Complex n-manifolds are orientable. A discussion from Stack Exchange Mathematics

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It is true that any complex n-manifold is orientable as a real manifold. One possible way of showing this, is to calculate the determinant of the Jacobians of the transition functions.

Suppose we have an n-dimensional complex manifold M. That is, M is a (first countable, Hausdorff) space such that every point $p \in M$ has an open neighbourhood V homeomorphic to an open subset $\Omega \subset \mathbb{C}^n$,

$$\phi \colon V \to \Omega \quad \text{homeomorphism}$$

and when two such neighbourhoods intersect, the transition functions are holomorphic,

$$\psi \circ \phi^{-1} \colon \phi(V \cap V') \to \psi(V \cap V') \quad \text{biholomorphism.}$$

By identifying

$$\mathbb{C}^n \quad \text{with} \quad \mathbb{R}^{2n}, \quad (z^1, \ldots, z^n) \leftrightarrow (x^1, y^1, \ldots, x^n, y^n)$$

where $z^k = x^k + iy^k$ is the real part/imaginary part decomposition, we get a real manifold structure on M. We can calculate the Jacobian matrices of the transition functions for both structures.

Let's set up some notation. The first coordinate chart will be

$$\phi \colon V \to \Omega, \quad p \mapsto (z^1(p), \ldots, z^n(p))$$

and the second will be

$$\psi \colon V' \to \Omega', \quad p \mapsto (Z^1(p), \ldots, Z^n(p)).$$

In the complex case, we have

$$\begin{aligned}\mathrm{Jac}_{\phi(p)}(\psi \circ \phi^{-1}) &= \left(\frac{\partial[\psi \circ \phi^{-1}]^k}{\partial z^l}\right)_{1 \le k,l \le n} = \left(\frac{\partial Z^k(z^1, \ldots, zn)}{\partial z^l}\right)_{1 \le k,l \le n} \\ &= \left(c^k_l\right)_{1 \le k,l \le n} \in \mathrm{GL}(n, \mathbb{C}),\end{aligned}$$

and in the real case you will find

$$\begin{aligned}\mathrm{Jac}_{\phi(p)}(\psi^{\mathbb{R}} \circ (\phi^{\mathbb{R}})^{-1}) &= \begin{bmatrix} \frac{\partial X^k(x^1,y^1,\ldots,x^n,y^n)}{\partial x^l} & \frac{\partial X^k(x^1,y^1,\ldots,x^n,y^n)}{\partial y^l} \\ \frac{\partial Y^k(x^1,y^1,\ldots,x^n,y^n)}{\partial x^l} & \frac{\partial Y^k(x^1,y^1,\ldots,x^n,y^n)}{\partial y^l} \end{bmatrix}_{1 \le k,l \le n} \\ &= \begin{bmatrix} \Re(c^k_l) & -\Im(c^k_l) \\ \Im(c^k_l) & \Re(c^k_l) \end{bmatrix}_{1 \le k,l \le n} \in \mathrm{GL}(2n, \mathbb{R}),\end{aligned}$$

using the Cauchy-Riemann equations. We will calculate the determinant of these matrices, show that it is always > 0, which is equivalent to $M^{\mathbb{R}}$ being orientable.

We move on to calculating the determinants of these. Consider the R-algebra homomorphism

$$\rho\colon M_n(\mathbb{C}) \to M_{2n}(\mathbb{R}), \quad \left(c_l^k\right)_{1\le k,l\le n} \mapsto \begin{bmatrix} \Re(c_l^k) & -\Im(c_l^k) \\ \Im(c_l^k) & \Re(c_l^k) \end{bmatrix}_{1\le k,l\le n}$$

Since it is $\mathbb{R}$-linear and the spaces involved are finite dimensional, it is continuous. Also, being an algebra homomorphism, we have

$$\det \rho(P^{-1}AP) = \det(\rho(P)^{-1}\rho(A)\rho(P)) = \det(\rho(A)).$$

Finally, the diagonalisable matrices are dense in $M_n(\mathbb{C})$, so we can restrict our calculations to diagonal matrices in $M_n(\mathbb{C})$. For these, the calculations are easy:

$$\begin{aligned}\det\left(\rho(\mathrm{Diag}(c_1,\dots,c_n))\right) &= \det\mathrm{diag}\left(\begin{bmatrix} \Re(c_1) & -\Im(c_1) \\ \Im(c_1) & \Re(c_1) \end{bmatrix},\cdots,\begin{bmatrix} \Re(c_n) & -\Im(c_n) \\ \Im(c_n) & \Re(c_n) \end{bmatrix}\right) \\ &= \prod_{i=1}^n \det\left(\begin{bmatrix} \Re(c_i) & -\Im(c_i) \\ \Im(c_i) & \Re(c_i) \end{bmatrix}\right) = \prod_{i=1}^n |c_i|^2 \\ &= |\det\mathrm{diag}(c_1,\dots,c_n)|^2,\end{aligned}$$

so we conclude that

$$\forall A \in M_n(\mathbb{C}), \quad \det\rho(A) = |\det A|^2.$$

Finally, we can conclude that the transition functions for the charts $\phi^{\mathbb{R}}$ for $M^{\mathbb{R}}$ have positive determinants, thus the real underlying manifold $M^{\mathbb{R}}$ is orientable.