Artinian modules and exact sequences

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Theorem 1. *Let M, M', M'' be R-modules and $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ a short exact sequence. Then M is artinian if and only if M' and M'' are artinian.*

Proof. ($\Rightarrow$): Suppose M is an artinian module and $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ is an exact sequence of modules. We want to show that this implies that M' and M'' are artinian. We begin by showing that M'' is artinian.

Suppose

$$X_0 \supset X_1 \supset \cdots \supset X_i \supset X_{i+1} \supset \cdots$$

is a descending chain of submodules of M''. Then

$$\pi^{-1}(X_0) \supset \pi^{-1}(X_1) \supset \cdots \supset \pi^{-1}(X_i) \supset \pi^{-1}(X_{i+1}) \supset \cdots$$

is a descending chain of submodules of M. Because M is artinian, there is an integer n such that $\pi^{-1}(X_i) = \pi^{-1}(X_n)$ for all $i \geq n$. The fact that $\pi\colon M \to M''$ is surjective implies that $\pi(\pi^{-1}(X_i)) = X_i$, for all i. In particular, $X_i = \pi\pi^{-1}(X_i)) = \pi(\pi^{-1}(X_n)) = X_n$ for all $i \geq n$. Hence, the fact that M is artinian and $\pi\colon M \to M'' \to 0$ is exact implies that M'' is artinian.

We now show that M' is artinian. Suppose

$$Y_0 \supset Y_1 \supset \cdots \supset Y_j \supset Y_{j+1} \supset \cdots$$

is a descending chain of submodules of M'. Then

$$\iota(Y_0) \supset \iota(Y_1) \supset \cdots \supset \iota(Y_j) \supset \iota(Y_{j+1}) \supset \cdots$$

is a descending chain of submodules of M. Because M is artinian, there is an integer n such that $\iota(Y_j) = \iota(Y_n)$ for all $j \geq n$. Because $\iota\colon M' \to M$ is a monomorphism, we know that $\iota^{-1}\iota(Y_j)) = Y_j$ for all j. In particular, $Y_j = \iota^{-1}(\iota(Y_j)) = \iota^{-1}(\iota(Y_n)) = Y_n$ for all $j \geq n$. Hence, the fact that $0 \to M' \xrightarrow{\iota} M$ is exact and M is artinian implies that M' is also artinian.

($\Leftarrow$): We have to show that if $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ is an exact sequence of R-modules with the property that M' and M'' are artinian modules, then M is also artinian. Suppose

$$X_0 \supset X_1 \supset \cdots \supset X_j \supset X_{j+1} \supset \cdots$$

is a descending chain of submodules of M. Then

$$\pi(X_0) \supset \pi(X_1) \supset \cdots \supset \pi(X_j) \supset \pi(X_{j+1}) \supset \cdots$$

is a descending chain of submodules of M'' whereas

$$\iota^{-1}(X_0) \supset \iota^{-1}(X_1) \supset \cdots \supset \iota^{-1}(X_j) \supset \iota^{-1}(X_{j+1}) \supset \cdots$$

is a descending chain of submodules of M'. Because M' and M'' are artinian, we know there is an integer n'' such that $\pi(X_j) = \pi(X_{n''})$ for all $j \geq n''$. Similarly, there is an integer n' such that $\iota^{-1}(X_j) = \iota^{-1}(X_{n'})$ for all $j \geq n'$. Therefore, if we let $n = \ max(n', n'')$, we have that $\pi(X_j) = \pi(X_n)$ and $\iota^{-1}(X_j) = \iota^{-1}(X_n)$ for all $j \geq n$.

Now for each $j \in \mathbb{N}$ define $g_j \colon X_j \to \pi(X_j)$ by $g_j(x) = \pi(x)$ for all $x \in X_j$. Clearly, g_j, is an epimorphism with kernel $K \cap X_j$, where$K = \ker \pi$. Because $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ is exact, $\iota \colon M' \to M$ is a monomorphism with $\iota(M') = K$. Therefore, $\iota^{-1}(X_j) = \iota^{-1}(K \cap X_j)$ and so $\iota(\iota^{-1}(X_j)) = K \cap X_j$. Hence, if we define $h_j \colon \iota^{-1}(X_j) \to X_j$ by $h_j(x) = \iota(x)$ for all $x \in \iota^{-1}(X_j)$, then the sequence $0 \to \iota^{-1}(X_j) \xrightarrow{h_j} X_j \xrightarrow{g_j} \pi(X_j) \to 0$ is exact for all $j \in \mathbb{N}$. Suppose $j \geq n$. Then we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \iota^{-1}(X_j) & \xrightarrow{h_j} & X_j & \xrightarrow{g_j} & \pi(X_j) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow \subset & & \Vert & & \\
0 & \longrightarrow & \iota^{-1}(X_n) & \xrightarrow[h_n]{} & X_n & \xrightarrow[g_n]{} & \pi(X_n) & \longrightarrow & 0
\end{array}$$

with exact rows. From this it follows that $\operatorname{im} h_j = \operatorname{im} h_n = \ker g_n$, so that X_j is a submodule of X_n containing $\ker g_n$. Therefore, we know that $g_n^{-1}(g_n(X_j)) = X_j$. But $g_n(X_j) = g_j(X_j) = \pi(X_j) = \pi(X_n)$, so that we also have $g_n^{-1}(g_n(X_j)) = g_n^{-1}(\pi(X_n)) = X_n$. Therefore, it follows that $X_j = X_n$ for each $j \geq n$. Thus, the descending chain of submodules of M

$$X_0 \supset X_1 \supset \cdots \supset X_j \supset X_{j+1} \supset \cdots$$

has the property that there is an integer $n \in \mathbb{N}$ such that $X_j = X_n$ for all $j \geq n$. Because this is true for any descending chain of submodules of M, we have shown that M is an artinian module if there is an exact sequence $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$ of R-modules with M' and M'' artinian modules. $\square$