Study note on K-Theory by M. F. Atiyah

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Introduction

This is my study note on "K-theory" by M. F. Atiyah. I tried to fill in the details while preserving the original treatment.

Caveat: the numbering of theorems, propositions etc. are altered due to the restriction of LaTeX.

The original introduction

These notes are based on the course of lectures I gave at Harvard I the fall of 1964. They constitute a self-contained account of vector bundles and K-theory assuming only the rudiments of point-set topology and linear algebra. One of the features of the treatment is that no use is made of ordinary homology or cohomology theory. In fact rational cohomology is defined in terms of K-theory.

The theory is taken as far as the solution of the Hopf invariant problem and a start is made on the J-homomorphism. In addition to the lecture notes proper two papers of mine published since 1964 have been reproduced at the end. The first, dealing with operations, is a natural supplement to the material in Chapter III. It provides an alternative approach to operations which is less slick but more fundamental than the Grothendieck method of Chapter III and it relates operations and filtration. Actually the lectures deal with compact spaces not cell-complexes and so the skeleton-filtration does not figure in the notes. The second paper provides a new approach to real K-theory and so fills an obvious gap in the lecture notes.

Chapter 1

Vector Bundles

1.1 Basic definitions

We shall develop the theory of *complex* vector bundles only, though much of the elementary theory is the same for real and symplectic bundles. Therefore, by vector space, we shall always understand complex vector space unless otherwise specified.

Let X be a topological space. A *family of vector spaces over* X is a topological space E, together with:

(i) a continuous map $p\colon E \to X$

(ii) a finite dimensional vector space structure on each

$$E_x = p^{-1}(x) \text{ for } x \in X,$$

compatible with the topology on E_x induced from E.

The map p is called the *projection map*, the space E is called the *total space* of the family, the space X is called the *base space* of the family, and if $x \in X$, E is called the fibre over x.

A *section* of a family $p\colon E \to X$ is a continuous map $s\colon X \to E$ such that $ps(x) = x$ for all $x \in X$.

A *homomorphism* from one family $p\colon E \to X$ to another family $q\colon F \to X$ is a continuous map $\varphi\colon E \to F$ such that:

(i) $q\varphi = p$

(ii) for each $x \in X$, $\varphi\colon E_x \to F_x$ is a linear map of vector spaces.

We say that φ is an *isomorphism* if φ is bijective and φ^{-1} is continuous. If there exists an isomorphism between E and F, we say that they are isomorphic.

Example 1.1.1. Let V be a vector space, and let $E = X \times V$, $p\colon E \to X$ be the projection onto the first factor. E is called the *product family* with fibre V. If F is any family which is isomorphic to some product family, F is said to be a *trivial* family.

If Y is a subspace of X , and if E is a family of vector spaces over X with projection p, $p\colon p^{-1}(Y) \to Y$ is clearly a family over Y. We call it the *restriction* of E to Y, and denote it by $E \mid Y$. More generally, if Y is any space, and $f\colon Y \to X$ is a continuous map, then we define the induced family $f^*(p)\colon f^*(E) \to Y$ as follows:

$f^*(E)$ is the subspace of $Y \times E$ consisting of all points (y, e) such that $f(y) = p(e)$, together with the obvious projection maps and vector space structures on the fibres. If $g\colon Z \to Y$, then there is a natural isomorphism $g^*f^*(E) \cong (fg)^*(E)$ given by sending each point of the form (z, e) into the point $(z, g(z), e)$, where $z \in Z, e \in E$. If $f\colon Y \to X$ is an inclusion map, clearly there is an isomorphism $E \mid Y \cong f^*(E)$ given by sending each $e \in E$ into the corresponding $(p(e), e)$.

A family E of vector spaces over X is said to be *locally trivial* if every $x \in X$ possesses a neighbourhood U such that $E \mid U$ is trivial. A locally trivial family will also be called a *vector bundle*. A trivial family will be called a *trivial bundle*. If $f\colon Y \to X$, and if E is a vector bundle over X, it is easy to see that $f^*(E)$ is a vector bundle over Y. We shall call $f^*(E)$ the *induced bundle* in this case.

Example 1.1.2. Let V be a vector space, and let X be its associated projective space. We define $E \subset X \times V$ to be the set of all (x, v) such that $x \in X, v \in V$, and v lies in the line determining x. We leave it to the reader to show that E is actually a vector bundle.

Notice that if E is a vector bundle over X, then $\dim(E_x)$ is a locally constant function on X, and hence is a constant on each connected component of X. If $\dim(E_x)$ is a constant on the whole X, then E is said to have a *dimension*, and the dimension of E is the common number $\dim(E)$ for all x. (Caution: the dimension of E so defined is usually different from the dimension of E as a topological space.)

Since a vector bundle is locally trivial, any section of a vector bundle is locally described by a vector valued function on the base space (See the remark below.)

Remark 1.1.3. Let $p\colon E \to B$ be a (real, for example) vector bundle. It is locally trivial, that is, if you take a small open set $U \subset B$, then you can find an isomorphism $\varphi\colon p^{-1}(U) \cong U \times \mathbb{R}^n$, compatible with p (meaning that $\varphi(\xi) = (p(\xi), \text{something})$).

Now a section $s\colon B \to E$ of p is a map such that $p \circ s = \mathrm{id}_B$. So if you restrict to the small open set U, then $s(b) \in p^{-1}(b) \subset p^{-1}(U)$, therefore $\varphi(s(b)) = (p(s(b)), \text{something}) = (b, \text{something})$. Call the "something" $\sigma(b)$ (it depends on b), then you get a map $\sigma\colon U \to \mathbb{R}^n$ (determined by s). This is what is meant by "locally a section is given by a vector valued function on the base space": the vector valued function is σ.

If E is a vector bundle, we denote by $\Gamma(E)$ the set of all sections of E. Since the set of functions on a space with values in a fixed vector space is itself a vector space, we see that $\Gamma(E)$ is a vector space in a natural way.

Suppose that V, W are vector spaces, and that $E = X \times V, F = X \times W$ are the corresponding product bundles. Then any homomorphism $\varphi\colon E \to F$

determines a map $\Phi\colon X \to \mathrm{Hom}(V, W)$ by the formula $\varphi(x, v) = (x, \Phi(x)v)$. Moreover, if we give $\mathrm{Hom}(V, W)$ its usual topology, then Φ is continuous; conversely, any such continuous map $\Phi\colon X \to \mathrm{Hom}(V, W)$ determines a homomorphism $\varphi\colon E \to F$. (This is most easily seen by taking bases $\{e_i\}$ and $\{f_j\}$ for V and W respectively. Then each $\Phi(x)$ is represented by a matrix $\Phi(x)_{i,j}$, where

$$\Phi(x)e_i = \sum_j \Phi(x)_{i,j} f_j.$$

The continuity of either φ or Φ is equivalent to the continuity of the functions $\Phi_{i,j}$.)

Let $\mathrm{Iso}(V, W) \subset \mathrm{Hom}(V, W)$ be the subspace of all isomorphisms between V and W. Clearly, $\mathrm{Iso}(V, W)$ is an open set in $\mathrm{Hom}(V, W)$. Further, the inverse map $T \mapsto T^{-1}$ gives us a continuous map $\mathrm{Iso}(V, W) \to \mathrm{Iso}(W, V)$. Suppose that $\varphi\colon E \to F$ is such that $\varphi_x\colon E_x \to F_x$ is an isomorphism for all $x \in X$. This is equivalent to the statement that $\Phi(X) \subset \mathrm{Iso}(V, W)$. The map $x \mapsto \Phi(x)^{-1}$ defines $\Psi\colon X \to \mathrm{Iso}(W, V)$, which is continuous. Thus the corresponding map $\psi\colon F \to E$ is continuous. Thus $\varphi\colon E \to F$ is an isomorphism if and only if it is bijective or, equivalently, φ is an isomorphism if and only if each φ_x is an isomorphism. Further, since $\mathrm{Iso}(V, W)$ is open in $\mathrm{Hom}(V, W)$, we see that for any homomorphism φ, the set of those points $x \in X$ for which φ_x is an isomorphism form an open subset of X. All these assertions are local in nature, and therefore are valid for vector bundles as well as for trivial families.

Remark 1.1.4. The finite dimensionality of V is basic to the previous argument. If one wants to consider infinite dimensional vector bundles, then one must distinguish between the different operator topologies on $\mathrm{Hom}(V, W)$.

1.2 Operations on vector bundles

Natural operations on vector spaces, such as direct sum and tensor product, can be extended to vector bundles. The only troublesome question is how one should topologise the resulting spaces. We shall give a general method for extending operations from vector spaces to vector bundles which will handle all of these problems uniformly.

Let T be a functor which carries finite dimensional vector spaces into finite dimensional vector spaces. For simplicity, we assume that T is a covariant functor of one variable. Thus, to every vector space V, we have an associated vector space $T(V)$. We shall say that T is a *continuous functor* if for all V and W, the map $T\colon \mathrm{Hom}(V, W) \to \mathrm{Hom}(T(V), T(W))$ is continuous.

If E is a vector bundle, we define the set $T(E)$ to be the union

$$\cup_{x \in X} T(E_x),$$

and, if $\varphi\colon E \to F$, we define $T(\varphi)\colon T(E) \to T(F)$ by the maps $T(\varphi_x)\colon T(E_x) \to T(F_x)$. What we must show is that $T(E)$ has a natural topology, and that, in this topology, $T(\varphi)$ is continuous.

We begin by defining $T(E)$ in the case that E is a product bundle. If $E = X \times V$, we define $T(E)$ to be $X \times T(V)$ in the product topology. Suppose that $F = X \times W$, and that $\varphi\colon E \to F$ is a homomorphism. Let $\Phi\colon X \to \operatorname{Hom}(V, W)$ be the corresponding map. Since, by hypothesis, $T\colon \operatorname{Hom}(V, W) \to \operatorname{Hom}(T(V), T(W))$ is continuous, $T\Phi\colon X \to \operatorname{Hom}(T(V), T(W))$ is continuous. Thus $T(\varphi)\colon X \times T(V) \to X \times T(W)$ is also continuous. If φ is an isomorphism, then $T\varphi$ will be an isomorphism since it is continuous and an isomorphism on each fibre.

Now suppose that E is trivial, but has no preferred product structure. Choose an isomorphism $\alpha\colon E \to X \times V$, and topologise $T(E)$ by requiring $T(\alpha)\colon T(E) \to X \times T(V)$ to be a homeomorphism. If $\beta\colon E \to X \times W$ is any other isomorphism, by letting $\varphi = \beta\alpha^{-1}$ above, we see that $T(\alpha)$ and $T(\beta)$ induce the same topology on $T(E)$, since $T(\varphi) = T(\beta)T(\alpha)^{-l}$ is a homeomorphism. Thus, the topology on E does not depend on the choice of α. Further, if $Y \subset X$, it is clear that the topology on $T(E) \mid Y$ is the same as that on $T(E \mid Y)$. Finally, if $\varphi\colon E \to F$ is a homomorphism of trivial bundles, we see that $T(\varphi)\colon T(E) \to T(F)$ is continuous, and therefore is a homomorphism.

Now suppose that E is any vector bundle. Then if $U \subset X$ is such that $E \mid U$ is trivial, we topologise $T(E \mid U)$ as above. We topologise $T(E)$ by taking for the open sets, those subsets $V \subset T(E)$ such that $V \cap (T(E) \mid U)$ is open in $T(E \mid U)$ for all open $U \subset X$ for which $E \mid U$ is trivial. The reader can now easily verify that if $Y \subset X$, the topology on $T(E \mid Y)$ is the same as that on $T(E) \mid Y$, and that, if $\varphi\colon E \to F$ is any homomorphism, $T(\varphi)\colon T(E) \to T(F)$ is also a homomorphism.

If $f\colon Y \to X$ is a continuous map and E is a vector bundle over X then, for any continuous functor T, we have a natural isomorphism

$$f^*T(E) \cong Tf^*(E).$$

The case when T has several variables both covariant and contravariant, proceeds similarly. Therefore we can define for vector bundles E, F corresponding bundles:

(i) $E \oplus F$, their direct sum

(ii) $E \otimes F$, their tensor product

(iii) $\operatorname{Hom}(E, F)$

(iv) E^*, the dual bundle of E

(v) $\lambda^i(E)$, where λ^i is the i^{th} exterior power.

We also obtain natural isomorphisms

(i) $E \oplus F \cong F \oplus E$

(ii) $E \otimes F \cong F \otimes E$

(iii) $E \otimes (F' \oplus F'') \cong (E \otimes F') \oplus (E \otimes F'')$

(iv) $\mathrm{Hom}(E, F) \cong E^* \otimes F$

(v) $\lambda^k(E \oplus F) \cong \oplus_{i+j=k}(\lambda^i(E) \otimes \lambda^j(F))$.

Finally, notice that sections of $\mathrm{Hom}(E, F)$ correspond in a $1-1$ fashion with homomorphisms $\varphi\colon E \to F$. We therefore define $\mathrm{HOM}(E, F)$ to be the vector space of all homomorphisms from E to F, and make the identification $\mathrm{HOM}(E, F) = \Gamma(\mathrm{Hom}(E, F))$.

1.3 Sub-bundles and quotient bundles

Let E be a vector bundle. A *sub-bundle* of E is a subset of E which is a bundle in the induced structure.

A homomorphism $\varphi\colon F \to E$ is called a *monomorphism* (respectively *epimorphism*) if each $\varphi_x\colon F_x \to E_x$ is a monomorphism (respectively epimorphism). Notice that $\varphi\colon F \to E$ is a monomorphism if and only if $\varphi^*\colon E^* \to F^*$ is an epimorphism. If F is a sub-bundle of E, and if $\varphi\colon F \to E$ is the inclusion map, then φ is a monomorphism.

Lemma 1.3.1. *If $\varphi\colon F \to E$ is, a monomorphism, then $\varphi(F)$ is a sub-bundle of E, and $\varphi\colon F \to \varphi(F)$ is an isomorphism.*

Proof. $\varphi\colon F \to \varphi(F)$ is a bijection, so if $\varphi(F)$ is a subbundle, φ is an isomorphism. Thus we need only show that $\varphi(F)$ is a sub-bundle.

The problem is local, so it suffices to consider the case when E and F are product bundles. Let $E = X \times V$ and let $x \in X$; choose $W_x \subset V$ to be a subspace complementary to $\varphi(F_x)$. $G = X \times W_x$ is a sub-bundle of E. Define $\theta\colon F \oplus G \to E$ by $\theta(a \oplus b) = \varphi(a) + i(b)$, where $i\colon G \to E$ is the inclusion. By construction, θ_x is an isomorphism. Thus, there exists an open neighbourhood U of x such that $\theta \mid U$ is an isomorphism. F is a sub-bundle of $F \oplus G$, so $\theta(F) = \varphi(F)$ is a sub-bundle of $\theta(F \oplus G) = E$ on U. □

Notice that in our argument, we have shown more than we have stated. We have shown that if $\varphi\colon F \to E$, then the set of points for which φ_x is a monomorphism form an open set. Also, we have shown that, locally, a sub-bundle is a direct summand. This second fact allows us to define quotient bundles.

Definition 1.3.2. If F is a sub-bundle of E, the *quotient bundle* E/F is the union of all the vector spaces E_x/F_x given the quotient topology.

Since F is locally a direct summand in E, we see that E/F is locally trivial, and thus is a bundle. This justifies the terminology.

1f $\varphi\colon F \to E$ is an arbitrary homomorphism, the function $\dim(\ker \varphi_x)$ need not be constant, or even locally constant.

Definition 1.3.3. $\varphi\colon F \to E$ is said to be a *strict* homomorphism if $\dim(\ker \varphi_x)$ is locally constant.

Proposition 1.3.4. *If $\varphi\colon F \to E$ is strict, then:*

(i) $\ker(\varphi) = \cup_x \ker(\varphi_x)$ *is a sub-bundle of F*

(ii) $\operatorname{im}(\varphi) = \cup_x \operatorname{im}(\varphi_x)$ *is a sub-bundle of E*

(iii) $\operatorname{coker}(\varphi) = \cup_x \operatorname{coker}(\varphi_x)$ *is a bundle in the quotient structure.*

Proof. Notice that (ii) implies (iii). We first prove (ii). The problem is local, so we can assume $F = X \times V$ for some V. Given $x \in X$, we choose $W_x \subset V$ complementary to $\ker(\varphi_x)$ in V. Put $G = X \times W_x$; then φ induces, by composition with the inclusion, a homomorphism, $\psi\colon G \to E$, such that ψ_x is a monomorphism. Thus ψ is a monomorphism in some neighbourhood U of x. Therefore, $\psi(G) \mid U$ is a sub-bundle of $E \mid U$. However, $\psi(G) \subset \varphi(F)$, and since $\dim(\varphi(F_y))$ is constant for all y, and

$$\dim(\psi(G_x) = \dim(\psi(G_x)) = \dim(\varphi(F_x)) = \dim(\varphi(F_y))$$

for all $y \in U, \psi(G) \mid U = \varphi(F) \mid U$. Thus $\varphi(F)$ is a sub-bundle of E.

Finally, we must prove (i). Clearly, $\varphi^*\colon E^* \to F^*$ is strict. Since $F^* \to \operatorname{coker}(\varphi^*)$ is an epimorphism, $(\operatorname{coker}(\varphi^*))^* \to F^*$ is a monomorphism. However, for each x we have a natural commutative diagram

$$\begin{array}{ccc} \ker(\varphi_x) & \longrightarrow & F_x \\ \downarrow & & \downarrow \\ \operatorname{coker}(\varphi_x^*)^* & \longrightarrow & F_x^{**} \end{array}$$

in which the vertical arrows are isomorphisms. Thus $\ker(\varphi) \cong (\operatorname{coker}(\varphi^*))^*$ and so, by Lemma 1.3.1, is a sub-bundle of F. □

Again, we have proved something more than we have stated. Our argument shows that for any $x \in X$, $\dim \varphi_x(F_x) \leq \dim \varphi_y(F_y)$ for all $y \in U$, U some neighbourhood of x. Thus, $\operatorname{rank}(\varphi_x)$ *is an upper semi-continuous function of* x.

Definition 1.3.5. A *projection operator* $P\colon E \to E$ is a homomorphism such that $P^2 = P$.

Notice that $\operatorname{rank}(P_x) + \operatorname{rank}(1 - P_x) = \dim E_x$ so that, since both $\operatorname{rank}(P_x)$ and $\operatorname{rank}(1 - P_x)$ are upper semicontinuous functions of x , they are locally constant. Thus both P and $1 - P$ are strict homomorphisms. Since $\ker(P) = (1 - P)E$, E is the direct sum of the two sub-bundles PE and $(1 - P)E$. Thus any projection operator $P\colon E \to E$ determines a direct sum decomposition $E = (PE) \oplus (1 - P)E)$.

We now consider metrics on vector bundles. We define a functor Herm which assigns to each vector space V the vector space $\operatorname{Herm}(V)$ of all Hermitian forms on V. By the techniques of Section 1.2, this allows us to define a vector bundle $\operatorname{Herm}(E)$ for every bundle E.

Definition 1.3.6. A *metric* on a bundle E is any section $h\colon X \to \mathrm{Herm}(E)$ such that $h(x)$ is positive definite for all $x \in X$. A bundle with a specified metric is called a *Hermitian bundle.*

Suppose that E is a bundle, F is a sub-bundle of E, and that h is a Hermitian metric on E. Then for each $x \in X$ we consider the orthogonal projection $P_x\colon E_x \to F_x$ defined by the metric. This defines a map $P\colon E \to F$ which we shall now check is continuous. The problem being local, we may assume F is trivial, so that we have sections $f_1, \cdots, f_n$ of F giving a basis in each fibre. Then for $v \in F_x$ we have

$$P_x(v) = \sum_i h_x(v, f_i(x)) f_i(x).$$

Since h is continuous this implies that P is continuous. Thus P is a projection operator on E. If $F_x^\perp$ is the subspace of E_x which is orthogonal to F_x under h, we see that $F^\perp = \cup_x F_x^\perp$ is the kernel of P, and thus is a sub-bundle of E, and that $E \cong F \oplus F^\perp$. Thus, a metric provides any sub-bundle with a definite complementary sub-bundle.

Remark 1.3.7. So far, most of our arguments have been of a very general nature, and we could have replaced “continuous” with “algebraic”, “differentiable”, “analytic’, etc. without any trouble. In the next section, our arguments become less general.

1.4 Vector bundles on compact spaces

In order to proceed further, we must make some restriction on the sort of base spaces which we consider. We shall assume from now on that our base spaces are *compact Hausdorff.* We leave it to the reader to notice which results hold for more general base spaces.

Recall that if $f\colon X \to V$ is a continuous vector-valued function, the support of f (written $\mathrm{supp}.f$) is the closure of $f^{-1}(V \setminus \{0\})$.

We need the following results from point set topology. We state them in vector forms which are clearly equivalent to the usual forms

Theorem 1.4.1 (Tietze Extension)**.** *Let X be a normal space, $Y \subset X$ a closed subspace, V a real vector space, and $f\colon Y \to V$ a continuous map. Then there exists a continuous map $g\colon X \to V$ such that $g \mid Y = f$.*

Theorem 1.4.2 (Existence of Partitions of Unity)**.** *Let X be a compact Hausdorff space, $\{U_i$ a finite open covering. Then there exist continuous maps $f_i\colon X \to \mathbb{R}$ such that:*

1. $f_i(x) \geq 0 \quad \forall x \in X$,
2. $\mathrm{supp}(f_i) \subset U_i$,
3. $\sum_i f_i(x) = 1 \quad \forall x \in X$.

Such a collection $\{f_i\}$ *is called a* partition of unity*.*

We first give a bundle form of the Tietze extension theorem.

Lemma 1.4.3. *Let* X *be compact Hausdorff,* $Y \subset X$ *be a closed subspace, and* E *be a bundle over* X*. Then any section* $s\colon Y \to E \mid Y$ *can be extended to* X*.*

Proof. Let $s \in \Gamma(E \mid Y)$. Since, locally, s is a vector-valued function, we can apply the Tietze extension theorem to show that for each $x \in X$, there exists an open set U containing x and $t \in \Gamma(E \mid U)$ such that $t \mid U \cap Y = s \mid U \cap Y$. Since X is compact, we can find a finite sub-cover $\{U_\alpha\}$ by such open sets. Let $t_\alpha \in \Gamma(E \mid U_\alpha)$ be the corresponding sections and let $\{p_\alpha\}$ be a partition of unity with $\operatorname{supp}(p_\alpha) \subset U_\alpha$. We define $S_\alpha \in \Gamma(E)$ by

$$S_\alpha = \begin{cases} p_\alpha(x) t_\alpha(x) & \text{if} \quad x \in U_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum S_\alpha$ is a section of E and its restriction to Y is clearly s. □

Lemma 1.4.4. *Let* Y *be a closed subspace of a compact Hausdorff space* X*, and let* E, F *be two vector bundles over* X*. If* $f\colon E \mid Y \to F \mid Y$ *is an isomorphism, then there exists an open set* U *containing* Y *and an extension* $f\colon E \mid U \to F \mid U$ *which is an isomorphism.*

Proof. f is a section of $\operatorname{Hom}(E \mid Y, F \mid Y)$, and thus, extends to a section of $\operatorname{Hom}(E, F)$. Let U be the set of those points for which this map is an isomorphism. Then U is open and contains Y. □

Lemma 1.4.5. *Let* Y *be a compact Hausdorff space,* $f_t\colon Y \to X$ $(0 \leq t \leq 1)$ *be a homotopy and* E *be a vector bundle over* X*. Then*

$$f_0^* E \cong f_1^* E.$$

Proof. Let I denote the unit interval and let $f\colon Y \times I \to X$ be the homotopy: $f(y, t) = f_t(y)$. Let $\pi\colon Y \times I \to Y$ denote the standard projection. Now apply Lemma 1.4.4 to the bundles $f^*E, \pi^* f_t^* E$ and the subspace $Y \times \{t\}$ of $Y \times I$, on which there is an obvious isomorphism s. By the compactness of Y we deduce that f^*E and $\pi^* f_t^* E$ are isomorphic in some strip $Y \times \delta t$ where δt denotes a neighbourhood of $\{t\}$ in I. Hence the isomorphism class of $f_t^* E$ is a locally constant function of t. Since I is connected this implies it is constant, whence

$$f_0^* E \cong f_1^* E.$$

□

We shall use $\operatorname{Vect}(X)$ to denote the set of isomorphism classes of vector bundles on X, and $\operatorname{Vect}_n(X)$ to denote the subset of $\operatorname{Vect}(X)$ given by bundles of dimension n. $\operatorname{Vect}(X)$ is an abelian semi-group under the operation $\oplus$. In $\operatorname{Vect}(X)$ we have one naturally distinguished element - the class of the trivial bundle of dimension n.

Lemma 1.4.6. 1. *If $f\colon X \to Y$ is a homotopy equivalence, then $f^*\colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is bijective.*

2. *If X is contractible, every bundle over X is trivial and $\mathrm{Vect}(X)$ is isomorphic to the non-negative integers.*

Lemma 1.4.7. *If E is a bundle over $X \times I$, and $\pi\colon X \times I \to X \times \{0\}$ is the projection, E is isomorphic to $\pi^*(E \mid X \times \{0\})$.*

Both these lemmas are immediate consequences of Lemma 1.4.5.

Suppose now Y is closed in X , E is a vector bundle over X and $\alpha\colon E \mid Y \to Y \times V$ is an isomorphism. We refer to α as a *trivialisation of E over Y*. Let $\pi\colon Y \times V \to V$ denote the projection and define an equivalence relation on $E \mid Y$ by

$$e \sim e' \Leftrightarrow \pi\alpha(e) = \pi\alpha(e').$$

We extend this by the identity on $E \mid (X\backslash Y)$ and we let E/α denote the quotient space of E given by this equivalence relation. It has a natural structure of a family of vector spaces over X/Y. We assert that E/α is in fact a vector bundle. To see this we have only to verify at the base point Y/Y of X/Y. Now by Lemma 1.4.4 we can extend α to an isomorphism $\widetilde{\alpha}\colon E \mid U \to U \times V$ for some open set U containing Y. Then $\widetilde{\alpha}$ induces an isomorphism

$$(E \mid U)/\alpha \cong (U/Y) \times V$$

which establishes the local triviality of E/α.

Suppose α_0, α_1 are homotopic trivialisations of E over Y. This means that we have a trivialisation β of $E \times I$ over $Y \times I \subset X \times I$ inducing α_0 and α_1 at the two end points of I. Let $f\colon (X \mid Y) \times I \to (X \times I)/(Y \times I)$ be the natural map. Then $f^*(E \times I/\beta)$ is a bundle on $(X/V) \times I$ whose restriction to $(X/Y) \times \{i\}$ is $E/\alpha_i (i = 0, 1)$. Hence by Lemma 1.4.5

$$E/\alpha_0 \cong E/\alpha_1.$$

To summarise we have established

Lemma 1.4.8. *A trivialisation α of a bundle E over $Y \subset X$ defines a bundle E/α over X/Y. The isomorphism class of E/α depends only on the homotopy class of α.*

Using this we shall now prove

Lemma 1.4.9. *Let $Y \subset X$ be a closed contractible subspace. Then $f\colon X \to X/Y$ induces a bijection $f^*\colon \mathrm{Vect}(X/Y) \to \mathrm{Vect}(X)$.*

Proof. Let E be a bundle on X. Then by Lemma 1.4.6 $E \mid Y$ is trivial. Thus trivialisations $\alpha\colon E \mid Y \to Y \times V$ exist. Moreover, two such trivialisations differ by an automorphism of $Y \times V$, i.e., by a map $Y \to \mathrm{GL}(V)$. But $\mathrm{GL}(V) = \mathrm{GL}(n, \mathbb{C})$ is connected and V is contractible. Thus α is unique up to homotopy and so the isomorphism class of $E \mid \alpha$ is uniquely determined by that of E. Thus we have constructed a map

$$\mathrm{Vect}(X) \to \mathrm{Vect}(X/Y)$$

and this is clearly a two-sided inverse for f^*. Hence f^* is bijective as asserted. □

Vector bundles are frequently constructed by a glueing or clutching construction which we shall now describe. Let

$$X = X_1 \cup X_2, \quad A = X_1 \cap X_2,$$

all the spaces being compact. Assume that E_i is a vector bundle over X_i and that $\varphi\colon E_1 \mid A \to E_2 \mid A$ is an isomorphism. Then we define the vector bundle $E_1 \cup_\varphi E_2$ on X as follows. As a topological space $E_1 \cup_\varphi E_2$ is the quotient of the disjoint sum $E_1 + E_2$ by the equivalence relation which identifies $e_1 \in E_1 \mid A$ with $\varphi(e_1) \in E_2 \mid A$. Identifying X with the corresponding quotient of $X_l + X_2$ we obtain a natural projection $p\colon E_1 \cup_\varphi E_2 \to X$, and $p^{-1}(x)$ has a natural vector space structure. It remains to show that $E_1 \cup_\varphi E_2$ is locally trivial. Since

$$(E_1 \cup_\varphi E_{2)} \mid (X \setminus A) = E_1 \mid (X_1 \setminus A) + E_2 \mid (X_2 \setminus A)$$

the local triviality at points $x \notin A$ follows from that of E_1 and E_2. Therefore, let $a \in A$ and let V_I be a closed neighbourhood of a in X_l over which E_1 is trivial, so that we have an isomorphism

$$\theta_1\colon E_1 \mid V_1 \to V_1 \times \mathbb{C}^n.$$

Restricting to A we get an isomorphism

$$\theta_1^A\colon E_1 \mid (V_1 \cap A) \to (V_1 \cap A) \times \mathbb{C}^n.$$

Let

$$\theta_2^A\colon E_2 \mid (V_1 \cap A) \to (V_1 \cap A) \times \mathbb{C}^n.$$

be the isomorphism corresponding to θ_1^A under φ. By Lemma 1.4.4 this can be extended to an isomorphism

$$\theta_2\colon E_2 \mid V_2 \to V_2 \times \mathbb{C}^n.$$

where V_2 is a neighbourhood of a in X_2. The pair θ_1, θ_2 then defines in an obvious way an isomorphism

$$\theta_1 \cup_\varphi \theta_2\colon (E_1 \cup_\varphi E_2) \mid (V_1 \cup V_2) \to (V_1 \cup V_2) \times \mathbb{C}^n.$$

establishing the local triviality of $E_1 \cup_\varphi E_2$. Elementary properties of this construction are. the following:

1. If E is a bundle over X and $E_i = E \mid X_i$, then the identity defines an isomorphism $I_A \colon E_1 \mid A \to E_2 \mid A$, and
$$E_1 \cup_{I_A} E_2 \cong E.$$
2. If $\beta_i \colon E_i \to E_i'$ are isomorphisms on X_i and $\varphi' \beta_1 = \beta_2 \varphi$, then
$$E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'.$$
3. If (E_i, φ) and (E_i', φ') are two "clutching data" on the X_i, then
$$(E_1 \cup_\varphi E_2) \oplus (E_1' \cup_{\varphi'} E_2') \cong (E_1 \oplus E_1') \cup_{\varphi \oplus \varphi'} (E_2 \oplus E_2'),$$
$$(E_1 \cup_\varphi E_2) \otimes (E_1' \cup_{\varphi'} E_2') \cong (E_1 \otimes E_1') \cup_{\varphi \otimes \varphi'} (E_2 \otimes E_2'),$$
$$(E_1 \cup_\varphi E_2)^* \cong E_1^* \cup_{(\varphi^*)^{-1}} E_2^*.$$

Moreover, we also have

Lemma 1.4.10. *The isomorphism class of $E_1 \cup_\varphi E_2$ depends only on the homotopy class of the isomorphism $\varphi \colon E_1 \mid A \to E_2 \mid A$.*

Proof. A homotopy of isomorphisms $E_1 \mid A \to E_2 \mid A$ means an isomorphism
$$\Phi \colon (\pi^* E_1 \mid A) \times I \to (\pi^* E_2 \mid A) \times I,$$
where I is the unit interval and $\pi \colon X \times I \to X$ is the projection. Let
$$f_t \colon X \to X \times I$$
be defined by $f_t(x) = x \times \{t\}$ and denote by
$$\varphi_t \colon E_1 \mid A \to E_2 \mid A$$
the isomorphism induced from Φ by f_t. Then
$$E_1 \cup_{\varphi_t} E_2 \cong f_t^*(\pi^* E_1 \cup_\Phi \pi^* E_2).$$
Since f_0 and f_1 are homotopic, it follows from Lemma 1.4.5 that
$$E_1 \cup_{\varphi_0} E_2 \cong E_1 \cup_{\varphi_1} E_2$$
as required. □

Remark 1.4.11. The "collapsing" and "clutching" constructions for bundles (on X/Y and $X_l \cup X_2$ respectively) are both special cases of a general process of forming bundles over quotient spaces. We leave it as an exercise to the reader to give a precise general formulation.

We shall denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$.

Lemma 1.4.12. *For any X, there is a natural isomorphism $\mathrm{Vect}_n(S(X) \cong [X, \mathrm{GL}(n, \mathbb{C})]$ where $S(X)$ denotes the suspension of X.*

Proof. Write $S(X)$ as $C^+(X) \cup C^-(X)$, where $C^+(X) = ([0,1/2] \times X)/\{0\} \times X$, $C^-(X) = ([1/2,1] \times X)/\{1\} \times X$. Then $C^+(X) \cap C^-(X) = X$. If E is any n-dimensional bundle over $S(X)$, $E \mid C^+(X)$ and $E \mid C^-(X)$ are trivial since $C^+(X)$ and $C^-(X)$ are contractible. Let $\alpha^\pm \colon E \mid C^\pm(X) \to C^\pm(X) \times V$ be such isomorphisms. Then $(\alpha^+ \mid X)(\alpha^- \mid X)^{-1} \colon X \times V \to X \times V$ is a bundle map, and thus defines a map α of X into $\mathrm{GL}(n, \mathbb{C}) = \mathrm{Iso}(V)$. Since both $C^+(X)$ and $C^-(X)$ are contractible, the homotopy classes of both α^+ and α^- are well defined, and thus the homotopy class of α is well defined. Thus we have a natural map $\theta \colon \mathrm{Vect}_n(S(X)) \to [X, \mathrm{GL}(n, \mathbb{C}]$. The clutching construction on the other hand defines by Lemma 1.4.10 a map

$$\varphi \colon [X, \mathrm{GL}(n, \mathbb{C})] \to \mathrm{Vect}_n(S(X)).$$

It is clear that θ and φ are inverses of each other and so are bijections. □

We have just seen that $\mathrm{Vect}_n(S(X))$ has a homotopy theoretic interpretation. We now give a similar interpretation to $\mathrm{Vect}_n(X)$. First we must establish some simple facts about quotient bundles.

Lemma 1.4.13. *Let E be any bundle over X. Then there exists a (Hermitian) metric on E.*

Proof. A metric on a vector space V defines a metric on the product bundle $X \times V$. Hence metrics exist on trivial bundles. Let $\{U_\alpha\}$ be a finite open covering of X such that $E \mid U_\alpha$ is trivial and let h_α be a metric for $E \mid U_\alpha$. Let $\{P_\alpha$ be a partition of unity with $\operatorname{supp} p_\alpha \subset U_\alpha$ and define

$$k_\alpha(x) = \begin{cases} p_\alpha h_\alpha(x) & \text{for } x \in U_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then k_α is a section of $\mathrm{Herm}(E)$ and is positive semi-definite. But for any $x \in X$ there exists α such that $p_\alpha(x) > 0$ (since $\sum_\alpha p_\alpha = 1$) and so $x \in U_\alpha$. Hence, for this α, $k_\alpha(x)$ is positive definite. Hence $\sum_\alpha k_{\alpha(x)}$ is positive definite for all $x \in X$ and so $k = \sum k_\alpha$ is a metric for E. □

Definition 1.4.14. A sequence of vector bundle homomorphisms

$$\to E \to F \to \cdots$$

is said to be *exact* if for each $x \in X$ the sequence of vectorÂ· space morphisms

$$\to E_x \to F_x \to \cdots$$

is exact.

Corollary 1.4.15. *Let*

$$0 \to E' \xrightarrow{\varphi'} E \xrightarrow{\varphi''} E'' \to 0$$

be an exact sequence of bundles over X. Then there exists an isomorphism $E \cong E' \oplus E''$.

Proof. Give E a metric. Then $E \cong E' \oplus (E')^{\perp}$. However, $(E')^{\perp} \cong E''$. $\square$

Definition 1.4.16. A subspace $V \subset \Gamma(E)$ is said to be *ample* if

$$\varphi\colon X \times V \to E$$

is a surjection, where $\varphi(x, s) = s(x)$.

Lemma 1.4.17. *If E is any bundle over a compact Hausdorff space X, then $\Gamma(E)$ contains a finite dimensional ample subspace.*

Proof. Let $\{U\}$ be a finite open covering of X such that $E \mid U_\alpha$ is trivial for each α, and let $\{p_\alpha$ be a partition of unity with $\operatorname{supp} p_\alpha \subset U_\alpha$. Since $E \mid U_\alpha$ is trivial we can find a finite dimensional ample subspace $V_\alpha \subset \Gamma(E \mid U_\alpha)$. Now define

$$\theta_\alpha\colon V_\alpha \to \Gamma(E)$$

by

$$\theta_\alpha v_\alpha(x) = \begin{cases} p_\alpha \cdot v_\alpha(x) & \text{if } x \in U_\alpha \\ 0 & \text{otherwise} \end{cases}$$

The θ_α define a homomorphism

$$\theta\colon \prod_\alpha V_\alpha \to \Gamma(E)$$

and the image of θ is a finite dimensional Subspace of $\Gamma(E)$; in fact, for each $x \in X$ there exists α with $p_\alpha(x) > 0$ and so the map

$$\theta_\alpha(V_\alpha) \to E_x$$

is surjective. $\square$

Corollary 1.4.18. *If E is any bundle, there exists an epimorphism $\varphi\colon X \times \mathbb{C}^m \to E$ for some integer m.*

Corollary 1.4.19. *If E is any bundle, there exists a bundle F such that $E \oplus F$ is trivial.*

We are now in a position to prove the existence of a homotopy theoretic definition for $\mathrm{Vect}_n(X)$. We first introduce Grassmann manifolds. If V is any vector space, and n any integer, the set $G_n(V)$ is the set of all subspaces of V of codimension n. If V is given some Hermitian metric, each subspace of V determines a projection operator. This defines a map $G_n(V) \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the set of endomorphisms of V. We give $G_n(V)$ the topology induced by this map.

Suppose that E is a bundle over a space X, V is a vector space, and $\varphi\colon X \times V \to E$ is an epimorphism. If we map X into $G_n(V)$ by assigning to x the subspace $\ker(\varphi_x)$, this map is continuous for any metric on V (here $n = \dim(E)$). We call the map $X \to G_n(V)$ the map induced by φ.

Let V be a vector space, and let $F \subset G_n(V)\times V$ be the sub-bundle consisting of all points (g, v) such that $v \in g$. Then, if $E = (G_n(V)\times V)/F$ is the quotient bundle, E is called the ***classifying bundle*** over $G_n(V)$.

Notice that if E' is a bundle over X, and $\varphi\colon X\times V \to E'$ is an epimorphism, then if $f\colon X \to G_n(V)$ is the map induced by φ, we have $E' \cong f^*(E)$, where E is the classifying bundle.

Suppose that h is a metric on V. We denote by $G_n(V_h)$ the set $G_n(V)$ with the topology induced by h. If h' is another metric on V, then the epimorphism $G_n(V_h)\times V \to E$ (where E is the classifying bundle) induces the identity map $G_n(V_h) \to G_n(V_{h'})$ Thus the identity map is continuous. Thus, the topology on $G_n(V)$ does not depend on the metric.

Now consider the natural projections

$$\mathbb{C}^m \to \mathbb{C}^{m-1}$$

given by $(z_1, \cdots, z_m) \mapsto (z_l, \cdots, z_{m-l})$. These induce continuous maps

$$\iota_{m-1}\colon G_n(\mathbb{C}^m) \to G_n(\mathbb{C}^{m-1}).$$

If $E_{(m)}$ denotes the classifying bundle over $G_n(\mathbb{C}^m)$ it is immediate that

$$\iota^*_{m-1}(E_m) \cong E_{(m-l)}.$$

Theorem 1.4.20. *The map*

$$\varinjlim_m [X, G_n(\mathbb{C}^m)] \to \mathrm{Vect}_n(X)$$

induced by $f \mapsto f*(E_{(m)})$ for $f\colon X \to G_n(\mathbb{C}^m)$, is an isomorphism for all compact Hausdorff spaces X.

Proof. We shall construct an inverse map. If E is a bundle over X, there exists (by 1.4.18) an epimorphism $\varphi\colon X\times\mathbb{C}^m \to E$. Let $f\colon X \to G_n(\mathbb{C}^m)$ be the map induced by φ. If we can show that the homotopy class of f (in $G_n(V^{m'})$) for m' sufficiently large does not depend on the choice of φ, then we construct our inverse map $\mathrm{Vect}_n(X) \to \varinjlim_m [X, G_n(V^m)]$ by sending E to the homotopy class of f.

Suppose that $\varphi_i\colon X\times\mathbb{C}^{m_i} \to E$ are two epimorphisms $(i = 0, 1)$. Let $g_i\colon X \to G_n(\mathbb{C}^{m_i})$ be the map induced by φ_1. Define $\psi_t\colon X\times\mathbb{C}^{m_0}\times\mathbb{C}^{m_1} \to E$ by $\psi_t(x, v_0, v_1) = (1-t)\varphi_0(x, v_0) + t\varphi_l(x, v_1)$. This is an epimorphism. Let $f_t\colon X \to G_n(\mathbb{C}^{m_0}\oplus\mathbb{C}^{m_1})$ be the map induced by ψ_t. If we identify $\mathbb{C}^{m_0}\oplus\mathbb{C}^{m_1}$ with $\mathbb{C}^{m_0}\times\mathbb{C}^{m_1}$ by

$$(z_1, \cdots, z_{m_0}) \oplus (u_1, \cdots, u_{m_1}) \mapsto (z_1, \cdots, z_{m_0}, \cdots, u_1, \cdots u_{m_1})$$

then we have

$$f_0 = j_0 g_0, \quad f_1 = T j_1 g_1$$

where $j_i\colon G_n(\mathbb{C}^{m_i}) \to G_n(\mathbb{C}^{m_0+m_1})$ is the natural inclusion and

$$T\colon G_n(\mathbb{C}^{m_0+m_1}) \to G_n(\mathbb{C}^{m_0+m_1})$$

is the map induced by a permutation of coordinates in $\mathbb{C}^{m_0+m_1}$, and so is homotopic to the identity. Hence j_1g_1 is homotopic to f_1 and hence to j_0g_0 as required. □

Remark 1.4.21. It is possible to interpret vector bundles as modules in the following way. Let $C(X)$ denote the ring of continuous complex-valued functions on X. If E is a vector bundle over X then $\Gamma(E)$ is a $C(X)$ - module under pointwise multiplication, i. e. ,

$$fs(x) = f(x)s(x) \quad f \in C(X), s \in \Gamma(E).$$

Moreover a homomorphism $\varphi\colon E \to F$ determines a $C(X)$ -module homomorphism

$$\Gamma_\varphi\colon \Gamma(C) \to \Gamma(F).$$

Thus Γ is a functor from the category $\mathscr{V}$ of vector bundles over X to the category $\mathscr{M}$ of $C(X)$ -modules. If E is trivial of dimension n. then $\Gamma(E)$ is free of rank n. If F is also trivial then

$$\Gamma\colon\ \mathrm{HOM}(E,F) \to \mathrm{Hom}_{C(X)}(\Gamma(E),\Gamma(F))$$

is bijective. In fact, choosing isomorphisms $E \cong X \times V, F \cong X \times W$ we have

$$\begin{aligned}\mathrm{HOM}(E,F) \cong \mathrm{Hom}_{\mathbb{C}}(V,W)^X &\cong C(X) \otimes \mathrm{Hom}_{\mathbb{C}}(V,W)\\ &\cong \mathrm{Hom}_{C(X)}(\Gamma(E),\Gamma(F)).\end{aligned}$$

Thus Γ induces an equivalence between the category $\mathscr{T}$ of trivial vector bundles to the category $\mathscr{F}$ of free $C(X)$-modules of finite rank. Let $\mathrm{Proj}(\mathscr{T})$ denote the sub-category of $\mathscr{V}$ whose objects are images of projection operators in $\mathscr{T}$, and let $\mathrm{Proj}(\mathscr{F}) \subset \mathscr{M}$ be defined similarly. Then it follows at once that Γ induces an equivalence of categories

$$\mathrm{Proj}(\mathscr{T}) \to \mathrm{Proj}(\mathscr{F}).$$

But, by (1.4.19), $\mathrm{Proj}(\mathscr{T}) = \mathscr{V}$. By definition $\mathrm{Proj}(\mathscr{F})$ is the category of finitely-generated projective $C(X)$-modules. Thus we have established the following:

Proposition 1.4.22. Γ *induces an equivalence between the category of vector bundles over X and the category of finitely generated projective modules over $C(X)$.*

1.5 Additional structures

In linear algebra one frequently considers vector spaces with some additional structure, and we can do the same for vector bundles. For example we have already discussed hermitian metrics. The next most obvious example is to consider non-degenerate bilinear forms. Thus if V is a vector bundle a non-degenerate bilinear form on V means an element T of $\mathrm{HOM}(V \otimes V, 1)$ which induces a non-degenerate element of $\mathrm{Hom}(V \otimes V, 1)$ for all $x \in X$. Equivalently T may be regarded as an element of $\mathrm{ISO}(V, V^*)$. The vector bundle V together with this isomorphism T will be called a *self-dual* bundle.

If T is symmetric, i. e., if T_x is symmetric for all $s \in X$, we shall call (V, T) an *orthogonal* bundle. If T is skew-symmetric, i, e,, if T_x is skew-symmetric for all $x \in X$, we shall call (V, T) a *symplectic* bundle.

Alternatively we may consider pairs (V, T) with $T \in \mathrm{ISO}(V, \overline{V})$ where $\overline{V}$ denotes the *complex conjugate bundle* of V (obtained by applying the "complex conjugate functor" to V). Such a (V, T) may be called a *self-conjugate* bundle. The isomorphism T may also be thought of as an anti-linear isomorphism $V \to V$. As such we may form T^2. If $T^2 = \mathrm{id}$ we may call (V, T) a *real* bundle. In fact the subspace $W \subset V$ consisting of all $v \in V$ with $Tv = v$ has the structure of a *real vector bundle* and V may be identified with $W \otimes_{\mathbb{R}} \mathbb{C}$, *complexification* of W. If $T^2 = -\mathrm{id}$ then we may call (V, T) a *quaternion* bundle. In fact, we can define a quaternion vector space structure on each V_x by putting $j(v) = Tv$ (the quaternions are generated over $\mathbb{R}$ by i, j with $ij = -ji, i^2 = j^2 = -1$).

Now if V has a hermitian metric h then this gives an isomorphism $\overline{V} \to V^*$ and hence turns a self-conjugate bundle into a self-dual one. We leave it as an exercise to the reader to examine in detail the symmetric[1] and skew-symmetric cases and to show that, up to homotopy, the notions of self-conjugate, orthogonal, symplectic, are essentially equivalent to self-dual, real, quaternion. Thus we may pick which ever alternative is more convenient at any particular stage. For example, the result of the preceding sections extend immediately to real and quaternion vector bundles although the extension of (1.4.5) for example to orthogonal or symplectic bundles is not so immediate. On the other hand the properties of tensor products are more conveniently dealt with in the framework of bilinear forms. Thus the tensor product of (V, T) and (W, S) is $(V \otimes W, T \otimes S)$ and the symmetry properties of $T \otimes S$ follow at once from those of T and S. Note in particular that the tensor product of two symplectic bundles is orthogonal.

A self-conjugate bundle is a special case of a much more general notion. Let F, G be two continuous functors on vector spaces. Then by an $F \to G$ bundle we will mean a pair (V, T) where V is a vector bundle and $T \in \mathrm{ISO}(F(V), G(V))$. Obviously a self-conjugate bundle arises by taking $F = \mathrm{id}, G = *$.
Another example of some importance is to take F and G to be multiplication by a fixed integer m , i. e.,

$$F(V) = G(V) = V \otimes V \otimes \cdots \otimes V \quad (m \text{ times}).$$

Thus an $m \to m$ bundle (or more briefly an m-bundle) is a pair (V, T) where $T \in \mathrm{Aut}(mV)$. The m-bundle (V, T) is *trivial* if there exists $S \in \mathrm{Aut}(V)$ so that $T = mS$.

In general for $F \to G$ bundles the analogue of (1.4.5) does not hold, i. e., homotopy does not imply isomorphism. Thus the good notion of equivalence must incorporate homotopy. For example, two m-bundles (V_0, T_0) and (V_1, T_1)

[1]The point is that $\mathrm{GL}(n, \mathbb{R})$ and $(n, \mathbb{C})$ have the same maximal compact subgroup $(n, \mathbb{R})$. Similar remarks apply in the skew case.

will be called *equivalent* if there is an m-bundle (W, S) on $X \times I$ so that

$$(V_1, T_i) \cong (W, S) \mid X \times i, \quad i = 0, 1.$$

Remark 1.5.1. An m-bundle over K should be thought of as a "mod m vector bundle" over $S(X)$.

1.6 G-bundles over G-spaces

Suppose that G is a topological group. Then by a *G-space* we mean a topological space X together with a given continuous action of G on X, i. e., G acts on X and the map $G \times X \to X$ is continuous. A *G-map* between G-spaces is a map commuting with the action of G. A G-space E is a G-vector bundle over the G-space X if

(i) E is a vector bundle over X,

(ii) the projection $E \to X$ is a G-map,

(iii) for each $g \in G$ the map $E_x \to E_{g(x)}$ is a vector space homomorphism.

If G is the group of one element then of course every space is a G-space and every vector bundle is a G-vector bundle. At the other extreme if X is a point then X is a G-space for all G and a G-vector bundle over X is just a (finite-dimensional) representation space of G. Thus G-vector bundles form a natural generalisation including both ordinary vector bundles and G-modules. Much of the theory of vector bundles over compact spaces generalises to G-vector bundles provided G is also compact. This however, presupposes the basic facts about representations of compact groups. For the present, therefore we restrict ourselves to *finite groups* where no questions of analysis are involved.

There are two extreme kinds of G-space:

(i) X is a free G-space if $g \neq 1 \Rightarrow g(x) \neq x$,

(ii) X is a trivial G-space if $g(x) = x$ for all $x \in X, g \in G$.

We shall examine the structure of G-vector bundles in these two extreme cases.

Suppose then that X is a free G-space and let X/G be the orbit space. Then $\pi \colon X \to X/G$ is a finite covering map. Let E be a G-vector bundle ewer X. Then E is necessarily a free G-space. The orbit space E/G has a natural vector bundle structure over X/G: in fact $E/G \to X/G$ is locally isomorphic to $E \to X$ and hence the local triviality of E implies that of E/G. Conversely, suppose V is a vector bundle over X/G. Then π^*V is a G-vector bundle over X; in fact, $\pi^*V \subset X \times V$ and G acts on $X \times V$ by $g(x, v) = (g(x), v)$. It is clear that $E \to E/G$ and $V \to \pi^*V$ are inverse functors. Thus we have

Proposition 1.6.1. *If X is G-free, G-vector bundles over X correspond bijectively to vector bundles over X/G by $E \to E/G$.*

Before discussing trivial G-spaces let us recall the basic feet about representations of finite groups, namely that there exists a finite set $V_1, \cdots, V_k$ of irreducible representations of G so that any representation V of G is isomorphic to a unique direct $\sum_{i=1}^{k} n_i V_i$ (Maschke's theorem).

We shall give a brief treatment of Maschke's theorem as follows.
Let U, V be G-modules over a field k and α be a mapping from U to V. We shall write $\alpha\colon U \to_k V, \alpha\colon U \to_G V$ to indicate that α is k-linear or a G-homomorphism respectively. Th|e space of all k-linear mappings from U to V is denoted by $\mathrm{Hom}_k(U, V)$ and the subspace of G-homomorphisms by $\mathrm{Hom}_G(U, V)$.

In the next lemma we shall (exceptionally) write mappings between right G-modules on the right, so that for $\alpha\colon U \to_k V$ the condition for a G-homomorphism is that

$$(ux)\alpha = (u\alpha)x \quad \text{for all } u \in U, x \in G.$$

Lemma 1.6.2 (Averaging lemma)**.** *Let G be a finite group and k a field of characteristic 0 or prime to $|G|$. Given any two G-modules U, V and $\alpha\colon U \to_k V$, the mapping*

$$\alpha'\colon u \mapsto |G|^{-1} \sum_x ((ux^{-1})\alpha)x \tag{1.6.3}$$

is a G-homomorphism from U to V. Moreover,

(i) if α is a G-homomorphism, then $\alpha^* = \alpha$,

(ii) if $\alpha\colon U \to_k V, \beta\colon V \to_G W$, then $(\alpha\beta)^* = \alpha^*\beta$,

(iii) if $\alpha\colon U \to_G V, \beta\colon V \to_k W$, then $(\alpha\beta)^* = \alpha\beta^*$.

Proof. Let us fix $a \in G$ and write $y = xa, x = ya^{-1}$. Then as one of x, y runs over G, so does the other. Now for $\alpha\colon U \to_k V$ we have

$$|G|.ua^*a = \sum_x ux^{-1}\alpha xa = \sum_y uay^{-1}\alpha y = |G|.ua\alpha^*. \tag{1.6.4}$$

This shows α^* to be a G-homomorphism. If α is a G-homomorphism, each term in the sum in 1.6.4 is $u\alpha a = ua\alpha$, so $\alpha^* = \alpha$ in this case and (i) follows. Now let $\beta\colon V \to_k V$; then

$$|G|.u(\alpha\beta)^* = \sum_x ux^{-1}\alpha\beta x = \sum_x ux^{-1}\alpha x\beta = |G|.u\alpha^*\beta.$$

Hence (ii follows; (iii) is proved similarly. □

We note that if neither α nor β is a G-homomorphism, there is nothing we can say. We can now prove the module form of Maschke's theorem, which states that every module extension splits, or equivalently, that the group algebra kG is semisimple.

Theorem 1.6.5 (Maschke's theorem)**.** *Let G be a finite group and k a field of characteristic 0 or prime to $|G|$. Then kG is semisimple.*

Proof. We shall show that every (finite-dimensional) G-module is semisimple, or equivalently, that every short exact sequence of G-modules

$$0 \to V' \xrightarrow{\alpha} V \xrightarrow{\beta} V'' \to 0. \tag{1.6.6}$$

splits. Such a sequence certainly splits as a sequence of k-spaces, for this just means that V' as k-subspace of V has a vector space complement. Thus we have a k-linear splitting map $\gamma\colon V \to V'$. We have $\alpha\gamma = 1_V$; therefore $1 = 1^* = (\alpha\gamma)^* = \alpha\gamma^*$, and so γ^* is the desired G-homomorphism splitting the sequence 1.6.6. □

Now for any two G-modules (i. e., representation spaces) V, W we can define the vector space $\mathrm{Hom}_G(V, W)$ of G-homomorphisms. Then we have

$$\mathrm{Hom}_G(V_i, V_j) \begin{cases} = 0 & i \neq j \\ \cong \mathbb{C} & i = j. \end{cases}$$

Hence for any V it follows that the natural map

$$\sum V_i \otimes \mathrm{Hom}_g(V_i, V) \to V$$

is a G-isomorphism. In this form we can extend the result to G-bundles over a trivial G-space. In fact, if E is any G-bundle over the trivial G-space X we can define the homomorphism $\mathrm{Av} \in \mathrm{End}\, E$ by

$$\mathrm{Av}(e) = \frac{1}{|G|} \sum_{g \in G} g(e) \quad e \in E$$

where $|G|$ denotes the order of G (This depends on the Averaging lemma 1.6.2 which states that, X being G-trivial, each $g \in G$ defines an endomorphism of E). It is immediate that Av is a projection operator for E and so its image, the invariant subspace of E, is a vector bundle. We denote this by E^G and call it the *invariant sub-bundle* of E. Thus if E, F are two G-bundles then $\mathrm{Hom}_G(E, F) = (\mathrm{Hom}(E, F))^G$ is again a vector bundle. In particular taking E to be the trivial bundle $\mathbb{V}_i = X \times V_i$ with its natural G-action we can consider the natural bundle map

$$\sum_{i=1}^{k} \mathbb{V}_i \otimes \mathrm{Hom}_G(V_i, F) \to F.$$

We have already observed that for a G-module F this is a G-isomorphism. In other words for any G-bundle F over X this is a G-isomorphism for all $\in X$. Hence it is an isomorphism of G-bundles. Thus every G-bundle F is isomorphic to a G-bundle of the form $\sum \mathbb{V}_i \otimes E_i$ where E_i is a vector bundle with trivial

G-action. Moreover the E_i are unique up to isomorphism. In fact we have

$$\begin{aligned}\operatorname{Hom}_G(\mathbb{V}_i, F) &\cong \sum_{i=1}^{k} \operatorname{Hom}_G(\mathbb{V}_i, \mathbb{V}_j \otimes E_j)\\ &\cong \sum_{i=1}^{k} \operatorname{Hom}_G(\mathbb{V}_i, \mathbb{V}_j) \otimes E_j\\ &\cong E_j.\end{aligned}$$

Thus we have established

Proposition 1.6.7. *Let X be a trivial G-space, $V_1, \dots, V_k$ a complete set of irreducible G-modules, $\mathbb{V}_i = X \times V_i$ the corresponding G-bundles. Then every G-bundle F over X is isomorphic to a direct sum $\sum_{i=1}^{k} \mathbb{V}_i \otimes E_i$ where the E_i are vector bundles with trivial G-action. Moreover the E_i are unique up to isomorphism and are given by $E_i = \operatorname{Hom}_G(\mathbb{V}_i, F)$.*

We return now to the case of a general (compact) G-space X and we shall show how to extend the results of 1.4 to G-bundles. Observe first that, if E is a G-bundle, G acts naturally on $\Gamma(E)$ by

$$(gs)(x) = g(s(g^{-1}(x))) \quad s \in \Gamma(E).$$

A section s is invariant if $gs = g$ for all $g \in G$. The set of all invariant sections forms a subspace $\Gamma(E)^G$ of $\Gamma(E)$. The averaging operator

$$\operatorname{Av} = \frac{1}{|G|} \sum g$$

defines as usual a homomorphism $\Gamma(E) \to \Gamma(E)^G$ which is the identity on $\Gamma(E)^g$.

Lemma 1.6.8. *Let X be a compact G-space $Y \subset X$ a closed sub G-space (i.e., invariant under the action of G) and let E be a G-bundle over X. Then any invariant section $s \colon Y \to E \mid Y$ extends to an invariant section over X .*

Proof. By 1.4.3 we can extend s to some section t of E over X. Then $\operatorname{Av}(t)$ is an invariant section of E over X, while over Y we have

$$\operatorname{Av}(t) = \operatorname{Av}(s) = s$$

since s is invariant. Thus $\operatorname{Av}(t)$ is the required extension. □

If E, F are two G-bundles then $\operatorname{Hom}(E, F)$ is also a G-bundle and we have

$$\Gamma(\operatorname{Hom}(E, F))^G \cong \operatorname{HOM}_G(E, F).$$

Hence the G-analogues of 1.4.4 and 1.4.5 follow at once from 1.6.8. Thus we have

Lemma 1.6.9. *Let Y be a compact G-space, X be a G-space, $f_t \colon Y \to X (0 \leq t \leq 1)$ be a G-homotopy and E be a G-vector bundle over X. Then $f_0^* E$ and $f_1^* E$ are isomorphic G-bundles.*

A G-homotopy means of course a G-map $F \colon Y \times I \to X$ where I is the unit interval with trivial G-action. A G-space is G-contractible if it is G-homotopy equivalent to a point. In particular, the cone over a G-space is always G-contractible. By a *trivial* G-bundle we shall mean a G-bundle isomorphic to a product $X \times V$ where V is a G-module. With these definitions 1.4.6 - 1.4.15 extend without change to G-bundles. We have only to observe that if h is a metric for E then $\mathrm{Av}(h)$ is an invariant metric .

To extend 1.4.17 we observe that if $V \subset \Gamma(E)$ is ample then $\sum_{g \in G} gV \subset \Gamma(E)$ is ample and invariant. This leads at once to the appropriate extension of 1.4.19.

In extending 1.4.20 we have to consider Grassmannians of G-subspaces of $m \sum_{i=1}^{k} V_i$ for $m \to \infty$, where as before $V_1, \cdots, V_k$ denote a complete set of irreducible G-modules. We leave the formulation to the reader.

Finally,consider the module interpretation of vector bundles. Write $A = C(X)$. Then if X is a G-space with G acting on A as a group of algebra automorphisms. If E is a G-vector bundle over X then $\Gamma(E)$ is a projective A-module and G acts on $\Gamma(E)$, the relation between the $A-$ and $G-$ actions being

$$g(as) = g(a)g(s) \quad a \in A, g \in G, s \in \Gamma(E).$$

We can look at this another way if we introduce the "twisted group algebra" B of G over A, namely elements of B are linear combinations $\sum_{g \in G} a_g g$ with $a \in A$ and the product is defined by

$$(ag)(a'g') = (ag(a'))gg'.$$

In fact, $\Gamma(E)$ is then just a B-module. We leave it as an exercise to the reader to show that the category of G-vector bundles over X is equivalent to the category of B-modules which are finitely generated and projective over A.

Chapter 2

K-Theory

2.1 Definitions

If X is any space, the set Vect (X) is defined by the structure of an abelian semigroup, where the additive structure is defined by direct sum. If A is any abelian semigroup, we can associate to A an abelian group $K(A)$ with the following property: there is a semigroup homomorphism $\alpha\colon A \to K(A)$ such that if G is any group, $\gamma\colon A \to G$ any semigroup homomorphism, there is a unique homomorphism $\chi\colon K(A) \to G$ such that $\gamma = \chi\alpha$. If such a $K(A)$ exists, it must be unique.

The group $K(A)$ is defined in the usual fashion. Let $F(A)$ be the free abelian group generated by the elements of A, let $E(A)$ be the subgroup of $F(A)$ generated by those elements of the form $a + a' - (a \oplus a')$, where $\oplus$ is the addition in A, $a, a' \in A$. Then $K(A) = F(A)/E(A)$ has the universal property described above, with $\alpha\colon A \to K(A)$ being the obvious map.

A slightly different construction of $K(A)$ which is sometimes convenient is the following. Let $\Delta\colon A \to A \times A$ be the diagonal homomorphism of semigroups, and let $K(A)$ denote the set of cosets of $\Delta(A)$ in $A \times A$. It is a quotient semi-group, but the interchange of factors in $A \times A$ induces an inverse in $K(A)$ so that $K(A)$ is a group. We then define $\alpha_A\colon A \to K(A)$ to be the composition of $a \mapsto (a, 0)$ with the natural projection $A \times A \to K(A)$ (we assume A has a zero for simplicity). The pair $(K(A), \alpha_A)$ is a functor of A so that if $\gamma\colon A \to B$ is a semi-group homomorphism we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & K(A) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle K(\gamma)} \\ B & \xrightarrow[\alpha_B]{} & K(B) \end{array}$$

If B is a group α_B is an isomorphism. That shows $K(A)$ has the required universal property.

If A is also a semi-ring (that is, A possesses a multiplication which is distributive over the addition of A) then $K(A)$ is clearly a ring.

If X is a space, we write $K(X)$ for the ring $K(\mathrm{Vect}(X))$. No confusion should result from this notation. If $E \in \mathrm{Vect}(X)$, we shall write $[E]$ for the image of E in $K(X)$. Eventually, to avoid excessive notation, we may simply write E instead of $[E]$ when there is no danger of confusion.

Using our second construction of K it follows that, if X is a space, every element of $K(X)$ is of the form $[E] - [F]$, where E, F are bundles over X. Let G be a bundle such that $F \in G$ is trivial. We write $\mathbf{n}$ for the trivial bundle of dimension n. Let $F \oplus G = \mathbf{n}$. Then $[E] - [F] = [E] + [G] - ([F] + [G]) = [E \oplus G] - [\mathbf{n}]$. Thus, every element of $K(X)$ is of the form $[H] - [\mathbf{n}]$.

Suppose that E, F are bundles such that $[E] = [F]$, then again from our second construction of K it follows that there is a bundle G such that $E \oplus G \cong F \oplus G$. Let G' be a bundle such that $G \oplus G' \cong \mathbf{n}$. Then $E \oplus G \oplus G' \cong F \oplus G \oplus G'$, so $E \oplus \mathbf{n} \cong F \oplus \mathbf{n}$. If two bundles become equivalent when a suitable trivial bundle is added to each of them, the bundles are said to be *stably equivalent*. Thus, $[E] = [F]$ if and only if E and F are stably equivalent.

Suppose $f \colon X \to Y$ is a continuous map. Then $f^* \colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ induces a ring homomorphism $f^* \colon K(Y) \to K(X)$. By 1.4.5 this homomorphism depends only on the homotopy class of f.

2.2 The periodicity theorem

The fundamental theorem for K-theory is the periodicity theorem. In its simplest form, it states that for any X, there is an isomorphism between $K(X) \otimes K(S^2)$ and $K(X \times S^2)$. This is a special case of a more general theorem which we shall prove.

If E is a vector bundle over a space X, and if $E_0 = E \setminus X$, where X is considered to lie in E as the zero section, the non-zero complex numbers act on E_0 as a group of fibre preserving automorphisms. Let $P(E)$ be the orbit space obtained from E_0 by the action of the complex number. $P(E)$ is called the *projective bundle associated to* E. If $p \colon P(E) \to X$ is the projection map, $p^{-1}(x)$ is a complex projective space for all $x \in X$. If V is a vector space, and W is a vector space of dimension one, V and $V \otimes W$ are isomorphic, but not naturally isomorphic. For any non-zero element $\omega \in W$ the map $\nu \mapsto \nu \otimes \omega$ defines an isomorphism between V and $V \otimes W$, and thus defines an isomorphism $(\omega) \colon P(V) \to P(V \otimes W)$. However, if ω' is any other non-zero element of W, $\omega' = \lambda\omega$ for some non-zero complex number λ. Thus $P(\omega) = (\omega')$, so the isomorphism between $P(V)$ and $P(V \otimes W)$ is natural. Thus, if E is any vector bundle, and L is a line bundle, there is a natural isomorphism $P(E) \cong P(R \otimes L)$.

If E is a vector bundle over X then each point $a \in P(E)_X = P(E_X)$ represents a one-dimensional subspace $H^*_X \subset E_X$. The union of all these defines a subspace $H^* \subset p^*E$, where $p \colon P(E) \to X$ is the projection. It is easy to check that H^* is a sub-bundle of p^*E. In fact, the problem being local we may assume E is a product and then we are reduced to a special case of the Grassmannian already discussed in §1.4. We have denoted our line-bundle by H^* because we want its dual H (the choice of convention here is dictated by algebro-geometric considerations which we do not discuss here).

We can now state the periodicity theorem.

Theorem 2.2.1. *Let L be a line bundle over X. Then, as a $K(X)$-algebra, $K(P(L \otimes 1))$ is generated by $[H]$, and is subject to the single relation $([H] - [l])([L][H] - [1]) = 0$.*

Before we proceed to the proof of this theorem, we would like to point out two corollaries. Notice that $P(1 \otimes 1) = X \times S^2$.

Corollary 2.2.2. *$K(S^2)$ is generated by $[H]$ as a $K(\text{point})$ module, and $[H]$ is subject to the only single relation $([H] - [1])^2 = 0$.*

Corollary 2.2.3. *IÂ£ X is any space, and if $\mu\colon K(X) \otimes K(S^2) \to K(X \times S^2)$ is defined by $\mu(a \otimes b) = (\pi_1^* a)(\pi_2^* b)$, where π_1, π_2 are the projections onto the two factors, then μ is an isomorphism of rings.*

The proof of the theorem will be broken down into a series of lemmas.

To begin, we notice that for any $x \in X$, there is a natural embedding $L_X \to P(L \oplus 1)$ given by the map $y \mapsto (y, 1)$. This map extends to the one point compactification of L_X, and gives us a homeomorphism of the one point compactification of L_X onto $P(L \oplus 1)_X$. If we map $X \to P(L \oplus 1)$ by sending x to the image of the "point at infinity" of the one point compactification of L, we obtain a section of $P(L \oplus 1)$ which we call the "section at infinity". Similarly, the zero section of L gives us a section of $P(L \oplus 1)$, which we call the *zero section* of $P(L \otimes 1)$.

We choose a metric on L , and we let $S \subset L$ be the unit circle bundle. We write ^{0} for the part of L consisting of vectors of length ≤ 1, and P^∞ for that part of $P(L \oplus 1)$ consisting of the section at infinity, together with all the vectors of length ≥ 1. We denote the projections $S \to X, P^0 \setminus X, P^\infty \setminus X$ by π, π_0, and π_∞ respectively.

Since π_0 and π_0^∞ are homotopy equivalences, every bundle on P^0 is of the form $\pi_0^*(E^0)$ and every bundle on P^∞ is of the form $\pi_\infty^*(E^0)$, where E^0 and E^∞ are bundles on X. Thus, any bundle$E$ on $P(L \oplus 1)$ is isomorphic to one of the form $(\pi_0^*(E^0), f, \pi_\infty^*(E^\infty))$, where $f \in \text{ISO}(\pi^\infty(E^0), \pi^*(E^\infty))$ is a clutching function. Moreover, if we insist that the isomorphism

$$E \to (\pi_0^* E^0, f, \pi_\infty^* E^\infty)$$

coincide with the obvious ones over the zero and infinite sections, it follows that *the homotopy class of f is uniquely determined by the isomorphism class of E.* This again uses the fact that the 0-section is a deformation retract of P^0 and the ∞-section a deformation retract of P^∞. We shall simplify our notation slightly by writing (E^0, f, E^∞) for $(\pi_0^*(E^0), f, \pi_\infty^*(E^\infty))$.

Our proof will now be devoted to showing that the bundles E^0 and E^∞ and the clutching function f can be taken to have a particularly simple form. In the special case that L is trivial, S is just $X \times S$, the projection $S \to S^1$ is a complex-valued function on S which we denote by z (here S^1 is identified with the complex numbers of unit modulus). This allows us to consider functions on S which are finite Laurent series in z whose coefficients are functions on X:

$$\sum_{k=-n}^{n} a_k(x)z^k$$

These finite Laurent series can be used to approximate functions on S in a uniform manner.

If L is not trivial, we have an analogue to finite Laurent series. Here z becomes a section in a bundle rather than a function. Since $\pi^*(L)$ is a subset of $S \times L$, the diagonal map $S \to S \times S \subset S \times L$ gives us a section of $\pi^*(L)$. We denote this section by z. Taking tensor products we obtain, for $k \geq 0$, a section $z - k$ of $(\pi^*(L))^k$, and a section z^{-k} of $(\pi^*(L^*))^k$. We write L^{-k} for $(L^*)^k$. Then, for any $k, k', L^k \otimes L'^k \cong L^{k+k'}$. Suppose that $a_k \in (L^{-k})$. Then$\pi^*(a_k) \otimes z^k \in \Gamma(\pi^*(1))$, and thus $\pi^*(a_k) \otimes z^k$ is a function on S. We write $a_k z^k$ for this function. By a finite Laurent series, we shall understand a sum of functions on S of the form

$$\sum_{k=-n}^{n} a_k z^k$$

where $a_k \in \Gamma(L^{-k})$ for all k.

More generally, if E^0, E^∞ are two vector bundles on X, and $a_k \in \Gamma \operatorname{Hom}(L^k \otimes E^0, E^\infty)$, then if we write $a_k z^k$ for $a_k \otimes z^k$, we see that any finite sum of the form

$$f = \sum_{k=-n}^{n} a_k z^k$$

is an element of $\Gamma(\pi^*(E^0), \pi^*(E^\infty))$. If $f \in \mathrm{ISO}(\pi^*(E^0), \pi^*(E^\infty))$, we call f a *Laurent clutching function* for (E^0, E^∞).

The function z is a clutching function for $(1, L)$. Further, $(1, z, L)$ is just the bundle H^* which we defined earlier. To see this, we first recall that H^* was defined as a sub-bundle of $\pi^*(L \oplus 1)$. For each $y \in P(L \oplus 1)_X, H^*_y$ is a subspace of $(L \oplus 1)_X$, and

$$H^*_\infty = L_X \oplus 0, \quad H^*_0 = 0 \oplus 1_X.$$

Thus, the composition

$$H^* \to \pi^*(L \oplus 1) \to \pi^*(1)$$

induced by the projection $L \oplus 1 \to 1$ defines an isomorphism:

$$f_0 \colon H^* \mid P^0 \to \pi_0^*(1).$$

Likewise, the composition

$$H^* \to \pi^*(L \oplus 1) \to \pi^*(L)$$

induced by the projection $L \oplus 1 \to L$ defines an isomorphism:

$$f_\infty \colon H^\infty \mid P^\infty \to \pi_0^\infty(L).$$

Hence $f = f_\infty f_0^{-1} \colon \pi^*(1) \to \pi^*(L)$ is a clutching function for H^*. Clearly, if $y \in S, f(y)$ is the isomorphism whose graph is H_y^*. Since H_y^* is the subspace of $L_X \oplus 1_X$ spanned by $y \oplus 1 (y \in S_X \subset L_X, 1 \in C)$, we see that f is exactly our section z. Thus

$$H^* \cong (1, z, L).$$

Therefore, for any integer k,

$$H^k \cong (1, z^{-k}, L^{-k}).$$

The next step in our classification of the bundles over P is to show that every clutching function can be taken to be a Laurent clutching function. Suppose that $f \in \Gamma \operatorname{Hom}(\pi^* E^0, \pi^* E^\infty)$ is any section. We define its Fourier coefficients

$$a_k \in \Gamma \operatorname{Hom}(L^k \otimes E^0, E^\infty)$$

by

$$a_k(x) = \frac{1}{2\pi i} \int_{S_x} f_x z_x^{-k-1} dz_x.$$

Here f_X, z_X denote the restrictions of f, z to S_Z, and dz_X is therefore a differential on S_X with coefficients in L_X. Let S_n be the partial sum

$$S_n = \sum_{k=-n}^{n} a_k z^k$$

and define the Cesaro means

$$f_n = \frac{1}{n} \sum_{0}^{n-1} S_k.$$

Then the proof of Fejer's theorem on the $(C, 1)$ summability of Fourier series extends immediately to the present more general case and gives

Lemma 2.2.4. *Let f be any clutching function for (E^0, E^∞), and let f_n be the sequence of Cesaro means of the Fourier series of f. Then f_n converges uniformly to f. Thus, for all large n, f_n is a clutching function for (E^0, E^∞), and $(E^0, f, E^\infty) \cong (E^0, f_n, E^\infty)$.*

Proof. Since $\mathrm{ISO}(E^0, E^\infty$ is an open subset of the vector space $\mathrm{HOM}(E^0, E^\infty)$, there exists an $\epsilon > 0$ such that $g \in \mathrm{ISO}(E^0, E^\infty)$ whenever $|f - g| < \epsilon$, where $||$ denotes the usual sup. norm with respect to fixed metrics in E^0, E^∞.

Since the f_n converge uniformly to f we have $|f - f_n| < \epsilon$ for large n. Thus, for $0 \le t \le 1, |tf + (l - t) f_n| \in \mathrm{ISO}(E^0, E^\infty)$. Thus, f and f_n are homotopic in $\mathrm{ISO}(E^0, E^\infty)$, so $(E^0, f, E^\infty) \cong (E^0, f_n, E^\infty)$. □

Next, consider a polynomial clutching function; that is, one of the form

$$p = \sum_{k=0}^{n} a_k z^k.$$

Consider the homomorphism

$$\mathcal{L}^n(p)\colon \pi^*(\sum_{k=0}^{n})L^k \otimes E^0) \to \pi^*(E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0)$$

given by the matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -z & 1 & & & & \\ & -z & 1 & & & \\ & & \dots & & & \\ & & & \dots & & \\ & & & & -z & 1 \end{bmatrix}$$

It is clear that $\mathcal{L}^n(p)$ is linear in z. Now, define the sequence $p_r(z)$ inductively by

$$p_0 = p, \quad zp_{r+1}(z) = p_r(z) - p_r(0).$$

Then we have the following matrix identity:

$$\mathcal{L}^n(p) = \begin{bmatrix} 1 & P_1 & P_2 & \dots & P_n \\ & 1 & & & \\ & & 1 & & \\ & & & & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} P & & & \\ & 1 & & \\ & & 1 & \\ & & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \dots & & \\ & & & -z & 1 \end{bmatrix}$$

or, more briefly

$$\mathcal{L}^n(p) = (1 + N_1)(p \oplus 1)(1 + N_2)$$

where N_1 and N_2 are nilpotent. If N is nilpotent, $1 + tN$ is nonsingular for $0 \le t \le 1$, so we obtain

Proposition 2.2.5. *$\mathcal{L}^n(p)$ and $p \oplus 1$ define isomorphic bundles on P, i.e.,*

$$\begin{aligned}(E^0, p, E^\infty) \oplus (\sum_{k=1}^{n} L^k \otimes E^0, 1, \sum_{k=1}^{n} L^k \otimes E^0) \\ \cong (\sum_{k=0}^{n} L^k \otimes E^0, \mathcal{L}^n(p), E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0)\end{aligned}$$

Remark 2.2.6. The definition of $\mathcal{L}^n(p)$ is, of course, modelled on the way one passes from an ordinary differential equation of order n to a system of first order equations.

For brevity, we write $\mathcal{L}^n(E^0, p, E^\infty)$ for the bundle

$$(\sum_{k=0}^{n} L^k \otimes E, \mathcal{L}^n(p), E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0).$$

Lemma 2.2.7. *Let p be a polynomial clutching function of degree $\leq n$ for (E^0, E^∞). Then*

(i) $\mathcal{L}^{n+1}(E^0, p, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{n+1} \otimes E^0, 1, L^{n+1} \otimes E^0)$

(ii) $\mathcal{L}^{n+1}(L^{-1} \otimes E^0, zp, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{-1} \otimes E^0, z, E^0)$

Proof. We have

$$\mathcal{L}^{n+1}(p) = \begin{bmatrix} \mathcal{L}^n(p) & & & 0 \\ 0 & \dots & -z & 1 \end{bmatrix}.$$

Multiplying the z on the bottom row by t gives us a homotopy between $\mathcal{L}^{n+1}(p)$ and $\mathcal{L}^n(p) \oplus 1$. This establishes the first part.

Similarly,

$$\mathcal{L}^{n+1}(zp) = \begin{bmatrix} 0 & a_0 & a_1 & \dots & a_n \\ -z & 1 & & & \\ & -z & 1 & & \\ & & -z & & \\ & & & -z & 1 \end{bmatrix}.$$

We multiply the 1 on the second row by t and obtain a homotopy between $\mathcal{L}^{n+1}(p)$ and $\mathcal{L}^n(p) \oplus (-z)$. Since $-z$ is the composition of z with the map -1, and since -1 extends E^0, $(L^{-1} \otimes E^0, -z, E^0) \cong (L^{-1} \otimes E^0, z, E^0)$. The second part is therefore proved. □

We shall now establish a simple algebraic formula in $K(p)$. We write $[E^0, p, E^\infty]$ for $[(E^0, p, E^\infty)]$.

Proposition 2.2.8. *For any polynomial clutching function p for E^0, E^∞, we have the identity*

$$([E^0, p, E^\infty] - [E^0, 1, E^0])([L][H] - [1]) = 0.$$

Proof. From the second part of the last lemma, together with the last proposition, we see that

$$\begin{aligned} (L^{-1} \otimes E^0, zp, E^\infty) \oplus (\sum_{k=0}^{n} L^k \otimes E^0, 1, \sum_{k=0}^{n} L^k \otimes E^0) \\ \cong (E^0, p, E^\infty) \oplus (\sum_{k=1}^{n} L^k \otimes E^0, 1, \sum_{k=1}^{n} L^k \otimes E^0) \\ \oplus (L^{-1} \otimes E^0, z, E^0). \end{aligned}$$

Thus in $K(P)$

$$[L^{-1} \otimes E^0, zp, E^\infty] \oplus [E^0, 1, E^0] = [E^0, p, E^\infty] \oplus [L^{-1} \otimes E^0, z, E^0].$$

Since $[1, z, L] = [H^{-1}]$,

$$[L^{-1}][H^{-1}][E^0, p, E^\infty] \oplus [E^0, 1, E^0] = [E^0, p, E^\infty] \oplus [L^{-1}][H^{-1}][E^0, 1, E^0]$$

In particular, if we put $E^0 = 1, p = z, E^\infty = L$, we obtain the formula

$$([H] - 1)([L][H] - [1]) = 0$$

which is part of our main theorem. □

We now turn our attention to linear clutching functions. First, suppose that T is an endomorphism of a finite dimensional vector space E, and let S be a circle in the complex plane which does not pass through any eigenvalue of T. Then

$$Q = \frac{1}{2\pi i} \int_S (z - T)^{-1} dz$$

is a projection operator in E which commutes with T. The decomposition $E = E_+ \oplus E_-, E_+ = QE, E_- = (1 - Q)E$ is therefore invariant under T, so that T can be written as $T = T_+ \oplus T_-$. Then T_+ has all its eigenvalues inside S, while $_-$ has all its eigenvalues outside S. This is, of course, just the spectral decomposition of T corresponding to the two components of the complement of S.

We shall now extend these results to vector bundles, but first we make a remark on notation. So far z and hence $p(z)$ have been sections over S. However, they extend in a natural way to sections over the whole L. It will also be convenient to include the ∞-section of P in certain statements. Thus, if we assert that $p(z)az + b$ is an isomorphism outside S, we shall take this to include the statement that a is an isomorphism.

Proposition 2.2.9. *Let p be a linear clutching function for (E^0, E^∞), and define endomorphisms Q^0, Q^∞ of (E^0, E^∞) by putting*

$$Q^0_x = \frac{1}{2\pi i} \int_{S_x} p_x^{-1} dp_x \quad Q^\infty_x = \frac{1}{2\pi i} \int_{S_x} dp_x p_x^{-1}$$

Then Q^0 and Q^∞ are projection operators, and

$$pQ^0 = Q^\infty p$$

Write $E^i_+ = Q^i E^i, E^i_- = (1 - Q^i)E^i, i = 0, \infty$, so that $E^i = E^i_+ \oplus E^i_-$. Then p is compatible with these decompositions, so that $= p_+ \oplus p_-$. Moreover, p_+ is an isomorphism outside S, and $p_-$is an isomorphism inside S.

Proof. It suffices to verify these statements at each point $x \in X$. In other words, we may assume that X is a point, $L = \mathbb{C}$, and z is just a complex number. Since $p(z)$ is an isomorphism for $|z| = 1$, we can find a real number $\alpha (> 1)$ such that $p(\alpha)\colon E^0 to E^\infty$ is an isomorphism. For simplicity of computation, we identify E^0 with E^∞ by this isomorphism. Next, we consider the conformal transformation

$$w = \frac{1 - \alpha z}{z - \alpha}$$

which preserves the unit circle and its inside. Substituting for z, we find (since we have taken $p(\alpha) = 1$)

$$p(z) = \frac{w - T}{w + \alpha}$$

where $T \in \operatorname{End}(E^0)$. Hence

$$\begin{aligned} Q^0 &= \frac{1}{2\pi i} \int_{|z|=1} p^{-1} dp \\ &= \frac{1}{2\pi i} \int_{|w|=1} (-(w + \alpha)^{-1} dw + (w - T)^{-1} dw) \\ &= \frac{1}{2\pi i} \int_{|w|=1} (w - T)^{-1} dw \quad \text{since } |\alpha| > 1. \end{aligned}$$

Similarly,

$$Q^\infty = \frac{1}{2\pi i} \int_{|w|=1} (dw)(w - T)^{-1} = Q^0,$$

so our assertions follow from the corresponding statements concerning a linear transformation T. □

Corollary 2.2.10. *Let p be as in 2.2.9, and write*

$$p_+ = a_+ z + b_+, \quad p_- = a_- z + b_-.$$

Then, if $p(t) = p_+(t) \oplus p_-(t)$, where

$$p_+(t) = a_+ z + t b_+, \quad p_-(t) = t a_- z + b_-, \quad 0 \le t \le 1,$$

we obtain a homotopy of linear clutching functions connecting p with $a_+ z \oplus b$. Thus

$$(E^0, p, E^\infty) \cong (E^0_+, z, L \otimes E^0_+) \oplus (E^0_-, 1, E^0_-).$$

Proof. The last part of the last lemma implies that $p_+(t)$ and $p_-(t)$ are isomorphisms on S for $0 \le t \le 1$. Thus, $p(t)$ is a clutching function for $0 \le t \le 1$. Thus,

$$\begin{aligned} (E^0, p, E^\infty) &\cong (E^0, p(1), E^\infty) \\ &\cong (E^0_+, a_+ z, E^\infty_+) \oplus (E^0_-, b, E^\infty_-). \end{aligned}$$

Since $a_+ : L \otimes E^0_+ \to E^\infty_+, b_- : E^0_- \to E^\infty_-$ are necessarily isomorphisms, we see that

$$(E^0_+, a_+ z, E^\infty_+) \cong (E^0_+, z, L \otimes E^0_+)$$
$$(E^0_-, b, E^\infty_-) \cong (E^0_-, 1, E^0_-).$$

□

Again, consider a polynomial clutching function p of degree $\leq n$. Then $\mathcal{L}^p(p)$ is a linear clutching function for (V^0, V^∞) where

$$V^0 = \sum_{k=0}^{\infty} L^k \otimes E^0, \quad V^\infty = E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0.$$

Hence, it defines a decomposition

$$V^0 = V^0_+ \oplus V^0_-$$

as above. To express the dependence of V^0_+ on p and n, we write

$$V^0_+ = V_n(E^0, p, E^\infty).$$

Note that this is a vector bundle on X. Let p_t be a homotopy of polynomial clutching functions of degree $\leq n$. By constructing V_n over $X \times I$, it follows that

$$V_n(E^0, p_0, E^\infty) \cong= V_n(E^0, p_1, E^\infty).$$

Hence, from the homotopies used in proving the two parts of 2.2.7, we obtain

$$V_{n+1}(E^0, p, E^\infty) \cong V_n(E^0, p, E^\infty),$$
$$V_{n+1}(L^{-1} \otimes E^0, zp, E^\infty) \cong V_n(E^0, p, E^\infty) \otimes (L^{-1} \otimes E^0)$$

or, equivalently

$$V_{n+1}(E^0, zp, L \otimes E^\infty) \cong L \otimes V_n(E^0, p, E^\infty) \oplus E^0.$$

Combining this with the above corollary and 2.2.5, we obtain the following formula in $K(P)$:

$$[E^0, p, E^\infty] + \{\sum_{k=1}^{n}[L^k \otimes E^0]\}[1] = [V_n(E^0, p, E^\infty)][H^{-1}]$$
$$+ \{\sum_{k=0}^{n}[L^k \otimes E^0] - [V_n(E^0, p, E^\infty)]\}[1]$$

and hence the formula

$$[E^0, p, E^\infty] = [V_n(E^0, p, E^\infty)]([H^{-1}] - [1]) + [E^0][1].$$

This shows that $[V_p^+] \in K[X]$ completely determines $[E^0, p, E^\infty] \in K(P)$. We can now prove our theorem.

Let t be an indeterminate over the ring $K(X)$. Then the map $t \mapsto H$ induces a $K(X)$-algebra homomorphism (recall that $((H-1)(LH-1) = 0)$

$$\mu\colon K(X)[t]/((t-1)([L]t-1)) \to K(P).$$

To prove that μ is an isomorphism, we explicitly construct an inverse.

First, suppose that f is a clutching function for (E^0, E^∞). Let f_n be the sequence of Cesaro means of its Fourier series, and put $p_n = z^n f_n$. Then, if n is sufficiently large, p_n is a polynomial clutching function (of degree $\leq 2n$) for $(E^0, L^n \otimes E^\infty)$. We define

$$\nu_n(f) \in K(X)[t]/((t-1)([L]t-1))$$

by the formula

$$\nu_n(f) = [V_{2n}(E^0, p_n, L^n \otimes E^\infty)](t^{n-1} - t^n) + [E^0]t^n.$$

Now, for sufficiently large n, the linear segment joining p_{n+1} and zp_n provides a homotopy of polynomial clutching functions of degree $\leq 2(n+1)$. Hence, by the formulae following 2.2.10

$$\begin{aligned} V_{2n+2}(E^0, p_{n+1}, L^{n+1} \otimes E^\infty) &\cong V_{2n+2}(E^0, zp_n, L^{n+1} \otimes E^\infty) \\ &\cong V_{2n+1}(E^0, zp_n, L^{n+1} \otimes E^\infty) \\ &\cong L \otimes V_{2n}(E^0, p_n, L^n \otimes E^\infty) \oplus E^0. \end{aligned}$$

Hence

$$\begin{aligned} \nu_{n+1}(f) &= \{[L][V_{2n}(E^0, p_n, L^n \otimes E^\infty)]\} + [E^0](t^n - t^{n-1}) + [E^0]t^{n+1} \\ &= \nu_n(f) \end{aligned}$$

since $(t-1)([L]t-1) = 0$. Thus, $\nu_n(f)$ is independent of n if n is sufficiently large, and thus depends only on f. We write it as $\nu(f)$. If g is sufficiently close to f, and n is sufficiently large, the linear segment joining f_n and g_n provides a homotopy of polynomial clutching functions of degree $\leq 2n$, and hence $\nu f = \nu_n(f) = \nu_n(g) = \nu(g)$. Thus, νf is a locally constant function of f, and hence depends only on the homotopy class of f. However, if E is any bundle on P, and f a clutching function defining E, we define $\nu(E) = \nu(f)$, and $\nu(E)$ will be well defined and depend only on the isomorphism class of E. Since $\nu(E)$ is clearly additive for $+$, it induces a group homomorphism

$$\nu\colon K(P) \to K(X)[t]/((t-1)([L]t-1)).$$

From our definition, it is clear that this is a $K(X)$-module homomorphism.

It remains to show that μ and ν are mutual inverses.
($\mu\nu$ is the identity:) With our notation as above,

$$\begin{aligned}\mu\nu(E) &= \mu\{[V_{2n}(E^0, p_n, L^n \otimes E^\infty)](t^{n-1} - t^n) + [E^0]t^n\} \\ &= [V_{2n}(E^0, p_n, L^n \otimes E^\infty)]([H]^{n-1} + [H]^n) + [E^0][H]^n \\ &= [E^0, p_n, L^n \otimes E^\infty][H]^n \\ &= [E^0, f_n, E^\infty] \\ &= [E^0, f, E^\infty] \\ &= E.\end{aligned}$$

Since $K(P)$ is additively generated by elements of the form $[E]$, this proves that $\nu\nu$ is the identity.
($\nu\mu$ is the identity:) Since $\nu\mu$ is a homomorphism of $K(X)$-modules, it suffices to show that $\nu\mu(t^n) = t^n$ for all $n \geq 0$. However,

$$\begin{aligned}\nu\mu(t^n) &= \nu(H^n) \\ &= \nu[1, z^{-n}, L^{-n}] \\ &= [V_{2n}(1,1,1)](t^{n-1} - t^n) + [1]t^n \\ &= t^n, \quad \text{since } V(1,1,1) = 0.\end{aligned}$$

2.3 $K_G(X)$

Suppose that G is a finite group and that X is a G-space. Let $\mathrm{Vect}_G(X)$ denote the set of isomorphism classes of G-vector bundles over X. This is an abelian semigroup under $\oplus$. We form the associated abelian group and denote it by $K_G(X)$. If $G = 1$ is the trivial group then $K_G(X) = K(X)$. If on the other hand X is a point then $K_G(X) \cong R(G)$ the character ring of G.

If E is a G-vector bundle over X then $P(E)$ is a G-space. If $E = L \oplus 1$ when L is a G-bundle then the zero and infinite sections $X \to P(E)$ are both G-sections. Also the bundle H over $P(E)$ is a G- line bundle. If we now examine the proof of the periodicity theorem which we have just given we see that we could have assumed a G-action on everything. Thus we get the periodicity theorem for K_G :

Theorem 2.3.1. *If X is a G-space, and if L is a G-line bundle over X, the map $t \mapsto [H]$ induces an isomorphism of $K_G(X)$- modules:*

$$K_G(X)[t]/(t[L] - 1)(t - 1) \to K_G(P(L \oplus 1)).$$

2.4 Cohomology property of K

We next define $K(X, Y)$ for a compact pair (X, Y). We shall then be able to establish, in a purely formal fashion, certain properties of K. Since the proofs are formal, the theorems are equally valid for any "cohomology theory" satisfying certain axioms. We leave this formalisation to the reader.

Let $\mathcal{C}$ denote the category of compact spaces, $\mathcal{C}^+$ the category of compact spaces with distinguished basepoint, and $\mathcal{C}^2$ the category of compact pairs. We define functors:

$$\mathcal{C}^2 \to \mathcal{C}^+$$
$$\mathcal{C} \to \mathcal{C}^2$$

by sending a pair (X, Y) to X/Y with basepoint y/Y (if $Y \neq \varnothing$, the empty set, $X?Y$ is understood to be the disjoint union of X and a point.) We send a space X to the pair $(X, \varnothing)$. The $\mathcal{C} \to \mathcal{C}^+$ is given by $X \mapsto X^+$, where X^+ denotes $X/\varnothing$.

If X is in $\mathcal{C}^+$, we define $\widetilde{K}(X)$ to be the kernel of the map $i^*\colon K(X) \to K(x_0)$ $i\colon x_0 \to X$ is the inclusion of the base-point. If $c\colon X \to x_0$ is the collapsing map then c^* induces a splitting $K(X) \cong \widetilde{K}(X) \oplus K(X_0)$. This splitting is clearly natural for maps in $^+$. Thus $\widetilde{K}$ is a functor on $\mathcal{C}^+$. Also, it is clear that $K(X) \cong \widetilde{K}(X^+)$. We define $K(X < Y)$ by $K(X < Y) \cong \widetilde{K}(X/Y)$. In particular $K(X, \varnothing) \cong K(X)$. Since $\widetilde{K}$ is a functor on $\mathcal{C}^+$ it follows that $K(X < Y)$ is a contravariant functor of (X, Y) in $\mathcal{C}^2$.

We now introduce the "smash product" operation in $\mathcal{C}^+$. If $X, Y \in \mathcal{C}^+$ we put

$$X \wedge Y = X \times Y / X * Y$$

where $X * Y = X \times y_0 \cup x_0 \times Y, x_0, y_0$ being the base-points of X, Y respectively. For any three spaces $X, Y, Z \in \mathcal{C}^+$ we have a natural homeomorphism

$$X \wedge (Y \wedge Z) \approx (X \wedge Y) \wedge Z$$

and we shall identify these spaces by the homeomorphism.

Let I denote the unit interval $[0, 1]$ and let $\partial I = \{0\} \cup \{1\}$ be its boundary. We take $I/\partial I \in \mathcal{C}^+$ as our standard model of the circle S^1. Similarly if I^n denotes the unit cube in $\mathbb{R}^n$ we take $I^n/\partial I^n$ as our model of the n-sphere S^n. Then we have a natural homeomorphism

$$S^n \approx S^1 \wedge S^1 \wedge \cdots \wedge S^1 \quad (n \text{ factors}).$$

For $X \in \mathcal{C}^+$ the space $S^1 \wedge X \in \mathcal{C}^+$ is called the *reduced suspension* of X, and often written as SX. The n-th iterated suspension $SS \cdots SX$ (n times) is naturally homeomorphic to $S^n \wedge X$ and is written briefly as $S^n \wedge X$.

Definition 2.4.1. For $n \geq 0$

$$\begin{aligned} \widetilde{K}^{-n}(X) &= \widetilde{K}(S^n X) && \text{for } X \in \mathcal{C}^+ \\ K^{-n}(X, Y) &= \widetilde{K}^{-n}(X, Y) = \widetilde{K}(S^n(X/Y)) && \text{for } (X, Y) \in \mathcal{C}^2 \\ K^{-n}(X) &= K^{-n}(X, \varnothing) = \widetilde{K}(S^n(X^+)) && \text{for } X \in \mathcal{C}. \end{aligned}$$

It is clear that all these are contravariant functors on the appropriate categories.

Before proceeding further we define the *cone* on X by

$$CX = (I \times X)/(\{0\} \times X)$$

Thus C is a functor $C \colon \mathcal{C}) \to \mathcal{C}^+$. We identify X with the subspace $\{1\} \times X$ of CX. The space $CX/X = (I \times X)/(\partial I \times X)$ is called the *unreduced suspension* of X. Note that this is a functor $\mathcal{X} \to \mathcal{C}^+$ whereas the reduced suspension S is a functor $\mathcal{C}^+ to \mathcal{C}^+$. If $X \in \mathcal{C}^+$ with base-point x_0 then we have a natural inclusion map

$$I \approx Cx_0/x_0 \to CX/X$$

and the quotient space obtained by collapsing I in CX/X is just SX. Thus by 1.4.9 $p \colon CX/X \to SX$ induces an isomorphism $K(SX) \cong K(CX/X)$ and hence also an isomorphism $\widetilde{K}(SX) \cong \widetilde{(}CX, X)$. Thus the use of the notation SX for both the reduced and unreduced suspensions leads to no problems.

If $(X, Y) \in \mathcal{C}^2$ we define $X \cup CY$ to be the space obtained from X and CY by identifying the sub spaces $Y \subset X$ and $\{1\} \times CY$. Taking the base-point of CY as base-point of $X \cup CY$ we have

$$X \cup CY \in \mathcal{C}^2.$$

We note that X is a subspace of $X \cup CY$ and that there is a natural homeomorphism

$$(X \cup CY)/X \approx CY/Y.$$

X Thus, if $Y \in \mathcal{C}^2$,

$$\begin{aligned} K(X \cup CY, X) &\cong K(CY, Y) \\ &\cong \widetilde{K}(SY) \\ &= \widetilde{K}^{-1}(Y). \end{aligned}$$

Now we begin with a simple lemma.

Lemma 2.4.2. *For $(X, Y) \in \mathcal{C}^2$ we have an exact sequence*

$$K(X, Y) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(Y)$$

where $i \colon Y \to X$ and $j \colon (X, \varnothing) \to X, Y)$ are the inclusions.

Proof. ($\operatorname{im} j^* \subset \ker i^*$): The composition $i^* j^*$ is induced by the composition $ji \colon (Y, \varnothing) \to (X, Y)$, and so factors through the zero group $K(X, Y)$. Thus $i^* j^* = 0$.

($\operatorname{im} j^* \supset \ker i^*$): Suppose now that $\xi \in \ker i^*$. We may represent ξ in the form $[E] - [n]$ where E is a vector bundle over X. Since $i^* \xi = 0$ by assumption it follows that $[E \mid Y] = [n]$ in $K(Y)$. This implies that for some integer m we have

$$(E \oplus m) \mid Y = n \oplus m$$

i. e., we have a trivialisation α of $(E \oplus m) \mid Y$. This defines a bundle $(E \oplus m)/\alpha$ on X/Y and so an element

$$\eta = [(E \oplus m)/\alpha] - [n \oplus m] \in \widetilde{K}(X/Y) = K(X,Y).$$

Then

$$j^*(\eta) = [E \oplus m] - [n \oplus m] = [E] - [n] = \xi.$$

Thus $\ker i^* = \operatorname{im} j^*$ and the exactness is established. □

Corollary 2.4.3. *If $(X,Y) \in \mathcal{C}^2$ and $Y \in \mathcal{C}^+$ (so that, taking the same base-point of X, we have $X \in \mathcal{C}^+$ also), then the sequence*

$$K(X,Y) \to \widetilde{K}(X) \to \widetilde{K}(Y)$$

is exact .

Proof. This is immediate from 2.4.2 and the natural isomorphisms

$$K(X) \cong \widetilde{K}(X) \oplus K(y_0)$$
$$K(Y) \cong \widetilde{K}(Y) \oplus K(y_0).$$

□

We are now ready for our main proposition:

Proposition 2.4.4. *For $(X,Y) \in \mathcal{C}^2$ there is a natural exact sequence (infinite to the left)*

$$\to K^{-2}(Y) \xrightarrow{\delta} K^{-1}(X,Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*}$$
$$K^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(X).$$

Proof. First we observe that it is sufficient to show that, for $(X,Y) \in \mathcal{C}^2$ and $y \in \mathcal{C}^+$, we have an exact sequence of five terms

$$\widetilde{K}^{-1}(X) \xrightarrow{i^*} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} \widetilde{K}^0(X,Y) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(Y) \qquad (2.4.5)$$

In fact, if this has been established, then we obtain an infinite sequence continuing 2.4.5 by replacing (X,Y) with (S^nX, S^nY) for $n = 1, 2, \ldots$. Then replacing (X,Y) by (X^+, Y^+) where (X,Y) is any pair in $\mathcal{C}^2$ we get the infinite sequence of the enunciation. Now 2.4.3 gives the exactness of the last three terms of 2.4.5. To get exactness at the remaining places we shall apply 2.4.3 in turn to the pairs $(X \cup CY, X)$ and $((X \cup CY) \cup CX, X \cup CY)$. First, taking the pair $(X \cup CY, X)$ we get an exact sequence (where k, m are the natural inclusions)

$$K(X \cup CY, X) \xrightarrow{m^*} \widetilde{K}(X \cup CY) \xrightarrow{k^*} \widetilde{K}(X).$$

Since CY is contractible 1.4.9 implies that

$$p^*\colon \widetilde{K}(X/Y) \to \widetilde{K}(X \cup CY)$$

is an isomorphism where

$$p\colon X \cup CY \to (X \cup CY)/CY = X/Y$$

is the collapsing map. Also the composition k^*p^* coincides with j^*. Let

$$\theta\colon K(X \cup CY, X) \to K^{-1}(Y)$$

be the isomorphism introduced earlier. Then defining

$$\delta\colon K^{-1}(Y) \to K(X, Y)$$

by $\delta = m^*\theta^{-1}$ we obtain the exact sequence

$$\widetilde{K}^{-1} \xrightarrow{\delta} K(X, Y) \xrightarrow{j^*} \widetilde{K}(X)$$

which is the middle part of 2.4.5.

Finally, we apply 2.4.3 to the pair

$$(X \cup C_1Y \cup C_2X, X \cup C_1Y)$$

where we have labelled the cones C_1 and C_2 in order to distinguish between them, (see figure 2.1). Then we obtain the following exact sequence

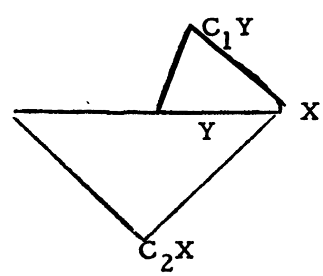


Figure 2.1:

$$K(X \cup C_1Y \cup C_2X, X \cup C_1Y) \to \widetilde{K}(X \cup C_1Y \cup C_2X) \to \widetilde{K}((X \cup C_1Y).$$

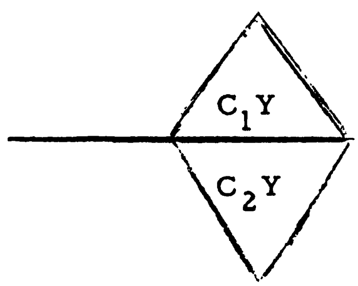
It will be sufficient to show that this sequence is isomorphic to the sequence obtained from the first three terms of 2.4.5. In view of the definition of δ it will be sufficient to show that the diagram

$$\begin{array}{ccc} K(X \cup C_1Y \cup C_2X, X \cup C_1Y) & \longrightarrow & \widetilde{K}(X \cup C_1Y \cup C_2X) \\ \| & & \| \\ \widetilde{K}(C_2X/X) & & \widetilde{K}(C_1Y/Y) \\ \| & & \| \\ K^{-1}(X) & \xrightarrow{\quad i^* \quad} & K^{-1}(Y) \end{array} \tag{2.4.6}$$

commutes up to sign. The difficulty lies, of course, in the fact that i^* is induced by the inclusion

$$C_2Y \to C_2X$$

and that in the above diagram we have C_1Y and not C_2Y. To deal with this situation we introduce the double cone on Y namely $C_1Y \cup C_2Y$. This fits into



the commutative diagram of maps

$$\begin{array}{ccccc} X \cup C_1Y \cup C_2X & \Longrightarrow & C_1Y/Y & \Longrightarrow & SY \\ \downarrow & \nwarrow\!\!\!\!\nwarrow \quad \nearrow\!\!\!\!\nearrow & & & \\ & C_1Y \cup C_2Y & & & \\ \downarrow & \swarrow \quad \searrow\!\!\!\!\searrow & & & \\ C_2X/X & \longleftarrow & C_2Y/Y & \Longrightarrow & SY \end{array} \tag{2.4.7}$$

where all double arrows $\Rightarrow$ induce isomorphism in K. Using this diagram we see that diagram 2.4.6 will commute up to sign provided the diagram induced by 2.4.7

$$\begin{array}{ccccc}
 & & K(C_1Y/Y) & \longleftarrow & \widetilde{K}(SY) \\
 & \swarrow & & & \Vert \\
K(C_1Y \cup C_2Y) & & & & \Vert \\
 & \nwarrow & & & \Vert \\
 & & K(C_2Y/Y) & \longleftarrow & \widetilde{K}(SY)
\end{array}$$

commutes up to sign. This will follow at once from the following lemma which is in any case of independent interest and will be needed later. □

Lemma 2.4.8. *Let, $T\colon S^1 \to S^1$ be defined by $T(t) = 1 - t, t \in I$ (we recall that $S^1 = I/\partial I$) and let $T \wedge 1\colon SY \to SY$ be the map induced by T on S^1 the identity on Y (for $Y \in \mathcal{C}^+$). Then $(T \wedge 1)^* y = -y$ for $y \in \widetilde{K}(SX)$.*

This lemma in turn is an easy corollary of the following:

Lemma 2.4.9. *For any map $f\colon Y \to \mathrm{GL}(n, \mathbb{C})$ let E_f denote the corresponding vector bundle over SY. Then $f \mapsto [E_f] - [n]$ induces a group isomorphism*

$$\lim_{n\to\infty} [Y, \mathrm{GL}(n.\mathbb{C})] \cong \widetilde{K}(SY)$$

where the group structure on the left is induced from that of $\mathrm{GL}(n, \mathbb{C})$.

The fact that this is in fact a group homomorphism follows from the homotopy connecting the two maps $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{C})$ given by

$$A \times B \to \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and

$$A \times B \to \begin{bmatrix} AB & 0 \\ 0 & 1 \end{bmatrix}$$

This homotopy is given explicitly by

$$\rho_t(A \times B) = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ \sin t & \cos t \end{bmatrix}$$

where $0 \le t \le \pi/2$.

From 2.4.4 we deduce at once:

Corollary 2.4.10. *If Y is a retract of X, then for all $n \ge 0$, the sequence $K^{-n}(X, Y) \to K^{-n}(X) \to K^{-n}(Y)$ is a split short exact sequence, and*

$$K^{-n}(X) \cong K^{-n}(X, Y) \oplus K^{-n}(Y).$$

Corollary 2.4.11. *If (X,Y) are two spaces with base-points, the projection maps $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ induce an isomorphism for all $n \geq 0$*

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y).$$

Proof. X is a retract of $X \times Y$, and Y is a retract of $(X \times Y)/Y$. The result follows by two applications of 2.4.10. □

Since $\widetilde{K}^0(X \wedge Y)$ is the kernel of $i_X^* \oplus i_Y^* : K^0(X \times Y) \to K^0(X) \oplus K^0(Y)$, the usual tensor product $K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$ induces a pairing $\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \to \widetilde{K}^0(X \wedge Y)$. Thus, we have a pairing

$$\widetilde{K}^{-n}(X) \otimes \widetilde{K}^{-m}(Y) \to \widetilde{K}^{-n-m}(X \wedge Y)$$

since $S^n \wedge S^m Y = S^n \wedge S^m \wedge X \wedge Y = S^{n+m} \wedge X \wedge Y$. Replacing X by X^+, Y by Y^+, we have

$$K^{-n}(X) \otimes K^{-m}(Y) \to K^{-n-m}(X \times Y).$$

Using this pairing, we can restate the periodicity theorem as follows:

Theorem 2.4.12. *For any space X and any $n \leq 0$, the map $K^{-2}(\text{point}) \otimes K^{-n}(X) \to K^{-n-2}(X)$ induces an isomorphism*

$$\beta \colon K^{-n}(X) \to K^{-n-2}(X).$$

Proof. $K^{-2}(\text{point}) = \widetilde{K}(S^2$ is the free abelian group generated by $[H] - [1]$. If $(X,Y) \in \mathcal{C}^2$ the maps in the exact sequence 2.4.5 all commute with the periodicity isomorphism β. This is immediate for i^* and j^* and is also true for δ since this was also induced by a map of spaces. In other words β shifts the whole sequence to the left by six terms. Hence if we define $K^n(X,Y)$ for $n > 0$ inductively by $K^{-n} = K^{-n-2}$ we can extend 2.4.5 to an exact sequence infinite in both directions. Alternatively using the periodicity β we can define an exact sequence of six terms

$$\begin{array}{ccccc} K^0(X,Y) & \longrightarrow & K^0(X) & \longrightarrow & K^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,Y) \end{array}$$

Except when otherwise stated we shall now always identify K^n and K^{n-2}. We introduce

$$K^*(X) = K^0(X) \oplus K^1(X).$$

Thus we define $K^*(X)$ to be $K^0(X) \oplus K^1(X)$. Then, for any pair (X,Y) we have an exact sequence

$$\begin{array}{ccccc} K^0(X,Y) & \longrightarrow & K^0(X) & \longrightarrow & K^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,Y) \end{array}$$

□

The form of the periodicity theorem given in 2.4.12 is a special case of a more general "Thom isomorphism theorem". If X is a compact space, and E is a real vector bundle over X, the *Thom complex* X of E is the one point compactification of the total space of E. Alternatively, if E is a complex bundle, $X^E = P(E \oplus 1)/P(E)$. Thus, we see that $\widetilde{K}(X^E)$ a module over $K(X)$. The Thom isomorphism theorem for complex line bundles can now be stated.

Theorem 2.4.13. *If L is a complex line bundle, $\widetilde{K}(X^L)$ is a free $K(X)$-module on one generator $\mu(L)$, and image of $\mu(L)$ $K(P(L \oplus 1))$ is $[H] - [L^*]$.*

Proof. This is immediate from our main theorem determining $K(P(L \oplus 1))$ and the exact sequence of the pair $P(L \oplus 1), P(L)$ (note that $P(L) = X$). □

We conclude this section by giving the following extension of 2.4.8 which will be needed later.

Lemma 2.4.14. *Let $T_\sigma \colon S^n X \to S^n X$ be the map induced by a permutation σ of the n factors in $S^n = S^1 \wedge S^1 \wedge \cdots \wedge S^1$. Then $(T_\sigma)^* x = \operatorname{sgn}(\sigma) x$ for $x \in \widetilde{K}(S^n X)$.*

Proof. Considering S^n as the one-point compactification of $\mathbb{R}^n$ we can make $\mathrm{GL}(n, \mathbb{R})$ act on it and hence on $\widetilde{K}(S^n X)$. This extends the permutation actions T_σ. Since $\mathrm{GL}(n, \mathbb{R})$ has just two components characterized by sgn det it is sufficient to check the formula $T^* x = -x$ for one $T \in \mathrm{GL}(n, \mathbb{R})$ with $\det T = -1$. But 2.4.8 gives this formula for

$$T = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

□

2.5 Computations of $K^*(X)$ for some X

From the periodicity theorem, we see that $\widetilde{K}(S^n) = 0$ if n is odd, and $\widetilde{K}(S^n) = \mathbb{Z}$ if n is even. This allows us to prove the Brouwer fixed point theorem.

Theorem 2.5.1 (Brouwer fixed point theorem)**.** *Let D^n be the unit disc in Euclidean n-space. If $f \colon D^n \to D^n$ is continuous, then for some $x \in D^n$, $f(x) = x$.*

Proof. Suppose $f(x) \neq x$ for all $x \in D^n$ and define $g \colon D^n \to S^{n-1}$ by $g(x) = (1 - \alpha(x)) f(x) + \alpha(x) x$, where $\alpha(x)$ is the unique function such that $\alpha(x) \geq 0, |g(x)| = 1$. If $f(x) \neq x$ for all x, clearly such a function $\alpha(x)$ exists. If $x \in S^{n-1}, \alpha(x) = 1$, and $g(x) = x$. Thus g is a retraction of D^n onto S^{n-1}. However, we have $\widetilde{K}^*(D^n) = 0$, and $\widetilde{K}^*(S^{n-1}) \neq 0$, a contradiction. Thus there must exist a point $x \in D^n$ such that $f(x) = x$. □

We will say that a space X is a *cell complex* if there is a filtration by closed sets $X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that each $X_k \setminus X_{k-1}$ is a disjoint union of open k-cells, and $X_{-1} = \varnothing$.

Proposition 2.5.2. *If X is a cell complex such that $X_{2k} = X_{2k+1}$ for all k, then*

1. $K^1(X) = 0$
2. *$K^0(X)$ is a free abelian group with generators in a one-one correspondence with the cells of X.*

Proof. We proceed by induction on n. Since X_{2n}/X_{2n-2} is a union of $2n$-spheres with a point in common, we have:

$$K^1(X_{2n}, X_{2n-2}) = 0$$
$$K^0(X_{2n}, X_{2n-2}) = \mathbb{Z}^k$$

where k is the number of $2n$-cells in X. The result for X_{2n} now follows from the inductive hypothesis and the exact sequence of the pair (X_{2n}, X_{2n-2}). $\square$

As examples of spaces to which this proposition applies, we may take X to be a complex Grassmann manifold, a flag manifold, a complex quadric (a space whose homogeneous defining equation is of the form $\sum z_i^2 = 0$). We shall return to the Grassmann and flag manifolds in more detail later.

Proposition 2.5.3. *Let $_1, \cdots, L_n$ be line bundles over X, and let H be the standard bundle over $P(L_1 \oplus \cdots L_n)$. Then, the map $t \mapsto [H]$ induces an isomorphism of $K(X)$-modules*

$$K(X)[t]/\prod_{i=1}^{n}(t - [L_i^*]) \cong K(P(L_1 \oplus \cdots L_n)).$$

Proof. First we shall show that we may take $L_n = 1$. In fact for any vector bundle E and line bundle L over X we have $P(E \otimes L) = P(E)$ and the standard line bundles G, H over $P(E \otimes L^*), P(E)$ are related by $G^* = H^* \otimes L$, i.e., $G = H \otimes L^*$. Taking $E = L_1 \oplus \cdots L_n$ and $L = L_n^*$ we see that the propositions for $L_1 \oplus \cdots L_n$ and for $M_1 \oplus \cdots M_n$ with $M_i = L_i \otimes L_n^*$ are equivalent. We shall suppose therefore that $L_n = 1$ and for brevity write

$$P_m = P(L_1 \oplus \cdots L_n) \quad \text{for } 1 \leq m \leq n$$

so that we have inclusions $X = P_1 \subset P_2 \subset \cdots \subset P_n$. If H_m denotes the standard line bundle over P_m then $H_m \mid P_{m-1} \cong H_{m-1}$. Now we observe that we have a commutative diagram

$$\begin{array}{ccc}
P_{n-1} & \xrightarrow{\ s\ } & P(H_{n-1}^* \oplus 1) \\
{\scriptstyle \pi_{n-1}}\downarrow & & \downarrow{\scriptstyle q} \\
P_1 & \xrightarrow[\ i_n\ ]{} & P_n
\end{array}$$

(π_{n-1} is the projection onto $X = P_1, i_n$ is the inclusion, s is the zero section) which induces a homeomorphism

$$P(H_{n-1}^*)/s(P_{n-1}) \to P_n/P_1.$$

Moreover $q^*(H_n) \cong G$, the standard line bundle over $P(H_{n-1}^*)$. Now $K(P(H_{n-1}^*))$ is a free $K(P_{n-1})$-module on two generators $[1]$ and $[G]$, and $[G]$ satisfies the equation $([G]-[1])([G]-[H_{n-1}]) = 0$. Since $s^*[G] = [1]$ it follows that $K(P(H_{n-1}^*), s(P_{n-1}))$ is the submodule freely generated by $[G]-[1]$ and that, on this submodule, multiplication by $[G]$ and $[H_{n-1}]$ coincide. Hence $K(P_n, P_1)$ is a free $K(P_{n-1})$-module generated by $([H_n]-[1])$ and this module structure is such that, for any $x \in K(P_n, P_1)$,

$$[H_{n-1}]x = [H_n]x.$$

Now assume the proposition established for $n-1$, so that

$$K(P_{n-1}) \cong K(X)[t]/\prod_{i=1}^{n-1}(L_i^*)$$

with t corresponding to $[H_{n-1}]$. Then it follows that $t \mapsto [H_n]$ induces an isomorphism of the ideal $(t-1)$ in

$$K(X)[t]/(t-1)\prod_{i=1}^{n-1}(t-[L_i^*])$$

onto $K(P_n, P_1)$. Since

$$K(P_n) \cong K(P_n, P_1) \oplus K(X)$$

and since $L_n = 1$ this gives the required result for $K(P_n)$ establishing the induction and completing the proof. □

Corollary 2.5.4. *$K(P(\mathbb{C}^n)) \cong \mathbb{Z}[t]/(t-1)^n$ under the map $t \mapsto [H]$.*

Proof. Take X to be a point. □

We could again have assumed that a finite group acted on everything, and we would have obtained

$$K_G(X)[t]/\prod_{i=1}^{n}(t-[L_i^*]) \cong K_G(P(L_1 \oplus \cdots L_n)).$$

2.6 Multiplications in $K^*(X, Y)$

We first observe that the multiplication in $K(X)$ can be defined "externally" as follows. Let E, F be two bundles over X, and let $E\widehat{\otimes}F$ be $\pi_1^*(E) \otimes \pi_2^*(F)$ over $X \times X$. If $\Delta\colon X \to X \times X$ is the diagonal, then $E \otimes F = \Delta^*(E\widehat{\otimes}F)$.

If E is a bundle on X, F a bundle on Y, let $E \widehat{\otimes} F = \pi_X^* \otimes \pi_Y^*(F)$ on $X \times Y$. This defines a pairing

$$K(X) \otimes K(Y) \to K(X \times Y).$$

If X, Y have base-points, $\widetilde{K}(X \wedge Y)$ is the kernel of $\widetilde{K}(X \times Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y)$. Thus, we have $\widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$. That is,

$$K(X, A) \otimes K(T, B) \to K(X \times Y, (X \times B) \cup (A \times Y)).$$

We define $(X, A) \times (Y, B)$ to be $(X \times Y, (X \times B) \cup (A \times Y))$.

In the special case that $X = Y$, we have a diagonal map $\Delta\colon (X, A \cup B) \to (X, A) \times (X, B)$. This gives us $K(X, A) \otimes K(X, B) \to K(X, A \cup B)$. In particular, taking $B = \varnothing$, we see that $K(X, A)$ is a $K(X)$-module. Further, it is easy to see that

$$K(X, A) \to K(X) \to K(A)$$

is an exact sequence of $K(X)$-modules.

More generally, we can define products

$$K^{-n}(X, A) \otimes K^{-m}(Y, B) \to K^{-n-m}((X, A) \times (Y, B))$$

for $m, n \leq 0$ as follows:

$$\begin{aligned} K^{-n}(X, A) &= \widetilde{K}(S^n \wedge (X/A)) \\ K^{-m}(Y, B) &= \widetilde{K}(S^m \wedge (Y/B)). \end{aligned}$$

Thus, we have

$$\begin{aligned} K^{-n}(X, A) \otimes K^{-m}(Y, B) &\to \widetilde{K}(S^n \wedge (X/A) \wedge S^m \wedge (Y, B)) \\ &= \widetilde{K}(S^n \wedge S^m \wedge (X/A) \wedge (Y, B)) \\ &= K^{-n-m}((X, A) \times (Y, B)). \end{aligned}$$

Thus, if we define $xy \in K^{-n-m}(X, A \cup B)$ for $s \in K^{-n}(X, A), y \in K^{-m}(X, B)$ to be $\Delta^*(x \otimes y)$, where $\Delta\colon (X, A \cup B) \to (X, A) \times (X, B)$ is the diagonal, then 2.4.14 shows that $xy = (-1)^{mn} yx$.

We define $K^{\#}(X, A)$ to be

$$\sum_{n=0}^{\infty} K^{-n}(X, A).$$

Then $K^{\#}(X)$ is a graded ring, and $K^{\#}(X, A)$ is a graded $K^{\#}$-module. If $\beta \in K^{-2}(\text{point})$ is the generator, multiplication by β induces an isomorphism $K^{-n}(X, A) \to K\ -n-2(X, A)$ for all n. We define $K^*(X, A)$ to be $K^{\#}(X, A)/(1-\beta)$. Then $K^*(X)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded ring, and $K^*(X, A)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded module over $K^*(X)$.

For any pair (X, A), each of the maps in the exact triangle

$$\begin{array}{ccc} K^*(X) & \longrightarrow & K^*(A) \\ & \nwarrow \quad \swarrow & \\ & K^*(X, A) & \end{array}$$

is a $K^*(X)$-module map. Only the coboundary δ causes any difficulty and so we need to prove

Lemma 2.6.1. $\delta\colon K^{-1}(Y) \to K^0(X, Y)$ *is a* $K(X)$*-module homomorphism.*

Proof. By definition δ is induced by the inclusion of pairs $j\colon (X \times \{1\} \cup Y \times I, Y \times \{0\}) \to (X \times \{1\} \cup Y \times I, Y \times \{0\} \cup X \times \{1\})$. (See Figure 2.2.) Hence

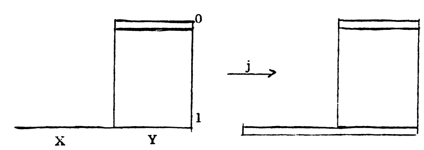


Figure 2.2:

$\delta = j^*$ is a module homomorphism over the absolute group

$$K(X \times \{1\} \cup Y \times I) \cong K(X).$$

It remains only to observe that the $K(X)$-module structures of the two groups involved are the standard ones. For $K^{-1}(Y)$ this is immediate and for $K(X, Y)$ we have only to observe that the projection $I \to \{1\}$ induces the isomorphisms

$$\begin{aligned} K(X, Y) &\to K(X \times \{1\} \cup Y \times I, Y \times \{0\}) \\ K(X) &\to K(X \times \{1\} \cup Y \times I). \end{aligned}$$

□

We shall now digress for some time to give an alternative and often illuminating description of $K(X, A)$ which has particular relevance for products.

If $n \geq 1$, we define $\mathcal{C}_n(X, A)$ to be a category as follows:
An object of $\mathcal{C}_n(X, A)$ is a collection $E_n, E_{n-1}, \cdots, E_0$ of bundles over X, together with maps $\alpha_i\colon E_i \mid A \to E_{i-1} \mid A$ such that

$$0 \to E_n \mid A \xrightarrow{\alpha_n} E_{n-1} \mid A \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} E_0 \mid A \to 0$$

is exact.
The morphisms $\varphi\colon E \to F$, where $E = (E_i, \alpha_i), F = (F_i, \beta_i)$, are collections of maps $\varphi_i\colon E_i \to F_i$ such that $\beta_i\varphi_i = \varphi_{i-1}\alpha_i$. In particular, $\mathcal{C}^1(X,A)$ consists of pairs of bundles E_1, E_0 over X and isomorphisms $\alpha\colon E_1 \mid A \cong R_0 \mid A$.

An elementary sequence in $\mathcal{C}_n(X,A)$ is a sequence of the form $0,\dots,0,E_p,E_{p-1},0,\dots,0$ where $E_p = E_{p-1}, \alpha =$, identity map. We define $E \sim F$ for some set of elementary objects $Q_1,\dots Q_n, P_1,\dots,P_m$,

$$E \oplus Q_1 \oplus \cdots \oplus Q_n \cong F \oplus P_1 \oplus \cdots \oplus P_m.$$

The set of such equivalence classes is denoted by $\mathcal{L}_n(X,A)$. It is clear that $\mathcal{L}_n(X,A)$ is a semigroup for each n.

There is a natural inclusion $\mathcal{C}_n(X,A) \subset \mathcal{C}_{n+1}(X,A)$ which induces a homomorphism $\mathcal{L}_n(X,A) \to \mathcal{L}_{n+1}(X,A)$. We denote by $\mathcal{C}_\infty(X,A)$ the union of all the $\mathcal{C}_n(X,A)$, and by $\mathcal{L}_\infty(X,A)$ the direct limit of the $\mathcal{L}_n(X,A)$.

The main theorem of this section is the following:

Theorem 2.6.2. *For all $n \geq 1$, the maps $\mathcal{L}_n(X,A) \to \mathcal{L}_{n+1}(X,A)$ are isomorphisms, and $\mathcal{L}_n(X,A) \cong K(X,A)$.*

We shall break up the proof of this theorem into a number of lemmas.

Consider first the special case $A = \varnothing, n = 1$. Then $\mathcal{C}_1(X,\varnothing)$ consists of all pairs E_1, E_0 of bundles. We see that $(E_1,E_0) \sim (F_1,F_0)$ if and only if there are bundles Q, P such that $E_1 \oplus Q \cong F_1 \oplus P, E_0 \oplus Q \cong F_0 \oplus P$. Then the map $\mathcal{L}_1(X,\varnothing) \to K(X), (E_1,E_0) \mapsto [E_0] - [E_1]$ is an isomorphism. In fact $\mathcal{L}_1(X,\varnothing)$ coincides with one of our definitions of $K(X)$.

Definition 2.6.3. An *Euler characteristic* χ_n for $\mathcal{L}_n$ is a transformation of functors

$$X_n\colon \mathcal{L}_n(X,A) \to K(X,A)$$

such that whenever $A = \varnothing, \chi(E_n, E_{n-1},\dots,E_0) = \sum(-1)^i[E_i]$.

To begin we need a simple lemma.

Lemma 2.6.4. *Let $A \subset X$, and let E, F be bundles over X. Let $\varphi\colon E \mid A \to F \mid A, \psi\colon E \to F$ be monomorphisms (resp. isomorphisms) and assume $\psi \mid A$ is homotopic to φ. Then φ extends to X as a monomorphism (resp. isomorphism).*

Proof. Let $Y = (A \times [0,1]) \cup (X \times [0])$. Then, if E', E' are the inverse images of E, F under the projection $Y \to X$, we can define $\Phi\colon E' \to F'$ which is a monomorphism. (resp. isomorphism) such that $\Phi \mid (A \times [1]) = \varphi, \Phi \mid (X \times [0]) = \psi$. We can extend Φ to $(U \times [0,1]) \cup (X \times [0])$ for some neighbourhood U of A. Let $f\colon X \to [0,1]$ be a continuous map such that $f(A) = 1, f(X \setminus U) = 0$. Let $\varphi_X = \Phi_{(x,f(x))}$. Then this extends φ to X. □

Lemma 2.6.5. *If X is a point,*

$$0 \to \mathcal{L}_1(X,A) \to \mathcal{L}_1(X) \to \mathcal{L}_1(A)$$

is exact. Thus, if χ_1 is an Euler characteristic for $\mathcal{L}_1$, $\chi_1\colon \mathcal{L}_1(X,A) \to (K,A)$ is an isomorphism when A is a point.

Proof. ($\mathcal{L}_1$ is half exact) : If E_1, E_0 represents an element of $\mathcal{L}_1(X)$ whose image in $\mathcal{L}_1(Z)$ is zero, E_1 and E_0 have the same dimension over A. Thus there is an isomorphism $\varphi\colon E_1 \mid A \to E_0 \mid A$. Thus we have exactness for $\mathcal{L}_1(X, A) \to \mathcal{L}_1(X) \to \mathcal{L}_1(A)$.
($\mathcal{L}_1$ is left exact) : If (E_1, E_0, φ) has image zero in $\mathcal{L}_1(X)$, there is a trivial P and an isomorphism $\psi/colon E_1 \oplus P \cong E_0 \oplus P$. $\psi(\varphi \oplus 1)^{-1}$ is an automorphism of $E_0 \oplus P \mid A$. Since A is a point any such automorphism must be homotopic to the identity and hence by 2.6.4 it extends to $\alpha E_0 \oplus P \cong E_0 \oplus P$. Thus, we have a commuting diagram:

$$\begin{array}{ccc} (E_1 \oplus P) \mid A & \xrightarrow{\varphi \oplus 1} & (E_0 \oplus P) \mid A \\ {\scriptstyle \psi|A}\downarrow & & \downarrow{\scriptstyle \alpha|A} \\ (E_0 \oplus P) \mid A & \xrightarrow[i]{} & (E_0 \oplus P) \mid A \end{array}$$

Thus (E_1, E_0, φ) represents 0 in $\mathcal{L}_1(x, a)$. Thus $\mathcal{L}_1(X, A) \to \mathcal{L}_1(X)$ is an injection. □

Lemma 2.6.6. $\mathcal{L}_1(X/A, A/A) \to \mathcal{L}_1(X, A)$ *is an isomorphism for all* (X, A). *ThusÂ» if* χ_1 *is an Euler characteristic,* $\chi_1\colon \mathcal{L}_1(X, A) \to K(X, A)$ *is an isomorphism for all* (X, A).

Proof. (χ_1 is injective) : Since the isomorphism $\mathcal{L}_1(X/A, A/A) \to K(X, A)$ factors through $\mathcal{L}_1(X, A)$, the map $\mathcal{L}_1(X/A, A/A) \to \mathcal{L}_1(X, A)$ is injective.
(χ_1 is onto) : Suppose that E_1, E_0 are bundles on $X, \alpha\colon E_1 \mid A \to E_0 \mid A$ is an isomorphism. Let P be a bundle on X such that there is an isomorphism $\beta\colon E_1 \oplus O \to F$, where F is trivial. Then (E_1, E_0, α) is equivalent to $(F, E_0 \oplus P, \gamma$ where $\gamma = (\alpha \oplus 1)\beta^{-1}$. Then, $(F, E_0 \oplus P, \gamma$ is the image of $(F, (E_0 \oplus P)/\gamma, \gamma/\gamma)$. Thus, $\mathcal{L}_1(X/A, A/A) \to \mathcal{L}_1(X, A)$ is onto. □

Lemma 2.6.7. *If* χ_1, χ_1' *are two Euler characteristics for* $\mathcal{L}_1, \chi_1 = \chi_1'$.

Proof. $\chi_1' \chi_1^{-1}$ is a transformation of functors from K to itself which is the identity on each $K(X)$. Since $K(X, A) = \widetilde{K}(X/A)$ is injected into $K(X/A)$, it is the identity on all $K(X, A)$. □

Lemma 2.6.8. *There exists an Euler characteristic* χ_1 *for* $\mathcal{L}_1$.

Proof. Suppose (E_1, E_0, α) represents an element of $\mathcal{L}_1(X, A)$. Let X_0, X_1 be two copies of X, and let $= X_0 \cup_A X_1$ be the space which results from identifying corresponding points of A. Then $[E_1, \alpha, X_0] \in K(Y)$. Let $\pi_i\colon Y \to X_i$ be the obvious retraction. Then $K(Y) = K(Y, X_i) \oplus K(X_i)$. The map $(X_0, A) \to (Y, X_1)$ induces an isomorphism $K(Y, X_1) \to K(X_0, A)$. Let $\chi_i(E_1, E_0, \alpha)$ be the image of the component of $[E_1, \alpha, E_0]$ which lies in $K(Y, X_1)$. If $A = \varnothing$, then $\chi(E_1, E_0, \alpha) = [E_0] - [E_1]$. One can easily verify that this definition is independent of the choices made. □

Corollary 2.6.9. *The class of (E_1, E_0, α) in $\mathcal{L}_1(X,A)$ depends only on the homotopy class of α.*

Proof. Let $Y = X \times [0,1], B = A \times [0,1]$. Let α_t be a homotopy with $\alpha_0 = \alpha$: Then α_t defines $\beta\colon \pi^*(E_1) \mid B \cong \pi^*(E_0 \mid B)$. Let $i_j\colon (X,A) \to (X \times [j], A \times [j])$. From the commuting diagram

$$\begin{array}{ccccc}
\mathcal{L}_\infty(X,A) & \xleftarrow{i_0^*} & \mathcal{L}_\infty(Y,B) & \xrightarrow{i_1^*} & \mathcal{L}_\infty(X,A) \\
{\scriptstyle \chi_1}\downarrow & & {\scriptstyle \chi_1}\downarrow & & \downarrow \\
K(X,A) & \xleftarrow[i_0^*]{} & K(Y,B) & \xrightarrow[i_1^*]{} & K(X,A)
\end{array}$$

we see that $(E_1, E_0, \alpha_0) = (E_1, E_0, \alpha_1)$, since every map is an isomorphism and $i_0^*(i_1^*)^{-1}$ is the identity. □

Lemma 2.6.10. *The map $\mathcal{L}_n(X,A) \to \mathcal{L}_{n+1}(X,A)$ is onto for $n \geq 1$.*

Proof. If $(E_{n+1}, \cdots, E_0; \alpha_{n+1}, \cdots, \alpha_1)$ represents an element of $\mathcal{L}_{n+1}(X,A))$, so does

$$(E_{n+1}, E_n \oplus R_{n+1}, E_{n-1} \oplus E_{n+1}, E_{n-2}, \cdots, E_0; \alpha_{n+1}, \alpha_n \oplus 1, \cdots, \alpha_1)$$

The two maps $\alpha_{n+1} \oplus 0\colon E_{n+1} \to E_n \oplus E_{n+1}$ and $0 \oplus 1\colon E_{n+1} \to E_n \oplus E_{n+1}$ are (linearly) homotopic as monomorphisms. Now, $0 \oplus 1$ extends to X, and thus by 2.6.4 $\alpha_{n+1} \oplus 0$ extends to a monomorphism $\beta\colon E_{n+1} \to E_n \oplus E_{n+1}$ on all X. Thus we can write $E_n \oplus E_{n+1}$ as $\beta(E_{n+1}) \oplus Q$. Then we see that, if $\gamma\colon Q \to E_{n-1} \oplus E_{n+1}$ is the resulting map, $(E_{n+1}, \cdots, E_0; \alpha_{n+1}, \cdots, \alpha_1)$ is equivalent to $(0, Q, E_{n-1} \oplus E_{n+1}, \cdots, E_0; 0, \gamma, \cdots, \alpha_1)$. Thus $\mathcal{L}_n(X,A) \to \mathcal{L}_{n+1}(X,A)$ is onto. □

Lemma 2.6.11. *The map $\mathcal{L}_n(X,A) \to \mathcal{L}_{n+1}(X,A)$ is an isomorphism for all $n \geq 1$.*

Proof. It suffices to produce a map $\mathcal{L}_{n+1}(X,A) \to \mathcal{L}_1(X,A)$ which is a left inverse of the map $\mathcal{L}_1(X,A) \to \mathcal{L}_{n+1}(X,A)$.

Let $(E_n, \cdots, E_0; \alpha_n, \cdots, \alpha_1)$ represent an element of $\mathcal{L}_{n+1}(X,A)$. Choose a Hermitian metric on each E_i. Let $\alpha_i'\colon E_{i-1} \mid A \to E_i \mid A$ be the Hermitian adjoint of α_i.

Put $F_0 = \sum E_{2i}, F_1 = \sum E_{2i+1}$, and define $\beta\colon F_1 \to F_0$ by $\beta = \sum \alpha_{2i+1} + \sum \alpha'_{2i+1}$. Then $(F_1, F_0, \beta) \in \mathcal{L}_1(X,A)$. This gives us a map $\mathcal{L}_n(X,A) \to \mathcal{L}_1(X,A)$. To see that it is well defined, we need only see that it does not depend on the choice of metrics. But all choices of metric are homotopic to one another, so that a change of metrics only changes the homotopy class of β. Thus this map is well defined. It clearly is a left inverse to $\mathcal{L}_1(X,A) \to \mathcal{L}_n(X,A)$. □

Corollary 2.6.12. *For each n there exists exactly one Euler characteristic $\chi_n\colon \mathcal{L}_n(X,A) \to K(X,A)$, and it is always an isomorphism. Thus, there exists $\chi\colon \mathcal{L}_\infty(X,A) \xrightarrow{\cong} K(X,A)$.*

We next want to construct pairings

$$\mathcal{L}_n(X,A)\otimes\mathcal{L}_m(X')\to\mathcal{L}_{n+m}((X,Y)\otimes(X',Y')$$

compatible with the pairings

$$K(X,A)\otimes K(X')\to K((X,Y)\otimes(X',Y').$$

To do this, we must consider chain complexes of vector bundles, i.e., sequences

$$0\to E_n\xrightarrow{\sigma_n}E_{n-1}\xrightarrow{\sigma_{n-1}}\cdots\xrightarrow{\sigma_1}E_0\to 0$$

where $\sigma_i\sigma_{i+1}=0$ for all i.

Lemma 2.6.13. *Let $E_0,\cdots E_n$ be vector bundles on X, and let $\sigma_i\colon E_i\mid Y\to E_{i-1}\mid Y$ be such that*

$$0\to E_n\mid Y\xrightarrow{\sigma_n}E_{n-1}\mid Y\xrightarrow{\sigma_{n-1}}\cdots\xrightarrow{\sigma_1}E_0\mid Y\to 0$$

Then the σ_i can be extended to $\rho_i\colon E_i\to E_{i-1}$ on X such that $\rho_i\rho_{i+1}=0$ for all i.

Proof. We shall show that there is some open neighbourhood U of Y in X and an extension τ_i of σ_i to U for a11 i such that

$$0\to E_n\mid U\xrightarrow{\tau_n}E_{n-1}\mid U\xrightarrow{\tau_{n-1}}\cdots\xrightarrow{\tau_1}E_0\mid U\to 0$$

is exact: the extension to the whole X is then achieved by replacing τ_i by $\rho\tau_i$ where ρ is a continuous function on X such that $\rho=$ on Y and $\operatorname{supp}\rho\subset U$.

Suppose that on some closed neighbourhood U_i of Y in X, we could extend $\sigma_1,\cdots\sigma_i$ to $\tau_1,\cdots,\tau_i$ such that,

$$0\to E_n\mid U_i\xrightarrow{\tau_i}E_{n-1}\mid U_i\xrightarrow{\tau_{n-1}}\cdots\xrightarrow{\tau_1}E_0\mid U_i\to 0$$

is exact. Let K_i be the kernel of τ_i on U_i. Then σ_{i+1} defines a section of $\mathrm{Hom}(E_{i+1},K_i)$ defined on Y. Thus, this section can be extended to a neighbourhood of Y in U_i, and thus $\sigma_{I+1}\colon E_{i+1}\to K_i$ can be extended to $\tau_{i+1}\colon E_{i+1}\to L_i$ on this neighbourhood. As σ_{+1} is a surjection on Y, τ_{i+1} will be a surjection on some closed neighbourhood U_{i+1} of Y in U_i. Thus, the lemma follows by induction on i. □

We introduce the set $\mathcal{D}_n(X,Y)$ of complexes of length n on X which are acyclic (i.e., exact) on Y. We say that two such complexes are *homotopic* if they are isomorphic to the restrictions to $X\times\{0\}$ and to $X\times\{1\}$ of an element in $\mathcal{D}_n(X\times I,Y\times I)$. There is a natural map

$$\Phi\colon\mathcal{D}_n(X,Y)\to\mathcal{L}_n(X,Y)$$

given by restriction of homomorphisms.

Lemma 2.6.14. *Φ induces a bijection of homotopy classes.*

Proof. The last lemma shows that Φ is surjective. To show that Φ is injective we have to show that any complex over $X \times \{0\} \cup X \times \{1\} \cup Y \times I$ which is acyclic over $Y \times I$ can be extended to a complex on the whole $X \times I$. We carry out this extension in three steps.
First we make the obvious extensions to $X \times [0, 1/4]$ and $X \times [3/4, 1]$.
Next we apply the preceding lemma to the pair $X \times [1/4, 3/4], Y \times [1/4, 3/4] \cup V \times \{1/4\} \cup V \times \{3/4\}$ where V is a closed neighbourhood of Y in X over which the given complexes are still acyclic. This gives a complex on $X \times [1/4, 3/4]$ which agrees with that already defined at the two thickened ends along the strips $V \times \{1/4\}$ and $V \times \{3/4\}$. Thus if we now multiply everything by a function ρ such that

$$\rho = \begin{cases} 1 & \text{on } X \times \{0\} \cup X \times \{1\} \cup U \times I \\ 0 & \text{on } (X \setminus V) \times \{1/4\} \cup (X \setminus V) \times \{3/4\}, \end{cases}$$

we obtain the desired extension (see figure 2.3: the dotted line indicates the support of ρ) . If $E \in \mathcal{D}_n(X, Y), F \in \mathcal{D}_m(X', Y')$ then $E \otimes F$ is a complex on

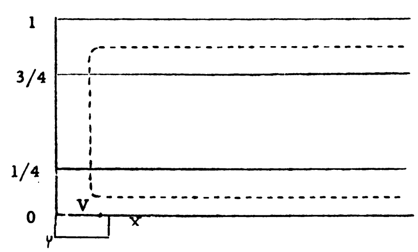


Figure 2.3:

$X \times X'$ which is acyclic on $(X \times Y') \cup (X' \times Y)$. Thus we have a natural pairing

$$\mathcal{D}_n(X, Y) \otimes \mathcal{D}_m(X', Y') \to \mathcal{D}_{n+m}((X, Y) \times (X', Y'))$$

which is compatible with homotopies. Thus, by means of Φ, it induces a pairing

$$\mathcal{L}_n(X, Y) \otimes \mathcal{L}_m(X', Y') \to \mathcal{L}_{n+m}((X, Y) \times (X', Y')).$$

□

Lemma 2.6.15. *For any classes $x \in \mathcal{L}_n(X, Y), x' \in \mathcal{L}_m(X', Y')$,*

$$\chi(x \times x') = \chi(x)\chi(x').$$

Proof. This clearly holds when $Y = Y' = \varnothing$. However, the pairing $K(X, Y) \otimes K(X', Y') \to K((X, Y) \times (X', Y'))$ which we defined earlier was the only natural pairing compatible with the pairings defined for the case $Y = Y' = \varnothing$. □

With this lemma we now have a very convenient description of the relative product. As a simple application we shall give a new construction for the generator of $\widetilde{K}(S^{2n})$.

Let V be a complex vector space and consider the exterior algebra $\wedge^*(V)$. We can regard this in a natural way as a complex of vector bundles over V. Thus we put $E_i = V \times \wedge^i(V)$, and define

$$V \times \wedge^i(V) \to V \times \wedge^{i+1}(V)$$

by

$$(v, w) \mapsto (v, v \wedge w).$$

If $\dim V = 1$ the complex has just one map and this is an isomorphism for $v \neq 0$. Thus it defines an element of $K(B(V), S(V)) \cong \widetilde{K}(S^2)$ where $B(V), S(V)$ denote the unit ball and unit sphere of V with respect to some metric. Moreover this element is, from its definition, the canonical generator of $\widetilde{K}(S^2)$ except for a sign -1.

Since

$$\wedge^*(V \oplus W) \cong \wedge^*(V) \otimes \wedge^*(W)$$

it follows that for any $V, \wedge^*(V)$ defines a complex over$V$ acyclic on $V \setminus \{0\}$, and that this gives the canonical generator of $\widetilde{K}(B(V), S(V)) = \widetilde{K}(S^{2n})$ except for a factor $(-1)^n$ (where $n = \dim V$).

More generally the same construction applies to a vector bundle V over a space X. Let us introduce the *Thom space* X^V defined as the one-point compactification of V or equivalently as $B(V)/S(V)$. Then $K(B(V), S(V)) \cong \widetilde{K}(X^V)$ and the exterior algebra of V defines an element of $\widetilde{K}(X^V)$ which we denote by λ_V. It has the two properties

(A) λ_V restricts to a generator of $\widetilde{K}(P^V)$ for each point $P \in X$.

(B) $\lambda_{V \oplus W} = \lambda_V \cdot \lambda_W$, where this product is from $\widetilde{K}(X^V) \times \widetilde{K}(X^W)$ to $\widetilde{K}(X^{V \oplus W})$.

A very similar discussion can be carried out for projective spaces. Thus if V is a vector bundle over X let $P = P(V \oplus 1)$ and let H be the standard line-bundle over P. By definition we have a monomorphism

$$H^* \to \pi^*(V \oplus 1)$$

when $\pi \colon P \to X$ is the projection. Hence tensoring with H we get a section of $H \otimes \pi^*(V \oplus 1)$. Projecting onto the first factor gives therefore a natural section

$$s \in \Gamma(H \otimes \pi^* V).$$

Consider the exterior algebra $\wedge^*(H \otimes \pi^* V)$. Each component is a vector bundle over P and exterior multiplication by s gives us a complex of vector bundles acyclic outside the subs pace where $s = 0$. But this is just the image of the natural cross-section $X \to P$. If we restrict to the complement of $P(V))$ in $P(V \oplus 1)$ then H becomes isomorphic to 1 and we recover the element which defines λ_V (identifying $P(V \oplus 1) \setminus P(V)$ with V in the usual way). This shows that the image of λ_V under the homomorphism

$$\widetilde{K}(X^V) = K(P(V \oplus 1), p(V)) \to K(P(V \oplus 1))$$

is the alternating sum

$$\sum (-1)^i [H]^i [\lambda^i V].$$

We conclude this section by remarking that everything we have been saying works equally well for G-spaces, G being a finite group. We have only used the basic facts about extensions of homomorphisms etc. which hold equally well for G bundles. Thus elements of $K_G(X, Y)$ maybe represented by G-complexes of vector bundles over X acyclic over Y. In particular the exterior algebra of a G-vector bundle V defines an element

$$\lambda_V \in \widetilde{K}_G(X^V)$$

as above.

2.7 The Thom isomorphism

If $E = \sum L_i$ is a decomposable vector bundle over X (i.e., a sum of line-bundles) then we have 2.5.3 determined the structure of $K(P(E))$ as a $K(X)$-algebra. Now for any space X we have a canonical isomorphism

$$K^*(X) \cong K(X \times S^1)$$

Also, if $\pi\colon X \times S^1 \to X$ is the projection, we have

$$P(E) \times S^1 = P(\pi^* E)$$

and so

$$K^*(P(E)) \cong K(P(\pi^* E)).$$

Thus replacing X by $X \times S^1$ in 2.5.3 gives at once

Proposition 2.7.1. *Let $E = \sum L_i$ be a decomposable vector bundle over X. Then $K^*(P(E))$, as a $K^*(X)$-algebra, is generated by $[H]$ subject to the single relation*

$$\prod([L_i][H] - 1) = 0.$$

Remark 2.7.2. As with 2.5.3 this extends at once to G-spaces giving $K^*_G(P(E))$ as a $K^*_G(X)$-algebra.

Now the Thorn space X^E may be identified with $P(E \oplus 1)/P(E)$, and at the end of §2.6 we saw that the image of λ_E in $K(P(E \oplus 1))$ is

$$\sum (-1)^i [H]^i [\lambda^i E] = \prod (1 - [L_i][H]).$$

Since this element generates (as an ideal) the kernel of

$$K^*(P(E \oplus 1)) \to K^*(P(E))$$

we deduce

Proposition 2.7.3. *Let E be a decomposable vector bundle over X. Then $\widetilde{K}^*(X^E)$ is a free $K^*(X)$-module on λ_E as generator.*

Remark 2.7.4. This "Thorn isomorphism theorem" for the decomposable case also holds as before for G-spaces. We now show how this fact can be put to use.

Corollary 2.7.5. *Let X be a G-space such that $K^1_G(X) = 0$ and let E be a decomposable G-vector bundle. Then, if $S(E)$ denotes the sphere bundle, we have an exact sequence*

$$0 \to K^1_G(S(E)) \to K^0_G \xrightarrow{\varphi} K^0_G(X) \to K^0_G(S(E)) \to 0$$

where φ is multiplication by

$$\lambda_{-1}[E] = \sum (-1)^i \lambda^i [E].$$

Proof. This follows at once by applying 2.7.3 in the exact sequence of the pair $(B(E), S(E))$. □

In order to apply this corollary when $X =$ point we need to verify

Lemma 2.7.6. $K^1_G(\text{point}) = 0$.

Proof. It is sufficient to show that

$$K_G(S^1) \to K_G(\text{point})$$

is an isomorphism. But, since G is acting trivially on S^1, we have

$$\begin{aligned} K_G(S^1) &\cong K(S^1 \otimes R(G)) \\ &\cong K(\text{point}) \otimes R(G) \\ &\cong K_G(\text{point}). \end{aligned}$$

□

Thus we can take $X =$ point in 2.7.5. Moreover if we take G abelian then E is necessarily decomposable. Thus we obtain

Corollary 2.7.7. *Let G be an abelian group, E a G-module. Then we have an exact sequence*

$$0 \to K^1_G(S(E)) \to R(G) \xrightarrow{\varphi} R(G) \to K^0_G(S(E)) \to 0$$

where φ is multiplication by

$$\lambda_{-1}[E] = \sum (-1)^i \lambda^i [E].$$

Suppose in particular that G acts freely on $S(E)$ (it is then necessarily cyclic), so that

$$K^*_G(S(E)) \cong K^*(S(E)/G).$$

Thus we deduce

Corollary 2.7.8. *Let G be a cyclic group, E a G-module with $S(E)G$-free. Then we have an exact sequence*

$$0 \to K^1(S(E)/G) \to R(G) \xrightarrow{\varphi} R(G) \to K^0(S(E)/G) \to 0$$

where φ is multiplication by $\lambda_{-1}[E]$.

Remark 2.7.9. A similar result will hold for other groups acting freely on spheres once the Thom isomorphism for K_G has been extended to bundles which are not decomposable. However, this will not be done in these notes.

As a special case of 2.7.8 take $G = \mathbb{Z}/2\mathbb{Z}, E = \mathbb{C}^n$ with the (-1) action. Then

$$S(E)/G = P_{2n-1}(\mathbb{R})$$

is a real projective space of odd dimension, and we have

$$R(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}[\rho]/(\rho^2 - 1)$$
$$\lambda_{-1}[E] = (1-\rho)^n.$$

Putting $\sigma = \rho - 2$ so that $\sigma^2 = -2\sigma$ and $\lambda_{-1}[E] = (-\sigma)^n$ we see that $\widetilde{K}^0(P_{2n-1}(\mathbb{R}))$ is cyclic of order 2^{n-1} while $K^1(P_{2n-1}(\mathbb{R}))$ is infinite cyclic. If we compare the sequences for n and $n+1$ we get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K^1(P_{2n+1}) & \longrightarrow & R(\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{(-\sigma)^{n+1}} & R(\mathbb{Z}/2\mathbb{Z}) \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle -\sigma} & & \big\downarrow{\scriptstyle 1} \\
0 & \longrightarrow & K^1(P_{2n-1}) & \longrightarrow & R(\mathbb{Z}/2\mathbb{Z}) & \xrightarrow[(\sigma)^n]{} & R(\mathbb{Z}/2\mathbb{Z})
\end{array}$$

But in $R(\mathbb{Z}/2\mathbb{Z})$ the kernel of $(-\sigma)^n$ (for $n \geq 1$) is $(2-\sigma)$ and so coincides with the kernel of $-\sigma$. Hence the map

$$K^1(P_{2n+1}) \to K^1(P_{2n-1})$$

is zero. From the exact sequences of the pairs $(P_{2n+1}, P_{2n}), (P_{2n}, P_{2n-1})$ we deduce that

$$K^1(P_{2n+1}) \to K^1(P_{2n})$$

is surjective, while

$$K^1(P_{2n}) \to K^1(P_{2n-1})$$

is injective. Hence

$$K^1(P_{2n}) = 0.$$

The exact sequence of the pair (P_{2n+1}, P_2) then shows that

$$K^0(P_{2n+1}) \to K^0(P_{2n})$$

is an isomorphism. Summarising we have established

Proposition 2.7.10. *The structure of $K^*(P_n(\mathbb{R}))$ is as follows*

$$\begin{aligned} K^1(P_{2n+1}) &= \mathbb{Z} \\ K^1(P_{2n}) &= 0 \\ \widetilde{K}^0(P_{2n+1}) &= \widetilde{K}^0(P_{2n}) = \mathbb{Z}/2^n\mathbb{Z}. \end{aligned}$$

We leave it as an exercise to the reader to apply 2.7.8 to other spaces.

We propose now to proceed to the general Thom isomorphism theorem. It should be emphasised at this point that the methods to be used do *not* extend to G-bundles. Entirely different methods are needed for G-bundles and we do not discuss them here.

We start with the following general result

Theorem 2.7.11. *Let $\pi\colon B \to X$ be a map of compact spaces, and let $\mu_1, \cdots, \mu_n$ be homogeneous elements of $K^*(B)$. Let M^* be the free ($\mathbb{Z}/2\mathbb{Z}$-) graded group generated by $\mu_1, \cdots, \mu_n$. Suppose that every point $x \in X$ has a neighbourhood U such that for all $V \subset U$, the natural map*

$$K^*(V) \otimes M^* \to K^*(\pi^{-1}(V))$$

is an isomorphism. Then, for any $Y \subset X$, the map

$$K^*(X, Y) \otimes M^* \to K^*(B, \pi^{-1}(Y))$$

is an isomorphism.

Proof. If $U \subset X$ has the property that, for all $V \subset U$,

$$K^*(V) \otimes M^* \cong K^*(\pi^{-1}(V)) \qquad (2.7.12)$$

we shall say that U is *good*. If U is good then, using exact sequences and the fact that $\otimes M^*$ preserves exactness (M^* being torsion free) we deduce

$$K^*(U, V) \otimes M^* \cong K^*(\pi^{-1}(U), \pi^{-1}(V)) \qquad (2.7.13)$$

Here we use of course the compatibility of σ with products (Lemma 2.6.1). What we have to show therefore is

$$X \text{ locally good } \Rightarrow \; X\text{good}.$$

Since X is compact it will be enough to show that

$$U_1, U_2 \text{ good } \Rightarrow \; U_1 \cup U_2 \text{ good}.$$

Now any $V \subset U_1 \cup U_2$ is of the form $V = V_1 \cup V_2$ with $V_i/subsetU_i$ (and so V_i is also good). Since

$$V/V_2 = V_1/(V_1 \cap V_2)$$

it follows that (2.7.13) holds for the pair (V, V_2). Since (2.7.12) holds for V_2 the exact sequence of (V, V_2) shows that (2.7.12) holds for V. Thus $U_1 \cup U_2$ is good and the proof is complete. □

Corollary 2.7.14. *Let $\pi\colon E \to X$ be a vector bundle, and let H be the usual line bundle over $P(E)$. Then $K^*(P(E))$ is a free $K^*(X)$-module on the generators $1, [H], [H]^2, \cdots, [H]^{n-1}$, where $[H]$ satisfies the equation $\sum(-1)^i[H]^i[\lambda^i E] = 0$.*

Proof. Since E is locally trivial it is in particular locally decomposable[1]. Hence, by (2.7.1), each point $x \in X$ has a neighbourhood U such that for all $V \subset U$, $K^*(P(E \mid V))$ is a free $K^*(V)$-module on generators $1, [H], [H]^2, \cdots, [H]^{n-1}$. Now apply (2.7.11). The equation for $[H]$ has already been established at the end of §2.6. □

Corollary 2.7.15. *If $\pi\colon E \to X$ is a vector bundle, and if $F(E)$ is the flag bundle of E with projection map $p\colon F(E) \to X$, then $p^*\colon K^*(X) \to K^*(f(E))$ is injective.*

Proof. $F(E)$ is the flag bundle over $P(E)$ of a bundle of dimension one less than $\dim(E)$. We proceed inductively on $\dim(E)$ using (2.7.14). □

Corollary 2.7.16 (The Splitting Principle)**.** *If $E_1, \cdots, E_n$ are vector bundles on X, then there exist a space F and a map $\pi\colon F \to X$ such that*

1) $\pi^*\colon K^*(X) \to K^*(F)$ is injective

2) Each $\pi^*(E_i)$ is a sum of line bundles.

Proof. We take F to be the flag bundle of $\oplus E_i$. The importance of the Splitting Principle is clear. It enables us to reduce many problems to the decomposable case. □

Corollary 2.7.17 (The Thom Isomorphism Theorem)**.** *If $\pi\colon E \to X$ is a vector bundle*

$$\Phi\colon K^*(X) \to \widetilde{K}^*(X^E)$$

defined by $\Phi(x) = \lambda_E x$ is an isomorphism.

Proof. This follows from (2.7.14) in the same way as (2.7.3) followed from (2.7.1). □

We leave the following propositions as exercises for the reader.

Proposition 2.7.18. *If $\pi\colon E \to X$ is a vector bundle, $L_1, \cdots, L_n$ the usual line bundles over $F(E)$, then the map defined by $t_i \mapsto [Li]$ defines an isomorphism of $K^*(X)$ modules*

$$K^*(X)[_1, \cdots t_n]/I \to K^*(F(E))$$

where I is the ideal generated by elements

$$\sigma^1(t_1, \cdots, t_n) - E, \sigma^2(t_1, \cdots, t_n) - \lambda^2(E), \cdots, \sigma^n(t_1, \cdots, t_n) - \lambda^n(E)$$

σ^i being the i-th elementary symmetric function.

[1] Caveat: This is the argument which does *not* generalise to G-spaces.

Proposition 2.7.19. *Let $\pi\colon E \to X$ be an n-dimensional vector bundle and let $G_k(E)$ be the Grassmann bundle (of k-dimensional subspaces) of E. Let F be the induced k-dimensional bundle over $G_k(E)$, F' the quotient bundle $p^*(E)/F$. Then the map defined by $t_i \mapsto \lambda^i(F), s_i \mapsto \lambda^i(F')$ defines an isomorphism of K^*-modules*

$$K^*(X)[t_1, \cdots, t_k, s_1, \cdots, s_{n-k}]/I \to K^*(G_k(E)),$$

where I is the ideal generated by the elements

$$(\sum_{i+j=\ell} t_i s_j) - \lambda^\ell(E) \quad \text{for all } \ell.$$

(Hint: Compare $G_k(E)$ with the flag bundle of E).

In particular, we see that if $G_{n,k}$ is the Grassmann manifold of k-dimensional subspaces of an n-dimensional vector space, $K^*(G_{n,k})$ is torsion free. This also follows from its cell decomposition. By induction we deduce K^* is torsion free for a product of Grassmannians.

Theorem 2.7.20. *Let X be a space such that $K^*(X)$ is torsion free, and let Y be a (finite) cell complex, $Y' \subset Y$ a subcomplex. Then the map*

$$K^*(X) \otimes K^*(Y, Y') \to K^*(X \times Y, X' \times Y')$$

is an isomorphism.

Proof. The theorem holds for Y a ball, Y' its boundary as a consequence of 2.7.3. It thus holds for any (Y, Y') by induction on the number of cells in Y. □

Corollary 2.7.21 (The Künneth Theorem)**.** *Let X be a space such that $K^*(X)$ is a finitely generated abelian group, and let Y be a cell complex. Then there is a natural exact sequence*

$$0 \to \sum_{i+j=k} K^i(X) \otimes K^j(X) \to K^k(X \times Y) \to \sum_{i+j=k} \operatorname{Tor}(K^i(X), K^j(Y)) \to 0$$

where all suffixes are in $\mathbb{Z}/2\mathbb{Z}$.

Proof. Suppose we can find a space Z and a map $F\colon X \to X$ such that $K^*(X)$ is torsion free, and f$f^*\colon K^*(Z) \to K^*(X)$ is surjective. Then from the exact sequence $K^*(Z/X)$ is torsion free. From the last theorem, $K^*(Z \times Y) = K^*(Z) \otimes K^*(Y)$, $K^*((Z/X) \times Y) = K^*(Z/X) \otimes K^*(Y)$. The result will then follow from the exact sequence for the pair $(Z \times Y, X \times Y)$. □

We now construct such a map $g\colon SX \to Z$. Let $a_1, \cdots, a_n$ generate $K^0(X)$, and let $b_1, \cdots, b_m$ $K^{-1}(X) = K(SX)$. Then each a_i determines a map $\alpha_i\colon X \to G_{r_i,s_i}$ for suitable r_i, s_i, and each b_i a map $\beta_i\colon SX \to G_{u_i,v_i}$. Let $\alpha = \alpha_1 \times \cdots \times \alpha_n\colon X \to G_{r_1,s_1} \times \cdots \times G_{r_n,s_n}$, and $\beta = \beta_1 \times \cdots \times \beta_m\colon SX \to G_{u_1,v_1} \times \cdots \times G_{u_m,v_m}$.

Then

$$\alpha^* \colon K^0(G') \to K^0(X) \quad \text{is subjective}$$
$$\beta^* \colon K^0(G'') \to K^0(SX) \quad \text{is subjective.}$$

Thus, if $f \colon (S\alpha) \times \beta \colon SX \to (SG') \times G'' = G$

$$f^* \colon K^*(G) \to K^*(SX) \quad \text{is surjective,}$$

and $K^*(G)$ is torsion free as required. This proves the formula for SX and this is equivalent to the formula for X. We next compute the rings $K^*(U(n))$, where $U(n)$ is the unitary group on n variables. Now for any compact Lie group G we can consider representations $\rho \colon G \to \mathrm{GL}(m, \mathbb{C})$ as defining elements $[\rho] \in K^1(G)$: we simply regard ρ as a map and disregard its multiplicative properties. Suppose now that α, β are two representations $G \to \mathrm{GL}(m, \mathbb{C}$ which agree on the closed subgroup H. Then we can define a map

$$\gamma \colon G/H \to \mathrm{GL}(m, \mathbb{C})$$

by $\gamma(gH) = \alpha(g)\beta(g)^{-1}$. This is well-defined because of the multiplicative properties of α, β. The map γ defines an element $[\gamma] \in K^1(G/H)$ whose image in $K^1(G)$ is just $[\alpha] - [\beta]$. As a particular case of this we take

$$G = U(n), \quad H = U(n-1), \quad G/H = S^{2n-1}.$$

For α, β we take the representations of G on the even and odd parts of the exterior algebra $\wedge^*(\mathbb{C}^n)$, and we identify these two parts by exterior multiplication with the n-th basic vector e_n of $\mathbb{C}$. Since $U(n-1)$ keeps e_n fixed, this identification is compatible with the action of $U(n-1)$. We are thus in the situation being considered and so we obtain an element

$$[\gamma] \in K^1(S^{2n-1}).$$

If we pass to the isomorphic group $\widetilde{K}(S^{2n})$ we see from its definition that $[\gamma]$ is just the basic element

$$\lambda_{\mathbb{C}^n} \in \widetilde{K}(S^{2n})$$

constructed earlier from the exterior algebra. Thus $[\gamma]$ is a generator of $K^1(S^(2n-1))$, and its image in $K^1(U(n))$ is $\sum(-1)^i[\lambda^i]$, where the λ^i are the exterior power representations. With this preliminary discussion we are now ready to prove:

Theorem 2.7.22. *$K^*(U(n))$ is the exterior algebra generated by $[\lambda^1], \cdots [\lambda^n]$, where λ^i is the i-th exterior power representation of $U(n)$.*

Proof. We proceed by induction on n. Consider the mapping

$$U(n) \to U(n)/U(n-1) = S^{2n-1}.$$

Since the restriction of λ^i to $U(n-1)$ is $\mu^i \oplus \mu^{i-1}$, where μ^i denotes the i-th exterior power representation of $U(n-1)$, the inductive hypothesis together with (2.7.11) imply that $K^*(U(n))$ is a free $K^*(S^{2n-1})$-module generated by the monomials in $[\lambda^1], \cdots [\lambda^{n-1}]$. But $K^*(S^{2n-1})$ is an exterior algebra on one generator $[\gamma]$ whose image in $K^*(U(n)))$ is

$$\sum_{i=0}^{n}(-1)^i[\lambda^i]$$

as shown above. Hence $K^*(U(n))$ is the exterior algebra on $[\lambda^1], \cdots, [\lambda^n]$ as required. □

Chapter 3

Operations

3.1 Exterior powers

By an operation F in K-theory, we shall mean a natural transformation $F_X : K(X) \to K(X)$. That is, for every space X, there is a (set) map $F_X : K(X) \to K(X)$, and if $f : X \to Y$ is any continuous map, $F_X^* f^* = f^* F_Y$.

Suppose that F and G are two operations which have the property that $F([E] - n) = G([E] - n)$ whenever E is a sum of line bundles and n is an integer. Then $F(x) = G(x)$ for all $x \in K(X)$, as we see immediately from the splitting principle of the last chapter.

There are various ways in which one can define operations using exterior power operations. The first of these which we shall discuss is due to Grothendieck
.

If V is a vector bundle over a space X, we define $\lambda_t[V] \in K(X)([t])$ to be the power series

$$\sum_{i=0} t^i [\lambda^i(V)]$$

The isomorphism

$$\lambda^k(V \oplus W) \cong \sum_{i+j=k} \lambda^i(V) \otimes \lambda^j(W)$$

gives us the formula

$$\lambda_t[V \oplus W] = \lambda_t(X)\lambda_t(W)$$

for any two bundles V, W. For any W the power series $\lambda_t[W]$ is a unit in $K(X)[[t]]$, because it has constant leading term 1.

Thus we have a homomorphism

$$\lambda_t : \mathrm{Vect}(X) \to 1 + K(X)[[t]]^+$$

of the additive semi-group $\mathrm{Vect}(X))$ into the multiplicative group of power series over $K(X)$ with constant term 1. By the universal property of $K(X)$ this extends uniquely to a homomorphism

$$\lambda_t \colon 1 + K(X)[[t]]^+.$$

Thus taking the coefficient of t^i we have operations

$$\lambda^i \colon K(X) \to K(X).$$

Explicitly therefore

$$\lambda_t([V] - [W]) = \lambda_t[V]\lambda_t[W]^{-1}.$$

In a very similar way we can treat the *symmetric powers* $W^i(V)$. Since

$$S^k(V \oplus W) \cong \sum_{i+j=k} S^i(V) \otimes S^j(W)$$

we obtain a homomorphism

$$S_t \colon K(X) \to 1 + K(X)[[t]]^+$$

whose coefficients define the operations

$$S^i \colon K(X) \to K(X).$$

Notice that if L is a line bundle,

$$\begin{aligned} \lambda_t(L) &= 1 + tL \\ S_t(L) &= 1 + tL + t^2 L + \cdots \\ &= (1 - tL)^{-1}. \end{aligned}$$

Thus

$$\lambda_t(L) S_t(L) = 1.$$

Thus, if V is a sum of line bundles, $\lambda_t[V]S_t[V] = 1$. Therefore, for any $x \in K(X)$, $\lambda_t(x)S_t(x) = 1$, and so

$$\lambda_t([V] - [W]) = \lambda_t[V]S_{-t}[W]$$

that is,

$$\lambda^k([V] - [W]) = \sum_{i+j=k} (-1)^j \lambda^i[V] S^j[W].$$

This gives us an explicit formula for the operations λ^i in terms of operations on bundles.

Now recall that, for any bundle $E, \dim E_x$ is a locally constant function of x. Since X is assumed compact

$$\dim E = \sup_{x \in X} \dim E_x$$

is finite. The exterior powers have the basic property that

$$\lambda^i E = 0 \quad \text{if} \quad i > \dim E.$$

Let us call an element of $K(X)$ *positive* (written $x \geq 0$) if it is represented by a genuine bundle, i.e., if it is in the image of $\mathrm{Vect}(X)$. Then

$$x \geq 0 \Rightarrow \lambda_y \in K(X)[t].$$

For many problems it is not $\dim E$ which is important but another integer defined as follows. First let us denote by $\operatorname{rank} R$ the bundle whose fibre at x is $C^{d(x)}$ where $d(x) = \dim E_x$: if X is connected then $\operatorname{rank} E$ is just the trivial bundle of dimension equal to $\dim E$. Then $E \to \operatorname{rank} E$ induces an (idempotent) ring endomorphism

$$\operatorname{rank}\colon K(X) \to K(X)$$

which is frequently referred to as the *augmentation*. The kernel of this endomorphism is an ideal denoted by $K_1(X)$. For a connected space with base-point we clearly have

$$K_1(X) = \widetilde{K}(X).$$

For any $x \in K(X)$ we have

$$x - \operatorname{rank} x \in K_1(X).$$

Now define $\dim_K x$, for any $x \in K_1(X)$, to be the least integer n for which

$$x - \operatorname{rank} x + n \geq 0$$

since every element of $K(X)$ can be represented in the form $[V] - n$ for some bundle V it follows that $\dim_K x$ is *finite* for all $x \in K(X)$. For a vector bundle E we clearly have

$$\dim_K [E] \leq \dim E.$$

Notice that

$$\dim_k x = \dim_K x_1$$

where $x_1 = x - \operatorname{rank} x$, so that $\dim_K K$ is essentially a function on the quotient $K_1(X)$ of $K(X)$.

It is now convenient to introduce operations γ^i which have the same relation to $\dim_K$ as the λ^i have to the dimension of bundles. Again following Grothendieck we define

$$\gamma_t(x) = \lambda_{t/(1-t)}(x) \in K(X)[[t]]$$

so that $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$. Thus for each i we have an operation

$$\gamma^i\colon K(X) \to K(X).$$

The γ^i are linear combinations of the λ^j for $j \leq i$ and vice-versa, in view of the formula

$$\lambda_s(x) = \gamma_{s/(1+s)}(x)$$

obtained by putting $s = t/(1-t), t = s/(1+s)$. Note that

$$\gamma_t(1) = (1-t)^{-1}$$

and for a line-bundle L

$$\gamma_t([L]-1) = 1 + t([L]-1)$$

Proposition 3.1.1. *Let $x \in K_1(X)$, then $\gamma_t(x)$ is a polynomial of degree $\leq \dim_K x$.*

Proof. Let $n = \dim_K x$, so that $x + n \geq 0$. Thus $x + n = [E]$ for some vector bundle E. Moreover $\dim E = n$ and so

$$\lambda^i(E) = 0 \quad \text{for} \quad i > n.$$

Thus $\lambda_t(x+n)$is a polynomial of degree $\leq n$. Now

$$\begin{aligned}
\gamma_t(x) &= \gamma t(x+n)\gamma_t(1)^{-n} \\
&= \gamma t/(1-t)(x+n)(1-t)^n \\
&= \sum_{i=0}^{n} \lambda^i(x+n)t^i(1-t)^{n-i}
\end{aligned}$$

and so is a polynomial of degree $\leq n$ as stated. □

We now define $\dim_\gamma x$ to be the largest integer n such that $\gamma^n(x - \operatorname{rank} x) \neq 0$, and we put

$$\begin{aligned}
\dim_K X &= \sup_{x \in K(X)} \dim_K x \\
\dim_\gamma X &= \sup_{x \in K(X)} \dim_\gamma x.
\end{aligned}$$

By 3.1.1 we have

$$\dim_\gamma x \leq \dim_K x, \quad \dim_\gamma X \leq \dim_K X.$$

We shall show that, under mild restrictions, $\dim_K X$ is finite. For this we shall need some preliminary lemmas on symmetric functions.

Lemma 3.1.2. *Let $x_1, \cdots, x_n$ be indeterminates. Then any homogeneous polynomial in $\mathbb{Z}[x_1, \cdots, x_n]$ of degree $n(n-1)$ lies in the ideal generated by the symmetric functions of $(x_1, \cdots, x_n)$ of positive degree.*

Proof. Let $\sigma_i(x_1, \cdots, x_n)$ be the i-th elementary symmetric function. Then the equation

$$x^n - \sigma_1 x^{n-1} + \sigma_2 c^{n-2} - \cdots + (-1)^n \sigma_n = 0$$

has roots $x = x_i$. Thus x_i^n is in the ideal generated by $\sigma_1, \cdots \sigma_n$. But any monomial in $x_1, \cdots, x_n$ of degree $> n(n-1)$ is divisible by x_i^n for some i and so is also in this ideal. □

Lemma 3.1.3. *Let $x_1, \cdots, x_n, y_1, \cdots, y_m$ be indeterminates and let*

$$a_i = \sigma_i(x_1, \cdots, x_n), \quad b_i = \sigma_i(y_1, \cdots, y_m)$$

be the elementary symmetric functions. Let I be any ideal in $\mathbb{Z}[a, b], J$ its extension in $\mathbb{Z}[x, y]$. Then

$$H \cap \mathbb{Z}[a, b] = I.$$

Proof. It is well-known that $\mathbb{Z}[x]$ is a free $\mathbb{Z}[a]$-module, with basis the monomials

$$x^r = x_1^{r_1} x_2^{r_2} \cdots x_{n-1}^{r_{n-1}} \quad 0 \leq r_i \leq n - i.$$

Hence $\mathbb{Z}[x, y] = \mathbb{Z}[x] \otimes \mathbb{Z}[y]$ is a free module over $\mathbb{Z}[a, b] = \mathbb{Z}[a] \otimes \mathbb{Z}[n]$ with basis the monomials $x^r u^s$. Then the ideal $J \subset \mathbb{Z}[x, y]$ consists of all elements F of the form

$$f = \sum f_{r,s} x^r y^s \quad \text{with} \quad f_{r,s} \in I.$$

Since the $x^r y^x$ are a free basis f belongs to $\mathbb{Z}[a, b]$ if and only if $f_{r,s} = 0$ for $r, s \neq (0, 0)$ in which case

$$f = f_{0,0} \in I.$$

Thus $J \cap \mathbb{Z}[a, b] = I$ as stated. □

Remark 3.1.4. This lemma is essentially an algebraic form of the splitting principle since it asserts that we can embed $\mathbb{Z}[a, b]/I$ in $\mathbb{Z}[x, y]/J$. It is of course purely formal in character and it seems preferable to use this rather than the topological splitting principle whenever we are dealing with formal algebraic results. The topological splitting principle depends of course on the periodicity theorem and should only be used when we are dealing with properties that lie at that depth.

Lemma 3.1.5. *Let K be a commutative ring (with 1) and suppose*

$$a(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$
$$b(t) = 1 + b_1 t + b_2 t^2 + \cdots + b_m t^m$$

are elements of $K[t]$ such that

$$a(t)b(t) = 1.$$

Then there exists an integer $N = N(n, m)$ such that any monomial

$$a_1^{r_1} a_2^{r_2} \cdots a_n^{r_n}$$

of weight $\sum j r_j > N$ vanishes.

Proof. Passing to the universal situation it is sufficient to prove that if $a_1, \cdots, a_n, b_1, \cdots, b_m$ are indeterminates, then any monomial α in the a_i of weight $\geq N$ lies in the ideal I generated by the elements

$$c_k = \sum_{i+j=k} a_i b_j \quad k = 1, \cdots, mn \quad (a_0 = b_0 = 1).$$

By (3.1.3), introducing indeterminates $x_1, \cdots, x_n, y_1, \cdots, y_m$, it is sufficient to prove that α belongs to the extended ideal J. But c_k is just the k-th elementary symmetric function of the $(m+n)$ variables $x_1, \cdots, x_n, y_1, \cdots, y_m$. The result now follows by applying (3.1.2) with $N = (m+n)(m+n-1)$. □

Remark 3.1.6. The value for $N(n, m)$ obtained in the above proof is not best possible. It can be shown by more detailed arguments that the best possible value is mn.

We now apply these algebraic results:

Proposition 3.1.7. *Let $x \in K_1(X)$. Then there exists an integer N, depending on x, such that any monomial*

$$\gamma^{i_1}(x)\gamma^{i_2}(x)\cdots\gamma^{i_k}(x)$$

of weight $\sum_{j=1}^{k} i_j > N$ is equal to zero.

Proof. We apply (3.1.5) to the polynomials $\gamma_t(x), \gamma_t(-x)$. Note therefore, that N depends on $\dim_\gamma x, \dim_\gamma(-x)$. □

Since $\gamma^1(x) = x$ we deduce:

Corollary 3.1.8. *Any $x \in K_1(X)$ is nilpotent.*

If we define the degree of each γ^i to be one, then for any monomial in the γ^i we have

$$\text{weight} \geq \text{degree}.$$

In view of (3.1.7), therefore, all monomials in $\gamma^i(x)$ of sufficiently high degree are zero if $x \in K_1(X)$. Thus we can apply a *formal power series*[1] in the γ^i to any $x \in K(X)$. Let us denote by $\mathrm{Op}(K_1, K)$ the set of all operations $K_1 \to K$. This has a ring structure induced by the ring structure of K (addition and multiplication of values). Then by what we have said we obtain a ring homomorphism

$$\varphi\colon \mathbb{Z}[[\gamma^1, \cdots, \gamma^n, \cdots]] \to \mathrm{Op}(K_1, K)$$

Theorem 3.1.9.

$$\varphi\colon \mathbb{Z}[[\gamma^1, \cdots, \gamma^n, \cdots]] \to \mathrm{Op}(K_1, K)$$

is an isomorphism.

Proof. Let $Y_{n,m}$ be the product of n copies of $P_m(\mathbb{C})$. Using the base point $P_0(\mathbb{C})$ of $P_m(\mathbb{C})$ the $Y_{n,m}$ form a direct system of spaces with inclusions

$$Y_{n,m} \to Y_{n',m'} \quad \text{for} \quad n' \geq n, m' \geq m.$$

[1] As usual a formal power series means a sum $f = /sum f_n$ where f_n is a homogeneous *polynomial* of degree n (and so involves only a finite number of the variables).

Then $K(Y_{n,m})$ is an inverse system of groups with

$$\begin{aligned} K(Y_{n,m}) &= \mathbb{Z}[x_1, \cdots, x_n]/(x_1^{m+1}, \cdots, x_n^{m+1}) \\ \varprojlim_m K(Y_{n,m}) &= \mathbb{Z}[[x_1, \cdots, x_n]] \\ \varprojlim_{m,n} K(Y_{n,m}) &= \varprojlim_n \mathbb{Z}[[x_1, \cdots, x_n]] \end{aligned}$$

Any operation will induce an operation on the inverse limits. Hence we can define a map

$$\psi\colon \operatorname{Op}(K_1, K) \to \varprojlim_n \mathbb{Z}[[x_1, \cdots, x_n]]$$

by $\psi(f) = \varprojlim f(x_1 + x_2 + \cdots + x_n)$. Since, in $K(Y_{n,m})$ we have

$$\gamma_t(x_1 + x_2 + \cdots + x_n) = \prod_{i=1}^{n}(a + x_i t)$$

it follows that

$$\psi\varphi(\gamma^i) = \varprojlim_n \sigma_i(x_1 + x_2 + \cdots + x_n)$$

where σ_i denotes the i-th elementary symmetric function. In particular, therefore $\psi\varphi$ is injective and so φ is injective. Moreover the image of $\psi\varphi$ is

$$\mathbb{Z}[[\sigma_1, \cdots, \sigma_n]]$$

which is the same as

$$\varprojlim_n \mathbb{Z}[[x_1, \cdots, x_n]]^{S_n}$$

where $[]^{S_n}$ denotes the subring of invariants under the action of symmetric group S_n. But, for all $f \in \operatorname{Op}(K_1, K)$,

$$\psi(f) = \varprojlim f(x_1 + x_2 + \cdots + x_n)$$

lies in this group. In other words

$$\operatorname{im}\psi\varphi = \operatorname{im}\psi.$$

To complete the proof it remains now to show that *psi* is injective. Suppose then that $\psi(f) = 0$. Since any line bundle over a space X is induced by a map into some ${}_n(\mathbb{C})$ it follows that

$$f([E] - n) = 0$$

whenever E is a sum of n line bundles. By the splitting principle this implies that

$$f(x) = 0 \quad \text{for all} \quad x \in K_1.$$

i. e. , f is the zero operation, as required. □

Let us define $H^0(X, \mathbb{Z})$ to be the ring of all continuous maps $X \to \mathbb{Z}$. Then we have a direct sum decomposition of groups

$$K(X) = K_1(X) \oplus H^0(X, \mathbb{Z})$$

determined by the rank homomorphism. It is easy to see that there are no non-zero natural homomorphisms

$$H^0(X, \mathbb{Z}) \to K_1(X)$$

and so $\mathrm{Op}(K) = \mathrm{Op}(K, K)$ differs from $\mathrm{Op}(K_1 K)$ only by $\mathrm{Op}(H^0(\mathbb{Z}))$ which is the ring of all maps $\mathbb{Z} \to \mathbb{Z}$. Thus (3.1.9) gives essentially a complete description of $\mathrm{Op}(K)$.

We turn now to a discussion of finiteness conditions on $K(X)$. First we deal with $H^0(X, \mathbb{Z})$.

Proposition 3.1.10. *The following are equivalent*

(A) $H^0(X, \mathbb{Z})$ *is a Noertherian ring,*

(B) $H^0(X, \mathbb{Z})$ *is a finite* $\mathbb{Z}$*-module.*

Proof. ((B) $\Rightarrow$ (A)): Obvious.
((A) $\Rightarrow$ (B)): Suppose $H^0(X, \mathbb{Z})$ is Noetherian. Assume if possible that we can find a strictly decreasing infinite chain of components (open and closed sets) of X

$$X = X_0 \supset X_1 \supset \cdots \supset X_N \supset X_{n+1} \supset \cdots .$$

Then for each n we can find a continuous map $f_{nm} \colon X \to \mathbb{Z}$ such that

$$f_n(X_{n+1}) = 0$$
$$f_n(X_n \setminus X_{n+1}) = 1.$$

Consider the ideal I of $H^0(X, \mathbb{Z})$ consisting of maps $f \colon X \to \mathbb{Z}$ such that $f(X_n = 0$ for some n. Since $H^0(X, \mathbb{Z})$ is Noetherian I is finitely generated and hence there exists N such that

$$f(X_n) = 0 \quad \text{for all} \quad f \in I.$$

But this is a contradiction because

$$f_N \in I, \quad f_N(X_N) \neq 0.$$

Thus X has only a finite number of components, so that

$$X = \sum_{i=1}^{n} X_i$$

with X_i connected. Hence $H^0(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^n$. ∎

Passing now to $K(X)$ we have

Proposition 3.1.11. *The following are equivalent*

(A) $K(X)$ is a Noertherian ring,

(B) $K(X)$ is a finite $\mathbb{Z}$-module.

Proof. ((A) $\Rightarrow$ (B)): Assume (A), then $H^0(X,\mathbb{Z})$ which is a factor ring of $K(X)$ is also Noetherian. Hence by (3.1.10), $H^0(X,\mathbb{Z})$ is a finite $\mathbb{Z}$-module. Now $K_1(X)$ is an ideal of $K(X)$ consisting of nilpotent elements (3.1.8). Since $K(X)$ is Noetherian it follows that $K_1(X)$ is a nilpotent ideal. For brevity put $I = K_1(X)$. Then $I^n = 0$ for some n and the $I^m/I^{m+1}, m = 0, 1, \cdots, n-1$ are all finite modules over $K/I = H^0(X,\mathbb{Z})$. Hence $K(X)$ is a finite $H^0(X,\mathbb{Z})$-module and so also a finite $\mathbb{Z}$-module. □

Examples of spaces X for which $K(X)$ is a finite $\mathbb{Z}$-module are cell-complexes.

Let us now define a filtration of $K(X)$ by the subgroups $K_n^\gamma(X)$ generated by all monomials

$$\gamma^{i_1}(X_1)\gamma^{i_2}(X_2)\cdots\gamma^{i_k}(X_k)$$

with $\sum_{j=1}^k i_j \geq n$ and $x_i \in K_1(X)$. Since $\gamma^1(x) = x$, we have $K_1^\gamma = K_1$. If $x \in K_n^\gamma(X)$ we say that x has *γ-filtration* $\geq n$ and write $F_\gamma(x) \geq n$.

Proposition 3.1.12. *Assume $K(X)$ is a finite $\mathbb{Z}$-module. Then for some n*

$$K_n^\gamma(X) = 0.$$

Proof. Let $x_1, \cdots, x_s$ be generators of $K_1(X)$ and let $N_j = N(x_j)$ be the integers given by (3.1.7). Because of the formula

$$\gamma_t(a+b) = \gamma_y(a)\gamma_t(b)$$

it will be sufficient to show that there exists N such that all monomials in the $\gamma^i(x_j)$ of total weight $> N$ are zero. But taking $N = \sum_{j=1}^s N_j$ we see that any such monomial must, for some j, have weight $> N_j$ in the $\gamma^i(x)$. Hence by (3.1.7) this monomial is zero. □

Corollary 3.1.13. *Assume $K(X)$ is a finite $\mathbb{Z}$-module. Then* $\dim_\gamma X$ *is finite.*

We call the reader's attention to certain further properties of the operations γ^i.

Proposition 3.1.14. *If V is a bundle of dimension n, $\lambda_{-1}[V] = (-1)^n\gamma^n([V]-n)$. Thus $\widetilde{K}^*(X^V)$ is a free K^* module generated by $\gamma^n([V]-n)$.*

Proposition 3.1.15. *There exist polynomials P_i, Q_{ij} such that for all x, y*

$$\gamma^i(xy) = P_1(\gamma^1(x),\gamma^1(y),\gamma^2(x),\gamma^2(y),\cdots,\gamma^i(x),\gamma^i(y))$$
$$\gamma^i(\gamma^j(x)) = Q_{ij}(\gamma^1(x),\cdots,\gamma^{ij}(x)).$$

We leave these proofs to the reader, who may verify them easily by use of the splitting principle.

3.2 The Adams operations

We shall now separate out for special attention some operations with particularly pleasing properties. These were introduced by J. F. Adams. We define $\psi^0(x) = \operatorname{rank}(x)$. In the ring $K(X)[[t]]$ we define $\psi_t(x) = \sum_{i=0} t^i\psi^i(x)$ by

$$\psi_t(x) = \psi^0(x) - t\frac{d}{dt}(\log \lambda_{-t}(x)).$$

Notice that since all of the coefficients of this power series are integers, this definition makes sense.

Proposition 3.2.1. *For any $x, y \in K(X)$*

1) $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$ for all k.

2) If x is a line bundle, $\psi^k(x) = x^k$.

3) Properties 1 and 2 uniquely determine the operations ψ.

Proof. $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$, so that $\psi^k(xy) = \psi^k(x) + \psi^k(y)$ for each k.
If x is a line bundle, $\lambda_{-t}(x) = 1 - tx$, so that

$$\begin{aligned}\frac{d}{dt}(\log(1-tx)) &= \frac{-x}{1-tx}\\ &= -x - tx^2 - t^2x^3 - \cdots .\end{aligned}$$

Thus $\psi_t(x) = 1 + tx + t^2]x^2 + \cdots$.
The last part follows from the splitting principle. □

Proposition 3.2.2. *For any $x, y \in K(X)$*

1. *$\psi^k(xy) = \psi^k(x)\psi^k(y)$ for all k.*
2. *$\psi^k(\psi^\ell(x)) = \psi^{k\ell}(x)$ for all k, ℓ.*
3. *If p is rime, $\psi^p(x) \equiv x^p \bmod p$.*
4. *If $u \in \widetilde{K}(S^{2n}), \psi^k(u) = k^n u$ for all k.*

Proof. The first two assertions follow immediately from the last proposition and the splitting principle. Also, from the splitting principle, $\psi^p(x) = x^p + pf(\lambda^1(x), \cdots, \lambda^p(x))$, where f is some polynomial with integral coefficients. Finally, if h is the generator of $\widetilde{K}(S^2)\psi^k(h) = kh$. Since $S^{2n} = S^2 \wedge \cdots \wedge S^2$, and $\widetilde{K}(S^{2n})$ is generated by $h \otimes h \otimes \cdots \otimes h$, the last assertion follows from the first. □

We next give an application of the Adams operations ψ^k. Suppose that $f\colon S^{4n-1} \to S^{sn}$ is any map. We define the *Hopf invariant* $H(f)$ as follows. Let X_f be the mapping cone of f. Let $i\colon S^{2n} \to X_f$ be the inclusion, and let $j\colon X_f \to S^{4n}$ collapse S^{2n}. Let u be the generator of $\widetilde{K}(S^{4n})$. From the exact sequence we see that there is an element $x \in \widetilde{K}(X_f)$ such that $i^*(x)$ generates $\widetilde{K}(S^{2n})$. $\widetilde{K}(X_f)$ is the free abelian group generated by x and $y = j^*(u)$. Since $(i^*(x))^2 = 0, x^2 = Hy$ for some H. This integer H we define as the Hopf invariant of f. Clearly, up to a minus sign, $H(f)$ is well defined. The following theorem was first established by J. F. Adams by cohomological methods called "secondary operations".

Theorem 3.2.3 (Hopf invariant one problem)**.** *If $H(f)$ is odd, then $n = 1, 2,$ or 4.*

Proof. Let $\psi^2(x) = 2^n x + ay, \psi^3(x) = 3^n x + by$. Since $\psi^2(x) \equiv x^2 \bmod 2$, a is odd. $\psi^k(y) = j^*(\psi^k(u)) = k^{2n} y$. Thus, we see that

$$\psi^6(x) = \psi^3(\psi^2(x)) = 6^n x + (2^n b + 3^{2n} a)y$$
$$\psi^6(x) = \psi^2(\psi^3(x)) = 6^n x + (2^{2n} b + 3^n a)y.$$

Thus $2^n b + 3^{2bn} a = 2^{2n} b + 3^n a$, or $2^n(2^n - 1)b = 3^n(3^n - 1)a$. Since a is odd, 2^n divides $3^n - 1$, which by elementary number theory can happen only if $n = 1, 2,$ or 4. □

If $n = 1, 2,$ or 4, the Hopf maps determined by considering S^{4n-1} as a subspace of the non-zero vectors in 2-dimensional complex, quaternionic, or Cayley space, and S^{2n} as the complex, quaternionic, or Cayley projective line all have Hopf invariant one. We leave the verification to the reader.

Proposition 3.2.4. *Let $x \in K(X)$ be such that $F_\gamma(x) \geq n$. Then for any k we have*

$$F_\gamma(\psi^k(x) - j^n x) \geq n + 1.$$

Proof. If $n = 0$ we have

$$\psi^k(x) = \psi^k(\operatorname{rank} x + x_1) = \operatorname{rank} x + \psi^k x_1.$$

Here x_1 and so $\psi^k x_1$ are in $K_1(X)$. Thus

$$\psi^k x - x = \psi^k x_1 - x_1 \in K_1(X) = K_1^\gamma(X).$$

Consider now $n > 0$. Since ψ^k is a ring homomorphism it is sufficient to prove that the composition $\psi^k \cdot \gamma^n - k^n \gamma^n$ (where $\psi^k \in \operatorname{Op}(X), \gamma^n \in \operatorname{Op}(K_1, K))$) is equal to a polynomial in the γ^i in which each term has weight $\geq n + 1$. As in (3.1.9) we have isomorphisms

$$\mathbb{Z}[[\gamma^1, \cdots]] \cong \operatorname{Op}(K_1, K) \cong \varprojlim_m \mathbb{Z}[x_1, \cdots, x_m]^{S_m}$$

in which γ^i corresponds to i-th elementary symmetric function σ_i of the x_j. Now

$$\psi^k(x_i) = (1 + x_i)^k - 1$$

and so

$$\begin{aligned}\psi^k(\sigma_n(x_1, \cdots)) &= \sigma_n((1+x_i)^k - 1, \cdots) \\ &= k^n \sigma_n(x) + f\end{aligned}$$

where f is a polynomial in the σ_i of weight $\geq n+1$. Since $\psi^k \cdot \gamma^n$ corresponds to $\psi^k(\sigma_n)$ by the above isomorphisms the proposition is established. □

Iterating (3.2.4) we obtain:

Corollary 3.2.5. *If* $K^\gamma_{n+1}(X) = 0$,

$$[\prod_{m=0}^{n} (\psi^{k_m} - (k_m)^m)] = 0$$

for any sequence of non-negative integers $k_0, k_1, \cdots, k_n$.

By (3.1.12) we can apply 3.2.5 in particular whenever $K(X)$ is a finite $\mathbb{Z}$-module.

Notice that ψ^k acts as a linear transformation on the vector space $K(X) \otimes \mathbb{Q}$. Taking $k = k_m$ for all m in (3.2.5) we see that

$$\prod_{m=0}^{n} (\psi^k - k^m) = 0 \quad \text{on} \quad K(X) \otimes \mathbb{Q}.$$

Thus the eigenvalues of each ψ^k are powers of k not exceeding k^m. Let $V_{k,i}$ denote the eigenspace of ψ^k corresponding to the eigenvalue k^i (we may have $V_{k,i} = 0$). Then if $k > 1$, we have an orthogonal decomposition of the identity operator 1 of $K(X) \otimes \mathbb{Q}$:

$$1 = \sum \Pi_i, \quad \Pi_i = \prod_{m \neq i} (\psi^k - k^m)/(k^i - k^m).$$

Thus $K(X) \otimes \mathbb{Q}$ is the direct sum of the $V_{k,i}$. Now put in (3.2.5),

$$k_i = \ell, \quad k_m = k \quad \text{for} \quad m \neq i$$

and we see that

$$(\psi^\ell - \ell^i)V_{k,i} = 0$$

and so $V_{k,i} \subset V_{\ell,i}$. Hence we deduce

Proposition 3.2.6. *Assume* $K(X)$ *has finite* γ*-filtration and let* $V_{k,i}$ *denote the eigenspace of* ψ^k *on* $K(X) \otimes \mathbb{Q}$ *corresponding to the eigenvalue* k*. Then if* $k, \ell > 1$ *we have*

$$V_{k,i} = V_{\ell,i}.$$

Since the subspace $V_{k,i}$ does not depend on k (for $k > 1$) we may denote it by a symbol independent of k. We shall denote it by $H^{2i}(X;\mathbb{Q}))$ and call it the $2i$-th *Betti group* of X. From (3.2.4) it follows that the eigenvalue $k^0 = 1$ occurs only in $H^0(X,\mathbb{Z}) \otimes \mathbb{Q}$. Thus our notation is consistent in that

$$H^0(X,\mathbb{Z}) \otimes \mathbb{Q} = H^0(X;\mathbb{Q}).$$

We define the odd Betti groups by

$$H^{2m+1}(X;\mathbb{Q}) = H^{2m+2}(SX^+;\mathbb{Q})$$

where $X^+ = X \cup$point and S denotes reduced suspension. If the spaces involved are finite-dimensional we put

$$B_k = \dim_{\mathbb{Q}} H^k(X;\mathbb{Q})$$

and the Euler characteristic $E(X)$ is defined by

$$E(X) = \sum (-1)^k B_k = \dim_{\mathbb{Q}}(K^0(X) \otimes \mathbb{Q}) - \dim_{\mathbb{Q}}(K^1(X) \otimes \mathbb{Q}).$$

Note that the Kunneth formula (when applicable) implies 5

$$E(X \times Y) = E(X)E(Y).$$

The following proposition is merely a reformulation of (3.2.4) in terms of the notation just introduced:

Proposition 3.2.7.

$$K_n^\gamma \otimes \mathbb{Q} = \sum_{m \geq n} H^{2m}(X;\mathbb{Q})$$

and so

$$\{K_n^{\gamma(X)}/K_{n+1}^\gamma(X)\} \cong H^{2n}(X;\mathbb{Q}).$$

Since $\psi^k u = ku$ for the generator u of $\widetilde{K}(S^2)$ it follows that

$$\psi^k \beta(x) = k\beta\psi^k(x)$$

where$\beta \colon K(X) \to K^{-2}(X)$ is the periodicity isomorphism. Thus β induces an isomorphism

$$H^{2m}(X;\mathbb{Q}) \cong H^{2m+2}(S^2X^+;\mathbb{Q}).$$

From the way the odd Betti groups were defined it follows that, for all k

$$H^k(X;\mathbb{Q}) \cong H^{k+1}(SX^+;\mathbb{Q}). \tag{3.2.8}$$

If we now take the exact K-sequence of the pair X, A, tensor with $\mathbb{Q}$, decompose under ψ^k and use (3.2.8) we obtain:

Proposition 3.2.9. *If $A \subset X$, and if both $K^*(X), K^*(A)$ are finite $\mathbb{Z}$-modules the exact sequence*

$$\cdots \to K^{i-1}(A) \xrightarrow{\delta} K^i(X,A) \to K^i(X) \to K^i(A) \xrightarrow{\delta} \cdots$$

induces an exact sequence

$$\cdots \to H^{i-1}(A;\mathbb{Q}) \xrightarrow{\delta} H^i(X,A;\mathbb{Q}) \to H^i(X;\mathbb{Q}) \to H^i(A;\mathbb{Q}) \xrightarrow{\delta} \cdots$$

We next give a second application of the operations ψ^k. Since $P_n(\mathbb{C})/P_{n-1}(\mathbb{C})$ is the sphere S^{2n}, we have an inclusion of S^{2n} into $P_{n+k}(\mathbb{C})/P_{n-1}(\mathbb{C})$ for a11 k. We should like to know for which values of n and k, S^{2n} is a retract of $P_{n+k}(\mathbb{C})/P_{n-1}(\mathbb{C})$. That is, we should like to know when can there exist a map $fcolonP_{n+k}(\mathbb{C})/P_{n-1}(\mathbb{C}) \to S^{2n}$ which is the identity on S^{2n}. We sha11 obtain certain necessary conditions on n and k for such an f to exist.

Theorem 3.2.10. *Assume a retraction*

$$f\colon P_{n+k}(\mathbb{C})/P_{n-1}(\mathbb{C}) \to P_n(\mathbb{C})/P_{n-1}(\mathbb{C}) = S^{2n}$$

exists. Then the coefficients of x^i for $i \le k$ in $(\frac{\log(1+x)}{x})^n$ are all integers.

Proof. Let ξ be the usual line-bundle over P_{n+k} and let $x = \xi - 1$. Then $K(P_{n+k})$ is a free abelian group on generators $x^s, 0 \le s \le +k$, and we may identify $K(P_{n+k}(\mathbb{C}), P_{n-1}(\mathbb{C})$ with the subgroup generated by x^s with $n \le s \le n+k$. In $K(P_{n+k}) \otimes \mathbb{Q}$ put $y = \log(1+x)$, so that $\xi = e^y$. Then

$$e^{ry} = \xi^r = \psi^r(e^y) = e^{\psi^r(y)},$$

so that $\psi^r(y) = ry$. Thus $H^{2s}(P_{n+k}(\mathbb{C})/P_{n-1}(\mathbb{C});\mathbb{Q})$, for $n \le s \le n+k$ is a one-dimensional space generated by y^s. Now let u generate $\widetilde{K}(S^{2n})$, and let

$$f^*(u) = \sum_{i=n}^{n+k} a_i x^i.$$

Since f is a retract we have $a_n = 1$. Since $\psi^k u = k^n u, f^*(u)$ must be a multiple of y^n, so that

$$\sum_{i=n}^{n+k} a_i x^i = \lambda y^n.$$

Restricting to S^{2n} we see that $\lambda = 1$, and so

$$y^n = (\log(1+x))^n$$

has all coefficients from x^n to x^{n+k} *integral* as required. □

Remark 3.2.11. It has been shown by Adams and Grant-Walker (Proc. Camb. PhilÂ· Soc. 61 (1965), 81-103) that (3.2.10) gives a sufficient condition for the existence of a retraction.

Suppose once more that we have a map $f\colon S^{2m+2n-1} \to S^{2m}$. Then we can attach to f an invariant $e(f) \in \mathbb{Q}/\mathbb{Z}$ in the following fashion.

Let X be the mapping cone of $f, i = S^{2m} \to X$ the inclusion, $j\colon X \to S^{2n+2m}$ the map which collapses S^{2m}. Let u generate $\widetilde{K}^0(S^{2m+2n})$, v generate $\widetilde{K}^0(S^{2m}))$, and let $x \in \widetilde{K}^0(X)$ be such that $i^*(x) = v$. Let $y = j^*(u)$. Then for any k,

$$\psi^k(x) = k^m x + a_k y.$$

As before, we know that $\psi^k \psi^\ell = \psi^\ell \psi^k$, so that

$$k^n(k^m - 1)a_\ell = \ell^n(\ell^m - 1)a_k.$$

Thus

$$e(f) = \frac{a_k}{k^n(k^m - 1)} \in \mathbb{Q}$$

is well defined once x is chosen. If x is changed by a multiple of y, $e(f)$ is changed by an integer, so that $e(f) \in \mathbb{Q}/\mathbb{Z}$ is well defined. We leave to the reader the elementary exercise that $e\colon \prod_{2n+2m-1}(S^{2m}) \to \mathbb{Q}/\mathbb{Z}$ is a group homomorphism. It turns out that this is a very powerful invariant.

3.3 The group $J(X)$

In this section we assume, for simplicity, that X is connected. One can introduce a notion of equivalence between vector bundles, known as *fibre homotopy equivalence*, which is of much interest in homotopy theory. Let E, E' be two bundles over a space X, and suppose that both E, E' have been given Hermitian metrics. Then E and E' are said to be *fibre homotopy equivalent* if there exist maps $f\colon S(E) \to S(E')$, $g\colon S(E') \to S(E)$, commuting with the projection onto X, and such that gf and fg are homotopic to the identity through fibre-preserving maps. Clearly this is an equivalence relation defined on the set of equivalence classes of vector bundles over X .

Fibre homotopy equivalence is additive; that is, if E, E' are fibre homotopy equivalent to F, F' respectively, then $E \oplus E'$ is fibre-homotopy equivalent to $F\oplus, F'$. This follows from the fact that $S(E \oplus E')$ may be viewed as the fibre-join of the two fibre spaces $S(E), S(E')$: in general the fibre-join of $\pi\colon Y \to X, \pi'\colon Y' \to X$ is defined as the space of triples y, t, y' where $t \in I, \pi(y) = \pi'(y')$ and we impose the equivalence relations

$$(y, 0, y_1') \sim (y, 0, y_2')$$
$$(y_2, 1, y') \sim (y_2, 0, y')$$

We say that two bundles E, E' are *stably* fibre-homotopy equivalent if there exist trivial bundles V, V' such that $E \oplus V$ is fibre-homotopy equivalent to $E' \oplus V'$. The set of all stable fibre-homotopy equivalence classes over x forms a semi-group which we denote by $J(X)$. Since every vector bundle E has a complementary bundle F so that $E \oplus F$ is trivial it follows that $J(X)$ is a group and hence the map

$$\mathrm{Vect}(X) \to J(X)$$

extends to an epimorphism

$$K(X) \to J(X)$$

which we also denote by J.

If we have two bundles E, E' and if $\pi\colon S(E) \to X, \pi'\colon S(E') \to X$ are the projection maps of the respective sphere bundles, the Thom complexes $X^E, X^{E'}$ are just the mapping cones of the maps π, π' respectively. Thus, we see that if E and $E^{[\prime]}$ are fibre homotopy equivalent, X^E and $X^{E'}$ have the same homotopy type. However, if E is a trivial bundle of dimension $n, X^E = S^{2n}(X^+)$. Thus, to show that $J(X) \neq 0$, it suffices to show that X^E does not have the same stable homotopy type as a suspension of X^+.

We shall now show how to use the operations ψ^j of §3.2 to give necessary conditions for $J(E) = 0$. By the Thom isomorphism (2.7.17) we know that $\widetilde{K}(X^E)$ is a free $K(X)$-module generated by λ_E. Hence, for any k, there is a unique element $\rho(E) \in K(X)$ such that

$$\psi^k(\lambda_E) = \lambda_E \rho^k(E).$$

The multiplicative property of the fundamental class λ_E, established in §3.2, together with the fact that ψ^k preserves products, shows that

$$\rho^k(E \oplus E') = \rho^k(E) \cdot \rho^k(E').$$

Also, taking $E =!$, and recalling that

$$\psi^k \cdot \beta = k\beta \cdot \psi^k$$

where β is the periodicity isomorphism, we see that

$$\rho^k(1) = k.$$

Now let $\mathbb{Q}_k = \mathbb{Z}[1/k]$ be the sub ring of $\mathbb{Q}$ consisting of fractions with denominators a power of k. Then if we put

$$\sigma^k(E) = k^{-n}\rho_k(E) \quad n = \dim E$$

we obtain a homomorphism

$$\sigma^k\colon K(X) \to G_k$$

where G_k is the *multiplicative group of units* of $K(X) \otimes \mathbb{Q}_k$. Suppose now E is fibre-homotopically trivial, then there exists $u \in \widetilde{K}(X^E)$ such that $\psi^k u = k^n u$. Putting $u = \lambda_E a$ we find that

$$\psi^k \lambda_E \cdot \psi^k a = k^n \lambda_E a$$

and so

$$\sigma^k(E) \cdot \psi^k(a) = a.$$

Moreover, restricting to a point, we see that a has augmentation 1, so a and $\psi^k(a)$ are both elements of G_k. Hence v/e may write

$$\sigma^k = \frac{a}{\psi^k(a)} \in G_k.$$

Since $\sigma^k(E))$ depends only on the stable class of R, we have established the following

Proposition 3.3.1. *Let $H_k \subset G_k$ be the subgroup generated by all elements of the form $a/\psi^k(a)$ with a a unit of $K(X)$. Then*

$$\sigma^k \colon K(X) \to G_k$$

maps the kernel of J into H_k, and so induces a homomorphism

$$J(X) \to G_k/H_k$$

In order to apply (3.3.1) it is necessary to be able to compute σ^k or equivalently ρ^k. Now

$$\rho^k \in \operatorname{Op} K$$

is an operation. Its augmentation is known so it remains to determine its value on combinations of line-bundles. Because of its multiplicative property, it is only necessary to determine $\rho^k(L)$ for a line-bundle L.

Lemma 3.3.2. *For a line-bundle L, we have*

$$\rho^k[L] = \sum_{j=0}^{k-1} [L]^j$$

Proof. By (2.7.1) and (2.7.3) we have a description of $\widetilde{K}(X^L)$ as the $K(X)$ sub-module of $K(P(L \oplus 1))$ generated by $n = 1 - [L][H]$. The structure of $K(P(L \oplus 1))$ is of course given by our main theorem (2.2.1). Hence

$$\begin{aligned} \psi^k(u) &= 1 - [L^k][H^k] \\ &= (1 - [L][H])\{\sum_{j=0}^{k-1} [L^j][H^j]\} \\ &= u \sum_{j=0}^{k-1} [L^j], \quad \text{since} \quad (1 - [L][H])(1 - [H]) = 0. \end{aligned}$$

Thus

$$\psi^k \lambda_L = \lambda_L \{\sum_{j=0}^{k-1} [L^j]\}$$

proving that

$$\rho^k(L) = \sum_{j=0}^{k-1}[L^j]$$

as required. □

As an example we take $X = P_{2n}(\mathbb{R})$, real projective $2n$-space. As shown in (2.7.10) $\widetilde{K}(X)$ is cyclic of order 2^n with generator $x = [L] - 1$, where L is the standard line-bundle. The multiplicative structure follows from the relation $[L]^2 = 1$ (since L is associated to the group $\mathbb{Z}/2\mathbb{Z}$). Now take $k = 3$, then

$$\psi^3(x) = [L^3] - 1 = x,$$

and so the group H_3 defined above is reduced to the identity. Using (3.3.2) we find

$$\begin{aligned}\sigma^3(mx) &= \rho^3(mx) = (\rho^3(x))^m = (\rho^3[L])^m \cdot 3^{-m}\\ &= 3^{-m}(1 + [L] + [L]^2)^m\\ &= (1 + x/3)^m\\ &= 1 + \sum_{i=1}^{m}(-1)^{i-1}\frac{2^{i-1}}{3^i}\binom{m}{i}x \quad (\text{since } x^2 = -2x)\\ &= 1 + \frac{1}{2}(1 - (1 - \frac{2}{3})^m)x\\ &= 1 + 3^{-m}(\frac{3^m - 1}{2})x.\end{aligned}$$

Thus if $J(mx) = 0$ we must have $3^m - 1$ divisible by 2^{n+1}. This happens if and only if 2^{n-1} divides m. Thus the kernel of

$$J\colon \widetilde{K}(P_{2n}(\mathbb{R})) \to J(P_{2n}(\mathbb{R}))$$

is at most of order 2. This result can in fact be improved by use of real K-theory and is the basis of the solution of the vector-field problem for spheres.

The problem considered in (3.2.10) is in fact a special case of the more general problem we are considering now. In fact, the space $P_{n+k(\mathbb{C})}/P_{n-1}(\mathbb{C})$ is easily seen to be the Thom space of the bundle nH over $P_k(\mathbb{C})$. The conclusion of (3.2.10) may therefore be interpreted as a statement about the order of $J[H] \in J(P_k(\mathbb{C}))$. The method of proof in (3.2.10) is essentially the same as that used in this section. The point is that we are now considering not just a single space but a whole class, namely Thom spaces, and describing a uniform method for dealing with all spaces of this class.

For further details of $J(X)$ on the preceding lines we refer the reader to the series of papers ‘On the groups $J(X)$’ by J. F. Adams (Topology 1964-).

Chapter 4

The space of Fredholm operators

In this appendix we shall give a Hilbert space interpretation[1] of $K(X)$. This is of interest in connection with the theory of the index for elliptic operators.

Let H denote a separable complex Hubert space, and let $\mathcal{A}(H)$ be the algebra of all bounded operators on H. We give $\mathcal{A}$ the norm topology. It is well-known that this makes $\mathcal{A}$ into a Banach algebra. In particular the group of units$\mathcal{A}^*$ of $\mathcal{A}$ forms an open set. We recall also that, by the closed graph theorem, any $T \in \mathcal{A}$ which is an algebraic isomorphism $H \to H$ is also topological isomorphism, i.e., T^{-1} exists in G and so $T \in \mathcal{A}^*$.

Definition 4.0.1. An operator $R \in \mathcal{A}(H)$ is a *Fredholm operator* if $\ker T$ and $\operatorname{coker} T$ are finite dimensional. The integer

$$\dim \ker T - \dim \operatorname{coker} T$$

is called the *index* of T .

We first observe that, for a Fredholm operator T, the image $T(H)$ is closed. In fact, since $T(H)$ is of finite codimension in H we can find a finite dimensional algebraic complement P. Then $T \oplus j \colon H \oplus P \to H$ (where $j \colon P \to H$ is the inclusion) is surjective, and so by the closed graph theorem the image of any closed set is closed. In particular $T(H) = T \oplus j(H \oplus 0)$ is closed.

Let $\mathcal{F} \subset \mathcal{A}$ be the subspace of all Fredholm operators. If T, S are two Fredholm operators we have

$$\dim \ker TS \leq \dim \ker T + \dim \ker S$$
$$\dim \operatorname{coker} TS \leq \dim \operatorname{coker} T + \dim \operatorname{coker} S$$

and so TS is again a Fredholm operator. Thus $\mathcal{F}$ is a topological space with an associative product $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$. Hence for any space X the set $[X, \mathcal{F}]$ of

[1] These results have been obtained independently by K. Janich (Bonn dissertation 1964).

homotopy classes of mappings $X \to \mathcal{F}$ is a semi-group. Our main aim will be to indicate the proof of the following:

Theorem 4.0.2. *For any compact space we have a natural isomorphism*

$$\text{index}[X, \mathcal{F}] \to K(X).$$

Remark 4.0.3. If XX is a point this means that the connected components of $\mathcal{F}$ are determined by an integer: this is in fact the index which explains our use of the word in the more general context of Theorem 4.0.2.

Theorem 4.0.2 asserts that $\mathcal{F}$ is a classifying or representing space for K-theory. Another closely related classifying space may be obtained as follows. Let $\mathcal{K} \subset \mathcal{A}$ denote all the compact operators. This is a closed two-sided ideal and the quotient $\mathcal{B} = \mathcal{A}/\mathcal{K}$ is therefore again a Banach algebra. Let $\mathcal{B}^*$ be the group of units of $\mathcal{B}$. It is a topological group and so, for any $X, [X, \mathcal{B}^*]$ is a group. Then our second theorem is:

Theorem 4.0.4. *$\mathcal{B}^*$ is a classifying space for K-theory, i.e., we have a natural group-isomorphism*

$$[Z, \mathcal{B}^*] \cong K(X).$$

We begin with the following lemma which is essentially the generalization to infinite dimensions of Proposition 1.3.4.

Lemma 4.0.5. *Let $T \in \mathcal{F}$ and let V be a closed subspace of H of finite codimension such that $V \cap \ker T = 0$. Then there exists a neighbourhood U of T in $\mathcal{A}$ such that, for all $S \in U$, we have*

(i) $V \cap \ker S = 0$,

(ii) $\cap_{S \in U} H/S(V)$ *topologised as a quotient space of $U \times H$ is a trivial vector bundle over U.*

Proof. Let $W = T(V)^\perp$ (the orthogonal complement of $T(V)$ in H.) Since $T \in \mathcal{F}$ and $\dim H/V$ is finite it follows that $\dim W$ is finite. Now define, for $S \in \mathcal{A}$,

$$\varphi_S \colon V \oplus W \to H$$

by $\varphi_S(V \oplus W)S(V) + W$. Then $S \to \varphi_S$ gives a continuous linear map

$$\varphi \colon \mathcal{A} \to \mathcal{L}(V \oplus W, H)$$

where $\mathcal{L}$ stands for the space of all continuous linear maps with the norm topology. Now φ_T is an isomorphism and the isomorphisms in $\mathcal{L}$ form an open set (like $\mathcal{A}^*$ in $\mathcal{A}$). Hence there exists a neighbourhood U of T in $\mathcal{A}$ so that φ_S is an isomorphism for all $S \in U$. This clearly implies (i) and (ii). □

Corollary 4.0.6. *$\mathcal{F}$ is open in $\mathcal{A}$.*

Proof. Take $V = (\ker T)^\perp$ in (4.0.5). □

Proposition 4.0.7. *Let $T\colon X \to \mathcal{F}$ be a continuous map with X compact. Then there exists $V \subset H$, closed and of finite codimension so that*

(i) $V \cap \ker T_x = 0$ *for all* $X \in X$.

(ii) Moreover, for any such V we have $\cup_{x \in X} H/T_x(V)$, topologised as a quotient space of $X \times H$, is a vector bundle over X.

Proof. For each $x \in X$ take $V_x = (\ker T_x)^{\perp}$ and let U_x be the inverse image under T of the open set given by (4.0.5). Let $K_i = U_{x_i}$ be a finite sub-cover of this family of open sets. Then $V = \cap_i V_{x_i}$ satisfies (i). To prove (ii) we apply (4.0.5) to each T_x, and deduce that$\cup_y H/T_y(V)$ is locally trivial near x, and hence is a vector bundle. □

For brevity we shall denote the bundle $\cup_{x \in X} H/T_x(V)$, occurring in (4.0.6), by $H/T(V)$. Just as in the finite-dimensional case we can split the map $\wp\colon X \times H \to H/T(V)$; more precisely we can find a continuous map

$$\varphi\colon H/T(X) \to X \times H$$

commuting with projection on X and such that

$$\wp\varphi = \mathrm{id}$$

One way to construct φ is to use the metric in H and map $H_t(V)$ onto the orthogonal complement $T(C)^{\perp}$ of $T(V)$. This is technically inconvenient since we then have to verify that $T(X)^{\perp}$ is a vector bundle. Instead we observe that, by definition, $\wp$ splits locally and so we can choose splittings φ_i over U_i, where U_i is a finite open covering of X. Then $\varphi_i - \varphi_j = \theta_{ij}$ is essentially a map $H/T(V) \mid U_i \cap U_j \to U_i \cap U_j \times V$. If ρ_i is a partition of unity subordinate to the covering we put, in the usual way

$$\theta_{ij} = \sum \rho_j \theta_{ij}$$

so that θ_{ij} is defined over all $_i$, and then $\varphi = \varphi_i - \theta_{ij}$ is independent of i and gives the required splitting.

We can now define $\operatorname{index} T$ for any map $T\colon X \to \mathcal{F}$ (X being compact). We choose V as in (4.0.7) and put

$$\operatorname{index} T = [H/V] - [H/T(V)] \in K(X),$$

where H/V stands for the trivial bundle $X \times H/V$. We must show that this is independent of the choice of V. If W is another choice so is $V \cap W$, so it is sufficient to assume $W \subset V$. But then we have the exact sequences of vector bundles

$$0 \to V/W \to H/W \to H/V \to 0$$
$$0 \to V/W \to H/T(W) \to H/T(V) \to 0$$

Hence

$$[H/V] - [H/W] = [V/W] = [H/T(V)] - [H/T(W)]$$

as required.

It is clear that our definition of index T is functorial . Thus if $f\colon Y \to X$ is a continuous map then

$$\operatorname{index} Tf = f^* \operatorname{index} T.$$

This follows from the fact that a choice of the subspace V for T is also a choice for Tf.

If $T\colon X \times I \to \mathcal{F}$ is a homotopy between T_0 and T_1, then index $T \in K(X \times I)$ restricts to index $T_i \in K(X \times \{i\}), i = 0, 1$. Since we know that

$$K(X \times I) \to K(X \times \{i\}) \cong K(X)$$

is an isomorphism, it follows that

$$\operatorname{index} T_0 = \operatorname{index} T_1.$$

Thus

$$\operatorname{index}\colon [X\mathcal{F}] \to K(X)$$

is well-defined.

Next we must show that "*Index*" is a homomorphism. Let $S\colon X \to \mathcal{F}, T\colon X \to \mathcal{F}$ be two continuous maps. Let $W \subset H$ be a choice for T. Replacing S by the homotopic map $\pi_W D$ (π_W denoting projection onto W) we can assume $S(H) \subset W$. Now let $V \subset H$ be a choice for S then it is also a choice for TS and we have an exact sequence of vector bundles over X

$$0 \to W/SV \xrightarrow{T} H/TSV \to H/TW \to 0.$$

Hence

$$\begin{aligned}
\operatorname{index} TS &= [H/V] - [H/TSV] \\
&= [H/V] - [W/SV] - [H/TW] \\
&= [H/V] - [H/SV] + [H/W] \\
&= \operatorname{index} S + \operatorname{index} T
\end{aligned}$$

as required.

Having now established that

$$\operatorname{index}\colon [X.\mathcal{F}] \to K(X)$$

is a homomorphism the next step in the proof of Theorem (4.0.2) is

Proposition 4.0.8. *We have an exact sequence of of semigroups*

$$[X, \mathcal{A}^*] \to [X, \mathcal{F}] \xrightarrow{\text{index}} K(X) \to 0$$

Proof. Consider first a map $T\colon X \to \mathcal{F}$ of index zero. This means that

$$[H/V] - [H/TV] = 0 \quad \text{in} \quad K(X).$$

Hence adding a trivial bundle P to both factors we have

$$H/V \oplus P \cong H/TV \oplus P.$$

Equivalently replacing V by a closed subspace W with $\dim V/W = \dim P$,

$$H/W \cong H/TW.$$

If we now split $X \times H \to H/W$ as explained earlier we obtain a continuous map

$$\varphi\colon X \times H/W \to X \times H$$

commuting with projection on X, linear on the fibres. If

$$\Phi\colon X \to \mathcal{L}(H/W, H)$$

is the map associated to φ, it follows from the construction of φ that

$$x \mapsto \Phi_x + T_x$$

gives a continuous map

$$X \to \mathcal{A}^*.$$

But if $0 \leq t \leq 1$, $T + t\Phi$ provides a homotopy of maps $X \to \mathcal{F}$ connecting T with $T + \Phi$. This proves exactness in the middle. □

It remains to show that the index is surjective. Let E be a vector bundle over X and let F be a complement so that $E \oplus F$ is isomorphic to the trivial bundle $X \times V$. Let $\pi_x \in \operatorname{End} V$ denote projection onto the subspace corresponding to E_x. Let $T_k \in \mathcal{F}$ denote the standard operator of index k, defined relative to an orthonormal basis $\{e_i\}(i = 1, 2, \dots)$ by

$$T_k(e_i) = \begin{cases} e_{i-k} & \text{if } i - k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then define a map

$$S\colon X \to \mathcal{F}(H \otimes V) \cong \mathcal{F}(H)$$

by $S_x = T_X \otimes \pi_x + T_0 \otimes (1 - \pi_x)$. We have $\ker S_X = 0$ for all x, and $(H \otimes V)S(H \otimes V) \cong E$. Hence

$$\operatorname{index} S = -[E].$$

The constant map $T_k\colon X \to \mathcal{F}$ given by $T_k(x) = T_k$ has index k and so

$$\operatorname{index} T_k S = k - [E].$$

Since every element of $K(X)$ is of the form $k - [E]$ this shows that the index is surjective and completes the proof of the proposition.

Theorem (4.0.2) now follows from (4.0.8) and the following:

Proposition 4.0.9.

$$[X, \mathcal{A}^*] = 1.$$

This proposition is due to Kuiper and we shall not reproduce the proof here (full details are in Kuiper's paper: Topology 3 (1964) 19-30). In fact, Kuiper actually shows that $\mathcal{A}^*$ is contractible.

We turn now to discuss the proof of (4.0.4). We recall first that

$$1 + \mathcal{K} \subset \mathcal{F}.$$

This is a standard result in the theory of compact operators: the proof is easy.

Proposition 4.0.10. *Let $\pi\colon \mathcal{A} \to \mathcal{B} = \mathcal{A}/\mathcal{A}$ be the natural map. Then*

$$\mathcal{F} = \pi^{-1}(\mathcal{B}^*).$$

Proof. ($\mathcal{F} \supset \pi^{-1}(\mathcal{B}^*)$): Let $T \in \mathcal{F}$ and let P, Q denote orthogonal projection onto $\ker T, \ker T^*$ respectively. Then $T^*T + P$ and $TT^* + Q$ are both in $\mathcal{A}^*$, and so their images by π are in $\mathcal{B}^*$. But $P, Q \in \mathcal{K}$ and so $\pi(T^*) \cdot \pi(T) \in \mathcal{B}^*, \pi(T) \cdot \pi(T^*) \in \mathcal{B}^*$. This implies that $\pi(T) \in \mathcal{B}^*$.
($\mathcal{F} \subset \pi^{-1}(\mathcal{B}^*)$): Let $T \in \pi^{-1}(\mathcal{B}^*)$, i.e., there exists $S \in \mathcal{A}$ with $ST \in 1 + \mathcal{K} \subseteq \mathcal{F}$ and $TS \in 1 + \mathcal{K} \subseteq \mathcal{F}$. Since $\dim\ker T \leq \dim\ker ST$

$$\dim \operatorname{coker} T \leq \dim \operatorname{coker} TS$$

it follows that $T \in \mathcal{F}$. □

Theorem (4.0.4) will now follow from (4.0.2) and the following general lemma (applied with $L = \mathcal{A}, M = \mathcal{B}, U = \mathcal{B}^*$.)

Lemma 4.0.11. *Let $\pi\colon L \to M$ be a continuous linear map of Banach spaces with $\pi(L)$ dense in M and let U be an open set in M. Then, for any compact X*

$$[X, \pi^{-1}(U)] \to [X, U]$$

is bijective.

Proof. First we shall show that if

$$\pi\colon L \to M$$

satisfies the hypotheses of the lemma, then for any compact X, the induced map

$$\pi^X\colon L^X \to M^X$$

also satisfies the same hypotheses. Since L^X, M^X are Banach spaces the only thing to prove is that $\pi^X(L^X)$ is dense in M^X. Thus, let $f\colon X \to M$ be given. We have to construct $g\colon X \to L$ such that $\|\pi g(x) - f(x)\| < \epsilon$ for all $x \in X$. Choose $a_1, \cdots a_n$ in $f(X)$ such that their $\frac{\epsilon}{3}$-neighbourhoods $\{U_i\}$ cover $f(X)$

and choose b_i such that $\|\pi(b_i - a_i\| < \epsilon/3$. Let $u_i(x)$ be a partition of unity of X subordinate to the covering $\{f^{-1}U_i\}$ and define $g\colon X \to L$ by

$$g(x) = \sum u_i(x)b_i.$$

This is the required map.

Hence replacing π by π^X and U by U^X (which is open in M^X) we see that it is only necessary to prove the lemma when X is a point, i.e., to prove that

$$\pi^{-1}(U) \to U$$

induces a bijection of path-components. Clearly this map of path-components is surjective: if $P \in U$ then there exists $P \in \pi(L) \cap U$ such that the segment $\overline{PQ}$ is entirely in U. To see that it is injective let $P_0, P_1 \in \pi^{-1}(U)$ and suppose $f\colon I \to U$ is a path with $f(0) = \pi(P_0), f(1) = \pi(P_1)$. By what we proved at the beginning there exists $g\colon I \to \pi^{-1}(U)$ such that

$$\|\pi g(t) - f(t)\| < \epsilon \quad \text{for all} \quad t \in T.$$

If ϵ is sufficiently small the segments joining $\pi g(i)$ to $f(i)$, for $i = 0, 1$, will lie entirely in U. This implies that the segment joining $g(i)$ to P_i, for $i = 0, 1$, lies in $\pi^{-1}(U)$. Thus P_0 can be joined to P_1 by a path in $\pi^{-1}(U)$ (see figure 4.1) and this completes the proof. □

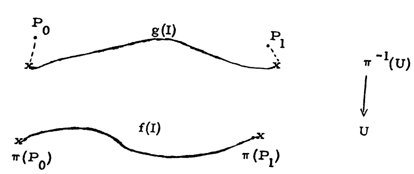


Figure 4.1: