Abelian Categories

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Contents

Chapter 1

Abelian categories

The objective of this chapter is the following:

- We will define the *kernel* and *cokernel* of a morphism in a category that has a *zero object*.
- We will define *monic* and *epi* morphisms in an arbitrary category.
- We will give an axiomatic definition of an *abelian category* and we will show that the category of right R-modules is the prototype for an abelian category.
- We will show that the category of complexes of right R-modules is also an abelian category.

In this chapter we assume the following conventions:

- Let R be a ring, not necessarily commutative.
- Let Mod_R denote the category of right R-modules.
- Let Comp_R denote the category of complexes of right R-modules.
- Let $\mathcal{C}$ be an arbitrary category.

1.1 The Category Mod_R

Let $M \xrightarrow{f} N$ be a morphism in Mod_R.

Definition 1.1.1. Recall that the *kernel* of f is

$$\ker f = \{m \in M \colon f(m) = 0\} \subseteq M$$

Definition 1.1.2. The *cokernel* of f is

$$\operatorname{coker} f = N / \operatorname{im} f$$

1.2 The Category Comp_R

Recall that in the category $Comp_R$, the morphisms are *chain maps.*

Definition 1.2.1. Given complexes $(C_\bullet, d_\bullet), (C'_\bullet, d'_\bullet)$, a *chain map*

$$f = f_\bullet \colon (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$$

is a sequence of maps $f_n \colon C_n \to C'_n$ for all $n \in \mathbb{Z}$ making the following diagramme commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

Definition 1.2.2. A chain complex $B_\bullet$ is called a *sub-complex* of a chain complex $C_\bullet$ if each B_n is a sub-module of $C_\bullet$, and the differential on $B_\bullet$ is the restriction of the differential on $C_\bullet$, that is when the inclusions $i_n \colon B_\bullet \to C_n$ constitute a chain map $B_\bullet \to C_\bullet$.

Example 1.2.3. Take any complex:

$$(C_\bullet, d_\bullet) = \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

The *zero complex* in Comp_R is

$$(0_\bullet, 0) = \cdots \to \{0\}_{n+1} \xrightarrow{0} \{0\}_n \xrightarrow{0} \{0\}_{n-1} \to \cdots$$

Consider the diagramme:

$$\begin{array}{cccccccccc}
(0_\bullet, 0) = & \cdots & \longrightarrow & \{0\}_{n+1} & \xrightarrow{0} & \{0\}_n & \xrightarrow{0} & \{0\}_{n-1} & \longrightarrow & \cdots \\
 & & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
(C_\bullet, d_\bullet) = & \cdots & \longrightarrow & C_{n+1} & \xrightarrow[d_{n+1}]{} & C_n & \xrightarrow[d_n]{} & C_{n-1} & \longrightarrow & \cdots
\end{array}$$

It is clear that this diagramme commutes. Therefore $(0_\bullet, 0)$ is a subcomplex of $(C_\bullet, d_\bullet)$, i.e. *the zero complex is a sub-complex of any complex.*

Definition 1.2.4. In the case of Definition 1.2.2, we assemble the quotient modules $\frac{C_n}{B_n}$ into a complex

$$\cdots \to \frac{C_{n+1}}{B_{n+1}} \xrightarrow{d'_{n+1}} \frac{C_n}{B_n} \xrightarrow{d'_n} \frac{C_{n-1}}{B_{n-1}} \to \cdots$$

denoted $\frac{C}{B}$ and called the *quotient complex.*

Remark 1.2.5. We must show that the induced differential

$$d'_n \colon c_n + B_n \mapsto d_n c_n + B_{n-1}$$

is well-defined.
Let $c_n + B_n = b_n + B_n$, i.e. $b_n - c_n \in B_n$. Then, since by definition the differential on $B_\bullet$ is the restriction of the differential on $C_\bullet$, applying d_n gives

$$\begin{aligned} b_n - c_n &\in B_n \\ \Rightarrow d_n(b_n - c_n) &\in B_{n-1} \\ \Rightarrow d_n(b_n) - d(c_n) &\in B_{b-1} \\ \Rightarrow d_n(b_n) + B_{n-1} &= d_n(c_n) + B_{n-1} \\ \Rightarrow d'_n(b_n) &= d'_n(c_n) \end{aligned}$$

so d' is well-defined, as required.

Remark 1.2.6. Let $f_\bullet \colon (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ be a chain map.
Define

$$\ker f = \cdots \to \ker f_{n+1} \xrightarrow{\delta_{n+1}} \ker f_n \xrightarrow{\delta_n} \ker f_{n-1} \to \cdots$$

where $\delta_n = d_n \mid_{\ker f_n}$. Consider the diagramme:

$$\begin{array}{ccccccccc}
\ker f = \cdots & \longrightarrow & \ker f_{n+1} & \xrightarrow{\delta_{n+1}} & \ker f_n & \xrightarrow{\delta_n} & \ker f_{n-1} & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
(C_\bullet, d_\bullet) = \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \big\downarrow f_{n+1} & & \big\downarrow f_n & & \big\downarrow f_{n-1} & & \\
(C'_\bullet, d'_\bullet) = \cdots & \longrightarrow & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

I claim that the maps in the top row send kernels into kernels as shown. We prove that $\delta_{n+1}(\ker f_{n+1}) \subseteq \ker f_n$. By applying the same argument in any degree, we get the result.

- Let $c_{n+1} \in \ker f_{n+1}$ be arbitrary, i.e. $f_{n+1}(c_{n+1}) = 0$.
- Then since d'_{n+1} is an R-map, we have

$$d'_{n+1} \circ f_{n+1}(c_{n+1}) = d'_{n+11}(f_{n+1}(c_{n+1})) = d_{n+1}(0) = 0.$$

- Since f is a chain map, all squares in the lower two rows commute, implying that

$$f_n(d_{n+1}(c_{n+1})) = f_n \circ d_{n+1}(c_{n+1}) = 0.$$

- Therefore $\delta_{n+1}(c_{n+1}) = d_{n+1}(c_{n+1}) \in \ker f_n$, as required.

- Now that we know that our top row is well-defined, it is clear from construction that the whole diagramme commutes.
- Since the top two rows commute, we have that $\ker f$ is a subcomplex of $(C_\bullet, d_\bullet)$ as claimed.

Remark 1.2.7. Define

$$\operatorname{im} f = \cdots \to \operatorname{im} f_{n+1} \xrightarrow{\Delta_{n+1}} \operatorname{im} f_n \xrightarrow{\Delta_n} \operatorname{im} f_{n-1} \to \cdots$$

where $\Delta_n = d_{n'} |_{\operatorname{im} f_n}$. Consider the diagramme:

$$\begin{array}{ccccccccc}
(C_\bullet, d_\bullet) = \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\operatorname{im} f = \cdots & \longrightarrow & \operatorname{im} f_{n+1} & \xrightarrow{\Delta_{n+1}} & \operatorname{im} f_n & \xrightarrow{\Delta_n} & \operatorname{im} f_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(C'_\bullet, d'_\bullet) = \cdots & \longrightarrow & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

I claim that the maps in the middle row send images into images as shown. We prove that $\Delta_{n+1}(\operatorname{im} f_{n+1}) \subseteq \operatorname{im} f_n$. By applying the same argument in any degree, we get the result.

- Let $c_{n+1} \in C_{n+1}$ be arbitrary, so that $f_{n+1}(c_{n+1}) \in \operatorname{im} f_{n+1}$ is arbitrary.
- Then since f is a chain map, all squares (ignoring the middle row) commute, implying that

  $$d'_{n+1}(f_{n+1}(c_{n+1})) = d'_{n+11} \circ f_{n+1}(c_{n+1}) = f_n \circ d_{n+1}(c_{n+1}) = f_n(d_{n+1}(c_{n+1}) \in \operatorname{im} f_n.$$

- Therefore $\Delta_{n+1}(f_{n+1}(c_{n+1})) = d'_{n+1}(f_{n+1}(c_{n+1})) \in \operatorname{im} f_n$, as required.
- Now that we know that our middle row is well-defined, it is clear from construction that the whole diagramme commutes.
- Since the bottom two rows commute, we have that $\operatorname{im} f$ is a subcomplex of $(C'_\bullet, d'_\bullet)$ as claimed.

Thus we have:

- The *First Isomorphism Theorem* holds.

  $$C_\bullet / ker f \cong \operatorname{im} f.$$

- The cokernels $\{\operatorname{coker}(f_n)\}$ similarly assemble to form a quotient complex of $(C'_\bullet, d'_\bullet)$, denoted $\operatorname{coker}(f)$.

1.3 Initial, Terminal and Zero Objects

Definition 1.3.1. An *initial object* is an object X such that for every object Y, there is a unique morphism $X \xrightarrow{i} Y$.

- In Set, the empty set is initial.
- In Grp, $\{1\}$ is initial.
- In Mod_R, $\{0\}$ is initial.
- In Comp_R, the zero complex is initial.

Definition 1.3.2. A *final object* (or *terminal object*) is an object such that for any object X, there is a unique morphism $X \xrightarrow{t} Y$.

Example 1.3.3.
- In Set, a set containing one element is terminal. So terminal objects (should they exist) need *not* be unique.
- In Grp, $\{1\}$ is a terminal object.
- In Mod_R, $\{0\}$ is a terminal object.
- In Comp_R, the zero complex is terminal.

Remark 1.3.4.
- If Y is terminal, then the only morphism $Y \xrightarrow{t} Y$ is the identity.
- Any two terminal objects of $\mathcal{C}$ are isomorphic in $\mathcal{C}$.

Definition 1.3.5. A *zero* (or *null*) object of $\mathcal{C}$ is an object which is both initial and terminal.

- In Set, there is no zero object.
- In Grp, f1g is a zero object.
- In Mod_R, $\{0\}$ is a zero object.
- In Comp_R, the zero complex is a zero object.
- In Ring (with unity), there is no zero object.

Thus every additive category has a zero object.

Remark 1.3.6. If $\mathcal{C}$ has a zero object Z, then for any objects X, Y of $\mathcal{C}$, there is a unique composition

$$X \xrightarrow{t} Z \xrightarrow{i} Y.$$

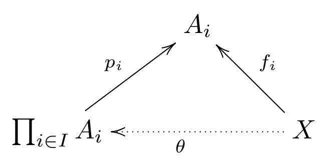
This is the *zero map* (or *zero morphism*) from X to Y.

1.4 Products and coproducts

Definition 1.4.1. Let $\{A_i : i \in I\}$ be a family of objects of $\mathcal{C}$. A *product* is an ordered pair

$$(C, \{p_i : C \to A_i\})$$

consisting of an object $C = \prod_{i \in I} A_i$ and a family $\{p_i : C \to X_i\}$ of *projection morphisms*, satisfying the following condition. For every object X equipped with morphisms $f_i : X \to A_i$, there exists a unique morphism $\theta : X \prod_{iinI} A_i$ making the following diagramme commute for each i:



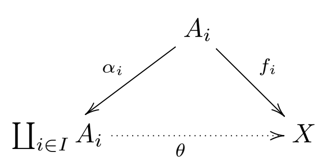
Example 1.4.2. In Mod_R, the product of a family of R-modules, $\{A_i : i \in I\}$ is the direct product.

Definition 1.4.3. Let $\{A_i : i \in I\}$ be a family of objects of $\mathcal{C}$. A *coproduct* is an ordered pair

$$(C, \{\alpha_i : A_i \to C\})$$

consisting of an object $C = \coprod_{i \in I} A_i$ and a family $\{\alpha_i : A_i to C\}$ of injection morphisms, satisfying the following condition.

For every object X equipped with morphisms $f_i : A_i \to X$, there exists a unique morphism $\theta : \coprod_{i \in I} A_i \to X$ making the following diagramme commute for each i :



Example 1.4.4. In Mod_R, the coproduct of a family of R-modules, $\{A_i : i \in I\}$ is the direct sum.

1.5 Pre-Additive and Additive Categories

Definition 1.5.1. Define an *Ab-category*, or *pre-additive category* to be a category $\mathcal{C}$ in which each Hom set is an additive abelian group and for which composition is bilinear with respect to this addition:

For morphisms $f, f' : A \to B$ and $g, g' : B \to C$,

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'$$

Lemma 1.5.2. *The category* Grp *is not pre-additive.*

Proof. For a contradiction, suppose that Grp is pre-additive. Then, by definition, for any group G, the Hom set $\mathrm{Hom}(G,G)$ is an additive abelian group. Let $G \xrightarrow{f}$G and $G \xrightarrow{g} G$ be any elements of $\mathrm{Hom}(G,G)$. Then $\mathrm{Hom}(G,G)$ is an abelian group with the operation $(f+g)(x) = f(x)+g(x)$. We are not assuming that G is abelian, so we re-write the RHS using multiplicative notation:

$$(f+g)(x) = f(x)g(x)$$

Since $\mathrm{Hom}(G,G)$ is a group with the above operation, it is closed with respect to the operation. In other words, the map $x \mapsto f(x)g(x)$ is an honest element of $\mathrm{Hom}(G,G)$, i.e. it is a group homomorphism from G to G.

Now let $G = \langle a,b\rangle$ be the free group on two generators. By the universal property of a free group, the following definitions yield group homomorphisms from G to G.

$$\begin{aligned} f\colon \langle a,b\rangle &\to \langle a,b\rangle \\ a &\mapsto a^2 \\ b &\mapsto b \\ g\colon \langle a,b\rangle &\to \langle a,b\rangle \\ a &\mapsto a \\ b &\mapsto b^2 \end{aligned}$$

Let $x = ab$. Then since $x \mapsto f(x)g(x)$ is a group homomorphism, we have

$$\begin{aligned} f(ab)g(ab) &= [f(a)g(a)][f(b)g(b)] \\ \Rightarrow [a^2b][ab^2] &= [a^2a][bb^2 \\ \Rightarrow a^2bab^2 &= a^3b^3 \end{aligned}$$

This is a contradiction, since both expressions are reduced but they are clearly not equal. □

Definition 1.5.3. An *additive category* is an Ab-category $\mathcal{C}$ with a zero object, and a product $A \times B$ for each pair of objects A, B from $\mathcal{C}$.

Example 1.5.4. Mod_R is an additive category.

1.6 Monics and Epis

Definition 1.6.1. A morphism $A \xrightarrow{i} B$ in an additive category $\mathcal{C}$ is *monic*, if, whenever $A' \xrightarrow{g} A$ is a morphism satisfying $i \circ g = 0$, then $g = 0$.

Remark 1.6.2. Monics can be *cancelled from the left.*

Example 1.6.3. In $\mathrm{Set}, \mathrm{Grp}$ and Mod_R, monics are just injective maps.

Definition 1.6.4. A morphism $CxrightarroweD$ in an additive category C is *epi*, if, whenever $D \xrightarrow{h} D'$ is a morphism satisfying $h \circ e = 0$, then $h = 0$.

Remark 1.6.5. Epis can be *cancelled from the right.*

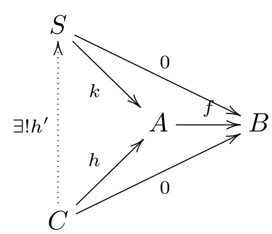
Example 1.6.6. In Set, Grp and Mod_R, epis are just surjective maps.

1.7 Kernels and Cokernels

Categorical Definition of Kernel

Definition 1.7.1. Let $\mathcal{C}$ have a zero object, so that for any objects X, Y of $\mathcal{C}$, there exists a zero map $X \xrightarrow{0} Y$. Let $f\colon A \to B$ be a morphism in $\mathcal{C}$. Then $k\colon S \to A$ is a *kernel of* $f\colon A \to B$ if

- $f \circ k = 0$
- Every $h\colon C \to A$ with $f \circ h = 0$ factors uniquely through k (as $h = k \circ h'$).



Remark 1.7.2.
- Kernels are not guaranteed to exist in general.
- If the kernel exists, then it is unique since it is defined by a universal property.
- In Grp, the usual definition of kernel, with the inclusion map into A, satisfies this universal property. So kernels always exist in Grp.
- In Ring, there is no zero object, so the kernel does not exist.
- In the category of *pointed topological spaces*, if $(X, x_0) \xrightarrow{f} (Y, y_0)$ is a continuous pointed map, then the preimage of the distinguished point y_0 of Y, K, is a subspace of X. The inclusion map of $K \xrightarrow{k} X$ is the categorical kernel of f.

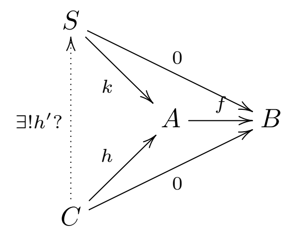
Proposition 1.7.3. *In* Mod_R*, the usual definition of kernel, with the inclusion map into* A*, satisfies this universal property. So kernels always exist in* Mod_R*.*

Proof. Let $A \xrightarrow{f} B$ be an arbitrary morphism in Mod_R. Define

$$S = \{a \in A \colon f(a) = 0\}$$

and let $S \xrightarrow{k} A$ be the inclusion map. Then it is clear that $f \circ k = 0$.

To verify the second property, start with the diagramme



Define

$$h' : C \to S$$
$$c \mapsto h(c)$$

This does define an honest map into S, since $f(h(c)) = f \circ h(c) = 0$, so that $h(c) \in S$.

Proof that $h = k \circ h'$: Let $c \in \mathcal{C}$ be arbitrary. Then $k \circ h' = k \circ h(c) = h(c)$, since k is just the inclusion.

Proof that h' is unique: Suppose that h'' also has the above properties. Then $h = k \circ h' = h''$. Therefore

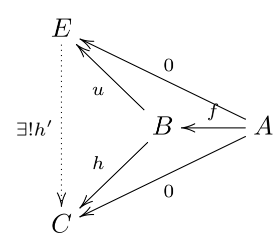
$$h(c) = k \circ h'(c) = h''(c) \Rightarrow 0 = k(h'(c) - h''(c))$$

and thus, since k is injective, we have $h'(c) = h''(c)$ as required. □

Categorical Definition of Cokernel

Definition 1.7.4. Let $\mathcal{C}$ have a zero object, so that for any objects X, Y of $\mathcal{C}$, there exists a zero map $X \xrightarrow{0} Y$. Let $f\colon A \to B$ be a morphism in $\mathcal{C}$. Then $u\colon B \to E$ is a *cokernel of* $f\colon A \to B$ if

- $u \circ f = 0$
- Every $h\colon B \to C$ with $h \circ f = 0$ factors uniquely through u (as $h = h' \circ u$).



Remark 1.7.5.

- Cokernel is the dual notion to kernel.
- Cokernels are not guaranteed to exist in general.

- If the cokernel exists, then it is unique since it is defined by a universal property.
- In Grp, the cokernel of a group homomorphism $G \xrightarrow{f} H$ is the quotient of H by the normal closure of the image of f.
- In Ab, the category of abelian groups, since every subgroup is normal, the cokernel is just H modulo the image of f.
- In Ring, there is no zero object, so the cokernel does not exist.

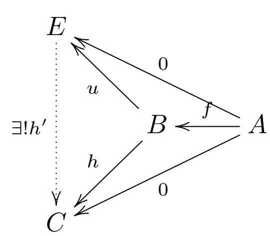
Proposition 1.7.6. *In* Mod_R*, the usual definition of* $\operatorname{coker} f = B/\operatorname{im} f$*, with the natural map onto* $B/\operatorname{im} f$*, satisfies this universal property. So cokernels always exist in* Mod_R*.*

Proof. Let $A \xrightarrow{f} B$ be an arbitrary morphism in Mod_R. Define

$$E = B/\operatorname{im} f$$

and let $B \xrightarrow{u} B/\operatorname{im} f$ be the natural map. Then it is clear that $u \circ f = 0$.

To verify the second property, start with the diagramme



Define

$$h' : E \to C$$
$$b + \operatorname{im} f \mapsto h(b)$$

We need to verify that h' is well-defined. Let $b+\operatorname{im} f = b'+\operatorname{im} f$, i.e. $b-b' \in \operatorname{im} f$. Write $b - b' = f(a)$, for some $a \in A$. Then applying h gives

$$\begin{aligned} h(b-b') &= h(f(a)) \\ h(b) - h(b') &= h \circ f(a) \\ &= 0 \\ \Rightarrow h(b) &= h(b') \\ \Rightarrow h'(b + \operatorname{im} f) &= h'(b' + \operatorname{im} f) \end{aligned}$$

so h' is well-defined as claimed. Then h' is an R-map since h is.

Proof that $h = h' \circ u$*:* Let $b \in B$ be arbitrary. Then $h' \circ u(b) = h'(u(b)) = h'(b + \operatorname{im} f) = h(b)$.

Proof that h' is unique: Suppose that h'' also has the above properties. Let $b \in B$ be arbitrary, so that $b + \operatorname{im} f \in E$ is arbitrary. Then

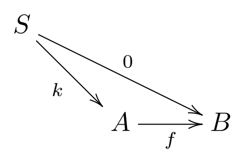
$$\begin{aligned} h''(b + \operatorname{im} f) &= h''(u(b)) \\ &= h(b) \\ &= h'(u(b)) \\ &= h'(b + \operatorname{im} f) \end{aligned}$$

□

Mod_R Monic Equivalences

Lemma 1.7.7. *In the additive category $\mathcal{C} = \mathrm{Mod}_R$, the notions of kernel, monics and monomorphisms are the same.*

Proof. Proof that kernel → monic: Suppose that $S \xrightarrow{k} A$ is a kernel, i.e. there is a morphism $A \xrightarrow{f} B$ with k as its kernel.



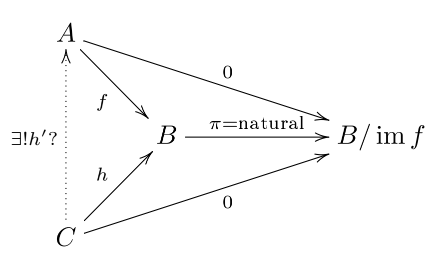
In Mod_R we may take

$$S = \{a \in A : f(a) = 0\} \subseteq A$$

and k to be the inclusion map. Let $A' \xrightarrow{g} A$ satisfy $k \circ g = 0$. Let $a' \in A'$ be arbitrary. Note that, since $S \xrightarrow{k} A$ is a kernel, $g(a') \in S$: see 1.7.3. Then $f(g(a')) = f \circ g(a') = 0 \Rightarrow g(a') \in \ker k$. But since k is the inclusion hence is injective, this says that $g(a') = 0$, i.e. $g = 0$ as required.

Proof that monic ⇒ monomorphism: Let $A \xrightarrow{f} B$ be monic. Then we have $\ker f \xrightarrow{k=\text{include}} A \xrightarrow{f} B$ and $f \circ k = 0$. Since f is monic, this implies that $k = 0$. Therefore $\ker f = \{0\}$, i.e. f is a monomorphism as required.

Proof that monomorphism ⇒ kernel: Let $A \xrightarrow{f} B$ be a monomorphism, i.e. $\ker f = \{0\}$. Consider the diagram



Define

$$\begin{aligned} h' \colon C &\to A \\ c &\mapsto a \in A \text{ such that } f(a) = h(c) \end{aligned}$$

I claim that we can always find such an $a \in A$ uniquely. We have $h(c) \in B$ and $\pi(h(c)) = \pi \circ h(c) = 0$, which says that $h(c) \in \operatorname{im} f$. So there always exists $a \in A$ such that $f(a) = h(c)$. Since f is a monomorphism, this a is unique, so the above function is well-defined.

I claim that h' is an R-map. Let $c, c' \in C$ and $r \in R$ be arbitrary. Then

$$\begin{aligned} h'(c) &= a \in A \text{ such that } f(a) = h(c) \\ h'(c') &= a' \in A \text{ such that } f(a') = h(c') \\ h'(c + c') &= a^* \in A \text{ such that } f(a^*) = h(c + c') \end{aligned}$$

Then

$$h(c + c') = h(c) + h(c') = f(a) + f(a') = f(a + a')$$

and since f is a monomorphism, $f(a^*) = h(c + c' = f(a + a')$ implies that $a^* = a + a'$, and thus $h'(c + c') = a^* = a + a' = h'(a) + h'(a')$. Keep the above notation. Then

$$h'(cr) = a^* \in A \text{ such that } f(a^*) = h(cr) = h(c)r$$

Then we have $f(ar) = f(a)r = h(c)r = h(cr) = f(a^*)$. Since f is a monomorphism, this implies that $ar = a^*$, and thus $h'(cr) = a^* = ar = h'(c)r$. So h' is an R-map as claimed.

The diagram commutes by the definition of h'.

Proof that h' is unique: Suppose that there exists an h'' with the above properties. Let $c \in C$ be arbitrary. Then

$$\begin{aligned} f(h''(c) - h'(c)) &= f(h''(c)) - f(h'(c)) \\ &= f \circ h''(c) - f \circ h'(c) \\ &= h(c) - h(c) \\ &= 0 \end{aligned}$$

and since f is injective, this implies that

$$h''(c) - h'(c) = 0 \Rightarrow h''(c) = h'(c).$$

So f is a kernel as required. □

Lemma 1.7.8. *In the additive category $\mathcal{C} = \mathrm{Mod}_R$, the notions of cokernel, epis and epimorphisms are the same.*

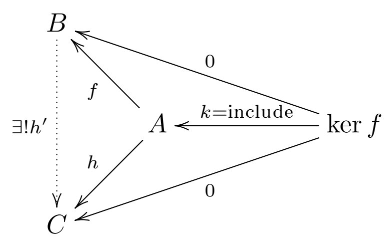
Proof. Proof that cokernel ⇒ epi: Suppose that $B \xrightarrow{u} E$ is a cokernel, i.e. there is a morphism $A \xrightarrow{f} B$ with u as its cokernel. In Mod_R we may take

$$E = B / \operatorname{im} f$$

and u to be the natural map. Let $E \xrightarrow{h} F$ satisfy $h \circ u = 0$. Let $b + \operatorname{im} f \in B/\operatorname{im} f$ be arbitrary. Then $u(b) = b + \operatorname{im} f$. Applying h to both sides gives $h \circ u(b) = h(b + \operatorname{im} f) \Rightarrow 0 = h(b + \operatorname{im} f)$. Thus $h = 0$ as required.

Proof that epi $\Rightarrow$ epimorphism: Let $A \xrightarrow{f} B$ be epi. Then we have $A \xrightarrow{f} B \xrightarrow{u} B/\operatorname{im} f$ and $u \circ f = 0$. Since f is epi, this implies that $u = 0$. Therefore $\operatorname{im} f = B$, i.e. f is an epimorphism as required.

Proof that epimorphism $\rightarrow$ cokernel: Let $A \xrightarrow{f} B$ be an epimorphism, i.e. it is surjective. Consider the diagram



Define

$$h' \colon B \rightarrow C$$
$$b \mapsto h(a) \text{ where } b = f(a)$$

We can always find such an $a \in A$ since f is surjective. I claim that the above definition does not depend on the choice of a. Suppose that $f(a) = b = f(a')$. Then $0 = f(a - a') \Rightarrow (a - a') \in \ker f$. Thus $h \circ k(a - a') = 0 \Rightarrow h \circ k(a) = h \circ k(a') \Rightarrow h(a) = h(a')$, since k is the inclusion map. Thus the above function is well-defined.

Note that h' is an R-map, since h is. The diagram commutes by the definition of h'.

Proof that h' is unique: Suppose we have an h'' satisfying the above properties. Let $b \in B$ be arbitrary. Since f is surjective, there exists an element $a \in A$ with $b = f(a)$. Then

$$\begin{aligned} h''(b) &= h''(f(a)) \\ &= h'' \circ f(a) \\ &= h(a) \\ &= h' \circ f(a) \\ &= h'(f(a)) \\ &= h'(b) \end{aligned}$$

So f is a cokernel as required. □

1.8 Abelian Categories

Definition 1.8.1. An *abelian category* $\mathcal{C}$ is an additive category satisfying the following conditions.

(AB1) Every morphism in $\mathcal{C}$ has a kernel and a cokernel.

(AB2) Every monic morphism in $\mathcal{C}$ is the kernel of its cokernel.

(AB3) Every epi morphism in $\mathcal{C}$ is the cokernel of its kernel.

Remark 1.8.2.

- Mod_R is the prototype for an abelian category. We showed that (AB1) holds immediately after defining kernels and cokernels. We proved (AB2) in Lemma 1.7.7 and (AB3) in Lemma 1.7.8.
- Taking $R = \mathbb{Z}$, we see that Ab, the category of abelian groups, is also an abelian category.
- Since Grp is not pre-additive by Lemma 1.5.2, it cannot be additive and therefore cannot be abelian.

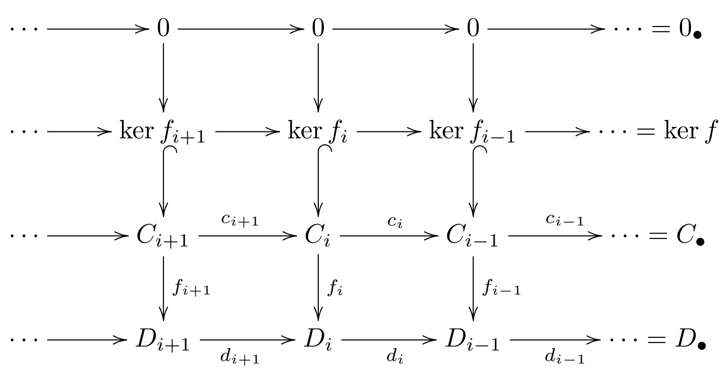
Comp_R Has Kernels and Cokernels

Lemma 1.8.3. *Suppose that* $\mathcal{C} = \mathrm{Comp}_R$ *and* $C_\bullet \xrightarrow{f} D_\bullet$ *is a chain map in* $\mathcal{C}$. *Then*

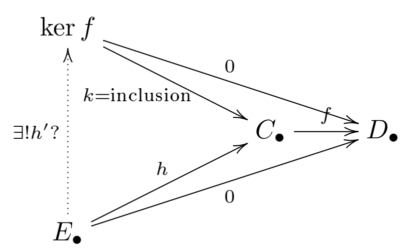
(1) The complex $\ker f$ *is a kernel of* f.

(2) The complex $\operatorname{coker} f$ *is a cokernel of* f.

Proof. (1): Consider the diagram:



and the diagram:



Define $h' \colon E_\bullet \to \ker f$ by

$$\begin{aligned} h'_i \colon E_i &\to \ker f_i \\ x_i &\mapsto h_i(x_i) \end{aligned}$$

This is a well-defined map into $\ker f_i$, since $f_i(h_i(x_i)) = f_i \circ h_i(x_i) = 0$, so $h_i(x_i) \in \ker f_i$. Then h'_i is an R-map since h_i is.

I claim that h' is a chain map. Consider the diagram:

$$\begin{array}{ccc} E_i & \xrightarrow{e_i} & E_{i-1} \\ {\scriptstyle h'_i}\downarrow & & \downarrow{\scriptstyle h'_{i-1}} \\ \ker f_i & \xrightarrow[\delta_i]{} & \ker f_{i-1} \end{array}$$

Let $x_i \in E_i$ be arbitrary. Recall that since h is a chain map, the following square commutes:

$$\begin{array}{ccc} E_i & \xrightarrow{e_i} & E_{i-1} \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle h_{i-1}} \\ C_i & \xrightarrow[c_i]{} & C_{i-1} \end{array}$$

Then we have

$$h'_{i-1} \circ e_i(x_i) = h'_{i-1}(e_i(x_i)) = c_i \circ h_i(x_i) = \delta_i \circ h_i(x_i) = \delta_i \circ h'_i(x_i)$$

So h' is a chain map as claimed. It is clear from the definition that h' makes the diagram commute.

Proof that h' is unique: Suppose that we have an h'' satisfying the above properties. Let $x_i \in E_i$ be arbitrary. Then

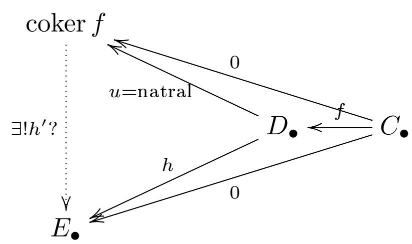
$$\begin{aligned} h''_i(x_i) &= k_i \circ h''_i(x_i) \\ &= h_i(x_i) \\ &= k_i \circ h'_i(x_i) \\ &= h'_i(x_i) \end{aligned}$$

Therefore $h'' = h'$.

(2): Consider the diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{i+1} & \xrightarrow{c_{i+1}} & C_i & \xrightarrow{c_i} & C_{i-1} & \xrightarrow{c_{i-1}} & \cdots = C_\bullet \\ & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\ \cdots & \longrightarrow & D_{i+1} & \xrightarrow[d_{i+1}]{} & D_i & \xrightarrow[d_i]{} & D_{i-1} & \xrightarrow[d_{i-1}]{} & \cdots = D_\bullet \\ & & \downarrow{\scriptstyle \text{natural}} & & \downarrow{\scriptstyle \text{natural}} & & \downarrow{\scriptstyle \text{natural}} & & \\ \cdots & \longrightarrow & \operatorname{coker} f_{i+1} & \longrightarrow & \operatorname{coker} f_i & \longrightarrow & \operatorname{coker} f_{i-1} & \longrightarrow & \cdots = \operatorname{coker} f \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots = 0_\bullet \end{array}$$

and the diagram:



Define $h'\colon \operatorname{coker} f \to E_\bullet$ by

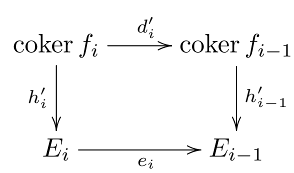
$$\begin{aligned} h'_i\colon \operatorname{coker} f_i &\to E_i \\ x_i + \operatorname{im} f_i &\mapsto h_i(x_i) \end{aligned}$$

This is a well-defined map. Let $d_i + \operatorname{im} f_i = d'_i + \operatorname{im} f_i$, so that $d_i - d'_i \in \operatorname{im} f_i$. Write $d_i - d'_i = f_i c_i$ for some $c_i / in C_i$. Applying h_i to both sides gives

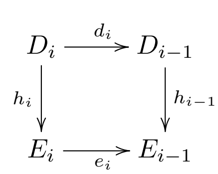
$$\begin{aligned} d_i - d'_i &= f(c_i) \\ h_i(d_i - d'_i) &= h(f_i(c_i)) \\ &= h_i - f_i(c_i) \\ h_i(d_i) - h(d'_i) &= 0 \\ h_i(d_i) &= h_i(d'_i) \\ \Rightarrow h'_i(d_i + \operatorname{im} f_i) &= h'_i(d'_i + \operatorname{im} f_i) \end{aligned}$$

So the map is well-defined as claimed. Then h'_i is an R-map since h_i is.

I claim that h' is a chain map. Consider the diagram:



Let $x_i \in D_i$ be arbitrary, so that $x_i + \operatorname{im} f_i \in \operatorname{coker} f_i$ is arbitrary. Recall that since h is a chain map, the following square commutes:



Then we have

$$e_i \circ h'_i(x_i + \operatorname{im} f_i) = e_i \circ h_i(x_i) = h_{i-1} \circ d_i(x_i) = h'_{i-1} \circ d_i(x_i + \operatorname{im} f_i)$$

So h' is a chain map as claimed. It is clear from the definition that h' makes the diagram commute.

Proof that h' is unique: Suppose that we have an h'' satisfying the above properties. Let $x_i \in D_i$ be arbitrary, so that $x_i + \operatorname{im} f_i \in \operatorname{coker} f_i$ is arbitrary. Then

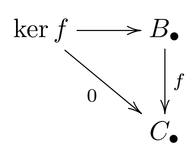
$$\begin{aligned} h_i''(x_i + \operatorname{im} f_i) &= h_i'' \circ u(x_i) \\ &= h_i(x_i) \\ &= h_i' \circ u_i(x_i) \\ &= h_i'(x_i + \operatorname{im} f_i) \end{aligned}$$

Therefore $h'' = h'$. □

Comp_R is Abelian

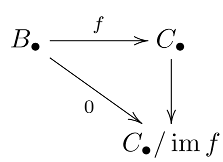
Theorem 1.8.4. Comp_R *is an abelian category.*

Proof. (AB1): This was Lemma 1.8.3 above.
(AB2): If $B_\bullet \xrightarrow{f} C_\bullet$ is a chain map, I claim that f is monic $\Leftrightarrow$ each $B_i \xrightarrow{f_i} C_i$ is monic.
$(Rightarrow)$ We have the composition



so if f is monic, then $\ker f \to B_\bullet$ is the zero map. This works for all i.
$(\Leftarrow)$ Trivial.

Thus if f is monic, then it is isomorphic to the kernel of $B_\bullet \to B_\bullet / C_\bullet$.
(AB3): Similarly, I claim that f is epi $\Leftrightarrow$ each $B_i \xrightarrow{f_i} C_i$ is an epi. $(\Rightarrow)$ We have the composition



if f is epi, then $C_\bullet \to C_\bullet / \operatorname{im} f$ is the zero map. This works for all i.
$(\Leftarrow)$ Trivial.

That is, f is isomorphic to the cokernel of $\ker f \to B_\bullet$. □