,1/2),1)$$

(where $(y,t,1/2) \in I^{r+2} = I^r \times \times I)$ is a homotopy $I^r \times I \to \mathbb{S}^n$ from ψ_1 to const. □

The Series $\{\pi_{n+k}(\mathbb{S}^n, \mathrm{ort}_1)\}$

Definition 5.2.1.5. The main content of Theorem 5.2.1.4 is that each of the series

$$\cdots \xrightarrow{\mathrm{sus}} \pi_r(\mathbb{S}^n, \mathrm{ort}_1) \xrightarrow{\mathrm{sus}} \pi_{r+1}(\mathbb{S}^{n+1}, \mathrm{ort}_1) \xrightarrow{\mathrm{sus}} \pi_{r+2}(\mathbb{S}^{n+2}, \mathrm{ort}_1) \xrightarrow{\mathrm{sus}} \cdots .$$

of homotopy groups of spheres, connected by the suspension, stabilises. That is to say, in the k-th series $\{\pi_{n+k}(\mathbb{S}^n, \mathrm{ort}_1)\}$, the groups $\pi_{n+k}(\mathbb{S}^n, \mathrm{ort}_1)$ with $n \geq k + 2$ are isomorphic via suspension. This canonical isomorphism enables us to identify the groups $\pi_{n+k}(\mathbb{S}^n, \mathrm{ort}_1)$, $n \geq k + 2$, with a single group, called the *stable group of the series* $\pi_{n+k}(\mathbb{S}^n, \mathrm{ort}_1)$; we denote it by $\mathrm{Stab}(k)$.

5.2.2 The Simplest Homotopy Groups of Spheres

Theorem 5.2.2.1. *The groups* $\pi_r(\mathbb{S}^n)$ *with* $r < n$ *are trivial. In particular,* $\mathrm{Stab}(k) = 0$ *whenever* $k < 0$.

Proof. This is a corollary of Theorems 2.3.2.4 and 5.1.3.8. □

The Homotopy Groups of the Circle

Theorem 5.2.2.2. *If* $r > 1$, $\pi_r(\mathbb{S}^1)$ *is trivial.* $\pi_1(\mathbb{S}^1, \mathrm{ort}_1)$ *is an infinite cyclic group with generator* sph_1.

Proof. The proof uses the covering $(\mathbb{R}, \mathrm{hel}, \mathbb{S}^1)$ [see Example 4.1.2.6]. First, notice that $\mathrm{hel}(0) = \mathrm{ort}_1$ and $\mathrm{hel}^{-1}(\mathrm{ort}_1) = \mathbb{Z}$. Since the line $\mathbb{R}$ is contractible, its homotopy groups are trivial, and hence, by Theorem 5.1.8.13 so is $\pi_r(\mathbb{S}^1, \mathrm{ort}_1)$ for any $r > 1$, and $\Delta\colon \pi_1(\mathbb{S}^1, \mathrm{ort}_1) \to \pi_0(\mathbb{Z}, 0) = \mathbb{Z}$ is invertible. Moreover, $(\mathbb{R}, \mathrm{hel}, \mathbb{S}^1)$ is obviously a principal bundle with structure group $\mathbb{Z}$, so that Δ is a group homomorphism (see Definition 5.1.9.15). Therefore, Δ is a group isomorphism, and it remains to observe that $\Delta(\mathrm{sph}_1) = -1$. □

Remark 5.2.2.3. The proof above computes the homotopy groups of the circle as quick as lightning by applying the general theory to the covering $(\mathbb{R}, \mathrm{hel}, \mathbb{S}^1)$. Such an approach is natural as the general theory is already available to us. However, it conceals the fact that the computation is in fact quite elementary.

Since the fundamental group of the circle has both a historical and intrinsic importance, we shall not spare a few more lines, and we shall redo the proof of the equality $\pi_1(\mathbb{S}^1, \mathrm{ort}_1) = \mathbb{Z}$ in an elementary way, which reveals those simple aspects of the general theory that are really necessary. Namely, we need only the following two facts: every path in $\mathbb{S}^1$ with origin ort_1 is covered by a path in $\mathbb{R}$ with origin 0; two paths in $\mathbb{R}$, with origin 0, which cover homotopic paths in $\mathbb{S}^1$ have the same end. (Cf. Definition 5.6.2.1).

Proof. Now to the proof: consider the powers of the fundamental loop $I\mathbb{S}\colon I \to \mathbb{S}^1$ (with the natural order of multiplication), and let u_n denote the n-th power ($n \in \mathbb{Z}$). Let $\widetilde{u}_n\colon I \to \mathbb{R}$ be the path with origin 0 covering the loop u_n. Obviously, $\widetilde{u}_n(1) = n$, and so the loops u_n are pairwise non-homotopic. Moreover, given any loop $u\colon I \to \mathbb{S}^1$ with origin ort_1, the covering path $\widetilde{u}\colon I \to \mathbb{R}$ with origin 0 ends at an integer. Consequently, $\widetilde{u}$ is homotopic to one of the paths $\widetilde{u}_n$, and thus u is homotopic to one of the loops u_n. In other words, the classes $(\mathrm{sph}_1)^n$ with $n \in \mathbb{Z}$ are pairwise distinct and exhaust $\pi_1(\mathbb{S}^1, \mathrm{ort}_1)$. ☐

Corollaries

Corollary 5.2.2.4. *The pair* $(\mathbb{D}^2, \mathbb{S}^1)$ *is simple.* $\pi_r(\mathbb{D}^2, \mathbb{S}^1)$ *is trivial for any* $r \neq 2$. $\pi_2(\mathbb{D}^2, \mathbb{S}^1, \mathrm{ort}_1)$ *is an infinite cyclic group with generator* kug_2.

Proof. These are all consequences of Theorem 5.2.2.2 and of the fact that $\mathbb{D}^2$ is contractible, which implies that $\partial\colon \pi_r(\mathbb{D}^2, \mathbb{S}^1, \mathrm{ort}_1) \to \pi_{r-1}(\mathbb{S}^1, \mathrm{ort}_1)$ is an isomorphism for any $r \geq 1$ (see Remark 5.1.6.7). ☐

Corollary 5.2.2.5. *It follows from Theorems 5.2.2.1 and 5.2.2.2 that for* $n \geq 1$, $\mathbb{S}^n$ *is a simple space. Similarly, Theorem 5.2.2.1 and Corollary 5.2.2.4 show that for* $n \geq 2$ *the pair* $(\mathbb{D}^n, \mathbb{S}^{n-1})$ *is simple.*

In particular, given any point $y \in \mathbb{S}^n$ *with* $n \geq 1$, *the group* $\pi_n(\mathbb{S}^n, y)$ *can be identified with* $\pi_n(\mathbb{S}^n, \mathrm{ort}_1)$, *and given any point* $y \in \mathbb{S}^{n-1}$ *with* $n \geq 2$, *the group* $\pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}, y)$ *can be identified with* $\pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}, \mathrm{ort})$. *Therefore, for* $r \geq 1$, *a continuous map* $f\colon \mathbb{S} \to X$, *with* X *a topological space, defines an element of* $\pi_r(X, x)$ *for any point* $x \in f(\mathbb{S}^r)$, *and not only for* $x = f(\mathrm{ort}_1)$. *Similarly, for* $r \geq 2$, *a continuous map* $f\colon (\mathbb{D}^r, \mathbb{S}^{r-1}) \to (X, A)$, *with* (X, A) *a topological pair, defines an element of* $\pi_r(X, A, x)$ *for any* $x \in f(\mathbb{S}^{r-1})$, *and not only for* $x = f(\mathrm{ort}_1)$.

The Groups $\pi_n(\mathbb{S}^n)$

Theorem 5.2.2.6. *For* $n \geq 1$, $\mathrm{sus}\colon \pi_n(\mathbb{S}^n, \mathrm{ort}_1) \to \pi_{n+1}(\mathbb{S}^{n+1}, \mathrm{ort}_1)$ *is an isomorphism and* $\mathrm{sus}(\mathrm{sph}_n) = \mathrm{sph}_{n+1}$.

Proof. By Theorem 5.2.1.4, $\mathrm{sus}\colon \pi_n(\mathbb{S}^n, \mathrm{ort}_1) \to \pi_{n+1}(\mathbb{S}^{n+1}, \mathrm{ort}_1)$ is an isomorphism for $n \geq 2$, and an epimorphism for $n = 1$. To prove that this epimorphism is also a monomorphism, we use the homotopy properties of the Hopf bundle $(\mathbb{S}^3, \mathrm{proj}, \mathbb{S}^2)$ [see Example 4.6.1.4]: the segment

$$\pi_2(\mathbb{S}^-3) = 0 \to \pi_2(\mathbb{S}^2) \to \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^3) = 0$$

of its homotopy sequence (see Theorem 5.2.2.1) demonstrates that $\pi_2(\mathbb{S}^2) = \pi_1(\mathbb{S}^1)$. Since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ (see Theorem 5.2.2.2), sus: $\pi_1(\mathbb{S}^1, \mathrm{ort}_1) \to \pi_2(\mathbb{S}^2, \mathrm{ort}_1)$ cannot have a non-trivial kernel. The equality $\mathrm{sus}(\mathrm{sph}_n) = \mathrm{sph}_{n+1}$ is obvious. ☐

Theorem 5.2.2.7. *If $n \geq 1$, $\pi_n(\mathbb{S}^n, \mathrm{ort}_1)$ is an infinite cyclic group with generator sph_n. In particular, $\mathrm{Stab}(O) = \mathbb{Z}$.*

Proof. For $n = 1$, this statement is a repetition of a part of Theorem 5.2.2.2, while for $n > 1$ it results from Theorems 5.2.2.2 and 5.2.2.6. ☐

Corollary 5.2.2.8. *If $n \geq 2$, $\pi_n(\mathbb{D}^n, \mathbb{S}^n, \mathrm{ort}_1)$ is an infinite cyclic group with generator kug_n.*

Remark 5.2.2.9. For each $n \geq 1$, Theorem 5.2.2.7 establishes a canonical isomorphism $\pi_n(\mathbb{S}^n) \to \mathbb{Z}$. In particular, it associates with each continuous map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ an integer, and it is not hard to see that this is nothing but the degree $\deg f$, as defined in Subsection 4.6.5. This is a consequence of three evident facts: $\deg(\mathrm{sus}\, f) = \deg f$; the class $k\, \mathrm{sph}_1$ is represented by the spheroid hel_k. (see Example 4.1.2.6); and $\deg(\mathrm{hel}_k) = k$.

Similarly, for each $n \geq 2$, Corollary 5.2.2.8 establishes a canonical isomorphism $\pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \to \mathbb{Z}$. In particular, it associates with each continuous map $f\colon (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ an integer, which coincides with $\deg f$, as defined in Subsection 4.6.5.

Further Information Obtained From the Hopf Bundles

Theorem 5.2.2.10. *If $r \geq 3$, the homomorphism $\mathrm{proj}_*\colon \pi_r(\mathbb{S}^3) \to \pi_2(\mathbb{S}^2)$ induced by the Hopf map $\mathrm{proj}\colon \mathbb{S}^3 \to \mathbb{S}^2$ is an isomorphism. In particular, $\pi_3(\mathbb{S}^2)$ is canonically isomorphic to $\mathbb{Z}$, and is generated by $\mathrm{proj}_*(\mathrm{sph}_3)$, i.e., by the class of the Hopf map itself.*

Proof. This is plain from the segment

$$\pi_r(\mathbb{S}^1) = 0 \to \pi_r(\mathbb{S}^3) \to \pi_r(\mathbb{S}^2) \to \pi_{r-1}(\mathbb{S}^1) = 0$$

of the homotopy sequence of the Hopf bundle $(\mathbb{S}^3, \mathrm{proj}, \mathbb{S}^2)$. ☐

Theorem 5.2.2.11. *If $r \geq 1$, the homomorphism $\mathrm{proj}_*\colon \pi_r(\mathbb{S}^7) \to \pi_r(\mathbb{S}^4)$ induced by the Hopf map $\mathrm{proj}\colon \mathbb{S}^7 \to \mathbb{S}^4$ maps $\pi_r(\mathbb{S}^7)$ isomorphically onto a subgroup of $pi_r(\mathbb{S}^4)$ which has a direct complement isomorphic to $\pi_{r-1}(\mathbb{S}^3)$. In particular, $\pi_7(\mathbb{S}^4) \cong \mathbb{Z} \oplus \pi_6(\mathbb{S}^3)$.*

Proof. This is a consequence of Theorem 5.1.8.11, when applied to the Hopf bundle $(\mathbb{S}^7, \mathrm{proj}, \mathbb{S}^4)$. ☐

Theorem 5.2.2.12. *If $r \geq 1$, the homomorphism $\mathrm{proj}_*\colon \pi_r(\mathbb{S}^{15}) \to \pi_r(\mathbb{S}^8)$ induced by the Hopf map $\mathrm{proj}\colon \mathbb{S}^{15} \to \mathbb{S}^8$ maps $\pi_r(\mathbb{S}^{15})$ isomorphically onto a subgroup of $pi_r(\mathbb{S}^8)$ which has a direct complement isomorphic to $\pi_{r-1}(\mathbb{S}^7)$. In particular, $\pi_{15}(\mathbb{S}^8) \cong \mathbb{Z} \oplus \pi_{14}(\mathbb{S}^7)$.*

Proof. This is a consequence of Theorem 5.1.8.11, when applied to the Hopf bundle $(\mathbb{S}^{15}, \mathrm{proj}, \mathbb{S}^8)$. ☐

Theorem 5.2.2.13. *The composite maps*

$$\pi_{r-1}(\mathbb{S}^1) \xrightarrow{\mathrm{sus}} \pi_r(\mathbb{S}^2) \xrightarrow{\Delta} \pi_{r-1}(\mathbb{S}^1), \quad \pi_{r-1}(\mathbb{S}^3) \xrightarrow{\mathrm{sus}} \pi_r(\mathbb{S}^4) \xrightarrow{\Delta} \pi_{r-1}(\mathbb{S}^3),$$
$$\pi_{r-1}(\mathbb{S}^7) \xrightarrow{\mathrm{sus}} \pi_r(\mathbb{S}^8) \xrightarrow{\Delta} \pi_{r-1}(\mathbb{S}^7),$$

where the homomorphisms Δ correspond to the Hopf bundles, coincide for any $r \geq 1$ with $\mathrm{id}_{\pi_{r-1}(\mathbb{S}^1)}$, $\mathrm{id}_{\pi_{r-1}(\mathbb{S}^3)}$, and $\mathrm{id}_{\pi_{r-1}(\mathbb{S}^7)}$, respectively.

Proof. Let q be 2, 4, or 8, and consider the map $\chi\colon \mathbb{D}^q \to \mathbb{S}^{q-1}$ given by

$$\chi(x_1, \ldots, x_q) = (x_1, \ldots, x_q, (1 - x_1^2 - \cdots - x_q^2), 0, \ldots, 0).$$

Its restriction to $\mathbb{S}^{q-1}$ is simply the inclusion $\mathbb{S}^{q-1} \to \mathbb{S}^{2q-1}$, while $\mathrm{proj} \circ \chi$, where proj is the Hopf map $\mathrm{proj}\colon \mathbb{S}^{2q-1} \to \mathbb{S}^q$, simply $\mathbb{DS}\colon \mathbb{D}^q \to \mathbb{S}^q$. Therefore, the diagram

$$\begin{tikzcd}
 & \pi_r(\mathbb{S}^{2q-1}, \mathbb{S}^{q-1}) \arrow[dl, "\mathrm{rel\,proj}_*"'] \arrow[dr, "\partial"] & \\
\pi_r(\mathbb{S}^q) & & \pi_{r-1}(\mathbb{S}^{q-1}) \\
 & \pi_r(\mathbb{D}^q, \mathbb{S}^{q-1}) \arrow[ul, "\mathrm{rel}\,\mathbb{DS}_*"] \arrow[ur, "\partial"'] \arrow[uu] &
\end{tikzcd}$$

commutes; here the vertical homomorphisms are induced by $\mathrm{rel}\,\chi\colon (\mathbb{D}^q, \mathbb{S}^{q-1}) \to (\mathbb{S}^{2q-1}, \mathbb{S}^{q-1})$. Moreover, $\mathrm{rel\,proj}_*$ and the lower ∂ are isomorphisms (see Lemma 5.1.8.1 and Remark 5.1.6.7), and from the above commutativity it follows that the composite homomorphism

$$\begin{aligned} \pi_{r-1}(\mathbb{S}^{q-1}) &\xrightarrow{\partial^{-1}} \pi_r(\mathbb{D}^q, \mathbb{S}^{q-1}) \xrightarrow{\mathrm{rel}_*\, \mathbb{DS}_*} \pi_r(\mathbb{S}^q) \\ &\xrightarrow{(\mathrm{rel\,proj}_*)^{-1}} \pi_r(\mathbb{S}^{2q-1}, \mathbb{S}^{q-1}) \xrightarrow{\partial} \pi_{r-1}(\mathbb{S}^{q-1}) \end{aligned}$$

equals $\mathrm{id}_{\pi_{r-1}(\mathbb{S}^{q-1})}$. To complete the proof, notice that $\mathrm{rel}\,\mathbb{DS}_* \circ \partial^{-1} = \mathrm{sus}$ (see Remark 5.2.1.2) and $\partial \circ (\mathrm{rel\,proj}_*)^{-1} = \Delta$ (see Definition 5.1.8.4). ☐

Remark 5.2.2.14. It follows from Theorem 5.2.2.13 and Remark 5.1.8.8 that $\mathrm{sus}\, \pi_5(\mathbb{S}^3) \to \pi_6(\mathbb{S}^4)$ and $\mathrm{sus}\, \pi_{13}(\mathbb{S}^7) \to \pi_{14}(\mathbb{S}^8)$ are isomorphisms, that is, in the two series $\{\pi_{n+2}(\mathbb{S}^n)\}$ and $\{\pi_{n+6}(\mathbb{S}^n)\}$ (as in the series $\{\pi_n(\mathbb{S}^n)\}$) stabilisation begins at least one step earlier than guaranteed by Theorem 5.2.1.4.

5.2.3 The Composition Product

Definition 5.2.3.1. Let X be a space with base point x_0. Given two spheroids, $\varphi \in \mathrm{Sph}_p^{\bigcirc}(X, x_0)$ and $\psi \in \mathrm{Sph}_q^{\bigcirc}(\mathbb{S}^p, \mathrm{ort}_1)$, the composition $\varphi \circ \psi(\mathbb{S}^q, \mathrm{ort}_1) \to (X, x_0)$ is a spheroid in $\mathrm{Sph}_q^{\bigcirc}(X, x_0)$, and the homotopy class of the latter is uniquely determined by the homotopy classes of φ and ψ. Therefore, for any two classes, $\alpha \in \pi_r(X, x_0)$ and $\beta \in \pi_q(\mathbb{S}^p)$, one may define the composition $\alpha \circ \beta \in \pi_q(X, x_0)$. Equivalently, we can set $\alpha \circ \beta = \varphi_*(\beta)$, where φ is any representative of α.

The following facts need no proof:

- if $\alpha \in \pi_p(X, x_0)$, then $\alpha \circ \mathrm{sph}_p = \alpha$;
- if $\alpha \in \pi_p(X, x_0)$, $\beta \in \pi_q(\mathbb{S}^p)$ and $f\colon X \to Y$ is continuous, then $f_*(\alpha \circ \beta) = (f_*(\alpha)) \circ \beta$;
- if $\alpha \in \pi_p(X, x_0)$ and $\beta \in \pi_q(\mathbb{S}^p)$ then $\mathrm{sus}(\alpha \circ \beta) = \mathrm{sus}\,\alpha \circ \mathrm{sus}\,\beta$;
- if $\alpha \in \pi_p(X, x_0)$ and $\beta_1, \beta_2 \in \pi_q(\mathbb{S}^p)$, then $\alpha \circ (\beta_1 + \beta_2) = \alpha \circ \beta_1 + \alpha \circ \beta_2$; [in particular, $\alpha \circ k\ \mathrm{sph}_p = k\alpha$ for all $\alpha \in \pi_p(X, x_0)$ and $k \in \mathbb{Z}$].

The last property is called *right* distributivity, to distinguish it from the left distributivity, which amounts to $(\alpha_1 + \alpha_2) \circ \beta = \alpha_1 \circ \beta + \alpha_2 \circ \beta$ for any $\alpha_1, \alpha_2 \in \pi_p(X, x_0)$ and $\beta \in pi_q(\mathbb{S}^p)$. In genera1, left distributivity does not hold; see Theorem 5.2.3.2 and Lemma 5.2.3.6, and also Exercise 5.2.9.1.

Theorem 5.2.3.2. *Given any* $\alpha_1, \alpha_2 \in \pi_p(X, x_0)$ *and* $\beta \in \pi_{q-1}(\mathbb{S}^{p-1})$,

$$(\alpha_1 + \alpha_2) \circ \mathrm{sus}\,\beta = \alpha_1 \circ \beta + \alpha_2 \circ \mathrm{sus}\,\beta. \tag{5.2.3.3}$$

In particular,

$$(k\,\mathrm{sph}_p) \circ \mathrm{sus}\,\beta = k\,\mathrm{sus}\,\beta$$

for any $\beta \in \pi_{q-1}(\mathbb{S}^{p-1})$ *and* $k \in \mathbb{Z}$.

Proof. Pick representatives $\varphi_1, \varphi_2 \in \mathrm{Sph}_p^{\bigcirc}(X, x_0)$ and $\psi \in \mathrm{Sph}_{q-1}^{\bigcirc}(\mathbb{S}^{p-1}, \mathrm{ort}_1)$ of the classes α_1, α_2 and β, respectively, and let p_1 and p_2 denote the projections $\mathrm{proj}\colon \mathbb{S}^{p-1} \times I \to \mathrm{sus}(\mathbb{S}^{p-1}, \mathrm{ort}_1) = \mathbb{S}^p$ and $\mathrm{proj}\colon \mathbb{S}^{q-1} \times I \to \mathrm{sus}(\mathbb{S}^{q-1}, \mathrm{ort}_1) = \mathbb{S}^q$. By definition (see Remark 5.1.10.2), the class $\alpha_1 + \alpha_2$ is represented by the spheroid

$$p_1(x,t) \mapsto \begin{cases} \varphi_1 \circ p_1(x, 2t), & \text{if} \quad 0 \le t \le 1/2, \\ \varphi_2 \circ p_2(x, 2t-1), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

while $\mathrm{sus}\,\beta$ is represented by the spheroid $p_2(x,t) \mapsto p_1(\psi(x), t)$ (see Remark 5.2.1.1). This shows that both sides of (5.2.3.3) are represented by the spheroid

$$p_2(x,t) \mapsto \begin{cases} \varphi_1 \circ p_1(\psi(x), 2t), & \text{if} \quad 0 \le t \le 1/2, \\ \varphi_2 \circ p_2(\psi(x), 2t-1), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

□

The Ring Stab

Lemma 5.2.3.4. *Let k, ℓ, and n be non-negative integers, with $n > 0$. Then for any $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ and $\beta \in \pi_{n+\ell}(\mathbb{S}^n)$,*

$$\operatorname{sus}^n \alpha \circ \operatorname{sus}^{n+k} \alpha = (-1)^{k\ell} \operatorname{sus}^n \beta \circ \operatorname{sus}^{n+\ell} \alpha.$$

Proof. Pick representatives $\varphi \in \operatorname{Sph}^{\bigcirc}_{n+k}(\mathbb{S}^n, \operatorname{ort}_1)$ and $\psi \in \operatorname{Sph}^{\bigcirc}_{n+\ell}(\mathbb{S}^n, \operatorname{ort}_1)$ of the classes α and β, and for non-negative integers, p and q, let $\operatorname{perm}(p, q)$ be the (auto)homeomorphism of the sphere $\mathbb{S}^{p+q} = (\mathbb{A}^1, \operatorname{ort}_1) \otimes \cdots \otimes (\mathbb{S}^1, \operatorname{ort}_1)$ (see Definition 1.2.8.9) which permutes the factors according to the rule

$$((1, \ldots, p+q) \mapsto (p+1, \ldots, p+q, 1, \ldots, p).$$

One may check directly that the following two compositions are equal:

$$\begin{aligned}
&\operatorname{sus}^n \varphi \circ \operatorname{perm}(n, n+k) \circ \operatorname{sus}^{n+k} \psi \circ \operatorname{perm}(n+k, n+\ell) \colon \mathbb{S}^{2n+k+\ell} \to \mathbb{S}^n,\\
&\operatorname{perm}(n, n) \circ \operatorname{sus}^n \varphi \circ \operatorname{perm}(n, n+\ell) \circ \operatorname{sus}^{n+\ell} \psi \colon \mathbb{S}^{2n+k+\ell} \to \mathbb{S}^n.
\end{aligned}$$

Since $\deg \operatorname{perm}(p, q) = (-1)^{pq}$, this yields

$$\begin{aligned}
\operatorname{sus} \alpha \circ [(-1)^{n(n+k)} \operatorname{sph}_{2n+k}] \circ \operatorname{sus}^{n+k} \beta \circ [(-1)^{(n+k)(n+\ell)} \operatorname{sph}_{2n+k+\ell}]\\
= [(-1)^{n^2} \operatorname{sph}_{2n}] \circ \operatorname{sus}^n \beta \circ [(-1)^{n(n+\ell)} \operatorname{sph}_{2n+\ell}] \circ \operatorname{sus}^{n+\ell} \alpha.
\end{aligned}$$

Using the right and left distributivities (see Definition 5.2.3.11 and Theorem 5.2.3.22), it is not hard to reduce this equality to the form

$$(-1)^{n(n+k)+(n+k)(n+\ell)} \operatorname{sus}^n \alpha \circ \operatorname{sus}^{n+k} \beta = (-1)^{n^2+n(n+\ell)} \operatorname{sus}^n \beta \circ \operatorname{sus}^{n+\ell} \alpha,$$

and now we note that $[n(n+k)+(n+k)(n+\ell)]-[n^2+n(n+\ell)] \equiv k\ell (\bmod 2)$. □

Definition 5.2.3.5. We set $\operatorname{Stab} = \oplus k = 0^\infty \operatorname{Stab}(k)$ and identify each group $\operatorname{Stab}(k)$ with its image under the natural embedding $\operatorname{Stab}(k) \to \operatorname{Stab}$. The operation $\circ$ transforms Stab into a ring: if $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ and $\beta \in \pi_{n+k+\ell}(\mathbb{S}^n)$ then $\operatorname{sus}(\alpha \circ \beta) = \operatorname{sus} \alpha \circ \operatorname{sus} \beta$ (see Definition 5.2.3.1). Therefore, $\circ$ is well defined as a distributive multiplication $\operatorname{Stab}(k) \times \operatorname{Stab}(\ell) \to \operatorname{Stab}(k+\ell)$ (see Definition 5.2.3.1 and Theorem 5.2.3.2), and can be extended bi-distributively to a multiplication $\operatorname{Stab} \times \operatorname{Stab} \to \operatorname{Stab}$. It results from Definition 5.2.3.1, Theorem 5.2.3.2, and Lemma 5.2.3.4 that the ring Stab is associative, has the identity $\operatorname{sph} = \operatorname{sph}_1 = \operatorname{sph}_2 = \cdots$ and is skew-commutative, meaning that $\beta \circ \alpha = (-1)^{k\ell} \alpha \circ \beta$ for any $\alpha \in \operatorname{Stab}(k)$ and $\beta \in \operatorname{Stab}(\ell)$.

An Application

Lemma 5.2.3.6. *The composition $(-\operatorname{sph}_2) \circ \operatorname{proj}_*(\operatorname{sph}_3)$, $\operatorname{proj} \colon \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map, equals $\operatorname{proj}_*(\operatorname{sph}_3)$.*

Proof. This is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r] \arrow[d, "\mathrm{proj}"'] & \mathbb{S}^3 \arrow[d, "\mathrm{proj}"] \\
\mathbb{S}^2 \arrow[r] & \mathbb{S}^2
\end{tikzcd}$$

where the horizontal maps are the spheroids $(x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, x_3, -x_4))$ and $(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$, which represent the classes sph_3 and $-\mathrm{sph}_2$, respectively. ☐

Theorem 5.2.3.7. *The group* Stab(1) *has at most two elements.*

Proof. Since $\pi_4(\mathbb{S}^3)$ is already stable, and $\mathrm{sus}\, \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is epimorphic (see Theorem 5.2.1.4), it suffices to show that $\ker \mathrm{sus}$ contains the $2[\mathrm{proj}\colon \mathbb{S}^3 \to \mathbb{S}^2]_*(\mathrm{sph}_3)$, i.e., twice the generator of $\pi_3(\mathbb{S}^2)$ (see Theorem 5.2.2.10). Indeed, since

$$\begin{aligned} 2\,\mathrm{proj}_*(\mathrm{sph}_3) &= \mathrm{proj}_*(\mathrm{sph}_3) + \mathrm{proj}_*(\mathrm{sph}_3) \\ &= \mathrm{sph}_2 \circ \mathrm{proj}_*(\mathrm{sph}_3) + (-\mathrm{sph}_2) \circ \mathrm{proj}_*(\mathrm{sph}_3) \end{aligned}$$

(see Lemma 5.2.3.6), we get

$$\mathrm{sus}(2\,\mathrm{proj}_*(\mathrm{sph}_3)) = \mathrm{sph}_3 \circ \mathrm{sus}(\mathrm{proj}_*(\mathrm{sph}_3)) + (-\mathrm{sph}_3 \circ (\mathrm{proj}_*(\mathrm{sph}_3))$$

(see Definition 5.2.3.1), and we note that the right-hand side of the last equality is 0 (according to Theorem 5.2.3.2). ☐

5.2.4 Information: Homotopy Groups of Spheres

Remark 5.2.4.1. For a long time the study and computation of homotopy groups of spheres was at the centre of the attention of topologists. It was hoped that one could succeed in solving this problem and that other, more difficult problems in homotopy theory could be reduced to a considerable extent to it. Deep results have actually been obtained in both these directions; the initial hopes, however, have not been realised. Gradually it became clear that from the homotopy point of view the sphere is not elementary, but rather an intricate, complicated object. On the other hand, the information acquired about the homotopy groups of spheres found unexpected applications, first of all in differential topology.

Below we discuss a (rather small) part of these results: general results in Remark 5.2.4.2, and those of tabular character in Remark 5.2.4.3. For more complete information, references, and proofs, see [7].

Remark 5.2.4.2. $\pi_{4m-1}(\mathbb{S}^{2m})$, $m = 1, 2, \dots$, are the only infinite groups among $\pi_r(\mathbb{S}^n)$ with $r > n$. Each of these infinite groups is isomorphic to a direct sum $\mathbb{Z} \oplus$ (finite group).

For an odd prime p, the order of the group $\mathrm{Stab}(2m(p-1)-1)$ with $1 \leq m \leq p-1$ is divisible by p, but not by p^2, while the order of $\mathrm{Stab}(k)$ with $k < 2p(p-1)-2$ is not divisible by p if $k \not\equiv -1 \bmod 2(p-1)$.

Remark 5.2.4.3. Among the groups $\pi_r(\mathbb{S}^n)$ which have been computed are all $\pi_{n+k}(\mathbb{S}^n)$ with $k \leq 22$, and all $\mathrm{Stab}(k)$ with $k \leq 7$. The group $\pi_{n+k}(\mathbb{S}^n)$ with $n \geq 2$ and $1 \leq k \leq 7$ are displayed in Table 5.1.

$n\diagdown k$	1	2	3	4	5	6	7
2	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
3	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/15\mathbb{Z}$
4		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}$	$\mathbb{Z}/15\mathbb{Z}$
5			$\mathbb{Z}/24\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/30\mathbb{Z}$
6				0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/60\mathbb{Z}$
7					0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/120\mathbb{Z}$
8						$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/120\mathbb{Z}$
9							$\mathbb{Z}/240\mathbb{Z}$

Table 5.1: $\pi_{n+k}(\mathbb{S}^n)$ with $n \geq 2$ and $1 \leq k \leq 7$

In Table 5.2, where proj always denotes one of the Hopf maps $\mathbb{S}^3 \to \mathbb{S}^2$, $\mathbb{S}^7 \to \mathbb{S}^4$, or $\mathbb{S}^{15} \to \mathbb{S}^8$, we indicate the generators of the groups $\mathrm{Stab}(k)$ with $k = 1, \ldots, 7$.

Groups	Generators
$\mathrm{Stab}(1) = \pi_4(\mathbb{S}^3)[\cong \mathbb{Z}/2\mathbb{Z}]$	$\mathrm{sus}(\mathrm{proj}_*(\mathrm{sph}_3))$
$\mathrm{Stab}(2) = \pi_6(\mathbb{S}^4)[\cong \mathbb{Z}/2\mathbb{Z}]$	$\mathrm{sus}^2(\mathrm{proj}_*(\mathrm{sph}_3)) \circ \mathrm{sus}^3(\mathrm{proj}_*(\mathrm{sph}_3))$
$\mathrm{Stab}(3) = \pi_8(\mathbb{S}^5)[\cong \mathbb{Z}/24\mathbb{Z}]$	$\mathrm{sus}(\mathrm{proj}_*(\mathrm{sph}_7))$
$\mathrm{Stab}(4) = \pi_{10}(\mathbb{S}^6)[= 0]$	-
$\mathrm{Stab}(5) = \pi_{12}(\mathbb{S}^7)[= 0]$	-
$\mathrm{Stab}(6) = \pi_{14}(\mathbb{S}^8)[\cong \mathbb{Z}/2\mathbb{Z}]$	$\mathrm{sus}^4(\mathrm{proj}_*(\mathrm{sph}_7)) \circ \mathrm{sus}^7(\mathrm{proj}_*(\mathrm{sph}_7))$
$\mathrm{Stab}(7) = \pi_{16}(\mathbb{S}^9)[\cong \mathbb{Z}/240\mathbb{Z}]$	$\mathrm{sus}(\mathrm{proj}_*(\mathrm{sph}_{15}))$

Table 5.2: Stab(k) and generators for Hopf fibrations

We add the relations

$$\mathrm{sus}^3(\mathrm{proj}_*(\mathrm{sph}_3)) \circ \mathrm{sus}^4(\mathrm{proj}_*(\mathrm{sph}_3)) \circ \mathrm{sus}^5(\mathrm{proj}_*(\mathrm{sph}_3)) = 12\,\mathrm{sus}(\mathrm{proj}_*(\mathrm{sph}_7)),$$
$$\mathrm{sus}^7(\mathrm{proj}_*(\mathrm{sph}_3)) \circ \mathrm{sus}^6(\mathrm{proj}_*(\mathrm{sph}_7)) \circ \mathrm{sus}^9(\mathrm{proj}_*(\mathrm{sph}_7)) = 120\,\mathrm{sus}(\mathrm{proj}_*(\mathrm{sph}_{15})),$$

which give, together with Table 1, a complete description of $\oplus_{k=1}^{7} \mathrm{Stab}(k)$, a part of Stab.

The groups $\mathrm{Stab}(k)$ with $k = 8, \ldots, 15$ are listed in Table 5.3.

k	$\operatorname{Stab} k$	k	$\operatorname{Stab} k$
8	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	12	0
9	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	13	$\mathbb{Z}/3\mathbb{Z}$
10	$\mathbb{Z}/2\mathbb{Z}$	14	$\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
11	$\mathbb{Z}/504\mathbb{Z}$	15	$\mathbb{Z}/480\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Table 5.3: $\operatorname{Stab}(k)$ with $k = 8, \ldots, 15$

5.2.5 The Homotopy Groups of Projective Spaces and Lenses

Theorem 5.2.5.1. *Let* $2 \leq n \leq \infty$*.* $\pi_1(\mathbb{R}P^n, (1:0:0:\cdots))$ *has two elements and is generated by the class of the loop* $\mathrm{rp}\colon I \to \mathbb{R}P^n$ *given by* $\mathrm{rp}(t) = (\cos \pi t : \sin \pi t : 0 : 0 : \cdots)$*.* $\pi_r(\mathbb{R}P^n, (1:0:0:\cdots))$ *is isomorphic to* $\pi_r(\mathbb{S}^n)$ *for all* $r \neq 1$ *[in particular, this group is trivial for* $n = \infty$*] and the isomorphism is induced by the projection* $\mathbb{S}^n \to \mathbb{R}P^n$*.* $\mathbb{R}P^n$ *is simple for* n *odd, and is not* n*-simple for* n *even.*

(The case $n = 1$ has been considered in Theorem 5.2.2.2.)

Proof. All assertions concerning the groups $\pi_r(\mathbb{R}P^n, (1:0:0:\cdots))$ follow from Theorem 5.1.8.13, when applied to the covering $(\mathbb{S}^n, \mathrm{proj}, \mathbb{R}P^n)$. Since the fundamental group of $\mathbb{R}P^n$ is Abelian $\mathbb{R}P^n$ is 1-simple. Now let $r \geq 2$, and consider the automorphism

$$T_{\mathrm{rp}}\colon \pi_r(\mathbb{R}P^n, (1:0:0:\cdots)) \to \pi_r(\mathbb{R}P^n, (1:0:0:\cdots)).$$

) Let $\widetilde{\mathrm{rp}}$ be the path in $\mathbb{S}^n$ which covers rp and has origin ort_1. Then from the (obvious) commutativity of the diagramme

$$\begin{tikzcd}
\pi_r(\mathbb{S}^n, \mathrm{ort}_1) \arrow[r, "T_{\widetilde{\mathrm{rp}}}"] \arrow[d, "\mathrm{proj}_*"'] & \pi_r(\mathbb{S}^n, -\mathrm{ort}_1) \arrow[d, "\mathrm{proj}_*"] & \pi_r(\mathbb{S}^n, \mathrm{ort}_1) \arrow[l, "(-\mathrm{id}_{\mathbb{S}^n})_*"'] \arrow[dl, "\mathrm{proj}_*"] \\
\pi_r(\mathbb{R}P^n, (1:0:0:\cdots)) \arrow[r, "T_{\mathrm{rp}}"'] & \pi_r(\mathbb{R}P^n, (1:0:0:\cdots)) &
\end{tikzcd}$$

If we make the identification $\pi_r(\mathbb{S}^n, -\mathrm{ort}_1) = \pi_r(\mathbb{S}^n, \mathrm{ort}_1) = \pi_r(\mathbb{S}^n)$ (see Corollary 5.2.2.5), then we see that T_{rp} is the identity if and only if

$$(-\mathrm{id}_{\mathbb{S}^n})_*\colon \pi_r(\mathbb{S}^{n)} \to \pi_r(\mathbb{S}^n)$$

is the identity. Finally, note that if n is odd, then $-\mathrm{id}_{\mathbb{S}^n}$ and $\mathrm{id}_{\mathbb{S}^n}$ are homotopic, and hence $(-\mathrm{id}_{\mathbb{S}^n})_*\colon \pi_r(\mathbb{S}^n) \to \pi_r(\mathbb{S}^n)$ is the identity automorphism for all r, while if n is even,

$$[(-\mathrm{id}_S^n)_*\colon (\mathbb{S}^n) \to \pi_r(\mathbb{S}^n)] = -\mathrm{id}_{\pi_n(\mathbb{S}^n)}\,.$$

$\square$

Theorem 5.2.5.2. *Let $1 \neq n \neq \infty$. $\pi_2(\mathbb{C}P^n, (1:0:0:\cdots))$ is isomorphic to $\mathbb{Z}$ and is generated by the class of the spheroid* $\mathrm{incl}\colon \mathbb{S}^2 = \mathbb{C}P^1 \to \mathbb{C}P^n$. *$\pi_r(\mathbb{C}P^n, (1:0:0:\cdots))$ is isomorphic to $\pi_r(\mathbb{S}^{2n+1})$ for any $r \neq 2$ [in particular, this group is trivial for $n = \infty$], and the isomorphism is induced by the projection $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$.*

(For $n = 1$ this theorem repeats Theorem 5.2.2.10.)

Proof. We make two claims:

- $\mathrm{incl}_*\colon \pi_2(\mathbb{C}P^1, (1:0)) \to \pi_2({}^n, (1:0:0:\cdots))$ is an isomorphism;
- $\mathrm{proj}_*\colon \pi_r(\mathbb{S}^{2n+1}, \mathrm{ort}_1) \to \pi_r(\mathbb{C}P^r, (1:0:0:\cdots))$ is an isomorphism for all $r \neq 2$.

The first follows from the 3-connectedness of the pair $(\mathbb{C}P^1, \mathbb{C}P^1)$ (see Theorem 2.3.2.4 and Remark 2.1.3.5), while the second is a consequence of the homotopy sequence of the bundle $(\mathbb{S}^{2n+1}, \mathrm{proj}, {}^n)$ and Theorem 5.2.2.2. ☐

Theorem 5.2.5.3. *Let $1 \neq n \neq \infty$ and $r \geq 1$. The homomorphism induced by the projection $\mathbb{S}^{4n+3} \to \mathbb{H}P^n$ maps $\pi_r(\mathbb{S}^{4n+3})$ isomorphically onto a subgroup of $\pi_r(\mathbb{H}P^n, (1:0:0:\cdots))$ which has a direct complement isomorphic to $\pi_{r-1}(\mathbb{S}^3)$. In particular, $\pi_r(\mathbb{H}P^n, (1:0:0:\cdots))$ isomorphic to $\pi_{r-1}(\mathbb{S}^3)$ for all $r \geq 1$.*

(For $n = 1$ this theorem repeats Theorem 5.2.2.11.)

Proof. We need only apply Theorem 5.1.8.11 to the bundle, $(\mathbb{S}^{4n+3}, \mathrm{proj}, \mathbb{H}P^n)$. ☐

Theorem 5.2.5.4. *The lenses $L(m; \ell_1, \dots, \ell_n)$ and $L(m; \ell_1, \ell_2, \dots)$ are simple. $\pi_1(L(m; \ell_1, \dots, \ell_n))$ and $\pi_1(L(m; \ell_1, \ell_2, \dots))$ are isomorphic to $\mathbb{Z}/m\mathbb{Z}$. If $r \geq 2$, $\pi_r(L(m; \ell_1, \dots, \ell_n))$ is isomorphic to $\pi_r(\mathbb{S}^{2n-1})$, and the isomorphism is induced by the projection $\mathbb{S}^{2n-1} \to L(m; \ell_1, \dots, \ell_n)$. $\pi_r(L(m; \ell_1, \ell_2, \dots))$ is trivial for all $r \geq 2$.*

Proof. The proof is clearly a generalisation of the first part of the proof of Theorem 5.2.5.1. ☐

5.2.6 The Homotopy Groups of Classical Groups

Theorem 5.2.6.1. *The inclusion homomorphism $\pi_r(\mathrm{SO}(n)) \to \pi_r(\mathrm{SO}(n+1))$ is an isomorphism for $r \leq n-2$ and an epimorphism for $r = n-1$.*

$$\pi_1(\mathrm{SO}(n)) \cong \begin{cases} \mathbb{Z} & \textit{for} \quad n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \textit{for} \quad n \geq 3, \end{cases}$$

and is generated by the class of the inclusion $\mathbb{S}^1 = \mathrm{SO}(2) \to \mathrm{SO}(n)$. $\pi_2(\mathrm{SO}(n))$ *is trivial for all* n.

$$\pi_3(SO(n)) \cong \begin{cases} 0 & \textit{for} \quad n \leq 2, \\ \mathbb{Z} & \textit{for} \quad n = 3, \\ \mathbb{Z} \oplus \mathbb{Z} & \textit{for} \quad n = 4, \\ (\mathbb{Z} \oplus \mathbb{Z})/(\textit{a cyclic subgroup}) & \textit{for} \quad n \geq 5. \end{cases}$$

Proof. The triviality of the groups $\pi_2(\mathrm{SO}(2))$, $\pi_3(\mathrm{SO}(2))$, $\pi_2(\mathrm{SO}(3))$, and $\pi_2(\mathrm{SO}(4))$, and the isomorphisms $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$, $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_3(\mathrm{SO}(3)) \cong \mathbb{Z}$, and $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$ all result from the equalities $\mathrm{SO}(2) = \mathbb{S}^1$, $\mathrm{SO}(3) = \mathbb{R}P^3$, and $\mathrm{SO}(4) = \mathbb{R}P^3 \times \mathbb{S}^3$ (see Remark 3.2.1.3, Theorem 3.2.3.1, and Remark 3.2.2.3), and Theorems 5.2.2.2, 5.2.2.7, and 5.2.5.1. The rest is a consequence of the homotopy sequence of the bundle $(\mathrm{SO}(n+1), \mathrm{proj}, \mathbb{S}^n)$ with base point $\mathrm{id} \in \mathrm{SO}(n+1)$; see Example 4.6.1.4. ☐

Theorem 5.2.6.2. *The inclusion homomorphism* $\pi_r * (\mathrm{U}(n)) \to \pi_r(\mathrm{U}(n+1))$ *is an isomorphism for* $r \leq 2n-1$ *and an epimorphism for* $r = 2n$. *If* $n \geq 1$, $\pi_1(\mathrm{U}(n))$ *is isomorphic to* $\mathbb{Z}$ *and is generated by the class of the inclusion* $\mathbb{S}^1 = \mathrm{U}(1) \to \mathrm{U}(n)$. $\pi_2(\mathrm{U}(n))$ *is trivial for all* n.

$$\pi_3(\mathrm{U}(n)) \cong \begin{cases} 0 & \textit{for} \quad n = 1, \\ \mathbb{Z} & \textit{for} \quad n \geq 2. \end{cases}$$

The inclusion homomorphism $\pi_1(\mathrm{U}(n)) \to \pi_r(\mathrm{SO}(2n))$ *is epimorphic for all* n.

Proof. These are corollaries of the equalities $[\mathrm{incl}\colon \mathrm{U}(1) \to \mathrm{SO}(2)] = \mathrm{id}$ and $\mathrm{U}(2) = \mathbb{S}^1 \times \mathbb{S}^3$, and of the homotopy sequence of the bundle $(\mathrm{U}(n+1), \mathrm{proj}, \mathbb{S}^{2n+1})$ with base point $\mathrm{id} \in \mathrm{U}(n+1)$; see Example 4.6.1.4. ☐

Theorem 5.2.6.3. *The inclusion homomorphism* $\pi_r(\mathrm{Sp}(n)) \to \pi_r(\mathrm{Sp}(n+1))$ *is an isomorphism for* $r \leq 4n+1$ *and an epimorphism for* $r = 4n+2$. *In particular, if* $r \leq 5$ *and* $n \geq 1$, $\pi_r(\mathrm{Sp}(n))$ *is isomorphic to* $\pi_r(\mathrm{Sp}(1) = \mathbb{S}^3)$.

Proof. This can be seen from the homotopy sequence of the bundle $(\mathrm{Sp}(n+1), \mathrm{proj}, \mathbb{S}^{4n+3})$ with base point $\mathrm{id} \in \mathrm{Sp}(n+1)$; see Example 4.6.1.4. ☐

Stabilisation

Definition 5.2.6.4. Theorems 5.2.6.1, 5.2.6.2 and 5.2.6.3 show that for $r \geq 1$, each series of groups

$$\begin{gathered} \pi_r(\mathrm{SO}(1)) \to \pi_r(\mathrm{SO}(2)) \to \pi_3(\mathrm{SO}(3)) \to \cdots, \\ \pi_r(\mathrm{U}(1)) \to \pi_r(\mathrm{U}(2)) \to \pi_r(\mathrm{U}(3)) \to \cdots, \\ \pi_r(\mathrm{Sp}(1)) \to \pi_r(\mathrm{Sp}(2)) \to \pi_r(\mathrm{Sp}(3)) \to \cdots, \end{gathered}$$

stabilises: the first one, starting with $\pi_r(\mathrm{SO}(r+2))$, the second one, with $\pi_r(\mathrm{U}([r+2)/2]))$, and the third one, with $\pi_r(\mathrm{Sp}([(r+2)/4]))$. The groups

$\pi_r(\mathrm{SO}(n))$ with $n \geq r+2$, $\pi_r(\mathrm{U}(n))$ with $n \geq [(r+2)/2]$, and $\pi_r(\mathrm{Sp}(n))$ with $n \geq [(r+2)/4]$ are said to be *stable*, and are denoted by $\pi_r(\mathrm{SO})$, $\pi_r(\mathrm{U})$, and $\pi_r(\mathrm{Sp})$, respectively. By Theorem 5.2.6.1, $\pi_r(\mathrm{SO}) \cong \mathbb{Z}$, $\pi_2(\mathrm{SO}) = 0$, and $\pi_3(\mathrm{SO}) \cong (\mathbb{Z} \oplus \mathbb{Z})/(\text{cyclic subgroup})$. By Theorem 5.2.6.2, $\pi_1(\mathrm{U}) \cong \mathbb{Z}$, $\pi_2(\mathrm{U}) = 0$, and $\pi_3(\mathrm{U}) \cong \mathbb{Z}$. Finally, by Theorem 5.2.6.3, $\pi_1(\mathrm{Sp}) = 0$, $\pi_2(\mathrm{Sp}) = 0$, and $\pi_3(\mathrm{Sp}) \cong \mathbb{Z}$.

The notations $\pi_r(\mathrm{SO})$, $\pi_r(\mathrm{U})$, and $\pi_r(\mathrm{Sp})$ have also a direct meaning: they represent the ordinary r-th homotopy groups of the limit spaces $\mathrm{SO} = \lim \mathrm{SO}(n)$, $\mathrm{U} = \lim \mathrm{U}(n)$, and $\mathrm{Sp} = \lim \mathrm{Sp}(n)$, respectively (see Theorem 5.1.11.5).

Information

Remark 5.2.6.5. The homotopy groups $\pi_r(\mathrm{SO})$, $\pi_r(\mathrm{U})$, and $\pi_r(\mathrm{Sp})$ have been explicitly computed. Namely, for any $r \geq 1$ there are canonical isomorphisms $\pi_r(\mathrm{SO}) \to \pi_{r+8}(\mathrm{SO})$, $\pi_r(\mathrm{Sp}) \to \pi_{r+8}(\mathrm{Sp})$, and $\pi_r(\mathrm{U}) \to \pi_{r+2}(\mathrm{U})$, and the first seven homotopy groups of SO and Sp, together with the first two homotopy groups of U are displayed in the following tables.

r	1	2	3	4	5	6	7	8
$\pi_r(\mathrm{SO})$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$\pi_r(\mathrm{Sp})$	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0

Table 5.4:

r	1	2
$\pi_r(\mathrm{U})$	$\mathbb{Z}$	0

Table 5.5:

For a proof, see [17].

There are also many unstable homotopy groups of the manifolds $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$ which have been computed. For example, $\pi_{2n}(\mathrm{U}(n)) \cong \mathbb{Z}/n!\mathbb{Z}$,

$$\pi_{4n+2}(\mathrm{Sp}(n)) \cong \begin{cases} \mathbb{Z}/(2n+1)!\mathbb{Z} & \text{for } n \text{ even}, \\ \mathbb{Z}/2[(2n+1)!]\mathbb{Z} & \text{for } n \text{ odd}. \end{cases}$$

For details and references for the proofs, see [7].

5.2.7 The Homotopy Groups of Stiefel Manifolds and Spaces

Lemma 5.2.7.1. *Let $k < n$. Then*

- *The manifold $V(n,k)$ is simple.*
- *The inclusion homomorphism $\pi_r(V(n,k)) \to \pi_r(V(n+1,k+1))$ is an isomorphism for $r < n-1$ and an epimorphism for $r = n-1$.*

- *If n is odd and $k = 1$, the last epimorphism is also an isomorphism.*
- *The manifolds $\mathbb{C}V(n,k)$ and $\mathbb{H}V(n,k)$ are all simple.*
- *The inclusion homomorphism $\pi_r(\mathbb{C}V(n,k)) \to \pi_r(\mathbb{C}V(n+1,k+1))$ is an isomorphism for $r < 2n$ and an epimorphism for $r = 2n$.*
- *The inclusion homomorphism $\pi_r(\mathbb{H}V(n,k)) \to \pi_r(\mathbb{H}V(n+1,k+1))$ is an isomorphism for $r < 4n+2$ and an epimorphism for $r = 4n+2$.*

Proof. The fact that the Stiefel manifolds are simple may be seen from the equalities $V(n,k) = \mathrm{SO}(n)/\mathrm{SO}(n-k)$, $\mathbb{C}V(n,k) = \mathrm{U}(n)/\mathrm{U}(n-k)$, and $\mathbb{H}V(n,k) = \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$ (see Remark 4.2.3.16 and Corollary 5.1.9.7). To prove the rest, use the homotopy sequences of the bundles $(V(n+1,k+1), \mathrm{proj}, \mathbb{S}^n)$, $(\mathbb{C}V(n+1,k+1), \mathrm{proj}, \mathbb{S}^{2n+1})$, and $(\mathbb{H}V(n+1,k+1), \mathrm{proj}, \mathbb{S}^{4n+3})$ described in Example 4.6.1.4, taking the inclusions $\mathbb{R}^k \to \mathbb{R}^n$, $\mathbb{C}^k \to \mathbb{C}^n$, and $\mathbb{H}^k \to \mathbb{H}^n$ as base points (in the respective total spaces). In the real case, we take advantage, in addition, of the fact that for n odd, the bundle $(V(n+1,2), \mathrm{proj}, \mathbb{S}^n)$ admits a section (see Example 3.1.4.9); for n odd and $k = 1$, this ensures that the first of the aforementioned homotopy sequences splits from the left at the terms $\pi_r(V(n+1,k+1))$ (see Theorem 5.1.8.9.) ☐

Corollary 5.2.7.2. *If $k < n$, then the manifold $V(n,k)$ is $(n-k-1)$-connected. $\pi_{n-k}(V(n,k))$ with $0 < k < n$ is cyclic and is generated by the class of the inclusion $\mathbb{S}^{n-k} = V(n-k+1,1) \to V(n,k)$; this group is infinite whenever $n-k$ is even or $k = 1$.*

Proof. This is a corollary of Lemma 5.2.7.1: when $r < n-k$,

$$\pi_r(V(n,k)) \cong \pi_r(V(n-1,k-1)) \cong \cdots \cong \pi_r(V(n-k+1,1)) = \pi_r(\mathbb{S}^{n-k}) = 0,$$

while in the sequence

$$\pi_{n-k}(\mathbb{S}^{n-k}) = \pi_{n-k}(V(n{-}k{+}1,1)) \to \pi_{n^-k}(V(n{-}k{+}2,2)) \to \cdots \to \pi_{n-k}, (V(n,k))$$

all the maps are isomorphisms, except for the first which is an isomorphism for $n-k$ even and an epimorphism for $n-k$ odd. ☐

Corollary 5.2.7.3. *The manifold $\mathbb{C}V(n,k)$ is $2(n-k)$-connected .*

$$\pi_{2n-2k+1}(\mathbb{C}V(n,k)) \cong \mathbb{Z}$$

is generated by the class of the inclusion

$$\mathbb{S}^{2n-2k+1} = \mathbb{C}V(n-k+1,1) \to \mathbb{C}V(n,k)$$

Proof. This is a corollary of Lemma 5.2.7.1 : when $r \leq 2n-2k+1$, in the sequence

$$\pi_r(\mathbb{S}^{2n-2k+1}) = \pi_r(\mathbb{C}V(n{-}k{+}1,1)) \to \pi_r(\mathbb{C}V(n{-}k{+}2,2)) \to \cdots \to \pi_r, (\mathbb{C}V(n,k))$$

all the arrows are isomorphisms. ☐

Corollary 5.2.7.4. *The manifold* $\mathbb{H}V(n,k)$ *is* $(4n-4k+2)$*-connected.*

$$\pi_{4n-4k+3}(\mathbb{H}V(n,k)) \cong \mathbb{Z}$$

is generated by the class of the inclusion

$$\mathbb{S}^{4n-4k+3} = \mathbb{H}V(n-k+1,1) \to \mathbb{H}V(n,k).$$

Proof. This is also a corollary of Lemma 5.2.7.1: if $r \leq 4n-4k+3$ (actually, if $r \leq 4n-4k+5$), then in the sequence

$$\pi_r(\mathbb{S}^{4n-4k+3}) = \pi_r(\mathbb{H}V(n{-}k{+}1,1)) \to \pi_r(\mathbb{H}V(n{-}k{+}2,2)) \to \cdots \to \pi_r,(\mathbb{H}V(n,k))$$

all the arrows are isomorphisms. □

Theorem 5.2.7.5. *The spaces* $V(\infty,k)$ *and* $\mathbb{C}V(\infty,k)$ *(see Definition 4.5.3.10), as well as*

$$\mathbb{H}V(\infty,k) = \varinjlim(\mathbb{H}V(n,k), \mathrm{incl}\colon \mathbb{H}V(n,k) \to \mathbb{H}V(n+1,k))$$

are ∞*-connected.*

Proof. This follows from Corollaries 5.2.7.2, 5.2.7.3, 5.2.7.4, and Theorem 5.1.11.5. □

5.2.8 The Homotopy Groups of Grassmann Manifolds and Spaces

Remark 5.2.8.1. In this subsection the computation of the most important homotopy groups of the Grassmann manifolds $G(n,k)$, $G_+(n,k)$, $\mathbb{C}G(n,k)$, and $\mathbb{H}G(n,k)$, and of the Grassmann spaces $G(\infty,k)$, $G_+(\infty,k)$, $\mathbb{C}G(\infty,k)$ (see Definition 4.5.3.2) and

$$\mathbb{H}G(\infty,k) = \varinjlim(\mathbb{H}G(n,k), \mathrm{incl}\colon \mathbb{H}G(n,k) \to \mathbb{H}G(n+1,k))$$

is reduced to the computation of the homotopy groups of the corresponding classical groups.

Grassmann manifolds and spaces are taken care of together, and thus n may also take the value ∞.

Theorem 5.2.8.2. *If* $k > 0$ *and* $0 < r < n-k$*, then* $\pi_r(G_+(n,k))$ *is isomorphic to* $\pi_{r-1}(\mathrm{SO}(k))$*, and the inclusion homomorphism* $\pi_r(G_+(n,k)) \to \pi_r(G_+(n',k))$ *is an isomorphism for all* $n' > n$*.*

Proof. The first claim results from Theorems Corollary 5.2.7.2 and Theorem 5.2.7.5, and the homotopy sequence of the bundle $(V(n,k), \mathrm{proj}, G_+(n,k))$, defined in Example 4.6.1.4, with the inclusion $\mathbb{R}^k \to \mathbb{R}^n$ as base point. The second claim results from the commutativity of the diagramme

$$\begin{tikzcd}
\pi_r(G_+(n,k)) \arrow[r, "\Delta"] \arrow[d, "\mathrm{incl}_*"'] & \pi_{r-1}(\mathrm{SO}(k)) \arrow[d, "\mathrm{incl}_*=\mathrm{id}"] \\
\pi_r(G_+(n',k)) \arrow[r, "\Delta"'] & \pi_{r-1}(\mathrm{SO}(k))
\end{tikzcd}$$

(see Theorem 5.1.8.7). □

Theorem 5.2.8.3.

$$\pi_r(G(n,k)) \cong \pi_r(G_+(n,k)), \quad 0<k<n \quad \textit{and} \quad r \geq 2,$$
$$\pi_1(G(n,k)) \cong \begin{cases} \mathbb{Z} & \textit{for} \quad n=2 \quad \textit{and} \quad r \geq 2, \\ \mathbb{Z}/2\mathbb{Z} & \textit{for} \quad 0<k<n \quad \textit{and} \quad n \geq 3. \end{cases}$$

Proof. Since $G(2,1)$ is homeomorphic to $\mathbb{S}^1$, Theorem 5.2.2.2 yields $\pi_1(G(2,1)) \cong \mathbb{Z}$. If we now apply Theorem 5.1.8.13 to the canonical two–sheeted covering $(G_+(n,k), \mathrm{proj}, G(n,k))$, the rest is plain. □

Theorem 5.2.8.4. *If* $0<r<2n-2k+1$*, then* $\pi_r(\mathbb{C}G(n,k)) \cong \pi_{r-1}(\mathrm{U}(k))$*, and the inclusion homomorphism* $\pi_r(\mathbb{C}G(n,k)) \to \pi_r(\mathbb{C}G(n',k))$ *is an isomorphism for all* $n'>n$*.*

Proof. The proof repeats that of Theorem 5.2.8.2 *mutatis mutandis.* □

Theorem 5.2.8.5. *If* $0<r<4n-4k+3$*, then* $\pi_r(\mathbb{H}G(n,k)) \cong \pi_{r-1}(\mathrm{Sp}(k))$*, and the inclusion homomorphism* $\pi_r(\mathbb{H}G(n,k)) \to \pi_r(\mathbb{H}G(n',k))$ *is an isomorphism for all* $n'>n$*.*

Proof. Again, the proof repeats that of Theorem 5.2.8.2 *mutatis mutandis.* □

5.2.9 Exercises

Exercise 5.2.9.1. Let $q=2,4,8$, and let $\mathrm{proj}\colon \mathbb{S}^{2q-1} to \mathbb{S}^q$ be the Hopf map. Show that for any integer k

$$(k\,\mathrm{sph}_q) \circ \mathrm{proj}_*(\mathrm{sph}_{2q-1}) = k^2\,\mathrm{proj}_*(\mathrm{sph}_{2q-1}).$$

Exercise 5.2.9.2. Show that for any positive integer n, $\mathbb{R}P^n$ is $(n+1)$-simple.

Exercise 5.2.9.3. Let n be even and k be odd. Show that $G(n,k)$ is simple.

Exercise 5.2.9.4. Let $3 \leq n \leq \infty$. Show that $G(n,2)$ is not 2-simple.

Exercise 5.2.9.5. Show that the inclusion homomorphisms

$$\pi_r(\mathrm{SO}(3)) \to \pi_r(\mathrm{SO}(4)), \quad \pi_r(\mathrm{SO}(7)) \to \pi_r(\mathrm{SO}(8)), \quad \pi_r(\mathrm{U}(1)) \to \pi_r(\mathrm{U}(2)),$$
$$\pi_r(\mathrm{U}(3)) \to \pi_r(\mathrm{U}(4)), \quad \pi_r(\mathrm{Sp}(1)) \to \pi_r(\mathrm{Sp}(2))$$

are monomorphic for any integer r.

Exercise 5.2.9.6. Consider the map $\mathbb{C}V(n,k) \to V(2n,2k-1)$ which takes each k-frame $(v_1,\dots,v_k)$ of $\mathbb{C}^n$ into $(v_1, iv_1, \dots, v_{k-1}, iv_{k-1}, v_k)$ of $\mathbb{C}^n$, considered as $\mathbb{R}^{2n}$. Show that the homomorphism

$$\pi_{2n-2k+1}(\mathbb{C}V(n,k)) \to \pi_{2n^2k+1}(V(2n,2k-1))$$

induced by this map takes the generator of $\pi_{2n-2k+1}(\mathbb{C}V(n,k))$ indicated in Corollary 5.2.7.3 into the generator of $\pi_{2n-2k+1}(V(2n,2k-1))$ indicated in Corollary 5.2.7.2.

5.3 HOMOTOPY GROUPS OF CELLULAR SPACES

5.3.1 The Homotopy Groups of One-dimensional Cellular Spaces

Definition 5.3.1.1. In this subsection we compute the homotopy groups of a bouquet $B = \bigvee_{\mu \in M}(S_\mu = \mathbb{S}^1, \mathrm{ort}_1)$ constructed from an arbitrary family, $\{S_\mu = \mathbb{S}^1\}_{\mu \in M}$ of circles. As usual, the base point bp will be the centre of the bouquet.

To simplify the exposition, we let u_μ and α_μ denote the loop defined by the inclusion $\mathrm{Imm}_\mu \colon \mathbb{S}^1 \to B$, i.e., the loop $\mathrm{Imm}_\mu \circ I\mathbb{S} \colon I \to B$, and the homotopy class of u_μ, i.e., $\mathrm{Imm}_{\mu *}(\mathrm{sph}_1)$, respectively. A loop will be referred to as *standard* if it is of the form $(\cdots((v_1 v_2)v_3)\cdots v_{n-1})v_n$, where each of the factors $v_1, \dots, v_n$ is either one of the loops u_μ or one of their inverses u_μ^{-1} and, in addition, two loops u_μ, u_μ^{-1} with the same μ are not allowed to be adjacent. The case $n = 0$ is not excluded: then, the product is simply the constant loop with origin bp.

Lemma 5.3.1.2. *There is a covering $(\widetilde{B}, p, B)$ with the following two properties:*

- *(i) $\widetilde{B}$ is contractible;*
- *(ii) the paths which cover standard loops and originate at some point x_0 of the fibre $F_0 = p^{-1}(\mathrm{bp})$ end at distinct points of F_0, and F_0 is exhausted by the ends of these paths.*

Proof. Let us agree to denote by $GF(M)$, as usual, the free group generated by the set M. We equip $GF(M)$ with the discrete topology, form the bouquet $A = \bigvee_{\mu \in M}(D_\mu = \mathbb{D}^1, 0)$, and then the product $A \times GF(M)$. Further, let $\wp$ be the partition of $A \times GF(M)$ into the pairs $\{(\mathrm{Imm}_{mu}(1), g), (\mathrm{Imm}_\mu(-1), g\mu)\}$ with $y \in M$ and $g \in GF(M)$, and the points which do not appear in any of these pairs, and denote by ρ the composition

$$A \times GF(M) \xrightarrow{\mathrm{proj}_1} A \xrightarrow{\bigvee_\mu (\mathbb{D}S_\mu = \mathbb{D}\mathbb{S})} B.$$

Then ρ is obviously constant on the elements of $\wp$. Now set

$$\widetilde{B} = [A \times GF(M)]/\wp, \quad p = [\mathrm{fact}\,\rho \colon \widetilde{B} \to B], \quad x_0 = \mathrm{proj}(a_0, e),$$

where a_0 is the centre of the bouquet A, $e = e_{GF(M)}$, and

$$\mathrm{proj} = [\mathrm{proj} \colon A \times GF(M) \to \widetilde{B}].$$

Then it is readily seen that $(\widetilde{B}, p, B)$ is a covering with $F_0 = \mathrm{proj}(a_0 \times GF(M))$ and $x_0 \in F_0$.

The contractibility of $\widetilde{B}$ follows from Lemma 2.3.3.4: in fact, the subspaces $\mathrm{proj}(A \times [GF(M) \setminus GF_{n-1}(M)])$ of $\widetilde{B}$, where $GF_n(M)$ is the part of $GF(M)$ consisting of words of length $\leq n$, satisfy the conditions of this lemma. The path with origin x_0 which covers the standard path

$$(\cdots(u_{\mu_1}^{\varepsilon_1} u_{\mu_2}^{\varepsilon_2})\cdots)u_{\mu_n}^{\varepsilon_n} \quad [\varepsilon_1, \dots, \varepsilon_n = \pm 1],$$

ends at the point $\operatorname{proj}(a_0, g)$, with $g = u_{\mu_1}^{\varepsilon_1} u_{\mu_2}^{\varepsilon_2} \cdots u_{\mu_n}^{\varepsilon_n}$. Clearly, the ends of these paths are pairwise distinct and exhaust F_0. □

Theorem 5.3.1.3. *The groups $\pi_r(B)$ with $r > 1$ are trivial, whereas $\pi_1(B, \mathrm{bp})$ is a free group with free generators α_μ .*

Proof. The proof is based on Lemma 5.3.1.2 and uses the same notation. Since $\widetilde{B}$ is contractible, all its homotopy groups are trivial, and hence so are the groups $\pi_r(B)$ with $r > 1$; moreover, the map $\Delta\colon \pi_1(B, \mathrm{bp}) \to \pi_0(F_0, x_0)$ is invertible (see Theorem 5.1.8.13). Combining the invertibility of Δ with property (ii) of the covering $(\widetilde{B}, p, B)$, we see that the homotopy classes of the standard loops are pairwise distinct and exhaust $\pi_1(B, \mathrm{bp})$. Consequently, $\pi_1(B, \mathrm{bp})$ is a free group with generators $\alpha_\mu, \mu \in M$. □

Corollary 5.3.1.4. *The fundamental group of a connected one-dimensional cellular space is free, whereas its higher homotopy groups are trivial.*

Proof. This is a corollary of Theorem 5.3.1.3, because every connected one-dimensional cellular space is homotopy equivalent to a bouquet of circles (see Theorem 2.3.3.6). □

5.3.2 The Effect of Attaching Balls

Remark 5.3.2.1. Let $X = A \cup_\varphi [\coprod_{\mu \in M} (D_\mu = \mathbb{D}^{k+1})]$, where A is a connected topological space, and φ is a continuous map $\coprod_{\mu \in M} (S_\mu = \mathbb{S}^k) \to A$ (see Lemma 2.3.2.1), and let $x_0 \in A$. In this subsection we exhibit a system of generators for the group $\pi_{k+1}(X, A, x_0)$ $[k \geq 1]$.

We remark that the homotopy groups $\pi_r(X, A)$ with $r \leq k$ are trivial (see Lemma 2.3.2.1), whereas for $r > k+1$, $\pi_r(X, A)$ is already a much more complicated object: in the simplest case, when A is just a point and the family $\{D_\mu\}$ consists of a single ball, $\pi_r(X, A)$ equals $\pi_r(\mathbb{S}^{k+1})$.

In Theorem 5.3.2.2 below, f_μ denotes the composite map

$$\mathbb{D}^{k+1} \xrightarrow{\mathrm{incl}_\mu} \coprod_\nu D_\nu \xrightarrow{\mathrm{Imm}_\mu} X,$$

and $\alpha_\mu \in \pi_{k+1}(X, A, f_\mu(\mathrm{ort}_1))$ is the class of the spheroid

$$f_\mu\colon (\mathbb{D}^{k+1}, \mathbb{S}^k, \mathrm{ort}_1) \to (X, A, f_\mu(\mathrm{ort}_1)).$$

Theorem 5.3.2.2. *Let $w_\mu\colon I \to A$ be an arbitrary path joining the points $f_\mu(\mathrm{ort}_1)$ and x_0. If $k \geq 1$, then $\pi_{n+k}(X, A, x_0)$ is generated over $\pi_1(A, x_0)$ by the classes $\beta_\mu = T_{w_\mu} \alpha_\mu$ [i.e., it is generated, in the usual sense, by the classes $T_\omega \beta_\mu$ with $\omega \in \pi_{k+1}(X, A, x_0)$].*

Proof. We claim that every element $\beta \in \pi_{k+1}(X, A, x_0)$ can be represented as

$$\beta = \prod_{i=1}^{m} [T_{\omega_i}(\beta_{\mu_i})]^{\pm 1} \quad (\mu_i \in M, \quad \omega_i \in \pi_1(A, x_0)). \tag{5.3.2.3}$$

By Lemma 2.3.2.1, there is a spheroid $g \in \mathrm{Sph}^{\bigcirc}_{k+1}(X, A, x_0)$ in the class μ, and similarity transformations $\sigma_1, \dots, \sigma_m$ mapping $\mathbb{D}^{k+1}$ onto pairwise disjoint balls $d_1, \dots, d_m \subset \operatorname{int} \mathbb{D}^{k+1}$, with the following properties:

- the point of d_i having the largest value of the first coordinate coincides with $\sigma_i(\mathrm{ort}_1)$;
- the segment joining this point with ort_1 lies in $C = \mathbb{D}^{k+1} \setminus \cup_{i=1}^m \operatorname{int} d_i$;
- the composition $\mathbb{D}^{k+1} \xrightarrow{\sigma_i} d_i \xrightarrow{\text{incl}} \mathbb{D}^{k+1} \xrightarrow{g} X$ is identical with one of the maps f_μ;
- $g(C) \subset A$.

Now it is clear that, if we suitably re-index the balls $d_1, \dots, d_m$, then X, A, x_0, g, and $d_1, \dots, d_m$ satisfy the conditions of Theorem 5.1.11.1, and hence we have, in the notation of this theorem, $\gamma = \prod_{i=1}^m T_{s_i}(\gamma_i)$. In our case, $\gamma = \beta$ and $\gamma_i = \alpha_{\mu_i}^{\pm 1}$; the last equality is a consequence of the fact that α_{μ_i} and γ_i are the elements of $\pi_{k+1}(X, A, f_{\mu_i}(\mathrm{ort}_1))$ represented by the spheroids f_{μ_i} and $g \circ \tau_i$, which are transformed one into another by the orthogonal transformations of $\mathbb{D}^{k+1}$, $\sigma_i^{-1} \circ \operatorname{abr} \tau_i$, $(\operatorname{abr} \tau_i)^{-1} \circ \sigma_i$ (which are inverses of one another). Consequently, $\beta = \prod_{i=1}^m T_{s_i}(\alpha_{\mu_i}^{\pm 1})$, and to obtain (5.3.2.3), we need only write ω_i for the class of the loop $w_{\mu_i}^{-1} s_i$. ☐

Corollary 5.3.2.4. *Under the hypotheses of Theorem 5.3.2.2, the inclusion homomorphism* $\pi_r(A, x_0) \to \pi_r(X, x_0)$ *is an isomorphism for* $r \leq k-1$, *and an epimorphism for* $r = k$. *The kernel of this epimorphism is generated over* $\pi_1(A, x_0)$ *by the classes* $\partial\beta_\mu = T_{w_\mu}(\partial\alpha_\mu)$ *[i.e., by the classes of the attaching spheroids* ∂f_μ, *translated to* x_0*].*

Theorem 5.3.2.5. *Let* (X, A) *be a cellular pair with base point* x_0. *If* A *is connected and* $A \supset \mathrm{skel}_k X$, *with* $k \geq 1$, *then* $\pi_r(X, A)$ *is trivial for all* $r \leq k$. *Moreover,* $\pi_{k+1}(X, A, x_0)$ *is generated over* $\pi_1(A, x_0)$ *by the classes of the characteristic maps of the* $(k+1)$*-cells in* $X \setminus A$ *(regarded as spheroids), translated to* x_0 *along arbitrary paths. The inclusion homomorphism* $\pi_r(A, x_0) \to \pi_r(X, x_0)$ *is an isomorphism for* $r \leq k-1$ *and an epimorphism for* $r = k$*; the kernel of the latter is generated over* $\pi_1(A, x_0)$ *by the classes of the attaching spheroids of the* $(k+1)$*-cells in* $X \setminus A$, *translated to* x_0 *along arbitrary paths.*

Proof. When $X \setminus A \subset \mathrm{skel}_{k+1}$, all these assertions follow from Remark 5.3.2.1, Theorem 5.3.2.2, and Corollary 5.3.2.4. The general case is reduced to this special situation by Theorem 2.3.2.6. ☐

5.3.3 The Fundamental Group of a Cellular Space

Remark 5.3.3.1. In this subsection we present an effective method for computing the fundamental group of a cellular space possessing a single 0-cell. This last condition is not a serious limitation, since, firstly, it is fulfilled in the most important cases and, secondly, every connected space can be transformed, by taking a rather simple quotient, into a homotopy equivalent space which meets our requirement (see Subsection 2.3.3). It is by no means difficult to generalise the computation scheme to arbitrary cellular spaces; however, the exposition is cumbersome.

Remark 5.3.3.2. Let X be a cellular space with a single 0-cell x_0. Since x_0 is also the unique 0-cell of $\operatorname{skel}_1 X$, this skeleton is homeomorphic to a bouquet of circles. Consequently, $\pi_1(\operatorname{skel} 1X, x_0)$ is the free group generated by the homotopy classes of the characteristic loops, i.e., of the characteristic maps of the 1 -cells (see Theorem 5.3.1.3).

According to Theorem 5.3.2.5, $\operatorname{incl}_* \colon \pi_1(\operatorname{skel} 1X, x_0) \to \pi_1(X, x_0)$ is an epimorphism whose kernel is generated over $\pi_1(\operatorname{skel}_1 X, x_0)$ by the homotopy classes of the attaching maps of the 2-cells of X, translated to x_0 along arbitrary paths. In our case, $\pi_1(\operatorname{skel}_1 X, x_0)$ acts as a group of inner automorphisms, and hence $\ker \operatorname{incl}_*$ is the smallest normal subgroup of $\pi_1(X, x_0)$ containing the above elements. Thus, the fundamental group that we want to compute is canonically isomorphic to the quotient group of $\pi_1(\operatorname{skel}_1 X, x_0)$ by this normal subgroup.

Remark 5.3.3.3. The discussion above shows that in order to compute $\pi_1(X, x_0)$ it suffices to know the 1-skeleton of X and the attaching maps of the 2-cells of X. Given these data, we can exhibit a system of generators and relations for $\pi_1(X, x_0)$: to each 1-cell corresponds a generator, namely the class of the respective characteristic loop; each 2-cells defines a relation, namely that the class of the attaching map of the given 2-cell, when translated to x_0 and expressed in terms of generators, must be equal to the identity element of $\pi_1(X, x_0)$. In a very simplified fashion, we may say that a set of generators of $\pi_1(X, x_0)$ consists of the 1-cells of X, while a system of relations consists of the 2-cells.

We remark that the system of relations is not entirely canonical, because it depends upon the choice of the paths along which we do the translation; consequently, the left-hand sides of the relations are determined only up to conjugation.

Lemma 5.3.3.4. *The fundamental group of a finite connected cellular space has a presentation given by a finite number of generators and relations.*

An Additional Theorem

Definition 5.3.3.5. If A and B are subspaces of the topological space X, with inclusions

$$\begin{tikzcd}
 & A \arrow[dr, "i'"] & \\
A \cap B \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\
 & B \arrow[ur, "j'"'] &
\end{tikzcd}$$

and $x_0 \in A \cap B$, then the rule $\alpha \star \beta \mapsto i'_*(\alpha)j'_*(\beta)$ defines a homomorphism $\pi_1(A, x_0) \star \pi_1(B, x_0) \to \pi_1(X, x_0)$ [$\star$ denotes the free product], whose kernel contains all the elements of the form $i_*(\delta)j_*(\delta)$ with $\delta \in \pi_1(A \cap B, x_0)$. Therefore, the same rule defines a homomorphism

$$[\pi_1(A, x_0) \star \pi_1(B, x_0)]/\operatorname{vk}(X, A, B, x_0) \to \pi_1(X, x_0), \tag{5.3.3.6}$$

where $\operatorname{vk}(X, A, B, x_0)$ designates the smallest normal subgroup of $\pi_1(A, x_0) \star \pi_1(B, x_0) \to \pi_1(X, x_0)$ containing the indicated elements (and is known as the *van Kampen subgroup*). Furthermore, homomorphism (5.3.3.6) is natural, meaning that the diagramme

$$\begin{tikzcd}
{[\pi_1(A, x_0) \star \pi_1(B, x_0)]/\operatorname{vk}(X, A, B, x_0)} \arrow[r] \arrow[d] & \pi_1(X, x_0) \arrow[d] \\
{[\pi_1(A', x'_0) \star \pi_1(B', x'_0)]/\operatorname{vk}(X', A', B', x'_0)} \arrow[r] & \pi_1(X', x'_0)
\end{tikzcd}$$

produced by a continuous map $(X, A, B, x_0) \to (X', A', B', x'_0)$ always commutes.

Theorem 5.3.3.7. *Let (X, A, B) be a cellular triad (i.e., a cellular space X with two subspaces, A and B, such that $A \cup B = X$), and let $x_0 \in D = A \cap B$. If A, B, and D are connected, then (5.3.3.6) is an isomorphism.*

(This theorem will be generalised in the next section; see Remark 5.4.3.12.)

Proof. Let us assume first that X has a single 0-cell x_0. By Remark 5.3.3.3, the fundamental group at x_0 of any of the spaces A, B, C, or D admits a presentation by generators and relations corresponding to its 1-cells and respectively its 2-cells. Therefore, the system of generators and relations of $\pi_1(X, x_0)$ ($\pi_1(D, x_0)$) is the union (respectively, intersection) of the systems of generators and relations of $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$, and the homomorphisms i_*, j_*, i'_* and j'_* from Definition 5.3.3.5 act as the identity on generators. Using the systems of generators and relations of $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ we may build a system of generators and relations of the group $\pi_1(A, x_0) \star \pi_1(B, x_0)$; however, the generators corresponding to the 1-cells in D must be counted twice. With this choice of generators and relations, the homomorphism

$$\pi_1(A, x_0) \star \pi_1(B, x_0) \to \pi_1(X, x_0), \quad \alpha \star \beta \mapsto i'_*(\alpha)j'_*(\beta),$$

is the identity on generators, and its kernel is generated by the elements obtained by identifying the generators corresponding to the 1 -cells in D. This completes the proof of the case that we considered.

To reduce the general case to this special one, we shall transfer the origin from x_0 to an arbitrary 0-cell e_0 in D (by a translation inside D), and then replace the quadruplet (X, A, B, e_0) by a homotopy equivalent quadruplet $(X', A', B', e_{0'})$ with a single 0-cell, e'_0, and such that $A' \cup B' = X'$. We exhibit such a quadruplet by taking quotients twice:

- first, the quotient of X by a contractible one-dimensional subspace of D containing $\operatorname{skel}_0 D$ (see Theorem 2.3.3.5 and Definition 1.3.3.7, and cf. Theorem 2.3.3.6),
- and subsequently the quotient of the resulting space by the union (bouquet) of contractible one-dimensional subspaces of the quotients of A and B containing the 0-skeletons of these quotients.

□

Corollary 5.3.3.8. *If A and B are cellular spaces with 0-cells a and b as base points, then* $\pi_1((A,a)\bigvee(B,b),\mathrm{bp})\cong\pi_1(A,a)\star\pi_1(B,b)$.

5.3.4 Homotopy Groups of Compact Surfaces

Remark 5.3.4.1. Recall that a sphere with handles and cross-caps and at one hole is homotopy equivalent to a bouquet of circles, the number of circles being $2g+\ell-1$ when g handles and ℓ holes are present, and $h+\ell-1$ when h cross-caps and ℓ holes are present (see Theorem 3.5.3.9). Thus, the fundamental group of such a surface is free, having $2g+\ell-1$ or $h+\ell-1$ generators, respectively, whereas the higher homotopy groups are trivial; see Theorem 5.3.1.3.

Below we shall discuss the homotopy groups of closed surfaces, i.e., of spheres with handles or cross-caps, but no holes. First (using the cellular decompositions indicated in Subsection 3.5.3, and Remark 5.3.3.3) we compute the fundamental groups, and then (by means of a simple device) we handle the higher homotopy groups. We disregard the sphere and the projective space, whose homotopy groups have been computed in the previous section (see Theorems 5.2.2.7, 5.2.2.10, Remark 5.2.4.3, and Theorem 5.2.5.1), and need no further comment.

The Fundamental Groups of Closed Surfaces

Remark 5.3.4.2. The cellular decomposition of a sphere with g handles, constructed in Remark 3.5.3.8, contains one 0-cell e_0, $2g$ 1-cells $a_1, b_1, \dots a_g, b_g$, and one 2-cell, whose attaching map takes ort_1 into e_0. Let $\alpha_1, \beta_1, \dots \alpha_g, \beta_g$ denote the generators of the fundamental group of the 1 -skeleton of the given surface which correspond to the 1-cells $a_1, b_1, \dots a_g, b_g$. The the homotopy class of the above attaching map (regarded as a loop) is the word

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}.$$

Therefore, the fundamental group of our surface at the point e_0 may be described as the group with generators $a_1, b_1, \dots a_g, b_g$, and the relation

$$a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}=1.$$

The cellular decomposition of a sphere with h cross-caps (see Remark 3.5.3.8) contains one 0-cell e_0, h 1-cells $c_1, \dots, c_h$, and one 2-cell. Repeating the previous argument with the obvious changes, we see that the fundamental group of this surface at e_0 can be described as the group with generators $c_1, \dots, c_h$ and the relation

$$c_1 c_1 \cdots c_h c_h = 1.$$

Remark 5.3.4.3. It is important to note that the groups computed in Remark 5.3.4.2 are pairwise non-isomorphic. (To see this, factor each fundamental group by its commutator subgroup: for a sphere with g handles this yields a free Abelian group of rank $2g$, while in the case of a sphere with h cross-caps the result is the direct sum of a free Abelian group of rank $h-1$ and a group of order 2.) In particular, the closed model surfaces are pairwise non-homeomorphic.

From this it is readily seen that the compact model surfaces are also pairwise non-homeomorphic: it suffices to seal up the holes by discs. The number of holes is a topological invariant, because it equals the number of components of the boundary (see Remark 3.1.1.4 and Theorem 4.6.5.13).

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Theorem 5.3.4.4. *Let P be a sphere with g handles. If $g \geq 1$ and $r \geq 2$, then $\pi_r(P) = 0$.*

Proof. It is clear that P admits as a covering space the infinite garland $\widetilde{P}$ constructed from $\mathbb{S}^1 \times \mathbb{R}$ by

- first removing small open discs centred at the points $(\mathrm{ort}_1, 2k)$ $[k = 0, \pm 1, \dots]$
- and then glueing a sphere with $g-1$ handles and one hole in the place of each such disc (see Fig. 5.3).

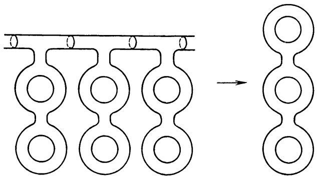

Figure 5.3: $(g = 3)$

Denote by $\widetilde{P}_n$ the finite garland constructed in the same fashion from the product $\mathbb{S}^1 \times [-2n-1, 2n+1]$. Obviously, $\widetilde{P}_n$ is a sphere with $(2n+1)(g-1)$ handles and two holes, so that $\pi_r(\widetilde{P}_n) = 0$ for all $r \geq 2$ (see Remark 5.3.4.1). Since $\widetilde{P} = \varinjlim(\widetilde{P}_n, \mathrm{incl}\colon \widetilde{P}_n \to \widetilde{P}_{n+1})$, we also have $\pi_r(\widetilde{P}) = 0$ for all $r \geq 2$ (see Theorem 5.1.11.5). Consequently, $\pi_r(P) = 0$ for all $r \geq 2$ (see Theorem 5.1.8.13). ☐

Corollary 5.3.4.5. *Let P be a sphere with h cross-caps. If $h \geq 2$ and $r \geq 2$, then $\pi_r(P) = 0$.*

Proof. Since P admits a sphere with $h-1$ as covering space (see Example 4.1.2.6), this is a corollary of Theorem 5.3.4.4. □

5.3.5 The Homotopy Groups of Bouquets

Remark 5.3.5.1. Suppose that we are given a family $\{(X_\mu, x_\mu)\}$ of pointed T_1-spaces, and consider the bouquet $B = \bigvee_{\mu \in M}(X_\mu, x_\mu)$. Then the formula $\mathrm{Imm}(\{\alpha_\mu\}_{\mu \in M}) = \sum_{\mu \in M} \mathrm{Imm}_{\mu *}(\alpha_\mu)$ defines a homomorphism

$$\mathrm{Imm}\colon \oplus_{\mu \in M}\, \pi_r(X_\mu, x_\mu) \to \pi_r(B, \mathrm{bp})$$

for any $r \geq 2$. This homomorphism is natural, i.e., if $B' = \bigvee_{\mu' \in M'}(X'_\mu, x'_\mu)$ is another bouquet of T_1-spaces, $\sigma\colon M' \to M$ is arbitrary, and $f_{\mu'}\colon (X', x') \to (X_{\sigma(\mu')}, x_{\sigma(\mu')})$ are continuous, then the following diagramme commutes

$$\begin{tikzcd}
\oplus_{\mu' \in M'}\pi_r(X'_\mu, x'_\mu) \arrow[r, "\mathrm{Imm}"] \arrow[d] & \pi_r(B', \mathrm{bp}) \arrow[d] \\
\oplus_{\mu \in M}\pi_r(X_\mu, x_\mu) \arrow[r, "\mathrm{Imm}"'] & \pi_r(B, \mathrm{bp})
\end{tikzcd} \tag{5.3.5.2}$$

where the left vertical map is the homomorphism

$$\{\alpha'_{\mu'}\}_{\mu' \in M'} \mapsto \left\{ \sum_{\mu' \in \sigma^{-1}(\mu)} (f_{\mu'})_*(\alpha'_{\mu'}) \right\}_{\mu' \in M'}$$

and the right vertical map is the homomorphism induced by the map

$$B' \to B, \quad \mathrm{Imm}_{\mu'}(y'_{\mu'}) \mapsto \mathrm{Imm}_{\sigma(\mu')} \circ f_\mu(y'_{\mu'}) \quad [y'_{\mu'} \in X'_{\mu'}, \mu' \in M'].$$

Lemma 5.3.5.3. *Given any $\alpha \in \pi_r(B, \mathrm{bp})$ [$r \geq 1$], there is Đ° finite set $M' \subset M$ such that*

- $\mathrm{proj}_{\mu'}(\alpha) = 0$ *if $\mu \in M \setminus M'$;*
- *α lies in the image of the homomorphism $\pi_r(B', \mathrm{bp}) \to \pi_r(B, \mathrm{bp})$ induced by the natural embedding of the bouquet $B' = \bigvee_{\mu \in M'}(X_\mu, x_\mu)$ in B.*

Proof. We only have to observe that for any spheroid $\varphi \in \mathrm{Sph}_r^(B, \mathrm{bp})$, $\varphi(I^r)$ is covered by a finite number of sets $\mathrm{Imm}_\mu(X)$. [Indeed, since $\varphi(I^r)$ is compact and every point of X is closed in X, we can choose a point in each non-empty intersection $\varphi(I^r) \cap \mathrm{Imm}_\mu(X_\mu \setminus x_\mu)$ and in this way produce a set which, being both discrete and compact, is finite.] □

Remark 5.3.5.4. Let $r \geq 2$ and define the map $\mathrm{Pr}\colon \pi_r(B, \mathrm{bp}) \to \oplus_{\mu \in M}\pi_r(X_\mu, x_\mu)$ by $\mathrm{Pr}(\alpha) = \{\mathrm{proj}_{\mu *}(\alpha)\}_{\mu \in M}$ (Lemma 5.3.5.3 shows that this definition is correct). Obviously, Pr is a homomorphism and diagramme (5.3.5.2) remains commutative when we replace $\xrightarrow{\text{Imm}}$ by $\xleftarrow{\text{Pr}}$.

$$\begin{tikzcd}
\oplus_{\mu' \in M'} \pi_r(X'_\mu, x'_\mu) \arrow[d] & \pi_r(B', \mathrm{bp}) \arrow[l, "\mathrm{Pr}"'] \arrow[d] \\
\oplus_{\mu \in M} \pi_r(X_\mu, x_\mu) & \pi_r(B, \mathrm{bp}) \arrow[l, "\mathrm{Pr}"]
\end{tikzcd} \tag{5.3.5.5}$$

If M is finite, then Pr equals the composition of the homomorphism

$$\pi_r(B, \mathrm{bp}) \to \pi_r(\prod_{\mu \in M} X_\mu, \{x_\mu\})$$

induced by the inclusion $B \to \prod_{\mu \in M} X_\mu$ (see Definition 1.2.8.3) with the canonical isomorphism $\pi_r(\prod_{\mu \in M} X_\mu, \{x_\mu\}) \to \oplus_{\mu \in M} \pi_r(X_\mu, x_\mu)$ (see Theorem 5.1.1.15).

Theorem 5.3.5.6. $\mathrm{Pr} \circ \mathrm{Imm}$ *equals the identity automorphism of the group* $\oplus_{\mu \in M} \pi_r(X_\mu, x_\mu)$. *In particular,* Pr *is epimorphic,* Imm *is monomorphic, and* $\pi_r(B, \mathrm{bp}) = \ker \mathrm{Pr} \oplus \mathrm{im}\, \mathrm{Imm}$.

Proof. Since $\mathrm{proj}_\mu \circ \mathrm{Imm}_\mu = \mathrm{id}_{X_\mu}$ and $\mathrm{proj}_\nu \circ \mathrm{Imm}_\mu(X_\nu) = x_\nu$ for $\nu \neq \mu$,

$$\begin{aligned}
\mathrm{Pr} \circ \mathrm{Imm}(\{\alpha_\mu\}_{\mu \in M}) &= \mathrm{Pr}[\sum_{\mu \in M} \mathrm{Imm}_{\mu *}(\alpha_\mu)] \\
&= \{\sum_{\mu \in M} (\mathrm{proj}_\nu \circ \mathrm{Imm}_\mu)(\alpha_\mu)\}_{\nu \in M} = \{\alpha_\nu\}_{\nu \in M}
\end{aligned}$$

for any $\alpha_\mu \in \pi_r(X_\mu, x_\mu)$ and $\mu \in M$. □

Theorem 5.3.5.7. *Let* (X_μ, x_μ) *be cellular pairs, and let* m, k_μ $(\mu in M)$ *be positive integers such that* $k_\mu + k_\nu \geq m$ *for* $\nu \neq \mu$, *and for any* ν

$$\pi_s(X_\mu, x_\mu) = 0 \quad \textit{for} \quad 1 \leq s \leq k_\mu.$$

If $2 \leq r \leq m$, *then*

$$\begin{aligned}
&\mathrm{Imm}\colon \oplus_{\mu \in M} \pi_r(X_\mu, x_\mu) \to \pi_r(B, \mathrm{bp}) \\
&\mathrm{Pr}\colon \pi_r(B, \mathrm{bp}) \to \oplus_{\mu \in M} \pi_r(X_\mu, x_\mu)
\end{aligned}$$

are isomorphisms.

This theorem will be generalised in the next section; see Definition 5.4.3.1.

Proof. Suppose first that M is finite. By Theorem 2.3.3.2 and the fact that Imm is natural (see Remark 5.3.5.1), we may assume that $\mathrm{skel}_{k_\mu} X_\mu$ reduces to the point x_μ for all $\mu \in M$. In this case, $B \supset \mathrm{skel}_{m+1} X$, where X is the cellular product of the spaces X_μ. By Theorem 2.3.2.4, $\mathrm{incl}_*\colon \pi_r(B, \mathrm{bp}) \to \pi_r(X, \mathrm{bp})$ is an isomorphism for $r \leq m$. According to Theorem 5.1.1.15,

$$\mathrm{incl}_* \circ \mathrm{Imm}\colon \oplus_{\mu\in M}\, \pi_r(X_\mu, x_\mu) \to \pi_r(B, \mathrm{bp})$$

is an isomorphism for any r. Therefore, Imm is an isomorphism for $r \leq m$.

In the general case, Lemma 5.3.5.3 shows that for any $\alpha \in \pi_r(B, \mathrm{bp})$ and $r \leq m$, there is a finite sub-bouquet $B' = \vee_{\mu'\in M'}, (X_\mu, x_\mu)$ of B such that α lies in the image of the homomorphism $\pi_r(B', \mathrm{bp}) \to \pi_r(B, \mathrm{bp})$ induced by the natural embedding $B' \to B$. Using Remark 5.3.5.1, this homomorphism is part of the commutative diagramme

$$\begin{tikzcd}
\oplus_{\mu'\in M'}\pi_r(X'_\mu, x'_\mu) \arrow[r, "\mathrm{Imm}"] \arrow[d] & \pi_r(B', \mathrm{bp}) \arrow[d] \\
\oplus_{\mu\in M}\pi_r(X_\mu, x_\mu) \arrow[r, "\mathrm{Imm}"'] & \pi_r(B, \mathrm{bp})
\end{tikzcd} \tag{5.3.5.8}$$

But we have already proved that the upper Imm is an isomorphism, so that α also lies in the image of our (lower) Imm. Therefore, the latter is an epimorphism and this, combined with Theorem 5.3.5.6, implies that Imm and Pr are isomorphisms. □

Corollary 5.3.5.9. *Let B be a bouquet of n-dimensional spheres, constructed from a family $\{(\S_\mu = \mathbb{S}^n, \mathrm{ort}_1)\}_{\mu\in M}$. If $n \geq 2$, then the groups $\pi_r(B)$ with $r < n$ are trivial, whereas $\pi_n(B, \mathrm{bp})$ is a free Abelian group with free generators* $\mathrm{Imm}_{\mu *}(\mathrm{sph}_n)$.

5.3.6 The Homotopy Groups of a k-Connected Cellular Pair

Lemma 5.3.6.1. *(A diagramme lemma) Consider the following commutative diagramme of groups and homomorphisms*

$$\begin{tikzcd}
\square \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \square \arrow[d, "\delta"] \\
\square \arrow[r, "\beta"'] & \square
\end{tikzcd}$$

If α, β, and γ are epimorphic and $\ker\beta \subset \alpha(\ker\gamma)$*, then δ is epimorphic,* $\ker\beta = \alpha(\ker\gamma)$*, and* $\ker\delta = \gamma(\ker\alpha)$.

Proof. Since the diagramme is commutative and α, β are epimorphic, δ is epimorphic. The commutativity of the diagramme implies also that $\alpha(\ker\gamma) \subset \ker\beta$ and $\gamma(\ker\alpha) \subset \ker\delta$. Let us verify that $\ker\delta \subset \gamma(\ker\alpha)$. Pick $d \in \ker\delta$. If $d = \gamma(a)$, then $\alpha(a) \in \ker\beta$ (again by commutativity), whence $\alpha(a) \in \alpha(\ker\gamma)$, i.e., there is $c \in \ker\gamma$ such that $\alpha(c) = \alpha(a)$. The last equality yields $ac^{-1} \in \ker\alpha$, and we have $d = \gamma(a) = \gamma(ac^{-1}) \in \gamma(\ker\alpha)$. □

Theorem 5.3.6.2. *Let* (X, A) *be a cellular pair with base point* $x_0 \in A$. *If* A *is connected and, for* $r \leq k$, $\pi_r(X, A) = 0$, *then* $\operatorname{proj}_* \colon \pi_{k+1} \colon (X, A, x_0) \to \pi_{k+1}(X/A, \operatorname{proj}(x_0))$ *is epimorphic, and for* $k \geq 1$ $\ker \operatorname{proj}_*$ *is the smallest subgroup of* $\pi_{k+1}(X, A, x_0)$ *containing all the "ratios"* $(T_\sigma \alpha)\alpha^{-1}$ *with* $\alpha \in \pi_{k+1}(X, A, x_0)$ *and* $\sigma \in \pi_1(A, x_0)$. *For* $k = 0$, *the situation is described by the commutative diagramme*

$$\begin{tikzcd}
\pi_1(A, x_0) \arrow[r, "\operatorname{incl}_*"] & \pi_1(X, x_0) \arrow[r, "\operatorname{rel}_*"] \arrow[dr, "\operatorname{abrs}\operatorname{proj}_*"'] & \pi_1(X, A, x_0) \arrow[d, "\operatorname{proj}_*"] \\
& & \pi_{k+1}(X/A, \operatorname{proj}(x_0))
\end{tikzcd} \tag{5.3.6.3}$$

where $\operatorname{abrs}\operatorname{proj}_*$ *and* rel_* *are also epimorphic, and* $\ker(\operatorname{abrs}\operatorname{proj}_*)$ *is the smallest normal subgroup of* $\pi_r(X, x_0)$ *which contains* $\ker \operatorname{rel}_* = \Im \operatorname{incl}_*$.

(This theorem will be generalised in the next section; see Remark 5.4.3.15.

Proof. The proof is quite involved.

PROOF OF THE CASE $k \geq 1$. Suppose first that $X \setminus A$ consists only of $(k+1)$-cells and, as a consequence, X/A is a bouquet of $(k+1)$–dimensional spheres. For each cell $e \in X \setminus A$, consider the homotopy class of its characteristic map (viewed as a spheroid of the pair (X, A)), and translate it to x_0, denoting the resulting element of $\pi_{k+1}(X, A, x_0)$ by α_e. Set $\beta_e = \operatorname{proj}_*(\alpha_e)$. By Theorem 5.3.2.5, the classes $(T_\sigma \alpha)$ with $\sigma \in \pi_1(A, x_0)$ form a system of generators of $\pi_{k+1}(X, A, x_0)$, and by Corollary 5.3.5.9, the classes β_e form a system of independent generators of the Abelian group $\pi_{k+1}(X/A, \operatorname{proj}(x_0))$. Moreover, it is obvious that $\operatorname{proj}_{r*}(T_\sigma \alpha) = \operatorname{proj}_*(\alpha)$ for all $\alpha \in \pi_{k+1}(X, A, x_0)$ and $\sigma \in \pi_1(A, x_0)$, and these facts will suffice to complete the proof of the theorem for $k \geq 1$. Since $\beta_e = \operatorname{proj}_*(\alpha_e)$ generate $\pi_{k+1}(X/A, \operatorname{proj}(x_0))$, proj_* is epimorphic. Further, since $\operatorname{proj}_*(T_\sigma \alpha) = \operatorname{proj}_*(\alpha)$, we have $(T_\sigma \alpha)\alpha^{-1} \in \ker \operatorname{proj}_*$. Let us show that the ratios $(T_\sigma \alpha)\alpha^{-1}$ generate $\ker \operatorname{proj}_*$. If $k > 1$ and the class $\xi \prod_{(e,\sigma)} (T_\sigma \alpha_e)^{\lambda(e,\sigma)}$ [with only a finite number of non-zero integers $\lambda(e, \sigma)$] belongs to $\ker \operatorname{proj}_*$, then $\sum_\sigma \lambda(e, \sigma) = 0$ for any cell e (because $\sum_e [\sum_\sigma \lambda(e, \sigma)] \beta_e = \operatorname{proj}_*(\alpha) = 0$), and thus $\xi = \prod_{(e,\sigma} [(T_\sigma \alpha_e)\alpha_e^{-1}]^{\lambda(e,\sigma)}$. When $k = 1$, this argument is valid only after we factor $\pi_{k+1}(X, A, x_0)$ by its commutator subgroup, and it only demonstrates that every element of $\ker \operatorname{proj}_*$ is a product of the above form multiplied by some commutators. However, since in $\pi_2(X, A, x_0)$ each commutator $\gamma^{-1}\delta\gamma\delta^1$ equals $(T_{\partial\gamma}\delta)\delta^{-1}$ (see Theorem 5.1.4.9), we obtain again the desired decomposition of ξ into ratios $(T_\sigma \alpha)\alpha^{-1}$.

In the general situation, we first transform (X, A) into a k-connected pair, removing those components of X which do not contain x_0, and then replace it by a homotopy equivalent pair (X', A') such that $\operatorname{skel}_k X' \subset A'$ (see Theorems 5.1.4.8 and 2.3.3.1). Thus, we may assume that $\operatorname{skel}_k \subset A$. Now set $Y =$

$A \cup \operatorname{skel}_{k+1} X$ and consider the commutative diagramme

$$\begin{tikzcd}
\pi_{k+1}(Y, A, x_0) \arrow[r, "i=\mathrm{incl}_*"] \arrow[d, "p'=\mathrm{proj}_*"'] & \pi_{k+1}(X, A, x_0) \arrow[d, "p=\mathrm{proj}_*"] \\
\pi_{k+1}(Y/A, \mathrm{proj}(x_0)) \arrow[r, "i'=\mathrm{incl}_*"'] & \pi_{k+1}(X/A, \mathrm{proj}(x_0))
\end{tikzcd}$$

Here i, i', and p' are epimorphic:

- i because $\operatorname{skel}_{k+1} X \subset Y$,
- i' because $\operatorname{skel}_{k+1}(X/A) \subset Y/A$,
- and p' because of the proof above.

We claim that our diagramme also satisfies the last condition of Lemma 5.3.6.1: $\ker i' \subset p'(\ker i)$.

To see this, note that every $(k+2)$-cell from $(X/A) \setminus (Y/A)$ is the image under p of some cell e from $X \setminus Y$, and its corresponding attaching map can be expressed as $\mathrm{proj} \circ \mathrm{att}_e$. By Theorem 5.3.2.5, this implies that

$$\ker i' \subset \mathrm{proj}_*(\ker \mathrm{incl}_*), \quad \begin{aligned} \mathrm{incl}_* &= [\mathrm{incl}_* \colon \pi_{k+1}(Y, x_0) \to \pi_{k+1}(X, x_0)], \\ \mathrm{proj}_* &= [\mathrm{proj}_* \colon \pi_{k+1}(Y, x_0) \to \pi_{k+1}(Y/A, \mathrm{proj}(x_0))]. \end{aligned}$$

Since the diagramme

$$\begin{tikzcd}
\pi_{k+1}(Y, y_0) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "\mathrm{rel}_*"'] & \pi_{k+1}(X, x_0) \arrow[d, "\mathrm{rel}_*"] \\
\pi_{k+1}(Y, A, x_0) \arrow[r, "i"'] & \pi_{k+1}(X, A, x_0)
\end{tikzcd}$$

commutes and $\mathrm{proj}_* = p' \circ [\mathrm{rel}_* \, colon \pi_{k+1}(Y, x_0) \to \pi_{k+1}(Y, A, x_0)]$, we see that $\mathrm{proj}_*(\ker \mathrm{incl}_*) \subset p'(\ker i)$, whence $\ker i' \subset p'(\ker i)$.

Applying Lemma 5.3.6.1, we conclude that p is epimorphic and $\ker p = i(\ker p')$. We have proved already that $\ker p'$ is generated by the ratios $(T_\sigma \alpha)\alpha^{-1})$ with $\alpha \in \pi_{k+1}(Y, A, x_0)$ and $\sigma \in \pi 1(A, x_0)$. Since $i((T_\sigma \alpha)\alpha^{-1}) = [T_\sigma(i(\alpha))](i(\alpha))^{-1}$ and i is epimorphic, $\ker p$ is generated by the ratios $(T_\sigma \alpha)\alpha^{-1})$ with $\alpha \in \pi_{k+1}(X, A, x_0)$ and $\sigma \in \pi 1(A, x_0)$.

PROOF OF THE CASE $k = 0$. The commutativity of (5.3.6.3) is obvious, while the fact that rel_* is epimorphic results from the connectedness of A. It remains to verify that abrs proj_* is an epimorphism with the indicated kernel. If x_0 is the unique 0-cell of X, this follows from Remark 5.3.3.3: indeed, the system of generators and relations for $(X/A, \mathrm{proj}(x_0))$ given in Remark 5.3.3.3 can be obtained from the system of generators and relations for $\pi_1(X, x_0)$, also appearing in Remark 5.3.3.3, by deleting the 1-cells and 2-cells of A. When x_0

is not the unique 0-cell of X, we may still reduce to this special case by translating inside A the origin at some 0-cell e_0, and subsequently replacing the triple (X, A, e_0) by a homotopy equivalent triple (X', A', e_0') having a single 0-cell e_0'. To produce such a triple, we take quotients twice:

- first the quotient of X by a one-dimensional contractible subspace of A containing $\operatorname{skel}_0 A$,
- and then the quotient of the resulting space by a one-dimensional contractible subspace which contains all its 0-cells

(cf. the proof of Theorem 5.3.3.7). □

Theorem 5.3.6.4. *Let X be a cellular space, and let A be a simply connected subspace of X. Then X/A is k-connected ($0 \leq k \leq \infty$) if and only if the pair (X, A) is k-connected. If this condition is fulfilled for some $k < \infty$, then* $\operatorname{proj}_*\colon \pi_{k+1}(X, A, x_0) \to \pi_{k+1}(X/A, \operatorname{proj}(x_0))$ *is an isomorphism.*

Proof. The second assertion is an obvious corollary of Theorems 5.3.6.2 and 5.1.4.8. The first assertion follows from the second by induction on k. However, note that to deduce the k-connectedness of X/A from the k-connectedness of (X, A), Theorems 2.3.3.1 and 2.3.3.2 suffice. □

Corollary 5.3.6.5. *If the cellular space X with the 0-cell x_0 as base point is k-connected, then* $\operatorname{sus}\colon \pi_{k+1}(X, x_0) \to \pi_{k+2}(\operatorname{sus}(X, x_0), \mathrm{bp})$ *is an isomorphism.*

Proof. This is a corollary of Theorem 5.3.6.4 (see Remark 5.2.1.2). □

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Remark 5.3.6.6. Under the assumptions of Theorem 5.3.6.4, if A is ℓ-connected $(1 \leq \ell \leq \infty)$, then $\operatorname{proj}_*\colon \pi_r(X, A, x_0) \to \pi_r(X/A, \operatorname{proj}(x_0))$ is an isomorphism for $r \leq k + \ell$ and an epimorphism for $r = k + \ell + 1$.

Under the assumptions of Corollary 5.3.6.5, $\operatorname{sus}\colon \pi_r(X, x_0) \to \pi_{r+1}(\operatorname{sus}(X, x_0), \mathrm{bp})$ is an isomorphism for $r \leq 2k$ and an epimorphism for $r = 2k + 1$ (cf. Theorem 5.2.1.4).

5.3.7 Spaces with Prescribed Homotopy Groups

Lemma 5.3.7.1. *Let π be a group and n a positive integer. If π is Abelian or if $n = 1$, then there is a connected cellular space X such that all the groups $\pi_r(X)$ with $r \neq n$ are trivial, whereas $\pi_n(X) \cong \pi$.*[1]

Proof. We proceed by induction and construct connected cellular spaces $X_0, X_1, \dots,$ with base points $x_0, x_1, \dots,$ and base-point preserving cellular embeddings $\varphi_0\colon X_0 \to X_1, \varpi_1\colon X_1 \to X_2, \dots$ such that:

---

[1] Translator's note. A space with such homotopy groups is known as a *cellular* $K(\pi, n)$-*space* or as a cellular space of type (π, n).

(i) the groups $\pi_r(X_k, x_k)$ with $r < n$ and $n < r \leq n + k$ are trivial;

(ii) $\pi_n(X_k, x_k)$ is isomorphic to π;

(iii) $\varphi_{k*} \colon \pi_n(X_K, x_k) \to \pi_n(X_{k+1}, x_{k+1})$ is an isomorphism.

Then the space $X = \varinjlim(X_k, \varphi_k)$ will have the desired properties (see Remark 2.1.5.7 and Theorem 5.1.11.5).

To produce (X_0, x_0), write π as a factor group F/F', where F is a free group if $n = 1$, and a free Abelian group if $n > 1$. Let B and B' be bouquets of n-dimensional spheres such that $\pi_n(B, \mathrm{bp}) = F$ and $\pi_n(B', \mathrm{bp}') = F'$ (see Theorem 5.3.1.3 and Corollary 5.3.5.9). Further, let $f \colon (B', \mathrm{bp}') \to (B, \mathrm{bp})$ be a continuous map such that $f_* \colon \pi_n(B', \mathrm{bp}) \to \pi_n(B, \mathrm{bp})$ equals the inclusion $F' \to F$ (one can construct such a map out of a family of spheroids whose classes in $\pi_n(B, \mathrm{bp}) = F$ constitute a free system of generators for F'). Now replace each sphere in B' by the ball that it bounds and take X_0 to be the result of attaching this new bouquet (of balls) to B by f. Theorem 5.3.1.3, Corollaries 5.3.5.9, and 5.3.2.4 show that X_0 and $x_0 = \mathrm{Imm}_1(\mathrm{bp})[= \mathrm{Imm}_2(\mathrm{bp})]$ satisfy conditions (i) and (ii) for $k = 0$.

Assume that for some $i \geq 1$, pointed spaces (X_k, x_k), $k < i$, and maps φ_k, $k < i - 1$, are already constructed and satisfy conditions (i), (ii), and (iii). Represent $\pi_{n+i}(X_{i-1}, x_{i-1})$ as the factor group of a free Abelian group, say G, and then construct a bouquet C of $(n+i)$-dimensional spheres, together with a map $g \colon (C, \mathrm{bp}) \to (X_{i-1}, x_{i-1})$ such that $g_* \colon \pi_{n+1}(C, \mathrm{bp}) \to \pi_{n+1}(X_{i-1}, x_{i-1})$ equals the projection $G \to \pi_{n+i}(X_{i-1}, x_{i-1})$. [To establish the existence of such a C and g, one may proceed as in the proof of the existence of B' and f above; however, here Theorem 5.3.1.3 is not necessary.] Now replace each sphere of C by the ball that it bounds and then attach the resulting bouquet to X_{i-1} by g to obtain X_i. Finally, set $x = \mathrm{Imm}_2(X_{i-1})$ and $\varphi_{i-1} = \mathrm{Imm}_2$. The fact that (X_i, x_i) satisfies (i), (ii) for $k = i$, and $\varphi_{ß-1}$ satisfies (iii) for $k = i - 1$, is a consequence of Corollary 5.3.2.4. $\square$

Theorem 5.3.7.2. *Given an arbitrary group* π_1 *and arbitrary Abelian groups* $\pi_2, \pi_3, \dots$, *there exists a connected cellular space* X *such that* $\pi_r(X) \cong \pi_r$ *(*$r = 1, 2, \dots$*).*

Proof. Let $X_1, X_2, \dots$ be connected cellular spaces with 0-cells $x_1, x_2, \dots$ as base points, such that the groups $\pi_r(X_k)$ are trivial for $r \neq k$, whereas $\pi_k(X_k) \cong \pi_k$ (see Lemma 5.3.7.1). Define inductively cellular spaces $Y_0, Y_1, \dots$ and cellular embeddings $\psi_k \colon Y_k \to Y_{k+1}$ by $Y_0 = \mathbb{D}^0$, $Y_{k+1} = Y_k \times_c X_{k+1}$, and $\psi_k(y) = (y, x_{k+1})$. Applying Theorems 5.1.1.15 and 5.1.11.5, the space $X = \varinjlim(X_k, \psi_k)$ has the desired properties. $\square$

5.3.8 Eight Instructive Examples

Example 5.3.8.1. If $r > 1$, then the r-th homotopy group of a finite, connected cellular space is not necessarily finitely generated (cf. Lemma 5.3.3.4). The bouquet $(\mathbb{S}^r, \mathrm{ort}_1) \vee (\mathbb{S}^1, \mathrm{ort}_1)$ is a simple illustration of this phenomenon: its

r-th homotopy group ($r > 1$) is a free Abelian group of infinite rank. Indeed, $(\mathbb{S}^r, \mathrm{ort}_1) \vee (\mathbb{S}^1, \mathrm{ort}_1)$ has a covering space which is homotopy equivalent to an infinite bouquet of r-dimensional spheres: to produce such a space, attach one copy of $\mathbb{S}^r$ in one point at each integer point of the real line $\mathbb{R}$.

INFORMATION. The homotopy groups of a finite cellular space with finite fundamental group are finitely generated. For a proof, see [19].

Example 5.3.8.2. Under the conditions of Theorem 5.1.6.8, the subgroup $\ker \mathrm{rel}_* = \mathrm{im}\, \mathrm{incl}_*$ of $\pi_1(X, x_0)$ is not necessarily normal (cf. 5.1.5.15).
Example: X is the bouquet of two circles, A the first circle, x_0 the centre of the bouquet, and ρ takes the second circle into x_0.

Example 5.3.8.3. Under the conditions of Theorem 5.1.6.9, the right splitting of the homotopy sequence of the pair (X, A) at $\pi_1(X, x_0)$ is not necessarily normal (cf. Theorem 5.1.5.18).
Example: $X = (\mathbb{D}^2, \mathrm{ort}_1) \vee (\mathbb{S}^1, \mathrm{ort}_1)$, $A = (\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^1, \mathrm{ort}_1)$, x_0 is the centre of both bouquets, and $h = [\mathrm{Imm}_2 \colon \mathbb{S}^1 \to X] \circ [\mathrm{proj}_2 \colon X \to \mathbb{S}^1]$.

Example 5.3.8.4. For any $k \geq 0$ there exist k-connected pairs (X, A) with A connected, which are not $(k+1)$-simple; moreover, under the conditions of Theorem 5.3.6.2, and for any $k \geq 0$, the epimornhism $\mathrm{proj}_* \colon pi_{k+1}(X, A, x_0) \to \pi_{k+1}(X/A, \mathrm{proj}(x_0))$ is not necessarily an isomorphism.
Example: $X = (\mathbb{S}^{k+1}, \mathrm{ort}_1) \vee (\mathbb{S}^1, \mathrm{ort}_1)$, $A = \mathrm{Imm}_2(\mathbb{S}^1)$ (cf. Examples 5.3.8.1 and 5.3.8.3).

Example 5.3.8.5. The second homotopy group of a pair (X, A) with connected A is not necessarily Abelian, even when $\pi?1(A)$ is Abelian and X is simply connected.
The simplest example: $A = \mathbb{S}^1 \times \mathbb{S}^1$, $x_0 = (\mathrm{ort}_1, \mathrm{ort}_1)$, and X is the result of attaching two copies of $\mathbb{D}^2$ to A by the maps $S^1 \to A$ given by $y \mapsto (y, \mathrm{ort}_1)$ and $y \mapsto (\mathrm{ort}_1, y)$. Then $\pi_1(A) = \mathbb{Z} \otimes \mathbb{Z}$, $\pi_2(A) = 0$, $\pi_1(X) = 0$, $\pi_2(X) \cong \mathbb{Z}$ (X is homotopy equivalent to $\mathbb{S}^2$), and we have the exact sequence

$$0 \xrightarrow{\mathrm{incl}_*} \mathbb{Z} \xrightarrow{\mathrm{rel}_*} \pi_2(X, A, x_0) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{incl}_*} 0 \qquad (5.3.8.6)$$

which shows, in particular, that ∂ is epimorphic. Assuming that $\pi_2(X, A, x_0)$ is Abelian, it follows from Theorem 5.1.4.9 that $\pi_1(A, x_0)$ acts identically on $\pi_2(X, A, x_0)$, whence, by Theorem 5.3.2.2, rank $\pi_2(X, A, x_0) \leq 2$. The latter contradicts the exactness of (5.3.8.6).

Example 5.3.8.7. There exist 1-connected pairs (X, A) such that

$$\mathrm{proj}_* \colon \pi_3(X, A, x_0) \to \pi_3(X/A, \mathrm{proj}(x_0))$$

is not even epimorphic.
Example: $X = \mathbb{D}^2$, A $= \mathbb{S}^1$, $x_0 = \mathrm{ort}_1$. Here $\pi_r(X, A) = 0$ for $r \neq 2$, whereas $\pi_3(X/A) \cong \mathbb{Z}$ (X/A is homeomorphic to S^2).

Example 5.3.8.8. For any $k \geq 2$, there are $(k-1)$-connected but not k-connected cellular pairs (X, A) with X and A connected and X/A contractible. To construct an example, let

$$\sigma \in \pi_1((\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1), \mathrm{bp}), \quad \alpha \in \pi_k((\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1), \mathrm{bp}),$$

designate the classes of the spheroids

$$\mathrm{Imm}_1\colon \mathbb{S}^1 \to (\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1), \quad \mathrm{Imm}_2\colon \mathbb{S}^k \to (\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1),$$

respectively. Next, attach $\mathbb{D}^{k+1}$ to $(\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1)$ by an arbitrary spheroid $\mathbb{S}^k \to (\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1)$ in the homotopy class $2\alpha - T_\sigma\alpha$. Take the resulting cellular space as X, and the circle $\mathrm{skel}_1 X$ as A. The quotient space X/A may be described as the result of attaching $\mathbb{D}^{k+1}$ to $\mathbb{S}^k$ by a map $\mathbb{S}^k \to \mathbb{S}^k$ homotopic to $\mathrm{id}_{\mathbb{S}^k}$, which implies that X/A is contractible (see Theorem 1.3.7.8). It is evident that X and A are connected and that (X, A) is $(k-1)$-connected; therefore, it remains to check that $\pi_k(X)$ is not trivial.
By Example 5.3.8.1 $\pi_k((\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1), \mathrm{bp})$ is a free Abelian group with free generators $\alpha_n = T_\sigma^n\alpha$ $(n = 0, \pm 1, \dots)$, while $\pi_k(X)$ is the factor group of $\pi_k((\mathbb{S}^1, \mathrm{ort}_1) \vee (\mathbb{S}^k, \mathrm{ort}_1), \mathrm{bp})$ by its subgroup generated by the elements $2\alpha_n - \alpha_{n+1}$. (see Theorem 5.3.6.2). Consequently, $\pi_k(X)$ is isomorphic to the additive group of binary rational numbers.

Example 5.3.8.9. The homomorphism $f_*\colon \pi_1(X, A, x_0) \to \pi_1(X', A', x')$ induced by a continuous map $f\colon (X, A, x_0) \to (X', A', x')$ is not necessarily an isomorphism even if all the homomorphisms

$$f_*\colon \pi_r(X, x_0) \to \pi_r(X', x_0'), \quad (\mathrm{abr}\, f)_*\colon \pi_r(A, x_0) \to \pi_r(A', x')$$

are isomorphisms.
Example: $X = X' = (\mathbb{S}^1, \mathrm{ort}_1) \vee (I, 0)$, $A = \mathrm{Imm}_1(\mathrm{ort}_2) \cup \mathrm{Imm}_2(1)$,
$A' = \mathrm{Imm}_1(\mathbb{S}^1) \cup \mathrm{Imm}_2(1)$, $x_0 = x_0' = \mathrm{Imm}_2(1)$, and $f = \mathrm{rel}\, \mathrm{id}_X$.

5.3.9 Exercises

Exercise 5.3.9.1. Consider the subset of ($\mathbb{C}^2$ defined by the equation $x_1^p = x_2^q$ where p and q are coprime integers, and intersect it with the sphere $\mathbb{S}^-3$. Show that the fundamental group of the complement of this intersection in $\mathbb{S}^3$ is isomorphic to the group with generators α_1, α_2, which are connected by the relation $\alpha_1^p = \alpha_2^q$.

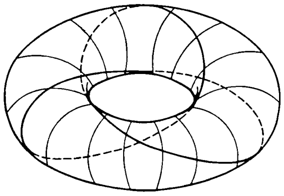

Figure 5.4: $p = 2$, $q = 3$

The above intersections (with various p, q) are all homeomorphic to a circle and lie on the torus $|x_1| = \sqrt{2}/2$, $|x_2| = \sqrt{2}/2$ which is contained in $\mathbb{S}^3$; they are called *torus knots*. A torus knot and the torus which carries it are depicted in Fig. 5.4 as they lie in $\mathbb{R}^3 = \mathbb{S}^3 \setminus \mathrm{bp}$.

Exercise 5.3.9.2. Consider the submanifold of $\mathbb{C}P^2$ defined by the equation $x_1^m + x_2^m + x_3^m = 0$ with m a positive integer (cf. Exercise 3.5.4.2). Show that the fundamental group of the complement of this submanifold is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.

Exercise 5.3.9.3. Consider the submanifold of non-zero vectors in the total space of the tangent bundle of a sphere with g handles, and show that its fundamental group is isomorphic to the group with generators $a_1, \dots a_g, b_1, b_1, \dots b_g, d$, which are connected by the relations

$$\begin{aligned} &a_1 b_1 a_{-1}^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = d^{2-2g}, \\ &a_1 d = d a_1, \dots, a_g d = d a_g, \quad b_1 d = d b_1, \dots, b_g d = d b_g. \end{aligned}$$

Exercise 5.3.9.4. Consider the submanifold of non-zero vectors in the total space of the tangent bundle of a sphere with h cross-caps, and show that its fundamental group is isomorphic to the group with generators $c_1, \dots, c_h, d$, which are connected by the relations

$$c_1 c_1 \cdots c_h c_h = d^{2-h}, \quad c_1 d = d c_1, \dots, c_h d = d c_h.$$

Exercise 5.3.9.5. Let π be any group which can be presented by a finite number of generators and relations. Show that there exists a smooth, closed 4-dimensional manifold whose fundamental group is isomorphic to π.

Exercise 5.3.9.6. Show that every smooth, closed, oriented manifold of dimension $\neq 0, 3$, is oriented cobordant to a simply connected manifold,

INFORMATION. This is also valid for dimension 3.

Exercise 5.3.9.7. Show that in Example 5.3.8.5, $\pi_2(X, A, x_0)$ is isomorphic to the group with generators a, b, c, which are connected by the relations

$$ac = ca, \quad bc = cb, \quad aba^{-1}b^{-1} = c.$$

5.4 WEAK HOMOTOPY EQUIVALENCE

5.4.1 Fundamental Concepts

Definition 5.4.1.1. If X and Y are topological spaces, a continuous map $f\colon X \to Y$ is called a *weak homotopy equivalence* if $f_*\colon \pi_r(X, x) \to \pi_r(Y, f(x))$ is an isomorphism for all $r \geq 0$ and all $x \in X$. To justify this term, we remark that every homotopy equivalence is a weak homotopy equivalence and that the converse is not true. The first fact was established in Theorem 5.1.3.7; for the second, see Definition 5.4.3.5.

The composition of two weak homotopy equivalences is obviously a weak homotopy equivalence.

Theorem 5.4.1.2. *Let $f\colon X \to Y$ be a weak homotopy equivalence. Then for any cellular pair (K, L) and continuous maps $\varphi\colon\ \colon K \to Y$ and $\psi\colon\ \colon L \to X$ with $f \circ \psi = \varphi|_L$ there is a continuous map $\chi\colon\ \colon K \to X$ such that $\chi|_L = \psi$ and $f \circ \chi$ is L-homotopic to φ. The converse is also true; moreover, if $f\colon X \to Y$ is continuous and has the property that for any continuous maps, $\varphi\colon \mathbb{D}^r \to Y$ and $\psi\colon \mathbb{S}^{r-1} \to X$ $(r \geq 0)$ with $f \circ \psi = \varphi|_{\mathbb{S}^{r-1}}$, there is a continuous map $\chi\colon \mathbb{D}^r \to X$ such that $\chi|_{\mathbb{S}^{r-1}} = \psi$ and $f \circ \chi$ is $\mathbb{S}^{r-1}$-homotopic to φ, then f is a weak homotopy equivalence.*

Proof. To prove the first part, consider the mapping cylinder $\operatorname{Cyl} f$ (see Definition 1.2.6.10). By Theorem 2.3.1.3, there exists a homotopy $\varphi\colon K \times I \to \operatorname{Cyl} f$ of the composition

$$K \xrightarrow{\varphi} Y \xrightarrow{\text{incl}} \operatorname{Cyl} f,$$

such that $\varphi(z, t) = \operatorname{Imm}_1(\psi(z), 1 - t)$ for all $z \in L$, $t \in I$. Since

$$\operatorname{rt} f \circ [\text{incl}\colon X \to \operatorname{Cyl} f] = f$$

and f is a weak homotopy equivalence, while $\operatorname{rt} f$ is a homotopy equivalence, we conclude that incl is a weak homotopy equivalence. Hence,

$$\text{incl}_*\colon \pi_r(X, x) \to \pi_r(\operatorname{Cyl} f, x)$$

is an isomorphism for all $r \geq 0$ and all $x \in X$. Thus, $(\operatorname{Cyl} f, X)$ is an ∞-connected pair (see Remark 5.1.6.7), and since $\varphi(L \times 1) \subset X$, there is an L-homotopy $K \times I \to \operatorname{Cyl} f$ such that $\psi(x, 0) = \varphi(x, 1)$ for all $x \in K$, and $\psi(K \times 1) \subset X$ (see Theorem 2.3.1.6). Now define $\chi\colon K \to X$ by $\chi(x) = \varphi(x, 1)$. Obviously, $\chi|_L = \psi$, and the product of the homotopies $\operatorname{rt} f \circ \varphi$ and $\operatorname{rt} f \circ \psi$ is an L-homotopy from $f \circ \chi$ to φ.

To prove the second part, we must check that $f_*\colon \pi_r(X, x) \to \pi_r(Y, f(x))$ is both epimorphic and monomorphic for all $r \geq 0$ and $x \in X$. To see that f_* is epimorphic, set $\psi(\mathbb{S}^{r-1}) = x$ and take φ to be some spheroid $(\mathbb{D}^r, \mathbb{S}^{r-1}) \to (Y, f(x))$ of an arbitrarily given class $\beta \in \pi_r(Y, f(x))$; the resulting spheroid $\chi\colon (\mathbb{D}^r, \mathbb{S}^{r-1}) \to (X, x)$ defines a class $\alpha \in \pi_r(X, x)$, and it is clear that $f_*(\alpha) = \beta$. Finally, to show that f_* is monomorphic, pick $\alpha \in \pi_r(X, x)$ with $f_*(\alpha) = 0$, take ψ to be some spheroid of class α, and take φ to be any continuous map $\mathbb{D}^{r+1} \to Y$ such that $\varphi|_{\mathbb{S}^r} = f \circ \psi$; the resulting map $\chi\colon \mathbb{D}^{r+1} \to X$ extends ψ, and hence $\alpha = 0$. ☐

Theorem 5.4.1.3. *If $f\colon X \to Y$ is a weak homotopy equivalence, then the mapping $\pi(\mathrm{id}, f)\colon \pi(M, X) \to \pi(M, Y)$ is invertible for any cellular space M.*

Proof. The first part of Theorem 5.4.1.2 shows that, given a class $\beta \in \pi(M, Y)$, there is an $\alpha \in \pi(M, X)$ such that $[\pi(\mathrm{id}, f)](\alpha) = \beta$: we need only set $K = M$, $L = \varnothing$, and take φ to be any map in the class β. Further, given arbitrary continuous maps $\varphi_0, \varphi_1\colon M \to X$, it follows from the same first part of Theorem 5.4.1.2 that if the compositions $f \circ \varphi_0$ and $f \circ \varphi_1$ are homotopic, then so are φ_0 and φ_1: indeed, take $K = M \times I$, $L = (M \times 0) \cup (M \times 1)$, and

$$\varphi\colon (M \times 0) \cup (M \times 1) \to X, \quad \varphi(x, 0) = \varphi_0(x), \quad \varphi(x, 1) = \varphi_1(x), \quad x \in M,$$

and take for ψp any homotopy $M \times I \to X$ from $f \circ \varphi_0$ to $f \circ \varphi_1$. ☐

The Case of Cellular Spaces

Theorem 5.4.1.4. *If X and Y are cellular spaces, then every weak homotopy equivalence $X \to Y$ is a homotopy equivalence.*

Proof. Suppose $f\colon X \to Y$ is a weak homotopy equivalence. By Theorem 5.4.1.3, the mapping $\pi(\mathrm{id}, f)\colon \pi(Y, X) \to \pi(Y, Y)$ is invertible, and hence there is a map $g\colon Y \to X$ whose homotopy class is taken by $\pi(\mathrm{id}, f)$ into the class of id_Y. That is to say, $f \circ g$ is homotopic to id_Y, and it remains to verify that $g \circ f$ is homotopic to id_X. The latter is a consequence of the invertibility of $\pi(\mathrm{id}, f)\colon \pi(X, X) \to \pi(X, Y)$, because this mapping takes the homotopy classes of $g \circ f$ and id_X into the same element (indeed, $f \circ g \circ f$ and f are homotopic). ☐

Remark 5.4.1.5. Theorem 5.4.1.4 states that two connected cellular spaces, X and Y, are homotopy equivalent whenever there is a continuous map $X \to Y$ which induces isomorphisms of the homotopy groups, but it certainly does not guarantee that X and Y are homotopy equivalent if their homotopy groups are just isomorphic. In fact, we have simple examples to show that the latter is not true. Take $X = \mathbb{S}^p \times \mathbb{R}P^q$, $Y = \mathbb{S}^q \times \mathbb{R}P^p$, and suppose that $1 < p < q$. By Theorems 5.2.5.1 and 5.1.1.15, $\pi_r(X) \cong \pi_r(Y)$ for all r. However, X and Y are not homotopy equivalent. Indeed, the map $\mathrm{proj}_1\colon \mathbb{S}^p \times \mathbb{R}P^q \to \mathbb{S}^p$ induces a group isomorphism $\pi_p(\mathbb{S}^p \times \mathbb{R}P^q) \to \pi_p(\mathbb{S}^p)$. We next show that there is no continuous map $f\colon \mathbb{S}^p \times \mathbb{R}P^q \to \mathbb{S}^p$ which induces a group isomorphism $\pi_p(\mathbb{S}^p \times \mathbb{R}P^q) \to \pi_p(\mathbb{S}^p)$. Assuming that such an f exists, the composition

$$\mathbb{S}^p \xrightarrow{\mathrm{proj}} \mathbb{R}P^p \xrightarrow{x \mapsto (\mathrm{ort}_1, x)} \mathbb{S}^q \times \mathbb{R}P^q \xrightarrow{f} \mathbb{S}^p \tag{5.4.1.6}$$

induces an automorphism of $\pi_p(\mathbb{S}^p)$. On the other hand (by Theorem 2.3.2.6), every continuous map $\mathbb{R}P^p \to \mathbb{S}^p$ is homotopic to a map which takes $\mathbb{R}P^{p-1}$ into ort_1, and thus (5.4.1.6) is homotopic to the composition of the composite projection $\mathbb{S}^p \to \mathbb{R}P^p/\mathbb{R}P^{p-1} = \mathbb{S}^p$ with some continuous map $\mathbb{S}^p \to \mathbb{S}^p$. Consequently, (5.4.1.6) cannot have degree ± 1, since the above composite projection has degree 0 when p is even and degree 2 when p is odd. A contradiction.

The following example illustrates the same phenomenon in the simply connected case. Set $X = \mathbb{S}^3 \times \mathbb{C}P^\infty$ and $Y = \mathbb{S}^2$. By Theorems 5.2.2.10, 5.2.5.2, and 5.1.1.15, $\pi_r(X) \cong \pi_r(Y)$ for all r. However, X and Y have not the same homotopy type. Indeed, $\operatorname{proj}_1\colon \mathbb{S}^3 \times \mathbb{C}P^\infty \to \mathbb{S}^3$ is not null homotopic (because it induces a group isomorphism $\pi_3(\mathbb{S}^3 \times \mathbb{C}P^\infty) = \mathbb{Z} \to \pi_3(\mathbb{S}^3) = \mathbb{Z}$). On the other hand, every continuous map $\mathbb{S}^2 \to \mathbb{S}^3$ is null homotopic.

Definition 5.4.1.7. We say that a topological space is *homotopy fit* if it is homotopy equivalent to a cellular space. From Theorem 5.4.1.4 it follows that if X and Y are homotopy fit, then every weak homotopy equivalence $X \to Y$ is a homotopy equivalence.

By Theorem 3.5.2.13, all smooth compact manifolds are homotopy fit.

An example of a space which is not homotopy fit was given in Exercise 2.3.5.4. This space is not connected. For an example of a connected (and even ∞-connected) space which is not homotopy fit, see Exercise 5.4.4.1 below.

INFORMATION. Every CNRS is homotopy fit, and the same holds true for every topological manifold (compact or not). A product of homotopy fit spaces is homotopy fit. If Y is homotopy fit, then $\mathcal{C}(X, Y)$ is homotopy fit for any compact space X. If Y has the homotopy type of a countable cellular space, then $\mathcal{C}(X, Y)$ has the homotopy type of a countable cellular space, for any compact space X with countable base. For proofs, see [16].

k-Equivalence

Definition 5.4.1.8. Let X and Y be topological spaces. A continuous map $f\colon X \to Y$ is a *k-equivalence* if, for all $x \in X$, $f_*\colon \pi_r(X, x) \to (Y, f(x))$ is an isomorphism for $r < k$ and an epimorphism for $r = k$. Here k is a nonnegative integer; sometimes, weak homotopy equivalences are referred to as ∞-equivalences.

A composition of two k-equivalences is obviously a k-equivalence.

Theorem 5.4.1.9. *Let $f\colon X \to Y$ be a k-equivalence. Then for any cellular pair (K, L) with $K \setminus L \subset \operatorname{skel}_k K$ and continuous maps $\varphi\colon K \to Y$ and $\psi\colon L \to X$ with $f \circ \psi = \varphi|_L$ there is a continuous map $\chi\colon K \to X$ such that $\chi|_L = \psi$ and $f \circ \chi$ is L-homotopic to φ. The converse is also true; moreover, if $f\colon X \to Y$ is continuous and has the property that for any continuous maps $\varphi\colon \mathbb{D}^r \to Y$ and $\psi\colon \mathbb{S}^{r-1} \to X$ $(0 \le r \le k)$ with $f \circ \psi = \varphi|_{\mathbb{S}^{r-1}}$ there is a continuous map $\chi\colon \mathbb{D}^r \to X$ such that $\chi|_{\mathbb{S}^{r-1}} = \psi$ and $f \circ \chi$ is $\mathbb{S}^{r-1}$-homotopic to φ, then f is a k-equivalence.*

Proof. The proof repeats that of Theorem 5.4.1.2, *mutatis mutandis*:

- in the first part, the pair $(\operatorname{Cyl} f, X)$ is now k-connected;

- in the second part, to see that f_* is epimorphic (monomorphic), we take $r \leq k$ (respectively, $r < k$).

□

Theorem 5.4.1.10. *If $f\colon X \to Y$ is a k-equivalence, then the mapping*

$$\pi(\mathrm{id}, f)\colon \pi(M, X) \to (M, Y)$$

is invertible (surjective) for any cellular space M with $\dim M < k$ *(respectively,* $\dim M = k$*) .*

Proof. The proof repeats that of Theorem 5.4.1.3, except that we need Theorem 5.4.1.9 instead of Theorem 5.4.1.2. □

Theorem 5.4.1.11. *If X and Y are cellular spaces with* $\dim X < k$ *and* $\dim Y \leq k$*, then every k-equivalence $X \to Y$ is a homotopy equivalence.*

Proof. The proof repeats that of Theorem 5.4.1.4, except that we need Theorem 5.4.1.10 instead of Theorem 5.4.1.3. □

The Relative Case

Definition 5.4.1.12. If (X, A) and (Y, B) are topological pairs, a continuous map $f\colon (X, A) \to (Y, B)$ is said to be a *weak homotopy equivalence* if $\mathrm{abrs}\, f\colon X \to Y$ and $\mathrm{abr}\, f (= \mathrm{abr}\,\mathrm{abrs}\, f)\colon A \to B$ are weak homotopy equivalences.

We remark that if $f\colon (X, A) \to (Y, B)$ is a weak homotopy equivalence, then $f_*\colon \pi_r(X, A, x) \to \pi_r(Y, B, f(x))$ is an isomorphism for all $r \geq 1$ and all $x \in A$. To see this, apply the 5-Lemma (see Theorem 5.1.5.20) to the homomorphism induced by f from the homotopy sequence of the pair (X, A) into the homotopy sequence of the pair (Y, B).

As another corollary of the 5-Lemma, we have the following result: suppose that

- $f\colon (X, A \neq \varnothing) \to (Y, B)$ is continuous,
- one of the maps $\mathrm{abrs}\, f\colon X \to Y$, $\mathrm{abr}\, f\colon A \to B$ is a weak homotopy equivalent
- for any $x \in A$ all the homomorphisms

  $$f_*\colon \pi_r(X, A, x) \to (Y, B, f(x)), \quad r \geq 1,$$
  $$(\mathrm{abrs}\, f)_*\colon \pi_0(X, x) \to \pi_0(Y, f(x)), \quad (\mathrm{abr}\, f)_*\colon \pi_0(A, x) \to \pi_0(B, f(x))$$

  are isomorphisms.

Then f is a weak homotopy equivalence.

Theorem 5.4.1.13. *If* $f\colon (X,A)\to(Y,B)$ *is a weak homotopy equivalence, then the mapping* $\pi(\mathrm{id},f)\colon \pi(M,N;X,A)\to\pi(M,N;Y,B)$ *is invertible for any cellular pair* (M,N).

Proof. Let us show first that every continuous map $\varphi\colon (M,N)\to(Y,B)$ is homotopic to the composition of some continuous map $(M,N)\to(Y,B)$ with f. By Theorem 5.4.1.2, there is a continuous map $\psi\colon N\to A$ whose composition with $\operatorname{abr} f\colon A\to B$ is homotopic to $\operatorname{abr}\varphi\colon N\to B$. Using Theorem 2.3.1.3, any homotopy from $\operatorname{abr}\varphi$ to $\operatorname{abr} f\circ\psi$ may be extended to a homotopy of φ, and hence there is a $\varphi'\colon (M,N)\to(Y,B)$ homotopic to φ and satisfying $\operatorname{abr}\varphi'=\operatorname{abr} f\circ\psi$. Finally, again using Theorem 5.4.1.2, we see that there is a continuous map $\chi\colon M\to X$ extending ψ and such that $f\circ\chi$ is N-homotopic to φ'. Obviously, $\chi(N) subset B$ and the maps $f\circ\chi,\varphi\colon (M,N)\to(Y,B)$ are homotopic.

To complete the proof, we have to show, given two continuous maps

$$\varphi_0,\varphi_1\colon (M,N)\to(X,A)$$

such that

- $f\circ\varphi_0$ and $f\circ\varphi_1$ are homotopic,
- φ_0 and φ_1 are also homotopic.

Let $\Phi\colon (M\times I,N\times I)\to(Y,B)$ be a homotopy from $f\circ\varphi_0$ to $f\circ\varphi_1$. By Theorem 5.4.1.2, there is a homotopy $Psi\colon N\times I\to A$ from $\operatorname{abr}\varphi_0\colon N\to A$ to $\operatorname{abr}\varphi_1\colon N\to A$, such that $\operatorname{abr} f\circ\Psi$ is $[(N\times 0)\cup(N\times 1)]$-homotopic to $\operatorname{abr}\varphi\colon N\times I\to B$. Further, by Theorem 2.3.1.3, every $[(N\times 0)\cup(N\times 1)]$-homotopy from $\operatorname{abr}\varphi$ to $\operatorname{abr} f\circ\Psi$ extends to a $[(M\times 0)\cup(M\times 1)]-$homotopy of φ. Consequently, there is a homotopy $\varphi'\colon\ \colon (M\times I,N\times I)\to(Y,B)$ from $f\circ\varphi_0$ to $f\circ\varphi_1$ such that $[\operatorname{abr}\varphi'\colon N\times I\to B]=\operatorname{abr} f\circ\Psi$. Finally, we apply Theorem 5.4.1.2 again to deduce that there exists a continuous map $\Xi\colon M\times I$ extending Ψ, such that $\xi(x,0)=\varphi_0(x)$, $\Xi(x,1)=\varphi_1(x)$ for all $x\in M$. In other words, Ξ is a homotopy from φ_0 to φ_1. □

Theorem 5.4.1.14. *If* (X,A) *and* (Y,B) *are cellular pairs, then every weak homotopy equivalence* $(X,A)\to(Y,B)$ *is a homotopy equivalence.*

Proof. The proof repeats that of Theorem 5.4.1.4, but one must refer to Theorem 5.4.1.13 instead of Theorem 5.4.1.3. □

5.4.2 Weak Homotopy Equivalence and Constructions

Remark 5.4.2.1. Many of the operations on maps which were described in §1.2 carry weak homotopy equivalences into weak homotopy equivalences. For example, it is clear that $\coprod_\mu f_\mu\colon \coprod_\mu X_\mu\to\coprod_\mu X'_\mu$ is a weak homotopy equivalence for any family $\{f_\mu\colon X_\mu\to X'_\mu\}_{\mu\in M}$ of weak homotopy equivalences; similarly, $f_1\times\cdots\times f_n\colon X_1\times\cdots\times X_n\to X'_1\times\cdots\times X'_n$ is a weak homotopy equivalence for any weak homotopy equivalences $f_1\colon X_1\to X'_1,\dots,f_n\colon X_n\to X'_n$. Also, the limit of a sequence of of weak homotopy equivalences is a weak homotopy equivalence.

Similar results for other constructions need supplementary arguments, or hold only under additional assumptions. The present subsection is devoted to such results.

Theorem 5.4.2.2. *Let* $f\colon (X, A, B) \to (X', A', B')$ *be a map of triads such that* $\operatorname{abr} f\colon A \to A'$, $\operatorname{abr} f\colon B \to B'$, *and* $\operatorname{abr} f\colon A \cap B \to A' \cap B'$ *are weak homotopy equivalences. If* $\operatorname{int} A \cup \operatorname{int} B = X$ *and* $\operatorname{int} A' \cup \operatorname{int} B' = X'$, *then* $\operatorname{abrs} f\colon X \to X'$ *is a weak homotopy equivalence.*

Proof. Using Theorem 5.4.1.2, and given continuous maps $\varphi\colon \mathbb{D}^r \to X'$ and $\psi\colon \mathbb{S}^{r-1} \to X$ with $f \circ \psi = \varphi|_{\mathbb{S}^{r-1}}$, it suffices to produce a continuous map $\chi\colon \mathbb{D}^r \to X$ such that $\chi|_{\mathbb{S}^{r-1}} = \psi$ and $f \circ \chi$ is $\mathbb{S}^{r-1}$-homotopic to φ. Obviously,

$$U = [\varphi^{-1}(\operatorname{int} A') \cap \operatorname{int} \mathbb{D}^r] \cup \psi^{-1}(\operatorname{int} A), \quad V = [\varphi^{-1}(\operatorname{int} B') \cap \operatorname{int} \mathbb{D}^r] \cup \psi^{-1}(\operatorname{int} B)$$

are open and cover $\mathbb{D}^r$. Therefore, there is an $\varepsilon > 0$ such that any subset of $\mathbb{D}^r$ with diameter less than ε is contained in U or V. Now triangulate $\mathbb{S}^{r-1}$ so that the diameter of each simplex is less than ε, and then extend the triangulation to $\mathbb{D}^r$, preserving this property. Let K (L) be the union of all simplices contained in U (respectively, V). It is clear that K and L are simplicial subspaces of $\mathbb{D}^r$ such that

$$\psi(K \cap \mathbb{S}^{r-1}) \subset \operatorname{int} A, \quad \psi(L \cap \mathbb{S}^{r-1}) \subset \operatorname{int} B, \quad \varphi(K) \subset \operatorname{int} A', \quad \varphi(L) \subset \operatorname{int} B'$$

By Theorem 5.4.1.2, there is a continuous map $\chi_0\colon K \cap L \to A \cap B$ such that $\chi_0|_{J \cap L \cap \mathbb{S}^{r-1}} = \operatorname{abr} \psi$ and the composition of χ_0 with $\operatorname{abr} f\colon A \cap B \to A' \cap B'$ is $(K \cap L \cap \mathbb{S}^{r-1})$-homotopic to $\operatorname{abr} \varphi\colon K \cap L \to A' \cap B'$. By Theorem 2.3.1.3, every $(K \cap L \cap \mathbb{S}^{r-1})$-homotopy from $\operatorname{abr} \varphi\colon K \cap L \to A' \cap B'$ to $\operatorname{abr} f \circ \chi_0$ extends to a $(K \cap \mathbb{S}^{r-1})$-homotopy of $\operatorname{abr} \varphi\colon K \to A'$ and to a $(L \cap \mathbb{S}^{r-1})$-homotopy of $\operatorname{abr}\colon L \cap B'$. The two resulting homotopies combine to define an $\mathbb{S}^{r-1}$-homotopy from φ to a map $\varphi'\colon \mathbb{D}^r \to X'$ which satisfies $\varphi|_{K \cap L} = (f|_{A \cap B}) \circ \chi_0$, $\varphi'(K) \subset A'$, and $\varphi'(L) \subset B'$. Finally, apply Theorem 5.4.1.2 again to deduce that there are continuous maps $\chi_1\colon K \to A$ and $\chi_2\colon L \to B$ with the following properties:

- $\chi_0 = \operatorname{abr} \chi_1$, $\chi_0 = \operatorname{abr} \chi_2$;
- the composition of χ_1 with $\operatorname{abr} f\colon A \to A'$ is $(K \cap L)$-homotopic to $\operatorname{abr} \varphi'\colon K \to A'$;
- the composition of χ_2 with $\operatorname{abr} f\colon B \to B'$ is $(K \cap L)$-homotopic to $\operatorname{abr} \varphi'\colon L \to B'$.

Now the desired map $\chi\colon \mathbb{D}^r \to X$ is obtained by combining χ_1 and χ_2. ☐

Theorem 5.4.2.3. *Suppose that* (X, C) *and* (X', C') *are Borsuk pairs,* Y *and* Y' *are topological spaces, and* $\varphi\colon C \to Y$, $\varphi'\colon C' \to Y'$, $f\colon (X, C) \to (X', C')$

and $g\colon Y \to Y'$ *are continuous, with* $g \circ \varphi = \varphi' \circ \operatorname{abr} f$*. If* f *and* g *are weak homotopy equivalences, then the formulae*

$$F \circ [\mathrm{Imm}_1\colon X \to Y \cup_\varphi X] = [\mathrm{Imm}_1\colon X' \to Y' \cup_{\varphi'} X'] \circ f,$$
$$F \circ [\mathrm{Imm}_2\colon Y \to Y \cup_\varphi X] = [\mathrm{Imm}_2\colon Y' \to Y' \cup_{\varphi'} X'] \circ g,$$

define a weak homotopy equivalence $F\colon Y \cup_\varphi X \to Y' \cup_{\varphi'} X'$*.*

Proof. Let us glue X, Y, and $C \times I$, identifying each point $(c, 0) \in C \times I$ with $c \in X$, and each point $(c, 1) \in C \times I$ with $\varphi(c) \in Y$. Let Z denote the resulting space and, to avoid confusion, denote the maps $\mathrm{Imm}_1\colon X \to Z$, $\mathrm{Imm}_2\colon Y \to Z$, and $\mathrm{Imm}_3\colon C \times I \to Z$ by α, β, and γ. First, we want to show that the formulae

$$h \circ \alpha = [\mathrm{Imm}_1\colon X \to Y \cup_\varphi X], \quad h \circ \beta = [\mathrm{Imm}_2\colon Y \to Y \cup_\varphi X],$$
$$h \circ \gamma = [\mathrm{Imm}_2\colon Y \to Y \cup_\varphi X] \circ \varphi \circ [\mathrm{proj}_1\colon C \times I \to C],$$

define a homotopy equivalence $h\colon Z \to Y \cup_\varphi X$. To this end, extend the the homotopy $\gamma\colon C \times I \to Z$ to a homotopy $\Gamma\colon X \times I \to Z$ of the map α and define $k\colon Y \cup_\varphi X \to Z$ by

$$k(\mathrm{Imm}_1(x)) = \Gamma(x, 1) \quad [x \in X], \quad k \circ \mathrm{Imm}_2 = \beta.$$

Then the formulae

$$\begin{aligned}
&H(\mathrm{Imm}_1(x), t) = h(\Gamma(x, t)) && \text{for} \quad x \in X, \quad t \in I,\\
&H(\mathrm{Imm}_2(x), t) = \mathrm{Imm}_2 && \text{for} \quad y \in Y, \quad t \in I,\\
&K(\alpha(x), t) = \Gamma(x, t) && \text{for} \quad x \in X, \quad t \in l,\\
&K(\beta(y), t) = \beta(y) && \text{for} \quad y \in Y, \quad t \in I,\\
&K(\gamma(c, t), u) = \gamma(c, tu - u + 1), && \text{for} \quad c \in C, \quad t \in I, \quad u \in I,
\end{aligned}$$

define a homotopy $H\colon (Y \cup_\varphi X) \times I \to Y \cup_\varphi X$ from $h \circ k$ to $\mathrm{id}_{(Y \cup_\varphi X)}$ and a homotopy $K\colon Z \times I \to Z$ from $k \circ h$ to id_Z. Consequently, k is a homotopy inverse to h.

Now repeat all this for $X', C', Y', \varphi', \dots$. We obtain a space Z', continuous maps $\alpha'\colon X' \to Z'$, $\beta'\colon X' \to Z'$, $\gamma'\colon C' \times I \to Z'$, and a homotopy equivalence $h'\colon Y' \cup_{\varphi'} X' \to Z'$. Let $G\colon Z \to Z'$ denote the map defined by

$$G \circ \alpha = \alpha' \circ f, \quad G \circ \beta = \beta' \circ g, \quad G \circ \gamma = \gamma' \circ (\operatorname{abr} f \times \mathrm{id}_I).$$

It is clear that G maps the triad

$$(Z, \alpha(X) \cup \gamma(C \times [0, 1/2]), \beta(Y) \cup \gamma(C \times I))$$

into the triad

$$(Z', \alpha'(X') \cup \gamma'(C' \times [0, 1/2]), \beta'(Y') \cup \gamma'(C' \times I)).$$

At the same time, all the conditions of Theorem 5.4.2.2 are fulfilled. Thus, applying this theorem, G is a weak homotopy equivalence. Finally, the commutativity of the diagramme

$$\begin{tikzcd}
Y \cup_\varphi X \arrow[r, "F"] \arrow[d, "h"'] & Y' \cup_{\varphi'} X' \arrow[d, "h'"] \\
Z \arrow[r, "G"'] & Z'
\end{tikzcd}$$

implies that F is a weak homotopy equivalence. □

Corollary 5.4.2.4. *Let* $f\colon (X, A, B) \to (X', A', B')$ *be a map of triads such that* $\operatorname{abr} f\colon A \to A'$, $\operatorname{abr} f\colon B \to B'$, *and* $\operatorname{abr} f\colon A \cap B \to A' \cap B'$ *are weak homotopy equivalences. If* $(A, A \cap B)$ *and* $(A', A' \cap B')$ *are Borsuk pairs, then* $\operatorname{abrs} f\colon X \to X'$ *is a weak homotopy equivalence.*

Proof. This is a corollary of Theorem 5.4.2.3, because

$$\begin{aligned} X &= B \cup_{\mathrm{incl}} A, \quad \mathrm{incl} = [\mathrm{incl}\colon A \cap B \to B], \\ X' &= B' \cup_{\mathrm{incl}} A', \quad \mathrm{incl} = [\mathrm{incl}\colon A' \cap B' \to B']. \end{aligned}$$

□

Theorem 5.4.2.5. *Let* (X, A) *and* (X', A') *be Borsuk pairs. If*

$$f\colon (X, A) \to (X', A')$$

is a weak homotopy equivalence, then so is $\operatorname{fact} f\colon X/A \to X'/A'$.

Proof. It suffices to apply Theorem 5.4.2.3 for $Y = \mathbb{D}^0$, $Y' = \mathbb{D}^0$. □

Theorem 5.4.2.6. *If* $f\colon X \to X'$ *is a weak homotopy equivalence, then so is* $\operatorname{sus}\colon \operatorname{sus} X \to \operatorname{sus} X'$.

Proof. It suffices to apply Theorem 5.4.2.5 to the map

$$\operatorname{rel} \operatorname{cone} f\colon (\operatorname{cone} X, X) \to (\operatorname{cone} X', X').$$

□

Theorem 5.4.2.7. *If* $f\colon X \to X'$ *and* $g\colon Y \to Y'$ *are weak homotopy equivalences, then so is* $f \star g\colon X \star Y \to X' \star Y'$.

Proof. Let A and B be the images of $X \times Y \times [0, 2/3]$ and $X \times Y \times [1/3, 1]$ under the projection $X \times Y \times I \to X \star Y$. Similarly, let A' and B' be the images of $X' \times Y' \times [0, 2/3]$ and $X' \times Y' \times [1/3, 1]$ under the projection $X' \times Y' \times I \to X' \star Y'$. Obviously, $(f \star g)(A) \subset A'$, $(f \star g)(B) \subset B'$, and

$$\operatorname{rel}(f \star g)\colon (X \star Y, A, B) \to (X' \star Y', A', B')$$

satisfies the conditions of Theorem 5.4.2.2. Therefore, $f \star g$ is a weak homotopy equivalence. □

Theorem 5.4.2.8. *If $f\colon X \to Y$ is a weak homotopy equivalence and K is a locally finite cellular space, then $\mathcal{C}(\mathrm{id}, f)\colon \mathcal{C}(K, X) \to \mathcal{C}(K, Y)$ is a weak homotopy equivalence.*

Proof. Given arbitrary continuous maps

$$\varphi\colon \mathbb{D}^r \to \mathcal{C}(K, Y), \quad \psi\colon \mathbb{S}^{r-1} \to \mathcal{C}(K, Y), \quad \text{with} \quad \mathcal{C}(\mathrm{id}, f) \circ \psi = \varphi|_{\mathbb{S}^{r-1}},$$

we need only exhibit a continuous map $\chi\colon \mathbb{D}^r \to \mathcal{C}(K, Y)$ such that $\chi|_{\mathbb{S}^{r-1}} = \psi$ and $\mathcal{C}(\mathrm{id}, f) \circ \chi$ is $\mathbb{S}^{r-1}$-homotopic to φ (see Theorem 5.4.1.2). Consider the maps $\widehat{\varphi}\colon \mathbb{D}^r \times K \to Y$, $\widehat{\psi}\colon \mathbb{S}^{r-1} \times K \to Y$ (see Theorems 1.2.7.6 and 2.1.4.3), which are continuous and satisfy $f \circ \widehat{\psi} = \widehat{\varphi}|_{\mathbb{S}^{r-1}}$. The first part of Theorem 5.4.1.2 applies to $\widehat{\varphi}$ and $\widehat{\psi}$ and yields a continuous map $\alpha\colon (\mathbb{D}^r \times K) \to X$ such that $\alpha|_{\mathbb{S}^{r-1} \times K} = \widehat{\psi}$ as well as an $(\mathbb{S}^{r-1} \times K)$-homotopy $h\colon (\mathbb{D}^r \times K) \times I \to Y$, from $\widehat{\varphi}$ to $f \circ \alpha$. Denote by H the composition of the canonical homeomorphism $(\mathbb{D}^r \times I) \times K \to (\mathbb{D}^r \times K) \times I$ with h, and set $\chi = \alpha^\vee$ (see again Theorem 1.2.7.6). It is clear that $\chi|_{\mathbb{S}^{r-1}} = \psi$ and that $H^\vee$ is an $\mathbb{S}^{r-1}$-homotopy from φ to $\mathcal{C}(\mathrm{id}, f) \circ \chi$. ☐

Corollary 5.4.2.9. *Let X_μ and X'_μ be topological spaces with base points x_μ and x'_μ such that (X_μ, x_μ) and (X'_μ, x'_μ) are Borsuk pairs. If*

$$f_\mu\colon (X_\mu, x_\mu) \to (X'_\mu, x'_\mu)$$

are weak homotopy equivalences, then so is $\vee_\mu f_\mu\colon\ \vee_\mu (X_\mu, x_\mu) \to \vee_\mu (X'_\mu, x'_\mu)$.

Proof. This is a corollary of Theorem 5.4.2.5. ☐

5.4.3 Cellular Approximations of Topological Spaces

Definition 5.4.3.1. A *cellular approximation of the topological space* X is any pair (K, φ) consisting of a cellular space K and a weak homotopy equivalence $\varphi\colon K \to X$.

Example: if X is a Hausdorff space with a cellular decomposition enjoying the property that each compact subset of X intersects only a finite number of cells, then $(\widetilde{X}, \mathrm{id}_X)$, where $\widetilde{X}$ is the cellular space obtained from X through the cellular weakening of its topology, is a cellular approximation of X. In particular, if $X_1, \dots, X_n$ are cellular spaces, then $(X_1 \times_c \cdots \times_c X_n, \mathrm{id})$ is a cellular approximation of $X_1, \dots, X_n$, while $(X_1 \star_c \cdots \star_c X_n, \mathrm{id})$ is a cellular approximation of $(X_1 \star \cdots \star X_n, \mathrm{id})$.

Theorem 5.4.3.2. *Every topological space admits cellular approximations.*

Proof. We observe first that cellular approximations of the components of a topological space yield a cellular approximation of the entire space, while the case of an empty space is trivial. Thus, we may assume that the space X we want to approximate is connected and non-empty. We shall construct a sequence $K_0, K_1, \dots$ of cellular spaces with 0-cells $y_0, y_1, \dots$ as base points, a sequence of continuous maps $\varphi_i\colon (K_i, y_i) \to (X, x_0)$, $i = 0, 1, \dots$ (here $x_0 \in X$ is an arbitrarily fixed point) and, finally, a sequence of cellular embeddings $\eta_i\colon (K_{i-1}, y_i) \to (K_i, y_i)$, $i = 1, 2, \dots$, such that φ_i is an i-equivalence and $\varphi_i \circ \eta_i = \varphi_{i-1}$. Then the pair $(\varinjlim K_i, \varinjlim \varphi_i)$ will be a cellular approximation of X (see Theorem 5.1.11.5).

We proceed by induction. Set $K_0 = \mathbb{D}^0$, $y_0 = \mathbb{D}^0$, $\varphi_0(K_0) = x_0$, and assume that K_i, y_i, φ_i, η_i, $i < r$, have been defined and enjoy the required properties. Pick a spheroid $f_\alpha\colon \mathbb{S}^r \to X$ in each homotopy class $\alpha \in \pi_r(X, x_0)$. Further, for any class $\beta \in N$, where $N = \ker[(\varphi_{r-1})_*\colon \pi_{r-1}(K_{r-1}, y_{r-1}) \to \pi_r(X, x_0)]$, pick a spheroid $g_\beta\colon \mathbb{S}^{r-1} \to K_{r-1}$ of class β, together with a continuous map $h_\beta\colon \mathbb{D}^r \to X$ satisfying $h|_{\mathbb{S}^{r-1}} = \varphi_{r-1} \circ g_\beta$. The maps f_α, g_β, and h_β (with $\alpha \in \pi_r(X, x_0)$ and $\beta \in N$) combine to define three other continuous maps:

$$f\colon \vee_{\alpha \in \pi_r(X,x_0)} (S_\alpha = \mathbb{S}^r, \mathrm{ort}_1) \to X,$$
$$g\colon \coprod_{\beta \in N} (S_\beta = \mathbb{S}^{r-1}) \to K_{r-1}, \quad g\colon \coprod_{\beta \in N} (D_\beta = \mathbb{D}^r) \to X.$$

Now set

$$K_r = [(K_{r-1} \cup_g (\coprod_{\beta \in N} D_\beta)), \mathrm{Imm}_2(y_{r-1})] \vee [(\vee_{\alpha \in \pi_r(X,x_0)} (S_\alpha, \mathrm{bp})), \mathrm{bp}]$$

and define η_r to be the composition

$$K_{r-1} \xrightarrow{\mathrm{Imm}_2} K_{r-1} \cup_g (\coprod_{\beta \in N} D_\beta) \xrightarrow{\mathrm{Imm}_1} K_r,$$

$\varphi_r\colon K_r \to X$ to be the map assembled from φ_{r-1}, h, and f, and y_r to be $\eta_r(y_{r-1})$. Applying Corollary 5.3.2.4 and Theorem 5.3.5.7, we see that φ_r is an r-equivalence, and it is clear that $\varphi_r \circ \eta_r = \varphi_{r-1}$ and that η_r is a cellular embedding. □

Theorem 5.4.3.3. *Let (K, φ) and (K', φ') be cellular approximations of topological spaces X and X', and let $f\colon X \to X'$ be an arbitrary continuous map. Then there is a continuous map $g\colon K \to K'$ such that the diagramme*

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, "\varphi"'] & K' \arrow[d, "\varphi'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

is homotopy commutative (i.e. , the maps $f \circ \varphi$ and $\varphi' \circ g$ are homotopic). This property uniquely defines the homotopy class of g.

Proof. The mapping $\pi(\mathrm{id}, \varphi')\colon \pi(K, K') \to \pi(K, X')$ is invertible (see Theorem 5.4.1.3), and thus in $\pi(K, K')$ there is a unique element which is taken by $\pi(\mathrm{id}, \varphi')$ into the class of $f \circ \varphi$. □

Theorem 5.4.3.4. *Let (K, φ) and (K', φ') be two cellular approximations of the same topological space X. Then there is a homotopy equivalence $g\colon K \to K'$ such that $\varphi' \circ g$ is homotopic to φ.*

Proof. To obtain g, apply Theorem 5.4.3.3 to $X' = X$ and $f = \mathrm{id}_X$. To get a homotopy inverse g' to g, interchange the roles of (K', φ') and (K, φ). Finally, apply Theorem 5.4.3.3 again to show that $g \circ g'$ and $g' \circ g$ are homotopic to $\mathrm{id}_{K'}$ and id_K, respectively. $\square$

Weak Homotopy Equivalence of an Equivalence Relation

Definition 5.4.3.5. Two topological spaces are said to be *weakly homotopy equivalent* if they admit cellular approximations (K, φ) and (L, ψ) with $K = L$. It is clear that this defines an equivalence relation (in the usual, set-theoretic sense).

Let X and Y be topological spaces such that there is a weak homotopy equivalence $f\colon X \to Y$. Then X and Y are weakly homotopy equivalent. Indeed, if (K, φ) is a cellular approximation of X, then $(K, f \circ \varphi)$ is a cellular approximation of Y. The converse is false: there are examples of weakly homotopy equivalent spaces X, Y such that there is no weak homotopy equivalence $X \to Y$ and no weak homotopy equivalence $Y \to X$. See Exercise 5.4.4.2 below for such an example.

Two homotopy equivalent spaces are certainly weakly homotopy equivalent. The converse is false: for example, every topological space is weakly homotopy equivalent to a cellular one (see Theorem 5.4.3.2), but not every topological space has the homotopy type of a cellular space (see Definition 5.4.1.7). On the other hand, Theorem 5.4.1.4 shows that two weakly homotopy equivalent cellular spaces are actually homotopy equivalent.

Weak Homotopy Equivalence of the Fibres of a Serre Bundle

Theorem 5.4.3.6. *Any two fibres of a Serre bundle with connected base are weakly homotopy equivalent.*

Proof. Let ξ be the given Serre bundle with $\mathrm{bs}\,\xi$ connected, and let $b_0, b_1 \in \mathrm{bs}\,\xi$. Pick a path $s\colon I \to \mathrm{bs}\,\xi$ with $s(0) = b_0$, $s(1) = b_1$, and a cellular approximation (K, φ) of the fibre $(\mathrm{proj}\,\xi)^{-1}(b_0)$. The map $\widetilde{f} = [\mathrm{incl}\colon (\mathrm{proj}\,\xi)^{-1}(b_0) \to \mathrm{tl}\,\xi] \circ \varphi$ and the homotopy $F\colon K \times I \to \mathrm{bs}\,\xi$, $F(x, t) = s(t)$, satisfy $F(x, 0) = (\mathrm{proj}\,\xi) \circ \widetilde{f}(x)$ $[x \in K]$, and hence there is a homotopy $\widetilde{F}\colon K \times I \to \mathrm{tl}\,\xi$ of $\widetilde{f}$ which covers F (see Theorem 4.1.3.6). Now define $\psi\colon K \to (\mathrm{proj}\,\xi)^{-1}(b_1)$ by $x \mapsto \widetilde{F}(x, 1)$ and check that ψ is a weak homotopy equivalence.

To do this, given any $x \in K$ and any spheroid $g \in \mathrm{Sph}_r(K, x)$, note that the formula $(y, t) \mapsto \widetilde{F}(g(y), t)$ defines a fibre homotopy (see Definition 5.1.7.1) from the spheroid $\varphi_\#(g)$ to the spheroid $\psi_\#(g)$ along the path $s\colon I \to \mathrm{tl}\,\xi$ given by $t \mapsto \widetilde{F}(g(\mathrm{ort}_1), t)$. Consequently, the diagramme

$$\begin{tikzcd}
& \pi_r(K, x) \arrow[dl, "\varphi_*"'] \arrow[dr, "\psi_*"] & \\
\pi_r((\operatorname{proj}\xi)^{-1}(b_0), \varphi_x) \arrow[rr, "T_{\widetilde{s}}"'] & & \pi_r((\operatorname{proj}\xi)^{-1}(b_1), \psi_x)
\end{tikzcd}$$

commutes (the translation $T_{\widetilde{s}}$ is defined in Definition 5.1.7.3). Since $T_{\widetilde{s}}$ is an isomorphism, the invertibility of φ_* implies the invertibility of ψ_*. $\square$

Cellular Approximations of Topological Pairs

Definition 5.4.3.7. A *cellular approximation of the topological pair* (X, A) is any pair $[(K, L), \varphi]$ consisting of a cellular pair (K, L) and a weak homotopy equivalence $\varphi\colon (K, L) \to (X, A)$.

When A and L are points, a cellular approximation $[(K, L), \varphi]$ of (X, A) is termed a *cellular approximation of the pointed space* (X, A).

Theorem 5.4.3.8. *Every topological pair (X, A) admits cellular approximations. Moreover, given any cellular approximation (L, ψ) of the subspace A, there is a cellular approximation $[(K, L), \varphi]$ of (X, A) with $\psi = \operatorname{abr}\varphi$.*

Proof. Let (M, χ) be a cellular approximation of X (see Theorem 5.4.3.2), and let $g\colon L \to M$ be a cellular map such that$\chi \circ g$ is homotopic to $[\mathrm{incl}\colon A \to X] \circ \psi$ (see Theorems 5.4.3.3 and 2.3.2.6). Set $K = \operatorname{Cyl} g$ and define φ to be the relativisation of the map $K \to X$ given by χ and some homotopy $L \times I \to X$ from $[\mathrm{incl}\colon A \to X] \circ \psi$ to $\chi \circ g$. Obviously, φ is a weak homotopy equivalence and $\operatorname{abr}\varphi = \psi$. $\square$

Theorem 5.4.3.9. *Let $[(K, L), \varphi]$ and $[(K', L'), \varphi']$ be cellular approximations of the topological pairs (X, A) and $(X',')$, and let $f\colon (X, A) \to (X', A')$ be an arbitrary continuous map. Then there is a continuous map $g\colon (K, L) \to (K', L')$ (unique up to homotopies) such that the diagramme*

$$\begin{tikzcd}
K, L \arrow[r, "g"] \arrow[d, "\varphi"'] & K', L' \arrow[d, "\varphi'"] \\
X, A \arrow[r, "f"'] & X', A'
\end{tikzcd}$$

is homotopy commutative. If $[(K, L), \varphi]$ and $[(K', L'), \varphi']$ are cellular approximations of the same topological pair, then there is a homotopy equivalence $g\colon (K, L) \to (K', L')$ such that $\varphi' \circ g$ is homotopic to $f \circ \varphi$.

Proof. The proof repeats the proofs of Theorems 5.4.3.3 and 5.4.3.4, except that one has to refer to Theorem 5.4.1.13 instead of Theorem 5.4.1.3. $\square$

Cellular Approximations and Constructions

Remark 5.4.3.10. It is clear that if (K_μ, φ_μ) are cellular approximations of the spaces X_μ $(\mu \in M)$, then $(\coprod_{\mu\in M} K_\mu, \coprod_{\mu\in M} \varphi_\mu)$ is a cellular approximation of $\coprod_{\mu\in M} X_\mu$. Also, applying Corollary 5.4.2.9, we see that if (X_μ, x_μ) are Borsuk pairs ($x_\mu \in X_\mu$ are base points) and $[(K_\mu, y_\mu), \varphi_\mu]$ are cellular approximations of these pointed spaces, then $[\vee_\mu(K_\mu, y_\mu), \vee_\mu \varphi_\mu]$ is a cellular approximation of the bouquet $\vee_\mu(X_\mu, x_\mu)$.

Further, if $(K_1, \varphi_1), \dots, (K_n, \varphi_n)$ are cellular approximations of $X_1, \dots, X_n$, then

$$(K_1 \times_c \cdots \times_c K_n, \varphi_1 \times \cdots \times \varphi_n)$$

is a cellular approximation of $X_1 \times \cdots \times X_n$ (see Remark 5.4.2.1 and Definition 5.4.3.1). In the same circumstances,

$$(K_1 \times_c \cdots \star_c K_n, \varphi_1 \star \cdots \star \varphi_n)$$

is a cellular approximation of $X_1 \star \cdots \star X_n$ (see Theorem 5.4.2.7). In particular, $(\operatorname{sus} K, \operatorname{sus} \varphi)$ is a cellular approximation of $\operatorname{sus} X$ whenever (K, φ) is a cellular approximation of X.

If $[(K, L), \varphi]$ is a cellular approximation of the Borsuk pair (X, A), then $(K/L, \operatorname{fact} \varphi\colon K/L \to X/A)$ is a cellular approximation of X/A This is a corollary of Theorem 5.4.2.5.

An Application: Generalisation of Theorems 5.3.3.7, 5.3.5.7, and 5.3.6.2

Lemma 5.4.3.11. *Let (X, A, B) be a triad with the property that either*

$$\operatorname{int} A \cup \operatorname{int} B = X \quad \textit{or} \quad (A, A \cap B)$$

is a Borsuk pair. Then there exist a cellular triad (K, L, M) and a continuous map $f\colon (K, L, M) \to (X, A, B)$ such that $\operatorname{abrs} f\colon K \to X$, $\operatorname{abr} f\colon L \to A$, *and* $\operatorname{abr} f\colon L \cap M \to A \cap B$ *are weak homotopy equivalences.*

Proof. By Theorem 5.4.3.2, $A \cap B$ has a cellular approximation, say (N, χ) and the latter can be extended to cellular approximations $[(L', N), \varphi]$ and $[(M', N), \psi]$ of the pairs $(A, A \cap B)$ and $(B, B \cap A)$, as shown by Theorem 5.4.3.8. Next, attach the cylinder $N \times I$ to $L' \amalg M'$ by the map

$$(N \times 0) \cup (N \times 1) \to L \amalg M, \quad (x, 0) \mapsto \operatorname{incl}_1(x), \quad (x, 1) \mapsto \operatorname{incl}_2(x),$$

and call the resulting cellular space K. Now identify $N \times I$, L', and M' with their images in K and set $L = (N \times I) \cup L'$, $M = (N \times I) \cup M'$. The composite maps

$$N \times I \xrightarrow{\operatorname{proj}_1} N \xrightarrow{\chi} A \cap B \xrightarrow{\operatorname{incl}} X,$$
$$L' \xrightarrow{\varphi} A \xrightarrow{\operatorname{incl}} X, \quad \text{and} \quad M' \xrightarrow{\psi} B \xrightarrow{\operatorname{incl}} X$$

jointly define the map $f\colon (K, L, M) \to (X, A, B)$. Obviously, $\operatorname{int} L \cup \operatorname{int} M = K$ and $\operatorname{abr} f\colon L \to A$, $\operatorname{abr} f\colon M \to B$, and $\operatorname{abr} f\colon L \cap M \to A \cap B$ are weak homotopy equivalences. Therefore, so is $\operatorname{abrs} f\colon K \to X$ (see Theorem 5.4.2.2 and Corollary 5.4.2.4) . ☐

Remark 5.4.3.12. The homomorphism

$$[\pi_1(A, x_0) \star \pi_1(B, x_0)]/\operatorname{vk}(X, A, B, x_0) \to \pi_1(X, x_0), \tag{5.4.3.13}$$

defined in Definition 5.3.3.5, is an isomorphism not only for a cellular triad (X, A, B) with A, B, $A \cap B$ connected (as asserted by Theorem 5.3.3.7), but also for any triad (X, A, B) such that A, B, $A \cap B$ are connected and either $\operatorname{int} A \cup \operatorname{int} B = X$ or $(A, A \cap B)$ is a Borsuk pair. In fact, this follows from Theorem 5.3.3.7 and Lemma reflem:05-4-3-11, since the homomorphism (5.4.3.13) is natural.

In particular, we see that the fundamental group of the bouquet of two spaces is canonically isomorphic to the free product of the fundamental groups of these spaces under the only assumption that each space forms, together with its base point, a Borsuk pair (cf. Corollary 5.3.3.8).

Remark 5.4.3.14. Concerning Theorem 5.3.5.7, we can weaken the demand that the pairs (X_μ, x_μ) be cellular and instead ask only that they be Borsuk pairs. That this is possible is guaranteed by Theorem 5.4.3.8, the discussion of bouquets in Definition 5.4.3.10, an the commutativity of diagramme (5.3.5.2) in Remark 5.3.5.1.

Remark 5.4.3.15. Theorem 5.3.6.2 and its corollary Theorem 5.3.6.4 are valid not only for cellular pairs, but also for arbitrary Borsuk pairs. This generalisation follows from Theorem 5.4.3.8 and the last statement on quotients in Definition 5.4.3.10.

5.4.4 Exercises

Exercise 5.4.4.1. Consider the union X of the graph of the function $x \mapsto \sin(1/x)$ on the interval $0 < x \leq 1/\pi$ and the broken line made of the four segments with the successive vertices $(1/\pi, 0)$, $(1/\pi, 2)$, $(-1, 2)$, $(-1, 0)$, and $(0, 0)$. Show that X is Â«-connected but not homotopy fit. (Cf. Definition 5.4.1.7)

Exercise 5.4.4.2. Let $A = \{0, 2^n | n \in \mathbb{Z}\} \subset \mathbb{R}$ (cf. Exercise 4.2.4.2). Show that the spaces$X = \mathbb{Z} \amalg (A \times \mathbb{S}^1)$ and $Y = A \amalg (\mathbb{Z} \times \mathbb{S}^1)$ are weakly homotopy equivalent, but there is no weak homotopy equivalence $X \to Y$, and no weak homotopy equivalence $Y \to X$. (Cf. Definition 5.4.3.5.)

Exercise 5.4.4.3. Let X denote the subset of $\mathbb{R}^3$ consisting of the segment I and the sequence of segments with endpoints $n \operatorname{ort}_1$, $\operatorname{ort}_1 + \operatorname{ort}_3 / n$ $(n = 1, 2, \dots)$. Show that $(X, (x_1, x_2, x_3) \to x_1, I)$ is a Serre bundle, but there exists fibres which are not homotopy equivalent.

Exercise 5.4.4.4. Suppose (X, x_0), (Y, y_0) are pointed topological spaces, Z is a cellular space with a 0-cell z_0 for base point, and $f\colon X \to Y$ is a weak homotopy equivalence with $f(x_0) = y_0$. Show that $\operatorname{abr} \mathcal{C}(\mathrm{id}, f)\colon \mathcal{C}(Z, x_0; X, x_0) \to \mathcal{C}(Z, z_0; Y, y_0)$ is a weak homotopy equivalence. (Cf. Theorem 5.4.2.8.) Â§

5.5 THE WHITEHEAD PRODUCT

5.5.1 The Class Whd(m, n)

Remark 5.5.1.1. In this section we define and study some of the properties of an operation on the elements of homotopy groups. In a certain sense, this operation generalises the action of the fundamental group on the homotopy groups. The definition assumes that a pair m, n of positive integers is given.

The present subsection is devoted to a very specific preliminary construction. Recall (see Remarks 2.1.3.2 and 2.1.5.2) that the cellular decomposition of $\mathbb{S}^m \times \mathbb{S}^n$, determined by the standard decompositions of $\mathbb{S}^m$ and $\mathbb{S}^n$ (each having two cells) consists of four cells: an $(m+n)$-cell and three other cells which form the bouquet $(\mathbb{S}^m, \mathrm{ort}_1) \vee (\mathbb{S}^n, \mathrm{ort}_1)$. We denote this bouquet by $B(m,n)$ or, simply, by B. The standard characteristic map of the $(m+n)$-cell is the composition of the canonical homeomorphism $\mathbb{D}^{m+n} \to \mathbb{D}^m \times \mathbb{D}^n$ (see Remark 1.2.6.9) with the map $\mathbb{DS} \times \mathbb{DS}$; it takes $\mathbb{S}^{m+n-1}$ into B, and takes the point $(\mathrm{ort}_1 + \mathrm{ort}_{m+1})/\sqrt{2}$ into $\mathrm{bp} = (\mathrm{ort}_1, \mathrm{ort}_1)$. Therefore, this characteristic map defines an element of the group $\pi_{m+n}(\mathbb{S}^m \times \mathbb{S}^n, B, \mathrm{bp})$ (see Corollary 5.2.2.5), which we call $\mathrm{Whd}(m,n)$ or, simply, Whd. Also, we write $\mathrm{whd}(m,n)$ or, simply, whd, for the element $\partial(\mathrm{Whd}) \in \pi_{m+n-1}(B, \mathrm{bp})$, i.e., the class of the attaching spheroid $\mathbb{S}^{m+n-1} \to B$.

We need two additional notations: θ for the homeomorphism $B(m,n) \to B(n,m)$ which permutes $\mathbb{S}^m$ and $\mathbb{S}^n$, and μ for the product of the spheroids $\mathrm{Imm}_1, \mathrm{Imm}_2 \colon (\mathbb{S}^n, \mathrm{ort}_1) \to (B(m,n), \mathrm{bp})$ when $m = n$.

Theorem 5.5.1.2. *The class* Whd *has infinite order.*

Proof. It is enough to establish that Whd is of infinite order and that

$$\partial \colon \pi_{m+n}(\mathbb{S}^m \times \mathbb{S}^n, B, \mathrm{bp}) \to \pi_{m+n-1}(B, \mathrm{bp})$$

is monomorphic The first is a consequence of the fact that the homomorphism

$$\mathrm{proj}_* \colon \pi_{m+n}(\mathbb{S}^m \times \mathbb{S}^n, B, \mathrm{bp}) \to \pi_{m+n}(\mathbb{S}^m \times \mathbb{S}^n)/B = \mathbb{S}^{m+n}, \mathrm{proj}(\mathrm{bp})) = \mathbb{Z}$$

takes Whd into a generator of the right-hand group. The second claim follows from the exactness of the homotopy sequence of the pair $(\mathbb{S}^m \times \mathbb{S}^n, B)$ with base point bp, because $\mathrm{incl}_* \colon \pi_{m+n}(B, \mathrm{bp}) \to \pi_{m+n}(\mathbb{S}^m \times \mathbb{S}^n, \mathrm{bp})$ is epimorphic (see Theorem 5.3.5.6). ☐

Theorem 5.5.1.3. *The isomorphism*

$$\theta_* \colon \pi_{m+n-1}(B(m,n), \mathrm{bp}) \to \pi_{m+n-1}(B(n,m), \mathrm{bp})$$

takes $\mathrm{whd}(m,n)$ *into* $(-1)^{mn}\,\mathrm{whd}(n,m)$.

Proof. This results from the commutativity of the diagramme

$$\begin{tikzcd}
\mathbb{S}^{m+n-1} \arrow[r] \arrow[d] & \mathbb{S}^{m+n-1} \arrow[d] \\
B(m,n) \arrow[r, "\theta"'] & B(n,m)
\end{tikzcd}$$

where the vertical maps are the attaching spheroids which represent the classes $\mathrm{whd}(m,n)$ and $\mathrm{whd}(n,m)$ (see Remark 5.5.1.1), while the upper horizontal map is given by $(x_1,\dots,x_{m+n}) \mapsto (x_{m+1},\dots,x_{m+n},x_1,\dots,x_m)$ (and its degree is $(-1)^{mn}$). □

Theorem 5.5.1.4. *If* $m = 1$, *then*

$$\mathrm{whd}(m,n) = \mathrm{Imm}_2(\mathrm{sph}_n)[T_{\mathrm{Imm}_1}(\mathrm{sph}_1)\,\mathrm{Imm}_{2*}(\mathrm{sph}_n)]^{-1}.$$

In particular, $\mathrm{whd}(1,1) = \alpha_2\alpha_1\alpha_2^{-1}\alpha_{1^{-1}}$, *where (as in Subsection 5.3.1),* α_1 , α_2 *denote the elements* $\mathrm{Imm}_{1*}(\mathrm{sph}_1), \mathrm{Imm}_{2*}(\mathrm{sph}_1) \in \pi_1(B(1,1),\mathrm{bp})$.

Proof. According to Remark 5.5.1.1, $\mathrm{whd}(1,n)$ is represented by the spheroid $\mathbb{S}^n \to B(1,n)$,

$$(x_1,\dots,x_{n+1}) \mapsto \begin{cases} \mathrm{Imm}_1 \circ \mathbb{DS}(\sqrt{2}x_1), & \text{if} \quad |x_1| \le 1/\sqrt{2}, \\ \mathrm{Imm}_2 \circ \mathbb{DS}(\sqrt{2}(x_2,\dots,x_{n+1})), & \text{if} \quad |x_1| \ge 1/\sqrt{2}. \end{cases}$$

This is obviously homotopic to the product of the spheroid

$$(x_1,\dots,x_{n+1}) \mapsto \begin{cases} \mathrm{bp}, & \text{if} \quad x_1 \le 1/\sqrt{2}, \\ \mathrm{Imm}_2 \circ \mathbb{DS}(\sqrt{2}(x_2,\dots,x_{n+1})), & \text{if} \quad x_1 \ge 1/\sqrt{2}, \end{cases} \tag{5.5.1.5}$$

with the spheroid obtained by translating the spheroid

$$(x_1,\dots,x_{n+1}) \mapsto \begin{cases} \mathrm{Imm}_2 \circ \mathbb{DS}(\sqrt{2}(x_2,\dots,x_{n+1})), & \text{if} \quad x_1 \le -1/\sqrt{2}, \\ \mathrm{bp}, & \text{if} \quad x_1 \ge -1/\sqrt{2}, \end{cases} \tag{5.5.1.6}$$

along the path $t \mapsto \mathrm{Imm}_1 \circ \mathbb{DS}(1-2t)$. Now it remains to observe that the class of (5.5.1.5) is $\mathrm{Imm}_{2*}(\mathrm{sph}_n)$, the class of (5.5.1.6) is $\mathrm{Imm}_{2*}(\mathrm{sph}_n)^{-1}$, and the class of the above path is $\mathrm{Imm}_{1*}(\mathrm{sph}_1)$. □

Theorem 5.5.1.7. *The class* $\mathrm{whd}(m,n)$ *belongs to the kernel of each of the following three homomorphisms:*

$$\begin{aligned} \mathrm{proj}_{1*}&\colon \pi_{m+n-1}(B,\mathrm{bp}) \to \pi_{m+n-1}(\mathbb{S}^m,\mathrm{ort}_1), \\ \mathrm{proj}_{2*}&\colon \pi_{m+n-1}(B,\mathrm{bp}) \to \pi_{m+n-1}(\mathbb{S}^n,\mathrm{ort}_1), \\ \mathrm{incl}_*&\colon \pi_{m+n-1}(B,\mathrm{bp}) \to \pi_{m+n-1}(\mathbb{S}^m \times \mathbb{S}^n,\mathrm{bp}). \end{aligned}$$

Proof. For incl_* this results from the exactness of the homotopy sequence of the pair $(\mathbb{S}^m \times \mathbb{S}^n, B)$ with base point bp. For the first and the second homomorphisms, use the equalities

$$\begin{aligned} [\mathrm{proj}_1\colon B \to \mathbb{S}^m] &= [\mathrm{proj}_1\colon \mathbb{S}^m \times \mathbb{S}^n \to \mathbb{S}^m] \circ \mathrm{incl}, \\ [\mathrm{proj}_2\colon B \to \mathbb{S}^n] &= [\mathrm{proj}_2\colon \mathbb{S}^m \times \mathbb{S}^n \to \mathbb{S}^n] \circ \mathrm{incl}\,. \end{aligned}$$

□

Theorem 5.5.1.8. *The homomorphism*

$$(\mathrm{id}_{\mathbb{S}^m} \vee \mu)_*\colon \pi_{m+n-1}(B(m,n), \mathrm{bp}) \to \pi_{m+n-1}((\mathbb{S}^m, \mathrm{ort}_1) \vee (B(n,n), \mathrm{bp}), \mathrm{bp})$$

takes $\mathrm{whd}(m,n)$ *into*

$$[(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1)_*(\mathrm{whd}(m,n))][(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2)_*(\mathrm{whd}(m,n))].$$

Proof. When $m = 1$ or $n = 1$ this follows from Theorems 5.5.1.4 and 5.5.1.3; now let $m > 1$ and $n > 1$. The bouquet $(\mathbb{S}^m, \mathrm{ort}_1) \vee (B(n,n), \mathrm{bp})$ is simply connected, and it yields the product $\mathbb{S}^m \times B(n,n)$ when we add two $(m+n)$-cells with attaching spheroids $\mathbb{S}^{m+n-1} \to (\mathbb{S}^m, \mathrm{ort}_1) \vee (B(n,n), \mathrm{bp})$ belonging to the classes $(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1)_*(\mathrm{whd}(m,n))$ and $(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2)_*(\mathrm{whd}(m,n))$. Consequently, the kernel of the homomorphism

$$\mathrm{incl}_*\colon \pi_{m+n-1}(\mathbb{S}^m, \mathrm{ort}_1) \vee (B(n,n), \mathrm{bp}), \mathrm{bp}) \to \pi_{m+n-1}(\mathbb{S}^m \times B(n,n), \mathrm{bp})$$

is generated by the indicated classes. This kernel contains also the class

$$(\mathrm{id}_{\mathbb{S}^m} \vee \mu)_*(\mathrm{whd}(m,n)).$$

To see this, note that $\mathrm{whd}(m,n)$ sits in the kernel of the homomorphism induced by the inclusion $B(m,n) \to \mathbb{S}^m \times \mathbb{S}^n$ (see Theorem 5.5.1.7), while $\mathrm{id}_{\mathbb{S}^m} \vee \mu$ is the compression of the map $\mathrm{id}_{\mathbb{S}^m} \times \mu\colon \mathbb{S}^m \times \mathbb{S}^n \to \mathbb{S}^m \times B(n,n)$. Therefore

$$\begin{aligned}&(\mathrm{id}_{\mathbb{S}^m} \vee \mu)_*(\mathrm{whd}(m,n)) = \\ &[(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1)_*(\mathrm{whd}(m,n))]^{k_1}[(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2)_*(\mathrm{whd}(m,n))]^{k_2}\end{aligned} \tag{5.5.1.9}$$

with $k_1, k_2 \in \mathbb{Z}$, and we shall presently show that $k_1 = k_2 = 1$.

The compositions $(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_1) \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mu)$ and $(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_2) \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mu)$, where proj_1, proj_2 are the projections of $B(n,n)$ onto $\mathbb{S}^n$, are both homotopic to $\mathrm{id}_{B(m,n)}$. At the same time,

$$(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_1) \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1) = (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_2) \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2) = \mathrm{id}_{B(m,n)}\,.$$

while both

$$(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_1) \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2), \quad (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_2) \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1)$$

equal the composition

$$B(m,n) \xrightarrow{\mathrm{proj}_1} \mathbb{S}^m \xrightarrow{\mathrm{Imm}_1} B(m,n).$$

Now applying the homomorphisms $(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_1)_*$ and $(\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{proj}_2)_*$ to both members of (5.5.1.9) and using Theorem 5.5.1.7, we get $\mathrm{whd}(m,n) = \mathrm{whd}(m,n)^{k_1}$, $\mathrm{whd}(m,n) = \mathrm{whd}(m,n)^{k_2}$. Finally, these equalities yield, by virtue of Theorem 5.5.1.2, $k_1 = 1$, $k_2 = 1$. □

Theorem 5.5.1.10. *The class* whd(m,n) *belongs to the kernel of the homomorphism*

$$\mathrm{sus}\colon \pi_{m+n-1}(B(m,n),\mathrm{bp}) \to \pi_{m+n}(\mathrm{sus}(B(m,n),\mathrm{bp}) = B(m+1,n+1),\mathrm{bp}).$$

Proof. By Remark 5.2.1.1, the diagramme

$$\begin{tikzcd}
\pi_{m+n-1}(\mathbb{S}^m,\mathrm{ort}_1) \arrow[d, "\mathrm{sus}"'] & \pi_{m+n-1}(B(m,n),\mathrm{bp}) \arrow[l, "\mathrm{proj}_{1*}"'] \arrow[r, "\mathrm{proj}_{2*}"] \arrow[d, "\mathrm{sus}"] & \pi_{m+n-1}(\mathbb{S}^n,\mathrm{ort}_1) \arrow[d, "\mathrm{sus}"] \\
\pi_{m+n}(\mathbb{S}^m,\mathrm{ort}_1) & \pi_{m+n}(B(m,n),\mathrm{bp}) \arrow[l, "\mathrm{proj}_{1*}"] \arrow[r, "\mathrm{proj}_{2*}"'] & \pi_{m+n}(\mathbb{S}^n,\mathrm{ort}_1)
\end{tikzcd}$$

commutes. This, combined with Theorem 5.5.1.7, shows that sus(whd(m,n)) belongs to the kernels of proj_{1*} and proj_{2*} and thus to the kernel of the homomorphism

$$\pi_{m+n}(B(m+1,n+1),\mathrm{bp}) \to \pi_{m+n}(\mathbb{S}^{m+1},\mathrm{ort}_1) \oplus \pi_{m+n}(\mathbb{S}^{n+1},\mathrm{ort}_1)$$

given by proj_{1*} and proj_{2*}. Finally, recall that the last homomorphism is an isomorphism (see Theorem 5.3.5.7). ☐

5.5.2 Definition and the Simplest Properties of the Whitehead Product

Definition 5.5.2.1. Let (X,x_0) be a pointed topological space, and let $\alpha \in \pi_m(X,x_0)$, $\beta \in \pi_n(X,x_0)$. Clearly, the homotopy class of the map

$$h\colon (B,\mathrm{bp}) \to (X,x_0)$$

defined by arbitrary spheroids $(\mathbb{S}^m,\mathrm{ort}_1) \to (X,x_0)$ and $(\mathbb{S}^n,\mathrm{ort}_1) \to (X,x_0)$ representing α and β is independent of the choice of these spheroids. Therefore, the element $h_*(\mathrm{whd}(m,n)) \in \pi_{m+n-1}(X,x_0)$ is determined solely by the classes α and β. This element is called the *Whitehead product* of α and β, denoted $[\alpha,\beta]$.

Notice that in terms of this definition, whd(m,n) itself is the Whitehead product of the classes of the spheroids

$$\mathrm{Imm}_1\colon (\mathbb{S}^m,\mathrm{ort}_1) \to (B,\mathrm{bp}), \quad \mathrm{Imm}_2\colon (\mathbb{S}^n,\mathrm{ort}_1) \to (B,\mathrm{bp}),$$

i.e.,

$$\mathrm{whd}(m,n) = [\mathrm{Imm}_{1*}(\mathrm{sph}_m), \mathrm{Imm}_{1*}(\mathrm{sph}_n)].$$

It is readily cheeked that $f_*([\alpha,\beta]) = [f_*(\alpha), f_*(\beta)]$ for any $\alpha \in \pi_m(X,x_0)$, $\beta \in \pi_n(X,x_0)$, and continuous $f\colon (X,x_0) \to (Y,y_0)$. Furthermore, $T_s([\alpha,\beta]) = [T_s\alpha, T_s\beta]$ for any $\alpha \in \pi_m(X,x_0)$, $\beta \in \pi_n(X,x_0)$, and any path $s\colon I \to X$ with $s(0) = x_0$.

Theorem 5.5.2.2. *If $\alpha \in \pi_m(X, x_0)$, $\beta \in \pi_n(X, x_0)$, then*

$$[\beta, \alpha] = (-1)^{mn}[\alpha, \beta].$$

Proof. Indeed, if $h\colon (B, \mathrm{bp}) \to (X, x_0)$ is the map defined by two spheroids, $(\mathbb{S}^m, \mathrm{ort}_1) \to (X, x_0)$ and $(\mathbb{S}^n, \mathrm{ort}_1) \to (X, x_0)$ which represent the classes α and β, then

$$\begin{aligned}[\beta, \alpha] &= (h \circ \theta)_*(\mathrm{whd}(n, m)) = h_*(\theta_*(\mathrm{whd}(n, m)) \\ &= h_*((-1)^{mn}\,\mathrm{whd}(m, n)) = (-1)^{mn}[\alpha, \beta]\end{aligned}$$

(see Theorem 5.5.1.3). ☐

Theorem 5.5.2.3. *If $\alpha \in (X, x_0)$ and $\beta_1, \beta_2 \in \pi_n(X, x_0$ with $n > 1$, then $[\alpha, \beta_1 + \beta_2] = [\alpha, \beta_1] + [\alpha, \beta_2]$. If $\alpha_1, \alpha_2 \in \pi_m(X, x_0)$ with $m > 1$ and $\beta \in \pi_n(X, x_0)$, then $[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta]$.*

Proof. Because of Theorem 5.5.2.2, one has to prove only the first equality. Consider the map $h\colon ((\mathbb{S}^m, \mathrm{ort}_1) \vee (B(n, n), \mathrm{bp}), \mathrm{bp}) \to (X, x_0)$ defined by arbitrary spheroids, $f\colon (\mathbb{S}^m, \mathrm{ort}_1) \to (X, x_0)$ and $g_1, g_2\colon (\mathbb{S}^n, \mathrm{ort}_1) \to (X, x_0)$, representing the classes α and β_1, β_2, respectively. Then the map $(B(m, n), \mathrm{bp}) \to (X, x_0)$ defined by f and g_1 equals $h \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1)$, the map defined by f and g_2 equals $h \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2)$, and finally the map defined by f and the product of the spheroids g_1 and g_2 equals $h \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mu)$. Hence,

$$\begin{aligned}[\alpha, \beta_1 + \beta_2] &= h_* \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mu)_*(\mathrm{whd}(m, n)) \\ &= h_*((\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1)_*(\mathrm{whd}(m, n)) + h_*((\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2)_*(\mathrm{whd}(m, n)) \\ &= (h_* \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_1))_*(\mathrm{whd}(m, n)) \\ &\quad + (h_* \circ (\mathrm{id}_{\mathbb{S}^m} \vee \mathrm{Imm}_2))_*(\mathrm{whd}(m, n)) \\ &= [\alpha, \beta_1] + [\alpha, \beta_2].\end{aligned}$$

☐

Theorem 5.5.2.4. *If $\alpha \in \pi_1(X, x_0)$, $\beta \in \pi_n(X, x_0)$ with $n \geq 1$, then $[\alpha, \beta] = \beta(T_\alpha\beta)^{-1}$. In particular, $[\alpha, \beta] = \beta\alpha\beta^{-1}\alpha^{-1}$ for any $\alpha, \beta \in \pi_1(X, x_0)$.*

Proof. This is a corollary of Theorem 5.5.1.4. ☐

Theorem 5.5.2.5. *For any $\alpha \in \pi_m(X, x_0)$, $\beta \in \pi_n(X, x_0)$, the product $[\alpha, \beta]$ belongs to the kernel of the homomorphism*

$$\mathrm{sus}\colon \pi_{m+n-1}(X, x_0) \to \pi_{m+n}(\mathrm{sus}(X, x_0), \mathrm{bp}).$$

Proof. This is a corollary of Theorem 5.5.1.10. ☐

Theorem 5.5.2.6. *For any $\alpha \in \pi_m(X, x_0)$, $\beta \in \pi_n(X, x_0)$ the product*

$$[\mathrm{Imm}_{1*}(\alpha), \mathrm{Imm}_{2*}(\beta)]$$

belongs to the kernel of each of the homomorphisms:

$$\begin{aligned}
&\mathrm{proj}_{1*}\colon \pi_{m+n-1}((X,x_0)\vee(Y,y_0),\mathrm{bp})\to\pi_{m+n-1}(X,x_0),\\
&\mathrm{proj}_{2*}\colon \pi_{m+n-1}((X,x_0)\vee(Y,y_0),\mathrm{bp})\to\pi_{m+n-1}(Y,y_0),\\
&\mathrm{incl}\,{*}\colon \pi_{m+n-1}((X,x_0)\vee(Y,y_0),\mathrm{bp})\to\pi_{m+n-1}(X\times Y,\mathrm{bp}).
\end{aligned}$$

Proof. This is a corollary of Theorem 5.5.1.7. □

Remark 5.5.2.7. Generally speaking, the Whitehead product is not associative. This was already implicit in Theorem 5.5.2.4: if we let (as in Subsection 5.3.1) α_1, α_2, α_3 denote the elements $\mathrm{Imm}_{1*}(\mathrm{sph}_1)$, $\mathrm{Imm}_{2*}(\mathrm{sph}_1)$, $\mathrm{Imm}_{3*}(\mathrm{sph}_1)$ of $\pi_1((\mathbb{S}^1,\mathrm{ort}_1)\vee(\mathbb{S}^1,\mathrm{ort}_1)\vee(\mathbb{S}^1,\mathrm{ort}_1),\mathrm{bp})$, then

$$\begin{aligned}
&[[\alpha_1,\alpha_2],\alpha_3]=\alpha_3\alpha_2\alpha_1\alpha_2^{-1}\alpha_1^{1}\alpha_3^{-1}\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1},\quad \text{whereas}\\
&[\alpha_1,[\alpha_2,\alpha_3]]=\alpha_3\alpha_2\alpha_3^{-1}\alpha_2^{-1}\alpha_1\alpha_2\alpha_3\alpha_2^{-1}\alpha_3^{-1}\alpha_1^{-1}
\end{aligned}$$

A second example can be found in Exercise 5.5.4.2 below,

The Case of H-Spaces

Theorem 5.5.2.8. *If X is a H-space, then $[\alpha,\beta]=0$ for any $x_0\in X$, $\alpha\in\pi_m(X,x_0)$, $\beta\in\pi_n(X,x_0)$.*

This generalises Theorem 5.1.9.11.

Proof. It is enough to consider the case where x_0 is the identity. Let

$$f\colon(\mathbb{S}^m,\mathrm{ort}_l)\to(X,x_0)\quad\text{and}\quad g\colon(\mathbb{S}^n,\mathrm{ort}_l)\to(X,x_0)$$

be spheroids in the classes α and β. Define $h\colon(\mathbb{S}^m\times\mathbb{S}^n,\mathrm{bp})\to(X,x_0)$ and spheroids $f_1\colon(\mathbb{S}^m,\mathrm{ort}_l)\to(X,x_0)$ and $g_1\colon(\mathbb{S}^n,\mathrm{ort}_l)\to(X,x_0)$ by

$$h(x,y)=f(x)g(y),\quad f_1(y)=f(y)x_0,\quad g_1(y)=g(y)x_0.$$

Obviously, f_1 and g_1 are homotopic to f and g, while the map $(B,\mathrm{bp})\to(X,x_0)$ defined by f_1 and g_1 equals $h|_B=h\circ[\mathrm{incl}\colon B\to\mathbb{S}^m\times\mathbb{S}^n]$. Consequently,

$$[\alpha,\beta]=(h\circ\mathrm{incl})_*(\mathrm{whd}(m,n))=h_*\circ\mathrm{incl}_*(\mathrm{whd}(m,n)).$$

Since $\mathrm{incl}_*(\mathrm{whd}(m,n))=0$, we get $[\alpha,\beta]=0$. □

5.5.3 Application

Theorem 5.5.3.1. *Let (X,x_0) and (Y,y_0) be pointed spaces, and let k and ℓ be non-negative integers. If X is k-connected, Y is ℓ-connected, and (X,x_0) and (Y,y_0) are Borsuk pairs, then the kernel of the homomorphism*

$$\mathrm{incl}_*\colon\pi_{k+\ell+1}((X,x_0)\vee(Y,u_0),\mathrm{bp})\to\pi_{k+\ell+1}(X\times Y,\mathrm{bp})$$

is generated, as a subgroup of $\pi_{k+\ell+1}((X,x_0)\vee(Y,y_0),\mathrm{bp})$, by the products $[\mathrm{Imm}_{1*}(\alpha),\mathrm{Imm}_{2*}(\beta)]$ with $\alpha\in\pi_{k+1}(X,x_0)$ and $\beta\in\pi_{\ell+1}(Y,y_0)$.

(Cf. Theorem 5.3.5.7 and Remark 5.4.3.14.)

Proof. By Theorem 5.5.2.6, $[\mathrm{Imm}_{1*}(\alpha), \mathrm{Imm}_{2*}(\beta)] \in \ker \mathrm{incl}_*$ for all $\alpha \in \pi_{k+1}(X, x_0)$ and $\beta \in \pi_{\ell+1}(Y, y_0)$. To see that these products actually generate $\ker \mathrm{incl}_*$, note that Remark 5.4.2.1, Corollary 5.4.2.9, and Theorem 2.3.3.2 together guarantee that it suffices to examine the case where (X, x_0) and (Y, y_0) are cellular spaces with $\mathrm{skel}_{k+\ell+1} X = x_0$ and $\mathrm{skel}_{k+\ell+1} Y = y_0$. Under these circumstances, $\mathrm{skel}_{k+\ell+1}(X \times Y) \subset (X, x_0) \vee (Y, y_0)$, and the classes of the attaching maps of the $(k+\ell+2)$-cells in $(X \times Y) \setminus [(X, x_0) \vee (Y, y_0)]$ are Whitehead products of the classes of the characteristic maps of the $(k+1)$-cells in $\mathrm{Imm}_1(X)$ and the classes of the characteristic maps of the $(\ell+1)$-cells in $\mathrm{Imm}_2(Y)$ (this is an immediate consequence of Definition 5.5.2.1). Therefore,

- when $k > 0$ and $\ell > 0$, $\ker \mathrm{incl}_*$ is generated by the classes
$$[\mathrm{Imm}_{1*}(\alpha), \mathrm{Imm}_{2*}(\beta)]$$
with $\alpha \in \pi_{k+1}(X, x_0)$ and $\beta \in \pi_{\ell+1}(Y, y_0)$ (see Theorem 5.3.2.5);
- when $k > 0$ and $\ell = 0$, $\ker \mathrm{incl}_*$ is generated by the classes
$$T_{\mathrm{Imm}_{2*}(\sigma)}[\mathrm{Imm}_{1_*}(\alpha), \mathrm{Imm}_{2*}(\beta)]$$
with $\alpha \in \pi_{k+1}(X, x_0)$ and $\beta, \sigma \in \pi_1(Y, y_0)$ (see Theorem 5.3.2.5);
- when $k = 0$ and $\ell > 0$, $\ker \mathrm{incl}_*$ is generated by the classes
$$T_{\mathrm{Imm}_{1*}(\sigma)}[\mathrm{Imm}_{1_*}(\alpha), \mathrm{Imm}_{2*}(\beta)]$$
with $\alpha, \sigma \in \pi_1(X, x_0)$ and $\beta, \sigma \in \pi_{\ell+1}(Y, y_0)$ (see Theorem 5.3.2.5);
- finally, when $k = \ell = 0$, $\ker \mathrm{incl}_*$ is generated by the classes
$$T_{\mathrm{Imm}_{1*}(\sigma_1)\,\mathrm{Imm}_{2*}(\omega_1 \cdots \mathrm{Imm}_{1*}(\sigma_q)\,\mathrm{Imm}_{2*}(\omega_q))}[\mathrm{Imm}_{1_*}(\alpha), \mathrm{Imm}_{2*}(\beta)]$$
with $\alpha, \sigma_1, \dots, \sigma_q \in \pi_1(X, x_0)$ and $\beta, \omega_1, \dots, \omega_q \in \pi_1(Y, y_0)$ (see Subsection 5.3.3).

Now all it remains is to observe that

(i)
$$\begin{aligned} &T_{\mathrm{Imm}_{1*}(\sigma)}[\mathrm{Imm}_{1_*}(\alpha), \mathrm{Imm}_{2*}(\beta)] = \\ &-[\mathrm{Imm}_{1*}(\sigma), \mathrm{Imm}_{2*}(\beta)] + [\mathrm{Imm}_{1*}(\alpha\sigma), \mathrm{Imm}_{2*}(\beta)] \end{aligned}$$
for any $\alpha, \sigma \in \pi_1(X, x_0)$ and $\beta \in \pi_{\ell+1}(Y, y_0)$ with $\ell > 0$;

(ii)
$$\begin{aligned} &T_{\mathrm{Imm}_{2*}(\sigma)}[\mathrm{Imm}_{1_*}(\alpha), \mathrm{Imm}_{2*}(\beta)] = \\ &-[\mathrm{Imm}_{1*}(\alpha), \mathrm{Imm}_{2*}(\sigma)] + [\mathrm{Imm}_{1*}(\alpha), \mathrm{Imm}_{2*}(\beta\sigma)] \end{aligned}$$
for any $\alpha, \sigma \in \pi_1(X, x_0)$ and $\beta \in \pi_{\ell+1}(Y, y_0)$ with $\ell > 0$;

(iii)

$$\begin{aligned}&T_{\mathrm{Imm}_{1*}(\sigma)\,\mathrm{Imm}_{2*}(\omega)}[\mathrm{Imm}_{1_*}(\alpha), \mathrm{Imm}_{2*}(\beta)] =\\&[\mathrm{Imm}_{1*}(\sigma), \mathrm{Imm}_{2*}(\omega\beta)]^{-1}[\mathrm{Imm}_{1*}(\alpha\sigma), \mathrm{Imm}_{2*}(\omega\beta)]*\\&[\mathrm{Imm}_{1*}(\sigma\alpha), \mathrm{Imm}_{2*}(\omega)]^{-1}[\mathrm{Imm}_{1*}(\sigma), \mathrm{Imm}_{2*}(\omega)]\end{aligned}$$

for any $\alpha, \sigma \in \pi_1(X, x_0)$ and $\beta \in \pi_1(Y, y_0)$;

(see Theorem 5.5.2.4). □

Theorem 5.5.3.2. *The class* $[\mathrm{sph}_n, \mathrm{sph}_n]$ *has infinite order in* $\pi_{2n-1}(\mathbb{S}^n)$ *for every even positive integer* n*. In particular, the groups* $\pi_{4k-1}(\mathbb{S}^{2k})$ *with* $k \geq 1$ *are infinite.*

(Cf. Subsections 5.2.2 and 5.2.4.)

Proof. Since

$$\begin{aligned}\mu_*[\mathrm{sph}_n, \mathrm{sph}_n] &= [\mu_*(\mathrm{sph}_n), \mu_*(\mathrm{sph}_n)]\\&= [\mathrm{Imm}_{1*}(\mathrm{sph}_n) + \mathrm{Imm}_{2*}(\mathrm{sph}_n), \mathrm{Imm}_{1*}(\mathrm{sph}_n) + \mathrm{Imm}_{2*}(\mathrm{sph}_n)]\\&= [\mathrm{Imm}_{1*}(\mathrm{sph}_n), \mathrm{Imm}_{1*}(\mathrm{sph}_n)] + [\mathrm{Imm}_{2*}(\mathrm{sph}_n), \mathrm{Imm}_{2*}(\mathrm{sph}_n)]\\&+ 2[\mathrm{Imm}_{1*}(\mathrm{sph}_n), \mathrm{Imm}_{2*}(\mathrm{sph}_n)]\\&= \mathrm{Imm}_{1*}[\mathrm{sph}_n, \mathrm{sph}_n] + \mathrm{Imm}_{2*}[\mathrm{sph}_n, \mathrm{sph}_n] + 2\,\mathrm{whd}(n, n),\end{aligned}$$

we obtain

$$2\,\mathrm{whd}(n, n) = y\mu_*[\mathrm{sph}_n, \mathrm{sph}_n] - \mathrm{Imm}_{1*}[\mathrm{sph}_n, \mathrm{sph}_n] - \mathrm{Imm}_{2*}[\mathrm{sph}_n, \mathrm{sph}_n].$$

Now assuming that $[\mathrm{sph}_n, \mathrm{sph}_n]$ has finite order, the class $\mathrm{whd}(n, n)$ would have finite order too, contradicting Theorem 5.5.1.2. □

Theorem 5.5.3.3. *The kernel of* $\mathrm{sus}\colon \pi_{2n-1}(\mathbb{S}^n) \to \pi_{2n}(\mathbb{S}^{2n+1})$ *is infinite for every positive integer* n*.*

Proof. This is a corollary of Theorems 5.5.3.2 and 5.5.2.5. □

5.5.4 Exercises

Exercise 5.5.4.1. Compute the third homotopy group of a bouquet of two-dimensional spheres.

Exercise 5.5.4.2. Show that if

$$\alpha = \mathrm{Imm}_{1*}(\mathrm{sph}_1) in \pi_1(B(1, 2), \mathrm{bp}), \quad \beta = \gamma = \mathrm{Imm}_{2*}(\mathrm{sph}_2) \in \pi_2(B(1, 2), \mathrm{bp}),$$

then

$$[[\alpha, \beta], \gamma] \neq [\alpha, [\beta, \gamma]].$$

Exercise 5.5.4.3. Show that

$$(-1)^{pm}[[\alpha, \beta], \gamma] + (-1^{mn}[[\beta, \gamma], \alpha] + (-1)^{np}[[\gamma, \alpha], \beta] = 0,$$

for any $\alpha \in \pi_m(X, x_0)$, $\beta \in \pi_n(X, x_0)$, and $\gamma \in \pi_p(X, x_0)$ with $m > 1$, $n > 1$, and $p > 1$.

5.6 CONTINUATION OF THE THEORY OF BUNDLES

5.6.1 Weak Homotopy Equivalence and Steenrod Bundles

Definition 5.6.1.1. Two Steenrod bundles, ξ_1 and ξ_2, with the same standard fibre F, are said to be *k-equivalent* if there exist a cellular space B and $\varphi_2^!(\xi_2)$ are F-equivalent. Here $0 \leq k \leq \infty$; the most important case is $k = \infty$, and two ∞-equivalent Steenrod bundles are also referred to as *weakly homotopy equivalent.*

Clearly, every bundle induced from a Steenrod bundle by a k-equivalence is k-equivalent with the original bundle. Moreover, bases of weakly homotopy equivalent Steenrod bundles are weakly homotopy equivalent, and by Theorem 5.1.5.9 this holds true for their total spaces, too. [To see this, consider the maps $\operatorname{adj}\varphi_1\colon \varphi_1^!\xi_1 \to \xi_1$ and $\operatorname{adj}\varphi_2\colon \varphi_2^!\xi_2 \to \xi_2$ and the homomorphisms that they induce from the homotopy sequences of the bundles $\varphi_1^!\xi_1$ and $\varphi_2^!\xi_2$ into the homotopy sequences of the bundles ξ_1 and ξ_2 respectively, and apply to these homomorphisms Theorem 5.1.5.20.]

Theorem 5.6.1.2. *Let ζ_1 and ζ_2 be Steenrod bundles with the same standard fibre F. Suppose that ζ_1 is universal. Then ζ_2 is universal if and only if it is weakly homotopy equivalent to ζ_1.*

Proof. We first show that the condition is sufficient, i.e., that given any Steenrod bundle ξ with standard fibre F and cellular base, any (cellular) subspace B of $\operatorname{bs}\xi$, and any continuous map $\varphi\colon B \to \operatorname{bs}\zeta_2$ such that $\varphi^!\zeta_2$ is F-equivalent to $\xi|_B$, there exists a continuous map $\psi\colon \operatorname{bs}\xi \to \operatorname{bs}\zeta_2$ such that $\psi|_B = \varphi$ and the bundle $\psi^!\zeta_2$ is F-equivalent to ξ (see Definition 4.4.2.2). By Definition 5.6.1.1, we can produce a cellular space K together with weak homotopy equivalences, $f_1\colon K \to \operatorname{bs}\zeta_1$ and $f_1\colon K \to \operatorname{bs}\zeta_1$, such that $f_1^!\zeta_1$ and $f_2^!\zeta_2$ are F-equivalent. Using Theorem 5.4.1.2, there is a continuous $g\colon B \to K$ such that $f_2 \circ g$ is homotopic to φ. Therefore, in the chain of F-bundles:

$$(f_1 \circ g)^!\zeta_1 = g^!(f_1^!\zeta_1), \quad g^!(f_1^!\zeta_2) = (f_2 \circ g)^!\zeta_2, \quad \varphi^!\zeta_2, \quad \xi|_B,$$

the adjacent bundles are F-equivalent, and hence so are $(f_1 \circ g)^!\zeta_1$ and $\xi|_B$. Since ζ_1 is universal, there is a continuous $h\colon \operatorname{bs}\xi \to \operatorname{bs}\zeta_1$ such that $h_B = f_1 \circ g$ and the bundles $h^!\zeta_1$ and ξ are F-equivalent (see again Definition 4.4.2.2). Further, since f is a weak homotopy equivalence, there is a continuous $k\colon \operatorname{bs}\xi \to K$ such that $k|_B = g$ and $f_1 \circ k$, h are homotopic (see Theorem 5.4.1.2). The restriction of $f_1 \circ k\colon \operatorname{bs}\xi \to \operatorname{bs}\zeta_2$ to B equals $f_2 \circ g$, and hence is homotopic to φ. Consequently, there is a continuous map $\psi\colon \operatorname{bs}\xi \to \operatorname{bs}\zeta_2$ homotopic to $f_2 \circ k$ such that $\psi|_B = \varphi$. Now the fact that the adjacent bundles in the chain

$$\psi^!\zeta_2, \quad (f_2 \circ k)^!\zeta_2 = k^!(f_2^!\zeta_1), \quad k^!(f_1^!\zeta_1 = (f_1 \circ k)^!\zeta_1, \quad h^!\zeta_1, \quad \xi,$$

are F-equivalent implies the F-equivalence of the bundles $\psi^!\zeta_2$ and ξ.

We may use what we just proved to show that the condition of the theorem is also necessary. In fact, assume that ζ_1 and ζ_2 are universal, and let (K, φ_1) and (K, φ_2) be cellular approximations of $\operatorname{bs}\zeta_1$ and $\operatorname{bs}\zeta_2$. Since $\varphi_1^!\zeta_1$ and $\varphi_2^!\zeta_2$ are weakly homotopy equivalent, they are both universal. Hence, K_1 and K_2 are homotopy equivalent (see Definition 4.4.2.2). ☐

Corollary 5.6.1.3. *If X and Y are classifying spaces of the same topological group, then X and Y are weakly homotopy equivalent.*

Theorem 5.6.1.4. *Given any topological group G, there is a universal G-bundle with cellular base.*

Proof. This follows from Remark 4.4.3.4, Theorems 5.4.3.2, and 5.6.1.2. ☐

Theorem 5.6.1.5. *A principal bundle is universal if and only if its total space is ∞-connected.*

Proof. First, let ξ be a universal G-bundle. Applying Theorem 5.6.1.2, K is weakly homotopy equivalent to $\operatorname{Mil} G$. Since $\operatorname{tl}\operatorname{Mil} G$ is ∞-connected (see Theorems 2.3.3.10, 5.4.2.7, and 5.1.11.5), $\operatorname{tl}\xi$ is also ∞-connected (see Definition 5.6.1.1).

Now let ξ be a Steenrod G-bundle with ∞-connected total space $\operatorname{tl}\xi$. Pick a cellular approximation (K, φ) of $\operatorname{bs}\xi$ and consider the G-bundle $\varphi^!\xi$. Since the bundle $\operatorname{Mil} G$ is universal, there is a continuous $\psi\colon K \to \operatorname{bs}\operatorname{Mil} G$ such that $\psi^!\xi$ is G-equivalent to $\psi^!\operatorname{Mil} G$. Furthermore, $\operatorname{tl}\operatorname{Mil} G$ and $\operatorname{tl}(\varphi^!\xi)$ are ∞Â»-connected, and the latter implies the ∞-connectedness of $\operatorname{tl}(\psi^!\operatorname{Mil} G)$. Applying Theorem 5.1.5.20 (to the homomorphism from the homotopy sequence of the bundle $\psi^!\operatorname{Mil} G$ into the homotopy sequence of the bundle $\operatorname{Mil} G$ induced by the map $\operatorname{adj}\psi\colon \psi^!\operatorname{Mil} G \to \operatorname{Mil} G$), we see that is a weak homotopy equivalence. Thus, ξ is weakly homotopy equivalent to $\operatorname{Mil} G$, and so ξ is universal (see Theorem 5.6.1.2). ☐

Theorem 5.6.1.6. *If X is a classifying space of the topological group G, then for any $r \geq 1$ the groups $\pi_r(X)$ and $\pi_{r-1}(G)$ are isomorphic.*

Proof. Indeed, by Theorem 5.6.1.5, the homomorphisms Δ figuring in the homotopy sequence of the universal G-bundle with base X are isomorphisms (see Remark 5.1.8.8). ☐

Theorem 5.6.1.7. *If X_1 and X_2 are classifying spaces of the topological groups G_1 and G_2, then $X_1 \times X_2$ is a classifying space of $G_1 \times G_2$.*

Proof. Given a universal G_1-bundle ξ_1 and a universal G_2-bundle ξ_2, it is enough to verify that the $(G_1 \times G_2)$-bundle $\xi_1 \times \xi_2$ is universal. But this follows from Theorems 5.6.1.5 and 5.1.1.15. ☐

Theorem 5.6.1.8. *Theorems 5.6.1.2 and 5.6.1.5 carry over to k-universal bundles. Thus, if ζ_1 and ζ_2 are Steenrod bundles with the same standard fibre and ζ_1 is k-universal, then ζ_2 is k-universal if and only if it is k-equivalent to ζ_1 and a principal bundle is k-universal if and only if its total space is k-connected.*

Proof. The proofs repeat those of Theorems 5.6.1.2 and 5.6.1.5, *mutatis mutandis.* ☐

An Application: Universal Principal Bundles for Finitely Generated Abelian Groups

Theorem 5.6.1.9. *For arbitrary n and $m_1, \dots, m_\ell$,*

$$(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n} \times \underbrace{\mathbb{S}^\infty \times \cdots \times \mathbb{S}^\infty}_{\ell}, \underbrace{\mathrm{hel} \times \cdots \times \mathrm{hel}}_{n} \times \underbrace{\mathrm{proj} \times \cdots \times \mathrm{proj}}_{\ell},$$
$$\mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \times L(m_1) \times \cdots \times L(m_\ell))$$

is a universal $\mathbb{Z}^n \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_\ell\mathbb{Z})$-bundle. In particular, $\mathbb{S}^1$ is a classifying space for $\mathbb{Z}$, while $L(m)$ is a classifying space for $\mathbb{Z}/m\mathbb{Z}$.

Proof. Since $\mathbb{R}$ and $\mathbb{S}^\infty$ are ∞-connected, this follows immediately from Theorems 5.6.1.5 and 5.1.1.15. ☐

5.6.2 Theory of Coverings

Definition 5.6.2.1. The main achievement of the present section is a clear enumeration of the covering spaces of non-pathological connected spaces and a criterion for the equivalence of two given coverings. Our instrument will be the fundamental group. The analysis is elementary enough: in fact, all we use from the whole theory of bundles may be concentrated in the following two theorems, where it is assumed that a covering ξ together with a base point $x \in \mathrm{tl}(\xi)$ are given. Then:

(i) every path in $\mathrm{bs}\,\xi$ with origin $(\mathrm{proj}\,\xi)(x)$ is covered by one and only one path in $\mathrm{tl}\,\xi$ with origin x;

(ii) if two paths in $\mathrm{bs}\,\xi$ with common origin $(\mathrm{proj}\,\xi)(x)$ are homotopic, then the paths in $\mathrm{tl}\,\xi$ which cover them are also homotopic and, in particular, have the same end.

(These assertions are straightforward corollaries of Theorems 4.1.3.6 and Proposition 4.1.3.8.)

It is true that here and there we do refer to other facts from the theory of bundles (for example, to Theorem 5.1.8.13), but all these can be readily deduced from (i) and (ii).

At the heart of the theory of coverings lies the definition of the *group of a covering*. Recall that, according to Theorem 5.1.8.13,

$$\mathrm{proj}_*\colon \pi_1(\mathrm{tl}\,\xi, x) \to \pi_1(\mathrm{bs}\,\xi, (\mathrm{proj}\,\xi)(x))$$

is a monomorphism for any covering ξ with base point $x \in \mathrm{tl}\,\xi$. We call the image of proj_* the *group of the covering ξ with base point x*, and denote it by $\mathrm{grp}\,\xi(x)$.

Theorem 5.6.2.2. *Let* $(\operatorname{proj}\xi)(x_1) = (\operatorname{proj}\xi)(x_0)$ *and let* s *be a path in* $\operatorname{tl}\xi$ *beginning at* x_0*and ending at* x_1*. Then* $\operatorname{grp}\xi(x_1) = \sigma[\operatorname{grp}\xi(x_0)]\sigma^{-1}$*, where* σ *is the homotopy class of the loop* $(\operatorname{proj}\xi)\circ s$*. In particular, if*

$$(\operatorname{proj}\xi)(x_0) = (\operatorname{proj}\xi)(x_1),$$

then $\operatorname{grp}(x_0)$ *and* $\operatorname{grp}\xi(x_1)$ *are conjugate subgroups of* $\pi_1(\operatorname{bs}\xi, x_0)$*. The converse is also true: the groups* $\operatorname{grp}\xi(x)$ *with* $x \in (\operatorname{proj}\xi)^1((\operatorname{proj}\xi)(x_0))$ *exhaust the subgroups of* $(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0)$ *which are conjugate to* $\operatorname{grp}\xi(x_0)$*.*

Proof. In fact,

$$\begin{aligned}\operatorname{grp}\xi(x_1) &= \operatorname{proj}\xi_*(\pi_1(\operatorname{tl}\xi, x_1)) = \operatorname{proj}\xi_*\circ T_s(\pi_1(\operatorname{tl}\xi, x_0))\\ &= T_{\operatorname{proj}\xi\circ s}\circ\operatorname{proj}\xi_*(\operatorname{tl}\xi, x_0)) = T_{\operatorname{proj}\xi\circ s}(\operatorname{grp}\xi(x_0)) = \sigma[\operatorname{grp}\xi(x_0)]\sigma^{-1}.\end{aligned}$$

These equalities demonstrate that for every $\sigma \in \pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0))$ the group $\sigma[\operatorname{grp}\xi(x_0)]\sigma^{-1}$ equals $\operatorname{grp}\xi(s_1)$, where s is a path in $\operatorname{tl}\xi$ with origin x_0 and which covers some path in the class σ. ☐

The Hierarchy of Coverings

Definition 5.6.2.3. We say that the covering ξ, with base point $x_0 \in \operatorname{tl}\xi$ is *subordinate* to the covering ξ' with base point $x'_0 \in \operatorname{tl}\xi'$, if $\operatorname{bs}\xi' = \operatorname{bs}\xi$ and there exists a continuous map $\varphi\colon \xi'\xi$ such that $\operatorname{bs}\varphi = \operatorname{id}_{\operatorname{bs}\xi}$, and $\operatorname{tl}\varphi(x'_0) = x_0$. In this case, the map φ is called a *subordination*.

Obviously, if $\varphi\colon \xi^{prime} \to \xi$ is a subordination, then $(\operatorname{tl}\xi', \operatorname{tl}\varphi, \operatorname{tl}\xi)$ is a covering.

Theorem 5.6.2.4. *If a subordination exists, then it is unique.*

Proof. Suppose that the covering ξ with base point x_0 is subordinate to the covering ξ' with base point x'_0, and let φ and ψ be two such subordinations. Then, if $x' \in \operatorname{tl}\xi'$ is arbitrary and $s\colon I \to \operatorname{tl}\xi'$ is such that $s(0) = x'_0$, $s(1) = x'$, then the paths $\operatorname{tl}\varphi\circ s$ and $\operatorname{tl}\psi\circ s$ cover the same path, $\operatorname{proj}\xi'\circ s$, in $\operatorname{bs}\xi$ and have the same origin. Therefore, $\operatorname{tl}\varphi\circ s = \operatorname{tl}\psi\circ s$ and $\operatorname{tl}\varphi(x') = \operatorname{tl}\varphi\circ s(1) = \operatorname{tl}\psi\circ s(1) = \operatorname{tl}\psi(x')$. ☐

Theorem 5.6.2.5. *If two coverings with base points are mutually subordinate, then the corresponding subordinations are equivalences which are inverses of one another.*

Proof. Let $\varphi\colon \xi' \to \xi$ and $\varphi'\colon \xi \to \xi'$ be subordinations. Then $\varphi'\circ\varphi\colon \xi' \to \xi'$ and $\varphi\circ\varphi'\colon \xi \to \xi$ are also subordinations. By Theorem 5.6.2.4, $\varphi'\circ\varphi = \operatorname{id}_{\xi'}$ and $\varphi\circ\varphi' = \operatorname{id}_\xi$. ☐

Lemma 5.6.2.6. *Let* ξ *and* ξ' *be coverings with base points* $x_0 \in \xi$ *and* $x'_0 \in \xi'$*, and such that* $\operatorname{bs}\xi' = \operatorname{bs}\xi$*,* $(\operatorname{proj}\xi')(x'_0) = (\operatorname{proj}\xi)(x_0)$*, and* $\operatorname{grp}\xi'(x'_0) \subset \operatorname{grp}\xi(x_0)$*. If the paths* $s'_1, s'_2\colon I \to \operatorname{tl}\xi^{prime}$ *have the common origin* x'_0 *and a common end, then the paths in* $\operatorname{tl}\xi$ *which cover* $\operatorname{proj}\xi'\circ s'_1$ *and* $\operatorname{proj}\xi'\circ s'_2$ *and have origin* x_0 *also have a common end.*

Proof. Let s_1 and s_2 be the path in $\operatorname{tl}\xi$, with origin x_0 which cover $\operatorname{proj}\xi \circ s'_1$ and $\operatorname{proj}\xi \circ s'_1$. The class of the loop $(\operatorname{proj}\xi \circ s'_1)(\operatorname{proj}\xi \circ s'_1)$ belongs to $\operatorname{grp}\xi'(x'_0)$, and hence to $\operatorname{grp}\xi(x_0)$. Consequently, the path in $\operatorname{tl}\xi$, with origin x_0 which covers this loop is closed; moreover, it can be expressed as the product of a path which covers $\operatorname{proj}\xi \circ s'_1$ (and which, having origin x_0, coincides with s_1) with a path which covers $(\operatorname{proj}\xi \circ s'_1)$ (and which, having origin x_0, coincides with s_2^{-1}). Therefore, s_1 and s_2 have the same end. □

Theorem 5.6.2.7. *Let ξ and ξ' be coverings with base points $x_0 \in \operatorname{tl}\xi$ and $x_0^{prime} \in \operatorname{tl}\xi'$, such that $\operatorname{bs}\xi' = \operatorname{bs}\xi$, $(\operatorname{proj}\xi')(x'_0) = (\operatorname{proj}\xi)(x_0)$.*

- *(i) If ξ is subordinate to ξ', then $\operatorname{grp}'(x'_0) \subset \operatorname{grp}\xi(x_0)$;*
- *(ii) If $\operatorname{grp}'(x'_0) \subset \operatorname{grp}\xi(x_0)$ and $\operatorname{bs}\xi$ is locally connected, then ξ is subordinate to ξ'.*

Proof. Assertion (i) is trivial, so let us prove (ii). For each point $x' \in \operatorname{tl}\xi'$, consider the common end of all paths in $\operatorname{tl}\xi$ which start at x_0 and cover paths of the form $\operatorname{proj}\xi' \circ u'$, where u' is a path in $\operatorname{tl}\xi'$ such that $u'(0) = x'_0$, $u'(1) = x'$ (see Lemma 5.6.2.6). This defines a map $F\colon \xi' \to \operatorname{tl}\xi$ satisfying $\operatorname{proj}\xi \circ F = \operatorname{proj}\xi'$, and all we have to check is the continuity of F. So let $x' \in \operatorname{tl}\xi'$ and U be a neighbourhood of $F(x')$. We shall produce a neighbourhood U' of x' such that $F(U') \subset U$.

To do this, let $W \subset \operatorname{bs}\xi$ be a neighbourhood of $\operatorname{proj}\xi'(x')$ such that there are neighbourhoods, V' of x' and V of $F(x')$, which are homeomorphically mapped by $\operatorname{proj}\xi'$ and respectively $\operatorname{proj}\xi$ onto W. Since $\operatorname{bs}\xi$ is locally connected, we can find a neighbourhood $W_1 \subset W$ of $\operatorname{proj}\xi'(x')$ such that every point of W_1 can be joined to $\operatorname{proj}\xi'(x')$ by a path in W. Now set $V'_1 = V' \cap (\operatorname{proj}\xi')^{-1}(W_1)$, $V_1 = V \cap (\operatorname{proj}\xi^{-1})(W_1)$, and $U' = V'_1 \cap (\operatorname{proj}\xi')^{-1}(\operatorname{proj}\xi(U))$. Obviously, $x' \in U'$. Let us check that $F(U') \subset U$.

Let $y' \in U'$. Since $\operatorname{ab}\operatorname{proj}\xi'\colon V' \to W$ is a homeomorphism, there is a path $v'\colon I \to \operatorname{tl}\xi'$ such that $v'(I) \subset V'$ and $v'(0) = x'$, $v'(1) = y'$. Define $v\colon I \to \operatorname{tl}\xi$ by

$$v(t) = (\operatorname{proj}\xi|_V)^{-1}(\operatorname{proj}\xi' \circ v'(t)),$$

pick a path $u'\colon I \to \operatorname{tl}\xi'$ with the property that $u'(0) = x_0$, $u'(1) = x'$, and consider the path $u\colon I \to \operatorname{tl}\xi$ with $u(0) = x_0$, $u(1) = F(x')$, and u covers the path $\operatorname{proj}\xi' \circ u'$. Clearly, $(u'v')(0) = x_0$, $(u'v')(1) = y'$, while the path uv covers $\operatorname{proj}\xi' \circ (u'v')$ and $(uv)(0) = x_0$. Therefore, $(uv)(1) = F(v')$, and hence

$$F(y') = v(1) = (\operatorname{proj}\xi|_V)^{-1}(\operatorname{proj}\xi'(y')) \in V \cap (\operatorname{proj}\xi)^{-1}(\operatorname{proj}\xi'(U')) \subset U.$$

□

Corollary 5.6.2.8. *Two coverings, ξ and ξ', with $\operatorname{bs}\xi' = \operatorname{bs}\xi$ a locally connected space, are equivalent if and only if for some points $x_0 \in \operatorname{tl}\xi$ and $x'_0 \in \operatorname{tl}\xi'$, such that $(\operatorname{proj}\xi')(x'_0) = (\operatorname{proj}\xi)(x_0)$, the groups $\operatorname{grp}\xi(x_0)$ and $\operatorname{grp}\xi'(x'_0)$ are conjugate in $\pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0))$.*

The Group of Automorphisms of a Covering

Remark 5.6.2.9. As with the automorphisms (i.e., self-equivalences) of an arbitrary bundle ξ, the automorphisms of a covering ξ, form a group $\operatorname{Aut}\xi$. [2] Its structure is described in Theorem 5.6.2.10 below, where x_0 designates, as usual, a base point in $\operatorname{tl}\xi$, and two new notations are used. Namely, let eval be the map from $\operatorname{Aut}\xi$ into the fibre $(\operatorname{proj}\xi)^{-1}(\operatorname{proj}\xi(x_0))$ given by $\operatorname{eval}(\varphi) = \operatorname{tl}\varphi(x_0)$, and let Reg be the set the set of all points $x \in (\operatorname{proj}\xi)^{-1}(\operatorname{proj}\xi(x_0))$ such that $\operatorname{grp}\xi(x) = \operatorname{grp}\xi(x_0)$. From Theorem 5.6.2.2 it follows that the preimage of Reg under the map

$$\Delta\colon \pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0)) \to \pi_0((\operatorname{proj}\xi)^{-1}(\operatorname{proj}\xi(x_0)), x_0) = (\operatorname{proj}\xi)^{-1}(\operatorname{proj}\xi(x_0)).$$

is nothing else but the normaliser $\operatorname{Nr}(\operatorname{grp}\xi(x_0))$ in $\pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0))$ of the group $\operatorname{grp}\xi(x_0)$. [Recall that, given a subgroup H of a group G, its normaliser $\operatorname{Nr}(H)$ is the set of all $g \in G$ such that $gHg^{-1} = H$; $\operatorname{Nr}(H)$ is a subgroup of G and contains H as a normal subgroup.] Therefore, Δ induces an invertible map

$$\operatorname{fact}\operatorname{abr}\Delta\colon\ \operatorname{Nr}(\operatorname{grp}\xi, (x_0))/\operatorname{grp}\xi(x_0) \to \operatorname{Reg}.$$

Theorem 5.6.2.10. eval *is injective,* $\operatorname{eval}(\operatorname{Aut}\xi) \subset \operatorname{Reg}$*, and the composition*

$$\operatorname{Aut} \xrightarrow{\operatorname{abr}\operatorname{eval}} \operatorname{Reg} \xrightarrow{(\operatorname{fact}\operatorname{abr}\delta)^{-1}} \operatorname{Nr}(\operatorname{grp}\xi, (x_0)/\operatorname{grp}\xi(x_0)$$

is an anti-homomorphism. If the base $\operatorname{bs}\xi$ *is locally connected, then*

$$\operatorname{eval}(\operatorname{Aut}\xi) = \operatorname{Reg},$$

and hence the group $\operatorname{Aut}\xi$ *is anti-isomorphic to* $\operatorname{Nr}(\operatorname{grp}\xi, (x_0)/\operatorname{grp}\xi(x_0)$. *If, in addition,* ξ *is regular (see Definition 5.6.2.11 below), then*

$$\operatorname{Nr}(\operatorname{grp}\xi(x_0)) = \pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0))$$

and $\operatorname{Aut}\xi$ *is anti-isomorphic* $\pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0))/\operatorname{grp}\xi(x_0)$.

Proof. Since every automorphism $\varphi \in \operatorname{Aut}\xi$ may be thought of as a subordination of the covering ξ with base point $\operatorname{tl}\varphi(x_0)$ to the covering ξ with base point x_0 , the injectivity of eval is immediate from Theorem 5.6.2.5. Similarly, the inclusion $\operatorname{eval}(\operatorname{Aut}\xi) \subset \operatorname{Reg}$ follows from part (i) of Theorem 5.6.2.7, while the equality $\operatorname{eval}(\operatorname{Aut}\xi) = \operatorname{Reg}$ results, when $\operatorname{bs}\xi$ is locally connected, from part (ii) of the same theorem. Finally, that $(\operatorname{fact}\operatorname{abr}\Delta) \circ \operatorname{eval}$ is an anti-homomorphism is plain. ☐

Regular Coverings

Definition 5.6.2.11. A covering ξ is *regular* if for some point $x_0 \in \operatorname{tl}\xi$, $\operatorname{grp}\xi(x_0)$ is a normal subgroup of $\pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0))$. In this case $\operatorname{grp}\xi(x)$ is

---

[2] Translator's note: the elements of $\operatorname{Aut}\xi$ are also known as *covering transformations* or *deck transformations.*

a normal subgroup of $\pi_1(\mathrm{bs}\,\xi, \mathrm{proj}\,\xi(x))$ for all $x \in \mathrm{tl}\,\xi$, as seen from Theorem 5.6.2.2.

If ξ, is regular, then again using Theorem 5.6.2.2, the groups $\mathrm{grp}\,\xi(x)$ with $(x \in \mathrm{proj}\,\xi)^{-1}(b)$ are all equal for each fixed $b \in \mathrm{bs}\,\xi$.

We remark that every two-sheeted covering is regular (because every subgroup of index 2 is normal). More examples of regular coverings are $(\mathbb{R}, \mathrm{hel}, \mathbb{S}^1)$, $(\mathbb{S}^1, \mathrm{hel}_m, \mathbb{S}^1)$ [see Example 4.1.2.6], $(\mathbb{S}^{2n-1}, \mathrm{proj}, L(m; \ell_1, \ldots, \ell_n))$ [see Remark 4.2.3.19] , and $(\mathbb{S}^3, \mathrm{proj}, \mathbb{S}^3/\widetilde{GP})$, where P is a tetrahedron, a cube, or a dodecahedron [see Example 4.2.3.21]. An example of a non-regular covering is given in Fig. 5.5 (where the two points marked A are identified, as are the two points marked B);

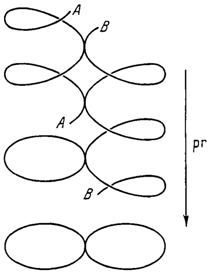

Figure 5.5:

its non-regularity is a result of the following theorem.

Theorem 5.6.2.12. *A covering* ξ, *is regular if and only if there is a point* $x_0 \in \mathrm{tl}\,\xi$, *with the property that given any loop* $s\colon I \to \mathrm{tl}\,\xi$, *with* $s(0) = x_0$ *and any point* $x \in (\mathrm{proj}\,\xi)^{-1}(\mathrm{proj}\,\xi, (x_0))$, *the path with origin* x *and covering* $\mathrm{proj}\,\xi \circ s$ *is closed.*

Proof. Indeed, the last condition means that each element of $\mathrm{grp}\,\xi(x_0)$ also belongs to every group $\mathrm{grp}\,\xi(x)$ with $x \in (\mathrm{proj}\,\xi)^{-1}(\mathrm{proj}\,\xi, (x_0))$, and this implies that every subgroup of $\pi_1(\mathrm{bs}\,\xi, \mathrm{proj}\,\xi, (x_0))$ which is conjugate with $\mathrm{grp}\,\xi(x_0)$ actually equals $\mathrm{grp}\,\xi(x_0)$ [see Theorem 5.6.2.2]. □

Theorem 5.6.2.13. *A covering* ξ *is regular if and only if it is equivalent to a principal bundle. The structure group of such a bundle is discrete and isomorphic to* $G = \pi_1(\mathrm{bs}\,\xi, \mathrm{proj}\,\xi(x_0))/\,\mathrm{grp}\,\xi(x_0)$. *Two Steenrod G-bundles which are equivalent to* ξ *are themselves* G*-equivalent.*

Proof. By Definition 4.3.2.10 and Remark 4.3.2.11, ξ is equivalent to a Steenrod G-bundle if and only if there exists a free, continuous right action of G on $\mathrm{tl}\,\xi$ whose orbits are precisely the fibres of ξ; in particular, G is necessarily discrete. Moreover, the transformation of $\mathrm{tl}\,\xi$ determined by such an action yield the automorphisms of ξ. Finally, by Remark 5.6.2.9, such an action exists if and only if ξ is regular and G is anti-isomorphic to $\operatorname{Aut}\xi$, i.e., is isomorphic to the group $\pi_1(\operatorname{bs}\xi, \operatorname{proj}\xi(x_0)/\operatorname{grp}\xi(x_0)$. ☐

Existence of Coverings

Definition 5.6.2.14. A topological space X is said to be *semi-locally simply connected* if every point $x \in X$ has a neighbourhood U such that $\operatorname{incl}_*\colon \pi_1(U,x) \to \pi_1(X,x)$ is trivial.

Obviously, the class of semi-locally simply connected spaces contains all simply connected spaces and all locally contractible spaces [and, among the latter, all locally Euclidean spaces, all CNRS's (see Theorem 1.3.6.8), in particular, all cellular spaces (see Theorem 2.1.4.5)].

Theorem 5.6.2.15. *Every space which has a simply connected covering space is semi-locally simply connected.*

Proof. In fact, let ξ be a covering projection with $\operatorname{tl}\xi$ simply connected and let $b \in \operatorname{bs}\xi$. Then the homomorphism $\operatorname{incl}_*\colon \pi_1(U,b) \to \pi_1(\operatorname{bs}\xi, b)$ is trivial for every neighbourhood U of b such that $\xi|_U$ is trivial. ☐

Theorem 5.6.2.16. *Let B be a connected, locally connected, and semi-locally simply connected space, let $b_0 \in B$, and let π be any subgroup of $\pi_1(B,b_0)$. Then there exists a covering ξ with base point $x_0 \in \operatorname{tl}\xi$ such that $\operatorname{bs}\xi = B$, $\operatorname{proj}\xi(x_0) = b_0$, and $\operatorname{grp}\xi(x_0) = \pi$.*

Proof. Consider $\mathcal{C}(I,0;B,b_0)$, identify any two paths s_1, s_2 in this set whenever $s_1(1) = s_2(1)$ and the homotopy class of $s_1 s_2^{-1}$ belongs to ξ, and denote the resulting quotient space by E. Further, given open subsets U of B and V of U, and any path $s\colon I \to B$, with $s(0) = b_0$, $s(1) \in V$, let $\operatorname{Nb}(U,V;s)$ be the subset of E consisting of the equivalence classes of the paths sw with $w(I) \subset U$ and $w(1) \in V$. The sets $\operatorname{Nb}(U,V;s)$ satisfy the conditions of Theorem 1.1.1.9, and thus yield a base for a topology on E. [Information: this topology coincides with the quotient topology that E inherits as a quotient of the topological space $\mathcal{C}(I,0;B,b_0)$.] The map $p\colon E \to B$ which takes each point from E into the common end of the paths which represent it is clearly continuous and open. We set $\xi = (E,p,B)$ and take for x_0 the point of ξ represented by the constant path. Then $p(x_0) = b_0$. Let us show that ξ is a covering and that $\operatorname{grp}\xi(x_0) = \pi$.

We first verify that 4ξ is a covering in the broad sense. Pick an arbitrary point $b \in B$ and find a neighbourhood U of b such that the inclusion homomorphism $\pi_1(U,b) \to \pi_1(B,b)$ is trivial, and then a neighbourhood V of b in U such that b can be joined to every point of V by a path in U. We claim that $\xi|_V$ is a trivial bundle with discrete fibre.

To see this, consider an arbitrary path $s\colon I \to B$ such that $s(0) = \mathrm{bp}$, $s(1) = b$, and for each coset $\alpha \in \pi_1(B,b_0)/\pi$ choose a loop $u\colon I \to B$ representing some element of α. The sets $\operatorname{Nb}(U,V;u_\alpha s)$ are open and pairwise disjoint [if the paths $(us_\alpha)w$ and $(us_{\alpha_1})w_1$ with $w(I) \subset U$, $w_1(I) \subset U$, define the same point of E, then $\alpha = \alpha_1$: indeed, the loop $((us_\alpha)w)(us_{\alpha_1})w_1)^{-1}$, and hence the loop

$u_\alpha u_{\alpha_1}^{-1}$ homotopic to it, are elements of a class that belongs to π] . Also, the sets $\mathrm{Nb}(U, V; u_\alpha s)$ cover $p^{-1}(V)$ [every path s' with $s'(0) = b_0$, $s'(1) \in V$, is homotopic to a path of the form $(us)w$, where u is a loop and $w(I) \subset U$: and example of such a path is $(((s'w^{-1})s^{-1})s)w$, where $w\colon I \to U$ is any path with $w(0) = b$, $w(1) = s'(1)$]. Finally, the maps $\mathrm{abr}\, p\colon \mathrm{Nb}(U, V; us) \to V$ are open (because p is) and invertible [if the paths $(u_\alpha s)w$, $(u_\alpha s)w_1$ with $w(I) \subset U$, $w_1(I) \subset U$, have the same end, then they are homotopic]; that is, $\mathrm{abr}\, p$ are homeomorphisms. Consequently, $\xi|_V$ is a trivial bundle with discrete fibre, as asserted.

The last thing to prove is that E is connected and that a path in E which has origin x_0 and covers a loop from a homotopy class $\sigma \in \pi_1(B, b_0)$ is closed if and only if $\sigma \in \pi$. Given $s\colon I \to B$ with $s(0) = b_0$, let $\widetilde{s}\colon I \to E$ be the path which takes each $t \in I$ into the point of E represented by the path $\tau \mapsto s(t\tau)$. Clearly, $s(0) = x_0$, $s(1)$ is the point of E represented by s, and $\widetilde{s}$ covers s. This has two consequences:

- every point of E can be joined to x_0 by a path,
- given a loop with origin b_0, the end of the path which starts at x_0 and covers it is precisely the point of E represented by the given loop.

The first shows that E is connected, while the second implies that, given a loop with origin b_0, the path which covers it and has origin b_0 ends at x_0 if and only if the homotopy class of the given loop belongs π. □

Definition 5.6.2.17. The previous theorem completes the theory of coverings with a fixed base. Combined with Theorem 5.6.2.7, it establishes a one-to-one correspondence between the equivalence classes of coverings over a connected, locally connected, and semi-locally simply connected space B with base point b_0 and the classes of conjugate subgroups of $\pi_1(B, b_0)$. It transforms the hierarchy of coverings into the usual, set-theoretic hierarchy of subgroups, and the normal subgroups correspond to the regular coverings. The covering corresponding to the trivial subgroup has a simply connected total space. Since every covering is subordinate to this one, it is called *universal*. (Warning: do not confuse this universality with the notion of universality defined in the theory of Steenrod bundles. Note, however, that among the universal Steenrod bundles one finds also universal coverings, namely the universal principal bundles with discrete structure groups; see Theorem 5.6.1.5 and cf. Theorem 5.6.2.18).

An Application: Classifying Spaces of Discrete Groups

Theorem 5.6.2.18. *In order that a connected topological space X be a classifying space of a given discrete group π, it is necessary that the groups $\pi_r(X)$ be trivial for all $r \geq 2$ and that $\pi_1(X)$ be isomorphic to π. If X is locally connected and semi-locally simply connected, then this condition is also sufficient.*

Proof. Necessity results from Theorems 5.6.1.5, 5.6.2.13, and Definition 5.1.9.15. Sufficiency results from Theorem 5.6.2.16, 5.1.8.13, and 5.6.1.5. □

Coverings Maps

Theorem 5.6.2.19. *Let ξ, and ξ' be coverings with base points $x_0 \in \mathrm{tl}\,\xi$, and $x'_0 \in \mathrm{tl}\,\xi'$, and let $f\colon (\mathrm{bs}\,\xi', \mathrm{proj}\,\xi'(x')) \to (\mathrm{bs}\,\xi, \mathrm{proj}\,\xi(x_0))$ be a continuous map. Then the inclusion $f_*(\mathrm{grp}\,\xi'(x'_0)) \subset \mathrm{grp}\,\xi(x_0)$ is a necessary condition for the existence of a continuous map $\varphi\colon \xi' \to \xi$ such that $\mathrm{bs}\,\varphi = f$ and $\mathrm{tl}\,\varphi(x') = x_0$. If $\mathrm{bs}\,\xi'$ is locally connected, then this condition is also sufficient. If such a φ exists, then it is unique.*

Proof. Assume that such a φ exists. Then from the commutative diagramme (see Theorem 5.1.8.7)

$$\begin{tikzcd}
\pi_1(\mathrm{tl}\,\xi', x_0) \arrow[r, "\mathrm{tl}\,\varphi_*"] \arrow[d, "\mathrm{proj}\,\xi'_*"'] & \pi_1(\mathrm{tl}\,\xi, x_0) \arrow[d, "\mathrm{proj}\,\xi_*"] \\
\pi_1(\mathrm{bs}\,\xi', \mathrm{proj}\,\xi'(x'_0)) \arrow[r, "\mathrm{bs}\,\varphi*"'] & \pi_1(\mathrm{bs}\,\xi, \mathrm{proj}\,\xi(x_0))
\end{tikzcd}$$

it follows that $f_*(\mathrm{grp}\,\xi'(x'_0)) \subset \mathrm{grp}\,\xi(x_0)$.

Now let us prove the converse. Consider the bundle $f^!\xi$ together with the map $\mathrm{adj}\,f\colon f^!\xi \to \xi$. Let $y'_0 \in (\mathrm{proj}\,f^!\xi)^{-1}(\mathrm{proj}\,\xi'(x'_0))$ be such that $\mathrm{tl}\,\mathrm{adj}\,f(y'_0) = x_0$. Further, let Y' be the component of $\mathrm{tl}(f^!\xi)$ containing y'_0, and let p' be the restriction of $\mathrm{proj}\,f^!\xi$ to Y'. Obviously, $(Y', p', \mathrm{bs}\,\xi')$ is a covering $\mathrm{tl}\,\mathrm{adj}\,f$ and f combine to define a map $(Y', p', \mathrm{bs}\,\xi') \to \xi$, and $\mathrm{tl}\,\mathrm{adj}\,f|_{Y'}$ is injective on every fibre of $(Y', p', \mathrm{bs}\,\xi')$. In the diagramme

$$\begin{tikzcd}
\pi_1(Y', y'_0) \arrow[r, "(\mathrm{proj})'_*"] \arrow[d, "\mathrm{abr}\,\mathrm{tl}\,\mathrm{adj}\,f_*"'] & \pi_1(\mathrm{bs}\,\xi', \mathrm{proj}\,\xi'(x'_0) \arrow[r, "\Delta"] \arrow[d, "f_*"] & \pi_0((p')^{-1}(\mathrm{proj}\,\xi'(x'_0), y'_0) \arrow[d, "(\mathrm{abr}\,\mathrm{tl}\,\mathrm{adj}\,f_*"] \\
\pi_1(\mathrm{tl}\,\xi, x_0) \arrow[r, "(\mathrm{proj}\,\xi_*"'] & \pi_1(\mathrm{bs}\,\xi, \mathrm{proj}(x_0)) \arrow[r, "\Delta"'] & \pi_0((\mathrm{proj}\,\xi)^{-1}(\mathrm{proj}\,\xi'(x_0), x_0))
\end{tikzcd}$$

the rows are exact, while the left vertical homomorphism is monomorphic. Consequently, $f_*^{-1}(\mathrm{im}((\mathrm{proj}\,\xi)_*)) = \mathrm{im}\,p'_*$, whence $\mathrm{im}\,p'_* \supset \mathrm{grp}\,\xi'(x'_0)$. Since $\mathrm{bs}\,\xi'$ is locally connected, the last inclusion shows that the covering $(Y', p', \mathrm{bs}\,\xi')$ with base point y'_0 is subordinate to the covering ξ' with base point x'_0. If ψ' is such a subordination, then $(\mathrm{tl}\,\mathrm{adj}\,f|_{Y'}) \circ \mathrm{tl}\,\psi'$ and f combine to define the desired map φ.

Finally, we claim that φ is unique. Indeed, suppose $\varphi_1\colon \xi' \to \xi$ is another map with $\mathrm{bs}\,\varphi_1 = f$ and $\mathrm{tl}\,\varphi_1(x'_0) = x_0$. Then for every point $x' \in \mathrm{tl}\,\xi'$ and every path $s'\colon I \to \mathrm{tl}\,\xi'$ with $s'(0) = x'_0$, $s'(1) = x'$, the paths $\mathrm{tl}\,\varphi \circ s$ and $\mathrm{tl}\,\varphi_1 \circ s'$ cover $f \circ \mathrm{proj}\,\xi' \circ s'$ and have the same origin x_0. Therefore, in these circumstances $\mathrm{tl}\,\varphi \circ s' = \mathrm{tl}\,\varphi_1 \circ s'$, and hence $\mathrm{tl}\,\varphi(x') = \mathrm{tl}\,\varphi \circ s'(1) = \mathrm{tl}\,\varphi_1 s'(1) = \mathrm{tl}\,\varphi_1(x')$. □

Theorem 5.6.2.20. *Suppose ξ is a covering with base point $x_0 \in \mathrm{tl}\,\xi$, Y is a locally connected space with base point y_0, and $f(Y, y_0) \to (\mathrm{bs}\,\xi, \mathrm{proj}\,\xi(x_0))$ is continuous. If $f_*(\pi_1(Y, y_0) \subset \mathrm{grp}\,\xi(x_0)$ [in particular, if Y is simply connected], then there is a map $F\colon (Y, y_0) \to (\mathrm{tl}\,\xi\pounds, x_0)$ which covers f.*

(Cf. Proposition 4.1.3.8.)

Proof. To see this, apply Theorem 5.6.2.19 to ξ and $\xi' = (Y, \mathrm{id}_Y, Y)$. □

Coverings and Additional Structures

Remark 5.6.2.21. There is an important general principle asserting that under favourable conditions an additional structure defined on the base of a given covering can be lifted to the covering space. To conclude our study of coverings, we apply this principle to three structures: differentiable, cellular, and simplicial. Further applications appear in Exercises 5.6.5.10 and 5.6.5.11.

Concerning differentiable structures, we restrict ourselves from the onset to manifolds, i.e., we assume that $\operatorname{bs}\xi$ is a $\mathcal{C}^r$-manifold $(1 \leq r \leq a)$ and that the number of sheets of ξ is countable. We know that every chart $\varphi \in \operatorname{Atl}\operatorname{bs}\xi$ such that the bundle $\xi|_{\operatorname{supp}\varphi}$ is trivial has a family of copies in $\operatorname{tl}\xi$ which cover $(\operatorname{proj}\xi)^{-1}(\operatorname{supp}\varphi)$. Obviously, these copies (taken for all $\varphi \in \operatorname{Atl}\operatorname{bs}\xi$) constitute a $\mathcal{C}^r$-atlas of the space $\operatorname{tl}\xi$. This atlas provides $\operatorname{tl}\xi$ with a $\mathcal{C}^r$-structure and turns it into a $\mathcal{C}^r$-manifold. The fundamental property of this lifted structure, and the one which defines it uniquely, is that relative to this structure $\operatorname{proj}\xi$ is a $\mathcal{C}^r$-submersion, i.e., that ξ is a $\mathcal{C}^r$-bundle. Let us add that the orientability of $\operatorname{bs}\xi$ implies the orientability of $\operatorname{tl}\xi$.

We can similarly lift a cellular structure. However, to lift the cells and their characteristic maps we need Theorem 5.6.2.20. As a result, the total space of the given covering with cellular base becomes a cellular space (rigged whenever the base is rigged), and the projection becomes a cellular map.

Lifting a simplicial structure may be viewed as a special case of lifting a cellular one. The total space of a covering with simplicial base thus becomes a simplicial space (ordered whenever the base is so), and the projection becomes a simplicial map.

Remark 5.6.2.22. In all the three cases considered above, the lifted structure is invariant under the automorphism of the given covering. It is clear, at least if the covering ξ, is regular, that every differentiable or cellular structure defined on $\operatorname{tl}\xi$ and invariant under the automorphisms of ξ can be lowered to $\operatorname{bs}\xi$, i.e., the given structure on $\operatorname{tl}\xi$ is the result of lifting a similar structure from $\operatorname{bs}\xi$ (this becomes true also for simplicial structures after we effect the barycentric subdivision twice). For example, the lenses and the quotient spaces of $\mathbb{S}^3$ by the binary tetrahedral, cube, or icosahedral groups (see Example 4.2.3.21) are regularly covered by the corresponding spheres (see Remark 4.3.2.11 and Theorem 5.6.2.13), and hence are $\mathcal{C}^a$-manifolds.

If the given covering ξ is a $\mathcal{C}^r$-bundle, then the $\mathcal{C}^r$-structure on $\operatorname{tl}\xi$ is automatically invariant under the automorphisms of ξ. In this case, the lowered $\mathcal{C}^r$-structure surely coincides with the original $\mathcal{C}^r$-structure on $\operatorname{bs}\xi$. This is exactly what happens to all the coverings described in Example 4.1.2.6 and, in particular, to $(\mathbb{R}, \mathrm{hel}, \mathbb{S}^1)$, $(\mathbb{S}^1, \mathrm{hel}_m, \mathbb{S}^1)$, and $(\mathbb{S}^n, \mathrm{proj}, \mathbb{R}P^n)$.

5.6.3 Orientations

Remark 5.6.3.1. In the present subsection the technique developed in the previous one is applied to the orientability problems considered in §§3.1 and 4.5 (see Subsections 3.1.3, 4.5.1, 4.5.4). We follow the recipe of Subsection 4.5.1, which applies to bundles over arbitrary spaces, in contrast to the recipe of Subsection 4.5.4), which yields the same results, but only for bundles over a cellular base.

Remark 5.6.3.2. Let ξ be an n-dimensional real vector bundle. Consider the associated principal bundle, $\operatorname{assoc}(\xi, \mathrm{GL}(n, \mathbb{R}))$, and construct the orbit space of the right action of $\mathrm{GL}_+(n, \mathbb{R})$ on $\operatorname{tl}\operatorname{assoc}(\xi, \mathrm{GL}(n, \mathbb{R}))$ obtained by the restriction to $\mathrm{GL}_+(n, \mathbb{R})$ of the canonical right action of $\mathrm{GL}(n, \mathbb{R})$ on $\operatorname{tl}\operatorname{assoc}(\xi, \mathrm{GL}(n, \mathbb{R}))$ [see Definition4.3.2.10]. Denote this orbit space by $\operatorname{bs}_+ \xi$, and observe that it is the total space of a bundle with base $\operatorname{bs} \xi$ and projection $\operatorname{fact}\operatorname{proj}\operatorname{assoc}(\xi, \mathrm{GL}(n, \mathbb{R}))$. It is clear that $(\operatorname{bs}_+ \xi, \operatorname{fact}\operatorname{proj}\operatorname{assoc}(\xi, \mathrm{GL}(n, \mathbb{R})), \operatorname{bs} \xi)$ is a two-sheeted covering in the broad sense; we denote it by $\operatorname{Or} \xi$.

One may view the points of $\operatorname{bs}_+ \xi$ as pairs (x, ε), where $x \in \operatorname{bs} \xi$ and ε is an orientation of the fibre $(\operatorname{proj} \xi)^{-1}(x)$. This permits us to look upon a simultaneous orientation of all the fibres of ξ as a map $s \colon \operatorname{bs} \xi \to \operatorname{bs}_+ \xi$ such that $\operatorname{proj}\operatorname{Or} \xi \circ s = \mathrm{id}_{\operatorname{bs} \xi}$. Clearly, the compatibility (in the sense of Remark 4.5.1.8) of the orientation given by s on the fibres of ξ is equivalent to the continuity of s. Thus, the orientations of the bundle ξ turn out to be the sections of the bundle $\operatorname{Or} \xi$.

We list the most important corollaries of the above discussion:

(i) a bundle ξ is orientable if and only if the bundle $\operatorname{Or} \xi$ is trivial;

(ii) if the base $\operatorname{bs} \xi$ is locally connected and the fundamental groups of its components have no subgroups of index 2 (the last happens, in particular, whenever the components of $\operatorname{bs} \xi$ are simply connected), then ξ is orientable.

(iii) if ξ is orientable, then the number of its orientations equals the number of maps $\operatorname{comp}\operatorname{bs} \xi \to \mathbb{S}^0$ (and, in particular, equals 2^m if $\operatorname{bs} \xi$ has $m < \infty$ components).

Remark 5.6.3.3. We would like to make another remark concerning the previous construction. Given a real vector bundle ξ, consider the bundle with base $\operatorname{bs}_+ \xi$, induced from ξ by the projection $\operatorname{proj}\operatorname{Or} \xi$. Here we note that ξ_+ possesses a canonical orientation. Namely, the orientation of its fibre $(\operatorname{proj} \xi_+)^{-1}(x, \varepsilon)$ (where $x \in \operatorname{bs} \xi$ and ε is an orientation of the fibre $(\operatorname{proj} \xi)^{-1}(x)$) is simply the orientation ε, transferred to $(\operatorname{proj} \xi_+)^{-1}(x, \varepsilon)$ by the isomorphism

$$\operatorname{abr}\operatorname{tl}\operatorname{adj}(\operatorname{proj}\operatorname{Or} \xi) \colon (\operatorname{proj} \xi_+)^{-1}(x, \varepsilon) \to (\operatorname{proj} \xi) - -1(x);$$

that these orientations are compatible in the sense of Remark 4.5.1.8 is plain.

The Case of Smooth Manifolds

Remark 5.6.3.4. Orienting a smooth manifold is the same as orienting its tangent bundle (see Remark 4.6.4.5), and hence the discussion in Remark 5.6.3.2 carries over to the orientability and orientations of smooth manifolds. In particular, a manifold X is orientable if and only if the bundle $\operatorname{Or} \operatorname{tang} X$ (which is a two-sheeted covering in the broad sense with base X) is trivial. Also, a smooth manifold is automatically orientable if the fundamental groups of its components have no subgroups of index 2 (as happens when all its components are simply connected).

Let us add that, according to Theorem 5.6.2.21, the space $\operatorname{tl} \operatorname{Or} \operatorname{tang} X$ is a smooth manifold. By Remark 5.6.3.3, the latter carries a canonical orientation. If X is connected and non-orientable, then $\operatorname{Or} \operatorname{tang} X$ is a (two-sheeted) covering; we call it the *orientation-covering* of the manifold X.

5.6.4 Some Bundles Over Spheres

Remark 5.6.4.1. In this subsection we present the most elementary results concerning orientable real vector bundles and complex vector bundles over lower-dimensional spheres. This topic has independent interest and, at the same time, illustrates the general theory.

Recall that the classes of $\mathrm{GL}_+ \mathbb{R}^n$-equivalent $\mathrm{GL}_+ \mathbb{R}^n$-bundles over a given cellular base are in a one-to-one correspondence with the homotopy classes of continuous maps of this base into $G_+(\infty, n)$. Similarly, the classes of $\mathrm{GL}\, \mathbb{C}^n$-equivalent $\mathrm{GL}\, \mathbb{C}^n$-bundles over a cellular base are in a one-to-one correspondence with the homotopy classes of continuous maps of this base into $\mathbb{C}G(\infty, n)$. [See Theorem 4.5.3.8.] If the base is $\mathbb{S}^r$, $r \geq 1$, then the fact that $\mathrm{GL}_+ \mathbb{R}^n$ and $\mathrm{GL}\, \mathbb{C}^n$ are simple spaces implies that the aforementioned homotopy classes may be thought of as elements of the groups $\pi_r(G_+(\infty, n))$ and $\pi_r(\mathbb{C}G(\infty, n))$, respectively. Since these groups are isomorphic to $\pi_{r-1}(\mathrm{SO}(n))$ and $\pi_{r-1}(\mathrm{U}(n))$ (see Theorems 5.2.8.2 and 5.2.8.4), it follows that the classes of $\mathrm{GL}_+ \mathbb{R}^n$-equivalent $\mathrm{GL}_+ \mathbb{R}^n$-bundles over $\mathbb{S}^r$ $(r \geq 1)$ are in a natural one-to-one correspondence with the elements of $\pi_{r-1}(\mathrm{SO}(n))$, while the classes of $\mathrm{GL}\, \mathbb{C}^n$-equivalent $\mathrm{GL}\, \mathbb{C}^n$-bundles over $\mathbb{S}^r$ $(r \geq 1)$ are in a natural one to - one correspondence with the elements of $\pi_{r-1}(\mathrm{U}(n))$. Since we already know $\pi_{r-1}(\mathrm{SO}(n))$ and $\pi_{r-1}(\mathrm{U}(n))$ for some small values of r (see Subsection 5.2.6), we get the classification of the corresponding bundles, which we shall outline with minimum of details below. Some supplements appear in Exercises 5.6.5.19-5.6.5.19.

We note that this method works in a considerably more general situation. Namely, let G be a topological group, and let F be an effective G-space. According to the general theory of bundles (see Subsection 4.4.2), the elements of $\mathrm{Stnrd}(\mathbb{S}^q, F)$ are in a natural one-to–one correspondence with the elements of $\pi(\mathbb{S}^q, X)$, where X is any classifying space of G. If X is q-simple (for example, if G is connected), then $\pi(\mathbb{S}^q, X)$ coincides with $\pi_q(X)$ and, for $q \geq 1$, with $\pi_{q-1}(G)$ (see Theorem 5.6.1.6).

The Real Oriented Case

Remark 5.6.4.2. Since $\pi_0(\mathrm{SO}(n))$ is trivial, every $\mathrm{GL}_+\,\mathbb{R}^n$-bundle over $\mathbb{S}^1$ is $\mathrm{GL}_+\,\mathbb{R}^n$-trivial, for any positive integer n.

Since $\pi_1(\mathrm{SO}(1))$ is trivial, $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$, and $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$, it follows that:

- every $\mathrm{GL}_+\,\mathbb{R}^1$-bundle over $\mathbb{S}^2$ is $\mathrm{GL}_+\,\mathbb{R}^1$-trivial;
- the pairwise $\mathrm{GL}_+\,\mathbb{R}^2$-non-equivalent $\mathrm{GL}_+\,\mathbb{R}^n$-bundles over $\mathbb{S}^2$ form an infinite and countable collection;
- for $n \geq 3$, the number of pairwise $\mathrm{GL}_+\,\mathbb{R}^n$-non-equivalent $\mathrm{GL}_+\,\mathbb{R}^n$-bundles over $\mathbb{S}^2$ is 2.

Since $\pi_2(\mathrm{SO}(n))$ is trivial, every $\mathrm{GL}_+\,\mathbb{R}^n$-bundle over $\mathbb{S}^3$ is $\mathrm{GL}_+\,\mathbb{R}^n$-trivial,for any positive integer n.

Since $\pi_3(\mathrm{SO}(1))$ and $\pi_3(\mathrm{SO}(2))$ are trivial, whereas $\pi_3(\mathrm{SO}(n))$, $n \geq 3$, is infinite and countable, it follows that:

- every $\mathrm{GL}_+\,\mathbb{R}^1$-bundle over $\mathbb{S}^4$ is $\mathrm{GL}_+\,\mathbb{R}^1$-trivial;
- every $\mathrm{GL}_+\,\mathbb{R}^2$-bundle over $\mathbb{S}^4$ is $\mathrm{GL}_+\,\mathbb{R}^2$-trivial;
- for $n \geq 3$, the pairwise $\mathrm{GL}_+\,\mathbb{R}^n$-non-equivalent $\mathrm{GL}_+\,\mathbb{R}^n$-bundles over $\mathbb{S}^4$ form an infinite and countable collection.

The Complex Case

Remark 5.6.4.3. Since $\pi_0(\mathrm{U}(n))$ is trivial, every $\mathrm{GL}\,\mathbb{C}^n$-bundle over $\mathbb{S}^1$is $\mathrm{GL}\,\mathbb{C}^n$-trivial, for any positive integer n.

Since $\pi_1(\mathrm{U}(n)) \cong \mathbb{Z}$ for all $n \geq 1$, the classes of pairwise $\mathrm{GL}\,\mathbb{C}^n$-non-equivalent $\mathrm{GL}\,\mathbb{C}^n$-bundles over $\mathbb{S}^2$ form an infinite and countable collection, for any positive integer n.

Since $\pi_2(\mathrm{U}(n))$ is trivial, every $\mathrm{GL}\,\mathbb{C}^n$-bundle over $\mathbb{S}^3$ is $\mathrm{GL}\,\mathbb{C}^n$-trivial, for any positive integer n.

Since $\pi_2(\mathrm{U}(1))$ is trivial and $\pi_3(\mathrm{U}(n)) \cong \mathbb{Z}$ for all $n \geq 2$, it follows that:

- every $\mathrm{GL}\,\mathbb{C}^n$-bundle over $\mathbb{S}^4$ is $\mathrm{GL}\,\mathbb{C}^n$-trivial;
- if $n \geq 2$, the pairwise $\mathrm{GL}\,\mathbb{C}^n$-non-equivalent $\mathrm{GL}\,\mathbb{C}^n$-bundles over $\mathbb{S}^4$ form an infinite and countable collection.

5.6.5 Exercises

Exercise 5.6.5.1. Show that a smooth two-dimensional manifold does not admit the disc $\mathbb{D}^2$ as a covering space (unless the given manifold is homeomorphic to $\mathbb{D}^2$, and then the covering projection is the corresponding homeomorphism).

Exercise 5.6.5.2. Show that a smooth, compact two-dimensional manifold does not admit the half plane $\mathbb{R}_-$ as a covering space.

Exercise 5.6.5.3. Show that a sphere with g handles admits a sphere with $\widetilde{g}$ handles as an m-sheeted covering space if and only if $\widetilde{g} - 1 = m(g-1)$. [Cf. Exercise 4.1.5.1.]

Exercise 5.6.5.4. Show that a sphere with h cross-caps admits a sphere with $\widetilde{h}$ cross-caps as an m-sheeted covering space if and only if $\widetilde{h} - 2 = m(h-2)$. [Cf. Exercise 4.1.5.2.]

Exercise 5.6.5.5. Show that a sphere with h cross-caps admits a sphere with g handles as a covering space if and only if m is even and $g - 1 = m(h-2)/2$.

Exercise 5.6.5.6. Show that the Klein bottle admits a topological space as a covering space with non-identical covering projection if and only if the given space is homeomorphic to the Klein bottle, to the interior of a Möbius strip, or to one of the products $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times \mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^1$.

Exercise 5.6.5.7. Let $\pi_1, \pi_2, \pi_3, \dots$ be arbitrary groups, with π_k Abelian for $k \geq 2$, and suppose that there is given a right group action of π_1 on π_k, $k^{\geq}2$. Show that there is a cellular space X with base point x_0 together with an isomorphism $f_1 \colon \pi_1 \to \pi_1(X, x_0)$, such that the group $\pi_k(X, x_0)$ is f-isomorphic to π_k for all $k \geq 2$.

Exercise 5.6.5.8. Let ξ be a covering with $\operatorname{bs}\xi$ an n-dimensional locally Euclidean space. Prove that $\operatorname{tl}\xi$ is an n-dimensional locally Euclidean space with boundary $\partial \operatorname{tl}\xi = (\operatorname{proj}\xi)^{-1}(\partial \operatorname{bs}\xi)$.

Exercise 5.6.5.9. Show that every covering space of a locally finite cellular space is locally finite (see Theorem 5.6.2.21).

Exercise 5.6.5.10. Let ξ be a covering with $\operatorname{bs}\xi$ a topological group. Prove that, given any point $x \in (\operatorname{proj}\xi)^{-1}(e_{\operatorname{bs}\xi})$, on $\operatorname{tl}\xi$ there is one and only one group structure, which turns $\operatorname{tl}\xi$ into a topological group with identity x, and turns $\operatorname{proj}\xi$ into a homomorphism.

Exercise 5.6.5.11. Let ξ be a covering with a finite number of sheets, such that a connected, compact topological group acts transitively on $\operatorname{bs}\xi$. Prove that one can define a transitive action of a connected, compact topological group on $\operatorname{tl}\xi$.

Exercise 5.6.5.12. Let ξ be a two-sheeted covering with $\operatorname{bs}\xi$ a non-orientable smooth manifold and $\operatorname{tl}\xi$ orientable. Show that ξ is equivalent to the orientation-covering of $\operatorname{bs}\xi$.

Exercise 5.6.5.13. Let X be a non-orientable smooth manifold and assume that X admits an orientable manifold Y as a covering space. Show that Y is also a covering space of $\operatorname{tl}\operatorname{Or}\operatorname{tang} X$.

Exercise 5.6.5.14. Show that:

(i) for each positive integer n there is a unique (up to $\operatorname{GL}\mathbb{R}^n$-equivalences) $\operatorname{GL}\mathbb{R}^n$-non-trivial $\operatorname{GL}\mathbb{R}^n$-bundle over $\mathbb{S}^1$ and that for $n = 1$ its total space is homeomorphic to the interior of a Möbius strip;

(ii) the pairwise $\operatorname{GL}\mathbb{R}^2$-non-equivalent $\operatorname{GL}\mathbb{R}^2$-bundles over $\mathbb{S}^2$ form an infinite and countable collection;

(iii) for $n \geq 3$, the number of pairwise $\mathrm{GL}\,\mathbb{R}^n$-non-equivalent $\mathrm{GL}\,\mathbb{R}^n$-bundles over $\mathbb{S}^2$ is equal to 2;

(iv) for any positive integer n, every $\mathrm{GL}\,\mathbb{R}^n$-bundle over $\mathbb{S}^3$ is $\mathrm{GL}\,\mathbb{R}^n$-trivial;

(v) every $\mathrm{GL}\,\mathbb{R}^2$-bundle over $\mathbb{S}^4$ is $\mathrm{GL}\,\mathbb{R}^2$-trivial;

(vi) for $n \geq 3$, the pairwise $\mathrm{GL}\,\mathbb{R}^n$-non-equivalent $\mathrm{GL}\,\mathbb{R}^n$-bundles over $\mathbb{S}^4$ form an infinite and countable collection.

Exercise 5.6.5.15. Show that for $r \geq 2$ every $\mathrm{GL}\,\mathbb{R}^1$-bundle over S is $\mathrm{GL}\,\mathbb{R}^1$-trivial.

Exercise 5.6.5.16. Let $r \geq 3$. Show that:

(i) every $\mathrm{GL}\,\mathbb{R}^2$-bundle over $\mathbb{S}^r$ is $\mathrm{GL}\,\mathbb{R}^2$-trivial;

(ii) every $\mathrm{GL}\,\mathbb{R}^2_+$-bundle over $\mathbb{S}^r$ is $\mathrm{GL}_+\,\mathbb{R}^2$-trivial;

(iii) every $\mathrm{GL}\,\mathbb{C}^1$-bundle over $\mathbb{S}^r$ is $\mathrm{GL}\,\mathbb{C}^1$-trivial.

Exercise 5.6.5.17. Show that two $\mathrm{GL}_+\,\mathbb{R}^2$-bundles over $\mathbb{S}^2$ become equivalent by extending the structure group $\mathrm{GL}\,\mathbb{R}^2_+$ to $\mathrm{GL}\,\mathbb{R}^2$ if and only if the corresponding elements of $\pi_1(\mathrm{SO}(2))$ [see Remark 5.6.4.1] are either equal or inverses of one another.

Exercise 5.6.5.18. Show that for every non-trivial $\mathrm{O}\,\mathbb{R}^2$-bundle ξ with $\mathrm{bs}\,\xi = \mathbb{S}^2$ the space $\mathrm{tl}\,\mathrm{assoc}(\xi, \mathbb{S}^1)$ is homeomorphic to one of the lenses $L(m; 1, 1)$ [here $\mathrm{O}(2)$ acts canonically on $\mathbb{S}^1$].

Exercise 5.6.5.19. Show that for $n \geq 2$ the number of pairwise $\mathrm{GL}\,\mathbb{C}^n$-non-equivalent $\mathrm{GL}\,\mathbb{C}^n$-bundles over $\mathbb{S}^5$ does not exceed 2.

INFORMATION. Actually, the last number is 2 if $n = 2$, and 1 if $n > 2$.

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