Jacobson's theorem on the commutativity of rings: case $x^3 = x$

Jacobson's theorem asserts

Theorem 1. *A ring R in which $x^n = x$, for some fixed $(2 \leq) n \in \mathbb{N}$ for every $x \in R$ is commutative.*

In general, this is hard to prove. But for $n = 2$, the proof is relatively easy. Then how about the case $n = 3$? I encountered an elementary proof several years ago as follows.

Theorem 2. *Let R be a ring with $x^3 = x$ for every $x \in R$. Then R is commutative.*

Proof. (Derek Holt) Recall that an element x is called *central* if $xy = yx$ for all $y \in R$. Note that the central elements form a subring of R.

(1) $xy = 0 \Rightarrow yx = 0$. For $yx = (yx)^3 = y(xy)^2x = 0$.

(2) $x^2 = x \Rightarrow x$ is central. For $x(y - xy) = xy - x^2y = xy–xy = 0$, so by (1), $(y - xy)x = 0$, and $yx = xyx$. Similarly, $(y - yx)x = 0 \Rightarrow x(y - yx) = 0 \Rightarrow xy = xyx$. Hence $xy = yx$ for all y.

(3) x^2 is central for all $x \in R$ by (2). For $(x^2)^2 = x^4 = x^2$.

(4) If $x^2 = nx$ for an integer n, then x is central. For $x = x^3 = qx^2$, which is central by (3).

(5) $x + x^2$ is central for all $x \in R$. For $(x + x^2)^2 = 2(x + x^2)$, so by (4) $x + x^2$.

(6) By (3) and (5), $x = (x + x^2) - x^2$ is central, completing the proof.

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