Notes on Cobordism Theory

Robert E. Stong

PREFACE

These notes represent the outgrowth of an offer by Princeton University to let me teach a graduate level course in cobordism theory. Despite the fact that cobordism notions appear in the earliest literature of algebraic topology, it has only been since the work of Thom in 1954 that more than isolated results have been available. Since that time the growth of this area has been phenomenal, but has largely taken the form of individual research papers. To a certain extent, the nature of cobordism as a classification tool has led to the study of many individual applications rather than the development of a central theory. In particular, there is no complete exposition of the fundamental results of cobordism theory, and it is hoped that these notes may help to fill this gap.

Being intended for graduate and research level work, no attempt is made here to use only elementary ideas. In particular, it is assumed that the reader knows algebraic topology fairly thoroughly, with cobordism being treated here as an application of topology. In many cases this is not the fashion in which development took place, for ideas from cobordism have frequently led to new methods in topology itself.

An attempt has been made to provide references to the sources of most of the ideas used. Although the main ideas of these sources are followed closely, the details have frequently been modified considerably. Thus the reader may find it helpful to refer to the original papers to find other methods which are useful. For example, the Adams spectral sequence gives a powerful computational tool which has been used in determining some theories and which facilitates low dimensional calculations, but is never used here. Many of the ideas which appear are of the "well known to workers in the field - but totally unavailable" type and a few ideas are my own.

The pattern of exposition follows my own prejudices, and may be roughly described as follows. There are three central ideas in cobordism theory:

1) Definition of the cobordism groups,

2) Reduction to a homotopy problem, and

3) Establishing cobordism invariants.

This material is covered in the first three chapters. Beyond that point, one must became involved with the peculiarities of the individual cobordism problem. This is begun in the fourth chapter with a survey of the literature, followed by detailed discussion of specific cobordism theories in the later chapters. Finally, two appendices covering advanced calculus and differential topology are added, this material being central to the 'reduction to a homotopy problem' but being of such a nature as to overly delay any attempt to get rapidly to the ideas of cobordism.

I am indebted to many people for leading me to this work and developing my ideas in this direction. Especially, I am indebted to Greg Brumfiel, Peter Landweber, and Larry Smith for numerous suggestions in preparing these notes, and to Mrs Barbara Duld for typing. I am indebted to Princeton University and the National Science Foundation for financial support. Finally I am indebted to my wife for putting up with the foul moods I developed during this work.

Contents

Chapter 1

Introduction - Cobordism Categories

In order to place the general notion of cobordism theory in mathematical perspective recall that differential topology is the study of the category of differentiable manifolds and differentiate maps, primarily in relation to the category of topological spaces and continuous maps. From a slightly less theoretical point of view, it is the study of differentiable manifolds by topologists using any methods they can find. The guiding principle is that one does not study imposed structures such as Riemannian metrics or connections and this distinguishes differential topology from differential geometry.

As in any subject, the primary problem is classification of the objects within isomorphism and determination of effective and computable invariants to distinguish the isomorphism classes. In the case of differentiable manifolds this problem is not solvable, since for any finitely presented group S one can construct a four dimensional manifold $M(S)$ with fundamental group S in such a way that $M(S)$ and $M(T)$ will be homeomorphic if and only if S and are isomorphic, but one cannot solve the word problem to determine whether two finitely presented groups are isomorphic (Markov [76]). In special cases one can solve the problem, but cobordism theory works in another way - by introducing an equivalence relation much weaker than isomorphism.

Briefly, two manifolds without boundary are called 'cobordant' if their disjoint union is the boundary of some manifold. It is worthwhile to note that every manifold M with empty boundary is the boundary of $M \times [0, \infty)$. To get a non-trivial theory it is standard to restrict attention to compact manifolds.

The first description of this equivalence relation was by H. Poincaré: Analysis Situs, Journal de l'École Polytechnique, 1 (l895), 1-121 (section 5, Homologies). His concept of homology is basically the same as the concept of cobordism used today.

The next development of cobordism theory was by L. S. Pontrjagin: Characteristic cycles on differentiate manifolds, Math. Sbor. (N.S.), 21 (63) (1947), 233-284 (Amer. Math. Soc. translations, series 1, no. 32). This paper shows that the characteristic numbers of a closed manifold vanish if the manifold is a boundary (providing the invariants for classification);

The cobordism classification of manifolds is reasonably elementary in dimensions 0,1, and 2, since manifolds are themselves classified in the dimensions. Using geometric methods the cobordism classification problem in dimension 3 was solved by V. A. Rohlin: A 3-dimensional manifold is the boundary of a 4-dimensional manifold, Doklady Akad. Nauk. S.S.S.R., 81 (1951), 355.

The first application of cobordism was by L. S. Pontrjagin: Smooth manifolds and their applications in homotopy theory, Trudy Mat. Inst, im Steklov no. 45, Izdat. Akad. Nauk. S.S.S.R. Moscow, 1955 (Amer. Math. Soc. translations, series 2, vol. 11, 1959). Pontrjagin attempted to study the stable homotopy groups of spheres as the groups of cobordism classes of 'framed' manifolds. This amounts to the equivalence of a homotopy problem and a cobordism problem. The lack of knowledge of manifolds has prevented this from being of use in solving the homotopy problem.

The major development of cobordism theory is the paper of R. Thorn: Quelques propriétés globales des variétés differenttables, Comm. Math. Helv. 28 (1954), 17-86. This paper showed that the problem of cobordism is equivalent to a homotopy problem. For many of the interesting manifold classification questions the resulting homotopy problem turns out to be solvable. Thus, Thom brought the Pontrjagin technique to the study of manifolds, largely reversing the original idea.

For a brief sketch of oobordism theory there are three survey articles of considerable interest. For an insight into the early development of the theory (up through Thom's work) see V. A. Rohlin: Intrinsic homology theories, Uspekhi Mat. Nauk., l4 (1959). 3-20 (Amer. Math. Soc. translations, series 2, 30 (1963), 255-271). A short article which covers many of the examples of cobordism classification problems is J. Milnor: A survey of Cobordism theory, Enseignement Mathematique,.8 (1962), 16-23. Contained in the survey of differential topology by Ð¡. Ð¢. Ð¡. Wall: Topology of smooth manifolds, Journal London Math. Soc, 40 (1965), 1-20, is a discussion of representative cobordism theories, with outlines of the methods by which these problems are solved.

1.1 Cobordism Categories

In order to formalise the notion of cobordism theory, it seems useful to set up a 'general nonsense' situation. As motivation, one may consider the properties of differentiable manifolds.

Let $\mathscr{D}$ denote the category whose objects are compact differentiable manifolds with boundary (of class $^\infty$) and whose maps are the differentiable maps (again C^∞) which take boundary into boundary. This category has finite sums given by the disjoint union and has an initial object given by the empty manifold. For each object of $\mathscr{D}$ one has its boundary, again an object of $\mathscr{D}$, and for each map the restriction of it to the boundary.

Further, the boundary of the boundary is always empty. This defines an additive functor $\partial\colon \mathscr{D} \to \mathscr{D}$. For any manifold M, the boundary of M is a subset whose inclusion is a differentiate map $i(M)\colon \partial M \to M$. This inclusion gives a natural transformation $i\colon \partial \to I$ of additive functors, $I\colon \mathscr{D} \to \mathscr{D}$ being the identity functor. Finally, the Whitney imbedding theorem shows that each differentiate manifold is isomorphic to a submanifold of countable dimensional Euclidean space. Thus $\mathscr{D}$ has a small subcategory $\mathscr{D}_0$ (suhmanifolds of $\mathbb{R}^\infty$) such that each object of $\mathscr{D}$ is isomorphic to an object of $\mathscr{D}_0$.

Abstracting these properties, one has:

Definition 1.1.1. A *cobordism category* $(\mathscr{C}, \partial, i)$ is a triple in which:

1) $\mathscr{C}$ is a category having finite sums and an initial object;
2) $\partial\colon \mathscr{C} \to \mathscr{C}$ is an additive functor such that for each object X of , $\partial\partial()$ is an initial object;
3) $i\colon \partial \to I$ is a natural transformation of additive functors from ∂ to the identity functor I; and
4) There is a small subcategory $\mathscr{C}_0$ of such that each object of $\mathscr{C}$ is isomorphic to an object of $\mathscr{C}_0$.

As noted in motivating this definition, $(\mathscr{D}, \partial, i)$ is a cobordism category. There are many more examples, and in fact the purpose of cobordism theory is to study the interesting examples. The precise choice of this formulation is based, somewhat vaguely, on the definition of 'adjoint functors'.

The purpose of this definition is not to establish a general nonsense structure; rather the definition will be used to follow the framework of previously developed theory and to try to unify the ideas. To begin, one has in any cobordism category the idea of a 'cobordism relation'.

Definition 1.1.2. If $(\mathscr{C}, \partial, i)$ is a cobordism category, one says that the objects X and of are *cobordant* if there exist objects U and V of such that the sum of X and ∂U is isomorphic to the sum of Y and ∂V. This will be written $X \equiv Y$.

One has easily:

Proposition 1.1.3. *a)* $\equiv$ *is an equivalence relation on the objects of* $\mathscr{C}$.

b) $X \equiv Y$ *implies* $\partial X \simeq \partial Y$.

c) For all X, $\partial X \simeq \varnothing$ *where* $\varnothing$ *is an initial object.*

d) If $X \equiv X'$, $Y \equiv Y'$ *and* Z *and* Z' *are sums of the pairs* (X, Y) *and* (X', Y') *respectively, then* $Z \equiv Z'$.

Proof. a) $X + \partial\varnothing \simeq X + \partial\varnothing$;
$X + \partial U \simeq Y + \partial V \to Y + \partial V \simeq X + \partial U$; and
$X + \partial U \simeq Y + \partial V$, $Y + \partial W \simeq +\partial$ implies
$X+\partial(U+W) \simeq X+\partial U+\partial W \simeq Y+\partial V+\partial W \simeq Z+\partial V+\partial T \simeq Z+\partial(V+T)$.

b) $X + \partial U \simeq Y + \partial V$ implies
$\partial X \simeq \partial X + \varnothing \simeq \partial X + \partial\partial U \simeq \partial Y + \partial\partial V \simeq \partial Y + \varnothing \simeq \partial Y$.

c) $\partial X + \partial\varnothing \simeq \varnothing + \partial X$ since $\partial\varnothing$ is initial.

d) $X+\partial U \simeq X'+\partial U'$, $Y+\partial V \simeq Y'+\partial V'$ gives $Z+\partial(U+W) \simeq Z'+\partial(U'+V')$ □

Remark 1.1.4. In all of the above $A + B$ denotes an object which is a sum for A and B.

Remark 1.1.5. If one is unhappy with equivalence relations on a category, one may reduce to considering $\equiv$ as an equivalence relation on the *set* of isomorphism classes of objects of $\mathscr{C}$. This is the reason for the assumption about existence of $\mathscr{C}_0$.

Definition 1.1.6. An object X of $\mathscr{C}$ is *closed* if ∂X is an initial object. An object X of $\mathscr{C}$ *bounds* if $X \equiv \varnothing$ where $\varnothing$ is an initial object.

Proposition 1.1.7. *a) X closed and $Y \equiv X$ implies Y closed,*

b) X and X' closed implies their sum is closed.

c) X bounds implies X is closed,

d) X and Y bound implies their sum bounds,

e) X bounds and $Y \equiv X$ implies Y bounds.

Proof. a) follows directly from b) of Proposition 1.1.3,

b) $\partial X \simeq \varnothing$, $\partial X' \simeq \varnothing$ implies $\partial(X + X') \simeq \varnothing + \varnothing \simeq \varnothing$.

c) $X \equiv \varnothing$ implies $\partial X \simeq \partial\varnothing \simeq \varnothing$.

d) $X \equiv \varnothing$, $Y \equiv \varnothing$ implies $X + Y \equiv \varnothing + \varnothing \simeq \varnothing$.

e) is immediate since $\equiv$ is an equivalence relation.

□

Proposition 1.1.8. *The set of equivalence classes of closed objects of $\mathscr{C}$ (under $\equiv$) has an operation induced by the sum in $\mathscr{C}$. This operation is associative, commutative, and has a unit (the class of any object which bounds).*

Proof. The existence of $\mathscr{C}_0$ form a set. That the sum in $\mathscr{C}$ follows immediately from the propositions 1.1.3 and 1.1.7. Associativity and commutativity hold for isomorphism classes of objects, hence also here. □

Definition 1.1.9. The *cobordism semigroup* of the cobordism category $(\mathscr{C}, \partial, i)$ is the set of equivalence classes of closed objects of $\mathscr{C}$ with the operation induced by the sum in $\mathscr{C}$. This semigroup will be denoted by $\Omega(\mathscr{C}, \partial, i)$.

Remark 1.1.10. 1) $\Omega(\mathscr{C}, \partial, i)$ may also be described as the semigroup of isomorphism classes of closed objects of $\mathscr{C}$ modulo the sub-semigroup of isomorphism classes of objects which bound.

2) The subgroup $\Omega(\mathscr{D}, \partial, i)$ is quite easily identifiable with Thom's cobordism group $\mathscr{N}_*$ of unoriented cobordism classes of closed manifolds. In order to clarify this slightly, in the usual expression for equivalence one has X equivalent to Y if there is a V with $\partial V = X \cup Y$. Then $X \cup \partial V \simeq Y \cup \partial(X \times I)$ giving $X \equiv Y$. The implication $X \cup \partial U \simeq Y \cup \partial V$ implies $X \cup Y = \partial T$ is an easy geometric argument by looking at components and piercing together manifolds with boundary by means of tubular neighbourhoods of their boundary components.

Within the literature of cobordism there are a few standard constructions performed. These may be generalised to the categorical situation as will now be shown.

Construction I Let $(\mathscr{C}, \partial i)$ be a cobordism category, $\mathscr{X}$ a category with finite sums and an initial object, and $F \colon \mathscr{C} \to \mathscr{X}$ an additive functor. For any object X of $\mathscr{X}$, form a category $\mathscr{X}/X$ whose objects are pairs (C, f) with C an object of $\mathscr{C}$ and $f \in \operatorname{Map}(F(C), X)$ and whose maps are given by letting $\operatorname{Map}((C, f), (C', f'))$ be the set maps $\phi \in \operatorname{Map}(C, C')$ such that the diagramme

$$\begin{array}{ccc} F(C) & \xrightarrow{F(\phi)} & F(C') \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

commutes.

If $\varnothing$ is an initial object of $\mathscr{C}$ mad $\phi \colon F(\varnothing) \to X$ is the unique map, then $(\varnothing, \phi)$ is an initial object of $\mathscr{C}/X$. If (D, g) and (D', g') are objects of $\mathscr{C}/X$ and $D + D'$ is a sum for D and D' in $\mathscr{C}$, then $F(D + D')$ is a sum for $F(D)$ and $F(D')$ in $\mathscr{X}$. The maps g and g' give a well defined map $g + g' \colon F(D + D') \to C$, and $(D + D', g + g')$ is the sum of (D, g) and (D', g') in $\mathscr{C}/X$.

Let $\widetilde{\partial}(c, f) = (\partial C, f \circ F(i_C))$ and $\widetilde{\partial}(\phi) = \phi \circ i_C$to define functor $\widetilde{\partial} \colon \mathscr{C}/X \to \mathscr{C}/X$. Define the natural transformation $\widetilde{i} \colon \widetilde{\partial} \to I$ by $\widetilde{i}_{C,f} = i_c \colon \partial C \to C$.

Then $(\mathscr{C}/X, \widetilde{\partial}, \widetilde{i})$ is a cobordism category.

Remark 1.1.11. 1) This is the algebraic-geometric (Grothendieck style) notion of the category of objects over a given object.

2) If one begins with the category $(\mathscr{D}, \partial, i)$ and takes $F \colon \mathscr{D} \to \mathscr{X}$ to be the forgetful functor to the category of topological spaces and continuous maps, then $\Omega(\mathscr{D}/X, \widetilde{\partial}, \widetilde{i})$ is the unoriented bordism group $\mathscr{N}_*(X)$ as originally formulated by M. F. Atiyah: *Bordism and cobordism*, Proc. Camb. Phil. Soc. 57 (1961), 200-208.

Construction II Let $\mathscr{A}$ be a small category, $(\mathscr{C}, \partial, i)$ cobordism category, and let $\mathrm{Fun}(\mathscr{A}, \mathscr{C})$ be the category whose objects are functors $\Phi\colon \mathscr{A} \to \mathscr{C}$ and whose maps are the natural transformations.

If $\varnothing$ is an initial object of $\mathscr{C}$, the constant functor $\varnothing\colon \mathscr{A} \to \mathscr{C}, \quad A \mapsto \varnothing$ is an initial object of $\mathrm{Fun}(\mathscr{A}, \mathscr{C})$. If $F, G\colon \mathscr{A} \to \mathscr{C}$ are functors, let $H\colon \mathscr{A} \to \mathscr{C}$ by letting $H(A)$ be a sum for $F(A)$ and $G(A)$ and let $(j_F)_A = j_{F(A)}\colon F(A) \to H(A)$ and let $(j_G)_A = j_{G(A)}\colon G(A) \to H(A)$ be the maps exhibiting $H(A)$ as the sum. Then j_F and j_G are natural transformations which exhibit H as a sum for F and G.

Let

$$\widetilde{\partial}\colon Fun(\mathscr{A}, \mathscr{C}) \to \mathrm{Fun}(\mathscr{A}, \mathscr{C}), \quad F \to \partial \circ F, \quad \lambda \to \partial(\lambda)$$

and let $\widetilde{i}\colon \widetilde{\partial} \to I$ the evaluation at any object A of $\mathscr{A}$ is $i_{F(A)}\colon \partial(F(A)) \to F(A)$.

Then $(\mathrm{Fun}(\mathscr{A}, \mathscr{C}, \widetilde{\partial}, \widetilde{i})$ is a cobordism theory.

Remark 1.1.12. Many standard examples fit this construction. Suppose $\mathscr{A}$ is the category with one object A whose maps are a finite group $G = \mathrm{Map}(A, A)$. A functor $F\colon \mathscr{A} \to \mathscr{D}$ is given by selecting a manifold $X = F(A)$ and a homomorphism $G \to \mathrm{Map}(X, X)$. Since G is finite , the induced map $G \times X \to X$ is a differentiable action of G on X. Thus $\Omega(\mathrm{Fun}(\mathscr{A}, \mathscr{C}), \widetilde{\partial}, \widetilde{i})$ is the unoriented cobordism group of (unrestricted) G-actions as defined by P. E. Conner and E. E. Floyd: “Differentiable Periodic Maps”, Springer, Berlin, 1964 (section 21).

1.2 Relative Cobordism

In order to study the relationship between two cobordism categories it is convenient to have available a ‘relative cobordism’ semigroup. In the geometric case this is made possible by joining together two manifolds with the same boundary to form a closed manifold. In the categorical situation, the idea is to replace a pair of objects having the same boundary by a pair of closed objects. For this one needs the idea of the Grothendieck group construction.

Recall that for any category with finite sums for which the isomorphism classes of objects form a set, $\mathscr{X}$, one defined $K(\mathscr{X})$, the Grothendieck group of $\mathscr{X}$, to be the set of equivalence classes of pairs (X, X') of objects of $\mathscr{X}$, where (X, X') is equivalent to (Y, Y') if there is an object A of $\mathscr{X}$ such that $X + Y' + A \simeq X' + Y + A$. $K(\mathscr{X})$ is an abelian group under the operation induced by the sum in$\mathscr{X}$.

Let $(\mathscr{C}, \partial, i)$ and $(\mathscr{C}', \partial', i')$ be two cobordism categories, $F\colon \mathscr{C} \to \mathscr{C}'$ an additive functor, and $t\colon \partial' \simeq F\partial$ a natural equivalence of additive functors such that the diagramme

$$\begin{array}{ccc} \partial' F(A) & \xrightarrow{\;t(A)\;} & F(\partial A) \\ & {\scriptstyle i'_{F(A)}} \searrow \quad \swarrow {\scriptstyle F(i_A)} & \\ & F(A) & \end{array}$$

commutes. Let $\mathscr{P}$ be the category whose objects are triples (X, Y, f) with $X \in \mathscr{C}'$, $Y \in \mathscr{C}$, Y closed, and $f \colon \partial' X \to FY$ an isomorphism and with $\mathrm{Map}((X, Y, F), (X', Y', f'))$ the set of $(\phi, psi) \in \mathrm{Map}(X, X') \times \mathrm{Map}(Y, Y')$ such that

$$\begin{array}{ccc} \partial' X & \xrightarrow{f} & FY \\ {\scriptstyle \partial'\phi}\downarrow & & \downarrow{\scriptstyle F\psi} \\ \partial' X' & \xrightarrow[f']{} & FY' \end{array}$$

commutes. Then $\mathscr{P}$ has finite sums and a small subcategory $\mathscr{P}_0(X \in \mathscr{C}'_0, Y \in \mathscr{C}_0)$ such that each object of $\mathscr{P}$ is isomorphic to an object of $\mathscr{P}_0$.

Let $\mathscr{S}$ be he collection of pairs $((X, Y, f), (X', Y', f'))$ of objects of $\mathscr{P}$ for which $Y \simeq Y'$. Let $(x, x') \sim (y, y')$ if there are objects u and v of $\mathscr{P}$ such that $x + u \simeq y + v$ and $x' + u \simeq y' + v$. Then the set of equivalence classes $\mathscr{S}/\sim$ forms an abelian group under the operation induced by the sum.

One has a homomorphism $\beta \colon K(\mathscr{C}'_{\mathrm{Cl}}) \to \mathscr{S}/\sim$, where $\mathscr{C}'_{\mathrm{Cl}}$ is the subcategory of closed objects of $\mathscr{C}'$ by setting (X, X') into $((X, \varnothing, j), (X', \varnothing, j'))$ where $\varnothing$ is an initial object of $\mathscr{C}$ and j, j' are the unique isomorphism of initial objects.

If one has a homomorphism

$$\alpha \colon \mathscr{S}/\sim \to K(\mathscr{C}'_{\mathrm{Cl}})/(\partial'_* K(\mathscr{C}') + F_* K(\mathscr{C}_{\mathrm{Cl}}))$$

such that the composition with β is the quotient homomorphism of $K(\mathscr{C}'_{\mathrm{Cl}})$, then on can define a relative cobordism semigroup as follows:

For objects (X, Y, f) and (X', Y', f') of $\mathscr{P}$, one writes $(X, Y, f) \equiv (X', Y', f')$ if there exist objects U and u' of $\mathscr{C}$ with $Y + \partial U \simeq Y' + \partial U'$ and for which

$$\alpha((X + FU, Y + \partial U, f + tU), (X' + FU', Y' + \partial U', f' + tU')) = 0.$$

Using the fact that α is a homomorphism one easily sees that $\equiv$ is an equivalence relation. The relative cobordism semigroup $\Omega(F, t, \alpha)$ is the set of equivalence classes under $\equiv$ of elements of $\mathscr{P}$ with the sum induced by the sum in $\mathscr{P}$.

One has homomorphisms

$$\begin{aligned} \partial \colon \Omega(F, t, \alpha) &\to \Omega(\mathscr{C}', \partial, i), \quad (X, Y, f) \mapsto Y, \\ F_* \colon \Omega(\mathscr{C}, \partial, i) &\to \Omega(\mathscr{C}', \partial', i'), \quad Y \mapsto FY, \quad \text{and} \\ i \colon \Omega(\mathscr{C}', \partial', i') &\to \Omega(Ft, \alpha), \quad X \mapsto (X, \varnothing, j) \end{aligned}$$

and the triangle

$$\begin{array}{ccc} \Omega(\mathscr{C}, \partial, i) & \xrightarrow{F_*} & \Omega(\mathscr{C}', \partial', i') \\ & {\scriptstyle \partial}\nwarrow \quad \swarrow{\scriptstyle i} & \\ & \Omega(F, t, \alpha) & \end{array}$$

is easily seen to have period 2 (i.e, $\partial i = iF_* = f_* = 0$).

In order to clarify the relationship between the homomorphism α and the joining of two manifolds along their common boundary, consider elements (X, Y, f) of $\mathscr{P}$ as a manifold with boundary together with additional structure on its boundary. For $((X, Y, f), (X', Y', f') \in \mathscr{S}$ choose an isomorphism $g\colon Y \xrightarrow{\simeq} Y'$ and let $\alpha(x, x')$ be the class of $X \cup_k (-X')$, where $-X'$ is X' with its opposite structure (e.g. orientation), and the boundaries of X and X' are identified vis $k = (f')^{-1}F(g)f$. This class does not depend on the choice of g, for if g' is another isomorphism one may attach $X' \times I$ to

$$(X \cup_k (-X')) \times I \cup [\pm(X \cup_{k'} (-X'))] \times I$$

so that the difference of two representatives is cobordant to $X \cup_{k''} (-X')$, where $k'' = f^{-1}F(g^{-1}g')f$. Identifying $\partial X \times 0$ with $\partial X \times 1$ using k'' in $X \times I$ gives a cobordism of $X \cup_{k''} (-X')$ and $\partial X \times I$ with ends of $Y \times I$ with ends identified using $g^{-1}g'$. Thus α does not depend on the choice of g.

With this choice of α, suppose one has $(X, Y, f) \equiv (X', Y', f')$. One may then find a cobordism of Y and Y', say $\partial V = Y - Y'$ so that $X \cup (-V) \cup (-X')$ is cobordant to a closed manifold D with additional structure. Thus one may find a cobordism of Y and Y', $U = V + D$, $\partial U = Y - Y'$, so that $X \cup (-U) \cup (-X')$ bounds. This is the usual geometric description for cobordism of manifolds with boundary.

Remark 1.2.1. One may let $\mathscr{C}$ be the subcategory of $\mathscr{C}'$ consisting of initial objects, with F the inclusion. Then β is epic, uniquely determining α. The relative cobordism semigroup in this case is then identifiable with the cobordism semigroup of $\mathscr{C}'$.

Chapter 2

Manifolds with Structure - the Pontrjagin-Thom theorem

2.1 (B, f) structures

2.2 Generalised Pontrjagin-Thom theorem

2.3 Tangential structures, sequences of maps, ring structure, relative groups

Chapter 3

Characteristic Classes and Numbers

As mentioned in the introduction, the determination of invariants which distinguish manifolds in one of the principal aims of differential topology. In the framework of cobordism theory, the use of characteristic classes provides invariants called *characteristic numbers* which are cobordism invariants. In order to set up the machinery for these invariants, the ideas of generalised cohomology theory play a central role, and for this basic reference is G. W. Whitehead: Generalized cohomology theories, Trans. Amer. Math. Soc., 102 (1962), 227-283.

3.1 Spectra

Definition 3.1.1. A *spectrum* E is a sequence $\{E_n | n \in \mathbb{Z}\}$ of spaces with base point together with a sequence of maps $e_n \colon \sum E_n \to E_{n+1}$, $\sum$ being the suspension. If $\underset{\sim}{F} = \{F_n, f_n\}$ is another spectrum, a map h from $\underset{\sim}{E}$ to $\underset{\sim}{F}$ is a sequence of maps $h_n \colon E_n \to F_n$ with $h_{n+1} \circ e_n = f_n \circ \sum h_n$.

$$\begin{array}{ccc} \sum E_n & \xrightarrow{e_n} & E_{n+1} \\ {\scriptstyle \sum h_n}\downarrow & & \downarrow{\scriptstyle h_{n+1}} \\ \sum F_n & \xrightarrow[f_n]{} & F_{n+1} \end{array}$$

3.2 Ring spectra

3.3 Thom class

3.4 Fundamental class

3.5 Characteristic class, characteristic number

3.6 Orientation and Thom isomorphism

3.7 Atiyah duality

3.8 Alexander and Spanier-Whitehead duality

Chapter 4

The Interesting Examples - A Survey of the Literature

Since cobordism theory is classification tool, the interest really lies in the investigation of specific classification problems. Numerous examples have been considered and hence a vast literature exists, with few really central theoretical tools, largely due to the idiosyncrasies inherent in the examples. The purpose of this chapter is to list many of these examples and indicate briefly what is known and where to find it in the literature.

4.1 Example 1: Framed cobordism Ω_*^{fr}

4.2 Example 2: Unoriented cobordism $\mathfrak{N}_*$

4.3 Example 3: Complex cobordism Ω_*^{U}

4.4 Example 4: Oriented cobordism Ω_*^{SO}

4.5 Example 5: w_1 spherical cobordism $\mathcal{W}_*$

4.6 Example 6: Bordism $\Omega_*(B, f)[X, A]$

4.7 Example 7: Special unitary cobordism Ω_*^{SU}

4.8 Example 8: c_1 spherical cobordism $\mathcal{W}_*^{\mathrm{U}}$

4.9 Example 9: Spin cobordism Ω_*^{Spin}

4.10 Example 10: $\mathrm{Spin}^{\mathbb{C}}$ cobordism $\Omega_*^{\mathrm{Spin}^{\mathbb{C}}}$

4.11 Example 11: Complex-Spin cobordism $\Omega_*^{\mathrm{Spin}^{\mathbb{C}-S}}$

4.12 Example 12: Symplectic cobordism Ω_*^{Sp}

4.13 Fifteen more examples and two pseudoexamples

Chapter 5

Cohomology of Classifying Spaces

In order to study the interesting examples of cobordism theories it is essential to have a detailed knowledge of the cohomology of the classifying space for the classical Lie groups.

5.1 Vector bundles

Let K be one of the fields $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), or $\mathbb{H}$ (quaternions). Let k be the dimension of K as vector space over the reals.

5.2 Definition of characteristic classes

5.3 Splitting lemma

5.4 Thom spaces

5.5 Ordinary cohomology of Grassmannians

5.6 Relationship between fields

5.7 Characteristic numbers of manifolds (projective spaces, Milnor hypersurfaces)

5.8 Cohomology of BO and BSO

5.9 Pontrjagin classes

5.10 Euler class

Chapter 6

Unoriented Cobordism

In many respects the most interesting cobordism theory is unoriented cobordism; i.e. the cobordism problem associated to the category $(\mathscr{O}, \partial, i)$ of all compact differentiable manifolds. It has additional interest in that its solution by Them [127] illustrates all of the basic techniques for dealing with cobordism problems, without encountering excessive technicality.

First note that $\Omega(\mathscr{O}, \partial, i)$ decomposes as a direct sum of semigroups $\Omega_n(\mathscr{O}, \partial, i)$, n being the dimension of the manifold. This semigroup is usually denoted $\mathfrak{N}_n$ with $\mathfrak{N}_*$ denoting the direct sum. The first structure theorem is:

Proposition 6.0.1. *$\mathfrak{N}_n$ is an abelian group in which every element has order 2. $\mathfrak{N}_*$ is a graded commutative ring, multiplication being induced by the product of manifolds, with unit, given by the cobordism class of a point.*

Proof. For any closed M, $M + M + \partial\varnothing \simeq \varnothing + \partial(M \times I)$ where $I = [0,1]$ so the class of M is its own inverse. If M, N_1 and N_2 are closed with $N_1 \cong N_2$, say $N_1 + \partial U_1 \simeq N_2 + \partial U_2$, then $M \times N_1 + \partial(M \times U_1) \simeq M \times N_2 + \partial(M \times U_2)$ so $M \times N_1 \cong M \times N_2$. Since $M \times (N_1 + N_2) \simeq M \times N_1 + M \times N_2$ and $M \times N \simeq N \times M$ this gives $\mathfrak{N}_*$ the structure of a graded commutative ring. If p is a point, $M \times p \simeq p \times M \simeq M$, so the class of p is a unit. □

Theorem 6.0.2. *The cobordism group $\mathfrak{N}_n$ is isomorphic to $\lim_{r\to\infty} \pi_{n+r}(\mathrm{TBO}_r, \infty)$. The ring structure in $\mathfrak{N}_*$ is induced by the maps $\mathrm{TBO}_r \wedge \mathrm{TBO}_s \colon \mathrm{TBO}_s \to \mathrm{TBO}_{r+s}$ obtained from the Whitney sum operation on vector bundles.*

The next step is clearly to try to solve the homotopy problem. It is here that the most ingenuity is required since the various cobordism theories differ widely at this point. The guidance one obtains from Thom's work is: Make use of the cohomology theories for which the manifolds in question are Orientable.

6.1 The mod2 Steenrod algebra $\mathcal{A}_2$

For oriented cobordism one makes use of ordinary cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients; i.e. the cohomology for the spectrum $\underset{\sim}{K}(\mathbb{Z}/2\mathbb{Z})$. One needs a knowledge of the operations in this theory,which may be summarised:

6.2 Adem relations

6.3 Cartan formula

6.4 Structure theorem for $\mathfrak{N}_2$

6.5 Wu classes v_k, $v(M)$

6.6 Wu relations on characteristic numbers

This completes the analysis of the unoriented cobordism ring. Beginning the pattern which will be followed throughout, one wishes to know the relationship with other cobordism theories and the structure of the related bordism theory.

6.7 Relation to framed cobordism: the Hopf invariant

Recall that a framed manifold is a manifold together with an equivalence class of trivialisations of the stable normal bundle. The cobordism corresponding is (B, f) cobordism with B_r a point and the cobordism group Ω_n^{fr} are identified with $\lim_{r\to\infty} \pi_{n+r}(\mathbb{S}^r, \infty)$. (Pontrjagin [101]).

6.8 Unoriented bordism: Steenrod representation

Let $\mathscr{T}$ denote the category of topological spaces and continuous maps and $F\colon \mathscr{D} \to \mathscr{T}$ the forgetful functor assigning to each differentiable manifold its underlying topological space. For any space X one may form the cobordism category $(\mathscr{D}/X, \widetilde{\partial}, \widetilde{i})$, obtained from Construction I. This gives rise to a cobordism semigroup $\mathfrak{N}_*(X)$ which was first defined by Atiyah [13] and which is called the *bordism group of* X.

Let (B, f) be the sequence of spaces and maps given by $B_r = X \times \mathrm{BO}_r$ and $f_r\colon B_r \to \mathrm{BO}_r$ the projection on the second factor. A (B, f) structure on a manifold is then a $(\mathrm{BO}, 1)$ structure together with a homotopy class of maps into X. Since homotopic maps define the same class in $\mathfrak{N}_*(X)$ one has induced a homomorphism $\Omega_*(B, f) \to \mathfrak{N}_*(X)$ which is clearly an isomorphism.

It is clear from the free $\mathcal{A}_2$ module structure of $\widetilde{H}^*((/A) \wedge \underset{\sim}{\mathrm{TBO}}; \mathbb{Z}/2\mathbb{Z})$ that all relations among these generalised Stiefel-Whitney numbers arise from teh Wu relation.

References: In addition to Atiyah's paper [13], one may find a discussion of unoriented bordism in Conner and Floyd [36]. The Steenrod representability is due to Thom [127].

Chapter 7

Complex Cobordism

Historically the next cobordism problem to be completely solved was the cobordism of stably almost complex manifolds. This was defined and completely determined by Milnor [81] and by Novlkov [93]. Specifically this is (B, f) cobordism in which $B_{2r} = B_{2r+1}$ is the classifying space BU_r for complex r-plane bundles. Since a complex vector bundle has a unique stable inverse, the objects are then manifolds with a chosen complex vector bundle structure on the normal or tangent bundle.

7.1 The $\mathrm{mod}\,p$ Steenrod algebra $\mathcal{A}_p$

7.2 Structure of Ω_*^{U}

7.3 Complex K-theory

7.4 Chern character

7.5 Calculation of K-theory characteristic numbers

7.6 Construction of almost complex manifolds with certain characteristic numbers

7.7 Ω_*^{U} is a polynomial.

7.8 Polynomial generators for Ω_*^{U}

7.9 Relations among characteristic numbers [Stong-Hattori theorem]

7.10 Relation to framed cobordism: the Adams invariant $e_{\mathbb{C}}$

7.11 Relation to unoriented cobordism

7.12 Complex bordism

Chapter 8

σ_1-Restricted Cobordism

Let K be one of the fields $\mathbb{R}$ or $\mathbb{C}$. If μ is an n-dimensional K vector bundle, the determinant bundle of μ, $\det \mu$, is the K line bundle $\wedge^n_K(\mu)$ given by the n-fold exterior power bundle over K of the bundle μ.

8.1 $\det(\mu)$, μ an n-plane bundle

8.2 $P(K^r)$-structure, $K = \mathbb{R}$ or $\mathbb{C}$

8.3 $\mathcal{W}_*(K, r)$

8.4 Semi-geometric methods: $\mathcal{W}_*(K, 2)$

8.5 Relation between $\mathcal{W}_*(K, 2)$ and Ω^{SG}_*: Semi-geometric methods

8.6 Relation to bordism groups

Chapter 9

Oriented Cobordism

With the exception of the unoriented oobordism problem, the most interesting manifold theoretic cohordism problem is the classification problem for “oriented” manifolds, where “oriented” is taken in the classical sense.

There are many equivalent descriptions of an “orientation” of a manifold, which may be given by:

a) A trivialisation of the determinant bundle of the tangent (or normal) bundle;

b) A reduction of the structural group of the tangent (or normal) bundle to the special orthogonal group;

c) An integral cohomology orientation of the tangent (or normal) bundle in the sense of Dold; or

d) A fundamental integral homology class giving an orientation in the sense of Whitehead.

In addition to the desire to classify “oriented” manifolds because of the classical interest, definition (d) indicates a relation between “oriented” bordism and integral cohomology and full exploration of this relationship is desirable for geometric understanding of integral homology.

The analysis of “oriented” cobordism is a very complicated problem, the major outline of its solution having been:

1) Reduction to a homotopy problem and rational structure by Them [127J;

2) Calculation of odd primary and mod torsion structure by Milnor [81], or Averbuh [21], and Novikov [93];

3) Calculation of 2 primary structure by Wall [130]; and

4) Analysis of oriented bordism by Conner and Floyd [36].

9.1 Structures of $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$

9.2 Torsion in Ω_*^{SO} is 2-primary.

9.3 $\Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{SO}}$ / Torsion is onto.

9.4 Ω_*^{SO} / Torsion is a polynomial.

9.5 Polynomial generators for Ω_*^{SO} / Torsion

9.6 All torsion in Ω_*^{SO} has order 2.

9.7 Pontrjagin and Stiefel-Whitney numbers determine classes in Ω_*^{SO}.

9.8 Image of $\Omega_*^{\mathrm{SO}} \to \mathfrak{N}_*$

9.9 Integrality theorem for oriented manifolds

9.10 Hirzebruch L class $\mathrm{L}(\xi)$

9.11 Relations among Pontrjagin numbers

9.12 The $\widehat{A}$ class

9.13 Oriented bordism

9.14 Relation to framed cobordism

9.15 The Pontrjagin numbers of an oriented manifold with framed boundary

9.16 Relation to unoriented cobordism

9.17 Relation to complex cobordism

9.18 The index (or signature)

9.19 The Hirzebruch index (or signature) theorem

Chapter 10

Special Unitary Cobordism

Having already built up the machinery to study special unitary cobordism, the 'oriented' analogue of complex cobordism, one may obtain much of the structure in fairly easy fashion. The only new feature which arises is the use of KO-theory characteristic numbers.

10.1 Structure of $\Omega_*^{\mathrm{SU}} \otimes \mathbb{Q}$

10.2 Torsion in Ω_*^{SU} is 2-primary.

10.3 Construction of SU-manifolds with certain characteristic numbers

10.4 $\Omega_*^{\mathrm{SU}} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial.

10.5 All torsion in Ω_*^{SU} has order 2.

10.6 Torsion in Ω_*^{SU}

10.7 KO-theory characteristic numbers

10.8 Chern numbers of SU-manifolds

10.9 Ω_*^{SU} is determined by integral cohomology and KO characteristic numbers.

10.10 Product in $\mathcal{W}_*(\mathbb{C}, 2)$

10.11 Relation to framed cobordism

10.12 Relation to complex cobordism

10.13 Relation to unoriented cobordism

10.14 Relation to oriented cobordism

Chapter 11

Spin, $\mathrm{Spin}^{\mathbb{C}}$, and Similar Nonsense

Among the (B, f) cobordism theories, the most interesting examples arise from the classical groups. The most difficult of these which have been successfully analysed are the theories given by the groups Spin and $\mathrm{Spin}^{\mathbb{C}}$. The group Spin arose classically in the study of Lie groups, being the simply connected covering group of the special orthogonal group.

11.1 Clifford algebra $\mathrm{Cliff}(V)$

11.2 $\mathrm{Spin}(k)$, $\mathrm{Spin}^{\mathbb{C}}(k)$

11.3 $\mathrm{Pin}(k)$, $\mathrm{Pin}^{\mathbb{C}}(k)$

11.4 $H^*(\mathrm{BSpin}; \mathbb{Z}/2\mathbb{Z})$

11.5 Connective covers of BO and BU

11.6 Filtration of $\mathrm{KO}^*(X)$ and $\mathrm{K}^*(X)$

11.7 Isomorphic homologies

11.8 2-primary analysis of MSpin and $\mathrm{MSpin}^{\mathbb{C}}$

11.9 Structure of $\Omega_*^{\mathrm{MSpin}}$ and $\Omega_*^{\mathrm{MSpin}^{\mathbb{C}}}$

11.10 KO-theory and $\mathrm{mod} 2$ cohomology characteristic numbers determine $\Omega_*^{\mathrm{MSpin}}$.

11.11 Ordinary $(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ cohomology characteristic numbers determine $\Omega_*^{\mathrm{MSpin}^{\mathbb{C}}}$.

11.12 Basis for $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Z}/2\mathbb{Z}$

11.13 $\Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{Spin}^{\mathbb{C}}}/$ Torsion is onto.

11.14 Relation to framed cobordism

11.15 Relation to unoriented cobordism

11.16 Relation to oriented cobordism

11.17 Relation to complex cobordism

11.18 Relation to Spin and $\mathrm{Spin}^{\mathbb{C}}$

Appendix A

Advanced Calculus

This appendix collects the results from standard advanced calculus which are needed for geometric arguments in cobordism theory. These results are lifted bodily from the following sources:

(1) Milnor, J.: Lectures on Characteristic Classes, mimeographed, Princeton University, Princeton, N. J., 1957.

(2) Milnor, J.: Topology from the Differentiable Viewpoint, The University Press of Virginia, Charlottesville, Va., 1965.

(3) Spivak, M.: Calculus on Manifolds, W. A. Benjamin, Inc., New York, New York, 1965.

(4) Steenrod, N.: The Topology of Fibre Bundles, Princeton University Press, Princeton, N. J. 1951.

(5) Sternberg, S.: Lectures on Differential Geometry, Prentice-Hall, New York, 1964.

A.1 Calculus

Definition A.1.1. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $a \in \mathbb{R}^n$ if there is a linear transformation $\lambda\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

Proposition A.1.2. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at at $a \in \mathbb{R}^n$, there is a unique linear transformation $\lambda\colon \mathbb{R}^n \to \mathbb{R}^m$ for which the above holds.*

Proof. If $\mu\colon \mathbb{R}^n \to \mathbb{R}^m$ is another such linear transformation, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then

$$\begin{aligned}
\frac{|\lambda(x) - \mu(x)|}{|x|} &= \lim_{t\to 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} \\
&= \lim_{t\to 0} \frac{|\lambda(tx) - f(a+tx) + f(a) + f(a+tx) - f(a) - \mu(tx)|}{|tx|} \\
&= \lim_{t\to 0} \frac{|f(a+tx) - f(a) - \lambda(tx)|}{|tx|} + \lim_{t\to 0} \frac{|f(a+tx) - f(a) - \mu(tx)|}{|tx|} \\
&= 0 + 0
\end{aligned}$$

So $\lambda(x) = \mu(x)$ for all x. □

Definition A.1.3. The linear transformation λ satisfying the above condition is denoted $Df(a)$ and is called the *derivative* of f at a.

Lemma A.1.4. *If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, there is a number M such that $|T(h)| \leq M|h|$ for all $h \in \mathbb{R}^n$.*

Proof. Let e_i^1, e_j^2 be the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and define $t_{ij} \in \mathbb{R}$ by $T(^1_i) = \sum t_{ij} e_j^2$. If $h = \sum h_i e_i^1$, then

$$\begin{aligned}
|T(h)| &= \sqrt{\sum_j (\sum h_i t_{ij})^2} \leq \sum_j |\sum_i h_i t_{ij}| \\
&\leq \sum_j \sum_i |t_{ij}| \cdot |h_i| \leq mn \sup_{i,j} |t_{i,}| \cdot |h|.
\end{aligned}$$

Thus it suffices to take $M = mn \sup_{i,j} |t_{ij}|$. □

Proposition A.1.5. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then f is continuous at a.*

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to a} |f(x) - f(a) - Df(a)(x-a)|/|x-a| = 0$, there is a $\delta_1 > 0$ such that $|x - a| < \delta_1$ implies

$$|f(x) - f(a) - Df(a)(x-a)| < (\varepsilon/2)(|x-a|).$$

By the lemma, there is an M such that $|Df(a)(h)| \leq M|h|$. Let $\delta = \min(\delta_1, \varepsilon/2M, 1)$. Then $|x - a| < \delta$ implies

$$\begin{aligned}
|f(x) - f(a)| &\leq |f(x) - f(a) - Df(a)(x-a)| + |Df(a)(x-a)| \\
&< (\varepsilon/2)|x-a| + M|x-a| \\
&\leq (\varepsilon/2) + M(\varepsilon/2M) \\
&= \varepsilon.
\end{aligned}$$

Hence f is continuous at a. □

Theorem A.1.6. *(Chain Rule) If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, and $g\colon \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $f(a) = b \in \mathbb{R}^m$, then $g \circ f\colon \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a, and*

$$D(g \circ f) = Dg(f(a)) \circ Df(a).$$

Proof. Define

$$\begin{aligned}\varphi(x) &= f(x) - f(a) - \lambda(x-a),\\ \psi(y) &= g(y) - g(b) - \mu(y-b).\end{aligned}$$

where $\lambda = Df(a)$, $\mu = Dg(f(a))$. Then

$$\begin{aligned}g(f(x)) - g(b) - \mu\nu(x-a) &= g(f(x)) - g(b) - \mu\nu(f(x) - f(a) - \varphi(x))\\ &= [g(f(x)) - g(b) - \mu(f(x) - b)] + \mu(\varphi(x))\\ &= \psi(f(x)) + \mu(\varphi(x)).\end{aligned}$$

By the lemma, there is an M_1 such that $|\mu(h)| \leq M_1|h|$, so

$$0 \leq \lim_{x\to a} \frac{|\mu(\varphi(x))|}{|x-a|} \leq M_1 \lim_{x\to a} \frac{|(\varphi(x))|}{|x-a|} = 0.$$

Now let $\varepsilon > 0$ and choose an M_2 such that $|\lambda(h)| \leq M_2|h|$. Since $\lim_{y\to b}|\psi_y|/|y-b| = 0$, there is a $\delta_1 > 0$ such that

$$|\psi(f(x))| < (\varepsilon/M_2)|f(x) - b|$$

if $f(x) - b| < \delta_1$. Since differentiability implies continuity, there is a $\delta_2 > 0$ such that $|x-a| < \delta_2$ implies $|f(x) - b| < \delta_1$. Thus if $|x - a| < \delta_2|$

$$\begin{aligned}|\psi(f(x))| &< (\varepsilon/M_2)|f(x) - b|\\ &= (\varepsilon/M_2)|\varphi(x) + \lambda(x-a)|\\ &\leq (\varepsilon/M_2)|\varphi(x)| + \varepsilon|(x-a)|\end{aligned}$$

and so

$$0 \leq \lim_{x\to a} \frac{|\psi(f(x))|}{|x-a|} \leq (\varepsilon/M_2) \lim_{x\to a} \frac{|\varphi(x)|}{|x-a|} + \varepsilon = \varepsilon,$$

and since this holds for all $\varepsilon > 0$

$$\lim_{x\to a} \frac{|g(f(x)) - g(b) - \mu(\lambda(x-a))|}{|x-a|} = 0.$$

□

Proposition A.1.7. *1) If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a constant function, then $DF(a) = 0$.*

2) If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $Df(a) = f$.

3) If $f\colon \mathbb{R}^n \to \mathbb{R}^m\colon x \mapsto (f^1(x), \dots, f^m(x))$, then f is differentiable at $a \in \mathbb{R}^n$ if and only if each f^i is differentiable at a and $Df(a) = (Df^1(a), \dots, Df^m(a))$.

4) If $f, g\colon \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at $a \in \mathbb{R}^n$, then $f + g\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ and

$$D(f+g)(a) = Df(a) + Dg(a).$$

5) If $f, g\colon \mathbb{R}^n \to \mathbb{R}$ are differentiable at $a \in \mathbb{R}^n$, then $f \cdot g\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ and

$$D(f \cdot g)(a) = f(a) \cdot Dg(a) + g(a) \cdot Df(a).$$

Proof. 1) If $f(x) = y$ for all x, then

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - 0|}{|h|} = \lim_{h\to 0} \frac{|y - y - 0|}{|h|} = 0.$$

2)

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - f(h)|}{|h|} = \lim_{h\to 0} \frac{|f(a) + f(h) - f(a) - f(h)|}{|h|} = 0.$$

3) If each f^i is differentiable and $\lambda = (Df^1(a), \dots, Df^m(a))$, then

$$f(a+h)-f(a)-\lambda(h) = (f^1(a+h)-f^1(a)-Df^1(a)(h), \dots, f^m(a+h)-f^m(a)-Df^m(a)(h))$$

so

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} \le \sum \lim_{h\to 0} \frac{|f^i(a+h) - f^i(a) - Df^i(a)(h)|}{|h|} = 0.$$

Conversely, f^i is the composition of f and the projection π_i which is linear, so $Df^i(a) = D(\pi_i \circ f)(a) = \pi_i Df(a)$.
4) Let $s\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $(x, y) \mapsto x + y$, and let $(f, g)\colon \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^m$, $a \mapsto (f(a), g(a))$. Then s is linear, so $Ds = s$ and by 3), $D(f, g) = (Df, Dg)$. By the chain rule,

$$\begin{aligned} D(f+g)(a) &= Ds(f(a), g(a)) \circ D(f, g)(a) \\ &= s(Df(a), Dg(a)) \\ &= Df(a) + Dg(a). \end{aligned}$$

5) Let $p\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto xy$. By the chain rule, it suffices to show that $Dp(a, b)(x, y) = bx + ay$. Letting $\lambda(x, y) = bx + ay$.

$$\lim_{(h,k)\to 0} \frac{|p(a+h, b+k) - p(a, b) - \lambda(h, k)|}{|(h, k)|} = \lim_{(h,k)\to 0} \frac{|hk|}{|(h, k)|}.$$

Since $|hk| \le \sup(|h|^2, |k|^2) \le |h|^2 + |k|^2$, one has

$$0 \le \lim_{(h,k)\to 0} \frac{|hk|}{|(h, k)|} \le \lim_{(h,k)\to 0} \frac{|(h, k)|^2}{|(h, k)|} = \lim_{(h,k)\to 0} |(h, k)| = 0.$$

$\square$

Proposition A.1.8. *If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ and has either a relative minimum or a relative maximum at a, then $Df(a) = 0$.*

Proof. Let $Df(a)(h) = th$ with $t \in \mathbb{R}$. If a is a relative maximum, then

$$f(a+h) - f(a) \leq 0$$

so if $th > 0$,

$$0 = \lim_{h\to 0, th \to 0} \frac{|f(a+h) - f(a) - th|}{|h|} \geq \lim_{h\to 0} \frac{|th|}{|h|} = |t|.$$

If a is a relative minimum, then $f(a+h) - f(a) \geq 0$ so if $th < 0$,

$$0 = \lim_{h\to 0, th \to 0} \frac{|f(a+h) - f(a) - th|}{|h|} \geq \lim_{h\to 0} \frac{|th|}{|h|} = |t|.$$

□

Theorem A.1.9 (Rolle)**.** *Let $[a,b] \subset \mathbb{R}$ and $f\colon [a,b] \to \mathbb{R}$ a continuous function with $f(a) = f(b) = 0$ and such that $Df(c)$ exists for all $a < c < b$. Then $Df(c) = 0$ for some $c \in (a,b)$.*

Proof. If f is not identically zero in which case $Df(c) = 0$ for all $c \in (a,b)$, then f has a positive maximum or a negative minimum which must occur at some $c \in (a,b)$. Thus c is either a relative maximum or relative minimum and so $Df(c) = 0$ by the proposition. □

Theorem A.1.10 (Mean Value)**.** *Let $[a,b] \subset \mathbb{R}$ and $f\colon [a,b] \to \mathbb{R}$ a continuous function which is differentiable at all points $c \in (a,b)$. Then there is a point $c \in (a,b)$ such that*

$$f(b) - f(a) = Df(c)(b-a)$$

Proof. Let $F(x) = f(x) - f(a) - [(f(b) - f(a))/(b-a)](x-a)$. Then F satisfies the conditions of Rolle's theorem, so for some $c \in (a,b)$

$$0 = DF(c) = Df(c) - [(f(b) - f(a))/(b-a)] \cdot \mathrm{id}_{\mathbb{R}}$$

where $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ is the identity function. □

Definition A.1.11. If $f\colon \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$, then the limit

$$\lim_{h\to 0} \frac{f(a^1, \ldots, a^{i-1}, a^i + h, \ldots, a^n) - f(a^1, \ldots, a^n)}{h}$$

is called the i-th *partial derivative* of f at a, denoted $D_i f(a)$, when it exists.

Theorem A.1.12. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has the property that all the partial derivatives $D_j f^i(x)$ exist in an open set containing a and are continuous at a, then $Df(a)$ exists.*

Proof. It suffices to show $Df^i(a)$ exists, so one may assume $m = 1$. Then

$$f(a+h) - f(a) =$$
$$\sum_{i=1}^{n}[f(a^1 + h^1, \ldots, a^i + h^i, \ldots a^{i+1}, \ldots, a^n) - f(a^1 + h^1, \ldots, a^{i-1} + h^{i-1}, \ldots, a^i, \ldots, a^n]$$
$$= \sum_{i=1}^{n} h^i D_i f(c_i)$$

for some point $c_i = (a^1 + h^1, \ldots, a^{i-1} + h^{i-1}, a^i + \theta_i h^i, a^{i+1}, \ldots, a^n)$ where $0 < \theta_i < 1$, by the mean value theorem. Hence

$$\begin{aligned}
\lim_{h\to 0} \frac{|f(a+h) - f(a) - \sum h^i \cdot D_i f(a)|}{|h|} &= \lim_{h\to 0} \frac{|\sum h^i [D_i f(c_i - D_i f(a)|}{|h|} \\
&\leq \lim_{h\to 0} \sum |[D_i f(c_i - D_i f(a)| \cdot \frac{|h^i|}{|h|} \\
&\leq \lim_{h\to 0} \sum |[D_i f(c_i - D_i f(a)| \\
&= 0
\end{aligned}$$

by continuity of $D_i f$ at a. Thus $Df(a)(h) = \sum D_i f(a) \cdot h^i$. □

Definition A.1.13. For $f\colon \mathbb{R}^n \to \mathbb{R}$, the function defined by

$$D_{i_1,\ldots,i_r} f = D_{i_1}(D_{i_2,\ldots,i_r} f)$$

is called an *r-th order partial derivative* of f. The function f is said to be of *class* C^∞ if all partial derivatives (of all orders) exist.

Theorem A.1.14. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ and $D_{i,j} f$ and $D_{j,i} f$ exist and are continuous in an open set containing $a \in \mathbb{R}^n$, then*

$$D_{i,j} f(a) = D_{j,i} f(a).$$

Proof. It suffices to consider the case $n = 2$. Let $a = (c, d)$ and let $(h, k) \in \mathbb{R}^2$ be small enough so that both $D_{1,2} f$ and $D_{2,1} f$ are defined on

$$\{(x, c)||x - c| \leq h, ||y - d| \leq k\}.$$

Let

$$\varphi(x) = f(x, d+k) - f(x, d), \quad \psi(y) = f(c+h, y) - f(c, y).$$

Then

$$\begin{aligned}
\alpha = f(c+h, d+k) - f(c, d+k) - f(c+h, d) + f(c, d) &= \varphi(c+h) - \varphi(c) \\
&= \psi(d+k) - \psi(d).
\end{aligned}$$

There is a $c' \in (c, c+h)$ with

$$\begin{aligned} \alpha &= \varphi(c+h) - \varphi(c) = D\varphi(c') \cdot h \\ &= [D_1 f(c', d+k) - D_1 f(c', d)]h \\ &= D_{2,1} f(c', d') hk \end{aligned}$$

for some $d' \in (d, d+k)$.

There is a $d'' \in (d, d+k)$ with

$$\begin{aligned} \alpha &= \psi(d+k) - \psi(d) = D\psi(d'') \cdot k \\ &= [D_2 f(c+h, d'') - D_2 f(c, d'')]k \\ &= D_{1,2} f(c'', d'') hk \end{aligned}$$

for some $c'' \in (c, c+h)$.

Thus every open set U containing a contains points p', p'' with

$$D_{1,2} f(p') = D_{2,1} f(p'').$$

By continuity of the $D_{i,j} f$ this gives $D_{1,2} f(a) = D_{2,1} f(a)$. □

Proposition A.1.15. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function and $x_0 \in \mathbb{R}^n$, there exist C^∞ functions $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, with $g_i(x_0) = \frac{\partial f}{\partial x_i}(x_0)$ such that*

$$f(x) = f(x_0) + \sum_{i=1}^{n} (x - x_0) \cdot g_i(x).$$

Proof. Define $h_x(t) = f(x_0 + t(x - x_0))$. Then $h_x(t)$ is a C^∞ function of t and

$$\begin{aligned} \int_0^1 \frac{dh_x}{dt} \cdot dt &= f_x(1) - h_x(0) \\ &= f(x) - f(x_0). \end{aligned}$$

By the chain rule,

$$\frac{dh_x}{dt} = \sum_j \frac{\partial f}{\partial x_j}(x_0 + t(x - x_0)) \cdot (x - x_0)_j$$

so

$$f(x) = f(x_0) + \sum_j^n (x - x_0)_j) \int_0^1 \frac{\partial f}{\partial x_j}(x_0 + t(x - x_0)) dt$$

and one may let $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_j}(x_0 + t(x - x_0)) dt$. Then

$$g_i(x_0) = \int_0^1 \frac{\partial f}{\partial x_j}(x_0) dt = \frac{\partial f}{\partial x_j}(x_0) \int_0^1 dt = \frac{\partial f}{\partial x_j}(x_0).$$

□

Lemma A.1.16. *Let $A \subset \mathbb{R}^n$ be a rectangle and $f\colon A \to \mathbb{R}^n$ continuously differentiable (i.e., each $D_j f^i(x)$ exists and is continuous on A). If there is a number M such that $|D_j f^i(x)| \leq M$ for all x in the interior of A, then*

$$|f(x) - f(y)| \leq n^2 M |x - y|$$

for all $x, y \in A$.

Proof. One has

$$\begin{aligned} f^i(y) - f^i(x) &= \sum_{j=1}^{n} [f^i(y^1, \ldots, y^j, x^{j+1}, \ldots, x^n) - f^i(y^1, \ldots, y^{j-1}, x^j, \ldots, x^n)] \\ &= \sum_{j=1}^{n} |y^j - x^j| \cdot D_j f^i(x_{ij}) \quad \text{for some } z_{ij} \in \text{ interior } A \\ &\leq \sum_{j=1}^{n} |y^j - x^j| \cdot M \\ &\leq nM|y - x| \end{aligned}$$

so

$$|f(x) - f(y)| \leq \sum_{i=1}^{n} |f^i(y) - f^i(x)| \leq n^2 M |x - y|.$$

□

Theorem A.1.17 (Inverse Function)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in an open set containing a, with $Df(a)$non-singular. Then there is an open set V containing a and an open set W containing $f(a)$ such that $f\colon V \to W$ has a continuous inverse $f^{-1}\colon V \to W$ which is differentiable and $Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$ for all $y \in W$.*

Proof. Let $\lambda = DF(a)$ and then

$$D(\lambda^{-1} \circ f)(a) = D(\lambda^{-1})(f(a)) \circ Df(a) = \lambda^{-1} \circ Df(a) = \mathrm{id}\,.$$

If g is an inverse for$\lambda^{-1} \circ f$, then $g \circ \lambda^{-1}$ is an inverse for f, and hence one may assume $\lambda = \text{id}$. Hence if $f(a+h) = f(a)$ one has

$$\frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = \frac{|h|}{|h|} = 1$$

but since

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

this means that $f(x) \neq f(a)$ if $x$is close to but not equal to a.

Thus there is an closed rectangle U containing a in its interior with

1. $f(x) \neq f(a)$ if $x \in U \setminus \{a\}$.

Since f is continuously differentiable in an open set containing a, one may also assume

2. $Df(x)$ is non-singular for all $x \in U$,

3. $|D_j f^i(x) - D_j f^i(a)| < (1/2)n^2$ for all i, j and $x \in U$.

Since $(D_j f^i(a))$ is the Kronecker delta δ_{ij}, the lemma applies to $g(x) = f(x) - x$ giving that for $x_1, x_2 \in U$

$$|f(x_1 - x_1 - (f(x_2) - x_2)| \leq (1/2)|x_1 - x_2|$$

so

$$|x_1 - x_2| - |f(x_1) - f(x_2)| \leq |f(x_1 - x_1 - (f(x_2) - x_2)| \leq (1/2)|x_1 - x_2|.$$

Hence

4. $|x_1 - x_2| \leq 2|f(x_1) - f(x_2)|$ if $x_1, x_2 \in U$.

Since f is continuous, $f(\partial U)$ is compact and by 1. cannot contain $f(a)$, so there is a $d > 0$ such that $|f(x) - f(a)| \geq d$ if $x \in \partial U$. Let $W = \{y||y - f(a)| < (d/2)\}$. If $y \in W$ and $x \in \partial U$ then

5. $|y - f(a)| < |y - f(x)|$ for

$$d \leq |f(x) - f(a)| \leq |y - f(x)| + |y - f(a)| < |y - f(x)| + (d/2).$$

Now let $y \in W$ and let $g\colon U \to \mathbb{R}$ by

$$g(x) = |y - f(x)|^2 = \sum_{i=1}^{n} (y^i - f^i(x))^2.$$

Then g is continuous so has a minimum on U, but by 5. $g(a) < g(x)$ for $x \in \partial U$, so the minimum of g must occur at an interior point of U, i.e., is a relative minimum. Thus there is a point $z \in$ interior U with $D_j g(z) = 0$ for all j, or

$$2\sum_{j=1}^{n} (y^i - f^i(z)) \cdot D_j f^i(z) = 0.$$

Since by 2. $Df(z)$ is non-singular, this gives $y^i - f^i(z) = 0$ or $y = f(z)$ for some $z \in$ interior U. By 4. this z is unique.

Letting $V =$ interior $U \cap f^{-1}(W)$, the function $f\colon V \to W$ has an inverse $f^{-1}\colon W \to V$, and rewriting 4. as $|f^{-1}(y_1) - f^{-1}(y_2)| \leq 2|y_1 - y_2|$ for $y_1, y_2 \in W$ proves continuity of f^{-1}.

To show that f^{-1} is differentiable, let $\mu = Df(x)$ and $y = f(x)$ and for $x_1 \in V$, let us define φ by

$$f(x_1) = f(x) + \mu(x_1 - x) + \varphi(x_1 - x)$$

so that

$$\lim_{x_1 \to x} \frac{|\varphi(x_1 - x)|}{|x_1 - x|} = 0.$$

Then

$$\mu^{-1}(f(x_1) - f(x)) = x_1 - x + \mu^{-1}(\varphi(x_1 - x))$$

and since every $y_1 \in W$ is of the form $f(x_1)$ with some $x_1 \in V$, one has

$$f^{-1}(y_1) = f^{-1}(y) + \mu^{-1}(y_1 - y) - \mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y))).$$

Since μ^{-1} is linear, there is an M with

$$\begin{aligned} \frac{|\mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y)))|}{|y_1 - y|} &\leq M \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y|} \\ &= M \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|f^{-1}(y_1) - f^{-1}(y)|} \cdot \frac{|f^{-1}(y_1) - f^{-1}(y)|}{|y_1 - y|} \\ &\leq 2M \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|f^{-1}(y_1) - f^{-1}(y)|} \end{aligned}$$

by equation 4. As $y_1 \to y$, continuity of f^{-1} gives $f^{-1}(y_1) \to f^{-1}(y)$, and by definition of φ, this term goes to zero. Thus μ^{-1} is a linear transformation of the form required to show f^{-1} is differentiable at y. □

Theorem A.1.18 (Implicit Function). *Let $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable in an open set containing (a, b), with $f(a, b) = 0$. Let M be the $m \times m$ matrix $(D_{n+j} f^i(a))$ $1 \leq i, j \leq m$. If M is non-singular, there is an open set $A \subset \mathbb{R}^n$ containing a and an open set $B \subset \mathbb{R}^m$ containing b, such that for each $x \in A$ there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. The function g is differentiable.*

Proof. Let $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$. Then $DF(a, b)$ is non-singular. There are then open sets $W \subset \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a, b) = (a, 0)$ and $V \subset \mathbb{R}^n \times \mathbb{R}^m$ containing (a, b), which may be a taken to be of the form $A \times B$, such that $F\colon V \to W$ has a differentiable inverse $h\colon W \to V = A \times B$. Clearly $h(x, y) = (x, k(x, y))$ since F has this form, where k is some differentiable function. Let $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m\colon (x, y) \mapsto y$ be the projection. Then

$$\begin{aligned} f(x, k(x, y)) &= f \circ h(x, y) = \pi \circ F \circ h(x, y) \\ &= \pi(x, y) = y \end{aligned}$$

so $f(x, k(x, 0)) = 0$ and one may let $g(x) = k(x, 0)$. □

Theorem A.1.19 (Rank Theorem). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable in an open set containing a, where $p \leq n$. If $f(a) = 0$ and $Df(a)$ is an epimorphism, there is an open set $A \subset \mathbb{R}^n$ and a differentiable function $h\colon A \to \mathbb{R}^n$ such that*

$$f \circ h(x_1, \ldots, x_n) = (x_{n-p+1}, \ldots, x_n).$$

Proof. Since $Df(a)$ has rank p, there are integers $1 \leq i_1 \leq \cdots \leq i_p \leq n$ such that the matrix $D_i f^j(a)$, $1 \leq j \leq p$, $i = i_1, \ldots, i_p$ is non-singular. Let $g\colon \mathbb{R}^n \to \mathbb{R}^n$ permute the coordinates so that $g(x^1, \ldots, x^n) = (\ldots, x^{i_1}, \ldots x^{i_p})$. Then $f \circ g\colon \mathbb{R}^n = \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$ has the matrix $(D_{n-p+j}(f \circ g)^j(g^{-1}(a))$ non-singular $1 \leq i, j \leq p$. As above, there is an $h\colon A \to \mathbb{R}^n$, $A \subset \mathbb{R}^n$ an open set with $(f \circ g) \circ h(x^1, \ldots, x^n) = (x^{n-p+1}, \ldots, x^n)$. This function $g \circ h$ satisfies the condition of the theorem. □

Lemma A.1.20. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable in an open set containing a, where $p \geq n$. If $Df(a)$ is monic, there is an open set $U \subset \mathbb{R}^p$ and a differentiable function $h\colon U \to \mathbb{R}^p$ with differentiable inverse such that*

$$h \circ f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

on some neighbourhood of a.

Proof. Since $(\frac{\partial f_i}{\partial x_j})$ has rank n, one may, by reordering coordinates in $\mathbb{R}^p$, assume $(\frac{\partial f_i}{\partial x_j})_{1 \leq i,j \leq n}$ is non-singular. Let $F\colon \mathbb{R}^p \times \mathbb{R}^{p-n} \to \mathbb{R}^p$ by

$$F(x_1, \ldots, x_p) = f(x_1, \ldots, x_n) + (0, \ldots, 0, x_{n+1}, \ldots, x_p).$$

Since $F(x_1, \ldots, x_n, 0, \ldots, 0) = f(x_1, \ldots, x_n)$, F extends f. $DF(a, 0)$ has

$$\begin{bmatrix} (\frac{\partial f_i}{\partial x_j}) & 0 \\ * & I \end{bmatrix}$$

as matrix so is non-singular. Hence F has an inverse h on a neighbourhood of $(a, 0)$, so

$$\begin{aligned} hf(x_1, \ldots, x_n) &= hF(x_1, \ldots, x_n, 0, \ldots, 0) \\ &= (x_1, \ldots, x_n, 0, \ldots, 0). \end{aligned}$$

□

A.2 Theorem of Sard and Its Consequences

Definition A.2.1. A *rectangle* in $\mathbb{R}^n$ is a set of the form $\prod_{i=1}^n [a_i, b_i]$ with $a_i \leq b_i$, $a_i, b_i \in \mathbb{R}$. The *volume* of the rectangle $S = \prod_{i=1}^n [a_i, b_i]$ is $v(S) = \prod_{i=1}^n |b_i - a_i|$.

Definition A.2.2. A subset $A \subset \mathbb{R}^n$ has (n-dimensional) *measure zero* if for every $\varepsilon > 0$ there is a countable collection B_i of rectangle with $A \subset \cup B_i$ and $\sum v(B_i) < \varepsilon$.

Theorem A.2.3. *A countable union of sets of measure zero is itself of measure zero.*

Proof. If $A = \cup A_i$, with each A_i of measure zero, let $\varepsilon > 0$ and choose families $B_{i,j}$ of rectangles with $A_i \subset \cup_j B_{i,j}$, $\sum v(B_{i,j}) < (\varepsilon/2^i)$. Then $A \subset \cup_{i,j} B_{i,j}$ and $\sum_{i,j} v(B_{i,j}) < \sum_i (\varepsilon/2^i) = \varepsilon$. □

Proposition A.2.4. *Let $\mathcal{U}$ be an open cover of the interval $[a, b]$ by intervals of length at most ε. Then there is a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ such that $\sum_{I_\alpha \in \mathcal{U}_0} v(I_\alpha) \leq 2(|b - a| + \varepsilon)$.*

Proof. Let $\mathcal{U}_1$ be a finite cover by elements of $\mathcal{U}$ and let $\mathcal{U}_0$ be the minimal family of $\mathcal{U}_1$ which cover. Order $\mathcal{U}_0$ by writing the elements of $\mathcal{U}_0$ as $I_j = (a_j, b_j)$ where $i < j$ if $a_i < a_j$. Then one has $\mathcal{U}_0$ $j = 1, \ldots, r$ and by minimality of the cover $a_i < a_{i+1} < b_i < b_{i+1}$ for each i and $a_1 < a < a_2$, $b_{r-1} < b < b_r$. The sum of the overlaps is at most

$$(a - a_1) + (b_1 - a_2) + \cdots + (b_i + a_{i+1} + \cdots + (b_{r-1} - a_r) + (b_r - b) \leq 2\varepsilon + |b - a|$$

since

$$a_1 < a < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_{r-1} < b_{r-2} < a_r < b_{r-1} < b < b_r,$$

and this gives the result. □

Theorem A.2.5 (Fubini)**.** *Let $A \subset \mathbb{R}^n$ be a compact set such that each set $A \cap (t \times \mathbb{R}^{n-1})$ has $(n-1)$-dimensional measure zero. Then A has measure zero.*

Proof. Since $A \subset [a, b] \times \mathbb{R}^{n-1}$ for some $a, b \in \mathbb{R}$. Let ε and choose $\varepsilon_1 > 0$ such that $2|b - a|\varepsilon_1 < \varepsilon$. For each $t \in [a, b]$, $A \cap (t \times \mathbb{R}^{n-1})$ has measure 0 so there is a countable collection of rectangles $B_{t,i} \subset \mathbb{R}^{n-1}$ such that $A \cap (t \times \mathbb{R}^{n-1}) \subset \cup_i t \times B^O_{t,i}$ and $\sum_i v(B_{t,i}) < \varepsilon_1$, where $B^O_{t,i}$ is the interior of $B_{t,i}$. Now $A \setminus \mathbb{R} \times \cup_i B^O_{t,i}$ is a compact set containing no point of the plane $t \times \mathbb{R}^{n-1}$ and hence there is a $(1/2) > \delta_t > 0$ such that

$$A \cap (t - \delta_t, t + \delta_t) \times \mathbb{R}^{n-1} \subset (t - \delta_t, t + \delta_t) \times \cup_i B^O_{t,i}.$$

The sets $(t - \delta_t, t + \delta_t)$ cover $[a, b]$ and by the proposition there is a finite family $t_1, \ldots, t_r$such that the intervals cover $[a, b]$ and have total length at most $2(|b - a| + 1)$. The countable family of all $(t_i - \delta_{t_i}, t_i + \delta_{t_i}) \times \cup_i B_{t_i,j}$ then covers A and has the sum of volumes at most $2(|b - a| + 1)\varepsilon_1 < \varepsilon$. □

Definition A.2.6. Let $f : U \to \mathbb{R}^p$ be a smooth (C^∞) map,U open in $\mathbb{R}^n$. A point $x \in U$ is a *critical point* if $Df(x)$ is not epic; it is a *regular point* if $Df(x)$ is epic. The *critical values* of f are the images under f of critical points; those points of $\mathbb{R}^p$ which are not the image of critical points are called *regular values*.

Theorem A.2.7 (Sard)**.** *Let $f : U \to \mathbb{R}^p$ be a C^∞ map, U open in $\mathbb{R}^n$, and let C be the set of critical points of f. Then $f(C) \subset \mathbb{R}^p$ has measure zero.*

Warning: The proof is rather involved.

Proof. The statement makes sense for $n \geq 0$, $p \geq 1$, with $\mathbb{R}^0$ being a single point. The proof is by induction on n, being obvious for $n = 0$.

Let $C_i \subset C$ denote the set of $x \in U$ such that all partial derivatives of f of order $\leq i$ are zero at x. For example, $C_1 = \{x \in U | Df(x) = 0\}$.

Step 1: The image $f(C \setminus C_1)$ has measure zero.

One may assume $p \geq 2$ for $C = C_1$ if $p = 1$.

Let $\overline{x} \in C \setminus C_1$. Since $\overline{x} \notin C_1$, there is some partial derivative, say $\frac{\partial f^1}{\partial x^1}$, which is non-zero at $\overline{x}$. Let $h \colon U \to \mathbb{R}^n$ by

$$h(x) = (f^1(x), x^2, \ldots, x^n).$$

Since $Df(\overline{x})$ is non-singular, h maps some neighbourhood V of $\overline{x}$ diffeomorphically onto an open set V' of $\mathbb{R}^n$. Then $g = f \circ h^{-1} \colon V' \to \mathbb{R}^p$. The set of critical points of g, C', is precisely $h(V \cap C)$, so $g(C') = f(V \cap C)$.

For each $(t, x^2, \ldots, x^n) \in V'$, $g(t, x^2, \ldots, x^n) \in t \times \mathbb{R}^{p-1} \subset \mathbb{R}^p$ or g takes hyperplanes to hyperplanes. Let

$$g^t \colon (t \times \mathbb{R}^{p-1}) \cap V' \to t \times \mathbb{R}^{p-1}$$

be the restriction of g. Since

$$\left(\frac{\partial g^i}{\partial x^j}\right) = \begin{bmatrix} 1 & 0 \\ * & \left(\frac{\partial g^t}{\partial x^j}\right) \end{bmatrix}$$

a point of $t \times \mathbb{R}^{p-1}$ is critical for g^t if and only if it is critical for g. By induction, the set of critical values of g^t has measure zero in $t \times \mathbb{R}^{p-1}$ and so $g(C')$ intersects each plane $t \times \mathbb{R}^{p-1}$ in a set of measure zero, or $f(V \cap C)$ intersects each plane $t \times \mathbb{R}^{p-1}$ in a set of measure zero.

Since $C \setminus C_1$ is a countable union of sets of the form $\widetilde{V} \cap C$ where $\widetilde{V}$ is a compact neighbourhood of $\overline{x}$, $\widetilde{V} \subset V$. Fubini's theorem shows that $f(C \setminus C_1)$ is a countable union of sets of measure zero, so has measure zero.

Step 2: The image $f(C_i \setminus C_{i+1})$ has measure zero, for $i \geq 1$.

For each $\in C_i \setminus C_{i+1}$ there is some $(i+1)$-st derivative $\frac{\partial^{i+1} f_r}{\partial x_{s_1} \cdots \partial x_{s_{k+1}}}$ which is non-zero. Thus

$$w(x) = \frac{\partial^k f_r}{\partial x_{s_2} \cdots \partial x_{s_{k+1}}}$$

vanishes at $\overline{x}$ but $\frac{\partial w}{\partial x_{s_1}}$ does not. Suppose $s_1 = 1$ for definiteness. Let

$$h \colon U \to \mathbb{R}^n, \quad x \mapsto (w(x), x^2, \ldots, x_n).$$

Then h carries a neighbourhood V of $\overline{x}$ diffeomorphically onto an open set V'. Also h takes $C_i \cap V$ into $0 \times \mathbb{R}^{n-1}$. Consider

$$g = f \circ h^{-1} \colon V' \to \mathbb{R}^p$$

and let $\overline{g}$ be the restriction of g to $(0 \times \mathbb{R}^{n-1}) \cap V'$. By induction, the set of critical values of $\overline{g}$ has measure zero in $\mathbb{R}^p$, but each point of $h(C_i \cap V)$ is a critical point of $\overline{g}$ (since) all derivatives of order $\leq i$ vanish). Thus

$$\overline{g}h(C_i \cap V) = f(C_i \cap V)$$

has measure zero. Since $C_i \setminus C_{i+1}$ is covered by countably many such sets V, it follows that $f(C_i \setminus C_{i+1})$ has measure zero.

Step 3: The image $f(C_k)$ has measure zero for k sufficiently large.

Let $I^n \subset U$ be a cube of edge δ. By Taylor's theorem, the compactness of I^n and the definition of C_k, one has

$$f(x+h) = f(x) + R(x,h)$$

where $|R(x,h)| \leq c|h|^{k+1}$ for $x \in C_k \cap I^n$, $x + h \in I^n$, c being a constant which depends only on f and I^n.

Subdivide I^n into r^n cubes of edge δ/r, and let I_1 be a cube of the subdivision which contains a point $x \in C_k$. Then any point of I_1 is $x+h$ with $|h| \leq (\delta/r)$. Since $|f(x+h) - f(x)| \leq c|h| - k + 1$, $f(I_1$ lies in a cube of edge a/r^{k+1} centred at $f(x)$, where $a = 2c(\sqrt{n}\delta)^{k+1}$ is constant. Thus $f(C_k \cap I^n)$ is contained in a union of at most r^n cubes having total volume

$$V \leq r^n (a/r^{k+1})^p = a^p r^{n-(k+1)p}.$$

If $k + 1 > n/p$, then $V \to 0$ as $r \to \infty$, so $f(C_k \cap I^n)$ has measure zero. □

Lemma A.2.8. *Let D, D' be two open rectangles in $\mathbb{R}^n$ with $\overline{D} \subset D'$. Then there is a real valued C^∞function g on $\mathbb{R}^n$ such that*

a) $0 \leq g(x) \leq 1$ *for all* x,

b) $g(x) = 1$ *for* $x \in D$, *and*

c) $g(x) = 0$ *for* $x \in \mathbb{R}^n \setminus D'$.

Proof. One may write $D = \prod(a_i, b_i)$, $D' = \prod(a'_i, b'_i)$ with $a'_i < a_i < b_i < b'_i$.

For any interval $[c,d] \subset \mathbb{R}$, let

$$\psi_{c,d}(x) = \begin{cases} \exp(-1/(x-c) + 1/(x-d)), & x \in [c,d], \\ 0 & x \notin [c,d]. \end{cases}$$

Then $\psi_{c,d}$ is C^∞ and $\psi_{c,d}(x) \geq 0$. Let

$$\varphi_{c,d} = \int_c^x \psi_{c,d}(x)dx / \int_c^d \psi_{c,d}(x)dx.$$

Then $\varphi_{c,d}$ is C^∞, $0 \leq \varphi_{c,d}(x) \leq 1$, $\varphi_{c,d} = 0$ if $x \leq c$, $\varphi_{c,d}(x) = 1$ if $x \geq d$.

For $a'_i < a_i < b_i < b'_i$, let

$$h_{a'_i,a_i,b_i,b'_i}(x) = \begin{cases} \varphi_{a'_i,a_i} & x \le b \\ 1 - \varphi_{b_i,b'_i} & x > b. \end{cases}$$

Then $h_{a'_i,a_i,b_i,b'_i}(x)$ is C^∞, $0 \le h_{a'_i,a_i,b_i,b'_i}(x) \le 1$, $h_{a'_i,a_i,b_i,b'_i}(x) = 1$ if $x \in [a,b]$, and $h_{a'_i,a_i,b_i,b'_i}(x) = 0$ if $x \notin [a',b']$.

Let $g(x) = \prod_{i=1}^n h_{a'_i,a_i,b_i,b'_i}(x_i)$. □

Lemma A.2.9. *Let U be an open set in $\mathbb{R}^n$ with $\overline{U}$ compact, and let V be an open set containing $\overline{U}$. Then there is a real valued C^∞ function $g\colon \mathbb{R}^n \to [0,1]$ such that*

$$g(x) = \begin{cases} 1 & \text{for} \quad x \in \overline{U}, \\ 0 & \text{for} \quad x \in \mathbb{R}^n \setminus V. \end{cases}$$

Proof. Since $\overline{U}$ is compact, there are a finite numbers of open rectangles $D_1, \dots, D_s$ with $\overline{D}_i \subset V$ covering $\overline{U}$. Let D'_i be an open rectangle containing $\overline{D}_i$ and contained in V. Let g_i be given as in the previous lemma for the pair D_i, D'_i. Then define g by

$$1 - g = (1 - g_1)(1 - g_2) \cdots (1 - g_s).$$

Then f is C^∞, $0 \le g(x) \le 1$ for all x. If $x \in \cup D_i$ then $g_j(x) = 1$ for some j so $1 - g(x) =$. Thus $g(x) = 1$ for $x \in \overline{U} \subset \cup D_i$. If $x \notin D'_i$ then $g_i(x) = 0$ for all i so $1 - g(x) = 1$. Thus $g(x) = 0$ if $x \in \mathbb{R}^n \setminus V \subset \mathbb{R}^n \setminus \cup D'_i$. □

Lemma A.2.10. *Let $F\colon W \to \mathbb{R}$, Z open in $\mathbb{R}^n$ be a continuous function of class C^∞ in an open set $U \subset W$. Let U', V' be open sets with $\overline{U}' \subset V' \subset \overline{V}' \subset W$, $\overline{U}'$, $\overline{V}'$ being compact. Let $\delta > 0$. Then there is a continuous function $G\colon W \to \mathbb{R}$ with $|G(x) - F(x)| < \delta$ for all $x \in W$, such that G is C^∞ in $U \cup U'$ and $F(x) = G(x)$ if $x \in W \setminus \overline{V}'$.*

Proof. By the Weierstrass approximation theorem there is a polynomial $H(x)$ such that $|H(x) - f(x)| < \delta$ for $x \in \overline{V}'$. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be C^∞, $0 \le g \le 1$ with $g|_{\overline{U}'} = 1$, $g|_{\mathbb{R}^n \setminus \overline{V}'} = 0$. Let

$$G(x) = g(x) \cdot H(x) + (1 - g(x))F(x)$$

for all $x \in W$. Then $G(x) = H(x)$ on U' and $G(x) = F(x)$ on $W \setminus V'$. On $\overline{V}'$,

$$|G(x) - F(x)| = |g(x)||H(x) - F(x)| < \delta.$$

Also $G(x)$ is C^∞ when F is, hence on U, so G is C^∞ on $U \cup U'$. □

Proposition A.2.11. *Let $f\colon E \to \mathbb{R}^k$ be a C^∞ function, W an open subset of $\mathbb{R}^n$, C a compact subset of W, V a neighbourhood of C with $\overline{V} \subset W$, and $\varepsilon > 0$. Then there exists a differentiable $g\colon E \to \mathbb{R}^k$ such that*

1) $g|_C$ has $0 \in \mathbb{R}^k$ as a regular value, *2)* $g = f$ *on* $W \setminus V$*,*

3) $|g_i(x) - f_i(x)| < \varepsilon$*,* $|(\partial g_i/\partial x_j)(x) - (\partial f_i/\partial x_j)(x)| < \varepsilon$*,*

for all$x \in W$*,* $a \leq i \leq k$*,* $1 \leq j \leq n$*.*

Proof. Let $\lambda\colon W \to \mathbb{R}$ be a C^∞ with $\lambda|_C = 1$, $\lambda|_{W\setminus V} = 0$ and $0 \leq \lambda(x) \leq 1$ for all x. If y is any regular value of f then

$$g(x) = f(x) - \lambda(x)y$$

satisfied conditions 1) and 2) above. By Sard's theorem, y may be chosen arbitrarily close to 0, and so 3) may be satisfied by taking y small enough. □

Proposition A.2.12. *Let* C *be a compact subset of* W*,* W *open in* $\mathbb{R}^n$ *and* $g\colon W \to \mathbb{R}^k$ *a* C^∞ *function such that* $g|_C$ *has* 0 *as regular value. Then there is an* $\varepsilon > 0$ *such that if* $h\colon W \to \mathbb{R}^k$ *with*

$$|h_i(x) - g_i(x)| < \varepsilon, \quad |\frac{\partial h_i}{\partial x_j}(x) - \frac{\partial g_i}{\partial x_j}(x)| < \varepsilon,$$

for all $x \in C$*,then* $h|_C$ *also has* 0 *as regular value.*

Proof. $\{x \in C|_x$ is critical for $g\}$ is closed so compact and the set of critical values of g is then closed. Thus there is an $\varepsilon_1 > 0$such that $|g_i(x)| < \varepsilon_1$ implies $x$is regular for g. In particular $Dg(x)$ is non-singular and there is an $\varepsilon_2 > 0$ such that $|A_{ij} - (\partial g_i/\partial x_j)(x)| < \varepsilon_2(x)$ implies (A_{ij}) is non-singular. On the set of x for which $|g_i(x)| \leq \varepsilon_{1/2}$ which is compact, there will be an ε_3 such that $\varepsilon_3 \leq \varepsilon_2(x)$ for all these x. Let $\varepsilon = \min(\varepsilon_1/2, \varepsilon_3)$. If

$$|h_i(x) - g_i(x)| < \varepsilon, \quad |\frac{\partial h_i}{\partial x_j}(x) - \frac{\partial g_i}{\partial x_j}(x)| < \varepsilon,$$

then implies $|g_i(x)| < \varepsilon \leq \varepsilon_{1/2}$ so $Dg(x)$ is non-singular and since

$$|\frac{\partial h_i}{\partial x_j}(x) - \frac{\partial g_i}{\partial x_j}(x)| < \varepsilon,$$

$Dh(x)$ is non-singular. Thus 0 is a regular value for h. □

Appendix B

Differentiable Manifolds

This appendix covers the basic notions of differentiable manifolds, tangent and normal bundles and proves the transverse regularity theorem which will be basic to the calculation of cobordism groups. In order to get this, one needs basic structure theorems for manifolds such as tubular neighbourhoods and imbeddabillty and these are also proved. Basic references are:

(1) Kelley, J. L. : General Topology, D. Van Nostrand Co., Inc. Princeton, N. J., 1955.

(2) Milnor, J.: Differential Topology, (mimeographed) Princeton University, 1958.

(3) Munkres, J. R. : Elementary Differential Topology, Princeton University Press, Princeton, N. J., 1966.

(4) Nomizu, K.: Lie Groups and Differential Geometry, Mathematical Society of Japan, 1956.

B.1 General Definition

Definition B.1.1. $\mathbb{H}^n \subset \mathbb{R}^n$ is the half space $\{(x_1, \dots, x_n \in \mathbb{R}^n) | x_n \geq 0\}$.

Definition B.1.2 (Differentiable Manifold with Boundary)**.** An *n-dimensional differentiable manifold with boundary* is a pair $(V, \mathcal{F})$ where V is a Hausdorff space with a countable base and $\mathcal{F}$ is a family of real-valued continuous functions on V satisfying:

1) $\mathcal{F}$ is local: if $f \colon V \to \mathbb{R}$ and for all $p \in V$ there is an open set $U_p \subset V$, and a function $g_p \in \mathcal{F}$ such that $f|_{U_p} = g|_{U_p}$, then $f \in \mathcal{F}$.

2) $\mathcal{F}$ is differentiably complete: if $f_1, \dots, f_k \in \mathcal{F}$ and $F \colon \mathbb{R}^k \to \mathbb{R}$ is C^∞, then $F \circ (f_1 \times \cdots \times f_k) \colon V \to \mathbb{R}$ belongs to $\mathcal{F}$.

3) For each point $p \in V$ there are n functions $f_1, \dots, f_n \in \mathcal{F}$ such that

$$f_1 \times \cdots \times f_n \colon V \to \mathbb{R}^n$$

is a homeomorphism of an open neighbourhood U of p onto an open set of $\mathbb{H}^n$. Further, every function $f \in \mathcal{F}$ agrees on U with a function of the form $F \circ (f_1 \times \cdots \times f_n)$ where $F \colon \mathbb{R}^n \to \mathbb{R}$ is C^∞

The functions $f \in \mathcal{F}$ are called the *differentiable functions* on V. A *chart* at $p \in V$ is a pair (U, h), where U is an open neighbourhood of p and $h \colon V \to \mathbb{R}^n$ is a function $f_1 \times \cdots \times f_n = h$ with $f_i \in \mathcal{F}$ mapping U homeomorphically onto an open subset of $\mathbb{H}^n$ as in 3).

B.2 Paracompactness and Partitions of Unity

Proposition B.2.1. *V is paracompact.*

Proof. Since $\mathbb{H}^n$ is locally compact, so is V, and there is a base $U_1, U_2, \dots$ for V with $\overline{U}_i$ compact for each i. There is a sequence $A_1, A_2, \dots$ of compact sets with union V and $A_i \subset$ interior A_{i+1}: let $A_1 = \overline{U}_1$ and if A_i is defined, there is a least integer $k = k(i)$ such that $A_i \subset U_1 \cup \cdots \cup U_k$. Then let $A_{i+1} = \overline{U_1 \cup \cdots \cup U_k}$.

Let $\mathcal{O}$ be any open cover of V. Cover the compact set $A_{i+1} \setminus$ interior A_i by a finite number of open sets $V_1, \dots V_r$ where V_j is contained in an element of $\mathcal{O}$ and in the open set interior$A_{i+2} \setminus A_{i-1}$. Let$\mathcal{P}_i$ denote the family $(V_1, \dots, V_r)$, and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots$. Then $\mathcal{P}$ refines $\mathcal{O}$, covers V and since any compact set $C$is contained in some A_i, C can intersect only finitely many elements of $\mathcal{P}$. Thus for $p \in V$, any compact neighbourhood of p meets only a finite number of elements of $\mathcal{P}$. □

Corollary B.2.2. *V is normal.*

Proof. a) First, we prove$V$ is regular. If $a \in V$, $B \subset V$, B closed and $a \notin B$, choose for each $b \in B$ open sets U'_b, V_b with $a \in U'_b$, $b \in V_b$ and $U'_b \cap V_b = \varnothing$. Let $U_b = U'_b \cap (V \setminus B)$. Then $a \in U_b$, $b \in V_b$ and $U_b \cap V_b = \varnothing$ and $U_b \subset V \setminus B$. Then $\{V \setminus a \setminus B, U_b, V_b\}_{b \in B}$ is an open covering of V, so has a locally finite refinement $\{C_\alpha\}_{\alpha \in I}$. Let $J = \{\alpha \in I | C_\alpha \cap B \neq \varnothing\}$, $W = \cup_{\alpha \in J} C_\alpha$. Then W is open and contains B. Let N be a neighbourhood of a meeting only a finite number of the sets C_α. There is a finite set $J_0 \subset J$ such that $\alpha \in J$, $N \cap C_\alpha \neq \varnothing$ implies $\alpha \in J_0$. For each $\alpha \in J_0$, $C_\alpha \cap B \neq \varnothing$, so there is a $b = b(\alpha) \in B$ with $C_\alpha \subset V_b$. Then T is open, $a \in T$ and $T \cap W = \varnothing$.

b) Now, we prove V is normal. Let $A, V \subset V$ be closed, $A \cap B = \varnothing$. For each $a \in A$ there are open sets U'_a, V'_a with $a \in U'_a$, $B \subset V'_a$ and $U'_a \cap V'_a = \varnothing$. Let $U_a = U'_a \cap (V \setminus B)$, $V_a = V'_a \cap (X \setminus A)$. Then $\{V \setminus A \setminus B, U_a, v_a\}_{a \in A}$ is an open cover of V so has a locally finite refinement $\{C_\alpha\}$. Let $J = \{\alpha | C_\alpha \cap A \neq \varnothing\}$. For each $b \in B$, there is a neighbourhood N_b of b meeting only a finite number of the sets C_α, $\alpha \in J$. Each such C_α is contained in some set U_a and the intersection of N_b with the corresponding sets V_a is a neighbourhood T_b of b not meeting any C_α with $\alpha \in J$. Then $B \subset T$, $A \subset W$ and $T \cap W = \varnothing$. Let $T = \cup_{b \in B} T_b$, $W = \cup_{\alpha \in J} C_\alpha$. Then $B \subset T$, $A \subset W$ and $T \cap W = \varnothing$. □

Lemma B.2.3 ("Shrinking"). *Let $\mathcal{U}$ bean open cover of V. Then there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that for each $x \in \mathcal{V}$ there is a set $Y \in \mathcal{U}$ with $\overline{X} \subset U$.*

Proof. Let $\mathcal{U}_0$ be a locally finite refinement of $\mathcal{U}$. Consider the set $\mathcal{A}$ of all functions F whose domain is a subfamily of $\mathcal{U}_0$, and for each U in the domain of F, $F(U)$ is an open set with closure contained in U, and such that

$$[\cup\{F(U)|U \in \text{ domain } F\}] \cup [\cup\{W \in \mathcal{U}_0 | W \notin \text{ domain } F\}] = V.$$

$\mathcal{A}$ is non-empty by normality of V. Partially order $\mathcal{A}$ by $F \leq G$ if G extends to F. If F_α is a linearly ordered family, let F be defined on $\cup\{\text{domain } F_\alpha\}$ by $F(U) = F_\alpha(U)$ if $U \in$ domain F_α. Let $x \in V$ and suppose $x \notin W$ for any $W \notin$ domain F. Thus if $x \in U$, $U \in \mathcal{U}_0$, then $U \in$ domain F. Since there are only a finite number of sets $U \in \mathcal{U}_0$ with $x \in U$,and each such $U \in$ domain F_α for some α, there is a β such that $x \in U$, $U \in \mathcal{U}_0$ implies $U \in$ domain F_β. Thus $x \in \cup\{F_\beta(U)|U \in \text{ domain } F_\beta\}$ so $x \in \cup\{F(U)|U \in \text{ domain } F\}$. Then $\mathcal{A}$ has a maximal element F and by normality of V, F must be defined on all $\mathcal{U}_0$. Thus $\mathcal{V} = \{F(U)|U \in \mathcal{U}_0\}$ suffices. □

Proposition B.2.4 (Partition of Unity). *Let $\mathcal{U}$ be any open cover of V. Then there is a differentiable partition of unity on V subordinate to $\mathcal{U}$, i.e., a collection $\Phi \subset \mathcal{F}$ such that*

1) *$\varphi \in \Phi$ implies $\varphi \colon V \to [0, 1]$.*
2) *The collection $\mathcal{V} = \{U_\varphi | \varphi \in \Phi\}$ is a locally finite refinement of $\mathcal{U}$ where $U_\varphi = \{x \in V | \varphi(x) > 0\}$.*
3) *For each $x \in V$, $\sum_{\varphi \in \Phi} \varphi(x) = 1$.*

Proof. Let $\mathcal{U}_1$ be the collection of open sets U such that there is a chart (U, h) and such that $U \subset U'$ for some $U' \in \mathcal{U}$. By the lemma, there is a locally finite refinement $\mathcal{U}_2$ of $\mathcal{U}_1$ such that for each $U_2 \in \mathcal{U}_2$ there is a $U_1 \in \mathcal{U}_1$ with $\overline{U}_2 \subset U_1$, and there is a refinement $\mathcal{U}_3$ of $\mathcal{U}_2$ such that for each $U_3 \in \mathcal{U}_3$ there is a $U_2 \in \mathcal{U}_2$ with $\overline{U}_3 \subset U_2$. In particular, there is a cover of V by sets U_3 such that $U_3 \in \mathcal{U}_3$, $\overline{U}_3 \subset U_2$, $U_2 \in \mathcal{U}_2$, $\overline{U}_2 \subset U_1$, $U_1 \in \mathcal{U}_1$ and the family of such sets U_1 is a locally finite refinement of $\mathcal{U}$. Let (U_1, h) be a chart and let $\psi_{U_3} \colon h(U_1 \to \mathbb{R}$ be C^∞, being 1 on $h(\overline{U}_3)$ and 0 outside $h(U_2)$, with $0 \leq \psi_{U_3} \leq 1$. Let φ'_{U_3}be $\psi_{U_3} \circ h$ on U_1 and 0 on $V \setminus \overline{U}_2$. Then being locally in $\mathcal{F}$, $\varphi'_{U_3} \in \mathcal{F}$. Finally let $\varphi_{U3}(x) = \varphi'_{U_3}(x) / \sum_{U3} \varphi'_{U_3}(x)$ and Φ the collection of φ_U. □

Corollary B.2.5. *Let U and W be open subsets of V with $\overline{U} \subset W$. There is an $f \in \mathcal{F}$ with $f(V) \subset [0, 1]$ such that $f|_{\overline{U}} = 1$,$f|_{V \setminus W} = 0$.*

Proof. $\{W, V \setminus \overline{U}\}$ is an open cover of V so there is a differentiable partition of unity Φ subordinate to this cover. If $\varphi \in_P hi$ and $\varphi(x) \neq 0$ for some $x \in \overline{U}$, then $U_\varphi \subset W$. Let f be the sum of those $\varphi \in \Phi$ which are non-zero on $\overline{U}$. □

B.3 Boundary, Interior and Submanifold

The set of points of V may be divided into two classes as follows. For each point $p \in V$, let (U, h) be a chart at p. Then either $h(p) \in \mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^n$ or $h(p)$ belongs to the interior of $\mathbb{H}^n$. If (U', h') is another chart at p and $h' \notin \mathbb{R}^{n-1} \times 0$, then

$$h \circ h'^{-1} \colon h'(U \cap U') \to h(U \cap U') \subset \mathbb{R}^n$$

is a C^∞ function with a C^∞ inverse, and by the inverse function theorem, $h \circ h'^{-1}$maps onto an open neighbourhood of $h(p)$ in $\mathbb{R}^n$. Thus $h(p) \notin \mathbb{R}^{n-1} \times 0$. Hence this property is independent of the choice of (U, h).

Definition B.3.1. The set of points $p \in V$ for which there is a chart (U, h) with $h(p)\mathbb{R}^{n-1} \times 0$ is called the *boundary of* V, and denoted ∂V. The complement of ∂V, $V \setminus \partial V$, is the *interior of* V.

Proposition B.3.2. *If* $(V, \mathcal{F})$ *is an* n*-dimensional differentiable manifold with boundary and* $\mathcal{F}|_{\partial V}$ *denotes the set of restrictions to* ∂V *of functions in* $\mathcal{F}$*, then* $(\partial V, \mathcal{F}|_{\partial V})$ *is an* $(n-1)$*-dimensional differentiable manifold (without boundary: i.e.,* $\partial(\partial V) = \varnothing$*).*

Proof. Clearly ∂V is Hausdorff and has a countable base, and properties 2) and 3) are clear. Suppose $f \colon \partial V \to \mathbb{R}$ is any function, and for each $p \in \partial V$ there is an open set $U_p \subset \partial V$ and $g_p \in \mathcal{F}|_{\partial V}$ such that $f|_{U_p} = g_p|_{U_p}$. There is then a function $f'_p \in \mathcal{F}$ and an open neighbourhood U'_p of $p$in V with $U'_p \cap \partial V = U_p$ and $g'_p|_{\partial V} = g_p$. Then $\{V \setminus \partial V, U'_p\}$ is an open cover of V and there is a partition of unity Φ subordinate to this cover. For each $\varphi \in \Phi$ such that $U_\varphi = \{x \in V | \varphi(x) > 0\}$ meets ∂V, there is a set U'_p with $U_\varphi \subset U'_p$. Let p_φ be one such. Then define $f' \colon V \to \mathbb{R}$ by $f'(x) = \sum_{\Phi'} \varphi(x) g'_p(x)$ where $\Phi' = \{\varphi \in \Phi | U_\varphi \cap \partial V \neq \varnothing\}$. f' is locally a finite sum of elements of $\mathcal{F}$, so belongs to $\mathcal{F}$. If $x \in \partial V$ and $\varphi(x) \neq 0$ then $x \in U'_{p_\varphi}$ so $g'_{p_\varphi} = f(x)$. Hence $f'(x) = f(x) \cdot \sum \varphi(x) = f(x)$. Thus $f = f'|_{\partial V}$ of $f \in \mathcal{F}|_{\partial V}$. □

Definition B.3.3. If $(V, \mathcal{F}(V))$ and $(W, \mathcal{F}(W))$ are differentiable manifolds with boundary, a function $f \colon V \to W$ is called *differentiable map* if for all $g \in \mathcal{F}(W)$, $g \circ f \in \mathcal{F}(V)$. f is a *diffeomorphism* if f has a differentiable inverse.

Proposition B.3.4. *If* $f \colon (V, \mathcal{F}(V)) \to (V, \mathcal{F}(V))$ *is a differentiable map and* $F(\partial V) \subset \partial W)$ *then* $f|_{\partial V} \colon (\partial V, \mathcal{F}V|_{\partial V}) \to (\partial W, \mathcal{F}W|_{\partial W})$ *is a differentiable map. The inclusion map* $i \colon (\partial V, \mathcal{F}V|_{\partial V}) \hookrightarrow (V, \mathcal{F})$ *is differentiable.*

Proposition B.3.5. *If* $(V, \mathcal{F}(V))$ *is a* n*-dimensional manifolds with boundary,* U*is an open subset of* V *and* $\mathcal{F}|_U$ *denotes the set of restrictions to* U *of functions in* $\mathcal{F}$*, then* $(U, \mathcal{F}|_U)$ *is an* n*-dimensional manifold with boundary, and the inclusion map is differentiable.*

B.4 Vector Bundles and Tangent Bundles

We discuss another definition of differentiable manifolds which is convenient for the discussion of tangent bundles.

Remark B.4.1. Let X be a set and suppose there is a countable collection $\mathcal{A} = \{(X_\alpha, h_\alpha)\}$ where $X_\alpha \subset X$ and $\cup_\alpha X_\alpha = X$ and h_α is a bijection of X_α with an n-dimensional manifold with boundary V_α such that for each pair α, β $h_\alpha(X_\alpha \cap X_\beta)$ is an open subset of V_α and

$$h_\beta \circ h_\alpha^{-1} \colon h_\alpha(X_\alpha \cap X_\beta) \to h_\beta(X_\alpha \cap X_\beta)$$

is differentiable. Then X may be given a topology and a differentiable structure such that X_α will be open and each function h_α is a diffeomorphism. X is then an n-dimensional manifold with boundary, and is uniquely determined within diffeomorphism.

For example, let $(V, \mathcal{F})$ and $(W, \mathcal{G})$ be n-dimensional and m-dimensional manifolds with boundary (∂W being empty). Let (U_i, h_i) and (T_i, g_i) be countable families of charts for V and W. Then the collection $\{(U_i \times T_j, h_i \times g_j)\}$ defines a differentiable structure on $V \times W$, giving the *product* manifold of dimension $n + m$. Then $\partial(V \times V)$ is diffeomorphic to $\partial V \times W$ (recall that ∂W has been supposed to be empty, so we don't have to worry about $\partial V \times \partial W$.).

Definition B.4.2. If $(V, \mathcal{F})$ id a differentiable manifold with boundary, a subset $A \subset V$ is called a *submanifold* of V if for each point $a \in A$ there is a chart (U, h) at a with $h(U \cap A) = h(U) \cap (0 \times \mathbb{R}^k)$. The collection $\mathcal{F}|_A$ of restrictions to A of functions of $\mathcal{F}$ is the family of differentiable functions on A.

Note: $\partial A = A \cap \partial V$, is then a submanifold of ∂V.

Definition B.4.3. A (real) *vector bundle* $\xi = (E, B, \pi, +, \cdot)$ is a 5-tuple where

1. E and B are topological spaces, called the *total space* and the *base space* of ξ,

2. $\pi \colon E \to B$ is a continuous map, called the *projection*,

3. $+\colon E + E = \{(e, e') \in E \times E | \pi e = \pi e'\}$ and
   $\cdot\colon \mathbb{R} \times E \to E$ are continuous maps such that $\pi \circ +(e, e') = \pi e = \pi e'$, $\pi \circ (r, e) = \pi e$ and the restriction to each *fibre* $\pi^{-1}(b)$ for $b \in B$ make $\pi^{-1}(b)$ into a real vector space.

Definition B.4.4. A *bundle map* $f \colon \xi \to \xi'$ is a pair f_E, f_B of continuous maps $f_E \colon E \to E'$, $f_E \colon E \to E'$ such that $\pi' \circ f_E = f_B \circ \pi$ and $f_E \circ + = +' \circ (f_E + f_{E'})$, $f_E \circ \cdot = \cdot \circ f_{E'}$ where $f_E + f_{E'}$ is the restriction to $E + E$ of $f_E + \times f_{E'}$. f is an *isomorphism* if there is a bundle map $g \colon \xi' \to \xi$ which is inverse to f.

For example one has the product bundle $(B \times \mathbb{R}^n, B, \pi, +, \cdot)$ where π is the projection of the product space.

Definition B.4.5. The bundle $\xi = (E, B, \pi, +, \cdot)$ is *locally trivial* if for each point $b \in B$ there is an open set U in B containing b and a bundle isomorphism $h_U|\xi|_U \to (U \times \mathbb{R}^n, U, \pi, +, \cdot)$ where $\xi|_U$ is the bundle $(\pi^{-1}(U), U, \pi|_{\pi^{-1}(U)}, +, \cdot)$ with induced operations with the induced map of base spaces being the identity map of U.

Definition B.4.6. A differentiable vector bundle is a vector bundle ξ for which the total space and base space are differentiable manifolds with boundary, the projection is a differentiable map and such that for each point $b \in B$, the open set U and map h_U may be chosen to give a diffeomorphism of total, spaces.

Note: $+$ and $\cdot$ are forced to be differentiable by the local triviality.

Definition B.4.7. Let $(V, \mathcal{F})$ be an n-dimensional manifold with boundary, and $v \in V$. A *tangent vector* X at v is a function $X : \mathcal{F} \to \mathbb{R}$ such that:

1) If $f, g \in \mathcal{F}$ and there is an open neighbourhood U of v with $f|_U = g|_U$, then $X(f) = X(g)$,

2) For $f, g \in \mathcal{F}$, $a, b \in \mathbb{R}$, $X(af + bg) = aX(f) + bX(g)$,

3) For $f, g \in \mathcal{F}$, $X(f \cdot g) = X(f) \cdot g(v) + f(v) \cdot X(g)$.

The set of tangent vectors at v forms a vector space induced from the additive structure in $\mathbb{R}$, called the *tangent space to V at v* and denoted τ_v.

Denote by $\tau(V)$ the union over all $v \in V$ of the sets τ_v and let $\pi : \tau(V) \to V$ be the function which sends each subset τ_v into the point v.

Proposition B.4.8. *Let $v \in V$ and let (U, h) be a chart at v, with $h = f_1 \times \cdots \times f_n$. Then*

$$\lambda_U : \pi^{-1}(U) \to U \times \mathbb{R}^n, \quad X \mapsto (\pi(X), (X(f_i)))$$

is a bijection. If (U', h') is another char at v, then

$$\lambda_U \circ \lambda_{U'^{-1}} : (U \cap U') \times \mathbb{R}^n \to (U \cap U') \times \mathbb{R}^n$$

is given by $\lambda_U \circ \lambda_{U'^{-1}}(u, \alpha) = (u, D(h \circ h'^{-1})(h'(u))(\alpha))$.

Proof. First note that if $X \in \tau_v$ then X annihilates constant functions. To see this, one has $X(c) = cX(1) = cX(1 \cdot 1) = c\{1X(1) + X(1) \cdot 1\} = 2cX(1)$. Thus $cX(1) = 2cX(1)$ must be zero, so $X(c) = 0$. Then let $f \in \mathcal{F}$ be any function. There is a C^∞ function $F : \mathbb{R}^n \to \mathbb{R}$ with $f = F \circ h$ and one may write

$$F(x) = F(h(v)) + \sum_{i=1}^{n} (x - h(v))_i g_i(x), \quad g_i(h(v)) = \frac{\partial F}{\partial x_i}(h(v))$$

with g_i being C^∞. Thus

$$f = f(v) + \sum_{i=1}^{n} (f_i - f_i(v)) \cdot (g_i \circ h)$$

so

$$X(f) = X(f(v)) + \sum_{i=1}^{n}\{X(f_i - f_i(v)) \cdot (g_i \circ h(v) + (f_i - f_i(v))X(g_i \circ h)\}$$
$$= \sum_{i=1}^{n} X(f_i) \cdot \frac{\partial F}{\partial x_j}(h(v)).$$

Thus λ_U is one-to-one, and letting

$$X_\alpha(f) = \sum \alpha_i \frac{\partial F}{\partial x_j}(h(v)), \quad \alpha \in \mathbb{R}^n$$

one has λ_U onto. Thus λ_U is a bijection.

Then $\lambda_U \circ \lambda_{U'}^{-1} = (u, (\lambda_{U'}^{-1}(u,\alpha)(f_i)))$ and

$$\lambda_{U'}^{-1}(u,\alpha)(f_i) = \sum_{j=1}^{n} \alpha_j \frac{\partial(\pi_i \circ h \circ h'^{-1})}{\partial x_j}(h'(v))$$
$$= [D(h \circ h'^{-1})(h'(v))(\alpha)]_i.$$

□

Proposition B.4.9. *$\tau = (\tau(V), V, \pi, +, \cdot)$ may be given the structure of a differentiable fibre bundle so that if (U,h) is a chart in V, λ_U is a local trivialisation of τ and $(\pi^{-1}(U), (h \times \mathrm{id}) \circ \lambda_U)$ is a chart of $\tau(V)$. The boundary of $\tau(V)$ is $\pi^{-1}(\partial V)$.*

Proposition B.4.10. *If $\varphi\colon (V, \mathcal{F}(V)) \to (W, \mathcal{F}(W))$ is a differentiable map, $v \in V$ and $X \in \tau_v$, let $\varphi_*(X)$ be defined by*

$$\varphi_*(X)(f) = X(f \circ \varphi), \quad f \in \mathcal{F}(W).$$

Then $\varphi_*\colon \tau(V) \to \tau(W)$ is a differentiable map covering φ and (φ_*, φ) is a differentiable bundle map.

B.5 Immersions and Imbeddings

Definition B.5.1. Let $M(p,n)$ denote the set of $p \times n$ matrices with differentiable manifold structure given by identification with $\mathbb{R}^{pn}$. Let $M(p,n;k)$ denote the subset consisting of matrices of rank k.

Lemma B.5.2. *$M(p,n;k)$ is a differentiable manifold of dimension $k(p+n-k)$, $k \leq \min(p,n)$.*

Proof. Let $E_0 \in M(p,n;k)$ and by reordering coordinate write

$$E_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$$

where A_0 is non-singular and $k \times k$. There is an $\varepsilon > 0$ such that if all entries of $A - A_0$ are less than ε, then A is non-singular. Let $U \subset M(p,n)$ consist of all

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with entries of $A - A_0$ less than ε. Then $E \in M(p,n;k)$ if and only if $D = CA^{-1}B$. To see this, note that

$$\begin{bmatrix} A & B \\ XA + C & XB + D \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ X & I_{p-k} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

has the same rank as E. If $X = -CA^{-1}$, this is

$$\begin{bmatrix} A & B \\ 0 & -CA^{-1}B + D \end{bmatrix}$$

so if $D = CA^{-1}B$ this has rank k, while if any element of $-CA^{-1}B + D$ is non-zero the rank is greater than k.

Let W be the open set in $\mathbb{R}^n$, $m = k(p+n-k) = pn - (p-k)(n-k)$, consisting of matrices

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

with all entries of $A - A_0$ less than ε. Then

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \to \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix}$$

maps W homeomorphically onto the neighbourhood $U \cap M(p,n;k)$ of E_0. □

Definition B.5.3. A differentiable map $\varphi\colon (V, \mathcal{F}(V)) \to (W, \mathcal{F}(W))$ is an *immersion* if φ_* is a monomorphism on each fibre of $\tau(V)$. It is an *imbedding* if it is also a homeomorphism into.

Proposition B.5.4. *Let U be an open subset in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^p$ a differentiable map with $p \geq 2n$. Given $\varepsilon > 0$, there is a $p \times n$ matrix A with all entries less than ε such that $g(x) - f(x) + Ax$ is an immersion.*

Proof. For any $p \times n$ matrix A, $Dg = Df + A$ and one wants to choose A so that Dg has rank n at all points of U, or equivalently, $A = Q - Df$ where Q has rank n.

Define $F_k\colon M(p,n;k) \times U \to M(p,n)$ by $(Q,x) \mapsto Q - Df(x)$. Then F_k is differentiable and domain F_k has dimension $k(p+n-k)+n < pn = \dim M(n,p)$. [Taking partials one has $p+n-2k$ so the dimension is a monotone function of k and for $k < n$ this is at most $(n-1)(p+n-(n-1))+n = (2n-p)+pn-i < pn$]. Thus for any chart (W,h) of $M(p,n;k) \times U$, $F_k \circ h^{-1}$ has no regular values. By Sard's theorem, $F_k(W) = F_k \circ h^{-1}(h(W))$ has measure zero but image F_k is a countable union of such sets so has measure zero. Hence there is an A arbitrarily near zero which is not in $\cup_{k<n}\text{image}F_k$. This A suffices. □

Remark B.5.5. If U were an open subset of $\mathbb{H}^n$ the same argument suffices since f is the restriction of a differentiable map from $\mathbb{R}^n$ into $\mathbb{R}^p$.

Theorem B.5.6 (Whitney Immersion)**.** *Given a differentiable map*

$$f\colon (V,\mathcal{F}(V)) \to \mathbb{R}^p, \quad p \geq 2n$$

and a continuous positive function δ on V, there is an immersion

$$g\colon (V,\mathcal{F}(V)) \to \mathbb{R}^p$$

such that $|g(v) - f(v)| < \delta(v)$. If f_* is monic on τ_v for all $v \in N$, N a closed subset of V, then one may let $g|_N = f|_N$.

Proof. Since $f_*|_{\tau_v}$ is monic for all $v \in N$, it is monic for all $V \in U$ where U is an open neighbourhood of N. One may then find a refinement of the open cover $\{V \setminus N, U\}$ by a locally finite countable family of sets V_i such that each set $\overline{V}_i$ is compact and such that each V_i is the underlying set of a chart (V_i, h_i). [There is a countable base consisting of sets W with $\overline{W}$ compact and (W,h) a chart. The proof that V is paracompact sows that one may find a countable locally finite refinement]. Index the set V_i so that the V_i contained in U have $i \leq 0$, while the remainder have $i > 0$, with $i \in \mathbb{Z}$. Applying the proof of the "shrinking lemma" (B.2.3) twice constructs open sets $W_i \subset \overline{W}_i \subset U_i \subset \overline{U}_i \subset V_i$ with $\{W_i\}$ being a cover of V.

Let $f_0 = f$ ans suppose $f_{k-1}\colon V \to \mathbb{R}^p$ is defined such that $(f_{k-1})_*|_{\tau_v}$ is monic for all $v \in N_{k-1} = \cup_{j<k}\overline{W}_j$. For any $p \times n$ matrix A let $F_A\colon h_k(V_k) \to \mathbb{R}^p$ be given by

$$F_A(x) = f_{k-1} \circ h_k^{-1}(x) + \varphi(x) \cdot A(x)$$

where φ is a C^∞ function from $\mathbb{R}^n \to I = [0,1]$ with

$$\varphi = \begin{cases} 1 & \text{on} \quad h_k(\overline{W}_k), \\ 0 & \text{on} \quad \mathbb{R}^n \setminus U_k. \end{cases}$$

First, one wants $DF_A(x)$ to have rank n on $K = h_k(N_{k-1} \cap \overline{U}_k)$. [$\overline{U}_k$ has a finite cover by open sets each meeting only finitely many $\overline{W}_j$ so $N_{k-1} \cap \overline{U}_k$ is compact] and

$$D(F_A)(x) = D(f_{k-1})h_k^{-1}(x)) + A(x) \cdot D\varphi(x) + \varphi(x) \cdot A$$

with $D(f_{k-1})h_k^{-1}(x))$ having rank n on K. This is a continuous function from$K \times M(p,n)$ to $M(p,n)$ sending $K \times 0$ onto $M(p,n;n)$, so if A is sufficiently small one has $K \times A$ mapped into $M(p,n;n)$. Assume A is small.

Next, choose A mall enough so that $|A(x)| < \varepsilon_\mu / 2^k$ where

$$\varepsilon_\mu = \inf\{\delta(x) | x \in \overline{U}_k\} \quad \forall x \in h_k(V_k).$$

Finally, as above A may be chosen arbitrarily small so that $f_{k-1}h_k^{-1}(x) + A(x)$ has rank n on $h_k(U_k)$.

Let A satisfy all these requirements.

Then define $f_k \colon V \to \mathbb{R}^p$ by

$$f_k(y) = \begin{cases} f_{k-1}(y) + \varphi(h_k(y))A(h_k(y)) & \text{if} \quad y \in V_k, \\ f_{k-1}(y) & \text{if} \quad y \in V \setminus \overline{U}_k. \end{cases}$$

These agree on the overlap $V_k \setminus \overline{U}_k$ so f_k is differentiable. By the first condition on A, DF_A has rank n on N_{k-1}, and by the third it has rank n on $\overline{W}_k$, hence $f_*|_{\tau_v}$ is injective for each $v \in N_k$. By the second condition, f_k is a $\delta/2^k$ approximation to f_{k-1}.

Since the cover V_i is locally finite, the f_k agree on any given compact set if k is sufficiently large, so $g(x) = \lim_{k\to\infty} f_k(x)$ exists and g is differentiable, $g_*|_{\tau_v}$ is monic for all $v \in V$, and g is a δ-approximation to f. □

Lemma B.5.7. *Let $(V, \mathcal{F})$ be a differentiable manifold with boundary and*

$$F \colon V \to \mathbb{R}^p$$

an immersion. Then for each point $a \in V$ there is an open set U containing a such that $f|_U$ is an imbedding.

Proof. Let (W, h) be a chart at a. Then $f \circ h^{-1} \colon h(W) \to \mathbb{R}^p$ extends to a differentiable map $k \colon \mathbb{R}^n \to \mathbb{R}^p$ with $Dk(h(a))$ monic. Thus there is an open set $T \subset \mathbb{R}^p$ containing $k(h(a))$ and a differentiable map $g \colon T \to \mathbb{R}^p$ with differentiable inverse such that $gk(x) = (x, 0)$ on a neighbourhood S of $h(a)$. Then

$$h^{-1}\pi f(y) = h^{-1}\pi f h h^{-1}(f(y)) = h^{-1}\pi gk(h(y)) = h^{-1}\pi(h(y), 0) = h^{-1}h(y) = y$$

for all y in a neighbourhood of a, $h^{-1}(S) = U$, where π projects $\mathbb{R}^p$ onto the first n coordinates. □

Lemma B.5.8. *If $p > 2n$ any immersion $f \colon (V, \mathcal{F}) \to \mathbb{R}^p$ may be δ-approximated by a $1-1$ immersion g. If f is $1-1$ on a neighbourhood U of the closed set N, one may choose $g|_N = f|_n$.*

Proof. Choose a covering of V by sets $\{U_\alpha\}$ such that $f|_{U_\alpha}$ is an imbedding for each α, refining the cover $\{U, V \setminus N\}$. Construct a countable locally finite refinement by sets V_i, of the cover $\{U_\alpha\}$, indexed so that the $V_i \subset U$ have $i \leq 0$. Applying the "shrinking lemma" (B.2.3) twice constructs open sets

$$W_i \subset \overline{W}_i \subset U_i \subset \overline{U}_i \subset V_i$$

and let $\varphi_i \colon V \to \mathbb{R}$ be a function of $\mathcal{F}$ such that $0 \leq \varphi_i \leq 1$, $\varphi_i(W_i) = 1$, $\varphi_i(V \setminus U_i) = 0$.

Let $f_0 = f$ and suppose the immersions $f_{k-1} \colon V \to \mathbb{R}^p$ is defined. Then f_k is defined by

$$f_k(x) = f_{k-1}(x) + \varphi_k(x) b_k$$

where $b_k \in \mathbb{R}^p$ is yet to be chosen. As above, for small b_k, f_k will be an immersion, so let b_k be this small. b_k may also be made small enough so that f_k is a $\delta/2^k$ approximation to f_{k-1}.

Finally, let N be the open subset of $V \times V$ consisting of pairs (x, x') with $\varphi_k(x) \neq \varphi_k(x')$, and let $\sigma\colon N \to \mathbb{R}^p$ by

$$\sigma(x, x') = -[f_{k-1}(x) - f_{k-1}(x')]/[\varphi_k(x) - \varphi_k(x')]$$

N is the union of the manifolds

$$(V \setminus \partial V) \times (V \setminus \partial V) \cap N, \partial V \times (V \setminus \partial V) \cap N, (V \setminus \partial V) \times (\partial V) \cap N, (\partial V) \times (\partial V) \cap N,$$

on each of which σ is differentiable and since each of these have dimension at most $2n < p$, $\sigma(N)$ has measure zero. Thus b_k may be chosen arbitrarily small and not in this image.

Then $f_k(x) = f_k(x')$ if and only if $\varphi_k(x) = \varphi_k(x')$ and $f_{k-1}(x) = f_{k-1}(x')$ for $k > 0$.

Let $g(x) = \lim_{k\to\infty} f_k(x)$. If $g(x) = g(x_0)$ and $x \neq x_0$ it follows that $f_{k-1}(x) = f_{k-1}(x_0)$ and $\varphi_k(x) = \varphi_k(x_0)$ for all $k > 0$. Thus $f(x) = f(x_0)$ so x and x_0 cannot belong to the same set V_i, and since $\varphi_k(x) = \varphi_k(x_0)$ for all $k > 0$ neither can belong to a set W_i with $i > 0$. Thus x and x_0 must lie in U, contradicting the fact that f is $1-1$ on U. □

Definition B.5.9. Let $f\colon (V, \mathcal{F}) \to \mathbb{R}^p$. The *limit set* $L(f)$ of f is the set of $y \in \mathbb{R}^p$ such that $y = \lim f(x_i)$ for some sequence $\{x_1, x_2, \dots\}$ in V which has no limit in V.

If V is compact, then $L(f)$ is void. For instance, if $V = \mathbb{R}$ and f is the identity, then $L(f)$ is void; but if f is a one-one mapping onto the open interval $0 < t < 1$, then $L(f)$ contains $t = 0$ and $t = 1$.

The mapping f is proper if $L(f) \cap f(V) = \varnothing$. If, for instance, f maps $\mathbb{R}$ into a figure 6 in $\mathbb{R}^2$, then $L(f)$ contains a point in $f(V)$, and f is not proper. It is easy to see that a one-one mapping f is proper if and only if the inverse f^{-1} is continuous in $f(V)$, or, if and only if f^{-1} carries compact sets into compact sets.

Proposition B.5.10. *$f(V)$ is a closed subset of $\mathbb{R}^p$ if and only if $L(f) \subset F(V)$.*

Proof. Let $y \in \overline{f(V)}$. Then there is a sequence of points $\{y_i\} \in f(V)$ with $\lim y_i = y$. Let $x_i \in V$ with $f(x_i) = y_i$. If the sequence $\{x_i\}$ has a limit point $x \in V$, then $f(x) = y$ by continuity of f. If the sequence $\{x_i\}$ has no limit point in V, then $y \in L(f)$ so $y \in f(V)$. Thus $y \in f(V)$, so $f(V)$ is closed. □

Proposition B.5.11. *f is a topological imbedding if and only if f is $1-1$ and $L(f) \cap f(V)$ is empty.*

Proof. Let $T \subset V$ be closed and $y \in \overline{f(T)} \cap f(V)$. Then there is a sequence $\{y_i\} \subset f(T)$ with $\lim y_i = y$. Let $x_i = f^{-1}(y_i) \in T$. If the sequence $\{x_i\}$ has no limit point then $y \in L(f)$, but $L(f) \cap f(V) = \varnothing$. Thus there is a limit point x

of the sequence $\{x_i\}$, and since T is closed, $x \in T$. By continuity of f, $f(x)$ is a limit point of the sequence $\{y_i\}$, and since y is the limit of the sequence of y, and $\mathbb{R}^p$ is Hausdorff, $y = f(x)$. Hence $f^{-1}\colon f(V) \to V$ is continuous, or f is a topological imbedding. □

Lemma B.5.12. *There is a differentiable map* $f\colon (V, \mathcal{F}) \to \mathbb{R}$ *with* $L(f) = \varnothing$*.*

Proof. Let V_i be a countable, locally finite cover of V by sets V_i with compact closure. Apply the "shrinking lemma" (B.2.3) twice constructs open sets

$$W_i \subset \overline{W}_i \subset U_i \subset \overline{U}_i \subset V_i$$

with $\{W_i\}$ a cover of V, and let $\varphi_i \in \mathcal{F}$ with that $0 \leq \varphi_i \leq 1$, $\varphi_i(\overline{W}_i) = 1$, $\varphi_i(V \setminus U_i) = 0$. This sum is finite for each x since V_i is locally finite. If $\{x_i\}$ is a sequence in V having no limit point, then only finitely many x_i can lie in any compact subset of V. Given any integer m, there is an integer $N(m)$ such that $i \geq N(m)$ implies $x_i \notin \overline{W}_1 \cup \cdots \cup \overline{W}_m$. Thus if $i \geq N(m)$, there is a $j > m$ with $x_i \in \overline{W}_j$, so $f(x_i) \geq> m$. Hence the sequence $\{f(x_i)\}$ can have no limit point. □

Corollary B.5.13 (Whitney Imbedding Theorem)**.** *Every n-dimensional differentiable manifold with boundary can be imbedded in* $\mathbb{R}^{2n+1}$ *as a closed subset.*

Proof. Let $f\colon (V, \mathcal{F}) \to \mathbb{R} \subset \mathbb{R}^{2n+1}$ be a differentiable map with $L(f) = \varnothing$ constructed as above. Let $\delta(x) = 1$ for all $x \in V$ and let g be a $1-1$ immersion of $(V, \mathcal{F})$ in $\mathbb{R}^{2n+1}$ with $|f(x) - f(x)| < \delta(x)$ for all $x \in V$. Let $\{x_i\}$ be any sequence in V having no limit point. Given any integer m there is an integer $P(m) = N(m+1)$ such that if $i \geq P(m)$, then $|g(x_i)| > m$ [Note that $|g(x_i)| \geq |f(x_i)| - 1 > m + 1 - 1$.] Thus the sequence $\{g(x_i)\}$ cannot have a limit point. Hence $L(g) \neq \varnothing$ and g is a topological imbedding as a closed subset. □

B.6 Normal Bundles and Tubular Neighbourhoods

Definition B.6.1. Let V_1, V_2 be differentiable manifolds, $F\colon V_1 \to V_2$ an immersion. The *normal bundle* of f, ν_f is defined as follows. Let τ_1 and τ_2 denote the tangent bundles of V_1 and V_2. Then $f_*\colon \tau_1 \to \tau_2$ induces a monomorphism into the bundle $f^!\tau_2$ over V_1, where $f^!\tau_2$ is the pull-back. The quotient bundle of $f^!\tau_2$ by τ_1 is a differentiable vector bundle over V_1 which is ν_f.

Now let $(V, \mathcal{F})$ be a differentiable manifold and let $g\colon V \to \mathbb{R}^p$ be an imbedding. Since the tangent bundle of $\mathbb{R}^p$ is trivial, i.e., the total space is $\mathbb{R}^p \times \mathbb{R}^p$ one may use the usual inner product in $\mathbb{R}^p$ to give an inner product in each fibre of $\tau(\mathbb{R}^p)$ and hence in $g^!(\tau(\mathbb{R}^p))$. The orthogonal complement of the image of each fibre of $\tau(V)$ in each fibre of $g^!(\tau(\mathbb{R}^p))$ is a subspace mapped isomorphically to the fibre of ν_g. The orthogonal complements fit together to form the total space of a differentiable vector bundle $\tau(V)^\perp$ over V isomorphic to ν_g, via the quotient map $\gamma = \beta \circ (\alpha|_{\tau(V)^\perp})^{-1}$ mapping $E(\nu_g)$ diffeomorphically onto the submanifold of $\mathbb{R}^p \times \mathbb{R}^p = E(\tau(\mathbb{R}^p))$ given by

$$\{(x,y) \in \mathbb{R}^p \times \mathbb{R}^p | x = g(v), \quad y \perp g_*(\tau_v), \quad v \in V\}$$

Let $e\colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p : (x,y) \mapsto (x+y)$.

Theorem B.6.2. *If $(V, \mathcal{F})$ is an n-dimensional differentiable manifold with $\partial V = \varnothing$ and $g\colon V \to \mathbb{R}^p$ is an imbedding, then the differentiable function $e \circ \gamma\colon E(\nu_g) \to \mathbb{R}^p$ maps an open neighbourhood of the zero section of ν_g diffeomorphically onto an open neighbourhood of $g(V)$ in $\mathbb{R}^p$.*

Proof. First we show that $e \circ \gamma$ is differentiable and has rank p at all points of the zero section.

To see this, let (U,h) be a chart on V such that ν_g is trivial over U. One then has a local trivialisation $k\colon h(U) \times E(\nu_g)$ with $(\pi^{-1}(U), k^{-1})$ a chart of $E(\nu_g)$. Then the function

$$\delta = e \circ \gamma \circ k\colon h(U) \times \mathbb{R}^{p-n} \to \mathbb{R}^p$$

is given by $\delta(x,\alpha) = g \circ h^{-1}(x) + \sum \alpha_i y_i(x)$ where for each $x \in h(U)$, $\{y_i(x)\}$ form a base for the orthogonal complement to $D(g \circ h^{-1})(x)[\mathbb{R}^n] = g_*(\tau_{h(x)})$. Then

$$D\delta(x,\alpha)\{y,\beta\} = D(g \circ h^{-1})(x)(y) + \sum \beta_i y_i(x) + \sum \alpha_i D y_i(x)(y)$$

where $\{y,\beta\} \in \mathbb{R}^n \times \mathbb{R}^{p-n} = \mathbb{R}^p$. For $\alpha = 0$, this gives

$$D\delta(x,0)\{y,\beta\} = D(g \circ h^{-1})(x)(y) + \sum \beta_i y_i(x)$$

which spans $\mathbb{R}^p$ as $\{y,\beta\}$ runs through $\mathbb{R}^p$ because of the choice of y_i.

Hence $e \circ \gamma$ has rank p in some neighbourhood of the zero section of $E(\nu_g)$, so that it is a local diffeomorphism at points of the zero section: i.e., it maps an open neighbourhood of each point x in the zero section diffeomorphically onto an open neighbourhood of $e \circ \gamma(x)$ in $\mathbb{R}^p$. □

To complete the proof it suffices to show:

Lemma B.6.3. *Let X and Y be Hausdorff space with countable bases and X locally compact. If $f\colon X \to Y$ is a local homeomorphism and the restriction of f to a closed subset A is a homeomorphism, then f is a homeomorphism on some neighbourhood V of A.*

Proof. The proof breaks up into three steps.

1) If A is compact, the lemma is true. If not, then every neighbourhood N of A contains a pair $\{x,y\}$ of points for which $f(x) = f(y)$. One may then find a countable family $\{N_i\}$ of compact neighbourhoods of A with $N_{i+1} \subset N_i^O$ and $\cap N_i = A$. For each i, let $x_i, y_i \in N_i^O$ with $f(x_i) = f(y_i)$. Since N_1

is compact, the sequence $\{x_i\}$ and $\{y_i\}$ have limit points x and y. Since $V \setminus N_{i+1}$ contains only a finite number of points x_j and y_j, one must have $x, y \in \cap N_i = A$. But $f(x) = \lim f(x_i) = \lim f(y_i) = f(y)$ contradicting the fact that $f|_A$ is a homeomorphism.

2) Let A_0 be a compact subset of$A$. Then there is a neighbourhood U_0 of A_0 such that $\overline{U}_0$ is compact and f is a homeomorphism on $\overline{U}_0 \cup A$. To see this, let V_0 be a neighbourhood of A_0 with $\overline{V}_0$ compact and $f|_{\overline{V}_0}$ $1-1$, which is possible by 1). If no neighbourhood of A_0 in V_0 satisfies the requirements for U_0, there is a sequence of points $\{x_n\} \subset X \setminus A$ converging to $s \in A_0$ with $\{f(x_n)\} \subset f(A)$.

3) Express A as the union of ascending sequence of compact sets $A_1 \subset A_2 \subset \ldots$. Suppose V_i is a neighbourhood of A_i with $\overline{V}_i$ compact and f is a homeomorphism on $\overline{V}_i \cup A$. Then $\overline{V}_i \cup A_i$ is a compact subset of $\overline{V}_i \cup A$ on which f is a homeomorphism and by 2 there is a neighbourhood V_{i+1} of $\overline{V}_i \cup A_i$ with $\overline{V}_{i+1}$ compact and f a homeomorphism on $\overline{V}_{i+1} \cup A$. Let $V = \cup V_i$. The sets $\{V_i\}$ are an ascending sequence of open sets so if $x, y \in V$ with $f(x) = f(y)$ then there is an i with $x, y \in V_i$, but $f|_{V_i}$ is $1-1$ on V so $x = y$. Thus f is $1-1$ on V and being a local homeomorphism, $f|_V$ is a homeomorphism.

□

Lemma B.6.4. *Let $(V, \mathcal{F})$ be a manifold with boundary. Then there is a differentiable function $g\colon V \to [0, \infty)$ such that $g(\partial V) = 0$ and $g_*|_{\tau_v}$ is non-zero for each $v \in \partial V$.*

Proof. Let (V_i, h_i) be a countable locally finite cover of V by charts and apply the shrinking lemma twice to get

$$W_i \subset \overline{W}_i \subset U_i \subset \overline{U}_i \subset V_i.$$

Let $\varphi_i \in \mathcal{F}$ with $0 \le \varphi_i \le 1$, $\varphi_i(\overline{W}_i) = 1$, $\varphi_i(V \setminus U_i) = 0$. Let K be the set of i such that $W_i \cap \partial V \ne \varnothing$. For each $i \in K$, $h_i\colon V_i \to \mathbb{H}^n$ is of the form $f_1^i \times \cdots \times f_n^i$ and $V_i \cap \partial V = h_i^{-1}(\mathbb{R}^{n-1} \times 0) = (f_n^i)^{-1}(0)$. Let $g(x) = \sum_{i \in K} \varphi_i(x) \cdot f_n^i(x)$. Then $g \in \mathcal{F}$ and $g_i\colon V \to [0, \infty)$ with $g(\partial V) = 0$.

Let $v \in \partial V$. There is then an $i \in K$ with $v \in W_i$. Let $\lambda\colon [0, \infty) \to \mathbb{R}^n$ by $\lambda(t) = h(v) + (0, \ldots, 0, t)$. Then there is an $\varepsilon > 0$ with $\lambda([0, \varepsilon)) \subset h(V_i)$. Then $h^{-1} \circ \lambda\colon [0, \varepsilon) \to V$ is a differentiable map and to show $g_*|_{\tau_v} \ne 0$ it suffices to prove that $\frac{d}{dt}(g \circ h^{-1} \circ \lambda) \ne 0$ at $t = 0$.

For the $i \in K$ used to define λ, we have $\varphi_i \circ h^{-1} \circ \lambda(t) = 1$ for all $t \in \lambda^{-1}(W_i)$ and $f_n^i \circ h^{-1} \circ \lambda(t) = t$ for all $t \in [0, \varepsilon)$. Thus $\frac{d}{dt}(\varphi_i \circ h^{-1} \circ \lambda) = 1$.

For any $i' \ne i$, $i' \in K$ with $v \in V_i$, one has

$$\frac{d}{dt}(\varphi_{i'} \cdot f_n^{i'} \circ h^{-1} \circ \lambda) = (\varphi_{i'} \circ h^{-1} \circ \lambda) \cdot \frac{d}{dt}(f_n^{i'} \circ h^{-1} \circ \lambda) + f_n^{i'} \circ h^{-1} \circ \lambda \cdot \frac{d}{dt}(\varphi_{i'} \circ h^{-1} \circ \lambda).$$

Now $\varphi_{i'} \circ h^{-1} \circ \lambda(0) \geq 0$, $f_n^{i'} \circ h^{-1} \circ \lambda(0) = 0$ and for $t > 0$ $f_n^{i'} \circ h^{-1} \circ \lambda(t) > 0$ in a neighbourhood of $t = 0$, and hence $\frac{d}{dt}(f_n^{i'} \circ h^{-1} \circ \lambda) \geq 0$.

Adding these up,on has $\frac{d}{dt}(g \circ h^{-1} \circ \lambda) \geq 1$. □

Theorem B.6.5 (Tubular Neighbourhood)**.** *Let $(V, \mathcal{F})$ be a differentiable manifold with boundary. There is an open neighbourhood U of ∂V in V such that $(U, \mathcal{F}|_U)$ is diffeomorphic to $\partial V \times [0, 1)$.*

Proof. Let $\overline{g}\colon V \to \mathbb{R}^p$ be an imbedding. Then$\overline{g}|_{\partial V}\colon \partial V \to \mathbb{R}^p$ is an imbedding so there is an pen neighbourhood N of ∂V in $\mathbb{R}^p$ diffeomorphic to a neighbourhood of the zero section in $E(\nu_{\overline{g}|_{\partial V}})$, with $\alpha\colon N \to E(\nu_{\overline{g}|_{\partial V}})$ the diffeomorphism into. Then $\pi \circ \alpha \circ \overline{g}\colon \overline{g}^{-1}(N) \to \partial V$ is a differentiable retraction of the open neighbourhood $\overline{g}^{-1}(N)$ of ∂V onto ∂V. Let $g\colon V \to [0, \infty)$ be as given previously. Then $r = (\pi \circ \alpha \circ \overline{g}) \times g\colon \overline{g}^{-1}(N) \to \partial V \times [0, \infty)$ is a differentiable map. For any $v \in \partial V$, the kernel of $g_*|_{\tau_v}$ contains the image of $\tau(\partial V)_v$, hence by dimension this is precisely the kernel. $(\pi \circ \alpha \circ \overline{g})_*$ maps the image of $\tau(\partial V)_v$ isomorphically. Thus $r_*|_{\tau_w}$ is monic for all w in some open neighbourhood W of ∂V,and so is a local diffeomorphism of W with an open neighbourhood of $\partial V \times 0$ in $\partial V \times [0, \infty)$, and is a homeomorphism of ∂V. Thus there is an open neighbourhood Q of ∂V in V diffeomorphic to an open neighbourhood of ∂V in $\partial V \times [0, \infty)$. By means of a countable locally finite cover of ∂V by charts, with compact closure, one may take a neighbourhood of∂V of the form $\{(x, y) \in \partial V \times [0, \infty) | y < \beta(x)\}$ for some $\beta \in \mathcal{F}(\partial V)$ with $\beta > 0$, within this neighbourhood. Sending $(x, y) \mapsto (x, y/\beta(x)))$ maps this diffeomorphically onto $\partial V \times [0, 1)$. □

Theorem B.6.6. *Let $(V, \mathcal{F})$ and $(W, \mathcal{G})$ be differentiable manifolds with boundary such that V is a submanifold of W with inclusion $i\colon V \hookrightarrow E$ and suppose there is a neighbourhood U of ∂W in W and a diffeomorphism*

$$f\colon (U, U \cap V) \to (\partial W \times [0, 1), \partial V \times [0, 1)).$$

Then there is an open neighbourhood of the zero section in ν_i.

Proof. Let $\alpha = \pi_1 \circ f\colon U \to \partial W$, $\beta = \pi_2 \circ f\colon U \to [0, 1)$. There is a function $\mu \in \mathcal{F}(W)$ with $0 \leq \mu \leq 1$, $\mu(\beta^{-1}([0, 3/4))) = 1$, $\mu(W \setminus U) = 0$ and a function $\nu \in \mathcal{F}(W)$ with $0 \leq \nu \leq 1$, $\nu(\beta^{-1}([0, 5/8))) = 0$, $\nu(W \setminus \beta^{-1}([0, 3/4)) = 1$ and so $\sigma = \mu \cdot \beta + \nu\colon W \to [0, \infty)$ is in $\mathcal{F}(W)$ and $\sigma|_{U'} = \beta|_{U'}$ where $U' = \beta^{-1}([0, 1/2)$.

Let $\varphi\colon [0, 1/2] \to [0, 1]$ be the C^∞ function with $\varphi[0, 1/4] = 0$, $\varphi[3/8, 1/2] = 1$ given by $\varphi_{1/4, 3/8}$. Let $q\colon E \to W$ be $f^{-1} \circ (\mathrm{id} \times u) \circ f$ on U' where $u(s) = \varphi(s) \cdot s$ and the identity on $W \setminus f^{-1}([0, 3/8))$.

Let $g\colon W \to \mathbb{R}^p$ be any imbedding and define $h = (g \circ q) \times \sigma\colon W \to \mathbb{H}^{p+1}$. h is easily seen to be an imbedding and $h \circ f^{-1}\colon \partial W \times [0, 1/2) \to \mathbb{H}^{p+1}$ agree with $g|_{\partial W} \times \mathrm{id}$.

The inner product on $\mathbb{R}^p$ gives inner products on $\tau(W)_w$ and $\tau(V)_v$ so that one may identify ν_i with

$$\{(x, y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{p+1} | x = hi(v), \quad y \in h_* \tau(W)_{i(v)}, \quad y \perp h_* i_* \tau(V)_v\}.$$

The evaluation map $e$sends this subset into $\mathbb{R}^{p+1}$, and by the agreement of $h \circ f^{-1}$ with $g|_{\partial W} \times \mathrm{id}$ on $\partial W \times [0, 1/2)$ will send $\{(x, y)|x = h(u'), u' \in U'\}$ into $\mathbb{H}^{p+1}$ (since y can have no component orthogonal to $\mathbb{R}^p \times 0$) and hence sends a neighbourhood of $h(i(V)) \times 0$ into $\mathbb{R}^{p+1}$. Since W is imbedded nicely by h, there is a retraction of a neighbourhood of W into W (as in the tubular neighbourhood theorem for closed manifolds in Euclidean space). The composite map of a neighbourhood of the zero section in $E(\nu_i)$ into W is of maximal rank at the zero section, and checking along the tubular neighbourhood of ∂V shows that this is a diffeomorphism of smaller neighbourhoods. □

Note: Such a nice tubular neighbourhood U seems to always exist if one has sufficient regularity at the intersection of V and ∂W. In particular, our definition of submanifold appears sufficiently restrictive to give this. No simple proof seems possible, and hoping that we won't need this existence, we will avoid the argument.

B.7 Transversality

Definition B.7.1. Let $f: M^m \to N^n$ be a differentiable map (between differentiable manifolds), N'^k a closed submanifold of N. $f$is said to be *transverse regular* toN'^k at $x \in M^m$ if

1) $f(x) \notin N'^k$, or

2) $f(x) \in N'^k$ and the composite

$$\tau(M)_x \xrightarrow{f_*} \tau(N)_{f(x)} \to \tau(N)_{f(x)}/i_*\tau(N')_{f(x)}$$

is epic, where $i: N' \hookrightarrow N$ is the inclusion.

f is said to be transverse regular on N'^k if f is transverse regular at reach point of M.

Proposition B.7.2. *The set of points $x \in M$ at which f is transverse regular to N' is open.*

Proof. $f^{-1}(N')$ is closed so the set of points of type 1 is open. Suppose x is of the second type and choose a chart at $f(x)$, (U, h), with $h(U \cap N') = h(U) \cap (0 \times \mathbb{R}^k)$. Let (V, k) be a chart at x with $V \subset f^{-1}(U)$. With coordinates u_i in $h(U)$, v_j in $k(V)$, one has $h \circ f \circ k^{-1}: k(V) \to h(U)$ and the transversality condition at x is the assertion that

$$\left(\frac{\partial u_i}{\partial v_j}\right)_{j=1,\dots,m}^{i=1,\dots,n-k}$$

has rank $n - k$ at $k(x)$. This matrix has rank $n - k$ in a neighbourhood of $k(x)$, so f is transverse regular on a neighbourhood of x. □

Proposition B.7.3. *If $f\colon M^n \to N^n$ is transverse regular to N'^k and the restriction of f to ∂M is also transverse regular to N'^k then $f^{-1}(N')$ is a submanifold of M of dimension $m-(n-k)$. Further, the normal bundle of $f^{-1}(N')$ in M is induced from the normal bundle to N' in N.*

Proof. Let f be transverse regular at $x \in f^{-1}(N')$ and choose charts (U,h) and (V,k) as above. By reordering coordinates in V, one may assume $(\partial u_i/\partial v_j)$ $i,j=1,\dots,n-k$ is non-singular at $k(x)$. Hence by the inverse function theorem the functions $(u_1,\dots,u_{n-k},v_{n-k+1},\dots,v_m)$ give a chart at $k(x)$ in $k(V)$ and hence a chart at x, (V',k') such that

$$k'(V' \cap f^{-1}(N')) = k'(V') \cap (0 \times \mathbb{R}^{m-(n-k)}).$$

If $x \in M \setminus \partial M$ this is a chart of the required type. If $x \in partial M$, then the condition on $f|_{\partial M}$ implies that in the reordering the function v_m is not replaced by any u_i, and hence that $k'(V') \subset \mathbb{H}^m$. Thus, the chart (V',k') is as required. The normal bundle condition is clear since the induced map is epic on fibres. □

Theorem B.7.4. *Let $f\colon M \to N$ be a differentiable map; let N' be a closed differentiable submanifold of N'. Let A be a closed subset of M such that the transverse regularity condition for f on N' is satisfied at all points of $A \cap f^{-1}(N')$. There exists a differentiable map $g\colon M \to N$ such that*

1) gis homotopic to f,

2) g is transverse regular on N',

3) $g|_A = f|_A$.

Proof. There is a neighbourhood U of A in M such that f satisfies the transverse regularity condition on U. Cover N by $N \setminus N' = Y_0$ and coordinate systems (Y_i,k_i) for $i>0$ with coordinate functions $(v_1,\dots,v_n)$ such that $N' \cap Y_i$ is mapped precisely to the set for which $v_1 = \dots = v_{n-k} = 0$. The sets $f^{-1}(Y_i)$ cover M, as do the sets U and $M \setminus A$. Let (V_j,h_j) a refinement of both coverings which is countable and locally finite, indexed so that $j \le 0$ if $V_j \subset U$ and the others have $j>0$. Apply the shrinking lemma twice to get

$$W_j \subset \overline{W}_j \subset U_j \subset \overline{U}_j \subset V_j.$$

and let $\varphi_j \in \mathcal{F}$, $0 \le \varphi_j \le 1$, $\varphi_j(\overline{W}_j) = 1$, $\varphi_j(M \setminus U_j) = 0$. For each j choose $i(j) \ge 0$ with $f(V_j) \subset Y_{i(j)}$.

Let $f_0 = f$ and suppose f_{k-1} has been defined, satisfies transverse regularity on $\cup_{h<k}\overline{W}_j$ with $f_{k-1}(\overline{U}_j) \subset Y_{i(j)}$ for each j. In particular, letting $i = i(k)$, $f_{k-1}(\overline{U}_k) \subset Y_{i(j)}$.

Consider the function $\pi k_i f_{k-1} \circ h_k^{-1}\colon h_k(U_k \to \mathbb{R}^{n-k}$ where π projects on the first $n-k$ coordinates. By the approximation of regular values theorem, there are arbitrarily small vectors $y \in \mathbb{R}^{n-k}$ such that $\pi k_i f_{k-1} \circ h_k^{-1} - (\varphi_k \circ h_k^{-1})y$ has the origin as regular value. We then define f_k by

$$f_k(x) = \begin{cases} k_i^{-1}\{k_i f_{k-1}(x) - \varphi_k(x)(y,0)\} & \text{for } x \text{ in a neighbourhood of } \overline{U}_k, \\ f_{k-1} & \text{if } x \in M \setminus U_k, \end{cases}$$

where $y \in \mathbb{R}^{n-k}$ is yet to be chosen.

First one needs y small enough that $k_i f_{k-1}(x) - \varphi_k(x)(y, 0)$ lies in $k_i(Y_i)$ for all $x \in \overline{U}_k$. If Y_i is a neighbourhood meeting ∂N then $(y, 0)$ is "parallel" to ∂N and one is not translated out of $k_i(Y_i)$ across ∂N. Hence for small y this holds and thus k_i^{-1} may be applied. Next y is chosen to give a $\delta/2^k$ approximation to f_{k-1}. Also y is chosen small enough that $f_K(\overline{U}_j) \subset Y_{i(j)}$ for each j. This is possible since only a finite number of sets $\overline{U}_j$ meets $\overline{U}_k$. Under these conditions f_k will be transverse regular on N' at each point of $f_k^{-1}(N') \cap \overline{W}_k$.

Now f_{k-1} is transverse regular on N' at each point of the compact set $\overline{U}_k \cap (\cup_{j<k} \overline{W}_j)$ and since small changes preserve regularity, for sufficiently small y, f_k will also be transverse regular on this set, hence on $\cup_{j<k} \overline{W}_j$.

After all these limitations, we have such a y and hence an f_k. Let $g(x) = \lim f_k(x)$. A homotopy from f_{k-1} to f_k is given by contracting y and a limit of these homotopies defines a homotopy from f to g.

□