Reading Notes on The Theory of Sheaves by Richard G. Swan

This is my study memo on "Theory of sheaves" by R. G. Swan. I tried to follow the original text in general, but, where felt appropriate, I modified the original terminology and/or wording to more modern ones. Since there are still few good textbooks on sheaf theory, this study memo may have some value (so I hope).

For the record, the original authour says:

Acknowledgement

These notes are based on a course of lectures given in Dr. I. M. James' seminar at Oxford in 1958. The original notes have been revised and rearranged by R. Brown and R. G. Swan; the latter has also added much material not in the lectures. We are grateful to the Mathematical Institute of the University of Oxford for permission to reproduce the notes here.

Much of the material was adapted from Seminaire Henri Cartan 1950-51 and A. Grothendieck - "Sur quelques points d'algèbre homologique" Tohoku Math. Journal, August, 1957. For help in editing the manuscript for publication, we should like to thank C. Procesi.

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Chapter 0

Introduction

This introduction is intended primarily for those readers who have no previous knowledge of the theory of sheaves. Such readers probably have already asked themselves at least two questions:

(1) What are sheaves?

(2) What are they good for?

I will try to answer these questions here, beginning with question (2). The obvious answer to this question is that sheaves are very useful in proving theorems, for example:

In topology

1. The theorem that singular and Čech cohomology agree for paracompact HLC spaces.

2. The duality theorems of Poincaré, Alexander, and Lefschetz, and hence their consequences such as the Jordan-Brouwer theorem.

3. The existence of a spectral sequence associated with a map and consequences of this, such as the Vietoris mapping theorem and the spectral sequence of a fibre space.

In differential and algebraic geometry

1. The theorems of de Rham and Dolbeaut concerning cohomology groups defined by means of differential forms.

2. The duality theorem of Serre and its consequences (see [7]).

3. The Riemann-Roch theorem for varieties of all dimensions (see [6]).

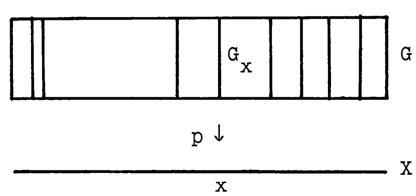
and many others.

Of course, many of these theorems can be proved without the use of sheaves. However, the proofs given by the theory of sheave are generally much simpler and clearer than the classical proofs and hold under much more general conditions, especially the proofs of the Poincaré, Alexander, and Lefschetz duality theorems.

At first sight, the theorems listed above may not appear to have much in common. However, a closer examination of the theorems and their proofs reveals some basic similarities. We start by assuming (or proving) that a space has certain local properties. We then express these properties of the space in terms of the properties of certain sheaves over the space. Finally, we apply the theory of sheaves to deduce global properties of the space. Thus, as far as applications to topology are concerned, the importance of the theory of sheaves is simply that it gives relations (quite strong relations, in fact) between the local and global properties of a space. Before explaining the nature of these relations, it is necessary to return to question (1): "What are sheaves?"

A sheaf is essentially a system of local coefficients over a space X. In the usual cohomology theories, we consider a space X and a group G and define cohomology groups $H\ n(X, G)$. In the theory of sheaves, we consider not a single group G, but a whole collection of groups G_x, one for each point $x \in X$. In the applications, of course, these groups will not be chosen arbitrarily but will be defined in some natural way in terms of properties of X. For convenience, we always regard the groups G_x as being disjoint. This can always be achieved by replacing each G_x by an isomorphic copy of G. The structure just defined is called a *protosheaf*[1] over X. The groups G_x are called the *stalks* of the protosheaf. We may picture a protosheaf as follows:

Let G be the union of all the G_x. Let $p\colon G \to X$ be defined by $p(g) = x$ if $g \in G_x$. Since the G_x are disjoint, this is well defined. Note that G is just a set with no topology so there is no question of continuity involved. This construction leads to the following picture, The vertical lines represent some of



the stalks of G. The analogy with sheaves and stalks in the agricultural sense is quite apparent.

It is possible to express quite a bit of information about the local structure of

[1]This term is due to Dr I. M. James.

X in terms of protosheaves. However, certain properties will be completely overlooked if we stick to using protosheaves with no additional structure. Namely, there is no relation between the different stalks of a protosheaf. This is best illustrated by means of examples.

The constant protosheaf

A very trivial example of a protosheaf is given by choosing some fixed group A and taking each G_x to be an isomorphic copy of A. In this protosheaf, any two stalks are canonically isomorphic. However, this is not apparent from the protosheaf structure itself. While it is obvious that two stalks are isomorphic, there will be many such isomorphisms in general and no reason to prefer one over another. The introduction of the canonical isomorphisms gives an additional element of structure.

The protosheaf of local homology groups

A more illuminating example is given as follows:

Let X be a topological n-manifold without boundary. Define a protosheaf G by $G_x = H_n(X, X \setminus x)$ (regarding these groups as disjoint). By excision, $G_X \cong \mathbb{Z}$ for all x. Therefore, this protosheaf is isomorphic to the constant protosheaf with stalk isomorphic to $\mathbb{Z}$. However, we can again introduce an additional element of structure in G. Suppose we choose some point $x \in X$ and a generator $z \in H_n(X, X \setminus x)$. Then z determines an orientation of X at x. Now, X may not be an orientable manifold, but in any case, every small enough neighbourhood of x is orientable. Therefore, the choice of an orientation at x gives a unique orientation at every point sufficiently near to x. This then gives us a canonical isomorphism, $G_x = G_y$, for all y sufficiently close to x.

The local system formed by the cohomology of fibres

Let $f\colon E \to X$ be a fibre bundle. Let F_x be the fibre over x for each $x \in X$. Define a protosheaf G over X by $G_x = H^n(F_x)$ for some fixed n. As usual, we regard these groups as disjoint. Since $f\colon E \to X$ is a fibre bundle, each $x \in X$ has a neighbourhood N_x such that $f^{-1}(N_x)$ is a product over N_x. This product structure then gives canonical isomorphisms $G_y \to G_x$ for all y near enough to x. (cf. Steenrod, *Topology of Fibre Bundles*, Part III, §30.2).

In all the examples so far considered, the additional structure imposed on G reduces to this: G has a topology which makes it a bundle of coefficients over X in the sense of Steenrod, *Fibre Bundles*, Part III, §30.1. Such a bundle of coefficients is a special case of a sheaf, called a *locally constant* (or *locally simple*, or *locally trivial*) sheaf. We now consider protosheaves with an additional element of structure which can*not* be expressed in this way.

The sheaf of germs of holomorphic functions

Let X be an open set in the complex plane, or, more generally a complex manifold of any dimension. Let G_x be the set of all power series convergent in a neighbourhood of x (the neighbourhood depending on power series). As usual, regard the G_x as being disjoint. Then G is a protosheaf over X. Now, if $g \in G_x$, then g is a power series convergent in a neighbourhood of x and therefore defines an analytic function in a neighbourhood N_x of x. At each point y of this neighbourhood N_x, we can expand this function in a power series, thus getting an element of G_y. Therefore, given an element $g \in G_x$, there are unique associated elements in all nearby stalks, that is, in all G such that $y \in N_x$.

Now, in this case, we cannot make G into a bundle of coefficients. The difficulty is that the neighbourhood N_x depends on the element g. There is no single N_x which will work for all $g \in G_x$. However, we still have enough structure to define a topology on G. In a bundle of coefficients, an element $h \in G_y$ is close to an element $g \in G_x$ if and only if y is close to x and h is the element corresponding to y under the canonical isomorphism $G_y \cong G_x$.

In the sheaf of germs of holomorphic functions, there is no canonical isomorphism $G_y \cong G_x$, but there is still a relation which tells us when $h \in G_y$ is the element associated with $g \in G_x$. Therefore, we can still define a topology on G by defining a neighbourhood of g to be the set of all elements associated with g and lying in stalks near g. In the particular sheaf under consideration here, this topology is simply that in which $g \in G_x$ and $h \in G_y$ are neighbouring elements if and only if x and y are close and g and h represent the same analytic function. The neighbourhoods just defined have an important property in common with the neighbourhoods of elements in a bundle of coefficients. Namely, every small enough neighbourhood of $g \in G_x$ meets every stalk G_y (with y near x) exactly once. Furthermore, if we have one such neighbourhood U of g, every smaller neighbourhood is obtained by taking only those elements of U which are in stalks G_y with y in some small neighbourhood of x. These properties are best expressed as properties of the projection $p\colon G \to X$. They are equivalent to the statement that every element $g \in G$ has a neighbourhood U which is mapped homeomorphically onto a neighbourhood of x. Such a map p is called a *local homeomorphism*.

We have now found almost all the properties used to define a sheaf. The final one we need is the continuity of addition. This is a standard axiom used in defining almost all structures which involve both algebra and topology. Since addition is defined only in the stalks, the property must be stated as follows:

If $g, h \in G_x, g', h' \in G_y, g$ is near g', and h is near h', then $g + h$ is near $g' + h'$.

We can now define a sheaf of abelian groups. It consists of

(1) a protosheaf of abelian groups $p\colon G \to X$, and

(2) a topology on G

such that

(a) p is a local homeomorphism, and

(b) addition in G is continuous.

There is no reason to consider only sheaves of abelian groups. We may also consider sheaves of modules over a ring K. The definition is almost the same except that the stalks are assumed to be K-modules rather than abelian groups and multiplication by elements of K is assumed continuous. In other words, if $k \in K, g \in G_x, g' \in G_y$, and g is near g', then kg is near kg'.

Similarly, we may define sheaves of any sort of algebraic structures, e. g., rings, non-abelian groups, etc. We simply assume that all G have the given structure, that p is a local homeomorphism, and that all algebraic operations are continuous.

It is now possible to outline the method by which we obtain relations between the local and global properties of a space. We consider a sheaf F of chain or cochain complexes associated with X. We then take the group of sections $\Gamma(F)$ of F, that is, the group of continuous functions $s\colon X \to F$ such that $ps =$ identity. This $\Gamma(F)$ will be, essentially, one of the ordinary chain or cochain complexes of X. Its homology or cohomology $H(\Gamma(F))$ will give certain homology or cohomology groups of X. Now, instead of applying Γ and then H, we can take homology (or cohomology) immediately. This gives a sheaf $H(F)$ of local homology (or cohomology) groups of X. The problem is to find relations between $H(F)$ and $H(\Gamma(F))$. This is a standard problem in homological algebra. It is solved by showing the existence of certain spectral sequences involving $F, H(F)$, and the derived functors of Γ. These derived functors of Γ play a central role in the theory of sheaves. If G is a sheaf over X, we define the i^{th} cohomology groups $H^i(X, G)$ of X with coefficients in G to be the result of applying the i^{th} right derived functor of Γ to G. It can be shown that, if X is paracompact and G is constant, these groups agree with the usual Čech cohomology groups of X.

There are two spectral sequences involving F. One relates $H(X, H(F))$ with a "hypercohomology group" of F. The other relates $H(H(, F))$, and in particular $H(\Gamma(F))$, with this same hypercohomology group. In order to get useful relations, it is necessary to compute the hypercohomology group. This can be done by means of the second spectral sequence provided certain cohomology groups are trivial. The importance of this is such that a good deal of the general theory of sheaves will be concerned with finding conditions under which sheaves have trivial cohomology.

In conclusion, I will outline the main parts of the general theory of sheaves. We first define sheaves, maps, sections, and various functors which can be applied to sheaves and then give a general method for constructing sheaves. We also show that sheaves form an exact category in the sense of Buchsbaum [5]. We then show that every sheaf is contained in an injective sheaf. As a consequence of this, we can apply the methods of homological algebra (injective resolution, derived functors, etc.) to the study of sheaves. As indicated above, we define the cohomology groups of X in terms of derived functors of Γ. It now becomes necessary to find general conditions under which these cohomology groups are trivial. This is the only part of the general theory which makes extensive use of geometrical arguments (covering theorems, paracompactness, etc.). Finally, we define the hypercohomology sequences and give applications to some of the theorems listed at the beginning of this introduction.

For other applications we refer the reader to the papers of S. S. Chern [1] and O. Zariski [2] and the bibliographies there.

Chapter 1

Algebraic Preliminaries

1.1 Categories and Functors

A *category* $\mathcal{A}$ is given by the following:

1. A collection of *objects* A.

2. A set $M(A,B)$ for any two objects $A, B \in \mathcal{A}$. The elements $g \in M(A,B)$ will be called *maps*. We shall frequently write $g\colon A \to B$ for $g \in M(A,B)$.

3. A function $M(B,C) \times M(A,B) \to M(A,C)$ for each triple of objects $A, B, C \in \mathcal{A}$. The image of $\psi \times \varphi$ in $M(A,C)$ will be denoted by $\psi\varphi$, the composition of ψ and φ.

These terms must satisfy the following axioms:

I. If $\alpha\colon A \to B, \beta\colon B \to C$ and $\gamma\colon C \to D$, then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$.

II. For each $A \in \mathcal{A}, \exists i_A\colon A \to A$ such that if $\beta\colon B \to A, \gamma\colon A \to C$ and $i_A\beta = \beta$ then $\gamma i_A = \gamma$.

It is easy to verify that i_A is unique.

A *covariant functor* $T\colon \mathcal{A} \to \mathcal{B}$ is a set theoretic function and also a collection of functions $T\colon M(A,B) \to M(T(A), T(B))$ satisfying

(1) $T(\alpha\beta) = T(\alpha)T(\beta)$, and

(2) $T(i_A) = I_T(A)$.

For two covariant functors $S, T\colon \mathcal{A} \to \mathcal{B}$ a *natural transformation* $f\colon S \to T$ shall be a function f, defined on $\mathcal{A}$, such that $f(A) \in M(S(A), T(A))$ and such that the following diagramme is commutative for any $A, B \in \mathcal{A}$ and map

$g\colon A \to B$:

$$\begin{CD} S(A) @>f(A)>> T(A) \\ @VS(g)VV @VVT(g)V \\ S(B) @>>f(B)> T(B) \end{CD}$$

For a *contravariant functor*, the only differences are that if $g\colon A \to B$, then $T(g) \in M(T(B), T(A))$ and $T(\alpha\beta) = T(beta)T(\alpha)$; and a natural transformation $f\colon S \to T$ of two contravariant functors will have $f(A) \in M(S(A), T(A))$ and the following diagramme commutative:

$$\begin{CD} S(A) @>f(A)>> T(A) \\ @AS(g)AA @AAT(g)A \\ S(B) @>>f(B)> T(B) \end{CD}$$

A natural transformation is an equivalence if it has an inverse.

A category $\mathcal{A}$ shall be called a *K-category* where K is a commutative ring with unit if it has a distinguished object 0, the zero object, and if it satisfies the following additional axioms:

I. $M(A, B)$ is a K-module (throughout these notes we shall assume that a K-module M is unitary: i.e., the unit of K acts as identity in M).

II. the function $M(B, C) \times M(A, B) \to M(A, C)$ is a bilinear map of K-modules.

III. $M(0, 0)$ is the zero module, also written 0.

In this case it is easy to verify that $M(0, A) = 0 = M(A, 0)$ for any $A \in \mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are K-categories, we generally require that a functor $T\colon \mathcal{A} \to \mathcal{B}$ should give a K-homomorphism $T\colon M(A, B) \to M(T(A), T(B))$ (or $M(T(B), T(A))$ if T is contravariant). Such a functor will be called a *K-functor* or *linear functor*.

In a K-category $\mathcal{A}$ we could forget about the operations of K; or K may be $\mathbb{Z}$, the ring of integers, and each module an abelian group. We shall then call $\mathcal{A}$ simply an *additive category* and the K-functors additive functors.

1.2 Universal Maps

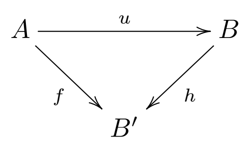
Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Let us be given a certain class of maps from objects in $\mathcal{A}$ to objects in $\mathcal{B}$ called $\mathcal{AB}$ maps, which satisfy

(1) if $f\colon A' \to A$ is in $\mathcal{A}$, $g\colon B \to B'$ is in $\mathcal{B}$, and $h\colon A \to B$ is an a $\mathcal{AB}$ map, then $hf\colon A' \to B, gh\colon A \to B'$ are defined and are $\mathcal{AB}$ maps;

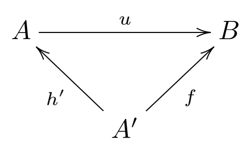
(2) if $f_1\colon A'' \to A', f_2\colon A' \to A$ are in $\mathcal{A}$, $g_1\colon B \to B', g_2\colon B' \to B''$ are in $\mathcal{B}$, and $h\colon A \to B$ is an $\mathcal{AB}$ map, then $h(f_2 f_1) = (hf_2)f_1, (g_2 g_1)h = g_2(g_1 h)$, and $g_1(hf_2) = (g_1 h)f_2$;

(3) if f, g are identities, then $hf = h, gh = h$.

Definition 1.2.1. An $\mathcal{AB}$ map $u\colon A \to B$ is *right-universal* if for any $f\colon A \to B'$ there exists a unique $h\colon B \to B'$ such that $hu = f$.

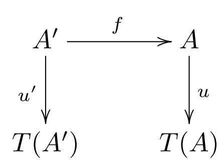


An $\mathcal{AB}$ map $u\colon A \to B$ is *left-universal* if for any $f\colon A' \to B$ there exists a unique $h'\colon A' \to A$ such that $uh' = f$.



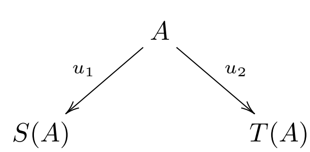
Suppose all objects of $\mathcal{A}$ have at least one associated right-universal map. Let T be a map from objects of $\mathcal{A}$ to objects of $\mathcal{B}$ constructed by choosing a universal map $u\colon A \to B$ for each $A \in \mathcal{A}$ and letting $T(A) = B$.

If $f\colon A' \to A$, we have the following diagramme:



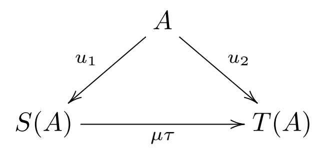
Since u' is right-universal, the map uf can be uniquely factored through $T(A')$ to give a map $T(f)\colon T(A') \to T(A)$. It is easily checked that, by the uniqueness property, T is a covariant functor.

Let the functor $S\colon \mathcal{A} \to \mathcal{B}$ be obtained by a similar choice of right-universal maps. Then the diagramme



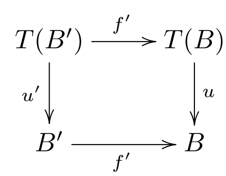
can be completed with maps $\tau(A)\colon S(A) \to T(A), \mu(A)\colon T(A) \to S(A)$ by the universal property of u_1, u_2 with $\tau u_1 = u_2, \mu u_2 = u_1$.

The diagramme



is commutative since $\mu\tau u_1 = \mu u_2 = u_1$ and hence $\mu\tau = \text{id}$. Similarly, $\tau\mu = \text{id}$; i.e., S and T are naturally equivalent.

Suppose each object $B \in \mathcal{B}$ has an associated left-universal map $u\colon A \to B$ for some $A \in \mathcal{A}$. Choose a particular such A for each $B \in \mathcal{B}$ and define $T(B) = A$. Then the following diagramme can, by the left-universal property, be uniquely completed and we define $T(f) = f'\colon T(B') \to T(B)$. We get a covariant functor $T\colon \mathcal{B} \to \mathcal{A}$, and by an argument similar to the earlier one, we can show that any two functors derived in this way are naturally equivalent.

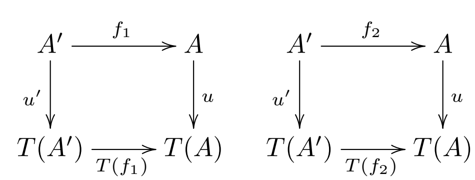


Remark

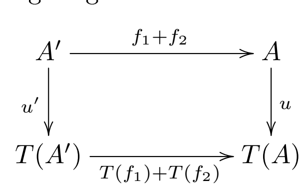
Suppose $\mathcal{A}$ and $\mathcal{B}$ are additive categories and that the given class of $\mathcal{AB}$ maps satisfies the following additional hypotheses:

1. if A is in $\mathcal{A}$, B in $\mathcal{B}$, the $\mathcal{AB}$ maps $h\colon A \to B$ form an abelian group.
2. if $f_1, f_2\colon A' \to A$ are in $\mathcal{A}$, $g_1, g_2\colon B \to B'$ are in $\mathcal{B}$, and $h\colon A \to B$ is an $\mathcal{AB}$ map, then $h(f_1 + f_2) = hf_1 + hf_2$ and $(g_1 + g_2)h = g_1h + g_2h$.

In this case the functors earlier defined by right (or left) universal maps are easily seen to be additive functors because, with the above notations, if the following diagrammes commute



then so does the following diagramme.

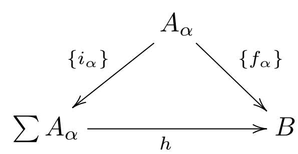


Therefore, by the uniqueness property, $T(f_1 + f_2) = T(f_1) + T(f_2)$.

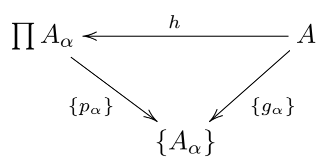
Examples

1. Direct sums Let $\mathcal{A}$ be the category of sets of groups $\{A_\alpha\}_{\alpha\in I}$ for some fixed indexing set I, in which the maps are sets of homomorphisms $g_\alpha\colon A_\alpha \to A'_\alpha$. Let $\mathcal{B}$ be the category of groups. Let an $\mathcal{AB}$ map be a set of maps $f_\alpha\colon A_\alpha \to B, B \in \mathcal{B}$.

Then the map $\{i_\alpha\}\colon \{A_\alpha\} \to \sum A_\alpha$, where the i_α are the injections into the direct sum, is a right-universal $\mathcal{AB}$ map, the map h being defined by $h = \sum f_\alpha \circ p_\alpha$ where p_α is the projection $\sum A_\alpha \to A_\alpha$



2. Direct products Let the categories $\mathcal{A}$ and $\mathcal{B}$ be as in (1). A $\mathcal{BA}$ map is now a set of homomorphisms $g_\alpha\colon B \to A_\alpha$. The set of projections of the direct product $p_\alpha\colon \prod A_\alpha \to A_\alpha$ is a left-universal $\mathcal{BA}$ map. By the definition of the direct product, there is for any $x_{alpha} \in A_\alpha$ a unique $x \in \prod A_\alpha$ such that $p_\alpha x = x_\alpha$.



So the $\{g_\alpha\}$ can be "lifted" to a map $h\colon A \to \prod A_\alpha$, which is easily seen to be a unique homomorphism.

3. Tensor products Let $\mathcal{B}$ be the category of abelian groups and homomorphisms. Let $\mathcal{A}$ be the category whose objects are cartesian products $A \times B$ where A is a right λ-module, B a left Λ-module, and whose maps are pairs (φ, ψ) of λ-homomorphisms where $\varphi\colon A \to A', \psi\colon B \to B'$. An $\mathcal{AB}$ map shall be a bilinear map $f\colon A \times B \to C$; i.e.,

$$\begin{aligned} f(a_1 + a_2, b) &= f(a_1, b) + f(a_2, b), \\ f(a, b_1 + b_2) &= f(a, b_1) + f(a, b_2), \quad \text{and} \\ f(a\lambda, b) &= f(a, \lambda b). \end{aligned}$$

Theorem 1.2.2. *Every object in $\mathcal{A}$ has an associated right-universal map.*

I do not give the proof. The corresponding group we write $A \otimes_\Lambda B$, the *tensor product* of A and B; it can be taken to be the factor group P/R where P is the free abelian group generated by pairs (a, b) $a \in A, b \in B$ and R is generated to relations

$$\begin{aligned}&(a,b)+(a,b')-(a,b+b'),\\&(a,b)+(a',b)-(a+a',b),\\&(a\tau,b)-(a,\tau b),\end{aligned}$$

where $a, a' \in A, b, b' \in B$, and $\tau \in \Lambda$.

The universal map $u\colon A \times B \to A \otimes_\Lambda B$ takes (a, b) into the coset of (a, b) mod R.

4. The group $\mathrm{Hom}_\Lambda(A, B)$ Let $\mathcal{A}$ be the category of abelian groups; let $\mathcal{M}$ be the category of left Λ-modules and Λ-homomorphisms. Let $\mathcal{M}^*$ be the dual category of $\mathcal{M}$; i.e., $\mathcal{M}^*$ has the same objects as $\mathcal{M}$ and its maps are the maps of $\mathcal{M}$, only written in the opposite direction. The product category $\mathcal{B} = \mathcal{M}^* \times \mathcal{M}$ has objects pairs of objects from $\mathcal{M}^*$ and $\mathcal{M}$ maps pairs $(\varphi, \psi)\colon (A, B) \to (A', B')$, where $\varphi\colon A \to A', \psi\colon B \to B'$ are maps of $\mathcal{M}^*$ and $\mathcal{M}$ respectively. φ is in fact a Λ-homomorphism $A' \to A$. By an $\mathcal{AB}$ map $C \to (A, B)$ we mean a bilinear map $C \times A \to B$.

Theorem 1.2.3. *Each object of $\mathcal{B}$ has an associated left-universal map.*

The corresponding group we write as $\mathrm{Hom}_\Lambda(A, B)$. As a representative we can take the group, say $\chi(A, B)$, of Λ-homomorphisms $A \to B$, with the map $u\colon \chi(A, B) \times A \to B$ defined by $u(g, a) = g(a)$.

5. Direct limits

Definition 1.2.4. An *ordered set* is a double $(I, <)$ where $<$ is a transitive and reflexive relation on the set I. A map of ordered sets $f\colon (I, <) \to (J, <)$, often written $f\colon I \to J$, is a function $f\colon I \to J$ such that $\alpha < p\beta \Rightarrow f(\alpha) < f(\beta)$.

Any subset $J \subset I$ has a natural ordering induced by that of I, making $J \to I$, by injection, a map of ordered sets.

The *product* $(I, <) \times (J, <)$, written $I \times J$, is $(I \times J, <<)$ where $(\alpha, \beta) << (\alpha', \beta')$ if and only if $a < a'$ and $\beta < \beta'$.

An ordered set $(I, <)$ is *directed* if for any $\alpha, \beta \in I, \exists \gamma \in I$ such that $\gamma < a$ and $\gamma < \beta$. As an example of a directed set, we may take the neighbourhoods of a point x of a topological space, ordered by inclusion.

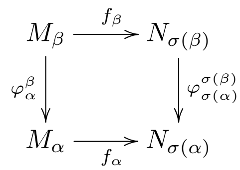
Definition 1.2.5. A *direct system of K-modules* is a directed set I; for each $\alpha \in I$, a K-module M_α; and for each two elements $\alpha, \beta \in I$ with $\alpha < \beta$ a K-homomorphism $\varphi_\alpha^\beta\colon M_\beta \to M_\alpha$ satisfying $\varphi_\alpha^\alpha = \mathrm{id}$, $\varphi_\alpha^\beta \varphi_\beta^\gamma = \varphi_\alpha^\gamma$ if $\alpha < \beta < \gamma$.

A *map of direct systems* $(M_\alpha, I) \to (N_\beta, J)$ is

(i) a map $\sigma\colon I \to J$ of order sets and

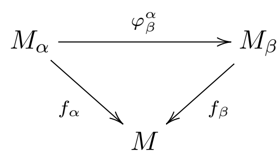
(ii) for each $\alpha \in I$, a K-homomorphism $f_\alpha\colon M_\alpha \to N_{\sigma(\alpha)}$

such that for each two elements $\alpha, \beta \in I$ with $\alpha < \beta$, the following diagramme commutes.



Note that if $J \subset I, M'_\alpha \in M_\alpha$ for each $\alpha \in I$, the natural inclusion $(M'_\alpha, J) \subset (M_\alpha, I)$ is a map of direct systems, if and only if when $\beta < \alpha$, $\varphi'^\beta_\alpha \colon M'_\beta \to M'_\alpha$ is the restriction to M'_β of $\varphi^\beta_\alpha \colon M_\beta \to M_\alpha$.

Let $\mathcal{A}$ be the category of direct systems of K-modules and $\mathcal{B}$ the category of K-modules. By an $\mathcal{AB}$ map $(M_\alpha, I) \to M$ we mean for each $\alpha \in I$, a K-homomorphism $f_\alpha \colon M_\alpha \to M$ such that if $\beta < \alpha$, $f_\alpha = f_\beta \varphi^\alpha_\beta$



Theorem 1.2.6. *Each object in $\mathcal{A}$ has an associated right-universal map.*

The associated functor is known as the *direct limit* of the direct system, and is written $\varinjlim_I M_\alpha$.

Proof. Without loss of generality we may suppose the M_α disjoint. Suppose $x_\alpha \in M_\alpha, y_\beta \in M_\beta$. We define an equivalence relation $\sim$ in $\cup M_{]\alpha}$ by $x_\alpha \sim y_\beta \Leftrightarrow \exists \gamma \in I$ such that $\gamma < \alpha, \gamma < \beta$ and $\varphi^\alpha_\gamma(x_\alpha) = \varphi^\beta_\gamma(y_\beta)$.

Let M be the set of equivalence classes. Let $(x_\alpha), (y_\beta) \in M$. Then $\exists \gamma < \alpha, \beta$. Let $x_\gamma = \varphi^\alpha_\gamma x_\alpha, y_\gamma = \varphi^\beta_\gamma x_\beta$. Then $x_\gamma, y_\gamma \in M_\gamma$ and $(x_\gamma) = (x_\alpha), (y_\gamma) = (y_\beta)$.

Define $\tau(x_\alpha) + \mu(y_\beta) = (\tau x_\gamma + \mu y_\gamma)$, $\tau, \mu \in K$. In this way, we give M the structure of a K-module. Let $u_\alpha \colon M_\alpha \to M$ be the natural map. Then $u = \{u_\alpha \colon (M_\alpha, I) \to M$ is an $\mathcal{AB}$ map. I assert

(1) u is epi. For if $x \in M$, then $\exists x_\alpha$ such that $u_\alpha(x_\alpha) = x$

(2) u is "as mono as possible;" i.e., if $u_\alpha(x_\alpha) = 0$, then $\exists \beta$, $\beta < \alpha$, such that $\varphi^\alpha_\beta(x_\alpha) = 0$.

Suppose now we have an $\mathcal{AB}$ map $(M_\alpha, I) \xrightarrow{f} N$. Then by (2) $\ker u \subset \ker f$, and so, since u is epi, we can uniquely factor f through u. We could define and prove existence in almost exactly the same way of direct systems of sets, i.e., assuming no algebraic structures in M_α. $\square$

Definition 1.2.7. Let $(I, <)$ be a directed set. $J \subset I$ is *cofinal* if for any $\alpha \in I$, $\exists \beta \in J$ such that $\beta < \alpha$.

Theorem 1.2.8. *Let* (M_α, I) *be a direct system,* J *be cofinal in* I*. Then the inclusion map* $(M_\alpha, J) \to (M_\alpha, I)$ *induce an isomorphism* $M_J = \varinjlim_J M_\alpha \cong \varinjlim_I M_\alpha = M_I$*.*

Proof. Let $i\colon M_J \to M_I$ be the induced map. Let $(x_\alpha) \in M_I$. Then $\exists \beta$ with $\beta < \alpha$; let $x_\beta = \varphi^\alpha_\beta x_\alpha$, so that $(x_\alpha) = (x_\beta)$. Define $j\colon M_I \to M_J$ by $j(x_\alpha) = (x_\beta) \in M_J$. Then j is well-defined, $ji = \mathrm{id}_{M_J}$, and $ij = \mathrm{id}_{M_I}$. □

Definition 1.2.9. A sequence $\to (M'_\alpha, I) \to (M_\alpha, I) \to (M''_\alpha, I) \to$ is *exact* at (M_α, I) if the two maps $I \to I$ shown are the identity and $M'_\alpha \to M_\alpha \to M''_\alpha$ is exact at M_α for all α.

Theorem 1.2.10. $\varinjlim$ *preserves exactness.*

Proof. Let $(M'_\alpha, I) \xrightarrow{i} (M_\alpha, I) \xrightarrow{j} (M''_\alpha, I)$ be exact. The theorem asserts that $M' \xrightarrow{\bar{i}} M \xrightarrow{\bar{j}} M''$ induced by $\varinjlim_I$ is exact.
($\ker \bar{j} \subset \operatorname{im} \bar{i}$): Let $(x_\alpha) \in \ker \bar{j}$. Then $\exists x_\beta \in (x_\alpha)$ such that $jx_\beta = 0$. Therefore $\exists y_\beta \in M'_\beta$ such that $iy_\beta = x_\beta$. Then $\bar{i}(y_\beta) = (x_\beta) = (x_\alpha)$.
($\operatorname{im} \bar{i} \subset \ker \bar{j}$): Let $(x_\alpha) \in \operatorname{im} \bar{i}$. Then $\exists x_\beta \in (x_\alpha)$ such that $x_\beta = iy_\beta$. Therefore $\bar{j}(x_\alpha) = \bar{j}(x_\beta) = (jiy_\beta) = 0$. □

Theorem 1.2.11. $\varinjlim$ *commutes* $\otimes$*; i.e., if* $u\colon (M_\alpha, I) \to M, v\colon (N_\alpha, J)$ *be the universal maps, then* $(M_\alpha \otimes N_\alpha, I \times J)$ *is also a direct system.*

Proof. The theorem asserts that $\varinjlim_{I\times J} M_\alpha \otimes N_\beta \cong M \otimes N$ and that the corresponding maps $M_\alpha \otimes N_\beta \to M \otimes N$ are $u_\alpha \otimes v_\beta$.

To prove this, consider the maps $(x_\alpha) \otimes (y_\beta) \leftrightarrows^i_j (x_\alpha \otimes y_\beta)$, show that they are well defined, and prove $ij = \mathrm{id}, ji = \mathrm{id}$. Details are left to the reader as an exercise. □

For later use, we mention the following: let M be a K-module and I the class of finite subsets of M, ordered by inclusion. For $\alpha \in I$, M_α shall be the submodule of M generated by elements of α. The inclusions $i_\alpha\colon M_\alpha \to M$ give a natural isomorphism $\varinjlim M_\alpha \cong M$.

Finally, to end this chapter, let us state a few theorems about $\otimes$.

Theorem 1.2.12. $\otimes$ *is right exact; i.e., if* $A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ *is exact, then so, for any* B*, is*

$$A' \otimes B \xrightarrow{i\otimes 1} A \otimes B \xrightarrow{j\otimes 1} A'' \otimes B \to 0$$

Proof. ($j \otimes 1$ is epic.): Let $a'' \otimes b \in A'' \otimes B$. Since j is onto, we have $a'' = ja$ for some $a \in A$. Therefore, $j \otimes 1(a \otimes b) = a'' \otimes b$, and $j \otimes 1$ is onto.
(Exactness at $A \otimes B$): Now $(j \otimes 1)(i \otimes 1) = (ji \otimes 1) = 0$, so we have a unique homomorphism $u\colon \operatorname{coker}(i \otimes 1) \to A'' \otimes B$. We prove this is an isomorphism. Let a bilinear map $g\colon A'' \times B \to \operatorname{coker}(i \otimes 1)$ be defined by $g(a'', b)$ is the image of $a \otimes b$ in $\operatorname{coker}(i \otimes 1)$, where $ja = a''$; this is independent of the choice of a. Then $\exists$ a unique homomorphism $v\colon A'' \otimes B \to \operatorname{coker}(i \otimes 1)$ such that $v(a'' \otimes b) = g(a'', b)$. Obviously vu and uv are identities. □

Definition 1.2.13. We say that the exact sequence $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ *splits* if $\exists k\colon A'' \to A$ such that $jk = \mathrm{id}_{A''}$.

In this case $\operatorname{im} k$ is a direct summand of A, the other summand being $\operatorname{im} i$. Hence there is a map $h\colon A \to A'$ which is 0 on $\operatorname{im} k$ and with $hi = \mathrm{id}_{A'}$. Finally, $kj + ih = \mathrm{id}_A$, and the sequence

$$0 \to A' \leftrightarrows_i^h A \leftrightarrows_j^k A'' \to 0$$

is exact both ways and splits.

Corollary 1.2.14. *If the exact sequence* $0 \to A' \to A \to A'' \to 0$ *splits, then* $0 \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ *is exact and splits.*

Suppose now K is a principal ideal domain. This implies that any finitely generated K-module is the direct sum of a finite number of K-modules with one generator. (cf. Bourbaki, Algebra.)

Definition 1.2.15. A K-module M is *torsion-free* if for $\lambda \in K$, $x \in M, \lambda x = 0 \Rightarrow \lambda = 0$ or $x = 0$.

Obviously a submodule of a torsion-free module is torsion-free. Also if K is a principal ideal domain (as we assume from now on), then any finitely generated torsion-free K-module is free, i.e., is isomorphic to the direct sum of copies of K.

Theorem 1.2.16. *If* $0 \to A' \to A \to A'' \to 0$ *is exact and* B *is torsion-free, then* $0 \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ *is exact.*

Proof. B is the direct limit of finitely generated submodules B_α; each $B_\alpha \cong \sum_i^h K$. Therefore $M \otimes B_\alpha \cong \sum_i^h M$; i.e., the sequence $0 \to A' \otimes B_\alpha \to A \otimes B_\alpha \to A'' \otimes B_\alpha \to 0$ is the direct sum of copies of $0 \to A' \to A \to A'' \to 0$ and so is exact. Since $\otimes$ commutes with direct limits, the theorem is proved. □

Theorem 1.2.17. *If* $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ *is exact,* A'' *torsion free and* B *arbitrary, then* $0 \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ *is exact.*

Proof. Again write $A'' = \varinjlim A''_\alpha$ where A''_α is finitely generated. Then $0 \to A' \xrightarrow{i} j^{-1}(A''_\alpha) \to A''_\alpha \to 0$ is exact. A''_α is free, so the sequence splits. Hence its exactness is preserved under tensor products. Proceeding to direct limits, we obtain the theorem. □

Corollary 1.2.18. *If* $\cdots \to A_n \xrightarrow{I_n} A_{n-1} \xrightarrow{i_{n-1}} A_{n-2} \to \cdots$ *is exact and* A_i *torsion-free for all* i*, then*

$$\cdots \to A_n \otimes B \xrightarrow{I_n \otimes 1} A_{n-1} \otimes B \xrightarrow{i_{n-1} \otimes 1} A_{n-2} \otimes B \to \cdots$$

is exact for arbitrary B.

Chapter 2

Sheaves

2.1 Definitions and examples

A *sheaf* is a triple (S, X, p) where S, the "sheaf", and X, the "base space", are topological spaces, and p, the "projection", is a continuous, onto map $S \to X$. (We shall often say: "let S be a sheaf" instead of the full notation "let (S, X, p) be a sheaf"). This shall have the additional structure:

i) Topological p is a local homeomorphism; i.e., any $x \in S$ has a neighbourhood U such that $p \mid U$ maps U homeomorphically onto a neighbourhood of $p(x)$. This implies p is an open map.

For the rest of the structure, we need some definitions.

Definition 2.1.1. The *stalk* over $x \in X$ is the discrete set $S_x = p^{-1}(x)$.

Definition 2.1.2. Given two sheaves $(S, X, p), (T, X, p^1)$, let $S + T = \{(x, y) \in S \times T; p(x) = p^1(y)\}$. Define $p'' \colon S + T \to X$ by $p''(x, y) = p(x) = p^1(y)$. Then $(S + T)_x = S_x \times T_x$.

ii) Algebraic For the algebraic structure we suppose given once and for all a commutative ring K with unit. Then:

a. Assume each stalk S_x is given the structure of a K-module.

b. Each $\lambda \in K$ gives a map $\overline{\lambda} \colon S \to S$ by the obvious multiplication $\lambda \colon S_x \to S_x$. The map $S_x \times S_x \to S_x$ given by $(s_1, s_2) \mapsto s_1 + s_2$.

gives a map $S + S \to S$.

We require both these maps to be continuous.

Definition 2.1.3. A triple (S, X, p) in which S is a set, p is a surjective function $S \to X$, and each $S_x = p^{-1}(x)$ has the structure of a K-module is called a *protosheaf*. Given any sheaf S we can form a protosheaf, written $\overline{S}$, by forgetting about the topology on S.

Much of the discussion to follow, e.g., definition of maps of sheaves, restrictions, extensions, tensor products, etc. applies in an even simpler manner to protosheaves. All we do is omit all considerations of topology.

Note that $(S+T, X, p'')$ is a sheaf. $(\overline{S+T}, X, p'')$ is a protosheaf. $\lambda\colon S+T \to S+T$ defined by $\lambda(a,b) = (\lambda a, \lambda b)$ is continuous and so is the composition

$$(S+T)+(S+T) \to (S+S)+(T+T) \to S+T$$

which is the addition in $S+T$. p'' is obviously a local homeomorphism.

Examples

i. Constant sheaf Let M be a K-module. Give M the discrete topology, let $S = X \times M$ and let $p\colon S \to X$ be the projection $p(x,m) = x$. The stalk over $y \in X$ is simply $y \times M$. This sheaf is usually denoted by M. The constant sheaf of the zero module is the "zero sheaf", written 0.

ii. Bundle of coefficients

Definition 2.1.4. A *bundle of coefficients* is a bundle of groups where the fibre is an abelian group and the group of the bundle is totally disconnected. This is a sheaf of $\mathbb{Z}$-modules.

As an example let $X = S^1, G = \mathbb{Z}/3\mathbb{Z}$ and let the sheaf F be the union of a circle A and the double covering of S^1 by a circle B. Each element of A shall be the zero in its stalk.

Definition 2.1.5. A map $\varphi\colon (F, X, p) \to (G, X, p')$ of sheaves (usually written $\varphi\colon F \to G$) is a continuous function $\varphi\colon F \to G$ such that

(i) $p = p' \cdot \varphi$ and

(ii) $\varphi_x\colon F_x \to G_x$ is a K-homomorphism, each $x \in X$ where $\varphi_x =: \varphi \mid F_x$.

$$\begin{array}{ccc} F & \longrightarrow & G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

Remark 2.1.6. Any sheaf map $\varphi F \to G$ is an open map of the topological spaces F and G.

Definition 2.1.7. A sheaf F is *trivial* if it is isomorphic to the constant sheaf.

If $\varphi\colon F \to G, \psi\colon L \to M$ are sheaf maps, we have a sheaf map $\varphi+\psi\colon F+L \to G+M$. We can define addition between maps $\varphi, \psi\colon F \to G$; $\varphi+\psi\colon F \to G$ is given by $(\varphi+\psi)_x = \varphi_x + \psi_x$.

For any $k \in K$, we can also define $k\varphi\colon F \to G$ by $(k\varphi)_x = k\varphi_x$. The set of sheaf maps $F \to G$ is a K-module written $\mathrm{Hom}(F, G)$. $\mathrm{Hom}(-,-)$ is easily seen to be a left-exact functor from sheaves to modules. So we can define the K-category F of sheaves and maps of sheaves. We shall be interested in constructing a homology theory for this category F. The construction of this homology theory will follow the well-worn lines of homological algebra. Such a homology could be described axiomatically, and its uniqueness proved, on any exact category; i.e., an additive category in which, roughly speaking, we have

(i) kernels, images, cokernels, and coimages of maps and

(ii) direct sums of two objects[1].

Cartan's original proof of the uniqueness of cohomology theory of sheaves used essentially an additive category satisfying (ii). Our proof (due to Grothendieck) will be somewhat simpler by using direct sums; though stated for sheaves, it can be generalised immediately to any exact category. The construction (also due to Grothendieck) will be both more general and simpler than Cartan's, mainly by using the tools of homological algebra as developed by Cartan and Eilenberg.

Definition 2.1.8. Let $(G, X, p), (F, X, p')$ be sheaves. If $G \subset F$ as a set and this inclusion map is a sheaf map, we say (G, X, p) is a *subsheaf* of (F, X, p') (often written simply $G \subset F$).

Proposition 2.1.9. *Let G be a subset of F where (F, X, p) is a sheaf. Then there is at most one sheaf structure on G making G a subsheaf of F. Such a structure exists if*

(1) $p(G) = X$;

(2) G is open in F; and

(3) G_x is a submodule of F_x, for all $x \in X$.

Proof. We leave the proof to the reader. □

Example 2.1.10. The sheaf F_u, where U is an open set in X and (F, X, p) is a sheaf, is defined by

$$(F_U)_x = \begin{cases} F_x & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

[1]For more details see [5]

Quotient sheaf Let F' be a subsheaf of F. Form a protosheaf $\overline{F}/\overline{F'}$, with projection p'' by letting $(\overline{F}/\overline{F'})_x = F_x/F'_x$. Define a map $\varphi\colon F \to \overline{F}/\overline{F'}$, by letting $\varphi \mid F_x$ be the natural map $F_x \to F_x/F'_x$; let F/F' be the protosheaf $\overline{F}/\overline{F'}$, together with the identification topology given by φ. The diagramme

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & F/F' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p''} & \\ & X & \end{array}$$

is commutative.

Proposition 2.1.11. *$(F/F', X, p'')$ is a sheaf; and φ is a sheaf map.*

Proof. Let $y \in F/F', \varphi(z) = y$, and $p(z) = x$. Let U be a neighbourhood of z which is mapped homeomorphically by p onto a neighbourhood V of x. Since φ is an open map, $\varphi(U)$ is a neighbourhood of y. But $\varphi \mid |U$ is 1-1, for no two elements of U belong to the same stalk, and so $\varphi \mid U$ is a homeomorphism. Therefore p'' maps $\varphi(U)$ homeomorphically onto V, a neighbourhood of x.

Finally we prove that addition and multiplication by $\lambda \in K$ in F/F' are continuous. To do this we need only consider the following commutative diagrammes in which φ and $\varphi \times \varphi$ are sheaf epimorphisms, and so identification maps. (For any spaces Y, Z a map $Y \to Z$ which is a continuous, open onto map is an identification map.)

$$\begin{array}{ccc} F/F' + F/F' & \longrightarrow & F/F' \\ {\scriptstyle \varphi+\varphi}\uparrow & & \uparrow{\scriptstyle \varphi} \\ F + F & \longrightarrow & F \end{array} \qquad \begin{array}{ccc} F/F' & \xrightarrow{\lambda} & F/F' \\ {\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \varphi} \\ F & \xrightarrow[\lambda]{} & F \end{array}$$

□

Example 2.1.12. Let A be a closed set in X. Define $F_A = F/F_{X\setminus A}$. Clearly

$$(F_A)_x = \begin{cases} F_x & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For any sheaf map $\varphi\colon F \to F$ we can define sheaves which we call $\operatorname{im}\varphi, \ker\varphi, \operatorname{coim}\varphi$, and $\operatorname{coker}\varphi$. We define $(\ker\varphi)_x = \varphi^{-1}(O_x)$. The set of zero elements of G is open in G, therefore $\ker\varphi$ is open in F, and by proposition 2.1.9, $\ker\varphi$ is a subsheaf of F. And $\operatorname{coim}\varphi$ is by definition $F/\ker\varphi$.

φ is an open map. Therefore $\operatorname{im}\varphi$ is a subsheaf of G. $\operatorname{coker}\varphi$ is by definition, $G/\operatorname{im}\varphi$. Direct sums of sheaves exist; cf. 2.4.1 below. Clearly both the categories of sheaves and of protosheaves are exact categories. We shall say that a sequence $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ is *exact* (at G) if $\operatorname{im}\varphi = \ker\psi$. A sequence

$$\to G_n \xrightarrow{\varphi_n} G_{n+1} \xrightarrow{\varphi_{n+1}} G_{n+2} \to \cdots$$

of a finite or infinite number of sheaves is exact if it is exact at each G_n. In the case of a finite sequence,

$$G_0 \xrightarrow{\varphi_0} G_1 \to \cdots \xrightarrow{\varphi_n} G_n,$$

this implies nothing about $\ker \varphi_0$ or $\operatorname{im} \varphi_n$.

2.2 Sections and Stacks

2.2.1 Sections

Let (F, X, p) be a sheaf and U an open set in X. A *section* over U is a continuous map $s\colon U \to F$ such that $ps = \mathrm{id}\colon U \to U$. We define the zero section over U to be the function $0\colon U \to F$ given by $0(x) = 0_x \in F_x$. Recall $\Gamma(U, F)$ is the set of sections $U \to F$.

For any $y \in F$, there exists a section over some $V \subset X$ passing through y. Take V to be a homeomorphic image under p of some neighbourhood W of y and let $s = (p \mid W)^{-1}$.

Lemma 2.2.1. *Let $s, t \in \Gamma(U, F)$. Then the set $A = \{x \in X \mid s(x) = t(x)\}$ is open, or equivalently if two sections agree at x they agree in a neighbourhood of x.*

Proof. Let W be a neighbourhood of $s(x)$ which is mapped homeomorphically by p onto a neighbourhood of x. There exists a neighbourhood V of x such that $s(V), t(V) \subset W$. If $y \in V$, then $s(y) = t(y)$ because there is only one point of W lying over y. □

Remark 2.2.2. If F is Hausdorff and two sections over a connected, open set $U \subset X$ agree at a point of U, then they agree over all U. For given two maps into a Hausdorff space, the set of points on which they agree is closed. If the base is Hausdorff and locally connected, the converse is true. (We leave the proof to the reader.)

If s, t are two sections over U, define $s+t$ by $(s+t)(x) = s(x)+t(x)$, $(ks)(x) = ks(x)$. Thus the set $\Gamma(U, F)$ of sections of F over U is a K-module. For a fixed $U, \Gamma(U, -)$ is a covariant K-functor from K-sheaves to K-modules.

For if $\varphi\colon F \to G$ is a map and $s\colon U \to F$ is a section of F over U, then $\varphi\colon UG$ is a section of G over U, giving a map

$$\Gamma(U, \varphi)\colon \Gamma(U, F) \to \Gamma(U, G).$$

If open $V \subset U$, define $\rho^U_V\colon \Gamma(U, F) \to \Gamma(V, F)$ by restriction; i.e., if we have section $s\colon U \to F$, $\rho^U_V(s) = s \mid V\colon V \to F$. If $\varphi\colon F \to G$ and $V \subset U$, the following diagramme is commutative.

$$\begin{array}{ccc} \Gamma(U, F) & \longrightarrow & \Gamma(U, G) \\ \downarrow & & \downarrow \\ \Gamma(V, F) & \longrightarrow & \Gamma(V, G) \end{array}$$

$\Gamma(-,F)$ is a contravariant functor from open sets and inclusion maps to K-modules, for clearly $\rho_U^U = \text{id}$, and if $W \subset V \subset U$, then $\rho_W^U = \rho_W^V \rho_V^U$.

2.2.2 Stacks (older name, pre-sheaf)

More generally, a *stack* is a contravariant functor from open sets and inclusion maps (of some fixed space X) to the category of K-modules.

A *map* of stacks is a natural transformation of functors. These maps $\underline{S} \to \underline{T}$ form a K-module by $(\alpha f + \beta g)(U) = \alpha f(U) + \beta g(U)$ for $\alpha, \beta \in K$ and $f, g\colon \underline{S} \to \underline{T}$.

The stacks with these maps form a K-category, which is also exact, and the functor $\Gamma(-,x)$ which takes F to $\Gamma(-,F)$ is a K-functor from sheaves to stacks.

Functor from stacks to sheaves

Given a stack $\underline{S}$, for each $x \in X$, define $S_x = \varinjlim_{U \ni x} \underline{S}(U)$ and $\varphi_x^U : \underline{S}(U) \to S_x$ to be the natural map. (See chapter 1)

Define a protosheaf $\overline{S} = \cup_{x\in X} S_x$ with projection p. For each $s \in \underline{S}(U)$ define a section $\overline{s}$ of $\overline{S}$ by $\overline{s}(x) = \varphi_x^U(s)$. Form a sheaf S from $\overline{S}$ by taking as a base for neighbourhoods in $\overline{S}$ the set $\{\overline{s}(U);\ \text{all } U,\ \text{all } s \in \underline{S}(U)\}$. p is onto; we show it is continuous and a local homeomorphism.

Let U be open in X, $y \in p^{-1}(U)$, and $p(y) = x$. Then from the construction of the direct limit, there exists $V \subset U, s \in \underline{S}(V)$ such that $\varphi_x^V(s) = y$. Then $y \in \overline{s}(V)$, since $\overline{s}(x) = \varphi_x^V(s) = y$. Therefore $p^{-1}(U)$ is open. If $p' = p \mid \overline{s}(V)$, $p'\overline{s} = \text{id}$ and $\overline{s}p' = \text{id}$. Therefore p' is a homeomorphism since $\overline{s}$ is continuous.

We leave to the reader the proof that addition and multiplication by elements of K are continuous.

Note that we could define stacks and sheaves of sets; i.e., no algebraic structure is assumed on the stalks, and we can still derive as above a construction of a sheaf of sets from a stack of sets. Or we may repeat the construction for any algebraic structure which admits direct limits.

Given any map $\underline{\eta}\colon \underline{S} \to \underline{T}$ of stacks, we obtain, by passage to direct limits (which we know is a functor) maps $\eta_x\colon S_x \to T_x$ for each $x \in X$. These can be stuck together to give a map $\overline{\eta}\colon \overline{S} \to \overline{T}$ of protosheaves. This map is in fact consistent with the topology given to $\overline{S}$ and $\overline{T}$; let $\eta(x) = y$, and $\overline{t}(U)$ any basic neighbourhood of y. Let $V \subset U$ be a neighbourhood of $p(x)$ so small that x has a preimage s in $\underline{S}(V)$. Then $\underline{\eta}(V)s$ and $\rho_V^U t$ are sections of $\overline{T}$ passing through y. Therefore they agree on a neighbourhood W of $p(y)$. So $\rho_W^V s(W)$ is a neighbourhood of x which is mapped into $\overline{t}(U)$ by η.

Thus we have a map $\eta\colon S \to T$ of sheaves, or, to summarise, there exists a covariant functor L from stacks to sheaves.

Relation of L and Γ

What is the relation of L and $\Gamma(-,*)$? We have the proposition:

Proposition 2.2.3. *$L\Gamma(-,*)$ is naturally equivalent to the identity. There exists a natural transformation:* $\mathrm{id} \to \Gamma(-,*)L$.

The latter natural transformation is $\underline{S}(U) \to \Gamma(U,S)$ where $\underline{s} \mapsto s$ by $s(x) = \varphi_x^U(\underline{s})$. For any sheaf F, we define maps $F \leftrightarrows_j^i L\Gamma(-,F) = F'$.

Proof. Let $y \in F$, s any section through y; i.e., $s \in \Gamma(U,F)$ for some U.

Define $i(y) = \varphi_{p(y)}^U(s) \in F'_{p(y)}$. Let $z \in F*'$. Take some $w \in \Gamma(U,F)$ such that $\varphi_x^U = z$.

Define $j(z) = w(x)$.

We leave the reader to check that these give well defined natural transformations of functors and that ij and ji are identity maps. □

Proposition 2.2.4. *L is an exact functor; $\Gamma(-,*)$ is left-exact.*

Proof. We leave the proof to the reader. □

Many of the most interesting sheaves, such as those of use in algebraic geometry, can be defined most naturally via stacks.

2.2.3 Examples

(vi) Sheaf of germs of functions Let M be a K-module; we define a stack $\underline{S}$ by letting $\underline{S}(U)$ be the set of functions $U \to M$ and give $\underline{S}(U)$ the natural K-module structure. If $V \subset U$, define $\varphi_V^U \colon S(U) \to S(V)$ by restriction. The associated sheaf S is the sheaf of germs of functions with values in M. That is to say, let $x \in X, U, V$ neighbourhoods of x, and f, g functions $f \colon U \to M, g \colon V \to M$. We say that f and g are *locally equal* if there is a neighbourhood W of $x, W \subset U \cap V$ such that $f \mid W = g \mid W$. This is an equivalence relation; the equivalence class of f is called the *germ* of f at x. The set of germs at x is the stalk S_x. In this case $\Gamma(-,S) = \underline{S}$. For the proof see Chapter 6.

(vii): generalisation of (vi) We can generalise the above as follows. If M is any K-module, U open in X: Let $\Psi^P(U,M)$ be the set of functions $U^{p+1} \to M$. If $V \subset U$, define $\Psi^P(U,M) \to \Psi^P(V,M)$ by restriction. $\Psi^P(U,M)$ has a natural K-module structure, and example (vi) is given by $p = 0$. The associated sheaf of $\Psi^p(-,M)$ we write C_M^p, the Alexander-Spanier sheaf of p-dimensional cochains.

Define a map the "coboundary" $\delta(u) \colon \Psi^P(U,M) \to \Psi^{P+1}(U,M)$ as follows: If $f \colon U^{P+1} \to M$, let

$$(\delta(U)f)(x_0, \ldots, x_{p+1}) = \sum_0^{p+1} (-1)^i f(x_0, \ldots, \widehat{x}_i, \ldots, X_{p+1}).$$

δ is a map of stacks, and $\delta\delta = 0$.

If $\Psi^{-1}(-,M)$ is the constant stack, define $\epsilon(U)\colon \Psi^{-1}(U,M) \to \Psi^0(U,M)$ by letting $\epsilon(U)(m)$ be the constant map with value m. This is called the *augmentation*. Note that $\delta\epsilon = 0$. So we get what is known as an *augmented cochain complex* of stacks, i.e., a sequence

$$M \xrightarrow{\epsilon} \Psi^0(-,N) \xrightarrow{\delta} \Psi^1(-,M) \xrightarrow{\delta} \Psi^2(-,M) \to \cdots$$

with $\delta^2 = 0, \delta\epsilon = 0$.

This sequence is in fact exact. We define maps $s\colon \Psi^{i+1}(U,M) \to \Psi^i(U,M)(i \geq 0)$ and $\tau\colon \Psi^0(U,M) \to \Psi^{-1}(U,M)$. Let $a \in U$. Define

$$(sf)(x_0,\ldots,x_i) = f(a,x_0,\ldots,x_i) \qquad \tau f = f(a).$$

The reader will easily prove that $s\delta + \delta s = 1$ except in $\Psi^0(U,M)$, where $s\delta = 1 - \epsilon\tau$.

By applying L we get an augmented, acyclic cochain complex of sheaves.

$$M \xrightarrow{\epsilon} C_M^0 \xrightarrow{\delta} C_M^1 \xrightarrow{\delta} C_M^2 \to \cdots$$

(viii) Sheaf of singular cochains Define $C^n(U,M)$ to be the module of singular n dimensional cochains of U. We have the sequence, where δ and ϵ are the usual coboundary and augmentation,

$$M \xrightarrow{\epsilon} C^0(U,M) \xrightarrow{\delta} C^1(U,M) \xrightarrow{\delta} C^2(U,M) \to \cdots$$

giving the sheaf of singular cochains

$$M \xrightarrow{\epsilon} C_M^0 \xrightarrow{\delta} C_M^1 \xrightarrow{\delta} C_M^2 \to \cdots$$

This sequence is not exact in general. If it is we say the space is *HLC*.

Sheaf of singular chains Let $C_n(X, X\backslash U; M)$ be the module of n-dimensional singular chains of $(X, X \setminus U)$ with coefficients in M. This is a stack; we have the sequence

$$M \xleftarrow{\epsilon} C_0(X, X \setminus U; M) \xleftarrow{\partial} C_1(X, X \setminus U; M) \xleftarrow{\partial} C_2(X, X \setminus U; M) \leftarrow \cdots$$

which gives the sheaf of singular chains.

We can also define this using locally finite singular chains. For this definition and further properties of the sheaf,, see Chapter 6.

2.3 Restriction and Prolongation

2.3.1 Restriction

Let $A \subset X$ be any subset of X. Let F be a sheaf over X. Define $F \mid A = p^{-1}(A)$. Then $F \mid A$ is a sheaf over A, the projection, topology, and algebraic structure being induced by that of F. We call $F \mid A$ the *restriction* of F to A.

2.3.2 Prolongation by Zero

Let $A \subset X$ be any subset of X and F' be a sheaf over A. We say that a sheaf F over X is a *prolongation of* F' *by zero* if $F \mid A = F'$ and $F \mid (X \setminus A) = 0$.

First we need a definition from general topology.

Definition 2.3.1. Let $A \subset X$ be spaces. A is *locally closed in* X if each $x \in A$ has a neighbourhood N_x such that $N_x \cap A$ is closed in N_x.

Proposition 2.3.2. *There is up to natural isomorphism, at most, one prolongation of* F' *by zero. Such a prolongation always exists if* A *is locally closed in* X.

Proof. Clearly the protosheaf $\overline{F}$ is unique and always exists. Suppose we have two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on $\overline{F}$ which make it a sheaf with the required properties. We show $\mathcal{T}_1 = \mathcal{T}_2$.

Let W_1', W_2' be sets containing $y \in \overline{F}_x$ and open in the respective topologies. We find a set open in both topologies containing y and contained in both W_1' and W_2'. Choose a neighbourhood of y such that $W_1 \subset W_1'$ and $p \mid W_1$ is a homeomorphism onto a neighbourhood U_1 of x. Similarly, choose a neighbourhood of y such that $W_2 \subset W_2'$ and $p \mid W_2$ is a homeomorphism onto a neighbourhood U_2 of x. Define sections $s_1 \in \Gamma(U_1, \overline{F})$ and $s_2 \in \Gamma(U_2, \overline{F})$ as the inverses of the homeomorphisms. Now $s_1 \mid A$ and $s_2 \mid A$ are continuous sections of F', so if $x \in A$, there exists a set V' open in A containing x and such that $s_1 \mid V' = s_2 \mid V'$. Then $V' = V \cap A$ where V is open in X. If $U = U_1 \cap U_2 \cap V$, then $s_1 \mid U = s_2 \mid U$, since $\overline{F} \mid X \setminus A = 0$. Also U is not empty, since $x \in U$ and is open in X. Then $s_1(U) = s_2(U)$ is contained in both W_1' and W_2', and is open in both topologies, since s_1 and s_2 are open maps in their respective topologies. If $x \notin A$, then y is the zero over x. The zero section over a sufficiently small neighbourhood of x will be open in both topologies and contained in both W_1' and W_2'. Therefore $\mathcal{T}_1 = \mathcal{T}_2$.

We now assume A is locally closed and define a topology on $\overline{F}$. Let $y \in \overline{F}$. If $x \notin A$, we define the basic neighbourhoods of y to be the zero sections over neighbourhoods of x. If $x \in A$, there is a neighbourhood N_x of x such that there is a section $s \in \Gamma(A \cap N, F)$ with $s(x) = y$. We define a basic neighbourhood (N_x, s) of y to be the union of $s(A \cap N_x)$ with the zero section over $N_x \setminus A$. Clearly the intersection of two such basic neighbourhoods of y contains a third, for two sections of $\overline{F}$ through y agree on small enough neighbourhoods of x. To show that these neighbourhoods define a topology on $\overline{F}$, we must show that if V is a neighbourhood of y it is also a neighbourhood of all points in some smaller neighbourhood W of y. This is trivial if V is a zero section. Let $V = (N_x, s)$ with $x \in A$. Since A is locally closed, there is an open neighbourhood M_x of x such that $M_x \subset N_x$ and $M_x \cap A$ is closed in M_x. Define $W = (M_x, s \mid M_x \cap A)$. If $z \in W$ and $p(z) \in A$, then W is also a neighbourhood of z. If $z \in W$ and $p(z) \notin A$, then $p(z)$ has a neighbourhood U disjoint from $M_x \cap A$. Therefore, $p^{-1}(U) \cap W$ is the zero section over U and so W is a neighbourhood of z. $\square$

Definition 2.3.3. If A is locally closed in X and F a sheaf over A, define F^X to be this unique prolongation by zero of F.

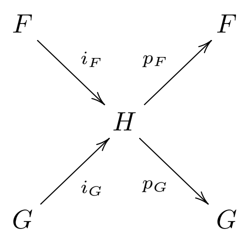
Definition 2.3.4. If A is locally closed in X, F a sheaf over X, define $F_A = (F \mid A)^X$. Note that this is consistent with our earlier definitions. For A is locally closed is equivalent to A being the intersection of an open and a closed set of X. Thus all open, and all closed sets are locally closed.

Remark 2.3.5. The above operations we can write functorially as $*_A, * \mid A, *^X$. They are all exact functors from sheaves to sheaves; in the last two cases the domain and range are sheaves over different spaces.

2.4 Various universal constructions

2.4.1 Direct sums

In any K-category a direct sum H of F and G is characterised by maps



such that $p_F i_F = \mathrm{id}, p_G i_G = \mathrm{id}, i_F p_F + i_G p_G = \mathrm{id}$.

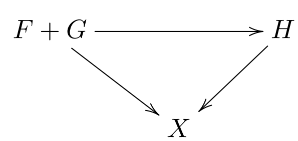
It follows that direct sums are preserved by K-functors. Now let $\underline{S}$ and $\underline{T}$ be stacks and let $\underline{S} + \underline{T}$ be defined by $(\underline{S} + \underline{T})(U) = \underline{S}(U) + \underline{T}(U)$ (direct sum). This is easily shown to be a direct sum of stacks. Since L is a K-functor, $L(\underline{S} + \underline{T}) = S + T$, where $S + T$ is a direct sum of sheaves. A direct definition of this is the one given earlier, viz. $S + T = \{(y, z) \in S \times T; p(y) = p(z)\}$. These definitions are, by the results of Chapter 1, equivalent.

2.4.2 Tensor products

Before we can define this, we must say what we mean by bilinear maps.

a. Let F, G, H be sheaves. A bilinear map $f \colon F + G \to H$ shall

1. be continuous,
2. make the diagramme



commutative, and

3. be such that $f \mid F_x + G_x$ is a bilinear map of K-modules.

b. Let $\underline{F}, \underline{G}, \underline{H}$ be stacks. A bilinear map

$$f \colon \underline{F} + \underline{G} \to \underline{H}$$

is a collection of bilinear maps $\underline{f}_U \colon \underline{F}(U) + \underline{G}(U) \to \underline{H}(U)$ such that if $V \subset U$, the following diagramme is commutative:

$$\begin{array}{ccc} \underline{F}(U) + \underline{G}(U) & \longrightarrow & \underline{H}(U) \\ \downarrow & & \downarrow \\ \underline{F}(V) + \underline{G}(V) & \longrightarrow & \underline{H}(V) \end{array}$$

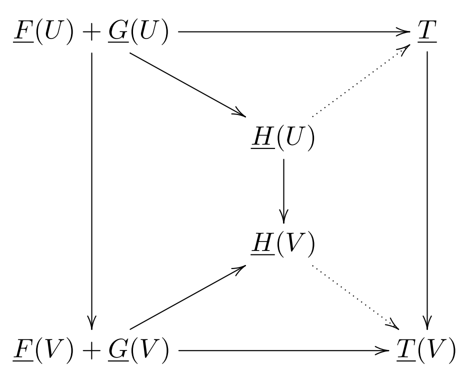
Since direct limits preserve direct sums and bilinearity, $\underline{f}$ gives rise to a corresponding bilinear map $\overline{f} \colon \overline{F} + \overline{G} \to \overline{H}$ of protosheaves and an argument exactly the same as before shows that f is consistent with the topology given to $\overline{F}, \overline{G}$, and $\overline{H}$. So we have a bilinear map of sheaves $f \colon F + G \to H$; i.e., L preserves bilinearity.

So, in fact, does $\Gamma(-,*)$. For if $f \colon F + G \to H$ is a bilinear map of sheaves, $\Gamma(-, F + G) = \Gamma(-, F) + \Gamma(-, G)$ since $\Gamma(-,*)$ is a K-functor and composition with f gives a bilinear map

$$\Gamma(-, F) + \Gamma(-, G) \to \Gamma(-, H).$$

Proposition 2.4.1. *If $\underline{F}, \underline{G}$ are stacks, there exists a universal bilinear map $\underline{i} \colon \underline{F} + \underline{G} \to \underline{H} = \underline{F} \otimes \underline{F}$.*

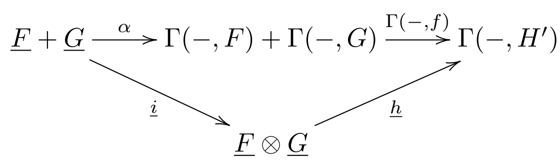
Proof. Let $\underline{H}(U) = \underline{F}(U) \otimes_K \underline{G}(U)$ and let the map $\underline{i}(U) \colon \; \colon \underline{F}(U) + \underline{G}(U)$ be the usual one. Then if $\underline{F} + \underline{G} \xrightarrow{f} \underline{T}$ is bilinear, we have for each open $U \subset X$ a unique factorisation $\underline{H}(U) \xrightarrow{h(U)} \underline{T}(U)$. By considering the diagramme,



where $V \subset U$, we easily see that $\underline{h} \colon \underline{H} \to \underline{T}$ is a natural transformation, i.e., is a map of stacks. □

Proposition 2.4.2. *If F, G are sheaves, there exists a universal bilinear map $i\colon F + G \to H = F \otimes H$.*

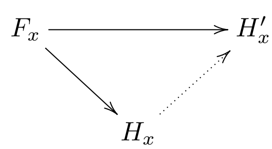
Proof. There exist stacks $\underline{F}, \underline{G}$ such that $L(\underline{F}) = F, L(\underline{G}) = G$. (e.g., $\underline{F} = \Gamma(-, F)$). Let $H = L(\underline{F} \otimes \underline{G})$. Let $f\colon F + G \to H'$ be bilinear. Consider the following diagramme:



α is the natural transformation given in proposition 2.2.3. Since $\underline{i}$ is universal, this diagramme has a unique completion $\underline{h}$ as shown. Applying L, we obtain a sheaf map

$$h\colon H \to H' \quad \text{such that} \quad f = h \cdot i.$$

Direct limits commute with tensor products. Therefore $H_x = F_x \otimes_K G_x$. But the factorisation



is unique. Therefore h is unique. □

Proposition 2.4.3. *$\otimes$ is right exact on the category of stacks; and so, by application of $L, \otimes$ is right exact on the category of sheaves.*

The proof is easy.

2.5 Supports

2.5.1 Definitions

Let F be a sheaf, $s \in \Gamma(U, F)$. We define the *support* of s, written $|s|$, to be the set $\{x \in U; s(x) \neq 0\}$. This is a closed set in U. For if $s(x) = 0$, since p is a local homeomorphism, there is a small neighbourhood of x mapped entirely to zero by s; i.e., $U \setminus |s|$ is open.

Let $f\colon F \to G$ be a sheaf morphism. We define $|f|$, *support* of f, to be the closure of $\{x \in X; f \mid F_x \neq 0\}$.

For protosheaves, we modify the above by saying that the support of a section shall be the closure of the set of x not mapped into 0.

Lemma 2.5.1. *Let $f\colon F \to G$ be a map that $|f| \subset C \subset X$ where C is closed in X. Then there is a unique factorisation*

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow & \swarrow & \\ F_C & & \end{array}$$

Note that F_C is a quotient sheaf of F; i.e., $F \to F_C$ is a universal map for maps $F \to G$ with support contained in C.

Proof. For $F \to F_C$ is epi, and f annihilates the kernel of this. □

2.5.2 Family of supports

A family Φ of subsets of X is called a *family of supports* if:

Axiom (1) If $A \in \Phi$, then A is closed.

Axiom (2) If $B \subset A$ is closed and $A \in \Phi$, then $B \in \Phi$.

Axiom (3) If $A, B \in \Phi$, then $A \cup B \in \Phi$.

Remark 2.5.2. Cartan used two additional restrictions in Φ. Φ is called *para-compactifying* (abbreviated here to PF) if in addition:

Axiom (4) If $A \in \Phi$, then A is paracompact.

Axiom (5) If $A \in \Phi$, then A has a neighbourhood in Φ.

A paracompact space is an Hausdorff space in which every covering has a locally finite refinement. By "A has a neighbourhood in Φ" we mean there is a $B \in \Phi$ such that $A \subset \operatorname{int} B$.

Definition 2.5.3. If F is a sheaf and Φ a family of supports, define $\Gamma_\Phi(F) = \{s \in \Gamma(X, F); |s| \in \Phi\}$.
Thus $\Gamma_\Phi(F)$ is the set of sections with support in Φ, and it is easily seen to be a submodule of $\Gamma(X, F)$.

Let $f\colon F \to G$. Then $|fs| \subset |s|$. So we have a map $\Gamma_\Phi(f)\colon \Gamma_\Phi(F) \to \Gamma_\Phi(G)$. Clearly $\Gamma_\Phi(-)$ is a covariant K-functor from sheaves to modules.

Definition 2.5.4. If F, G are sheaves, define $\operatorname{Hom}_\Phi(F, G)$ to be the set of sheaf maps $F \to G$ with support in Φ. $\operatorname{Hom}_\Phi(-,-)$ is a K-functor of two variables from sheaves to modules, covariant in one variable, contravariant in the other.

Proposition 2.5.5. $\operatorname{Hom}_\Phi(-,-)$ *is left-exact in either variable.*

Proof. Let $0 \to F' \xrightarrow{i} F \xrightarrow{j} F'' \to 0$ be an exact sequence of sheaves. Let G be a sheaf. Consider the induced sequence,

$$0 \to \mathrm{Hom}_\Phi(G, F') \xrightarrow{i'} \mathrm{Hom}_\Phi(G, F) \xrightarrow{j'} \mathrm{Hom}_\Phi(G, F'').$$

Let $s' \in \mathrm{Hom}_\Phi(G, F')$.
i' is mono. For

$$\begin{aligned} i's' = 0 &\Rightarrow (i(s'(y)) = 0 \text{ for all } y \in G \\ &\Rightarrow s'(y) = 0 \Rightarrow s' = 0 \end{aligned}$$

since i is mono.
Clearly, $(j'i'(s'))(y) = (ji)(s'(y)) = 0$ since $ji = 0$.
Let $s \in \mathrm{Hom}_\Phi(G, F)$ and $j's = 0$. Then $j'(s(y)) = 0$, $y \in G$. Since i is mono and the first sequence is exact, there exists a unique $s'(y) \in F'$ such that $i(s'(y)) = s(y)$.

The reader can check that s' is a sheaf morphism. $|s'| = |s| \Rightarrow s' \in \Phi$ since i is mono. Therefore $s' \in \mathrm{Hom}_\Phi(G, F')$ with $i'(s') = s$. Therefore the sequence above is exact.

In a similar way, we prove $\mathrm{Hom}_\Phi(-,-)$ is left-exact in the other variable. □

Corollary 2.5.6. $\Gamma_\Phi(-,-)$ *is left-exact.*

Proof. Obvious since $\Gamma_\Phi(F) = \mathrm{Hom}_\Phi(K, F)$ naturally. □

2.6 Functor from protosheaves to sheaves

Finally, we define a functor from protosheaves to sheaves. If $\overline{M}$ is a protosheaf, $\Gamma(-, \overline{M})$ is a stack. We write $\widetilde{\overline{M}} = L\Gamma(-, \overline{M})$.

Lemma 2.6.1. *If F is a sheaf and $\overline{M}$ a protosheaf, then there is a natural support preserving isomorphism*

$$\Gamma\colon\ \mathrm{Hom}(\overline{F}, \overline{M}) \approx \mathrm{Hom}(F, \widetilde{\overline{M}});$$

i.e., if $f\colon \overline{F} \to \overline{M}$ is a morphism of protosheaves, $|f| = |\Gamma(f)|$.

This shows incidentally that $\sim$ is a functor. We express the lemma by saying that $-$ and $\sim$ are *adjoint functors*. Another example of adjoint functors are Ω and E as functors of topological spaces; for $\mathrm{Map}(X, \Omega Y) = \mathrm{Map}(SX, Y)$. For more details on adjoint functors see Kan [3].

Proof. of Lemma:
We construct K-homomorphisms

$$\mathrm{Hom}(\overline{F}, \overline{M}) \underset{\xi}{\overset{\eta}{\leftrightarrows}} \mathrm{Hom}(F, \widetilde{\overline{M}}).$$

i. Let $f \in \mathrm{Hom}(F, \overline{M})$. The composition (for each $U \subset X$)

$$\Gamma(U, F) \xrightarrow{i} \Gamma(U, \overline{F}) \xrightarrow{\Gamma(U,\overline{f})} \Gamma(U, \overline{M})$$

where i is the inclusion map gives by application of L, a map

$$g = \eta(\overline{f})\colon F \to \widetilde{\widetilde{\overline{M}}}.$$

Explicitly, let $\alpha \in F_x$ and s be a local section through α. Then $g(\alpha) = \varphi_x^U(fs)$ for some neighbourhood U of x. Suppose $x \notin |\overline{f}|$, then $\overline{f} \mid F_y = 0$ for any y in a neighbourhood of x.

Therefore $\overline{f}s = 0$ in a neighbourhood of x. Thus $\varphi_x^U(\overline{f}s) = 0$; i.e., $g(\alpha) = 0$ and $g \mid F_x = 0$. So $|g| \subseteq |\overline{f}|$.

ii. Any $x \in U$ defines a map $\Gamma(U, \overline{M}) \to \overline{M}_x$ by $s \mapsto s(x)$. By passage to direct limits, we have a map $\widetilde{\widetilde{\overline{M}}}_x \to \overline{M}_x$ and so a map of protosheaves $\chi\colon \overline{\widetilde{\widetilde{\overline{M}}}} \to \overline{M}$. If $g \in \mathrm{Hom}(F, M)$, define $\xi(g) = \chi\overline{g}$ where $\overline{F} \xrightarrow{\overline{g}} \overline{\widetilde{\widetilde{\overline{M}}}} \xrightarrow{\chi} \overline{M}$.

Now $|\chi\overline{G}| \subset |g|$. For if $x \notin |g|$, and $\alpha \in \overline{F}_x$, then $\chi\overline{g}(\alpha) = \chi(0) = 0$. Explicitly, let $\alpha \in \overline{F}_x$, and let t_α be a section of $\overline{M}$ so that $\varphi_x\, U(t_\alpha) = g(\alpha)$ for some neighbourhood U of x. Then $\chi\overline{g}(\alpha) = t_\alpha(x)$.

iii. We show $\xi\eta = \mathrm{id}$ and $\eta\xi = \mathrm{id}$. Let $\eta(f) = g, \xi(g) = h$. Let α, t_α be as in (ii). Then $h(\alpha) = t_\alpha(x) = (fs)(x)$ (for some section s, as in (i)) $= f(s(x)) = f(\alpha)$. Therefore $h = f$ and $\xi\eta = \mathrm{id}$.

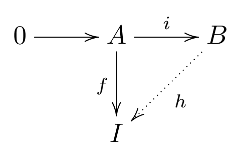
Let $\xi(g) = f$ and $\eta(f) = h$. Let $\alpha \in F_x$. Let s be a local section of F through α. Then fs is a local section of $\overline{M}$ and $h(x) = \varphi_x^U(fs)$ for some neighbourhood U of x. But we can choose s so that $\varphi_x^U(fs) = g(\alpha)$. Thus $h(\alpha) = g(\alpha)$.

The naturality of η follows from its functorial definition.

□

2.7 Injective sheaves

Definition 2.7.1. We say a sheaf (or module) I is *injective* if, for any exact sequence $0 \to A \xrightarrow{i} B$ and map $f\colon A \to I$, there is a map $h\colon B \to I$ such that $hi = f$.



Lemma 2.7.2. *There is always a completion h with $|h| \subset |f|$.*

Proof. The diagramme

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & {\scriptstyle f}\downarrow & & \\ & & I & & \end{array}$$

has by lemma 2.5.1 with $C = |f|$ a factorisation given by a diagramme

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & & \longrightarrow & & B \\ & & & \searrow & & & \downarrow \\ & & & & I & & \downarrow \\ & & & \nearrow & & & \downarrow \\ 0 & \longrightarrow & A_C & & \longrightarrow & & B_C \end{array}$$

Since I is injective, there is a completion $h' \colon B_C \to I$. If h is the composition $B \to B_C I$, then $|h| \subset C$. □

Theorem 2.7.3. *Any module can be imbedded in an injective module.*

For the proof we refer the reader to Cartan and Eilenberg, p. 9 or 51, or Eckmann-Schöpf [4].

Theorem 2.7.4. *Every sheaf can be imbedded in an injective sheaf.*

Proof. Let F be a sheaf. A protosheaf is injective if and only if its stalks are injective.

Lemma 2.7.5. *If a protosheaf $\overline{I}$ is injective, then $\tilde{\overline{I}}$ is injective.*

Proof. of 2.7.5:
Let the row of

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B \\ & & {\scriptstyle f}\downarrow & & \\ & & \tilde{\overline{I}} & & \end{array}$$

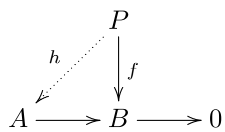
be exact. Now the diagramme

$$\begin{array}{ccccc} 0 & \longrightarrow & \overline{A} & \xrightarrow{g} & \overline{B} \\ & & {\scriptstyle f'}\downarrow & & \\ & & \tilde{\overline{I}} & & \end{array}$$

certainly has a completion $\overline{h} \colon \overline{B} \to \overline{I}$, since $\overline{I}$ is injective. By lemma 2.6.1 there is a completion of the first diagramme by $h \colon B \to \tilde{\overline{I}}$. □

Now, choose $\overline{I}$ to be a protosheaf of injective modules such that $\overline{I}_x \supset F_x$. The morphism $\overline{F} \xrightarrow{i} \overline{I}$ gives, by lemma 2.6.1, a morphism $F \to \widetilde{\overline{I}}$. Let $G = \ker(F \to \widetilde{\overline{I}})$. Then the composition $\overline{G} \to \overline{F} \xrightarrow{i} \overline{I}$ is zero. But I is mono. Therefore $\overline{G} = 0$, hence $G = 0$. □

Remark 2.7.6. We could define a projective sheaf P to be one such that for any exact sequence $A \to B \to 0$ and any morphism $f\colon P \to B$, there is a morphism $h\colon P \to A$ making the diagramme



commutative. But if the base space X is *not* discrete, I know of no examples of projective sheaves except the zero sheaf.

Chapter 3

Homological Algebra

To construct a homology theory for sheaves and to prove its uniqueness we need a number of concepts and propositions from homological algebra; we assume a certain familiarity with the techniques involved. We consider two exact K-categories $\mathcal{A}$ and $\mathcal{B}$ (e.g., sheaves, modules). Functors will always, unless stated otherwise, be K-functors $\mathcal{A} \to \mathcal{B}$. We refer to Cartan and Eilenberg's "Homological Algebra" as CE.

3.1 δ functors

A *covariant δ-functor* T^* is a series of covariant functors $T^i, a < i < b$, and for each exact sequence $0 \to A' \to A \to A'' \to 0$ a map

$$\delta \colon T^i(A'') \to T^{i+1}(A') \quad a < i < b-1$$

such that

(i) δ is natural. By this we mean the following:

A map of exact sequences $E \to E'$ is a commutative diagramme

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

where E and E' are the upper and lower exact sequences of this diagramme.

Condition (i) states that the induced diagramme

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow[\delta]{} & T^{i+1}(B') \end{array}$$

shall be commutative.

(ii) From an exact sequence $0 \to A' \to A \to A'' \to 0$ we obtain a sequence

$$\cdots \xrightarrow{\delta} T^i(A') \to T^i(A) \to T^i(A'') \xrightarrow{\delta} T^{i+1}(A') \to \cdots$$

We require that the composition of any two adjacent maps in this sequence shall be zero.

Definition 3.1.1. A *contravariant δ-functor* T^* is a series of contravariant functors $T^i, a < i < b$, and for each exact sequence $0 \to A' \to A \to> A'' \to 0$ a natural map $\delta\colon T^i(A') \to T^{i+1}(A'')$ such that in the sequence

$$\cdots \to T^i(A'') \to T^i(A) \to T(A') \xrightarrow{\delta} T^{i+1}(A'') \to \cdots$$

the composition of two adjacent maps is zero.

Assume from now on that δ-functors are covariant.

Definition 3.1.2. A *map of δ-functors* $(T^i_{a,ib}$ and $(S^i)_{a<i<b}$ is a set of natural transformations $f^i\colon T^i \to S^i$ which preserve δ; i.e., if $0 \to A' \to A \to A'^{prime} \to 0$ is exact, the following diagramme is commutative:

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta} & T^{i+1}(A') \\ \downarrow{\scriptstyle f^i(A'')} & & \downarrow{\scriptstyle f^{i+1}(A')} \\ S^i(A'') & \xrightarrow[\delta]{} & S^{i+1}(A') \end{array}$$

Definition 3.1.3. A δ-functor T^* is *exact* if the sequence in (ii) is exact.

Definition 3.1.4. Let S be a (covariant) K-functor. An *augmented* δ-functor T^* over S is a δ-functor $(T^i)_{0\le i<a}$ together with a natural transformation $\epsilon\colon S \to T^0$.

A map of augmented δ-functors $\{U^i\}_{0\le i<a}$ over R, $\{V^i\}_{0\le i<a}$ over S, over a natural transformation $\tau\colon R \to S$ is a map f of δ-functors such that the following diagramme commutes:

$$\begin{array}{ccc} R & \xrightarrow{\epsilon} & U^0 \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle f^0} \\ S & \xrightarrow[\epsilon]{} & V^0 \end{array}$$

Definition 3.1.5. A δ-functor $T^* = (T^i)_{0\le i<a}$ is *universal* if for any δ-functor $S^* = (S^i)_{0\le i<a}$ and any natural transformation $f^0\colon T^0 \to S^0$ there exists a unique map $g\colon T^* \to S^*$ of δ-functors such that $g^0 = f^0$.

Definition 3.1.6. A functor F is *effaceable* if for each $A \in \mathcal{A}$, there is a monomorphism $i\colon A \to M$ such that $F(i) = 0$.

A monomorphism $i\colon A \to M$ with $F(i) = 0$ is called an *effacement* of F at A.

Definition 3.1.7. A *cohomological δ-functor* T^* over S is an augmented δ-functor over S which is exact, and such that T^i is effaceable for $i > 0$ and $\epsilon\colon S \to T^0$ is a natural equivalence.

Theorem 3.1.8. *If $T^* = (T^i)_{0 \le i < a}$ is an exact, δ-functor with T^i effaceable for $i > 0$, then $T^*$is universal.*

Proof. We first define a map $f^1\colon T^1 \to S^1$ extending any natural transformation $f^0\colon T^0 \to S^0$ where S is a δ-functor.

Take any $A \in \mathcal{A}$, and let $i\colon A \to M$ be an effacement of T^1. Let $0 \to A' \xrightarrow{i} M \to \overline{A} \to 0$ be exact. Then we have the diagramme

$$\begin{array}{ccccccccc}
T^0(A) & \longrightarrow & T^0(M) & \xrightarrow{\lambda} & T^0(\overline{A}) & \xrightarrow{\delta} & T^1(A) & \xrightarrow{T^1(i)} & T^1(M) \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^0} & & \vdots{\scriptstyle f^1} & & \\
S^0(A) & \longrightarrow & S^0(M) & \xrightarrow[\mu]{} & S^0(\overline{A}) & \xrightarrow[\delta]{} & S^1(A) & \longrightarrow & S^1(M)
\end{array}$$

By "diagramme chasing", since the top sequence is exact and $T^1(i) = 0$, there is a unique extension $f^1(A)\colon T^1(A) \to S^1(A)$. For this statement and for many later ones, we need the easy lemma:

Lemma 3.1.9. *Consider the diagramme*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{j} & A'' & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle h} & & & \\
 & & & & B & & & &
\end{array}$$

If the top sequence is exact and $gi = 0$, there exists a unique $h\colon A'' \to B$ such that $hj = g$.

Using this lemma, I shall give the above diagramme chasing in detail. The sequence $0 \to \ker\delta \to T^0(A) \to T^1(A) \to 0$ is exact. Since T is an exact δ-functor, the row of the diagramme

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im}\lambda & \xrightarrow{j} & T^0(\overline{A}) & \longrightarrow & T^1(A) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \delta f^0} & & & & \\
 & & & & S^1(A) & & & &
\end{array}$$

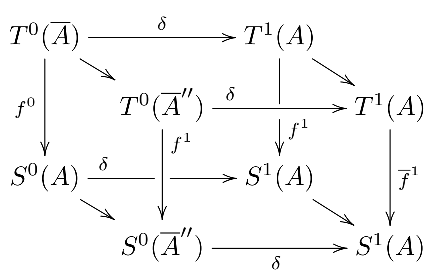
is exact. Now $\delta f^0 \lambda = \delta\mu f^0 = 0$ since $\delta\mu = 0$ by definition. Therefore $\delta f^0 j = 0$.

Thus by the lemma, we have a unique map $f^1(A)\colon T^1(A) \to S^1(A)$ as required.

We show f^1 is independent of the choice of M and i. Let $i\colon A \to M, i'\colon A \to M'$ be two effacements of T^1 at A. Then $i'' = (i, i')\colon A \to M'' = M + M'$ is an effacement of T^1 at A. Consider the diagramme, where the rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\;i\;} & M & \longrightarrow & \overline{A} & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow{\overline{j}} & & \Big\downarrow{r} & & \\
0 & \longrightarrow & A & \xrightarrow[i'']{} & M'' & \xrightarrow[j]{} & \overline{A}'' & \longrightarrow & 0
\end{array}$$

Here $\overline{j}$ is the injection, so that the first square is commutative. $j\overline{j}i = ji'' = 0$. Therefore there exists a unique $r\colon A \to A''$ making the last square commutative. This map of exact sequences gives a 3-dimensional diagramme:



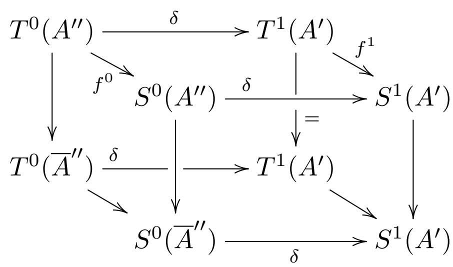
(3.1.10)

Let $f^1, \overline{f}^1$ be obtained from M' and M'' respectively, by the same constructions as before. All faces of the cube are commutative except possibly that involving f^1 and $\overline{f}^1$. Therefore this face is commutative on $\operatorname{im}(T^0(A) \xrightarrow{\delta} T^1(A))$. But this map is onto. Therefore $f^1 = \overline{f}^1$. Similarly $\widetilde{f}^1 = \overline{f}^1$, therefore $\widetilde{f}^1 = f^1$.

We show f^0, f^1 preserves δ. Let $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ be exact, and $k\colon A \to M$ be an effacement of T^1 at A. Then $ki\colon A' \to M$ is an effacement of T^1 at A' since $T(ki) = T(k)T(i) = 0$ and ki is mono. Consider the map of exact sequences in which we construct the map $A'' \to \overline{A}''$ by the usual method.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{\;i\;} & A & \xrightarrow{\;j\;} & A'' & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow{k} & & \Big\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{\;i\;} & M & \longrightarrow & \overline{A}'' & \longrightarrow & 0
\end{array}$$

This map induces a cubical diagramme in which we have to show the top face commutes:



(3.1.11)

The bottom face of this commutes since it is derived from an effacement of A'. The other faces commute (except the top one) by naturality and such like. Therefore any paths linking the opposite vertices $T^0(A'')$ and $S^1(A')$ of the cube give the same map. As the vertical map $S^1(A') \to S^1(A')$ is an identity, the top face commutes.

We show naturality; i.e., let $g\colon A \to B$ be a map and $i\colon A \to M, j\colon B \to N$ be effacements of T^1; we must show that the following diagramme commutes:

$$\begin{array}{ccc} T^1(A) & \xrightarrow{T^1(g)} & T^1(B) \\ \downarrow & & \downarrow{\scriptstyle f^1} \\ S^1(A) & \xrightarrow[S^1(A)]{} & S^1(B) \end{array}$$

Now $(i, jg)\colon A \to M + N = P$ is an effacement of T^1 at A. We have a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & M+N & \longrightarrow & \overline{A} & \longrightarrow & 0 \\ & & {\scriptstyle g}\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & N & \longrightarrow & \overline{B} & \longrightarrow & 0 \end{array}$$

where $M + N \to N$ is the projection. We get another cube, and the proof goes rather similarly to the earlier ones.

We construct the transformations $f^2, f^3, \ldots$ in an entirely similar manner. □

Corollary 3.1.12. *A cohomological δ-functor is universal as an augmented δ-functor; i.e., let $U^* = U^i_{0\leq i<a}$ be a cohomological δ-functor over R, $V^* = V^i_{0\leq i<a'}$ an augmented δ-functor over S, and let $f\colon R \to S$ be a natural transformation. Then there is a unique map $U^* \to V^*$ of augmented δ-functors.*

Here the composition $U^0 \xrightarrow{\epsilon^{-1}} R \xrightarrow{f} S \xrightarrow{\epsilon} V^0$ starts the construction.

3.2 Injective objects

Definition 3.2.1. An object $I \in \mathcal{A}$ is called *injective* if for any diagramme with row exact

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle h} & \\ & & I & & \end{array}$$

there is an $h\colon B \to I$ making the diagramme commute.

Remark 3.2.2. Let I be injective. If $0 \to I \to M$ is exact, I is a direct summand of M. For the completion of the diagramme

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & M \\ & & {\scriptstyle =}\downarrow & & \\ & & I & & \end{array}$$

is the required projection.

Assumption 3.2.3. Assume that every object in $\mathcal{A}$ can be embedded in an injective one; i.e., if $A \in \mathcal{A}$, there is a mono $i\colon A \to I$ with I injective. This is true for modules, protosheaves, and sheaves.

Then the converse of our above remark is also true. Let $I \xrightarrow{i} Q \xrightarrow{p} I$ be a representation of I as the direct summand of an injective Q. Let $0 \to A \to B$ be exact, $f\colon A \to I$ a map. Then since Q is injective, $i \cdot f$ can be factored through B.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ & & I & \xrightarrow[i]{} & Q \end{array}$$

ph is the required map $B \to I$. Thus I is injective. We also have the lemma:

Lemma 3.2.4. *Let $\mathcal{M}$ be a collection of objects of $\mathcal{A}$ such that*

(1) for every $A \in \mathcal{A}$, there is a mono $i\colon A \to M$, for some $M \in \mathcal{M}$,

(2) $M \in \mathcal{M}, N$ direct summand of $N \to N \in \mathcal{M}$.

If I injective, then $I \in \mathcal{M}$

Proof. There is a mono $i\colon I \to M$ for some $M \in \mathcal{M}$. I is injective and so a direct summand of M. Therefore $I \in \mathcal{M}$. □

Theorem 3.2.5. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is effaceable if and only if $F(I) = 0$ for all injective I.*

Proof. (i) Let I be injective. There is a mono $i\colon I \to M$ with $F(I) = 0$. By our remark, $M = I + J$. The inclusion $F(I) \to F(I) + F(J)$ is zero. Therefore $F(I) = 0$.

(ii) Assume $F(I) = 0$ all injective I. For any $A \in \mathcal{A}$, there is a mono $i\colon A \to I$, where I is injective. Certainly $F(i) = 0$.

□

Definition 3.2.6. By $\mathrm{Hom}(A, B)$ we mean the K-module of maps $A \to B$.

Proposition 3.2.7. *I is injective if and only if $\mathrm{Hom}(-, I)$ is exact.*

The proof again is easy.

Proposition 3.2.8. *A direct sum is injective if and only if each summand is injective.*

The proof is easy.

Definition 3.2.9. A *right-complex* C over $A \in \mathcal{A}$ is a sequence

$$0 \to A \xrightarrow{\epsilon} C^0 \xrightarrow{\delta} C^1 \to C^2 \to \cdots \quad (C^i = 0, i < 0)$$

such that $\delta^2 = 0$, and $\delta\epsilon = 0$. δ is called the *coboundary*, ϵ the *augmentation*. Such a complex is called *acyclic* if the sequence is exact. It is called O-injective (Object-Injective) if each C^i is injective. Define $Z^i(C) = \ker(C^i \to C^{i+1})$. In a particular category $Z^i(C)$ will be called the set of i-dimensional *cocycles* of C.

Definition 3.2.10. If $f\colon A \to \overline{A}$ is a map, a map g over f of right-complexes C, and $\overline{C}$ over A, and $\overline{A}$ is a commutative diagramme,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{\epsilon} & C^0 & \xrightarrow{\delta} & C^1 & \xrightarrow{\delta} & C^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^1} & & \downarrow{\scriptstyle g^2} & & \\
0 & \longrightarrow & \overline{A} & \xrightarrow[\epsilon]{} & \overline{C}^0 & \xrightarrow[\delta]{} & \overline{C}^1 & \xrightarrow[\delta]{} & \overline{C}^2 & \longrightarrow & \cdots
\end{array}$$

Definition 3.2.11. Two such maps, g and h, over f are *chain homotopic* if there are maps $s^i\colon C^i \to \overline{C}^{i-1}$ such that $g^i - h^i = \delta s^i + s^{i+1}\delta$ (which we write $g - h = \delta s + s\delta$). We have the fundamental theorem of homological algebra.

Theorem 3.2.12. *Let C be an acyclic right complex over A, $\overline{C}$ an O-injective right complex over $\overline{A}$, and $f\colon A \to \overline{A}$ any map. Then*

(i) there is a map $g\colon C \to \overline{C}$ over f,

(ii) any such maps are chain homotopic.

For proof see CE, p. 78.

Definition 3.2.13. An acyclic right-complex C over A is called a *resolution* of A. C is called an *injective resolution* if it is O-injective. By the fundamental theorem, an injective resolution is unique up to chain homotopy equivalence.

Theorem 3.2.14. *There is an injective resolution over any $A \in \mathcal{A}$.*

For proof see CE p. 80, though there it is proved for projective resolutions only. A curious consequence of this theorem is that if $f \colon S \to T$ is a natural transformation of left-exact K-functors such that $f(I)$ is an isomorphism for I injective, then f is a natural equivalence. We give the simple proof later when this fact is needed.

Definition 3.2.15. Let T be a covariant K-functor. Let $A \in \mathcal{A}$ and let

$$0 \to A \xrightarrow{\epsilon} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \cdots$$

be an injective resolution of A. Consider the complex $T(C)$

$$0 \to T(C^0) \xrightarrow{T(\delta)} T(C^1) \xrightarrow{T(\delta)} \cdots$$

Define the *right-derived functors* R^iT of T by

$$R^iT.(A) = H^iT(C)).$$

These are functors. For if $f \colon A \to \overline{A}$ is a map and $C, \overline{C}$ injective resolutions of A and $\overline{A}$, by the fundamental theorem there is a map $g \colon C \to \overline{C}$ such that the following diagramme commutes

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{A} & \longrightarrow & \overline{C} \end{array}$$

Any two such g induce the same map of homology, so we have a unique map $R^iT.(f)$.

It is easy to see that R^iT is unique up to natural isomorphisms. If C, C' are two injective resolutions of A, the maps $C \leftrightarrows C'$ which cover the identity map of A furnish these isomorphisms.

Remark 3.2.16. 1. $R^iT = 0$ for $i < 0$.

2. R^iT are effaceable for $i > 0$. For if A is injective, $0 \xrightarrow{\approx} A \to A \to 0 \to \cdots$ is an injective resolution of A.

3. The R^*T form an exact covariant δ-functor. We prove this later.

3.3 Cochain complexes

Starting with an exact category $\mathcal{A}$ satisfying assumption 3.2.3, let $\mathcal{C}(\mathcal{A})$ which we abbreviate to $\mathcal{C}$, be the category of cochain complexes of objects in $\mathcal{A}$ and of cochain maps. A complex here is an infinite sequence

$$\xrightarrow{\delta} A^{-n} \xrightarrow{\delta} \cdots \xrightarrow{\delta} A^{-1} \xrightarrow{\delta} A^0 \xrightarrow{\delta} A^1 \xrightarrow{\delta} A^2 \xrightarrow{\delta} \cdots$$

with $\delta\delta = 0$. This is clearly a K-category, and is easily seen to be exact.

Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ such that $C \in \mathcal{C}_0 \Rightarrow C^i = 0$ for $i < 0$.

Remark 3.3.1. By a *full subcategory* of a category $\mathcal{A}$ we mean a subcategory $\mathcal{B}$ of $\mathcal{A}$ in which, if $A, B \in \mathcal{B}$, the set of $\mathcal{A}$-maps $A \to B$ is the same as the set of $\mathcal{B}$ maps $A \to B$. Let $\mathcal{C}^N$ be the full subcategory of $\mathcal{C}$ such that $C \in \mathcal{C}^N \Rightarrow C^i = 0$ for $i > N$. Let $\mathcal{C}_0^N = \mathcal{C}_0 \cap \mathcal{C}^N$. These are all exact K-categories.

In each of these categories we can define injective objects, complexes, resolutions, and so on. Theorem 3.2.12 is also true. To prove this we need the following theorem.

Theorem 3.3.2. *A complex $C \in \mathcal{C}$ is injective* $\Leftrightarrow$

1. *Each C^i is injective.*

2. *Each $Z^i(C)$ is a direct summand of C^i.*

3.
$$H^i(C) = 0 \quad \textit{for all} \begin{cases} i & \textit{if } C \in \mathcal{C} \textit{ or } \mathcal{C}^N, \\ i \neq 0 & \textit{if } C \in \mathcal{C}_0, \mathcal{C}_0^N. \end{cases}$$

Finally for every complex C there is a mono $j\colon C \to I$, with I an injective complex.

Proof. We prove the implication $\Leftarrow$ first. Each $C^i = Z^i + D^i$. So C is the sum of complexes E_i of the form

$$\cdots \to 0 \to \cdots \to 0 \to D^i \to Z^{i+1} \to 0 \to \cdots$$

Thus it is sufficient to show E_i is injective if D^i, Z^{i+1} are injective. If $X \in \mathcal{C}$, a map $g\colon X \to E_i$ is completely determined by a map $g^{i+1}\colon X^{i+1} \to Z^{i+1}$. This is obvious from the diagramme

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \longrightarrow & X^{i+1} & \longrightarrow & X^{i+2} & \longrightarrow & \cdots \\ & & & & & & \big\downarrow{\scriptstyle g^{i+1}} & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & D^i & \underset{\approx}{\longrightarrow} & Z^{i+1} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Therefore if Z^{i+1} is injective, so is E_i, and hence so is C. $\square$

Remark 3.3.3. In the case $C \in \mathcal{C}_0$ the elementary complex E_{-1} will be of the form

$$\cdots \to 0 \to Z^0 \to 0 \to \cdots$$

Next we prove for every complex there is a mono $j\colon C \to I$ and I is injective. Let $X \in \mathcal{C}$. For each i, choose a mono $j\colon X^i \to I^i$, I^i injective. Let I_i be the elementary complex

$$\cdots \to 0 \to I^i \xrightarrow{=} I^i \to 0 \to \cdots$$

There is a map $X \to I_i$ by

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & X^{i-2} & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & I^i & \longrightarrow & I^i & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Let I be the product of these complexes I_i. Then we have a mono $X \to I$. We leave to the reader the examination of the cases $\mathcal{C}^N, \mathcal{C}_0$, and $\mathcal{C}_0^N$. Finally, let $\mathcal{M}$ be the class of complexes satisfying (1), (2), and (3). Then $\mathcal{M}$ satisfies the conditions of lemma 3.2.4. Therefore I injective $\Rightarrow I \in \mathcal{M}$.

Let $\mathcal{E}$ be the K-category of short exact sequences of objects of $\mathcal{A}$. This is not an exact category. But $\mathcal{E}$ is a full subcategory of $\mathcal{C}_0^2$. So we can define exact sequences in $\mathcal{E}$, and the conclusions of the theorem hold in $\mathcal{C}_0^2$.

Proposition 3.3.4. *Every short exact sequence can be embedded in an injective short exact sequence. (Here we mean injective as an object of $\mathcal{C}_0^2$.)*

Proof. Let $0 \to A' \to A \to A'' \to 0$ be exact. Let $i'\colon A' \to I', i''\colon A'' \to I''$ be monos into injective objects.
Let $I = I' + I''$, and let $0 \to I' \to I \to I'' \to 0$ be built of an injection and a projection. Complete the following diagramme.

$$\begin{array}{ccccc}
0 & \longrightarrow & A' & \longrightarrow & A \\
 & & {\scriptstyle i'}\downarrow & & \vdots{\scriptstyle k} \\
0 & \longrightarrow & I' & \xrightarrow[=]{} & I'
\end{array}$$

Then the following is a map of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \xrightarrow{p} & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle (k,0)} & & \downarrow{\scriptstyle i''} & & \\
0 & \longrightarrow & I' & \longrightarrow & I & \longrightarrow & I'' & \longrightarrow & 0
\end{array}$$

To prove that injective resolutions exist in $\mathcal{E}$, we need the following lemma.

Lemma 3.3.5. *(The 9-lemma)*[1]
In an exact category $\mathcal{A}$ let the following be a commutative diagramme with exact rows.

$$\begin{array}{ccccccccc}
 & & (1) & & (2) & & (3) & & \\
 & & & & & & & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then the following holds.

(a) Columns (1) and (2) are exact $\Rightarrow$ column (3) is exact,

(b) Columns (2) and (3) are exact $\Rightarrow$ column (1) is exact.

(c) Columns (1) and (3) are exact and compositions in column (2) are zero $\Rightarrow$ column (2) is exact.

We leave the proof to the reader.

Corollary 3.3.6. *If $E, E' \in \mathcal{E}$, and $\varphi\colon E \to E'$ is a map, then*

(a) φ is mono $\Rightarrow \operatorname{coker}\varphi \in \mathcal{E}$,

(b) φ is epi $\Rightarrow \ker\varphi \in \mathcal{E}$.

The proof is an easy application of the 9-lemma. □

The standard process of building up an injective resolution now works in $\mathcal{E}$.

Theorem 3.3.7. *If T is a covariant K-functor, the right derived functors form an exact covariant δ-functor.*

Proof. Let $0 \to A' \to A \to A'' \to 0$ be exact. Let $0 \to C' \to C \to C'' \to 0$ be an injective resolution in $\mathcal{E}$ of this exact sequence. Now each C'^i, C^i, C''^i is

[1] The 3×3 lemma would be a more suitable name.

injective. Therefore the sequence $0 \to C'^i \to C^i \to C''^i \to 0$ splits. Any K-functor preserves split exact sequences. So $0 \to T(C') \to T(C) \to T(C'') \to 0$ is exact. By a standard homology argument, we obtain an exact sequence

$$\to H^i(T(C')) \to H^i(T(C)) \to H^i(T(C'')) \xrightarrow{\delta} H^{i+1}(T(C')) \to \cdots ,$$

i.e., an exact sequence

$$\to R^iT.(A') \to R^iT.(A)) \to R^iT.(A'') \xrightarrow{\delta} R^{i+1}T.(C') \to \cdots .$$

The naturality of δ follows from the homology construction. (For details, see CE, lemma 3.3, p. 40). $\square$

Remark 3.3.8. The construction of δ for an exact category follows from the following lemma[2].

Lemma 3.3.9. *In an exact category $\mathcal{A}$ let the following be a commutative diagramme with the columns and the middle two rows exact. (The $*$ represent some objects of $\mathcal{A}$.)*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A & \xrightarrow{j} & B & \xrightarrow{j} & C & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then there is a map $k\colon C \to A'$ such that the following sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} A' \xrightarrow{i'} B' \xrightarrow{j'} C'$$

is exact, k is unique and natural with respect to maps of these diagrams.

We can also use the 12-Lemma to prove the following lemma.

[2]This is a slightly modified version of the snake lemma.

Lemma 3.3.10. *Let $0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ be a short exact sequence of cochain complexes. Let $Z^n(C)$, as usual, denote the set of n-cocycles of C. Then there is exact sequence*

$$0 \to Z^n(C') \xrightarrow{i} Z^n(C) \to Z''(C'') \to H^{n+1}(C)$$
$$\xrightarrow{i*} H^{n+1}(C) \xrightarrow{j_*} H^{n+1}(C'') \xrightarrow{\delta} H^{n+1}(C').$$

Note that this sequence is the ordinary cohomology sequence from $H^{n+1}(C')$ on.

Remark 3.3.11. The set R^* could be considered as a functor from the category of functors and natural transformations to the category of δ-functors and maps of δ-functors.

We have:

Proposition 3.3.12. *R^*T is an augmented δ-functor over T. We define a map $\epsilon\colon T \to R^0T$ as follows:*

Let $0 \to A \xrightarrow{\epsilon} I$ be an injective resolution of A. The map $T(\epsilon)\colon T(A) \to T(I^0)$ defines, since $\delta\epsilon = 0$, a map $T(A) \to Z^0T(I) \approx H^0T(I) = R^0T.(A)$. Let $\epsilon(A)$ be this composition $T(A) \to R^0T.(A)$. ϵ is obviously a natural transformation.

Proposition 3.3.13. *ϵ is an isomorphism $\Leftrightarrow$ T is left-exact.*

Proof. ($\Rightarrow$): Suppose $\epsilon(A)$ is an isomorphism for each $A \in \mathcal{A}$. To prove the proposition we need only show R^0T is left-exact.

Let $0 \to A' \to A \to A'' \to 0$ be exact. Then the sequence

$$0 \to R^0T.(A') \to R^0T.(A) \to R^0T.(A'') \to R^1T.(A) \to \cdots$$

is exact; i.e., R^0T is left-exact.
($\Leftarrow$): Suppose T is left-exact. Let $0 \to A \xrightarrow{\epsilon} I$ be an injective resolution of A. Then $0 \to T(A) \xrightarrow{T(\epsilon)} T(I^0) \to T(I^1)$ is exact; i.e., $T(A) \approx Z^0(T.(I)) \approx H^0(T(I)) = R^0T(A)$. Therefore ϵ is an isomorphism. □

Corollary 3.3.14. *If T is left-exact, R^*T is a cohomological δ-functor over T.*

Corollary 3.3.15. *If T is left-exact, any cohomological δ-functor over T is naturally equivalent to R^*T.*

What we need now is an easier method of computation for the right-derived functors. Injective resolutions suffer from the disadvantage that injective objects are usually too big. In particular we wish to know for what objects M and functors T does $R^iT.(M) = 0$. For sheaves, with $T = \Gamma_\Phi$ this question will be gone into in great detail later on. For the present, we note the following proposition and lemma.

Proposition 3.3.16. *Let T be left-exact and $0 \to A \to C$ an acyclic complex over A such that $R^iT.C^j) = 0, i > 0$, all j. Such a complex we call a T-*resolution *of A.*

Then there is a natural isomorphism

$$H^iT(C) \approx R^iT.(A)$$

which preserves ϵ and δ; i.e., if $0 \to A \to C$, and $0 \to A' \to C'$ are T-resolutions and

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C' \end{array}$$

is a map of augmented complexes, then the following diagrams commute:

$$\begin{array}{ccc} & & H^0T(C) \\ & \overset{T(\epsilon)}{\nearrow} & \\ T(A) & & \downarrow \approx \\ & \underset{\epsilon}{\searrow} & \\ & & R^0T.(A) \end{array} \qquad \begin{array}{ccc} H^iT(C) & \xrightarrow{\approx} & R^iT.(A) \\ \downarrow & & \downarrow \\ H^iT(C') & \xrightarrow[\approx]{} & R^iT.(A') \end{array}$$

The reader will easily see what preservation of δ means.

Proof. Since $0 \to A \to C$ is acyclic, it is built up out of exact sequences

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow Z^1 \longrightarrow 0$$

$$0 \longrightarrow Z^1 \longrightarrow C^1 \longrightarrow Z^2 \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow Z^j \longrightarrow C^j \longrightarrow Z^{j+1} \longrightarrow 0$$

Applying R^*T, we have an exact sequence

$$\to R^{i-1}T.(C^j) \to R^{i-1}T.(Z^{j+1}) \to R^iT.(Z^j) \to R^iT.(C^j) \to \cdots$$

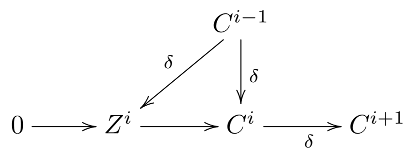
By assumption $R^iT.(C^j) = 0$. Therefore $R^iT.(Z^j) \approx R^{i-1}T.(Z^{j+1})$. Therefore

$$R^iT.(A) = R^iT.(Z^0) \approx R^{i-1}T.(Z^1) \approx \cdots \approx R^1T.(Z^{i-1}).$$

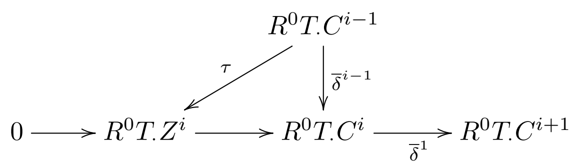
Prom the exact sequence, we have

$$R^0T.(C^{i-1}) \xrightarrow{\tau} R^0T.(Z^i) \to R^1T.(Z^{i-1}) \to R^1T.(C^{i-1}) = 0,$$
$$R^1T.(Z^{i-1}) \approx \operatorname{coker} \tau.$$

Consider the commutative diagram, with exact row



This gives a diagram with exact row



Then $\operatorname{im} \tau \approx \operatorname{im} \overline{\delta}^{i-1}$, and $R^0T.(Z^i) \approx \ker \overline{\delta}^i$. Therefore $\operatorname{coker} \tau \approx H^i(R^0T.(C)) \approx H^i(T(C))$, since T is left-exact.

Naturality follows from the naturality of all the isomorphisms considered.

We leave the reader to prove that the isomorphism of the proposition preserves ϵ and δ. □

Definition 3.3.17. An element $A \in \mathcal{A}$ is called *T-acyclic* if $R^iT.(A) = 0$ when $i > 0$.

Lemma 3.3.18. *Let $0 \to A \to A_1 \to \cdots \to A_n \to B \to 0$ be exact, with $A_1, \ldots, A_n$ T-acyclic. Then for $p \geq 1, R^PT.(B) \approx R^{p+n}T.(A)$. This is natural with respect to maps of exact sequences of the type considered.*

Proof. Split the exact sequence into $0 \to A \to A_1 \to Z_2 \to 0$, $0 \to Z_2 \to A_2 \to Z_3 \to 0$, etc. and use the initial argument of the previous proposition. □

Lemma 3.3.19. *Let T be a covariant K-functor and let $\mathcal{M}$ be a class of objects of $\mathcal{A}$ such that:*

1. *If I injective, then $I \in \mathcal{M}$.*

2. *If $0 \to M'M \to A \to 0$ is exact and $M', M \in \mathcal{M}$, then $A \in \mathcal{M}$.*

3. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of elements of $\mathcal{M}$, then $0 \to T(M') \to T(M) \to t(M'') \to 0$ is exact.*

Then $R^iT.(M) = 0, i > 0$, and all $M \in \mathcal{M}$.

Proof. Let $0 \to M \stackrel{\epsilon}{\to}\to I$ be an injective resolution of M. This is built up of short exact sequences:

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow Z^1 \longrightarrow 0$$

$$0 \longrightarrow Z^1 \longrightarrow I^1 \longrightarrow Z^2 \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow Z^i \longrightarrow I^i \longrightarrow Z^{i+1} \longrightarrow 0$$

Now $I^i \in \mathcal{M}$. By induction, using (2), $Z^i \in \mathcal{M}$. Therefore by (3), we get short exact sequences:

$$0 \longrightarrow T(M) \longrightarrow T(I^0) \longrightarrow T(Z^1) \longrightarrow 0$$

$$0 \longrightarrow T(Z^1) \longrightarrow T(I^1) \longrightarrow T(Z^2) \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow T(Z^i) \longrightarrow T(I^i) \longrightarrow T(Z^{i+1}) \longrightarrow 0 \qquad \cdots$$

Pitting these together we get an exact sequence

$$0 \to T(M) \to T(I^0) \to T(I^1) \to T(I^2) \to \cdots$$

which is just $0 \to T(M) \xrightarrow{T(\epsilon)} T(I)$. Therefore $R^iT.(M) = 0$. □

3.4 Natural resolutions

This section will be of use mainly in the chapter on spectral sequences. Let T he a K-functor $\mathcal{A} \to \mathcal{B}$.

Definition 3.4.1. A *natural resolution*[3] N on $\mathcal{A}$ consists of:

1. a K-functor $N\colon \mathcal{A} \to \mathcal{C}_0(\mathcal{A})$; each $N(A)$ is a complex

$$\cdots \to 0 \to N^0(A) \stackrel{\delta}{\to} N^1(A) \stackrel{\delta}{\to} N^2(A) \stackrel{\delta}{\to} \cdots$$

and N^i is a K-functor $\mathcal{A} \to \mathcal{A}$;

[3]This name is due to R. Brown.

2. a natural transformation $\epsilon\colon \mathrm{id} \to N^0$; satisfying the following conditions:

   a. N is an exact functor;

   b. the sequence $0 \to A \xrightarrow{\epsilon} N^0(A) \xrightarrow{\delta} N^1(A) \cdots$ is exact.

Definition 3.4.2. A *natural T-resolution* N is a natural resolution such that, in addition:

$$R^iT.N^j(A) = 0, \quad i > 0, \quad A \in \mathcal{A}.$$

Definition 3.4.3. A *resolvent functor $\mathcal{F}$ over T* consists of:

1. A K-functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{C}_0(\mathcal{B})$; each $\mathcal{F}(A)$ is a complex

$$\cdots \to \mathcal{F}^0(A) \xrightarrow{\delta} \mathcal{F}^1(A) \xrightarrow{\delta} \mathcal{F}^2(A) \to \cdots$$

   and $\mathcal{F}^i$ is a K-functor $\mathcal{A} \to \mathcal{B}$;

2. a natural transformation $\epsilon\colon T \to \mathcal{F}^0$ satisfying the following conditions:

   i. $\mathcal{F}$ is an exact functor;

   ii. $\epsilon T(A) \approx H^0\mathcal{F}(A)$, or equivalently the sequence

$$0 \to T(A) \xrightarrow{\epsilon} \mathcal{F}^0(A) \to \mathcal{F}^1(A)$$

   is exact;

   iii. If I is injective in $\mathcal{A}$ then

$$0 \to T(I) \to \mathcal{F}^0(I) \to \mathcal{F}^1(I) \to \cdots$$

   is exact.

Proposition 3.4.4. *Let N he a natural T-resolution, where T is left-exact. Then TN is a resolvent functor over T.*

Proof. Since T is left-exact, condition (2ii) is satisfied.

If $0 \to A' \to A \to A'' \to 0$ is exact, then so is $0 \to N^j(A') \to N^j(A) \to N^j(A'') \to 0$. Therefore,

$$0 \to R^0T.N^j(A') \to R^0T.N^j(A) \to R^0T.N^j(A'') \to R^1T.N^j(A') \to \cdots$$

is exact. But $R^0T = T$ and $R^1T.N^j(A') = 0$. Therefore TN is an exact functor.

Let I be injective in $\mathcal{A}$. We must show that

$$0 \to T(I) \xrightarrow{T(\epsilon)} TN^0(I) \xrightarrow{T(\delta)} TN^1(I) \xrightarrow{T(\delta)} \cdots$$

is acyclic. It is acyclic at $TN^0(I)$ since T is left-exact. For the other terms, we note that by proposition 3.3.16, $H^iTN(I) \approx R^iT.(I) = 0$ for $i > 0$, since I is injective. □

Proposition 3.4.5. *If T is left-exact and $\mathcal{F}$ is a resolvent functor over T, then H^* is a cohomological δ-functor over T.*

Proof. If $0 \to A' \to A \to A'' \to 0$ is exact, then so is $0 \to \mathcal{F}(A') \to \mathcal{F}(A) \to \mathcal{F}(A'') \to 0$. Taking cohomology, we have an exact δ-functor. This is augmented and ϵ is, by (2ii), a natural equivalence. If I is injective, $H^i\mathcal{F}(I) = 0$, for $i > 0$, by (2iii), and so $H^i\mathcal{F}$ is effaceable. □

Corollary 3.4.6. *Under the conditions of proposition 3.4.5, there is a natural isomorphism of augmented δ-functors $H^*\mathcal{F} \approx R^*T$.*

Definition 3.4.7. Let N, N' be natural resolutions. A *map of natural resolutions* $g\colon N \to N'$ is a natural transformation of functors such that if $A \in \mathcal{A}$ the following diagram commutes:

$$\begin{array}{ccc} & & N^0(A) \\ & \overset{\epsilon}{\nearrow} & \\ A & & \Big\downarrow g^0(A) \\ & \underset{\epsilon}{\searrow} & \\ & & N'^0(A) \end{array}$$

Definition 3.4.8. Let $\mathcal{F}, \mathcal{F}'$ be resolvent functors over S, T, respectively. A *map of resolvent functors $g\colon \mathcal{F} \to \mathcal{F}'$ over a natural transformation $f\colon S \to T$* is a natural transformation of functors such that the following diagram commutes:

$$\begin{array}{ccc} S(A) & \xrightarrow{\ \epsilon\ } & \mathcal{F}^0(A) \\ {\scriptstyle f(A)}\Big\downarrow & & \Big\downarrow{\scriptstyle g^0(A)} \\ T(A) & \xrightarrow[\ \epsilon\ ]{} & \mathcal{F}'0(A) \end{array}$$

Lemma 3.4.9. *If T is left-exact, N, N' are natural T-resolutions, and $g\colon N \to N'$ is a map of natural resolutions. Then $Tg\colon TN \to TN'$ is a map of resolvent functors.*

The proof is easy.

Chapter 4

The Cohomology of Sheaves

4.1 Generalities

A cohomology theory for sheaves is a cohomological δ-functor over Γ_Φ from the category of (K-) sheaves to the category of (K-) modules. By the results of the last chapter, such a homology theory exists and is unique up to natural equivalence. We can thus talk about *the* cohomology theory of sheaves.

Definition 4.1.1. $H^i_\Phi(X, -) = R^i\Gamma_\Phi(-)$.

A partial cohomology theory for sheaves is an augmented δ-functor over Γ_Φ. There is a unique map (as a map of augmented δ-functor over Γ_Φ) from the cohomology theory to any partial cohomology theory.

Note that $H^0_\Phi(X, F) \approx \Gamma_\Phi(F)$ naturally.

For each sheaf $A, \mathrm{Hom}_\Phi(A, -)$ is a covariant, left-exact K-functor from sheaves to modules.

Definition 4.1.2. $\mathrm{Ext}^i_\Phi = R^i \mathrm{Hom}_\Phi(A, -)$.

$Ext^i_\Phi(-, -)$ is in fact a functor of two variables, contravariant in the first, covariant in the second. For if $f\colon A \to A'$ is a map, $\mathrm{Hom}(f, 1)\colon \mathrm{Hom}_\Phi(A', -) \to \mathrm{Hom}_\Phi(A, -)$ is a natural transformation of functors.
So we have a map $\mathrm{Ext}^i_\Phi(f, 1)\colon \mathrm{Ext}^i_\Phi(A', -) \to \mathrm{Ext}^i_\Phi(A, -)$.

Remark 4.1.3. $\mathrm{Ext}^i_\Phi(A, B) = \begin{cases} 0 & i < 0 \\ \mathrm{Hom}_\Phi(A, B) & i = 0. \end{cases}$

Remark 4.1.4. For fixed A, $\mathrm{Ext}^*_\Phi(A, -)$ is an exact covariant δ-functor.

Remark 4.1.5. For fixed B, $\mathrm{Ext}^*_\Phi(-, B)$ is an exact contravariant δ-functor.

Proof. Let $0 \to A' \to A \to A'' \to 0$ be exact and let $0 \to B \to I$ be an injective resolution of B. Then we have an exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_\Phi(A'', I) \to \mathrm{Hom}_\Phi(A, I) \to \mathrm{Hom}_\Phi(A', I) \to 0.$$

(Here $\mathrm{Hom}_\Phi(A, I)$, stands for the cochain complex whose objects are $\mathrm{Hom}_\Phi(A, I^i)$, and similarly for the others.) On taking homology, we have an exact sequence:

$$\cdots \to \mathrm{Ext}^i_\Phi(A'', B) \to \mathrm{Ext}^i_\Phi(A, B) \to> \mathrm{Ext}_\Phi(A', B) \xrightarrow{\delta} \mathrm{Ext}^i_\Phi(A'', B) \to \cdots$$

□

Remark 4.1.6. If B is injective, then $\mathrm{Ext}^i_\Phi(A, B) = 0 \quad i > 0,$ all A.

Remark 4.1.7. If A is projective, then $\mathrm{Ext}^i_\Phi(A, B) = 0 \quad i > 0,$ all B.

4.2 Cohomology theory without using injective sheaves

We now show how a cohomology theory can be defined without using injective sheaves. The main advantage of this approach is that we can construct a natural Γ_Φ-resolution for sheaves, whereas injective resolutions cannot easily be given in this way. By proposition 3.4.4 and the corollary to proposition 3.4.6 we can use any natural Γ_Φ-resolution to calculate cohomology; its existence will be of great use in constructing the spectral sequences.

Definition 4.2.1. A sheaf F is *flabby*[1] if every section over an open set of X extends to a section over X.

Theorem 4.2.2. *Let $\mathcal{M}$ be the class of flabby sheaves. Then $\mathcal{M}$ satisfies the conditions of lemma 3.3.19 for $T = \Gamma_\Phi$ and any family of supports Φ.*

Corollary 4.2.3. *If F is flabby, then $H^i_\Phi(X, F) = 0$ for $i > 0$ (i.e., F is acyclic, which we abbreviate to F is* Φ-acyclic*).*

Remark 4.2.4. $\mathcal{M}$ satisfies the following condition stronger than (3.3.19, 3).

(3′) If $0 \to M' \xrightarrow{i} A \xrightarrow{j} B \to 0$ is exact and $M' \in \mathcal{M}$ then $0 \to \Gamma_\Phi(M') \to \Gamma_\Phi(A) \to \Gamma_\Phi(B) \to 0$ is exact.

Proof. We first prove (1); i.e. that all injective sheaves are in $\mathcal{M}$.
1: Let I be injective. Then the sequence

$$0 \to \mathrm{Hom}(K_{X\setminus U}, I) \to \mathrm{Hom}(K, I) \to \mathrm{Hom}(K_U, I) \to 0$$

is exact. But $\mathrm{Hom}(K_U, I) \approx \Gamma(U, I)$ naturally. Therefore the map $\rho^X_U : \Gamma(X, I) \to \Gamma(U, I)$ is epi. i.e., a section over U extends to a section over X.
(3′): Let $s \in \Gamma_\Phi(B)$. Consider all sections t of A over open sets U and such that $jt = s \mid U$. We define an ordering in this set by $t_0 > t_1$, if t_0 is an extension of t_1.

[1]The French term “flasque” for this kind of sheaf is due to Godement. “Flabby” is a literal translation of this, and has about it the same sort of feeling.

Any chain $\{t_i\}$ has an upper bound. For if t_i is defined on U_i, define t on $\cup U_i$ by $t \mid U_i = t_i$.

Start with the zero section $0 \in \Gamma(X \setminus |s|, A)$ and take a maximal $t > 0$. Certainly $|t| \subset |s| \in \Phi$ and so $|t| \in \Phi$. Therefore $t \in \Gamma_\Phi(U, A)$ for some open U.

Suppose $U \neq X$, and let $x \in X \setminus U$. Since j is onto, there is a neighbourhood N_x of x and a section $r \in \Gamma(N_x, A)$ such that $jr = s \mid N_x$. Let $r \mid (N_x \cap U) - t \mid (N_x \cap U) = \sigma \in \Gamma(N_x \cap U, \mathcal{A})$. Then $j\sigma = 0$. By the left-exactness of $\Gamma(U, -)$, there is a section u of M' such that $iu = \sigma$. Extend u to X (by the flabbiness of M') and then let w' be the restriction of this section to N_x. Finally, let $w = r - iw' \in \Gamma(N_x, A)$. Then $jw = s \mid N_x$ and w and t agree on $N_x \cap U$. Thus t is not maximal, as we assumed.

1: Let $0 \to M' \xrightarrow{i} M \xrightarrow{j} F \to 0$ be exact with $M, M' \in \mathcal{M}$. If we show F is flabby, then the proof is complete. This is done by the next lemma. □

Lemma 4.2.5. *If M' is flabby and U is open in X, then $M' \mid U$ is flabby.*

Proof. For any section of $M' \mid U$ is a section of M'. Let $s \in \Gamma(U, F)$. The sequence $0 \to M' \mid U \to M \mid U \xrightarrow{j'} F \mid U \to 0$ is exact. By ($3'$), there is a $t' \in \Gamma(U, M \mid U)$ such that $j't' = s$. Extend t' to a section $t \in \Gamma(X, M)$ (M is flabby). Then jt is an extension of s to X. Therefore F is flabby. □

4.3 Some lemmas on protosheaves and sheaves

We need a few lemmas about protosheaves and sheaves.

Lemma 4.3.1. *$-$ and $\sim$ are exact functors.*

Proof. It is obvious that $-$ is an exact functor. Let $0 \to \overline{P}' \to \overline{P} \to \overline{P}'' \to 0$ be an exact sequence of protosheaves. $\Gamma(-, *)$ is left-exact. Therefore

$$0 \to \Gamma(-, \overline{P}') \to \Gamma(-, \overline{P}) \to \Gamma(-, \overline{P}'')$$

is an exact sequence of stacks. But the last map is onto, since any section of $\overline{P}''$ can be lifted back, point by point, to a section of $\overline{P}$. □

Applying L, we have the lemma

Lemma 4.3.2. *If $\overline{\mathcal{P}}$ is a protosheaf, $\tilde{\tilde{\overline{P}}}$ is flabby.*

Proof. By definition,we have

$$\Gamma(U, \tilde{\tilde{\overline{P}}}) \approx \mathrm{Hom}(K_U, \tilde{\tilde{\overline{P}}}) \approx \mathrm{Hom}(\overline{K}_U, \overline{P}).$$

Any map $\overline{K}_U \overline{P}$ extends by zero to a map $\overline{K} \to \overline{P}$. Tracing this back through the isomorphisms, $\tilde{\overline{P}}$ is flabby. □

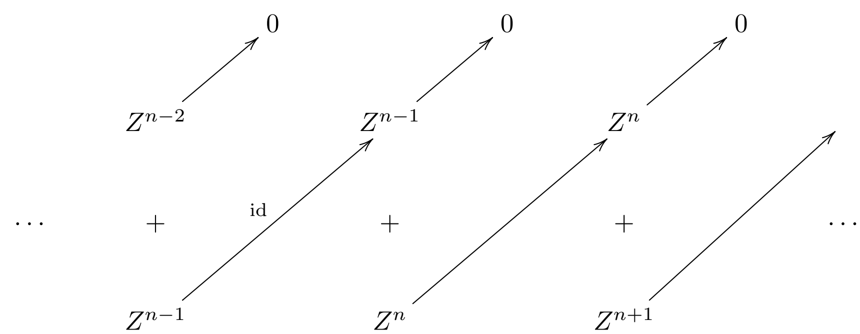
Lemma 4.3.3. *There is a natural inclusion $i\colon F \subset \overline{\widetilde{\overline{F}}}$, for all sheaves F, such that F_x is naturally a direct summand of $\widetilde{\overline{F}}_x$; i.e., $\overline{F}$ is naturally a direct summand of $\overline{\widetilde{\overline{F}}}$.*

Proof. Under the isomorphism of lemma 2.6.1, let $i\colon F \to F$ correspond to the identity map $F \to F$.

For convenience, let $G = \widetilde{\overline{F}}$. We wish to define a projection $r\colon \overline{G} \to \overline{F}$ such that $r\bar{i} = \text{id}$. Let r correspond to the identity map $G \to \widetilde{\overline{F}}$.

The composition $F \xrightarrow{\bar{i}} \widetilde{\overline{F}}$ is simply i. By lemma 4.3.2, the composition $F \xrightarrow{\bar{i}} \overline{\widetilde{\overline{F}}} \xrightarrow{r} \overline{F}$ is the identity. □

Definition 4.3.4. An acyclic complex $\cdots \xrightarrow{\delta} F^n \xrightarrow{\delta} F^{n+1} \cdots$ *splits* if each Z^n is a direct summand of F^n. In this case $F^n/Z^n \approx Z^{n+1}$ and the complex is isomorphic to one built up out of elementary complexes $\cdots 0 \to Z^n \to Z^{n+1} \to 0 \cdots$ in the following way:



Equivalently, the complex splits if there are maps $s\colon F^{n+1} \to F^n$ such that $\delta s + s\delta = \text{id}$. For complex $0 \to F \xrightarrow{\epsilon} C^0 \xrightarrow{\delta} C^1 \to \cdots$ we have instead maps $s\colon C^{i+1} \to C^i$, $\eta\colon C^0 \to F$ such that

$$\begin{aligned} \delta s + s\delta &= \text{id} && \text{in} \quad \dim \neq 0 \\ s\delta &= 1 - \epsilon\eta && \text{in} \quad \dim = 0. \end{aligned}$$

Definition 4.3.5. A complex $\cdots \to F^n \xrightarrow{\delta} F^{n+1} \to \cdots$ is *algebraically split* if the complex of protosheaves $\cdots \to \overline{F}^n \xrightarrow{\bar{\delta}} \overline{F}^{n+1} \to \cdots$ splits.

Lemma 4.3.6. *If $\cdots \to F^n \to F^{n+1} \to \cdots$ is algebraically split and G is any sheaf, then $\cdots \to F^n \otimes G \xrightarrow{\delta\otimes 1} F^{n+1} \otimes \to \cdots$ is algebraically split.*

Proof. Obvious. □

Now if F is a sheaf, let $P(F) = \widetilde{\overline{F}}$ and $Q(F) = \operatorname{coker} i = \widetilde{\overline{F}}/F$. Then $0 \to F \xrightarrow{i} P(F) \to Q(F) \to 0$ is algebraically split (by lemma 4.3.3). So is

$0 \to Q(F) \to PQ(F) \to Q^2(F) \to 0$. Continuing, and sticking the sequences together in the standard way, we obtain a complex

$$0 \to F \to P(F) \to PQ(F) \to PQ^2(F) \to \cdots$$

which is naturally, algebraically split. This complex is a natural Γ_Φ-resolution.

i. The sequence is exact, since it is algebraically split.

ii. PQ^n is a functor; moreover an exact functor. $\overline{P(F)} \approx \overline{F} \oplus \overline{Q(F)}$ naturally. P and $-$ are exact functors. Therefore $\overline{Q}$ is exact, hence Q is exact,

iii. $PQ^n(F)$ is flabby, by lemma 4.3.2. Therefore $R^i\Gamma_\Phi(F) = H^i_\Phi(X, PQ^n(F)) = 0$.

This complex is called the *canonical flabby resolution* of F.

Remark 4.3.7. Let M be a constant sheaf. Then $\widetilde{\overline{M}}$ is the sheaf of germs of functions with values in the module M.

Lemma 4.3.8. *If* $0 \to A \xrightarrow{i'} B \xrightarrow{j'} C \to 0$ *is algebraically split, there is a map*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i'} & B & \xrightarrow{j'} & C & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \longrightarrow & P(A) & \longrightarrow & Q(A) & \longrightarrow & 0
\end{array}$$

Proof. Let $k\colon \overline{B} \to \overline{A}$ be a splitting of $O \to \overline{A} \to \overline{B} \to \overline{C} \to 0$. Since $\mathrm{Hom}(\overline{B}, \overline{A}) = \mathrm{Hom}(B, \widetilde{\overline{A}})$, k induces a map $f\colon B\widetilde{\overline{A}} = P(A)$. Since $ki' = \mathrm{id}$, $fi' \in \mathrm{Hom}(A, \widetilde{\overline{A}})$ corresponds to $\mathrm{id} \in \mathrm{Hom}(\overline{A}, \overline{A})$, thus $fi' = i$.

The map g is obtained by taking quotients. □

Proposition 4.3.9. *Any algebraically split resolution of A can be mapped into the canonical flabby resolution of A.*

Proof. Let $0 \to A \to C^0 \to C^1 \to \cdots$ be the resolution. Then $0 \to A \to C^0 \to Z^1 \to 0$, $0 \to Z^1 \to C^1 \to Z^2 \to$ etc., are algebraically split.

We use lemma 4.3.8 recursively to get maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & C^0 & \longrightarrow & Z^1 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle f^0} & & \big\downarrow{\scriptstyle g^1} & & \\
0 & \longrightarrow & A & \longrightarrow & P(A) & \longrightarrow & Q(A) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X^i & \longrightarrow & C^i & \longrightarrow & Z^{i+1} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle g^i} & & \big\downarrow{\scriptstyle f^i} & & \big\downarrow{\scriptstyle g^{i+1}} & & \\
0 & \longrightarrow & Q^iA & \longrightarrow & PQ^i(A) & \longrightarrow & Q^{i+1}(A) & \longrightarrow & 0
\end{array}$$

These fit together to give a map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & \cdots \\
 & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \\
0 & \longrightarrow & A & \longrightarrow & P(A) & \longrightarrow & PQ(A) & \longrightarrow & \cdots
\end{array}$$

□

Remark 4.3.10. The sheaf of Alexander-Spanier Cochains is algebraically split. For the homotopies defined in chapter 2, example (vii) in 2.2.3 gives the splitting.

4.4 Cup products

4.4.1 Genralities on cup products

Let $\mathcal{A}$, $\mathcal{B}$ be exact K-categories admitting bilinear maps and tensor products. (We do not formulate exactly what this would mean; we might take $\mathcal{A}$ to be, e.g., the category of sheaves, $\mathcal{B}$ that of protosheaves, stacks, or modules.)

Let T_1, T_2, T be covariant K-functors $\mathcal{A} \to \mathcal{B}$ and assume that for $A, A' \in \mathcal{A}$ there is a natural map $T_1(A) \otimes T_2(A*') \xrightarrow{\eta_{A,A'}} T(A \otimes A')$. I.e., if $f\colon A \to B, g\colon A' \to B'$ are maps in $\mathcal{A}$, the following diagramme shall commute:

$$\begin{array}{ccc}
T_1(A) \otimes T_2(A') & \xrightarrow{\eta_{A,A'}} & T(A \otimes A') \\
{\scriptstyle T_1(f)\otimes T_2(g)}\downarrow & & \downarrow{\scriptstyle T(f\otimes g)} \\
T_1(B) \otimes T_2(B') & \xrightarrow[\eta_{B,B'}]{} & T(B \otimes B')
\end{array}$$

Let T_1^*, T_2^*, T^* be augmented δ-functors over T_1, T_2 and T respectively. A *cup product* for these functors (and η) shall be a natural map

$$\eta^{p,q}_{A,A'}\colon T_1^p(A) \otimes T_2^q(A') \to T^{p+q}(A \otimes A')$$

for each $A, A' \in \mathcal{A}$ which shall satisfy the following axioms:

I Let $p = q = 0$. Then the following diagramme commutes:

$$\begin{array}{ccc}
T_1^0(A) \otimes T_2^0(A') & \xrightarrow{\eta^{0,0}} & T^0(A \otimes A') \\
{\scriptstyle \epsilon\otimes\epsilon}\uparrow & & \uparrow{\scriptstyle \epsilon} \\
T_1(A) \otimes T_2(A') & \xrightarrow[\eta]{} & T(A \otimes A')
\end{array}$$

II Let $0 \to A' \to A \to A'' \to 0$ be exact. Let $B \in \mathcal{A}$ and consider the commutative diagramme:

$$\begin{array}{ccccccccc} & & A' \otimes B & \longrightarrow & A \otimes B & \longrightarrow & A'' \otimes B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} T_1^p(A'') \otimes T_2^q(B) & \xrightarrow{\delta \otimes 1} & T_1^{p+1}(A'') \otimes T_2^q(B) \\ \downarrow & & \downarrow \\ T^{p+q}(C'') & \xrightarrow[\delta]{} & T^{p+q+1}(C') \end{array}$$

The vertical maps are cup products followed by maps of the first diagramme.

III Let $0 \to B' \to B \to B'' \to 0$ be exact in $\mathcal{A}$. Let $A \in \mathcal{A}$ and consider the commutative diagramme:

$$\begin{array}{ccccccccc} & & A \otimes B' & \longrightarrow & A \otimes B & \longrightarrow & A \otimes B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

We use the notation w for the map $T^* \to T^*$ which is $(-l)^p$ in dimension p. Then we require the following diagramme to be commutative:

$$\begin{array}{ccc} T_1^p(A) \otimes T_2^q(B'') & \xrightarrow{w \otimes \delta} & T_1^p(A'') \otimes T_2^{q+1}(B) \\ \downarrow & & \downarrow \\ T^{p+q}(C'') & \xrightarrow[\delta]{} & T^{p+q+1}(C') \end{array}$$

where the vertical maps are similar to those in II.

We wish to prove the theorem.

Theorem 4.4.1. *Let T_1^*, T_2^*, T^* be cohomological δ-functors over T_1, T_2 and T admitting cup products. Let $S_1^*, S_{,}^*$ and S be augmented δ-functors over S_1, S_2 and S admitting cup products, and let $f_1 \colon T_1 \to S_1$, $f_2 \colon T_2 \to S_2$ and $f \colon T \to S$ be natural transformations such that the following diagramme commutes:*

$$\begin{array}{ccc} T_1(A) \otimes T_2(A') & \xrightarrow{\eta_{A,A'}} & T(A \otimes A') \\ {\scriptstyle f_1 \otimes f_2}\downarrow & & \downarrow{\scriptstyle f} \\ S_1(A) \otimes S_2(A') & \xrightarrow[\eta_{A,A'}]{} & S(A \otimes A') \end{array}$$

Then the unique extensions $f_1^*: T_1^* \to S_1^*, f_2^*: T_2^* \to S_2^*$, and $f^*: T^* \to S^*$ given by theorem III preserve cup products; i.e., the following diagramme commutes:

$$\begin{array}{ccc} T_1^p(A) \otimes T_2^q(A') & \longrightarrow & T^{p+q}(A \otimes A') \\ \downarrow & & \downarrow \\ S_1^p(A) \otimes T_2^q(A') & \longrightarrow & S^{p+q}(A \otimes A') \end{array}$$

Unfortunately, to prove this we need to make an assumption about the category $\mathcal{A}$. We assume that for any $A \in \mathcal{A}$ and each T_i there is a short exact sequence $0 \to A \to P \to Q \to 0$ such that

(i) $T_i^j(P) = 0$ for $j > 0$,

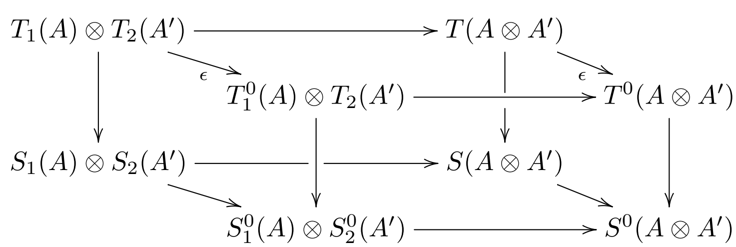
(ii) for any $B \in \mathcal{A}$, $0 \to A \otimes B \to P \otimes B \to Q \otimes B \to 0$ is exact.

The exact sequence $0 \to A \to P \to Q \to 0$ may be different for T_1 and T_2. This assumption is certainly true for sheaves, when $T_1 = \Gamma_{\Phi_1}, T_2 = \Gamma_{\Phi_2}$, and $T = \Gamma_\Phi$ where $\Phi = \Phi_1 \cap \Phi_2$, since for any sheaf F we can take the short exact sequence to be $0 \to F \to P(F) \to Q(F) \to 0$ (see later for more details). It also works for modules over group rings, when the T^*'s are cohomologies of groups; cf. CE Chapter XII, Section 5.

Proof. Let (p, q) denote the commutativity of the diagram of the theorem for the case p, q.

We prove $(0, 0)$, and then $(p, q) \Rightarrow (p + 1, q)$ and $(p, q) \Rightarrow (p, q + 1)$.

a. $(0, 0)$:
Consider the following cube:

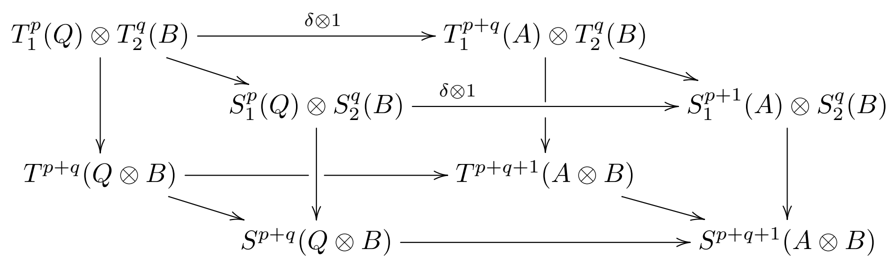


(4.4.2)

We wish to show the front face commutes. By the assumptions of the theorem, all the other faces commute. But the maps written above as ϵ are isomorphisms. Therefore the front face commutes,

b. $(p, q) \Rightarrow (p + 1, q)$:
Let $0 \to A \to P \to Q \to 0$ satisfy our assumptions (i) and (ii) for T_1^*. Let $B \in \mathcal{A}$. Then from the identity map of $0 \to A \otimes B \to P \otimes B \to Q \otimes B \to 0$

we obtain a cube:



(4.4.3)

We must show the right-hand face commutes. By our assumption and by naturality and such, all the others do. Therefore the diagram commutes as far as paths starting from $T_1^p(Q) \otimes T_2^q(B)$ are concerned. But, since $T_1^i(P) = 0$ and $\otimes$ is right-exact, $\delta \otimes 1$ is epi. Therefore the right-hand face commutes.

c. $(p, q) \Rightarrow (p, q+1)$:
The argument is almost exactly the same as (b) with a map $w \otimes \delta$ replacing $\delta \otimes 1$ and with care for signs. We leave this to the reader. The theorem now follows by induction.

□

4.4.2 Cup products for sheaves

Now let $\mathcal{A}$ be the category of sheaves and $\mathcal{B}$ the category of modules. Let Φ_1, Φ_2 be families of supports for X. Then $\Phi = \Phi_1 \cap \Phi_2$ is also a family of supports. Let $T_1 = \Gamma_{\Phi_1}, T_2 = \Gamma_{\Phi_2}$, and $T = \Gamma_\Phi$. If F, G are sheaves, there is a natural pairing $\Gamma_{\Phi_1}(F) \otimes \Gamma_{\Phi_2}(G) \to \Gamma_\Phi(F \otimes G)$. For if $s \in \Gamma_{\Phi_1}(F), t \in \Gamma_{\Phi_2}(G)$, define $s \otimes t \in \Gamma_\Phi(F \otimes G)$ by $(s \otimes t)(x) = s(x) \otimes t(x)$. Clearly $|s \otimes t| \subset |s| \cap |t|$ and so $|s \otimes t| \in \Phi$. The pairing is obviously natural. We can take the T_α^*'s to be $H_\Phi(X, -)$ with $\alpha = 1, 2$ or nothing. Cup products would now be pairings:

$$H^p_{\Phi_1}(X, F) \otimes H^q_{\Phi_2}(X, G) \to H^{p+q}_\Phi(X, F \otimes G).$$

The last theorem shows that if cup products exist, they are unique; our assumption about the category $\mathcal{A}$ is satisfied by taking the short exact sequence to be $0 \to F \to P(F) \to Q(F) \to 0$. This is algebraically split, and so satisfies condition (b). $P(F)$ is flabby, so for any family of supports $H^p_\Phi(X, P(f)) = 0$, for $p > 0$.

Theorem 4.4.4. *There are cup products over the above functors and maps.*

For a sheaf F, let $C(F)$ be its canonical flabby resolution. If F, G are sheaves, the complex $0 \to F \otimes G \xrightarrow{\epsilon \otimes epsilon} C(F) \otimes C(G)$ is an algebraically split right-complex over $F \otimes G$. For if we use η's to denote the splitting maps of both $C(F)$ and $C(G)$, the splitting maps for $C(F) \otimes C(G)$ are $\eta \otimes \eta$ and $s \otimes 1 + \epsilon\eta \otimes s$.

Proof. Let $0 \to F \otimes G \to I$ be an injective resolution of $F \otimes G$. Then we have a map, unique up to homotopy

$$\begin{array}{ccccc} & & & & C(F)\otimes C(G) \\ & & & \nearrow & \downarrow \\ 0 & \longrightarrow & F\otimes G & & \downarrow \\ & & & \searrow & \downarrow \\ & & & & I \end{array}$$

So we have natural maps

$$\Gamma_{\Phi_1}(C(F)) \otimes \Gamma_{\Phi_2}(C(G)) \to \Gamma_{\Phi}(C(F) \otimes C(G)) \to \Gamma_{\Phi}(I).$$

Taking cohomologies and using the natural map of the Künneth formulas which injects a product of homologies into the homology of the product, we obtain a well-defined map,

$$H^p_{\Phi_1}(X, F) \otimes H_{\Phi_2}(X, G) \to H^{p+q}_{\Phi}(X, F \otimes G).$$

This map is easily seen to be natural. We verify the axioms for cup products.
I: We have a cochain map

$$\begin{array}{ccccc} 0 & \longrightarrow & \Gamma_{\Phi_1}(F) \otimes \Gamma_{\Phi_2}(G) & \longrightarrow & \Gamma_{\Phi_1}(C(F)) \otimes \Gamma_{\Phi_2}(C(G)) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma_{\Phi}(f \otimes G) & \longrightarrow & \Gamma_{\Phi}((C(F) \otimes C(G)) \end{array}$$

This induces

$$\begin{array}{ccccc} 0 & \longrightarrow & \Gamma_{\Phi_1}(F) \otimes \Gamma_{\Phi_2}(G) & \longrightarrow & H^0(\Gamma_{\Phi_1}(C(F)) \otimes \Gamma_{\Phi_2}(C(G))) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma_{\Phi}(F \otimes G) & \longrightarrow & H^0(C(F) \otimes C(G)) \end{array}$$

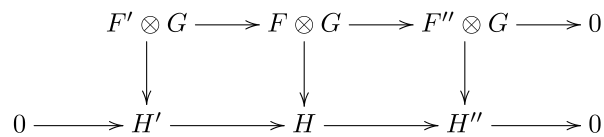
The result follows from the diagramme.

$$\begin{array}{ccccc} & & & & \Gamma_{\Phi}(C(F)\otimes C(G)) \\ & & & \nearrow & \downarrow \\ 0 & \longrightarrow & \Gamma_{\Phi}(F\otimes G) & & \downarrow \\ & & & \searrow & \downarrow \\ & & & & \Gamma_{\Phi}(I) \end{array}$$

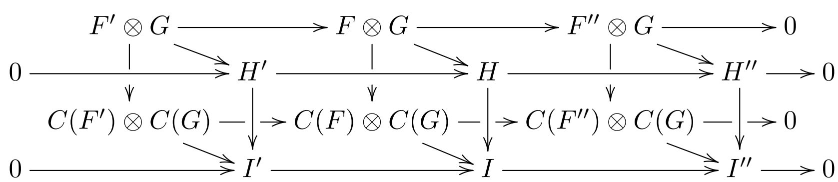
Note the fact that

$$H^0(\Gamma_{\Phi_1}(C(F)) \otimes \Gamma_{\Phi_2}(C(G))) \approx H^0_{\Phi_1}(F) \otimes H^0_{\Phi_2}(G)$$

II: Let $0 \to F' \to F \to F'' \to 0$ be exact. Consider the commutative diagramme



If we take injective resolutions I', I, I'' of H', H, H'', we can, by theorem 3.2.12, obtain a commutative diagram



It is easily seen that this induces the correct commutativity relation for δ and $\delta \times 1$.

III: We leave the verification to the reader. □

Chapter 5

Some Classes of Φ-acyclic Sheaves

5.1 Φ-acyclic Sheaves

A sheaf $\mathscr{F}$ is called Φ-*acyclic* if $H^p_\Phi(X, F) = 0$ for all $p \neq 0$. We have already met two classes of sheaves with this property, namely, injective sheaves and flabby sheaves. These are Φ-acyclic for all Φ. We now introduce a larger class of sheaves, which we will call Φ-*soft*. In contrast to the definition of flabby sheaves, the definition of Φ-soft sheaves will involve a specific family Φ. These sheaves will not be Φ-acyclic for all families Φ. We must assume that Φ satisfies the conditions of the following definition:

Definition 5.1.1. Φ is *paracompactifying* (French: paracompactifiante) if

(a) Each $A \in \Phi$ has a neighbourhood in Φ; i.e., there is a $B \in \Phi$ such that $A \subset \operatorname{int} B$, and

(b) Each $A \in \Phi$ is paracompact.

I will generally abbreviate this by writing is "Φ is PF". Note that in Cartan's theory of sheaves, only paracompactifying families of supports were considered.

Before defining Φ-soft sheaves, I will give some definitions and lemmas which are basic in all arguments involving paracompactifying families of supports. By covering, I always mean an open covering.

Definition 5.1.2. A covering $\{U_\alpha\}$ of X is called *shrinkable* if there is another covering $\{V_\alpha\}$ such that $\overline{V}_\alpha \subset U_\alpha$ for all α.

A classical theorem states that every point-finite covering of a normal space is shrinkable.

Definition 5.1.3. A Φ-covering of X is a *locally finite* covering $\{U_\alpha \mid \alpha \in I\}$ of X such that there is a set $C \in \Phi$ and an element $\alpha_0 \in I$ with the property that $U_\alpha \subset C$ for all $\alpha \neq \alpha_0$. We refer to U_{α_0} as the *exceptional set*.

Remark 5.1.4. U_{α_0} is unique unless $X \in \Phi$ in which case any a will do. In this case we assume some definite choice for α_0.

Lemma 5.1.5. *If Φ is PF, any Φ-covering is shrinkable.*

Proof. Let the notation be as in the definition of Φ-coverings. $\{U_\alpha \cap C\}$ is a locally finite covering of C. But, C is paracompact and hence normal. Therefore, we can shrink this covering to $\{V_{\alpha'}\}$. Define $V_\alpha = V'_\alpha$ for $\alpha \neq \alpha_0$ and $V_{\alpha_0} = V'_{\alpha_0} \cup X \setminus C$. Then $\{V_\alpha\}$ is the required shrinking of $\{U_\alpha\}$. □

Remark 5.1.6. A covering obtained by shrinking a Φ-covering is obviously again a Φ-covering.

Lemma 5.1.7. *Let Φ be PF.*

Let $\mathcal{M}$ be any covering of X which includes a set $X \setminus A$ with $A \in \Phi$. Then $\mathcal{M}$ is refined by a Φ-covering the exceptional set being contained in $X \setminus A$.

Proof. A has a neighbourhood C in Φ. For each $x \in X$, choose a neighbourhood N_x such that N_x is contained in some set of $\mathcal{M}$. If $x \in \operatorname{int} C$, take N_x so small that $N_x \subset \operatorname{int} C$. Now, $\{N_x \cap C\}$ covers C which is paracompact. Take a locally finite refinement $\mathcal{N}$ of this covering. Throw away all sets of $\mathcal{N}$ not contained in $\operatorname{int} C$. Then add $X \setminus A$ to get the required Φ-covering, the exceptional set being $X \setminus A$. □

This lemma is usually applied to the following situation;

We have a sheaf F over X, a covering of X and a section $s_\beta \in \Gamma(N_\beta, F)$ for each β. We assume that some has the form $X \setminus A$ with $A \in \Phi$ and that $s_\beta = 0$.

We want to fit the sections s_β together to get a section $s \in \Gamma(X, F)$ with support in Φ. To do this, we take a Φ-covering $\{U_\alpha\}$ which refines $\{N_\alpha\}$. For each α, we choose a β such that $U_\alpha \in N_\beta$ and define $t_\alpha = s_\beta \mid U_\alpha$. Of course,we must be careful to choose N_{β_0} to be the set containing the exceptional U_{α_0}. This insures that t_{α_0} is zero. We now make use of the local finiteness of $\{U_\alpha\}$ to fit the t_α's together. The fact that $t_\alpha = 0$ while the U_α with $\alpha \neq \alpha_0$ are in C insures that the resulting section has its support contained in C. Therefore this support will be in Φ.

A simple example of this argument is given by the proof of the next lemma. In this lemma, no family of supports is used, but except for this, the argument is exactly as above.

Lemma 5.1.8. *Let X be paracompact and A a closed subset of X. Let F be any sheaf over X. Then any section of F over A can be extended to a neighbourhood of A.*

Proof. Assume $s \in \Gamma(A, F)$ is given. Every point $x \in A$ has a neighbourhood N_x so small that there is a section $s_x \in \Gamma(N_x, F)$ such that s_x and s agree over $N_x \cap A$. To find s_x, it is sufficient to find a section t_x over some neighbourhood M of x such that $t_x(x) = s(x)$ and then take N to be $(M_x \setminus A) \cup \{y \in A \mid t_x(y) = s(y)\}$.

For $x \notin A$, choose $N_x = X \setminus A$ and $s_x = 0$. Then $\{N_x\}$ covers X. Take a locally finite refinement $\{U_\alpha\}$ and apply the above argument to get sections $t_\alpha \in \Gamma(U_\alpha, F)$ such that $t_\alpha \mid U_\alpha \cap A = s \mid U_\alpha \cap A$.

Shrink $\{U_\alpha\}$ to a covering $\{V_\alpha\}$ such that $\overline{V}_\alpha \subset U_\alpha$. Let W be the set of x such that the $t_\alpha(x)$ have the same value for all α such that $x \in \overline{V}_\alpha$. If $x \in W$, let $t(x)$ be this common value, i.e., $t(x) = t_\alpha(x)$ for any α such that $x \in \overline{V}_\alpha$.

Obviously, $W \supset A$ and $t \mid A = s$. We must show W is open and t is continuous. If $x \in W$, some neighbourhood meets only a finite number of U_α. A smaller neighbourhood meets only those $\overline{V}_\alpha$ such that $x \in \overline{V}_\alpha$. At points y in this neighbourhood, only those α such that $x \in \overline{V}_\alpha$ are used to decide whether $y \in W$ and to determine $t(y)$. Now, these t_α agree at x and therefore in a neighbourhood of x. This neighbourhood is contained in W. Therefore W is open. Finally, in this neighbourhood, t is obtained by piecing together a finite number of t_α's each defined over a set relatively closed in this neighbourhood (i.e., the intersection of the neighbourhood with a $\overline{V}_\alpha$). Therefore t is continuous. □

Corollary 5.1.9. *Let A be closed in X and have a paracompact neighbourhood in X. Let F be any sheaf over X. Then any section of F over A extends to a neighbourhood of A.*

Proof. If B is a paracompact neighbourhood of A, we apply the lemma with B in place of X. □

5.2 Φ-soft Sheaves

We now define Φ-soft sheaves. There are four possible definitions. We will show that these all coincide if Φ is PF or, more generally, if every set of Φ has a neighbourhood in Φ (the sets of Φ not being assumed paracompact).

Definition 5.2.1. Let Φ be any family of supports in X. Let F be a sheaf over X. Then

(a) F is Φ-soft_1 if, whenever A is a closed set of X and s is a section of F over A with support in Φ, we can extend s to a section over X with support in Φ;

(b) F is Φ-soft_2 if, whenever $A \in \Phi$ and s is a section of F over A, we can extend s to a section over X with support in Φ;

(c) F is Φ-soft_3 if, whenever $A \in \Phi$ any section of F over A extends to a section of F over X;

(d) F is Φ-soft_4 if, whenever $A, B \in \Phi$ and $A \subset B$, any section of F over A extends to a section of F over B.

Remark 5.2.2. Clearly Φ-soft_1 ⇒ Φ-soft_2 ⇒ Φ-soft_3 ⇒ Φ-soft_4. Grothendieck defines Φ-soft to be Φ-soft_2.

Proposition 5.2.3. *Suppose every set in Φ has a neighbourhood in Φ. Then all four definitions of Φ-soft agree.*

Proof. It is sufficient to show that Φ-soft$_4 \Rightarrow \Phi$-soft$_1$. Let A be closed in X and $s \in \Gamma(A, F)$ with support $|s| \in \Phi$. Let $B \in \Phi$ be a neighbourhood of $|s|$ and let $C \in \Phi$ be a neighbourhood of B. The set $D = (A \cap C) \cup (C \setminus \operatorname{int} B)$ is a closed subset of C and so is in Φ. We define a section s' over D by $s' \mid (A \cap C) = s \mid (A \cap C)$ and $s' \mid (C \setminus \operatorname{int} B) = 0$. Then s' is obviously continuous since s is zero outside $\operatorname{int} B$.

Extend s' to a section s'' over C using the property Φ-soft$_4$. Then extend s'' to a section t over X by $t \mid C = s''$, $t \mid (X \setminus \operatorname{int} B) = 0$. Then t is continuous because s'' is zero outside $\operatorname{int} B$. Clearly $|t| \subset B$ and so $|t| \in \Phi$. Finally, $t \mid (A \cap C) = s \mid (A \cap C)$ and $t \mid (A \setminus \operatorname{int} B) = 0 = s \mid (A \setminus \operatorname{int} B)$. Therefore $t \mid A = s$. □

Proposition 5.2.4. *If Φ is PF, flabby implies Φ-soft.*

Proof. Let F be flabby. We show F is Φ-soft$_3$. Let s be a section of F over $A \in \Phi$. Since Φ is PF, A has a paracompact neighbourhood in X. Therefore, s extends to a section s' over an open neighbourhood U of A. We then extend s' to X by using the flabbiness of F. □

Corollary 5.2.5. *If Φ is PF, injective implies Φ-soft.*

Proof. Injective always implies flabby. □

Proposition 5.2.6. *Assume Φ is PF. Then the following hold:*

(1) If $0 \to F' \xrightarrow{i} F \xrightarrow{j} F'' \to 0$ is exact and F' is Φ-soft, then $0 \to \Gamma_\Phi(F') \to \Gamma_\Phi(F) \to \Gamma_\Phi(F'') \to 0$ is exact,

(2) Assume $0 \to F' \to F \to F'' \to 0$ is exact with F' and F be Φ-soft. Then F'' is Φ-soft.

Proof. We first show that (1) implies (2).

Let $A \in \Phi$ and $s \in \Gamma(A, F'')$.

The sequence $0 \to F' \mid A \to F \mid A \to F'' \mid A \to 0$ is exact and $F' \mid A$ is Φ_A-soft where $\Phi_A = \{C \in \Phi \mid C \subset A\}$. To see this, use property Φ-soft$_3$. Since Φ_A is cleanly PF, we can find, using (1), $t \in \Gamma(A, F \mid A)$ such that $j(t) = s$. We now extend t to X, using the fact that F is Φ-soft. Finally, we get the required extension of s by applying j to this extension of t.

We must now prove (1). Since Γ_Φ is left-exact, we only have to show $\Gamma_\Phi(F) \to \Gamma_\Phi(F''))$ is an epimorphism.

Let $s \in \Gamma_\Phi(F'')$. Let $A = |s|$. For each $x \in X$, there is a neighbourhood N_x and a section $s_X \in \Gamma(N_x, F)$ such that $js_x = s \mid N_x$. If $x \notin A$, we choose $N_x = X \setminus A$ and $s_x = 0$. Applying the standard argument, we find a Φ-covering $\{U_\alpha\}$ and sections $t_\alpha \in \Gamma(U_\alpha, F)$ such that $jt_\alpha = s \mid U_\alpha$ and such that $t_{\alpha_0} = 0$ for the exceptional α_0.

Shrink $\{U_\alpha\}$ to a covering $\{V_\alpha\}$. Consider sets which are unions of some of the $\overline{V}_\alpha$'s. These are closed since $\{\overline{V}_\alpha\}$ is locally finite. Consider sections t over such sets such that $t \mid \overline{V}_\alpha - t_\alpha$, when defined, is a section of $F' \subset F$.

Order these t's by extension. Zorn's lemma applies because the local finiteness of $\{\overline{V}_\alpha\}$ insures that the upper bound of an increasing chain of t's will be continuous.

Start with the zero section over $\{\overline{V}_\alpha\}$ and take a maximal extension t. Suppose we can show that t is defined over all of X. Then $|t| \in \Phi$ because t is zero over the exceptional set $\overline{V}_{\alpha_0}$ and so $|t| \subset C$ where $C \in \Phi$ is the set considered in the definition of a Φ-covering. Also, $j(t) \cdot (x) = j(t_\alpha +$ section of $F')$. $(x) = j(t_\alpha) \cdot (x) = s(x)$ (where $x \in V_\alpha$). Therefore, we have only to show that t is defined over all of X.

Let D be the domain of t. Suppose $\overline{V}_{\alpha'} \not\subset D$. Then $\alpha' \neq \alpha_0$, therefore $V_{\alpha'} \subset C$. Thus $\overline{V}_{\alpha'} \in \Phi$. And so $D \cap \overline{V}_{\alpha'} \in \Phi$ also.

Now, $t \mid D \cap \overline{V}_{\alpha'} - t_{\alpha'} \mid D \cap \overline{V}_{\alpha'}$, is a section of F'. Extend it to $\overline{V}_{\alpha'}$, using the Φ-softness of F'. This gives a section r of F' over $\overline{V}_{\alpha'}$. But, $t_{\alpha'} + r$ is a section of F over $\overline{V}_{\alpha'}$, which agrees with t over $D \cap \overline{V}_{\alpha'}$. Therefore we can extend t to $D \cup \overline{V}_{\alpha'}$ by taking t over D and $t_{\alpha'} + r$ over $\overline{V}_{\alpha'}$. This gives a contradiction since t was assumed maximal. □

Corollary 5.2.7. *If Φ is PF and F is Φ-soft, then F is Φ-acyclic.*

Proof. This follows from the proposition and the fact that injective implies Φ-soft. We use lemma 3.3.19, the same lemma which was used to show that flabby implies Φ-acyclic. □

5.3 Fine Sheaves

If Φ is PF, another class of Φ-acyclic sheaves is the class of fine sheaves.

Definition 5.3.1. A sheaf F over X is fine if, for every locally finite covering Â£ua3 of X, there exist endomorphisms la :F -Â» F such that

(a) $|1_\alpha| \subset \overline{U}_\alpha$,

(b) $\sum 1_\alpha = \mathrm{id}$.

Note that (a) and the local finiteness of $\{U_\alpha\}$ imply that the $\sum 1_\alpha$ is well defined.

The existence of many fine sheaves is given by the following proposition.

Proposition 5.3.2. *If $\overline{M}$ is any protosheaf, then $\widetilde{\overline{M}}$ is fine.*

Proof. Let $\{U_\alpha\}$ be a locally finite covering of X. For each x, choose some α_x such that $x \in U_{\alpha_x}$. Define endomorphisms $1'_\alpha \colon \overline{M} \to \overline{M}$ as follows:

$$1_{\alpha'} \mid \overline{M}_x = \begin{cases} \mathrm{id} & \text{if } \alpha = \alpha_x \\ 0 & \text{if } \alpha \neq \alpha_x \end{cases}$$

These l'_α's induce maps $l_\alpha\colon \overline{M} \to \widetilde{\widetilde{M}}$ since $\sim$ is a functor. Suppose $x \notin \overline{U}_\alpha$. Since $1'_\alpha$ is zero on all $\overline{M}_y$ with $y \in X \setminus \overline{U}_\alpha$, $1'_\alpha$ is zero over a neighbourhood of x. Therefore the same is true of 1_α and so, $x \notin |1_\alpha|$. This shows $|1_\alpha| \subset \overline{U}_\alpha$. In a small neighbourhood of any x, only a finite number of $1'_\alpha$ will be non-zero and the sum of the $1_{\alpha'}$ will be the identity. Consequently, $\sum 1_\alpha = \mathrm{id}$ also. □

Corollary 5.3.3. *Every sheaf F can be imbedded in a fine sheaf.*

Proof. Take the canonical imbedding $F \to \widetilde{\widetilde{F}}$. □

Lemma 5.3.4. *If $F \subset G$ is a direct summand of G and G is a fine sheaf, then F is fine.*

Proof. Let $r\colon G \to F$ be a "retraction", i.e., a sheaf map such that $r \mid F = \mathrm{id}$. Let $i\colon F \to G$ be the inclusion. Let $\{U_\alpha\}$ be a locally finite covering of X. Find endomorphisms $1_\alpha\colon G \to G$ such that $|1_\alpha| \subset \overline{U}_\alpha$ and $\sum 1_\alpha = \mathrm{id}$. Then the maps $r1_\alpha\colon F \to F$ obviously satisfy $|r1_\alpha i| \subset \overline{U}_\alpha$ and $\sum r1_\alpha i = ri = \mathrm{id}$. □

Corollary 5.3.5. *Every injective sheaf is fine.*

Proof. Let I be injective. Imbed I in a fine sheaf G. Since I is injective, it is a direct summand of G. □

Proposition 5.3.6. *If Φ is PF and F is fine, then F is Φ-soft.*

Proof. Let $s \in \Gamma(A, F)$ with $A \in \Phi$. For each $x \in A$, there is a neighbourhood N_x and a section $s_x \in \Gamma(N_x, F)$such that $s_x \mid N_x \cap A = s_x \mid N_x \cap A$. For $x \notin A$, choose $N_x = X \setminus A$ and $s_x = 0$. In the usual way, we find a Φ-covering $\{U_\alpha\}$ of X and sections $t \in \Gamma(U, F)$ such that $t\alpha \mid U_\alpha \cap A = s \mid U_\alpha \cap A$ and such that $t_{\alpha_0} = 0$ for the exceptional α_0.

Shrink $\{U_\alpha\}$ to $\{v_\alpha\}$. Using the fineness of F, we get endomorphisms $1_\alpha\colon F \to F$ such that $|1_\alpha| \subset \overline{V}_\alpha$ and $\sum 1_\alpha = \mathrm{id}$. Define $t(x) = \sum 1_\alpha t_\alpha(x)$. This makes sense because t_α is defined over U_α while 1_α is zero outside $\overline{V}_\alpha \subset U_\alpha$. Obviously t is continuous and $t \mid A = s$. This shows that F is Φ-soft$_3$. Note that $|t| \subset C$ (the set occurring in the definition of Φ-coverings). Therefore $|t| \in \Phi$, and so we have shown directly that F is Φ-soft$_2$- □

Corollary 5.3.7. *If Φ is PF and F is fine, then F is Φ-acyclic.*

It is possible to prove a generalisation of this corollary which will be useful in the applications involving singular chains.

Definition 5.3.8. Let F be a sheaf over X. Let $\mathcal{C}$ be a collection of endomorphisms of F. We say that F is $\mathcal{C}$-fine if, given any locally finite covering $\{U_\alpha\}$ of X, there are endomorphisms $1_\alpha\colon F \to F$ such that

(a) $|1_\alpha| \subset \overline{U}_\alpha$,

(b) $\sum 1_\alpha \in \mathcal{C}$

For example, if $\mathcal{C}$ contains only the identity map, then $\mathcal{C}$-fine is the same as fine.

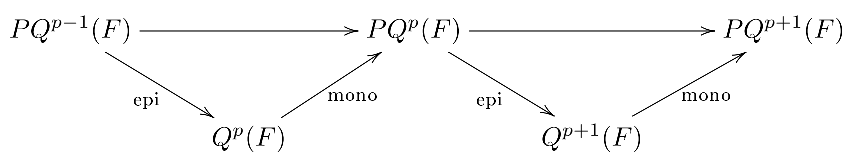
Proposition 5.3.9. *Let Φ be PF. Let F be $\mathcal{C}$-fine. Then, for $p > 0$, every element of $H^p_\Phi(X, F)$ is annihilated by a map in $\mathcal{C}$. That is, if $u \in H^p_\Phi(X, F)$, there is a map $f \in \mathcal{C}$ such that $f_*(u) = 0$.*

Proof. The proof is based on two lemmas. We recall the definitions of the functors P and Q. If F is any sheaf, $P(F) = \widetilde{\overline{F}}$ and $Q(F) = P(F)/F$. Note that these functors preserve supports; i.e., if $f \colon F \to G$, then $|P(f)| \subset |f|$ and $|Q(f)| \subset |f|$. This follows from the trivial fact that the functors and $\tilde{}$ have this property. Obviously, any composition of P's and Q's will again have this property.

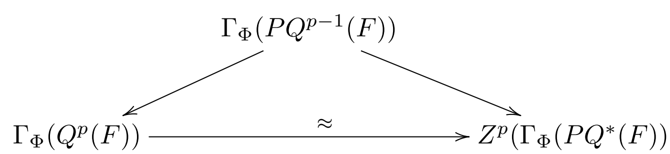
We also recall the canonical flabby resolution of F, $0 \to F \to P(F) \to PQ(F) \to PQ^2(F) \to \cdots$ (or $PQ^*(F)$ for short) and the fact that if we apply to this resolution (omitting the term F), we get a cochain complex whose cohomology groups are naturally isomorphic to $H^*_\Phi(X, F)$.

Lemma 5.3.10. *Let F be any sheaf and Φ any family of supports. Then for $p > 0$, there is a natural exact sequence $\Gamma_\Phi(PQ^{p-1}(F)) \xrightarrow{j} \Gamma_\Phi(PQ^p(F)) \to H^p_\Phi(X, F) \to 0$ where j is induced by the natural epimorphism $PQ^{p-1}(F) \to Q^p(F)$ given by the fact that $Q^p(F) = QQ^{p-1}(F)$ is defined as a quotient of $PQ^{p-1}(F)$.*

Proof. We can factor $PQ^{p-1}(F) \to PQ^{p+1}(F)$ as follows:



Since Γ_Φ is left exact, $0 \to \Gamma_\Phi(Q^p(F)) \to \Gamma_\Phi(PQ^p(F)) \to \Gamma_\Phi(PQ^{p+1}(F))$ is exact. Therefore $\Gamma_\Phi(Q^p(F)) \xrightarrow{\approx} Z^p(\Gamma_\Phi(PQ^*(F))$. Now,



and the image of the second vertical map is $B^p\Gamma_\Phi(PQ^*F)$. Therefore $H^p_\Phi(X, F) = Z^p/B^p$ is naturally isomorphic to the cokernel of $\Gamma_\Phi(PQ^{p-1}(F)) \to \Gamma_\Phi(Q^p(F))$. □

Lemma 5.3.11. *Let S and T be covariant linear functors from sheaves on X to sheaves on X which preserve supports (in the sense mentioned above in connection with P and Q). Let $j \colon S \to T$ be a natural transformation.*

Assume Φ is PF, F is $\mathcal{C}$-fine, and $j_F \colon S(F) \to T(F)$ is an epimorphism.

Then, every element of the cokernel of $\Gamma_\Phi(S(F)) \to \Gamma_\Phi(T(F))$ is annihilated by some map in $\mathcal{C}$.

Remark 5.3.12. Proposition 5.3.9 follows immediately from this lemma with $S = PQ^{p-1}$ and $T = Q^p$.

Remark 5.3.13. This lemma is a generalization of the fundamental theorem of Cartan's theory of sheaves.

Proof. of lemma:
Let $s \in \Gamma_\Phi(T(F))$ represent an element of the cokernel. Let $A = |s|$. For each $x \in X$, choose a neighbourhood N_x and section $s_x \in \Gamma(N_x, S(F))$ such that $j(s_x) = s \mid N_x$. If $x \notin A$, choose $N_x = X \setminus A$ and $s_x = 0$. In the usual way, we find a Φ-covering $\{U_\alpha\}$ and sections $t \in \Gamma(U_\alpha, S(F))$ such that $j(t_\alpha) = s \mid U_\alpha$ and $t_{\alpha_0} = 0$ for the exceptional α_0.

Shrink $\{U_\alpha\}$ to $\{V_\alpha\}$ and find endomorphisms 1_α of F such that $|1_\alpha| \subset \overline{V}_\alpha$ and $\sum 1_\alpha = f \in \mathcal{C}$. Define $t(x) = \sum S(1_\alpha)t_\alpha(x)$. This makes sense because $|S(1_\alpha)| \subset |1_\alpha| \subset \overline{V}_\alpha$. t is obviously continuous and has support in Φ because $t_{\alpha_0} = 0$ and so $|t| \subset C$ (the set used in defining Φ-coverings).

But,

$$j(t)(x) = \sum jS(1_\alpha)t_\alpha(x) = \sum T(1_\alpha)jt_\alpha(x) = \sum T(1_\alpha)s(x) = T(f)s(x).$$

Therefore $\Gamma_\Phi T(f) \cdot (s)$ is in the image of $\Gamma_\Phi(j)$. In other words, f annihilates the element of the cokernel represented by s. □

Thus the proposition is proved. □

Chapter 6

The Sections of Sheaves

6.1 Formulation of the problem

The general problem we will consider here is the following:

Let $\underline{S}$ be a stack with "restriction" maps φ^U_V and $S = L(\underline{S})$ its sheaf. We then have the natural map $\varphi\colon \underline{S}(X) \to \Gamma(X, S)$. The problem is first to define supports in $\underline{S}(X)$ so that φ preserves supports, then to find the kernel of φ, and finally to decide when $\varphi\colon \underline{S}_\Phi(X) \to \Gamma_\Phi(S)$ is an epimorphism. (See below for definition of $\underline{S}_\Phi$.)

The first two parts of the problem are quite easily solved.

Definition 6.1.1. Let S be a stack over X. If $s \in \underline{S}(U)$, define the support $|s|$ of s as follows:

A point $x \in U$ does not belong to $|s|$ if and only if x has an open neighbourhood N_x such that $\varphi^U_{N_X}(s) = 0$.

Obviously $|s|$ is closed in U because $x \notin |s|$ implies that all points in N_x are not in $|s|$.

Lemma 6.1.2. *Let S be the sheaf of $\underline{S}$ and $\varphi\colon \underline{S}(U) \to \Gamma(U, S)$ be the natural map. Then $|\varphi(s)| = |s|$ for all $s \in \underline{S}(U)$.*

Proof. ($x \notin |s| \Rightarrow x \notin |\varphi(s)|$): If $x \notin |s|$, then $\underline{S}(U) \to \underline{S}(N_x) \to S_x$ annihilates s. Consequently, $\varpi(s) \cdot (x) = 0$ and so $x \notin |\varphi(s)|$.
($x \notin |\varphi(s)| \Rightarrow x \notin |s|$): If $x \notin |\varphi(s)|$, then $\underline{S}(U) \to S_x$ annihilates s. Since S_x is the direct limit of $S(N_x)$ over neighbourhoods of x, there is an N_x such that $\underline{S}(X) \to \underline{S}(N_x)$ annihilates s. Therefore $x \notin |s|$. □

Corollary 6.1.3. *The kernel of $\varphi\colon \underline{S}(U) \to \Gamma(U, S)$ is $\{s \mid |s| = \varnothing\}$. Such s's are called* locally zero elements.

Proof. This corollary follows immediately from the lemma and the observation that an element $t \in \Gamma(U, S)$ is zero if and only if $|t| = 0$. □

Definition 6.1.4. If Φ is a family of supports in X, let $\underline{S}_\Phi(X) = \{\underline{S}(X) \mid |s| \in \Phi\}$.

It follows from lemma 6.1.2 that $\varphi\colon \underline{S}_\Phi(X) \to \Gamma_\Phi(S)$ and the kernel of φ consists of locally zero elements.

The final part of our problem is to decide when this map is an epimorphism. We note first of all that it is not necessary to worry about supports in solving this problem. That is, suppose $s \in \Gamma_\Phi(S)$, and suppose we can find $t \in \underline{S}(X)$ such that $\varphi(t) = s$. It then follows automatically that $|t| = |\varphi(t)| = |s| \in \Phi$. Therefore $t \in \underline{S}_\Phi(X)$.

Suppose now that we are given $s \in \Gamma_\Phi(S)$. For each $x \in X$, there is an open neighbourhood N_x and an element $s_x \in \underline{S}(N_x)$ such that $\varphi(s_x) = s \mid N_x$. If $x \notin |s|$, we can obviously choose $N_x = X \setminus |s|$ and $s_x = 0$. We now consider either the covering $\{N_x\}$ itself or a suitable refinement. In this way we get a covering $\{U_\alpha\}$ of X and sections $s \in \underline{S}(U_\alpha)$ such that $\varphi(s_\alpha) = s \mid U_\alpha$. This last condition implies that the differences

$$\varphi^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) - \varphi^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$$

are locally zero. We want to piece together the s_α to get an element $t \in \underline{S}(X)$ such that $\varphi^X_{U_\alpha}(t) - s_\alpha$ is locally zero. This is usually much easier to do if we assume that the differences

$$\varphi^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) - \varphi^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$$

are zero and not just locally zero.

Definition 6.1.5. We say a stack $\underline{S}$ has the *collation property* if the following property holds for all coverings $\{U_\alpha\}$ of X: Suppose we have elements $s_\alpha \in \underline{S}(U_\alpha)$ such that

$$\varphi^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \varphi^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$$

for all α, β. Then there is an element $t \in \underline{S}(X)$ such that $\varphi^X_{U_\alpha}(t) = s_\alpha$ for all α.

If we weaken the conclusion to read "$\varphi^X_{U_\alpha}(t) - s_\alpha$ is locally zero for all α," e then say that $\underline{S}$ has the *approximate collation property.*

Finally, if we assume the above property holds only for coverings of a certain class (e.g., locally finite coverings or Φ-coverings), we say $\underline{S}$ has the *collation property* (or *approximate collation property*) *with respect to this class.*

The next proposition follows immediately from the above definition and the remarks preceding it.

Proposition 6.1.6. *Let $\underline{S}$ be a stack with the collation property. Assume that for every $U, \underline{S}(U)$ has no locally zero elements except zero. Then $\underline{S}_\Phi(X) \to \Gamma_\Phi(S)$ is an epimorphism (and, in fact, an isomorphism).*

Remark 6.1.7. If $\underline{S}(X)$ is a stack over X and U is open in X, we can define a stack $\underline{S} \mid U$ over U by considering only those $\underline{S}(V)$ for $V \subset U$. If S is the sheaf of $\underline{S}$, then the sheaf of $\underline{S} \mid U$ is clearly $S \mid U$. The above proposition shows that a necessary and sufficient condition for $\underline{S}$ to be the stack of its sheaf (i.e., for $\varphi\colon \underline{S}(U) \to \Gamma(U, S)$ to be isomorphic for all U) is that $\underline{S}$ has the following two properties:

1. Each $\underline{S} \mid U$ has the collation property.

2. Each $\underline{S}(V)$ has no locally zero elements except zero.

It is obvious that $\Gamma(U, F)$ has these properties for any sheaf F. Conversely, if $\underline{S}$ has these properties the above proposition, applied to U instead of X, shows that $\underline{S}(U) \to \Gamma(U, S \mid U) = \Gamma(U, S)$ is an isomorphism. (Of course, we take Φ to be all closed sets of U.)

The two conditions on $\underline{S}$ are obviously satisfied by stacks of functions as well as by stacks of sections of vector bundles.

If a stack contains non-trivial locally zero elements, we cannot apply the above proposition. If, however, we assume Φ is PF, we can dispense with the condition that $\underline{S}$ has no non-trivial locally zero elements.

Proposition 6.1.8. *Let $\underline{S}$ be a stack having the approximate collation property with respect to Φ-coverings. Assume Φ is PP. Then $\underline{S}_\Phi(X) \to \Gamma_\Phi(S)$ is an epimorphism.*

Proof. The proposition follows immediately from the definition of the approximate collation property and the following lemma. □

Lemma 6.1.9. *Let $\underline{S}$ be any stack and S its associated sheaf. Assume Φ is PP. Let $s \in \Gamma_\Phi(S)$. Then there is a Φ-covering $\{W_\beta\}$ of X and elements $t_\beta \in \underline{S}(W_\beta)$ such that*

(a) $\varphi(t_\beta) = s \mid W_\beta$,

(b) $\varphi^{W_\beta}_{W_\beta \cap W_\gamma}(T_\beta) = \varphi^{W_\gamma}_{W_\beta \cap W_\gamma}(T_\gamma)$,

(c) $t_\beta = 0$ *if* W_{β_0} *is the exceptional set of the Φ-covering.*

Remark 6.1.10. (c) is not needed for the proposition.

Proof. We follow the method indicated in the remarks preceding the definition of the collation property. Since Φ is PP, the covering $\{N_x\}$ is refined by a Φ-covering $\{U_\alpha\}$. We define $s_\alpha \in \underline{S}(U_\alpha)$ to be $\varphi^{N_x}_{U_\alpha}(s_x)$ for some x with $U_\alpha \subset N_x$. For the exceptional U_{α_0} we choose $N_x = X \setminus |s|$, so $s_{\alpha_0} = 0$. Let $C \in \Phi$ be such that $U_\alpha \subset C$ for $\alpha \neq \alpha_0$. This C is given by the definition of a Φ-covering.

Shrink $\{U_\alpha\}$ to $\{V_\alpha\}$. Let $x \in X$ be any point. Some neighbourhood of x meets only a finite number of U_α's. Take a smaller neighbourhood G_x such that

(i) G_x meets $\overline{V}_\alpha$ if and only if $x \in \overline{V}_\alpha$,

(ii) $x \in U_\alpha$ implies $G_x \subset U_\alpha$,

(iii) $x \in V_\alpha$ implies $G_x \subset V_\alpha$,

(iv) If $x \in \overline{V}_\alpha$ and $x \in \overline{V}_\beta$, then $\varphi^{U_\alpha}_{G_x}(s_\alpha) = \varphi^{U_\beta}_{G_x}(s_\beta)$.

All of these but (iv) are obviously satisfied by all small enough G_x. To satisfy (iv), consider all α such that $x \in \overline{V}_\alpha$. There are only a finite number of such α and each corresponding t_α is defined in a neighbourhood of x (namely U_α). All give the same element of the direct limit S_x of $\underline{S}$ (neighbourhoods of x). If follows from the properties of direct limits that (iv) is satisfied by all small enough G_x.

Now, for each G_x choose some V_α with $x \in \overline{V}_\alpha$ and define $t_x = \varphi^{U_\alpha}_{VG_x}(s_\alpha)$. It follows from (iv that t_x does not depend on the choice of α.

I claim that

$$\varphi^{G_x}_{G_x \cap G_y}(t_x) = \varphi^{G_y}_{G_x \cap G_y}(t_y)$$

for a11 x and y.

To prove this it is sufficient to consider the case where $G_x \cap G_y \neq \varnothing$. Therefore there is $z \in G_x \cap G_y$. For some α, $z \in V_\alpha$. Therefore G_x and G_y meet $\overline{V}_\alpha$. It now follows from (i) that $x, y \in \overline{V}_\alpha$. Consequently, we may define both t_x and t_y using the same α. The result clearly follows from this.

We can now add another set $G_0 = X \backslash C$ to the covering and define $t_0 \in \underline{S}(G_0)$ to be zero. This does not spoil the property just proved. In other words,

$$\varphi^{G_0}_{G_0 \cap G_x}(t_0) = \varphi^{G_0}_{G_0 \cap G_x}(t_x)$$

To see this, we merely observe that if $G_0 \cap G_x \neq \varnothing$, then $G_x \notin C$. Since all U_α with $\alpha \neq \alpha_0$ are in C, the only α we can use to define t_x is α_0. Since $s_{\alpha_0} = 0$, we have $t_x = 0$.

Since $C \in \Phi$, the covering $\{G_x, G_0\}$ is refined by a Φ-covering $\{W_\beta\}$. If $\beta \neq \beta_0$, the exceptional index, we choose a $G_x \supset W_\beta$ and define $t_\beta = \varphi^{G_x}_{W_\beta}(t_0) = 0$. If $\beta = \beta_0$, we choose $G_0 \supset W_\beta$ and define $t_{\beta_0} = \varphi^{G_0}_{W_{\beta_0}}(t_0) = 0$. The required property of the $\{t_\beta\}$ follows from the corresponding property of the t_x's and t_0. □

6.2 Stacks of chains and cochains

We now consider stacks of chains and cochains and show that they have the collation property. We start with an indexed collection $\{D_i\}_{i \in I}$ of subsets of X. Assume that for each i there is also given a "base point" $d_i \in D_i$. For example, we can choose a fixed "test space" T with a base point t and let the index set I be the set of all maps $T \to X$. The set D_i will then be defined to be the image of map i or its closure and the point d_i will be defined to be the image of t under the map i. If we let T be the standard n-simplex, then I is the set of all singular n-simplexes of X. If we let T be an ordered set of $n+1$ points, then I is the set of all Vietoris n-simplexes of X.

Define $I_U = \{i \in I \mid D_i \subset U\}$. Let M be a module. Define $C(U, M)$ to be the set of functions on I_U with values in M. More generally, if $\overline{M}$ is a protosheaf over X, we let $C(U, \overline{M})$ be the set of functions defined on I_U and such that $f(i) \in \overline{M}_{d_i}$. These $C(U, \overline{M})$ with the obvious restriction maps φ_V^U form a stack $C(*, \overline{M})$.

Lemma 6.2.1. *The stack $C(x, \overline{M})$ has the collation property.*

Proof. Let $\{U_\alpha\}$ cover X. Let $t_\alpha \in C(U, M)$ be such that

$$\varphi^{U_\alpha}_{U_\alpha \cap U_\beta}(t_\alpha) = \varphi^{U_\beta}_{U_\alpha \cap U_\beta}(t_\beta).$$

Define $t \in C(X, M)$ as follows: If D_i is in some U_α, choose one and let $t(i) = t_\alpha(i)$. This is obviously independent of the choice of α. If D_i is in no U_a, let $t(i) = 0$. Clearly $\varphi_V(t) = t_\alpha$ for all α. □

Corollary 6.2.2. *If C_M is the sheaf defined by the stack $C(*, \overline{M})$, then $\Gamma_\Phi(C_M) = C_\Phi(X, M)/\{$locally zero elements$\}$ provided Φ is PF.*

I do not know if this is true when Φ is not PF.

Remark 6.2.3. If we start with the Vietoris n-simplexes (defined above), and if M is a module then $C_\Phi(X, M)/\{$locally zero elements$\}$ is the classical module of Alexander-Spanier n-cochains with coefficients in M.

If we start with the singular n-simplexes, and if M is a module, then the classical module of singular n-cochains with coefficients in M and supports in Φ is just $C_\Phi(X, M)$. I will show later that the cochain complex of locally zero singular cochains with coefficients in M has zero cohomology. Therefore, factoring it out does not affect the cohomology groups of $C_\Phi(X, M)$. In other words, $C_\Phi(X, M)/\{$locally zero elements$\}$ has the same cohomology as $C_\Phi(X, M)$. To see this, we consider the exact cohomology sequence associated with the short exact sequence

$$0 \to 0(X, M) \to C_\Phi(X, M) \to C_\Phi(X, M)/0(X, M) \to 0$$

where $0(X, M)$ is the set of locally zero singular cochains.

The sheaf C_M defined by $C(*, M)$ (using any system $\{D_i\}_{i \in I}$ and any protosheaf $\overline{M}$) has another important property.

Proposition 6.2.4. *The sheaf C_M is fine.*

Proof. Let $\{U_\alpha\}$ be a locally finite covering of X. Define a partition of unity $\{w_\alpha\}$ as follows: For each $x \in X$, choose some α_x such that $x \in U_{\alpha_x}$. Let

$$w_\alpha(x) = \begin{cases} 1 & \text{if } \alpha = \alpha_x \\ 0 & \text{if } \alpha \neq \alpha_x \end{cases}$$

Define $1'_\alpha \colon C(U, M) \to C(U, M)$ by $1'_\alpha(t) \cdot (i) = w_\alpha(d_i) \cdot t(i)$. These $1'_\alpha$ are stack endomorphisms and so induce $1_\alpha \colon C_M \to C_M$.

If $x \notin \overline{U}_\alpha$, then x has a neighbourhood N_x not meeting $\overline{U}_\alpha$. If $D_i \subset N_x$, then $d_i \notin U_\alpha$. Consequently $w_\alpha(d_i) = 0$ and so $1'_\alpha \mid C(N_x, M) = 0$. Therefore 1_α is zero on the stalk of C_M over x. This shows that $|1_\alpha| \subset \overline{U}_\alpha$.

Finally, each $x \in X$ has a neighbourhood N_x meeting only a finite member of U_α. The argument just given shows that $1'_\alpha \mid C(N_x, M) = 0$ except for this finite set of α. Since $\sum w_\alpha = 1$, it follows that $\sum 1'_\alpha \mid C(N, M) = \text{id}$. Therefore $\sum 1_\alpha = \text{id}$ in the stalk of C_M over x. □

We now come to the stack of chains. Let the system $\{D_i\}_{i \in I}$ be as before. Let $\overline{M}$ be a protosheaf over X. We define $C_*(X, \overline{M})$ to be the set of all finite or infinite formal sums $\sum m_i i$ where $m_i \in \overline{M}_{d_i}$ and $\{D_i \mid m_i \neq 0\}$ forms a locally finite system. We refer to these as *locally finite sums.* If A is any subset of X, let $C_*(A, M)$ be the submodule of $C_*(X, M)$consisting of those sums in which $D_i \subset A$ for all i such that $m_i \neq 0$. Define

$$C_*(X, A; M) = \frac{C_*(X, M)}{C_*(A, M)}.$$

If $B \subset A$, there is an obvious quotient map $C_*(X, B; M) \to C_*(X, A; M)$. Therefore, we can define a stack $\underline{S}$ by letting $\underline{S}(U, M) = C_*(X, X \setminus U; M)$.

Recall that if $\underline{S}$ is a stack and U is open in X, then $\underline{S} \mid U$ is the stack over U defined by those $\underline{S}(V)$ with $V \subset U$.

Proposition 6.2.5. *Let $\underline{S}$ be the stack defined above. Assume all are connected. Then $\underline{S}$ has the collation property.*

Proof. Let $\{U_\alpha\}$ be an open covering of X. Let $t_\alpha \in S(U_\alpha, M)$ be elements such that

$$\varphi^{U_\alpha}_{U_\alpha \cap U_\beta}(t_\alpha) = \varphi^{U_\beta}_{U_\alpha \cap U_\beta}(t_\beta).$$

We say that an index i is *essential* in an open set $V \subset X$ if $D_i \cap V \neq \varnothing$. Two elements $t, t' \in \underline{S}(V)$ are clearly equal if and only if every i essential in V has the same coefficient in t and t'.

I claim that each i has the same coefficient in all t for which i is essential in U_α.

To prove this, let i be essential in U_α and U_β. There are two cases to consider: *Case 1*: i is essential in $U_\alpha \cap U_\beta$.

Obviously, i has the same coefficient in $t\alpha$ as in $\varphi^{U_\alpha}_{U_\alpha \cap U_\beta}(t_\alpha)$. A similar result holds for t_β. But,

$$\varphi^{U_\alpha}_{U_\alpha \cap U_\beta}(t_\alpha) = \varphi^{U_\beta}_{U_\alpha \cap U_\beta}(t_\beta).$$

Case 2: The general case.

Since D_i is connected, we can find a "chain" $U_\alpha = U_{\gamma_0}, U_{\gamma_1}, \ldots, u_{\gamma_k} = U_\beta$ such that $U_{\gamma_i} \cap U_{\gamma i+1} \cap D_i \neq \varphi$. But now, case 1 shows that i has the same coefficient in t_{γ_i} and in $t_{\gamma_{i+1}}$.

We now define $t = \sum m_i \cdot i$ by choosing m_i to be the coefficient of i in any t_α for which i is essential in t_α. We must show that $\{D_i \mid m_i \neq 0\}$ is locally finite. Let $x \in X$. Then $x \in U_\alpha$ for some α. Therefore, we only have to worry about those which meet U_α. In other words, those i which are essential in U_α. But, these occur in t with the same coefficients as in t_α. Since t_α is a locally finite sum, x has a neighbourhood which meets only a finite number of D_i such that $m_i \neq 0$.

This shows that $t \in \underline{S}(X)$. Clearly $\varphi^X_{U_\alpha}(t) = t_\alpha$. □

Proposition 6.2.6. *Assume all D_i are connected.*

Let S_M be the sheaf defined by the stack $\underline{S}$. Then $\varphi\colon \underline{S}(X, M) \to \Gamma(X, S_M)$ is an isomorphism.

Proof. In view of the preceding proposition, it is sufficient to show that $\underline{S}(U, M)$ never has non-trivial locally zero elements, which we show in 6.2.7 below. Note that this does not require any condition on the D_i. □

Lemma 6.2.7. *For all U, the only locally zero element of $\underline{S}(U, M)$ is zero itself.*

Proof. Let $s = \sum m_i \cdot i$ represent a locally zero element of $\underline{S}(U, M)$. Suppose i is essential in U. Then there is a point $x \in U \cap \overline{D}_i$. Since s is locally zero, there is a neighbourhood N_x of x such that $\varphi^U_{N_x}(s) = 0$. Since i is essential in N_x, m_i must be zero. Therefore every i essential in U has coefficient zero in s. Thus s is zero in $\underline{S}(U, M)$. □

For certain applications, it is useful to have a slightly different definition of the sheaf S_M. Define $C'_*(X, M)$ to be the submodule of $C_*(X, M)$ consisting of those $\sum m_i \cdot i$ such that $m_i = 0$ except for a finite number of i. We then define $C'_*(X, A; M)$ and $\underline{S}'(U, M)$ in terms of $C'_*(X, M)$ in exactly the same way that $C_*(X, A; M)$ and $\underline{S}(U, M)$ are defined in terms of $C_*(X, M)$. Obviously $\underline{S}'$ is a substack of $\underline{S}$. Let $\underline{S}'_M$ be the sheaf it defines.

Lemma 6.2.8. *The inclusion $\underline{S}' \to \underline{S}$ induces an isomorphism $S'_M \to S_M$.*

Proof. Since the inclusion is a monomorphism, it induces a monomorphism of sheaves. Now, let $y \in (S_M)_x$. Choose a representative $s = \sum m_i \cdot i \in S(N_x, M)$ where N_x is some neighbourhood of x. Since s is a locally finite sum, some smaller neighbourhood U_x meets only a finite number of the D_i for which $m_i \neq 0$. Consequently, $\varphi^{N_x}_{U_x}(s) \in S'(U_x, M)$; but this element also represents y. □

Corollary 6.2.9. *If M is a module (regarded as a constant sheaf), then $S'(U, M) \cong S'(U, K) \otimes M$ where K is the ground ring.*

Proof. It is obvious that $S'(U, M) \cong S'(U, K)$ because $S'(U, K)$ is a free module (since only finite sums occur). □

Another useful property is the following.

Lemma 6.2.10. *Let $s = \sum m_i \cdot i \in \underline{S}(U, M)$. Then $|s| = \cup \overline{D}_i$, the union being taken over those i which are essential in U and such that $m_i \neq 0$.*

Proof. Suppose $x \notin |s|$, then there is a neighbourhood N_x such that $\varphi^U_{N_x}(s) = 0$. Therefore, no i such that $m_i \neq 0$ can be essential in N_x. This shows that $x \notin \cup \overline{D}_i$.

Conversely, suppose $x \notin \cup \overline{D}_i$. Then there is a neighbourhood N_x disjoint from $\overline{D}_i$ because the union of a locally finite system of closed sets is closed. No i with $m_i \neq 0$ is essential in N_x. Therefore, $\varphi^U_{N_x}(s) = 0, x \notin |s|$. □

6.3 Singular chains and cochains

To conclude this chapter, I will prove some additional properties of the sheaves and stacks of singular chains and cochains. The stacks of n-dimensional singular chains and cochains are defined, as indicated above, by taking the set of indices I to be the set of singular n-simplexes of X. If we take M to be a module, we can define boundary and coboundary operators by the usual formulae. In this way we arrive at chain and cochain complexes of stacks and sheaves. We will need the following lemma of singular homology theory:

Lemma 6.3.1. *Let $\mathcal{U} = \{U_\alpha\}$ be a collection of subsets of X whose interiors cover X. Let $\underline{S}'_{\mathcal{U}}(X) = C'_{\mathcal{U}}(X, K)$ be the set of finite singular chains (with coefficients in the ground ring K) which contain only simplexes i for which $\mathrm{im}(i)$ is contained in some set of $\mathcal{U}$. Let $i\colon \underline{S}'_{\mathcal{U}}(X) \to \underline{S}'(X)$ be the inclusion map.*

Then, there is a chain map $j\colon \underline{S}'(X) \to \underline{S}'_{\mathcal{U}}(X)$ and a chain homotopy $D\colon \underline{S}'(X) \to \underline{S}'(X)$ such that

(a) $ji = \mathrm{id}$,

(b) $\mathrm{id} - ij = \partial D + D\partial$, and

(c) j and D (and i) preserve supports, i.e. $|js| \subset |s|$ and $|Ds| \subset |s|$.

Proof. We refer the reader to Eilenberg and Steenrod, "Foundations of Algebraic Topology", pp. 207-8. □

It is now quite easy to prove the following proposition which was mentioned earlier.

Proposition 6.3.2. *Let M be any module and $0(X, M)$ the set of all locally zero singular cochains of X with coefficients in M. Then $H(0(X, M)) = 0$.*

Proof. Let $v \in 0(X, M)$ be a cocycle. Since v is locally zero, there is a covering $\mathcal{U}$ of X such that $\varphi^X_{U_\alpha}(v) = 0$ for all α. Clearly, v annihilates S. Let i, j, D be the maps defined in lemma 6.3.1. Let $i^\#, j^\#, D^\#$ be the maps induced on $\mathrm{Hom}(\underline{S}'(X), M)$ and $\mathrm{Hom}(\underline{S}'(X), M) = C(X, M)$. Then, $i^\#(v) = 0$. Therefore $v = (\mathrm{id} - ij)^\# v = (\partial D + D\partial)^\# v = (\delta D^\# + D^\# \delta)v = \delta D^\# v$. This shows that every cocycle of $0(X, M)$ cobounds. □

We now state and prove an important property of the sheaf S_K of singular chains.

Definition 6.3.3. Let F be a chain (or cochain) complex of sheaves. We say F is *homotopically fine* if, for every locally finite covering $\{U_\alpha\}$ of X, we can find endomorphisms 1_α and D of F such that

(a) $|1_\alpha| \subset \overline{U}_\alpha$, and

(b) $\sum 1_\alpha = \mathrm{id} - \partial D - D\partial$.

Remark 6.3.4. This is equivalent to saying that F is $\mathcal{C}$-fine where $\mathcal{C}$ is the class of endomorphisms of F which are chain homotopic to the identity. Note that the 1_α are not required to be chain maps.

Proposition 6.3.5. *The sheaf S_K of singular chains is,homotopically fine.*

Proof. □

Corollary 6.3.6. *S_M is homotopically fine for all modules M.*

Proof. □

Our last lemma shows that singular homology based on finite chains is the same as singular homology with compact supports.

Lemma 6.3.7.

Proof. □

Chapter 7

Miscellaneous Results

7.1 Induced Sheaves

Let X and Y be spaces and $f\colon X \to Y$ any map. Let F be a sheaf over X and G a sheaf over Y.

Definition 7.1.1.

An example of such a mapping is given by the canonical mapping from a sheaf G to the induced sheaf $f^{-1}(G)$ defined as follows.

Definition 7.1.2.

7.2 Subspaces

7.3 Continuity properties

7.4 Alexander-Spanier cohomology

7.5 Singular cohomology

In the next chapter, we shall see that the same is true for theČech groups.

Chapter 8

Čech Cohomology

I will first define the Čech cohomology groups of X with coefficients in a stack. These groups are defined for all families Φ and all directed systems of coverings. There are two ways to define these groups, depending on how we define supports in the cochain modules. The natural definition of supports make use of the concept of the support of an element of a stack, as defined in the previous section. This leads to a system of cohomology groups $\check{H}^*_\Phi(X, \underline{S})$ which are augmented but do not form a δ-functor. To remedy this, we use a cruder definition of supports. This gives a system $\check{H}^*_\Phi(X, \underline{S})'$ which is an exact δ-functor but is not augmented in general. The two definitions agree if Φ is the collection of all closed subsets of X.

8.1 Coverings and nerves

Chapter 9

The Spectral Sequences

9.1 Preliminaries

The spectral sequences will be defined in terms of double complexes. We will first remind the reader of some properties of these complexes. Full details may be found in CE, chapter IV, section 4, and chapter XV, section 6.

9.2 The Spectral Sequences

I will define the spectral sequences using a resolvent functor and show that they are independent of the choice of this functor. At the end of this section, I will prove that these spectral sequences agree with those defined by CE chapter XVII. This fact will not be used in the applications however.

When I say two spectral sequences are *isomorphic*, I will mean that they are isomorphic from the term E_2 on. It does not matter if the E_0 and E_1 terms are isomorphic or not.

9.3 Comparison with the Cartan-Eilenberg theory

The results of this section will not be used anywhere else in these notes. Consequently the reader interested only in the applications may skip this section.

In chapter XVII of CE, spectral sequences are defined for a cochain complex M and a functor such as T. These spectral sequences are defined in terms of an injective resolution of M. The only property of this resolution we use here is that it is a T-resolution in the sense of the following definition.

Chapter 10

The Spectral Sequences of a Map

Lemma 10.0.1.

Proof. □

Chapter 11

The Duality Theorems

11.1 Singular Homology

Let X be any space. Let S be the sheaf of singular chains over X with coefficients in the ground ring K.

Definition 11.1.1. If G is any sheaf over X, we define $H_i^\Phi(X, G) = H_i(\Gamma_\Phi(S \otimes G))$.

Remark 11.1.2. If G is a constant sheaf, we have seen that $\Gamma_\Phi(S \otimes G)$ is naturally isomorphic to the usual complex of singular chains with coefficients in G and supports in Φ. Therefore $H_i^\Phi(X, G)$ agrees with the classical singular homology groups if G is constant and Φ is the collection of compact subsets of X. (X being assumed Hausdorf here.)

11.2 the Duality Theorems

We now assume X to be an n-manifold, possibly with boundary. By this, we mean a Hausdorff space such that every point has a neighbourhood homeomorphic to a closed n-cell. A point is called an *interior point* if it has a neighbourhood homeomorphic to an open n-cell. Otherwise, it is called a *boundary point.* The set of boundary points is denoted by $\dot{X}$.

Chapter 12

Cup and Cap Products

12.1 Cup products

1.1

I will here show how to define cup products in the spectral sequences defined in Chapters 9 and 10. I will also give a general definition of cap-products and an application of these to the Poincaré and Alexander duality theorems. Both of these products will he defined by first defining them for natural resolutions and resolvent functors, and constructing them from a weaker kind of product, the *qip*[1] product. This last product is easy to derive in the case of sheaves.

12.2 Cap products

[1] This name is due to Miss M. Rochat.

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