Homological Algebra

Peter Jørgensen

Contents

Chapter 1

Modules

1.1 Definition

Recall the concept of a vector space V over the real numbers $\mathbb{R}$. V is an abelian group, and there is a scalar multiplication

$$\mathbb{R} \times V \to V, \quad (r, v) \mapsto rv$$

satisfying some conditions:

$$1v = v, (r + s)v = rv + sv, r(v + w) = rv + rw, r(sv) = (rs)v.$$

Example 1.1.1. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Its elements are n-tuples $(r_1, \ldots, r_n)$ of real numbers. Its abelian group structure is defined by

$$(r_1, \ldots, r_n) + (s_1, \ldots, s_n) = (r_1 + s_1, \ldots, r_n + s_n).$$

Its scalar multiplication is defined by $r(r_1, \ldots, r_n) = (rr_1, \ldots, rr_n)$.

There is nothing to stop us from making the same definition over any ring R. The concept thus defined is called a left-R-module.

Definition 1.1.2. A *left-R-module* is an abelian group M with a scalar multiplication

$$R \times M \to M, \quad (r, m) \mapsto rm$$

satisfying the same conditions as before:

$$1m = m, (r + s)m = rm + sm, r(m + n) = rm + rn, r(sm) = (rs)m.$$

Example 1.1.3.

- A vectorspace over $\mathbb{R}$ is an $\mathbb{R}$-module. In particular, $\mathbb{R}^n$ is an $\mathbb{R}$-module.
- If R is any ring then R^n is an R-module, in the same way that $\mathbb{R}^n$ is a $\mathbb{R}$-module.

- An abelian group A can be viewed as a $\mathbb{Z}$-module with the scalar multiplication

$$na = \begin{cases} \overbrace{a + \cdots + a}^{n \text{ times}} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \underbrace{(-a) + \cdots + (-a)}_{-n \text{ times}} & \text{if } n < 0. \end{cases}$$

- Let k be a field. Then the set Λ of 2 by 2 upper triangular matrices over k is a ring:

$$\Lambda = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$$

There are right-Λ-modules

$$P = (0\ k), \quad Y = (k\ k), \quad I = \cdots .$$

The scalar multiplication of Λ on P and Y is given by matrix multiplication. Any finitely generated right-Λ-module is isomorphic to a direct sum of copies of the indecomposable modules P, Y, and I.

1.2 Homomorphisms

Definition 1.2.1. Let M and N be left-R-modules. A *homomorphism of R-modules* from M to N is an R-linear map $\frac{1}{4}\colon M \to N$. That is, $\mu(m_1 + m_2) = \mu(m_1) + \mu(m_2)$ and $\frac{1}{4}(\mu m) = r\mu(m)$.

Example 1.2.2.
- An $\mathbb{R}$-linear ma@ of -vector space is a homomorppshim of $\mathbb{R}$-module.
- A homomorphism of abelian groups is a homomorphism of $\mathbb{Z}$-modules, if the abelian groups are viewed as $\mathbb{Z}$-modules
- If N is any left-R-module and $n_1, \ldots, n_s$ are elements of N, then $\rho\colon R^s \to N$ given by $\rho(r_1, \ldots, r_s) = r_1 n_1 + \cdots + r_s n_s$ is a homomorphism of R-modules.
- Recall that over

$$\Lambda = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$$

There are right-Λ-modules

$$P = (0\ k), \quad Y = (k\ k), \quad I = \cdots .$$

There is an injective homomorphism

$$P \to Y, \quad (0\ p) \mapsto (0\ p).$$

There is a surjective homomorphism

$$Y \to I$$

(more about this later!).

1.3 Kernels, Images, Cokernels

Definition 1.3.1. A bijective homomorphism is called an *isomorphism.*

Isomorphic modules are algebraically equivalent.

Module homomorphisms are a natural generalisation of linear maps of vector spaces. Other notions can also be generalised. In particular, submodules and quotient modules are defined analogously to sub vector spaces and quotient vector spaces. Recall that a module M is an abelian group with a scalar multiplication.

Definition 1.3.2. A *submodule* M' of M is a subgroup closed under scalar multiplication.
The *quotient module* M/M' is the quotient group equipped with the scalar multiplication

$$r(m + M') = rm + M'.$$

Let us check that this is well defined: If $m_1+M' = m_2+M'$, then m_1-m_2 is in M'. Then $rm_1-rm_2 = r(m_1-m_2)$ is also in M', whence $rm_1+M' = rm+M'$.

Example 1.3.3. Over the upper triangular matrix ring Λ, the injective homomorphism $P = (0\ k) \to (k\ k) = Y$ means that P can be viewed as a submodule of Y. The third, "mysterious", module I is just Y/P.

Definition 1.3.4. Let $\mu : M \to N$ be a module homomorphism. The *kernel* of μ is the submodule $\ker\mu = \{m \in M \mid \mu(m) = 0\}$. There are homomorphisms

$$\ker\mu \hookrightarrow M \xrightarrow{\mu} N.$$

The *image* of μ is the submodule $\operatorname{im}\mu = \{\mu(m) \mid m \in M\}$.
There are homomorphisms

$$M \twoheadrightarrow \operatorname{im}\mu \hookrightarrow N$$

which compose to μ.
The *cokernel* of μ is the quotient module $\operatorname{coker}\mu = N/\operatorname{im}\mu$.
There are homomorphisms

$$M \xrightarrow{\mu} N \twoheadrightarrow \operatorname{coker}\mu.$$

Example 1.3.5.

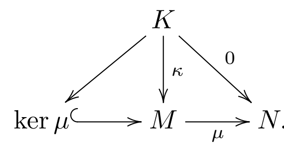
- Consider the morphism of $\mathbb{Z}$-modules $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$.
  - $\ker = 0$ (the homomorphism is injective).
  - $\operatorname{im} = 2\mathbb{Z}$.
  - $\operatorname{coker} = \mathbb{Z}/2\mathbb{Z}$.
- The ring $R = k[X]/(X^2)$ is a left-module over itself, and $R \xrightarrow{\cdot X} R$ is a homomorphism. The ring is spanned over k by 1 and X, and the homomorphism is determined by $1 \mapsto X$ and $X \mapsto X^2 = 0$.
  - $\ker = (X)$.

- im $= (X)$.
- coker $= R/(X)$.

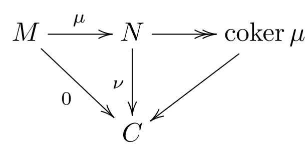
These three modules are 1-dimensional, and they are in fact isomorphic.

Kernel and cokernel have so-called universal properties.

Theorem 1.3.6. *If $K \xrightarrow{\kappa} M$ satisfies $\mu\kappa = 0$, then κ factors uniquely through the kernel.*



If $N \xrightarrow{\nu} C$ satisfies $\nu\mu = 0$, then ν factors uniquely through the cokernel.



1.4 Short Exact Sequences

Definition 1.4.1. A short exact sequence is a diagramme of module homomorphisms of the form

$$M \xrightarrow{\nu} N \xrightarrow{\nu} P$$

where μ is a kernel of ν (in the sense of having the universal property) and ν is a cokernel of μ (in the sense of having the universal property).

Note that in particular, μ is injective and ν is surjective!
Short exact sequences (extensions) are often written in the form

$$0 \to M \to N \to P \to 0.$$

The existence of such a short exact sequence means precisely that there exists a module N which contains a submodule isomorphic to M such that the quotient module is isomorphic to P.

Example 1.4.2.

- There is a short exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

- If $m \leq n$ then there is an injective homomorphism of R-modules $R^m \to R^n$ given by

$$(r_1, \ldots, r_m) \mapsto (r_1, \ldots, r_m, 0, \ldots, 0).$$

  The quotient R^n/R^m is obtained by discarding the m first coordinates, so it can be identified with R^{n-m}. There is a short exact sequence

$$0 \to R^m \to R^n \to R^{n-m} \to 0.$$

- Over the 2 by 2 upper triangular matrix ring Λ, there is a short exact sequence of right-modules

$$0 \to P \to Y \to I \to 0.$$

Chapter 2

Classical Homological Algebra

2.1 Complexes and Chain Maps

Definition 2.1.1. A *complex* of R-modules is a diagramme of R-modules and homomorphisms of the form

$$\cdots \to M^{-2} \xrightarrow{\partial^{-2}} M^{-1} \xrightarrow{\partial^{-1}} M^{0} \xrightarrow{\partial^{0}} M^{1} \xrightarrow{\partial^{1}} M^{2} \to \cdots$$

where $\partial^i \partial^{i-1} = 0$ for each i.
This equation means $\operatorname{im} \partial^{i-1} \subseteq \ker \partial^i$, and the *ith cohomology* of the complex is

$$H^i(M) = \ker \partial^i / \operatorname{im} \partial^{i-1}.$$

We have $H^i(M) = 0$ if and only if $\operatorname{im} \partial^{i-1} = \ker \partial^i$, and then M is called *exact* in degree i.

Example 2.1.2.

- If the complex

  $$0 \to M \xrightarrow{\mu} N \xrightarrow{\nu} P \to 0$$

  is exact then it is a short exact sequence. Exactness at M means that $0 = \ker \mu$, that is, μ is injective. Exactness at P means that $\operatorname{im} \nu = P$, that is, ν is surjective. Exactness at N means that $\operatorname{im} \mu = \ker \nu$. But μ can be seen as identifying M with $\operatorname{im} \mu$, that is, as identifying M with $\ker \nu$, so μ is a kernel of ν. And ν can be seen as identifying $N/\ker \mu$ with P, that is, as identifying $N/\operatorname{im} \mu$ with P, so ν is a cokernel of μ.

- Over the ring $R = k[X]/(X^2)$, there is an exact complex

  $$\cdots \to R \xrightarrow{\cdot X} R \xrightarrow{\cdot X} R \xrightarrow{\cdot X} R \to \cdots$$

  since each of the homomorphisms has kernel and image (X).

- Over the integers $\mathbb{Z}$, there is a complex

$$P = \cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0 \to \cdots$$

with $H^0(P) = \mathbb{Z}/2\mathbb{Z}$ and all other cohomology groups 0.

- Over the ring $R = k[X, Y]$, there is a complex

$$Q = \cdots \to 0 \to R \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R \to 0 \to \cdots$$

where the maps are given by the matrices

$$\varphi = (X \ Y), \quad \text{and} \quad \psi = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

The zeroth cohomology group is $H^0(Q) = k$ and all the other cohomology groups are 0.

Definition 2.1.3. A *chain map* of complexes of R-modules involves two complexes of R-modules and homomorphisms,

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & M^{-2} & \xrightarrow{\partial_M^{-2}} & M^{-1} & \xrightarrow{\partial_M^{-1}} & M^0 & \xrightarrow{\partial_M^0} & M^1 & \xrightarrow{\partial_M^1} & M^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mu^{-2}} & & \downarrow{\scriptstyle \mu^{-1}} & & \downarrow{\scriptstyle \mu^{0}} & & \downarrow{\scriptstyle \mu^{1}} & & \downarrow{\scriptstyle \mu^{2}} & & \\
\cdots & \longrightarrow & N^{-2} & \xrightarrow[\partial_N^{-2}]{} & N^{-1} & \xrightarrow[\partial_N^{-1}]{} & N^0 & \xrightarrow[\partial_N^0]{} & N^1 & \xrightarrow[\partial_N^1]{} & M^2 & \longrightarrow & \cdots
\end{array}$$

and vertical R-module homomorphisms.
The diagram is commutative:

$$\mu^{i+1} \circ \partial_M^i = \partial_M^i \circ \mu^i$$

Commutativity implies that $\mu(\ker \partial_M) \subseteq \ker(\partial_N)$ and $\mu(\operatorname{im} \partial_M) \subseteq \operatorname{im}(\partial_N)$. Hence there is a well-defined induced homomorphism

$$\ker \partial_M^i / \operatorname{im} \partial_M^{i-1} \to \ker \partial_N^i / \operatorname{im} \partial_N^{i-1},$$

that is,

$$H^i(M) \to H^i(N).$$

This homomorphism is denoted $H^i(\mu)$.
It depends functorially on μ: If μ and ν are consecutive chain maps, then

$$H^i(\nu \circ \mu) = H^i(\nu) \circ H^i(\mu).$$

Example 2.1.4. Consider the complexes

$$P = \cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to 0 \to \cdots$$

and

$$Q = \cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot q} \mathbb{Z} \to 0 \to \cdots$$

with cohomology $H^0(P) = \mathbb{Z}/p\mathbb{Z}$ and $H^0(Q) = \mathbb{Z}/q\mathbb{Z}$.
If a is an integer such that $q \mid pa$, then $pa = qb$ and there is a chain map μ given by

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow{\cdot b} & & \downarrow{\cdot a} & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot q} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

The induced homomorphism

$$H^0(\mu)\colon H^0(P) \to H^0(Q)$$

is

$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z}/q\mathbb{Z}.$$

2.2 Free Modules

Recall that if R is a ring, then R^s, the set of s-tuples of R-elements, is a left-R-module. Such a module (and any module isomorphic to it) is called *free*.

Free modules have a special property with regard to homomorphisms. Denote by e_i the tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith position.

Theorem 2.2.1. *If N is any left-R-module and $n_1, \ldots, n_s$ are in N, then there is precisely one homomorphism $\rho\colon R^s \to N$ with $\rho(e_i) = n_i$ for each i.*

Proof.

$$\begin{aligned} \rho((r_1, \ldots, r_s)) &= \rho(r_1 e_1 + \cdots + r_s e_s) \\ &= r_1 \rho(e_1) + \cdots + r_s \rho(e_s) \\ &= r_1 n_1 + \cdots + r_s n_s. \end{aligned}$$

□

"Homomorphisms from R^s are determined by the images of the basis elements $e_1, \ldots, e_s$, and the images can be chosen freely."
More generally, if I is any index set, then we can think of basis elements e_i for i in I and form the free left-R-module F of formal linear combinations of the e_i. So the elements of F are finite linear combinations $r_1 e_{i_1} + \cdots + r_t e_{i_t}$ with the r_j in R.
The module F has the same property with respect to homomorphisms: If N is any left-R-module and ni are elements of N indexed by I, then there is precisely one homomorphism $\varphi\colon F \to N$ with $\varphi(e_i) = n_i$ for each i. Namely,

$$\varphi(r_1 e_{i_1} + \cdots + r_t e_{i_t}) = r_1 n_{i_1} + \cdots + r_t n_{i_t}.$$

Note that the basis elements of a free module are linearly independent:

$$r_1 e_{i_1} + \cdots + r e_{i_t} = 0$$

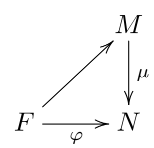
implies $r_1 = \cdots = r_t = 0$.

Example 2.2.2.

- Any vectorspace V over R is a free R-module. If V has a finite generating system, then by dropping vectors from the system, we will eventually arrive at a linearly independent generating system. Using this system as the e_i shows that V is free. For a general V, basis elements e_i can be found using Zorn's Lemma.
- The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is not free. It has no basis. The module has elements 0 and 1. But $1 \cdot 0 = 0$ and $2 \cdot 1 = 0$ so it is impossible to find a linearly independent set.

Free modules have the following property:

Theorem 2.2.3. *Let $\mu\colon M \to N$ be a surjective module homomorphism. Let F be a free module with basis elements e_i. Let $\varphi\colon F \to N$ be a module homomorphism.*



Then there is a homomorphism φ' with $\mu \circ \varphi' = \varphi$.

Proof. For each $\varphi(e_i) = n_i$, since μ is surjective, we can pick m_i in M with $\mu(m_i) = n_i$. Define $\varphi'\colon F \to M$ by $\varphi'(e_i) = m_i$. Then $\mu \circ \varphi'(e_{i)} = n_i = \varphi(e_i)$ so $\mu \circ \varphi' = \varphi$. □

If N is a left-R-module, then a system of elements n_i indexed by i in I is said to *generate* N if N is equal to the set of all linear combinations $r_1 n_{i_1} + \cdots + r_t n_{i_t}$. For instance, taking all the elements of N obviously gives a system of generators. Given a system of generators n_i, we can consider the free left-R-module F with basis elements e_i, and the homomorphism

$$\varphi\colon F \to N, \quad \varphi(e_i) = n_i.$$

It is clearly surjective since

$$\varphi(r_1 e_{i_1} + \cdots + r_t e_{i_t}) = r_1 n_{i_1} + \cdots + r_t n_{i_t}.$$

Example 2.2.4. Over $\mathbb{Z}$, the module $\mathbb{Z}/2\mathbb{Z}$ has the system of generators $\{1\}$. There is hence a surjective homomorphism from a free module with a single basis element, e_1, given by $e_1 \mapsto 1$.
Such a free module is given by $\mathbb{Z}^1 = \mathbb{Z}$, and the surjective homomorphism $\varphi\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is

$$\varphi(x) = \varphi(x \cdot e_1) = x \cdot 1 = \text{ the residue class of } x \text{ in } \mathbb{Z}/2\mathbb{X}.$$

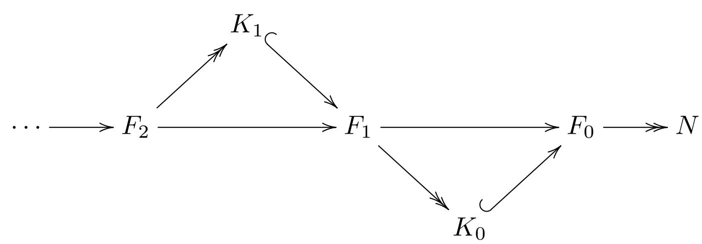
Note that $\ker \varphi = 2\mathbb{Z}$.

2.3 Free and Projective Resolutions

Let us recall theorem 2.2.1 in the following form.

Theorem 2.3.1. *Any module N permits a surjective homomorphism $F \to N$ with F free.*

The homomorphism induces an isomorphism $F/K \cong N$ where K is the kernel.
We can iterate this construction: Choose a surjection $F_0 \to N$. Let K_0 be the kernel and choose a surjection $F_1 \to K_0$. Let K_1 be the kernel and choose a surjection $F_2 \to K_1$. Continue to get



Definition 2.3.2. The complex $\cdots \to F_2 \to F_1 \to F_0 \to 0 \to \cdots$ is called a *free resolution* of the module N.

Example 2.3.3.

- The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ permits the free resolution

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0 \to \cdots .$$

- Over the ring $R = k[X, Y]$, we have seen the complex

$$\cdots \to 0 \to R \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R \to 0 \to \cdots$$

  where the homomorphisms were given by certain matrices. This is in fact a free resolution of the R-module k.

Note that we used homological (lower) indices in the free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to 0 \to \cdots \quad \text{of} \quad N.$$

As a generel rule, they relate to cohomological (upper) indices via $F_i = F^{-i}$. We have

$$H_i(F) = \begin{cases} F_0/K_0 \cong N & i = 0 \\ 0 & i \neq 0. \end{cases}$$

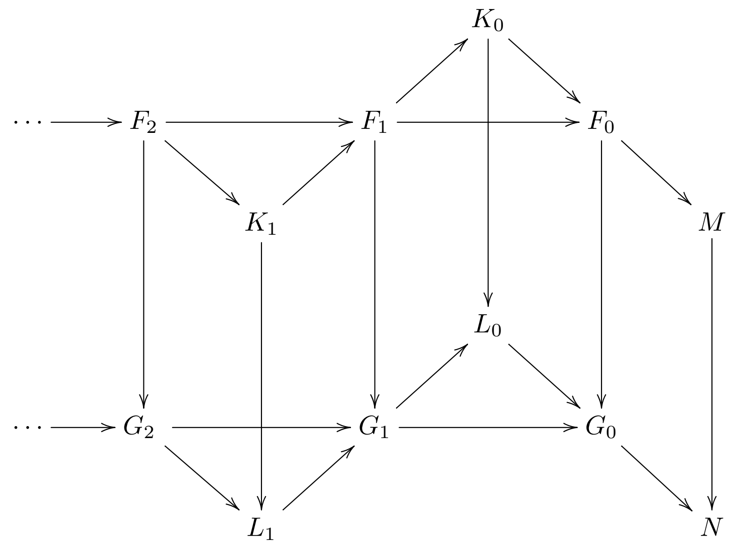
Let

$$\cdots \to F_2 \to F_1 \to F_0 \to 0 \to \cdots$$

and

$$\cdots \to G_2 \to G_1 \to G_0 \to 0 \to \cdots$$

be free resolutions of M and N, and let $\mu\colon M \to N$ be a homomorphism. We can lift the homomorphism to the free resolutions:



So we get a chain map

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow & & \\
\cdots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The induced homomorphism $H_0(\varphi)\colon H_0(F) \to H_0(G)$ is the given homomorphism $M \to N$.

The lifting is not unique, but it is "unique up to homotopy". Any two liftings, φ and φ', differ by a chain homotopy:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & F_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & \overset{\theta_1}{\swarrow} & \downarrow & \overset{\theta_0}{\swarrow} & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & G_0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

We have $\varphi_i - \varphi'_{ui} = \partial^G_{i+1}\theta_i + \theta_{i-1}\partial^F_i$.

If F and G are both free resolutions of M, then the identity on M lifts to chain maps $F \to G$ and $G \to F$.

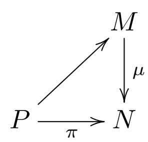
The compositions $F \to G \to F$ and $G \to F \to G$ also lift the identity on M, so they are homotopic to the identity chain map. We say that F and G are *homotopy equivalent*.

As a final observation on free modules, note that a direct summand of a free module is known as a *projective* module.

Example 2.3.4. The upper triangular ring $\Lambda = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ is a free right-Λ-module. It is the direct sum of the right-Λ-modules $Y = (k\ k)$ and $P = (0\ k)$, so Y and P are projective right-Λ-modules (and they are not free).

One of the key properties of free modules remains true for projective modules:

Theorem 2.3.5. *Let $\mu\colon M \to N$ be a surjective module homomorphism. Let P be a projective module. Let $\pi\colon P \to N$ be a module homomorphism.*



Then there is a homomorphism π' with $\mu \circ \pi' = \pi$.

As a consequence, we can replace free resolutions with projective resolutions above:
Homomorphisms of modules can be lifted to chain maps of projective resolutions, and the liftings are unique up to chain homotopy.

2.4 Ext and Extensions

Let M and N be left-R-modules. By $\mathrm{Hom}(M, N)$ is denoted the set of homomorphisms of R-modules from M to N. It is clearly an abelian group. If

$$\mu\colon M \to M'$$

is a homomorphism, then there is an induced homomorphism

$$\mathrm{Hom}(\mu, N)\colon \mathrm{Hom}(M', N) \to \mathrm{Hom}(M, N)$$

given by $\mu' \mapsto \mu' \circ \mu$.

Hence $\mathrm{Hom}(-, N)$ sends left-R-modules to abelian groups, and it sends homomorphisms to homomorphisms. It is a functor from left-R-modules to abelian groups since it satisfies $\mathrm{Hom}(\mathrm{id}, N) = \mathrm{id}$ and $\mathrm{Hom}(\mu' \circ \mu, N) = \mathrm{Hom}(\mu, N) \circ \mathrm{Hom}(\mu', N)$.

Let M be a left-R-module with projective resolution

$$P = \cdots \to P_2 \to P_1 \to P_0 \to 0 \to \cdots .$$

Applying the functor $\mathrm{Hom}(-, N)$ produces a new complex $\mathrm{Hom}(P, N)$ which is

$$\cdots \to 0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \mathrm{Hom}(P_2, N) \to \cdots$$

and we define

$$\mathrm{Ext}^i(M, N) = H^i(\mathrm{Hom}(P, N)).$$

This is independent of the choice of P, because different choices of P are homotopy equivalent

Example 2.4.1. Consider the modules $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ over $\mathbb{Z}$. We have the free resolution

$$P = \cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z} \to 0 \to \cdots$$

of $\mathbb{Z}/6\mathbb{Z}$. Hence

$$\operatorname{Hom}(P, \mathbb{Z}/8\mathbb{Z}) = \cdots \to 0 \to \mathbb{Z}/8\mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z}/8\mathbb{Z} \to 0 \to \cdots .$$

The Ext groups become

$$\operatorname{Ext}^0(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}) = \ker(\cdot 6) = \{0, 4\} \cong \mathbb{Z}/2\mathbb{Z}$$

and

$$\operatorname{Ext}^1(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}) = (\mathbb{Z}/8\mathbb{Z})/\operatorname{im}(\cdot 6) = (\mathbb{Z}/8\mathbb{Z})/\{0, 2, 4, 6\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Let $\mu\colon M \to M'$ be a homomorphism. If P and P' are projective resolutions of M and M', then μ lifts to a chain map

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} & & \downarrow & \\ \cdots & \longrightarrow & P_2' & \longrightarrow & P_1' & \longrightarrow & P_0' & \longrightarrow & 0 & \longrightarrow \cdots . \end{array}$$

Applying the functor $\operatorname{Hom}(-, N)$ gives a chain map

$$\begin{array}{ccccccc} \cdots \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}(P_0', N) & \longrightarrow & \operatorname{Hom}(P_1', N) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}(P_0, N) & \longrightarrow & \operatorname{Hom}(P_1, N) & \longrightarrow \cdots \end{array}$$

Taking H^i gives an induced homomorphism $\operatorname{Ext}^i(M', N) \to Ext^i(M, N)$ which is denoted $\operatorname{Ext}^i(\mu, N)$. It is well defined because π is unique up to homotopy. Ext^i becomes a functor in the first variable.

Example 2.4.2. Consider the canonical homomorphism $\mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}$. There is an induced chain map of free resolutions

$$\begin{array}{ccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 12} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow & & \downarrow{\scriptstyle \cdot 2} & & \downarrow{\scriptstyle \cdot 1} & & \downarrow & \\ \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\cdot 6]{} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

Applying the functor $\operatorname{Hom}(-, \mathbb{Z}/8\mathbb{Z})$ gives

$$\begin{array}{ccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/8\mathbb{Z} & \xrightarrow{\cdot 6} & \mathbb{Z}/8\mathbb{Z} & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow & & \downarrow{\scriptstyle \cdot 1} & & \downarrow{\scriptstyle \cdot 2} & & \downarrow & \\ \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/8\mathbb{Z} & \xrightarrow[\cdot 12]{} & \mathbb{Z}/8\mathbb{Z} & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

The induced maps on

$$\operatorname{Ext}^i(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}) \to \operatorname{Ext}^i(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/8\mathbb{Z})$$

for $i = 0, 1$ are both the canonical injection $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$.

If

$$\nu \colon N \to N'$$

is a homomorphism, then there is an induced homomorphism

$$\operatorname{Hom}(M, \nu)\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N')$$

given by $\mu \mapsto \nu \circ \mu$. In other words, Hom is also a functor in its second variable.

In fact, Hom is even a *bifunctor*: If $\mu \colon M \to M'$ is also a homomorphism, then there is a commutative square of induced homomorphisms

$$\begin{array}{ccc} \operatorname{Hom}(M', N) & \longrightarrow & \operatorname{Hom}(M, N) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(M', N') & \longrightarrow & \operatorname{Hom}(M, N') \end{array}$$

Let M have the projective resolution

$$P = \cdots \to P_2 \to P_1 \to P_0 \to 0 \to \cdots .$$

Then $\nu \colon N \to N'$ induces a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}(P_0, N) & \longrightarrow & \operatorname{Hom}(P_1, N) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}(P_0, N') & \longrightarrow & \operatorname{Hom}(P_1, N') & \longrightarrow & \cdots \end{array}$$

Taking H^i gives an induced homomorphism $\operatorname{Ext}^i(M, N) \to \operatorname{Ext}^i(M, N')$ which is denoted $\operatorname{Ext}^i(M, \nu)$. So Ext^i becomes a functor in the second variable.

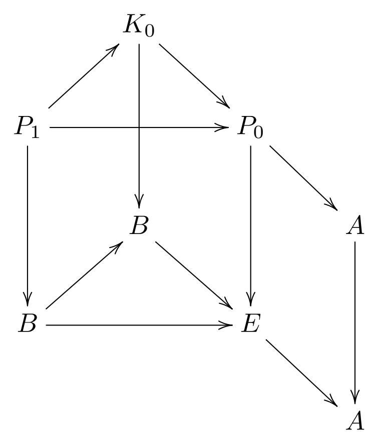
In fact, it is not hard to verify that $\operatorname{Ext}^i(-, -)$ is a bifunctor.

Definition 2.4.3. Extensions $0 \to B \to E \to A \to 0$ and $0 \to B \to E' \to A \to 0$ are called *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B' & \longrightarrow & E' & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

Theorem 2.4.4. *Given left-R-modules A and B, there is a bijection between the set of equivalence classes of extensions $0 \to B \to E \to A \to 0$ and $\operatorname{Ext}^1(A, B)$.*

Proof. (*Sketch*):
The bijection comes as follows: Given an extension $0 \to B \to E \to A \to 0$. Let $P_1 \to P_0 \to A \to 0$ be an exact sequence where P_1 and P_0 are the first terms of a projective resolution of A. We can lift as follows, using that P_0 and P_1 are projective:



Now $P_1 \to B$ represents an element of $\mathrm{Ext}^1(A, B)$. □

Chapter 3

The Derived Category

This chapter gives a sketch of the derived category. Details are coming up in parts **??** and **??**.

3.1 Abstract Categories

Let R be a ring. We can consider the collection of all left-R-modules. For each pair M and N of modules, there is the set $\mathrm{Hom}_R(M,N)$ of R-homomorphisms $M \to N$. Composing homomorphisms give new homomorphisms. Abstracting:

Definition 3.1.1. A *category* $\mathcal{C}$ is a collection of objects and, for each pair of objects x and y, a set $\mathrm{Hom}_{\mathcal{C}}(x,y)$ of morphisms. For each object x, there is an identity morphism id_x in $\mathrm{Hom}_{\mathcal{C}}(x,x)$. There is also a rule for composing morphisms,

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \times \mathrm{Hom}_{\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathcal{C}}(x,z), \quad (f,g) \mapsto g \circ f$$

and these data satisfy

$$(h \circ g) \circ f = h \circ (g \circ f), \quad id_y \circ f = f, \quad g \circ id_x = g.$$

Example 3.1.2.

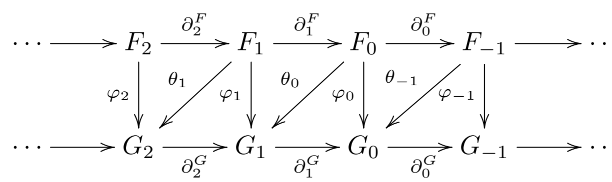
- As we saw, when R is a ring, there is a category $\mathrm{Mod}(R)$ (or ${}_R\mathrm{Mod}$) whose objects are the left-R-modules and whose morphisms are the R-homomorphisms.

- There is a category Top of topological spaces. The objects are the topological spaces and the morphisms are the continuous maps.

- Let R be a ring. There is a category $C(R)$ (of ${}_R\mathrm{Comp}$) of complexes of left-R-modules. The objects are the chain complexes of left-R-modules and the morphisms are the chain maps.

3.2 The Homotopy Category

Definition 3.2.1. The chain map $\varphi\colon F \to G$ is *null homotopic* if there is a chain homotopy θ for which $\varphi = \partial\theta + \theta\partial$, i.e., $\varphi_i = \partial^G_{i+1}\theta_i + \theta_{i-1}\partial^G_i$ for each i.



Definition 3.2.2. Set

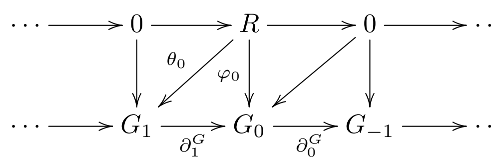
$$\operatorname{Hom}_{K(R)}(F,G) = \operatorname{Hom}_{C(R)}(F,G)/\{\text{null homotopic chain maps}\}.$$

It is easy to check that if φ is null homotopic, so is any composition $\psi \circ \varphi$ and $\varphi \circ \chi$, so composition of chain maps induces a well defined composition

$$\operatorname{Hom}_{K(R)}(F,G) \times \operatorname{Hom}_{K(R)}(G,H) \to \operatorname{Hom}_{K(R)}(F,H).$$

Hence there is a category $K(R)$ whose objects are the complexes and whose morphism sets are the $\operatorname{Hom}_{K(R)}(F,G)$. This is known as the *homotopy category of complexes over* R.

Example 3.2.3. Consider R as a complex concentrated in degree 0 and let G be a complex.

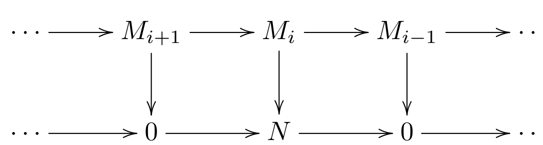


Since R is free with generator 1, the homomorphism φ_0 is determined by its value on 1. Since φ is a chain map, $\varphi_0(1)$ must be in $\ker \partial^G_o$.

The φ for which it is possible to find a chain homotopy θ are precisely the ones where $\varphi_0(1)$ is in $\operatorname{im} \partial^G_1$. So

$$\operatorname{Hom}_{K(R)}(R,G) = \ker \partial^G_0 / \operatorname{im} \partial^G_1 = H_0(G).$$

Example 3.2.4. Let N be a module and view N as a complex concentrated in degree 0. There is an obvious operation of shifting complexes one step to the left; let us denote it by Σ (for technical reasons we will also let Σ change the sign of the differential). Then $\Sigma^i N$ is N viewed as a complex concentrated in (homological) degree i. Let M be a complex and consider morphisms $M \to \Sigma^i N$.



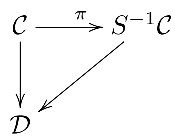
An argument similar to the previous one shows

$$\operatorname{Hom}_{K(R)}(M, \Sigma^i N) \cong H^i \operatorname{Hom}_R(M,N).$$

3.3 Localisation

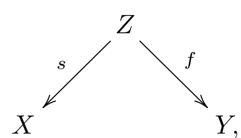
Definition 3.3.1. An *isomorphism* f in a category $\mathcal{C}$ is a morphism with an inverse g such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Definition 3.3.2. Let $\mathcal{C}$ be a category with a class S of morphisms. A *localisation* of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$ and a functor $\pi\colon \mathcal{C} \to S^{-1}\mathcal{C}$ with the following universal property: If $\mathcal{C} \to \mathcal{D}$ is a functor sending the elements of S to isomorphisms, then there is a unique functor $S^{-1}\mathcal{C} \to \mathcal{D}$ making the diagram commutative.

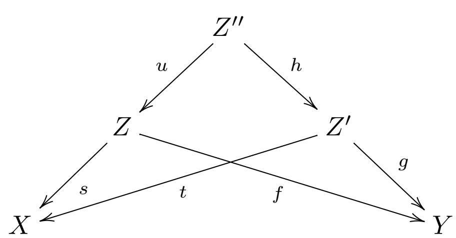


Example 3.3.3. If S itself consists of isomorphisms, then $S^{-1}\mathcal{C} = \mathcal{C}$ and $\pi = \mathrm{id}_{\mathcal{C}}$ satisfy the universal property, so the localisation does not change $\mathcal{C}$.

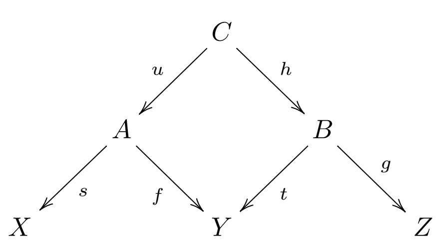
If S is a so-called *multiplicative set*, satisfying axioms reminiscent of the Ore conditions from ring theory, then $S^{-1}\mathcal{C}$ can be constructed as follows.
The objects are the same as in $\mathcal{C}$. The morphisms from X to Y are equivalence classes of diagrams of the form



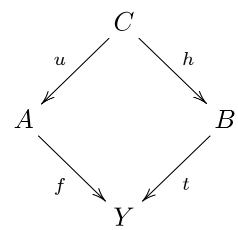
where s is in S. The diagram represents $f \circ s^{-1}$.
The equivalence relation is that diagrams representing fs^{-1} and gt^{-1} are equivalent when they embed in a bigger commutative diagram as follows:



where $su = th$ is in S.
Morphisms fs^{-1} and gt^{-1} are composed as follows.



The existence of the commutative square

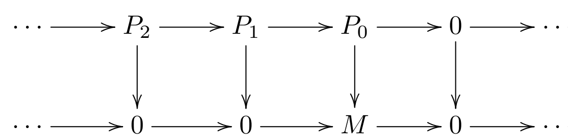


is one of the axioms of multiplicative sets.
Recall that a chain map $\varphi\colon F \to G$ induces homomorphisms of homology groups $H_i(\varphi)\colon H_i(F) \to H_i(G)$. It is an easy computation to check that if φ is null homotopic, then $H_i(\varphi) = 0$ for each i.
If f is in $\mathrm{Hom}_{K(R)}(F, G)$, then pick a chain map φ in the homotopy class f and set $H_i(f) = H_i(\varphi)$. By the above, this is well defined.

Definition 3.3.4. The homotopy class f is called a *quasi-isomorphism* if each $H_i(f)$ is bijective.

Definition 3.3.5. Let S be the class of quasi-isomorphisms in $K(R)$. The localisation $S^{-1}K(R)$ is called the *derived category* of R. It is denoted $D(R)$.

Example 3.3.6. Let P be a projective resolution of the module M. We can view M as a complex concentrated in degree 0 and construct a chain map as follows.



The homomorphism $P_0 \to M$ is just the one which exists since P is a projective resolution of M.
It is not hard to verify that the homotopy class of this chain map is a quasi-isomorphism. Hence it becomes an isomorphism in the derived category. In the derived category, a module is identified with its projective resolution, up to isomorphism.

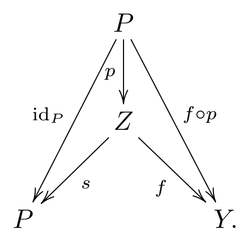
3.4 Morphisms

Definition 3.4.1. A complex P of left-R-modules is called *K-projective* if $\mathrm{Hom}_{K(R)}(P, -)$ sends quasi-isomorphisms to bijections.

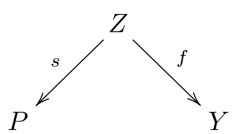
Theorem 3.4.2. *If P is the projective resolution of a left-R-module, then P is K-projective. More generally, if P is a right-bounded complex of projective left-R-modules, then P is K-projective.*

Proof. If P is a K-projective complex of left-R-modules and $Z \xrightarrow{s} P$ is a quasi-isomorphism in $K(R)$, then the induced map $\mathrm{Hom}_{K(R)}(P, Z) \to \mathrm{Hom}_{K(R)}(P, P)$ is bijective. Hence there exists an element p in $\mathrm{Hom}_{K(R)}(P, Z)$ which maps to id_P in $\mathrm{Hom}_{K(R)}(P, P)$. □

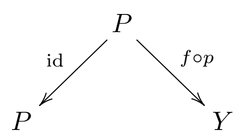
Unravelling this, there is a morphism $P \xrightarrow{p} Z$ such that $s \circ p = \mathrm{id}_P$. If $Z \xrightarrow{f} Y$ is also given, then there is a commutative diagram



Hence the two diagrams



and



are equivalent, so represent the same morphism in $D(R)$. This shows that when mapping out of P, it is not necessary to invert s.

Elaborating the argument gives a proof of the following.

Theorem 3.4.3. *Let P be a K-projective complex of left-R-modules. For any complex Y of left-R-modules, there is an isomorphism*

$$\mathrm{Hom}_{D(R)}(P, Y) \cong \mathrm{Hom}_{K(R)}(P, Y).$$

Example 3.4.4.

- Recall that when viewing R as a complex concentrated in degree 0, we get $\mathrm{Hom}_{K(R)}(R, Y) \cong H_0(Y)$. But R viewed as a complex concentrated in degree 0 is a projective resolution of the module R, so it is K-projective, so

$$\mathrm{Hom}_{D(R)}(R, Y) \cong \mathrm{Hom}_{K(R)}(R, Y) \cong H_0(Y).$$

- Let M and N be modules, and let P be a projective resolution of M. View M and N as complexes concentrated in degree 0. Recall that there is a quasi-isomorphism $P \to M$. Due to localisation, this becomes an isomorphism in the derived category so

$$\mathrm{Hom}_{D(R)}(M, \Sigma^i N) \cong \mathrm{Hom}_{D(R)}(P, \Sigma^i N) = (*).$$

But P is K-projective, so we get

$$(*) \cong \mathrm{Hom}_{K(R)}(P, \Sigma^i N) = (**).$$

Finally, by 3.2.4 and 2.4,

$$(**) \cong H^i \operatorname{Hom}_R(P, N) = Ext^i_R(M, N).$$

The homomorphisms of the derived category capture the Ext groups of classical homological algebra.

3.5 Example: Upper Triangular Matrices

Recall that over the upper triangular matrix ring $\Lambda = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, any finitely generated right-module is a direct sum of copies of the indecomposable modules

$$P = (0\ k), \quad Y = (k\ k), \quad I = Y/P.$$

We can give a similar description of the derived category $D(\Lambda)$, or rather of the full subcategory $D^f(\Lambda)$ of bounded complexes of finitely generated modules.
It turns out that in $D^f(\Lambda)$, each object is the direct sum of copies of indecomposable objects of the form $\Sigma^i P, \Sigma^i Y$, or $\Sigma^i I$. In other words, each object is the direct sum of complexes which are just P, Y, or I concentrated in a single degree.
The inclusion $P \hookrightarrow Y$ and the surjection $Y \twoheadrightarrow I$ are module homomorphisms. They induce chain maps when the modules are viewed as complexes concentrated in degree zero, and morphisms in $K(\lambda)$ and $D^f(\lambda)$ ensue. Using Σ^i gives morphisms

$$\Sigma^i P \to \Sigma^i Y \quad \text{and} \quad \Sigma^i Y \to \Sigma^i I.$$

The short exact sequence

$$0 \to P \to Y \to I \to 0$$

corresponds to an element in $Ext^1_\Lambda(I, P)$, and we know that this group is isomorphic to $\operatorname{Hom}_{D^f}(\Lambda)(I, \Sigma P)$. So the short exact sequence gives a morphism $I \to \Sigma P$ in $D^f(\Lambda)$, and using Σ^i gives morphisms

$$\Sigma^i I \to \Sigma^{i+1} P.$$

We can organise the morphisms we have found graphically as follows:

$$\begin{array}{ccccccccccccccccccc}
 & \nearrow & \Sigma^{-1}P & \searrow & & \nearrow & \Sigma^{-1}I & \searrow & & \nearrow & Y & \searrow & & \nearrow & \Sigma P & \searrow & & \nearrow & \Sigma I & \searrow & \\
\cdots & & & & \Sigma^{-1}Y & & & & P & & & & I & & & & \Sigma Y & & & & \cdots
\end{array}$$

This is the so-called *Auslander-Reiten* quiver of $D^f(\Lambda)$. It contains all the indecomposable objects of $D^f(\Lambda)$.
It turns out that each morphism between indecomposables is a linear combination of compositions of morphisms in the quiver, and that each composition of two consecutive morphisms in the quiver is zero.
Combined with the knowledge that each object is a direct sum of indecomposable objects, this gives a very precise "picture" of the category $D^f(\Lambda)$.