Notes on Algebraic Topology

Preface

This is an authour's trial to compile an *accessible* account of algebraic topology in the spirit of John F. Adams as suggested in his "Algebraic Topology: a Student's Guide".

Thus the purpose is to provide a sufficient background, from the modern viewpoint, to read Milnor's "Differential Topology" and "Lectures on Characteristic Classes" published some sixty years ago.

These notes should hopefully cover at least the following topics:

1. Spaces and maps
2. Homotopy sets
3. CW complexes
4. Cofibration
5. Fibration
6. Vector bundles and fibre bundles
7. Axiomatic treatment of homology/cohomology
8. Spectral sequences and their applications
9. Eilenberg-Mac Lane spaces and Postnikov systems
10. Homotopy groups of spheres
11. Spectra

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Chapter 1

Homotopy Theory I

1.1 Motivation

This section gives a motivation for studying algebraic topology. First, let us recall some notions from general topology.

Definition 1.1.1. A topological space X is said to be *disconnected* if it is the union of two disjoint non-empty open sets. Otherwise, X is said to be *connected.*

A path-connected space is a stronger notion of connectedness, requiring the structure of a path.

Definition 1.1.2. A *path* from a point x to a point y in a topological space X is a continuous function f from the unit interval $I = [0, 1]$ to X with $f(0) = x$ and $f(1) = y$. A *path-component* of X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y.

The space X is said to be *path-connected* (or *pathwise connected*) if there is exactly one path-component, i.e. if there is a path joining any two points in X.

Next, notions important in algebraic topology.

Definition 1.1.3. Let X be a topological space and let $x \in X$. We say that X is *locally connected at* x if for every open set V containing x there exists a connected, open set U with $x \in U \subseteq V$. The space X is said to be *locally connected* if it is locally connected at x for all $x \in X$.

Remark 1.1.4. Note that local connectedness and connectedness are not related to one another; a space may possess one or both of these properties, or neither.

Example 1.1.5. $\mathbb{Q}$ with the usual metric topology is neither connected nor locally connected: no point in $\mathbb{Q}$ has any connected neighbourhood.

Lemma 1.1.6. *If X is locally connected, then its connected components are open.*

Proof. This is almost immediate; if U is a connected open the of X, then U is inside the connected component of x. □

Definition 1.1.7. Let X be a topological space and let $x \in X$. We say that X is *locally path connected at* x if for every neighbourhood U of x there exists a path connected neighbourhood V of x such that $x \in V \subseteq U$. X is said to be *locally path connected (on all of* X*)* if X is locally path connected at every $x \in X$.

If X is locally path-connected, then its path components are open.

Lemma 1.1.8. *If X is locally path connected, then the components of X coincide with the path components of X - that is,*

$$\sim_{conn} = \sim_{path} .$$

Proof. Since paths themselves are connected, clearly if two points belong to the same path component they belong to the same connected component. Now suppose X is locally path connected, let C be a connected component of X. Then C is a union of path components:

$$C = \cup_{i \in I} P_i$$

But since X is locally path-connected, each P_i is open. That means C has been written as a disjoint union of open sets, and so C is not connected, a contradiction. □

Corollary 1.1.9. *If X is locally-path connected, then it is connected if and only if it is path connected.*

Now we begin the algebraic topology portion of the course. As you will see, it has a pretty different flavour from what has come up to this point, and the main reason for that is the emphasis on *homotopy*: when one continuous map can be deformed into another, we regard them as equivalent, and that gets us into a much more tenable situation when it comes to classifying spaces.

Henceforth, we denote the closed interval $I \in \mathbb{R}$ by I. Here is what you might regard as the fundamental definition of the whole field:

Definition 1.1.10 (Homotopy)**.** Let X and Y be spaces and let $f, g\colon X \to Y$ be continuous functions. Then a *homotopy* from f to g is a continuous function

$$H\colon X \times I \to Y$$

such that

$$H|_{X \times \{0\}} = f, \quad H|_{X \times \{1\}} = g.$$

If there exists a homotopy from f to g, we say f and g are *homotopic*.

Remark 1.1.11. We think of H as like a "movie" of continuous maps that starts at f and ends at g.

$$I \to (X \to Y) \quad t \mapsto h_t \in (X \to Y) = Y^X, \quad h_0 = f, h_1 = g$$

We will be back with the mapping spaces like $(X \to Y) = Y^X$ later.

Example 1.1.12. Suppose X is a space and $f, g\colon X \to \mathbb{R}^n$ are continuous functions. Then if we define $H\colon X \times I \to \mathbb{R}^n$ by

$$H(x,t) = (1-t)f(x) + tg(x)$$

then H is a homotopy from f to g. So any pair of maps into $\mathbb{R}^n$ are homotopic.

Example 1.1.13. Let $*$ denote the one-point space; then for any space Y, a map $* \to Y$ is just given by a point $y \in Y$. We denote this map by i_y. Moreover, a homotopy from $i_{y_0}\colon * \to Y$ to $i_{y_1}\colon * \to Y$ is just a path from y_0 to y_1. We conclude that two maps $* \to Y$ are homotopic if and only if they pick out points in the same path component.

Theorem 1.1.14. *Let X and Y be topological spaces. If $f, g\colon X \to Y$ are homotopic, we write $f \sim g$. Then $\sim$ is an equivalence relation on* $\mathrm{Map}(X, Y)$.

Proof. There are three conditions to verify:

- $f \sim f$. Let $p_X\colon X \times I \to X$ denote the projection $(x,t) \mapsto x$. The constant homotopy
$$f \circ p_X\colon X \times I \to Y, \quad (f \circ p_X)(x,t) = f(x)$$
is a homotopy from f to itself.

- If H is a homotopy from f to g, then
$$H'\colon X \times I \to Y, \quad H'(x,t) = H(x, 1-t)$$
is a homotopy from g to f.

- Also, we can compose homotopies, just like we can compose paths. If H_0 is a homotopy from f to g and H_1 is a homotopy from g to h, then we can define
$$H_{01}\colon X \times I \to Y$$
by
$$H_{01}(x,t) = \begin{cases} H_0(2t) & t \le 1/2, \\ H_1(2t-1) & t \ge 1/2. \end{cases}$$
Then H_{01} is a homotopy from f to h. (You just have to prove that H_{01} is continuous; this is similar to the proof that the composition of two paths is continuous. If you're interested, it follows from the pasting lemma on p. 108 of Munkres, copied below for your convenience.)

□

Lemma 1.1.15. *Let X be a topological space and $X = A \cup B$, where A and B are closed in X. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g can be combined to give a continuous function $h\colon X \to Y$, defined by setting*

$$h(x) = \begin{cases} f(x) & \text{if} \quad x \in A, \quad \text{and} \\ g(x) & \text{if} \quad x \in B. \end{cases}$$

Proof. Let C be a closed subset of Y. Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C),$$

by elementary set theory. Since f is continuous, $f^{-1}(C)$ is closed in A and, therefore, closed in X. Similarly, $g^{-1}(C)$ is closed in B and therefore closed in X. Their union $h^{-1}(C)$ is thus closed in X. □

Now let us get back to homotopy. An equivalence class for the equivalence relation $\sim$ is called a *homotopy class.* The set of homotopy classes of maps from X to Y is denoted $[X, Y]$.

Lemma 1.1.16. *Homotopy respects composition of functions. Suppose $f_0, f_1\colon X \to Y$ and $g_0, g_1\colon Y \to Z$ are maps such that $f_0 \sim f_1$ and $g_0 \sim g_1$. Then*

$$g_0 \circ f_0 \sim g1 \circ f_1.$$

Proof. Let H be a homotopy from f_0 to f_1 and let J be a homotopy from g_0 to g_1. We have to use H and J to build a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$. Let's define

$$K\colon X \times I \to Z$$

by

$$K(x, t) = J(H(x, t), t).$$

Then

$$K(x, 0) = J(H(x, 0), 0) = J(f_0(x), 0) = g_0(f_0(x)),$$

and similarly

$$K(x, 1) = J(H(x, 1), 1) = J(f_1(x), 1) = g_1(f_1(x)),$$

so K is the desired homotopy. □

Here is the thing somewhat surprising at first: although homotopy is a notion of equivalence for functions, it also gives rise to a new notion of equivalence for spaces.

Definition 1.1.17. Let $f\colon X \to Y$ be a map. Then a *homotopy inverse* for f is a map $g\colon Y \to X$ such that

$$g \circ f \sim \mathrm{id}_X, \quad f \circ g \sim \mathrm{id}_Y .$$

That is, g is a homotopy inverse of f if there are continuous maps $F\colon X\times I\to X$ and $G\colon Y\times I\to Y$ such that $F|_{X\times\{0\}}=g\circ f$, $F|_{X\times\{1\}}=\mathrm{id}_X$ and $G|_{Y\times\{0\}}=f\circ g$, $G|_{Y\times\{1\}}=\mathrm{id}_Y$ respectively. This is a natural weakening of the definition of a topological inverse (= homeomorphism).

Clearly g is a homotopy inverse to f if and only if f is a homotopy inverse to g. If f has a homotopy inverse, we say f is a *homotopy equivalence*. If there exists a homotopy equivalence from X to Y, we say X and Y are *homotopy equivalent*.

The goal of algebraic topology is arguably to classify spaces up to homotopy equivalence.

Example 1.1.18. Let $p\colon\mathbb{R}^n\to *$ be the unique map. Then any map from $*$ to $\mathbb{R}^n$, say i_0, is a homotopy inverse to p. Indeed, clearly $p\circ i_0=\mathrm{id}_*$, and $i_0\circ p$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$ since, as we just saw in Example 1.1.12, any two maps into $\mathbb{R}^n$ are homotopic. (For an explicit homotopy, we can use $H(x,t)=tx$.)

Definition 1.1.19. If $p\colon X\to *$ is a homotopy equivalence, we say X is *contractible* - hopefully the previous example should make it clear why this is the terminology, since the homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to the constant map at 0 was like a "contraction" of $\mathbb{R}^n$.

Lemma 1.1.20. *Every contractible space is path connected.*

Proof. Let X be a contractible space and let H be a homotopy from id_X to the constant map at some point $p\in X$. Then for any $x\in X$, the map $H_x\colon I\to X$, $H_x(t)=H(x,t)$ is a path from x to p. Since every point has a path to p, X is path connected. □

So at least some spaces are not contractible, hence homotopy theory is not completely vacuous. How about connected spaces which are not contractible? We'll have to wait a little while to be able to prove that. But here's another interesting example of a pair of homotopy equivalent spaces:

Lemma 1.1.21. $\mathbb{R}^n\setminus\{0\}$ *is homotopy equivalent to* $\mathbb{S}^{n-1}$.

Proof. Let $i\colon\mathbb{S}^{n-1}\to(\mathbb{R}^n\setminus\{0\})$ be the inclusion of the sphere, and define $p\colon(\mathbb{R}^n\setminus\{0\})\to\mathbb{S}^{n-1}$ by

$$p(v)=\frac{v}{|v|}.$$

(That's why we had to take out 0 - we wouldn't have known what to do with it in this formula.) Then we claim that i and p are homotopy inverses. Indeed, clearly

$$p\circ i=\mathrm{id}_{\mathbb{S}^{n-1}},$$

and if define $H\colon(\mathbb{R}^n\setminus\{0\})\times I\to(\mathbb{R}^n\setminus\{0\})$ by

$$H(v,t)=\left(1+\left(\frac{1}{|v|}-1\right)t\right)v,$$

then H is a homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to $i\circ p$. □

We are almost ready to define the underlying set of the fundamental group. Of course, the group structure is the really important thing, and we will have to briefly go over some group theory before we can define that. We need to introduce one more piece of homotopy-theoretic terminology before we can do that:

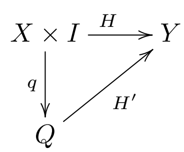
Definition 1.1.22. Suppose X is a space and $Z \subseteq X$ is a subspace. Then if $f, g\colon X \to Y$ are a pair of maps, a *homotopy from f to g relative to Z* is a homotopy H from f to g such that if $z \in Z$, then $H(z,t)$ is independent of t. In other words, points of Z stay where they are. In particular, if f and g are homotopic relative to Z (we say $f \sim_Z g$), then

$$f|_Z = g|_Z.$$

Remark 1.1.23. If H is a homotopy from f to g, then H is a homotopy relative to Z if and only if H factors through the quotient space

$$Q = (X \times I)/\{(z,t) = (z,t')|t,t' \in I, z \in Z\}$$

(by which we mean there is some $H'\colon Q \to Y$ such that $H = H' \circ q$.)



Definition 1.1.24. Let X be a space and $A \subset X$ a subspace, $i\colon A \to X$ the inclusion map..

- A is a *retract* of X if there exists a continuous map (a *retraction*) $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$, that is, $ri = \mathrm{id}_A$.
- A is a *deformation retract* of X if the identity map of X is homotopic rel A to a retraction $r\colon X \to A$ of X onto A. A *deformation retraction* is such a homotopy $X \times I \to X$ rel A between the identity on X and the retraction. Since $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_A$ rel A, we see in particular that A and X are homotopy equivalent spaces.

Note that a retraction of X onto A is a left inverse to the inclusion of A into X. A retraction can also be defined as an idempotent map, a map such that $r^n = r$ for all $n \geq 1$.

The subspace A is a deformation retract of X if one of the following equivalent conditions hold

- there exists a map, the deformation retraction, $R\colon X \times I \to X$ such that $R(x,0) = x$ and $R(x,1) \in A$ for all $x \in X$ while $R(a,t) = a$ for all $a \in A$, $t \in I$,
- there exists a map $r\colon X \to A$ such that $ri = \mathrm{id}_A$ and $ir \sim \mathrm{id}_X$ rel A (where i is the inclusion map).

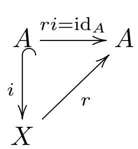
Proposition 1.1.25. *Let X be a space and $A \subset X$ a subspace. Then the following holds.*

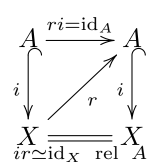
1. *A is a retract of $X \Leftrightarrow$ Any map on A extends to X*
2. *A is a deformation retract of $X \Leftrightarrow$ Any map on A extends uniquely up to homotopy relative to A to X*

Proof. First assertion:
$\Rightarrow$: Let $r\colon X \to A$ be a retraction of X onto A. If $f\colon A \to Y$ is a map defined on A then $fr\colon X \to Y$ is an extension of f to X.
$\Leftarrow$: The identity map of A extends to a map $r\colon X \to A$ defined on X.
Second assertion:
$\Rightarrow$: Let $r\colon X \to A$ be a retraction of X onto A such that $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$ rel A. Let $f\colon A \to Y$ be a map defined on A. Since A is a retract of X, f extends to X. Suppose that $f_0, f_1\colon X \to Y$ are two extensions of f. Then $f_0 = f_0 \circ \mathrm{id}_X \simeq f_0 ir$ rel A and $f_1 = f_1 \circ \mathrm{id}_X \simeq f_1 ir$ rel A. As $f_0 i = f_1 i$, this says that $f_0 \simeq f_1$ rel A.
$\Leftarrow$: The identity map of A extends to a map $r\colon X \to A$ defined on X and $ir \simeq \mathrm{id}_X$ rel A as both ir and id_X are extensions of the inclusion of A into X. □

The following diagrammes epitomises the difference between a retract and a deformation retract.

Retract: Deformation retract:





We already noted that if A is a deformation retract of X, then the inclusion of A into X is a homotopy equivalence. The converse does not hold in general. If the inclusion map is a homotopy equivalence, there exists a map $r\colon X \to A$ such that $ri \simeq \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$ but r may not fix the points of A and, even if it does, the points in A may not be fixed under the homotopy from ri to the identity of A. Surprisingly enough, however, the converse does hold if the pair (X, A) has a sufficiently nice property (see §1.4.)

Example 1.1.26. $\mathbb{S}^2$ is a retract of $\mathbb{S}^2 \vee \mathbb{S}^1$ and a deformation retract of $\mathbb{S}^2 \vee I$.

Any retract A of a Hausdorff space X is closed for $A = \{x \in X | r(x) = x\}$ is the equaliser of two continuous maps.

If A is a deformation retract of X, then the inclusion map $i\colon A \hookrightarrow X$ is a homotopy equivalence. Conversely, if the inclusion map is a homotopy equivalence, there exists a map $r\colon X \to A$ such that $ri \sim \mathrm{id}_A$ and $ir \sim \mathrm{id}_X$. This is *not* quite the same as saying that A is a deformation retract of X since r may not fix the points of A and, even if it does, the points in A may not be fixed under the homotopy from ri to the identity of A. However, surprisingly enough, the converse does hold if the pair (X, A) is sufficiently well behaved, namely, a cofibration (Proposition **??**).

Definition 1.1.27. Let X be a topological space and let $x_0 \in X$ be a point. Then the *fundamental group of X based at x_0*, denoted $\pi_1(X, x_0)$, is the set

$$\{p\colon I \to X | p(0) = p(1) = x_0\}/\{\text{homotopy relative to } \partial I = \{0, 1\}\}$$

Remark 1.1.28. As we know from our study of quotient spaces, a map from I which sends 0 and 1 to the same point is the same as a map from $\mathbb{S}^1$, so we could also formulate the definition in those terms. However, it will be useful to us to think of these elements as maps from the interval.

1.2 Construction of Spaces

Mapping cylinder and mapping cone

We introduce constructions called *mapping cylinder* and *mapping cone* which are extensively used in algebraic topology.

Definition 1.2.1. The mapping cylinder M_f of a continuous function f between topological spaces X and Y is the quotient

$$M_f := ((I \times X) \coprod Y)/\sim$$

where the $\coprod$ denotes the disjoint union, and $\sim$ is the equivalence relation generated by

$$(0, x) \sim f(x) \quad \text{for each} \quad x \in X.$$

That is, the mapping cylinder M_f is obtained by gluing one end of $X \times I$ to Y via the map f. Notice that the "top" of the cylinder $\{1\} \times X$ X is homeomorphic to X, while the "bottom" is the space $f(X) \subset Y$. Sometimes we write Mf for M_f, and to use the notation $\sqcup_f$ or $\cup_f$ for the mapping cylinder construction. That is, one writes

$$Mf = (I \times X) \cup_f Y$$

with the subscripted cup symbol denoting the equivalence.

Note that the mapping cylinder deformation retracts onto its subspace Y. (Set $r(x, t) = f(x)$ for $x \in X$, $t \in I$, and $r(y) = y$ for $y \in Y$.)

The mapping cylinder may be viewed as a way to replace an arbitrary map by an equivalent cofibration (see §**??**), in the following sense: Given a map $f\colon X \to Y$, the mapping cylinder is a space M_f, together with a cofibration $\widetilde{f}\colon X \to M_f$ and a surjective homotopy equivalence $M_f \to Y$ (indeed, Y is a

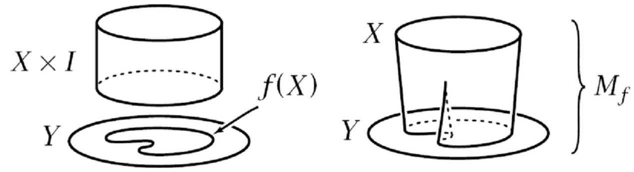
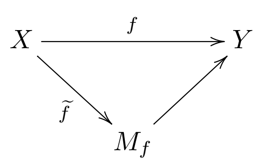


Figure 1.1: A mapping cylinder

deformation retract of M_f, such that the composition $X \to M_f \to Y$ equals f.



Thus the space Y gets replaced with a homotopy equivalent space M_f, and the map f with a lifted map $\widetilde{f}$. Equivalently, the diagram

$$f\colon X \to Y$$

gets replaced with a diagram

$$\widetilde{f}\colon X \to M_f$$

together with a homotopy equivalence between them.

The construction serves to replace any map of topological spaces by a homotopy equivalent cofibration. Note that pointwise, a cofibration is a closed inclusion.

The construction of a mapping cylinder M_f of a continuous map $f\colon X \to Y$ is an example of the coarse type of gluing and pasting constructions we are allowed to do once we go beyond manifolds. In this section we will introduce more such constructions, and introduce a class of spaces which is very convenient for algebraic topology.

Definition 1.2.2. Given a map $f\colon X \to Y$, the *mapping cone* C_f is defined to be the quotient space of the mapping cylinder $(X \times I) \sqcup_f Y$ with respect to the equivalence relation $\forall x, x' \in X,\ (x,0) \sim (x',0),\ (x,1) \sim f(x)$. Here I denotes the unit interval $[0,1]$ with its standard topology. Note that some authors (like J. Peter May) use the opposite convention, switching 0 and 1.

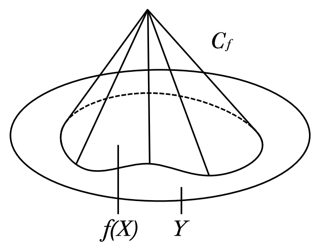


Figure 1.2: A mapping cone

Visually, one takes the cone on X (the cylinder $X \times I$ with one end (the 0 end) identified to a point), and glues the other end onto Y via the map f (the identification of the 1 end).

There is a sequence of maps (called "Puppe sequence")

$$X \xrightarrow{f} Y \to C_f \to SX \xrightarrow{S_f} SY \to C_{S_f} \to SSX \to \cdots$$

where the map $C_f \to SX$ is collapse of $Y \subset C_f$.

Proposition 1.2.3. *Any map factors as an inclusion map followed by a homotopy equivalence.*

Proof. For any map $f\colon X \to Y$ there is a commutative diagramme using the mapping cylinder

$$\begin{array}{ccc} & & M_f \\ & \overset{x \mapsto (x,1)}{\nearrow} & \simeq \big\downarrow (x,t) \mapsto f(x) \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the slanted map is an inclusion map and the vertical map is a homotopy equivalence (the target is deformation retract of the mapping cylinder). □

Example 1.2.4. (Wedge sum and smash product of pointed spaces) Let (X, x_0) and (Y, y_0) be pointed spaces. The *wedge sum* and the *smash product* of X and Y are

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y, \quad X \wedge Y = (X \times Y)/(X \vee Y)$$

The *reduced suspension* of the pointed space (X, x_0) is the smash product

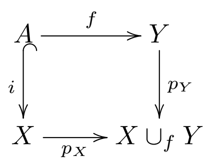
$$\sum X = X \wedge \mathbb{S}^1 = X \wedge (I/\partial I) = (X \times I)/(X \times \partial I \cup x_0 \times I)$$

of X and a pointed circle $(\mathbb{S}^1, 1) = (I/\partial I, \partial I/\partial I)$.

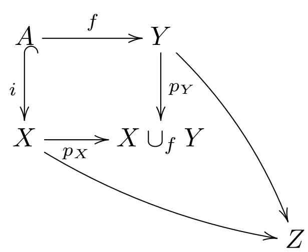
Example 1.2.5. (The mapping cylinder for the degree n map on the circle) Let $n > 0$ and let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the map $f(z) = z^n$ where we think of the circle as the complex numbers of modulus 1. Let $C_n = \langle t | t^n \rangle$ be the cyclic group of order En. The mapping cylinder M_f of f is quotient space of $\bigvee_{C_n} I) \times I$ by the equivalence relation $\sim$ that identifies $(s, x, 0) \sim (st, x, 1)$ for all $s \in C_n$, $x \in I$.

Example 1.2.6. (Adjunction spaces) Let X and Y be two disjoint topological spaces and $f\colon A \to Y$ a continuous map defined on a closed subspace A of X. Define $X \cup_f Y$ to be the quotient of $X \amalg Y$ (disjoint union) by the smallest equivalence relation such that $a \in A$ and $f(a) \in Y$ are equivalent for all points $a \in A$. (To picture this, tie an elastic band from each point a of A to its image $f(a)$ in Y and let go!) The equivalence classes, $[y] = f^{-1}(y) \cup \{y\}$ for $y \in Y$ and $[x] = \{x\}$ for $x \in X \setminus A$, are represented by points in Y or in $X \setminus A$. Let $p\colon X \amalg Y \to X \cup_f Y$ be the quotient map; p_X the restriction of p to X and p_Y the restriction of p to Y.

The adjunction space $X \cup_f Y$ fits into a commutative diagramme



called a *push-out diagramme* because of this universal property: If $X \to Z$ and $Y \to Z$ are continuous maps that agree on A then there is a unique continuous map $X \cup_f Y \to Z$ such that the diagram



commutes. (This is just the universal property for quotient spaces in this particular situation.)

Here are the main properties of adjunction spaces.

Lemma 1.2.7. *Let $p\colon X \amalg Y \to X \cup_f Y$ be the quotient map.*

(1) The quotient map p embeds Y into a closed subspace of $X \cup_f Y$. (We therefore identify Y with its image $p_Y(Y)$ in the adjunction space.)

(2) The quotient map p embeds $X \setminus A$ into the open subspace $(X \cup_f Y) \setminus Y$ of the adjunction space.

(3) If X and Y are normal, also the adjunction space $X \cup_f Y$ is normal.

(4) The projection map $p\colon X \amalg Y \to X \sup_f Y$ *is closed if and only if* f *is closed.*

Proof. (1): The map $p_Y = p|_Y\colon Y \to X \cup_f Y$ is closed for closed sets $B \subset Y \subset X \amalg Y$ have closed saturations $f^{-1}(B) \amalg B$. Since p_Y is also injective it is an embedding.
(2) The map $p_X|_{X\setminus A}\colon X \setminus A \to (X \cup_f Y)$ is open because the saturation of any (open) subset U of $X \setminus A$ is $U \cup \varnothing \subset X \cup Y$ itself. Since $p_X|_{X\setminus A}$ is also injective it is an embedding.
(3): Points are closed in the quotient space $X \cup_f Y$ because the equivalence classes are closed in $X \cup Y$. Let C and D be two disjoint closed subspaces of $X \cup_f Y$. We will show that there is a continuous map $X \cup_f Y \to [0,1]$ with value 0 on C and value 1 on D. Since Y is normal, there exists a Urysohn function $g\colon Y \to [0,1]$ such that $g(Y\ capC) = \{0\}$ and $g(Y \cap D) = \{1\}$. Since X is normal, by the Tietze extension theorem, there is a continuous map $X \to [0,1]$ which is 0 on $p_X^{-1}(C)$, 1 on $p_X^{-1}(D)$, and is $g \circ f$ on A. By the universal property for adjunction spaces (2), there is a map $X \cup_f Y \to [0,1]$ that is 0 on C and 1 on D. This shows that C and D can be separated by a continuous function and that $X \cup_f Y$ is normal.
(4): Closed subsets of Y always have closed saturations as we saw in item (1). If f is closed then also the saturation, $B \cup f^{-1}f(A \cap B) \cup f(A \cap B) \subset X \cup Y$, of a closed subset $B \subset X$ is closed. (Since closed quotient maps (surjective closed maps) preserve normality, this gives an easy proof of (3) under the additional assumption that $f\colon A \to Y$ be a closed map.) □

1.3 CW Complexes

CW complexes: construction

A *CW complex*, is a topological space constructed from disks (called *cells*), step by step increasing in dimension. The basic procedure in the construction is called "attaching an n-cell". An n-cell is the interior e^n of a closed disk D^n of dimension n. How to attach it to a space X? Simply glue D^n to X with a continuous map $\varphi\colon \mathbb{S}^{n-1} \to X$, forming:

$$X \sqcup D^n / \{x \sim \varphi(x) : x \in \partial D^n\}.$$

The result is a topological space (with the quotient topology), but as a set, is the disjoint union $X \sqcup e^n$.

The attaching process can be expressed more formally (pedantically?) as follows:
Let X be a space and $\phi\colon \amalg \mathbb{S}^{n-1}_\alpha \to X$ a map from a disjoint union of spheres

into X. The adjunction space (= the mapping cone on ϕ)

$$\begin{array}{ccc} \coprod \mathbb{S}^{n-1}_\alpha & \xrightarrow{\ \phi\ } & X \\ \big\downarrow & & \big\downarrow \\ \coprod \mathbb{D}^n_\alpha & \xrightarrow[\ \overline{\phi}\ ]{} & X \cup_\phi \coprod \mathbb{D}^n_\alpha \end{array}$$

is called the *n-cellular extension* of X with *attaching map* ϕ and *characteristic map* $\overline{\phi}$.

Building a cell complex X A CW-complex is a space X with a sequence of subspaces (called *skeleta*)

$$\varnothing = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n = X = \cup X^n$$

constructed in the following way:

- Start with a discrete set X^0, whose points we view as 0-cells.
- Inductively form the n-skeleton X^n from X^{n-1} by attaching a set of n-cells $\{e^n\}$ to X^{n-1}. I.e, X^n is (homeomorphic to) an n-cellular extension of X^{n-1} for $n \geq 1$.
- Either set $X = X^n$ for some $n < \infty$, or set $X = \cup_n X^n$, where in the infinite case we use the *weak topology*. I.e, the topology on X is coherent with the filtration in the sense that

  $$A \text{ is closed (open) in } X \Leftrightarrow A \cap X^n \text{ is closed (open) in } X^n \text{ for all } n$$

  for any subset A of X.

The second item of the definition means that for every $n \geq 0$ there are *attaching maps* $\varphi_\alpha \colon \mathbb{S}^{n-1} \to X^{n-1}$ and *characteristic maps* $\Phi_\alpha \colon \mathbb{D}^n \to X^n$ such that

- The n-skeleton
  $$X^n = X^{n-1} \cup_{\coprod_\alpha \varphi_\alpha} \coprod_\alpha \mathbb{D}^n_\alpha$$
  is the n-cellular extension of the $(n-1)$-skeleton by the attaching maps.
- The complement in the n-skeleton of the $(n-1)$-skeleton,

  $$X^n \setminus X^{n-1} = \coprod_\alpha e^n_\alpha \xleftarrow{\coprod \Phi_\alpha} \coprod_\alpha \operatorname{int} \mathbb{D}^n$$

  is the disjoint union of its connected components $e^n_\alpha = \Phi_\alpha(\operatorname{int} \mathbb{D}^n)$, the open n-cells of X. An open n-cell is open in X^n but maybe not in X.

- A CW-complex is the disjoint union

$$X = \bigcup_{n=-1}^{\infty} Xn = \bigcup_{n=0}^{\infty} (X^n \setminus X^{n-1}) = \coprod_{n=0}^{\infty} \coprod_{\alpha} e_{\alpha}^{n}$$

  of its open cells. This is a disjoint union of sets (but usually not of topological spaces).

- The quotient of the n-skeleton by the $(n-1)$-skeleton,

$$X^n/X^{n-1} = \bigvee_{\alpha} (\mathbb{D}^n/\mathbb{S}^{n-1}) = \bigvee_{\alpha} S^n$$

  is a wedge sum (or bouquet) of n-spheres.

X^1 is a topological space since it is a 1-cellular extension of the topological space X^0. In fact, all the skeleta X^n are topological spaces and X^i is a closed subspace of X^j for $i \leq j$. The purpose of the third item of the definition is to equip the union of all the skeleta with the largest topology making all the inclusions continuous.

A CW-complex X is finite-dimensional if $X = X^n$ for some n *Caveat*: CW-decompositions are not unique; there are generally many CW-decompositions of a given space X. We will see $\mathbb{S}^2$ has two distinct CW-decompositions.

Example 1.3.1. The 1-skeleton of a cell complex is a graph, and may have loops.

Example 1.3.2. (Compact surfaces as CW-complexes)

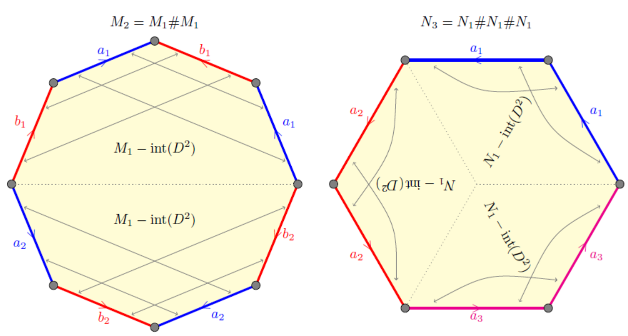


Figure 1.3: Surfaces as CW-complexes

The closed orientable surface $M_g = (\mathbb{S}^1 \times \mathbb{S}^1) \# \cdots \# (\mathbb{S}^1 \times \mathbb{S}^1)$ of genus $g \geq 1$ is a CW-complex

$$M_g = \bigvee_{1 \leq i \leq g} \mathbb{S}_{a_i}^1 \vee \mathbb{S}_{b_i} \cup_{\prod [a_i, b_i]} \mathbb{D}^2$$

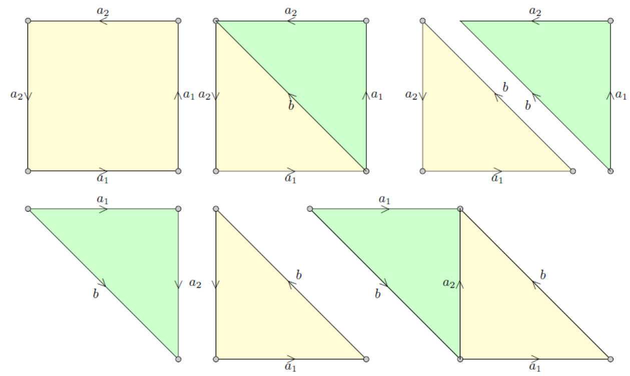


Figure 1.4: Two representations of the Klein bottle N_2

with a single 0-cell, $2g$ 1-cells, and a single 2-cell (see the left side of Figure 1.3).

One sees immediately from this representation that to puncture such a surface at a single point would render it homotopy equivalent to a "wedge" of $2g$ circles, i.e. the disjoint union of $2g$ circles where $2g$ points, one from each circle, are identified.

The closed nonorientable surface $N_g = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ of genus $h \geq 1$ is a CW-complex

$$N_g = \bigvee_{1 \leq i \leq h} \mathbb{S}^1_{a_i} \cup_{\prod a_i^2} \mathbb{D}^2$$

with a single 0-cell, g 1-cells, and a single 2-cell. (See Fiure 1.4 for $g = 2$.)

Example 1.3.3. (Spheres as CW-complexes: 1) The n-sphere $\mathbb{S}^n$ may be expressed as a cell complex with a single 0-cell and a single n-cell. So $\mathbb{S}^n = e^0 \sqcup e^n$.

Example 1.3.4. (Spheres as CW-complexes: 2) There is another way of decomposing n-sphere into a CW-complex: we can think $\mathbb{S}^n$ an be obtained from $\mathbb{S}^{n-1}$ by attaching two n-cells (the Northern and Southern hemispheres) as follows: Points on the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ have coordinates of the form (x, u). Let $\mathbb{D}^n_\pm$ be the images of the embeddings $\mathbb{D}^n \to \mathbb{S}^n$: $x \mapsto (x, \pm\sqrt{1 - |x|^2})$. Then

$$\mathbb{S}^n = \mathbb{S}^{n-1} \cup \mathbb{D}^n_+ \cup \mathbb{D}^n_- = \mathbb{S}^{n-1} \cup_{\mathrm{id} \coprod \mathrm{id}} (\mathbb{D}^n \coprod \mathbb{D}^n)$$

is obtained from $\mathbb{S}^{n-1}$ by attaching two n-cells. Thus $\mathbb{S}^n$ is a finite CW-complex with two cells in each dimension 0 through n.

The infinite sphere S^∞ is an infinite dimensional CW-complex

$$\mathbb{S}^0 \subset \mathbb{S}^1 \subset \cdots \subset \mathbb{S}^{n-1} \subset \mathbb{S}^n \subset \cdots \subset \mathbb{S}^\infty = \bigcup_{n=0}^{\infty} \mathbb{S}^n$$

with two cells in each dimension. A subspace A of $\mathbb{S}^\infty$ is closed if and only if $A \cap \mathbb{S}^n$ is closed in $\mathbb{S}^n$ for all n.

Example 1.3.5. (Projective spaces as CW-complexes) The projective spaces are $\mathbb{R}P^n = S(\mathbb{R}^{n+1})/S(\mathbb{R})$, $\mathbb{C}P^n = S(\mathbb{C}^{n+1}) = S(\mathbb{C})$, and $\mathbb{H}P^n = S(\mathbb{H}^{n+1}) = S(\mathbb{H})$. In each case there are maps

$$\mathbb{D}^n = D(\mathbb{R}^n) \overset{x\mapsto(x,\sqrt{1-|x|^2})}{\hookrightarrow} S(\mathbb{R}^{n+1}) = \mathbb{S}^{n+1} \overset{p_n}{\twoheadrightarrow} \mathbb{R}P^n$$
$$\mathbb{D}^{2n} = D(\mathbb{C}^n) \overset{x\mapsto(x,\sqrt{1-|x|^2})}{\hookrightarrow} S(\mathbb{C}^{n+1}) = \mathbb{S}^{2n+1} \overset{p_n}{\twoheadrightarrow} \mathbb{C}P^n$$
$$\mathbb{D}^{4n} = D(\mathbb{H}^n) \overset{x\mapsto(x,\sqrt{1-|x|^2})}{\hookrightarrow} S(\mathbb{H}^{n+1}) = \mathbb{S}^{4n+3} \overset{p_n}{\twoheadrightarrow} \mathbb{H}P^n$$

We note that each map $D(\mathbb{F}^n) \hookrightarrow S(\mathbb{F}^{n+1}) \twoheadrightarrow \mathbb{F}P^n$ is

- surjective,
- restricts to the projection $p_{n-1}\colon S(\mathbb{F}^n) \to \mathbb{F}P^{n-1}$ on the boundary $S(\mathbb{F}^n)$ of the disc $D(\mathbb{F}^n)$,
- injective on the interior $D(\mathbb{F}^n) \setminus S(\mathbb{F}^n)$ of the disc,

where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. To prove the first item observe that any point in projective space is represented by a point on the sphere with last coordinate ≥ 0. This means that $\mathbb{R}P^n$ consists of $\mathbb{R}P^{n-1}$ together with the n-disc $D(\mathbb{R}^n)$ with identifications on the boundary. In other words

$$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_{p_{n-1}} \mathbb{D}^n, \quad \mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_{p_{n-1}} \mathbb{D}^{2n}, \quad \mathbb{H}P^n = \mathbb{H}P^{n-1} \cup_{p_{n-1}} \mathbb{D}^{4n}.$$

Consequently, $\mathbb{R}P^n$ is a finite CW-complex with one cell in every dimension between 0 and n, $\mathbb{C}P^n$ is a finite CW-complex with one cell in every even dimension between 0 and $2n$, $\mathbb{H}P^n$ is a finite CW-complex with one cell in every dimension divisible by 4 between 0 and $4n$,

In particular, $\mathbb{R}P^0 = *, \mathbb{C}P^0 = *, \mathbb{H}P^0 = *$, and $\mathbb{R}P^1 = \mathbb{S}^1, \mathbb{C}P^1 = \mathbb{S}^2, \mathbb{H}P^1 = \mathbb{S}^4$. The Hopf maps are the projection maps

$$\mathbb{S}^0 \to \mathbb{S}^1 \xrightarrow{p_1} \mathbb{R}P^1 = \mathbb{S}^1, \quad \mathbb{S}^1 \to \mathbb{S}^3 \xrightarrow{p_1} \mathbb{C}P^1 = \mathbb{S}^2, \quad \mathbb{S}^3 \to \mathbb{S}^7 \xrightarrow{p_1} \mathbb{H}P^1 = \mathbb{S}^4, \tag{1.3.6}$$

obtained when $n = 1$.

Definition 1.3.7. Let A be any topological space. A *relative CW-complex on* A is a space X with an ascending filtration of subspaces (called *skeleta*)

$$A = X^{-1} \subset X^0 \subset X^1 subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X = \bigcup X^n$$

such that

- X^0 is the union of A and a discrete set of points,
- X^n is (homeomorphic to) an n-cellular extension of X^{n-1} for $n \geq 1$,
- the topology on X is coherent with the filtration in the sense that B is closed (open) in X $\Leftrightarrow$ $B \cap X^n$ is closed (open) in X^n for all n for any subset B of X.

Topological properties of CW-complexes.

We shall see that CW-complexes have convenient topological properties.

Proposition 1.3.8. *Any CW-complex is a Hausdorff, even normal, topological space.*

Proof. See Lemma 1.2.7. □

Lemma 1.3.9. *The closure of the open n-cell $e^n_\alpha = \Phi_\alpha(\operatorname{int}\mathbb{D}^n)$ is $\overline{e}^n_\alpha = \Phi_\alpha(\mathbb{D}^n)$.*

Proof. The image $\Phi_\alpha(\mathbb{D}^n)$ of the compact space $\mathbb{D}^n)$ is compact and therefore closed in the Hausdorff space X. Thus $e^n_\alpha \subset \Phi_\alpha(\mathbb{D}^n)$. On the other hand, we have

$$\Phi_\alpha(\mathbb{D}^n) = \Phi_\alpha(\overline{\mathbb{D}^n \setminus \mathbb{S}^{n-1}}) \subset \overline{\Phi_\alpha(\mathbb{D}^n \setminus \mathbb{S}^{n-1})} = \overline{e}^n_\alpha$$

simply because Φ_α is continuous. □

Proposition 1.3.10. *Any compact subspace of a CW-complex X is contained in a skeleton.*

Proof. Let X be a CW-complex and C a compact subspace of X. Choose a point t_n in $C \cap (X^n \setminus X^{n-1})$ for all n where this intersection is nonempty. Let $T = \{t_n\}$ be the subspace of these points. For all n, $T \cap X^n$ is finite and hence closed in X since points are closed in X (Proposition 1.3.8). Thus T is closed since X has the coherent topology. In fact, any subspace of T is closed by the same argument. In other words, T has the discrete topology. As a closed subspace of the compact space C, T is compact. Thus T is compact and discrete. Then T is finite. □

Subcomplexes.

We define what we mean by a *subcomplex*.

Definition 1.3.11. A subcomplex of a CW-complex is a closed subspace that is a union of open cells.

If A is subcomplex then the closure of any open cell in A is still in A since A is closed.

If A is a subcomplex of the CW-complex X then

- A is a CW-complex with n-skeleton $A^n = A \cap X^n$,
- (X, A) is a relative CW-complex,
- (X, A) has the homotopy extension property (see Section 1.4),
- X/A is a CW-complex and the quotient map $X \to X/A$ is cellular.

Example 1.3.12. The n-skeleton of X is always a subcomplex of X.

Consider $X = \mathbb{S}^1 \vee \mathbb{S}^2$ as a CW-complex with one 0-cell, one 1-cell, and one 2-cell attached at a point different from the 0-cell. Then closed subspace $\mathbb{S}^1$ is subcomplex of X. The closed subspace $\mathbb{S}^2$ is not a subcomplex since it is not the union of open cells.

Products of CW-complexes.

We shall now discuss the product of two CW-complexes. A slight complication will arise because product topologies and infinite union (= colimit) topologies do not in general commute.

Definition 1.3.13. Let (X, A) and (Y, B) be two CW pairs — a CW pair is (CW complex, its subcomplex). The product of two pairs is defined as

$$(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$$

where $(X \times Y) \setminus (A \times Y \cup X \times B) = (X \setminus A) \times (Y \setminus B)$.

For example, if I^n is the unit cube in $\mathbb{R}^n$ then clearly

$$(I^n, \partial I^n) = (I^i, \partial I^i) \times (I^j, \partial I^j)$$

whenever $i, j \geq 0$ and $i + j = n$. Since $(\mathbb{D}^n, \mathbb{S}^{n-1})$ and $(I^n, \partial I^n)$ are homeomorphic pairs, we have just seen that

$$(\mathbb{D}^n, \mathbb{S}^{n-1}) = (\mathbb{D}^i, \mathbb{S}^{i-1}) \times (\mathbb{D}^j, \mathbb{S}^{j-1})$$

where $i, j \geq 0$ and $i + j = n$ and the equality sign means that the two sides are homeomorphic. We make this observation because by convention we build CW-complexes from discs rather than cubes.

Let $X = A \cup_\varphi \mathbb{D}^i$, $Y = B \cup_\psi \mathbb{D}^j$, be an i-cellular and a j-cellular extension with characteristic maps $\Phi\colon (\mathbb{D}^i, \mathbb{S}^{i-1}) \to (X, A)$, $\Psi\colon (\mathbb{D}^j, \mathbb{S}^{j-1}) \to (Y, B)$ and open cells $e^i = X \setminus A$ and $f^j = Y \setminus B$. The product $X \times Y$ is an $(i + j)$-cellular extension (see below Definition 1.3.14)

$$X \times Y = (A \times Y \cup X \times B) \cup_{(\Phi\times\Psi)|_{\mathbb{S}^{i+j-1}}} (\mathbb{D}^i \times \mathbb{D}^j)$$

with one open cell

$$X \times Y \setminus (A \times Y \cup X \times B) = (X \setminus A) \times (Y \setminus B) = e^i \times f^j$$

which is the product of the open cells in X and Y.

The characteristic map of $X \times Y$ is the product

$$\Phi \times \Psi\colon (\mathbb{D}^i, \mathbb{S}^{i-1}) \times (\mathbb{D}^j, \mathbb{S}^{j-1}) \to (X, A) \times (Y, B)$$

of the characteristic maps and the attaching map

$$(\Phi \times \Psi)|_{\mathbb{S}^{i+j-1}}\colon \mathbb{D}^i \times \mathbb{S}^{j-1} \cup \mathbb{S}^{i-1} \times \mathbb{D}^j \to X \times B \cup A \times Y$$

is the restriction of $\Phi \times \Psi$ to the sphere $\mathbb{S}^{i+j-1} = \mathbb{D}^i \times \mathbb{S}^{j-1} \cup \mathbb{S}^{i-1} \times \mathbb{D}^j$.

Definition 1.3.14. Let X and Y be CW-complexes with characteristic maps $\Phi_\alpha\colon (\mathbb{D}^i, \mathbb{S}^{i-1}) \to (X^i, X^{i-1})$ and $\Phi_\beta\colon (\mathbb{D}^j, \mathbb{S}^{j-1}) \to (Y^j, Y^{i-1})$. The product CW-complex has n-skeleton

$$(X \times_{\mathrm{CW}} Y)^n = \bigcup_{i+j=n} X^i \times Y^j$$

The characteristic maps for the n-cells are products of characteristic maps

$$\begin{aligned}\Phi_\alpha \times \Phi_\beta \colon (\mathbb{D}^i, \mathbb{S}^{i-1}) \times (\mathbb{D}^j, \mathbb{S}^{j-1}) &\to (X^i, X^{i-1}) \times (Y^j, Y^{i-1}) \\ &\subset ((X \times_{\mathrm{CW}} Y)^n, (X \times_{\mathrm{CW}} Y)^{n-1})\end{aligned}$$

for all $i, j \geq 0$ and $i + j = n$. The attaching maps for the n-cells are the restrictions

$$\mathbb{D}^i \times \mathbb{S}^{j-1} \cup \mathbb{S}^{i-1} \times \mathbb{D}^j \to (X^i \times Y^{j-1} \cup X^{i-1} \times Y) \subset (X \times Y)^{n-1}$$

of the characteristic maps. $(X \times_{\mathrm{CW}} Y)$ has the topology coherent with the skeleta.

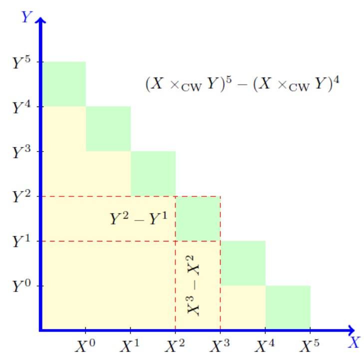


Figure 1.5: Skeleta in product CW complex

There is a commutative diagram

$$\begin{array}{ccc} (X \times_{\mathrm{CW}} Y)^{n-1} \amalg \coprod_{i+j=n} (D^i_\alpha \times D^j_\beta) & \xrightarrow{\text{incl} \amalg \coprod (\Phi^i_\alpha \times \Phi^j_\beta \times)} & (X \times_{\mathrm{CW}} Y)^n \\ \downarrow & \nearrow & \\ (X \times_{\mathrm{CW}} Y)^{n-1} \cup_\rho \coprod_{i+j=n} (D^i_\alpha \times D^j_\beta) & & \end{array}$$

The horizontal map is closed and the slanted map, produced by the universal property, is a homeomorphism (because it is a closed continuous bijection). This shows that $(X \times_{\mathrm{CW}} Y)^n$ is an n-cellular extension of $(X \times_{\mathrm{CW}} Y)^{n-1}$. Thus $X \times_{\mathrm{CW}} Y$ is a CW-complex. The open n-cells of the product CW-complex,

$$\begin{aligned}(X \times_{\mathrm{CW}} Y)^n \setminus (X \times_{\mathrm{CW}} Y)^{n-1} &= \coprod_{i+j=n} (X^i \setminus X^{i-1}) \times (Y^j \setminus Y^{j-1}) \\ &= \coprod_{i+j=n} \left(\coprod_\alpha e^i_\alpha \times \coprod_\alpha e^i_\alpha \right) = \coprod_{i+j=n,\alpha,\beta} e^i_\alpha \times f^j_\beta\end{aligned}$$

are the products of the open cells e^i_α in X with the open cells f^j_β in Y for all $i, j \geq 0$ with $i + j = n$.

The topology on $X \times_{\mathrm{CW}} Y$, defined to be the topology coherent with the ascending skeletal filtration, is finer than the product topology. We cite, but will not prove, the following:

Theorem 1.3.15. *There is a bijective continuous map $X \times_{\mathrm{CW}} Y \to X \times Y$. This map is a homeomorphism if X and Y have countably many cells.*

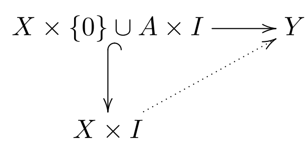
A proof can be found in the appendix of Hatcher "Algebraic Topology".

In all cases relevant for us, $X \times_{\mathrm{CW}} Y$ and $X \times Y$ are homeomorphic.

1.4 The Homotopy Extension Property

The Homotopy Extension Property will be very important to algebraic topology.

Definition 1.4.1. Let X be a space with a subspace $A \subset X$. The pair (X, A) has the *Homotopy Extension Property* (*HEP* for short) if any partial homotopy $A \times I \to Y$ of a map $X \to Y$ into any space Y can be extended to a (full) homotopy of the map. That is, if it is always possible to complete the diagramme



for any space Y and any partial homotopy of a map $X \to Y$.

The pair $(X, \varnothing)$ always has the HEP. A *nondegenerate base point* is a point $x_0 \in X$ such that $(X, \{x_0\})$ has the HEP.

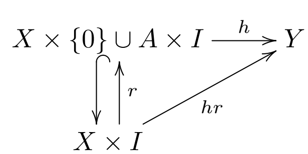
Proposition 1.4.2. *Let X be a space and $A \subset X$ be a subspace. The following three conditions are equivalent*

(1) (X, A) has the HEP.

(2) The partial cylinder $X \times \{0\} \cup A \times I$ is a retract of the cylinder $X \times I$.

(3) The partial cylinder $X \times \{0\} \cup A \times I$ is a deformation retract of the cylinder $X \times I$.

Proof. If (X, A) has the HEP then the identity map of the partial cylinder $X \times \{0\} \cup A \times I$ extends to a retraction of the cylinder $X \times I$ onto the partial cylinder. Conversely, if the inclusion of the partial cylinder into the cylinder has a left inverse r then it is very easy



to find an extension of any partial homotopy h. This shows that (1) $\Leftrightarrow$ (2).
It is clear that (3 $\Rightarrow$ (2).
To prove that (2 $\Rightarrow$ (3) let $r\colon X \times I \to X \times \{0\} \cup A \times I$ be a retraction. Define a homotopy $H\colon X \times I \times I \to X \times I$ by

$$H(x, t, s) = (\pi_1 r(x, st), (1 - s)t + \pi_2 r(x, t))$$

where $X \xleftarrow{\pi_1} X \times I \xrightarrow{\pi_2} X \times I$ are the projections. Then $H(x, t, 0) = (x, t)$, $H(x, 1) = r(x, t)$, $H(x, 0, s) = (x, 0)$, and $H(a, t, s) = (a, t)$ for all $a \in A$. Thus H is a deformation retraction of the cylinder $X \times I$ onto the partial cylinder $X \times \{0\} \cup A \times I$. □

What is the HEP good for?

The next theorem explains what the HEP can do for you.

Theorem 1.4.3. *Suppose that* (X, A) *has the HEP.*

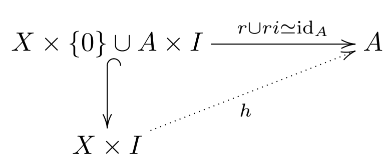
(1) If the inclusion map has a homotopy left inverse then A *is a retract of* X*.*

(2) If the inclusion map is a homotopy equivalence then A *is a deformation retract of* X*.*

(3) If A *is contractible then the quotient map* $X \to X/A$ *is a homotopy equivalence.*

(4) The homotopy type of the adjunction space $Y \cup_\varphi X$ *only depends on the homotopy class of the attaching map* $\varphi\colon A \to Y$ *for any space* Y *and any map* $\varphi\colon A \to Y$*.*

Proof. (1): Assume that $r\colon X \to A$ is a map such that $ri \simeq \mathrm{id}_A$. We must change r on A so that it actually fixes points of A. There is a map $X \times \{0\} \cup A \times I \to A$ which on $X \times \{0\}$ is r and on $A \times I$ is a homotopy from ri to the identity of A. Using the HEP we may complete the commutative diagramme

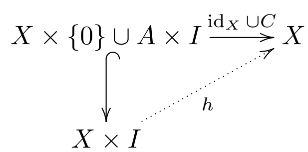


and get a homotopy $h\colon X \times I \to A$. The end-value of this homotopy is a map $h_1\colon X \to A$ such that $h_1 i = \mathrm{id}_A$ (a retract).
(2): Let $i\colon A \to X$ be the inclusion map. The assumption is that there exists a map $r\colon X \to A$ such that $ri \simeq \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$. By point (1) we can assume that $ri = \mathrm{id}_A$, i. e, that A is a retract of X. Let $G\colon X \times I \to X$ be a homotopy with start value $G_0 = \mathrm{id}_X$ and end value $G_1 = ir$. For $a \in A$, $G(a, 0) = a$ and $G(a, 1) = a$ but we have no control of $G(a, t)$ when $0 < t < 1$. We want to modify G into a deformation retraction, that is a homotopy from id_X to ir *relative to*

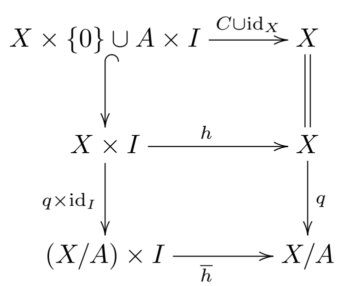
A. Since (X, A) has the HEP so does $(X, A) \times (I, \partial I) = (X \times I, A \times I \cup X \times \partial I)$ (Proposition 1.4.12, (3)). Let $H\colon X \times I \times I \to X \times I$ be an extension (a homotopy of homotopies) of the map $X \times I \times \{0\} \cup A \times I \times I \cup X \times \partial I \times I$ given by

$$\begin{aligned} H(x,t,0) &= G(x,t), \\ H(a,t,s) &= G(a, t(1-s)) \quad \text{for} \quad a \in A, \\ H(x,0,s) &= x, \\ H(x,1,s) &= G(ir(x), 1-s). \end{aligned}$$

Note that H is well-defined since the first line, $H(x,1,0) = G(x,1) = ir(x)$, and the fourth line, $H(x,1,0) = G(ir(x),1) = irir(x) = ir(x)$, yield the same result. The end value of $H\colon (x,t) \mapsto H(x,t,1)$, is a homotopy rel A of $H(x,0,1) = x$ to $H(x,1,1) = G(ir(x),0) = ir(x)$. This is a homotopy rel A since $H(a,t,1) = G(a,0) = a$ for all $a \in A$.
(3): We need to show that there is a homotopy inverse to the projection map $q\colon X \to X/A$ and this is more or less the same thing as a homotopy $X \times I \to X$ from the identity to a map that collapses A inside A. Note that we can get such a homotopy precisely because of the HEP! (In fact, this could be used as the motivation for HEP.) Let $C\colon A \times I \to A \subset X$ be a contraction of A, a homotopy of the identity map to a constant map. Use the HEP to extend the contraction of A and the identity on X



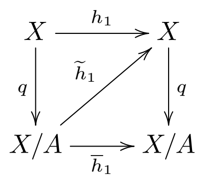
to a homotopy $h\colon X \times I \to X$ such that h_0 is the identity map of X, h_t sends A to A for all $t \in I$, and h_1 sends A to a point of A. By the universal property of quotient maps, the homotopy h induces a homotopy $\overline{h}$ and the map h_1 induces a map $\widetilde{h}_1$ such that the following diagramme commutes.



$$h_0 = \mathrm{id}_X, \quad h_1(A) = *$$

$$\overline{h}_0 = \mathrm{id}_{X/A}, \quad \overline{h}_1(*) = *$$

Note that the product map $q \times \mathrm{id}_X\colon X \times I \to (X/A) \times I$ is quotient since I is locally compact Hausdorff.) Since h_1 takes A to a point, it factors through the quotient space X/A. The lower square considered only at time $t = 1$ can be enlarged to a commutative diagramme



$$\widetilde{h}_1 q = h_1 \simeq h_0 = \mathrm{id}_X$$

$$q\widetilde{h}_1 = \overline{h}_1 \simeq \overline{h}_0 = \mathrm{id}_{X/A}$$

with a diagonal map $\widetilde{h}_1$ - and this is the homotopy inverse to q that we are looking for!

(4): Let $\varphi_0\colon A \to Y$ and $\varphi_1\colon A \to Y$ be two attaching maps. Suppose that $\varphi\colon A \times I \to Y$ is a homotopy from φ_0 to φ_1. We want to show that $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are homotopy equivalent. The point is that both $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are deformation retracts of $Y \cup_\varphi (X \times I)$. We get the deformation retractions of $Y \cup_\varphi (X \times I)$ onto $Y \cup_{\varphi_0} X$ or $Y \cup_{\varphi_1} X$ from the deformation retractions of Proposition 1.4.2 (3) of $X \times I$ onto $A \times I \cup X \times \{0\}$ or $A \times I \cup X \times \{1\}$. The idea behind the proof is indicated in Figure 1.6. We intend to show that the

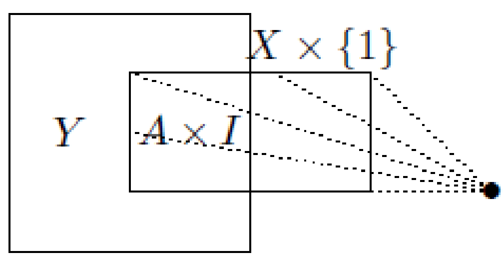
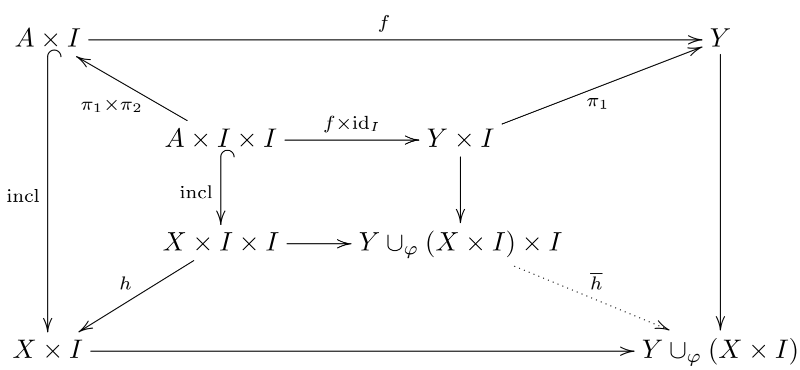


Figure 1.6: A deformation retraction $Y \cup_\varphi (X \times I) \times I \to \cup_\varphi (X \times I)$ of $Y \cup_\varphi (X \times I)$ onto $Y \cup_\varphi (X \times I)$

inclusions

$$Y \cup_{\varphi_0} X = Y \cup_{\varphi_0} (X \times \{0\} \cup A \times I) \subset Y \cup_\varphi (X \times I) \supset Y \cup_{\varphi_0} (X \times \{0\} \cup A \times I) = Y \cup_{\varphi_1} X$$

are homotopy equivalences. (The equality signs are there because all points of $A \times I$ have been identified to points in Y.) The left inclusion is a homotopy equivalence because the subspace is a deformation retract of the big space. The deformation retraction $\overline{h}$ of $Y \cup_\varphi (X \times I)$ onto $Y \cup_{\varphi_0} X$ is induced by the universal property of adjunction spaces as in the diagramme



from a deformation retraction $h\colon X\times I\times I\to X\times I$ of $X\times I$ onto $X\times\{0\}\cup A\times I$ (Proposition 1.4.2 (3)). Here, the outer square is the push-out diagram for $Y\cup_\varphi (X\times I)$ and the inner square is just this diagramme crossed with the unit interval. The homotopy $h\colon X\times I\times I\to X\times I$ starts as the identity map, is constant on the subspace $X\times\{0\}\cup A\times I\subset X\times I$, and ends as a retraction of $X\times I$ onto this subspace. The induced homotopy

$$\overline{h}\colon Y\cup_\varphi (X\times I)\times I\to Y\cup_\varphi (X\times I)$$

starts as the identity map, is constant on the subspace

$$Y\cup_\varphi (X\times\{0\}\cup A\times I)=Y\cup_\varphi X,$$

and ends as a retraction onto this subspace. We conclude that $Y\cup_\varphi X$ deformation retracts onto its subspace $Y\cup_{\varphi_1} X$. Similarly, $Y\cup_\varphi X$ deformation retracts onto its subspace $Y\cup_{\varphi_1} X$. Thus $Y\cup_{\varphi_0} X$ and $Y\cup_{\varphi_1} X$ are homotopy equivalent spaces. (Note that we proved this by construction a zig-zag

$$Y\cup_{\varphi_0} X\to Y\cup_\varphi (X\times I)\leftarrow Y\cup_{\varphi_1} X$$

of homotopy equivalences, not by constructing a direct homotopy equivalence between the two spaces.) □

Are there any pairs of spaces that have the HEP?

Our work on HEP pairs would be futile if there weren't any pairs that enjoying this property. But we shall next see that pairs with the HEP are ubiquitous: It is difficult, but not impossible, to find a pair that does not have the HEP.

Corollary 1.4.4. *The pair* $(\mathbb{D}^n,\mathbb{S}^{n-1})$ *has the HEP for all* $n\geq 1$. *In fact,* (CX,X) *has the HEP for all spaces* X.

Proof. For instance, for $n=1$, $\mathbb{D}^1\times I\subset\mathbb{R}\times I\subset\mathbb{R}^2$ (deformation) retracts onto $\mathbb{D}^1\times\{0\}\cup\mathbb{S}^0\times I$ by radial projection from $(0,2)$ as indicated in Picture 1.7: In

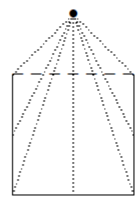


Figure 1.7:

fact, $\mathbb{D}^n \times I \subset \mathbb{R}^n \times I \subset \mathbb{R}^{n+1}$ (deformation) retracts onto $\mathbb{D}^n \times \{0\} \subset \mathbb{S}^{n-1} \times I$ by a radial projection from $(0, \dots, 0, 2)$.

More generally, for any space X, the pair (CX, X) has the HEP because $CX \times \{0\} \cup X \times I$ is a retract of $CX \times I$. Picture 1.8 indicates a retraction $R\colon I \times I \to \{0\} \cup I \times \{0\}$, sending all of $\{1\} \times I$ to the point $(1, 0)$. The map

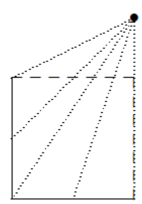


Figure 1.8:

$\mathrm{id}_X \times R\colon X \times I \times I \to X \times \{0\} \times I \cup X \times I \times \{0\}$ factors through

$$\begin{array}{ccc} X \times I \times I & \xrightarrow{\mathrm{id}_X \times R} & X \times \{0\} \times I \cup X \times I \times \{0\} \\ \downarrow & & \downarrow \\ (X \times I)/(X \times \{1\}) \times I & \dashrightarrow & X \times \{0\} \times I \cup (X \times I)/(X \times \{1\}) \times \{0\} \end{array}$$

to give the required retraction $CX \times I \to X \times I \cup CX \times \{0\}$. □

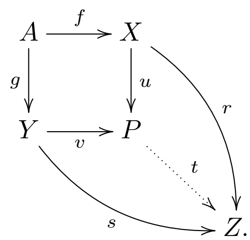
We introduce a terminology to explain the example involving mapping cylinder below:

Definition 1.4.5. Given spaces and maps

$$Y \xleftarrow{g} A \xrightarrow{f} X \tag{1.4.6}$$

a *pushout* of 1.4.6 consists of a space P and maps $u\colon X \to P$ and $v\colon Y \to P$ such that $uf = vg$. In addition, we require the following universal property. If

Z is any space and if $r\colon X \to Z$ and $s\colon Y \to Z$ are maps such that $rf = sg$, then there is a unique map $t\colon P \to Z$ such that $tu = r$ and $tv = s$,



We call either P or the triple (P, u, v), the *pushout* of 1.4.6. The square diagram above is then called a *pushout square*.

We show that any two pushouts of 1.4.6 are homeomorphic. Suppose that (P, u, v) and (P', u', v') are both pushouts of 1.4.6. Since P is a pushout, there is a map $t\colon P \to P'$ such that $tu = u'$ and $tv = v'$. Since P' is a pushout, there is a map $t'\colon P' \to P$ such that $t'u' = u$ and $t'v' = v$. Therefore $t'tu = u$ and $t'tv = v$. By the uniqueness of pushout maps, $t't = \mathrm{id}_P$. Similarly $tt' = \mathrm{id}_{P'}$, and so t is a homeomorphism with inverse t'.

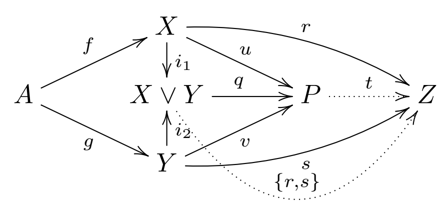
We now show the existence of pushouts.

Proposition 1.4.7. *Given 1.4.6, there exists a pushout* (P, u, v).

Proof. Consider $X \vee Y$, regarded as a subspace of $X \times Y$, and introduce the equivalence relation on $X \vee Y$ defined by $(f(a), *) \sim (*, g(a))$, for every $a \in A$. Set $P = X \vee Y/\sim$ and let $q\colon X \vee Y \to P$ be the quotient map. Define u and v by $u = qi_1$ and $v = qi_2$, where $i_1\colon X \to X \vee Y$ and $i_2\colon Y \to X \vee Y$ are the two injections. Clearly $uf = vg$. Now we show that (P, u, v) is a pushout of 1.4.6. If $r\colon XtoZ$ and $s\colon Y \to Z$ are maps with $rf = sg$, then there is a map $\{r, s\}\colon X \vee Y \to Z$ and

$$\{r, s\}(f(a), *) = rf(a) = sg(a) = \{r, s\}(*, g(a)).$$

Thus $\{r, s\}$ induces $t\colon P \to Z$ such that $tu = r$ and $tv = s$.



To prove uniqueness of t, let $m\colon P \to Z$ be a map such that $mu = r$ and $mv = s$. Then

$$mqi_1 = mu = r = tu = tqi_1,$$

and similarly, $mqi_2 = tqi_2$. Therefore, $mq = tq$, and so $m = t$. □

Example 1.4.8 (Neil Strickland and Charles Rezk). Suppose we have a map $f\colon X \to Y$ and we form the mapping cylinder M_f. Then the pair $(M_f, X \cup Y)$ satisfies the homotopy extension property. Equivalently we could find a retraction of $M_f \times I$ to $M_f \times \{0\} \cup (X \cup Y) \times I$.

Solution by Neil Strickland

Let us assume the convention where M_f is $(X \times I) \cup Y$ with $(x, 0)$ attached to $f(x)$. Now $M_f \times I = (X \times I^2) \cup (Y \times I)$ with $(x, 0, t)$ attached to $(f(x), t)$. We want to retract this onto the space

$$Q = (M_f \times \{0\}) \cup (((X \times \{1\}) \cup Y) \times I).$$

Note that $X \times \{0\} \times I$ gets identified with part of $Y \times I$ and so is contained in Q. Thus $Q = (X \times U) \cup (Y \times I)$, where

$$U = (\{0, 1\} \times I) \cup (I \times \{0\}),$$

and again $(x, 0, t)$ is attached to $(f(x), t)$. Now let r be a retraction from $I \times I$ onto U, say by radial projection from the point $(1/2, 1)$. We can then fit $1 \times r\colon X \times I^2 \to X \times U$ together with the identity map on $Y \times I$ to get the required retraction of $M_f \times I$ onto Q.

Solution by Charles Rezk

It may be useful to note that you can obtain results like this from a combination of some "easier" facts:

- The pair $(I, \{0, 1\})$ has the HEP.
- If (L, K) has the HEP where K and L are locally compact Hausdorf, and if Z is any space, then $(Z \times L, Z \times K)$ has the HEP.
- If (U, A) has the HEP, and $g\colon A \to B$ is any map, then (V, B) has the HEP, where V is the pushout of U along g.

Apply the second one with $(L, K) = (I, \{0, 1\})$ and note that M_f can be obtained from $X \coprod Y$ by gluing it to a copy of $X \times I$ along $X \times \{0, 1\}$.

Proposition 1.4.9. *If (X, A) has the HEP and X is Hausdorff, then A is a closed subspace of X.*

Proof. $X \times \{0\} \subset A \times I$ is a closed subspace of $X \times I$ since it is a retract. Now look at X at level $\frac{1}{2}$ inside the cylinder $X \times I$. □

See Ronald Brown, 'Topology and Groupoids" for more (either necessary or sufficient) conditions for an inclusion to have the HEP.

Example 1.4.10 (A closed subspace that does not have the HEP.). (I, A) where $A = \{0\} \cup \{\frac{1}{n} | n = 1, 2, \dots\}$ does *not* have the HEP since $I \times \{0\} \cup A \times I$ is not a retract of $I \times I$.

Indeed, assume that $r\colon I \times I \to A \times I$ is a retraction. For each $n \in \mathbb{N}_+$, the map $t \mapsto r(t \times 1)$, $\frac{1}{n+1} \leq t \leq \frac{1}{n}$, is a path in A from $\frac{1}{n+1} \times 1$ to $\frac{1}{n} \times 1$ and its image under the retraction, $t \mapsto r(t \times 1)$, $\frac{1}{n+1} \le t \le \frac{1}{n}$, is a path in A connecting the same two points.

Such a path must pass through all points of

$$(\frac{1}{n+1}, \frac{1}{n}) \times \{0\} \subset I \times \{0\}$$

because the projection $\pi_1 r([\frac{1}{n+1}, \frac{1}{n}] \times \{1\}) \supset [\frac{1}{n+1}, \frac{1}{n}]$ by connectedness. Thus there is a point $t_n \in (\frac{1}{n+1}, \frac{1}{n})$ such that $r(t_n \times 1) \in (\frac{1}{n+1}, \frac{1}{n}) \times \{0\}$. This contradicts continuity of r for $t_n \times 1$ converges to 0×1 and $r(t_1 \times 1)$ converges to $0 \times 0 \neq r(0 \times 1)$.

A similar (but simpler) argument shows that there is no retraction $A \times I \to A \times \{0\} \cup \{0\} \times I$ so that 0 is a "degenerate" base point of A.

Example 1.4.11. Let Y be the quasi-circle, a closed subspace of $\mathbb{R}^2$ consisting of

A the segment $[-1, 1]$ in the y axis,

B the arc connecting these two pieces,

C the portion of the graph of $y = \sin(1/x)$,

thus $Y = A \cup B \cup C$ (see Fiure 1.9.) Induce a map $f: W \to \mathbb{S}^1$ by collapsing

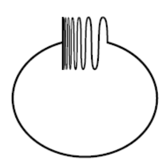
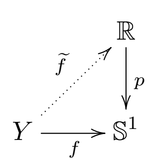


Figure 1.9: A quasi-circle

the interval $A = [-1, 1]$. Then f does not lift to the covering space $\mathbb{R} \to \mathbb{S}^1$, even though $\pi_1(W) = 0$. (since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ and $\pi_1(C)$ is trivial) even though A is contractible.

Indeed, we have a quotient map $q: Y \to Y/A$ by collapsing A to a point, which we denote by a. Then we have a map $g: Y/A \to \mathbb{S}^1$ by leaving a and B fixed, and projecting down the graph of $\sin(1/x)$ to the x-axis. By doing a rotation if necessary, we can assume that $g(a) = 1$ (we are thinking of $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$). Then the map f is the composition $f = g \circ q$.

Let $p: \mathbb{R} \to \mathbb{S}^1$ be the usual covering map $t \mapsto e^{it}$, and suppose there is a lift $\widetilde{f}: Y \to \mathbb{R}$.



For $\epsilon > 0$, define

$$U_\epsilon = \{y \in Y : \operatorname{dist}(y, A) > \epsilon\}$$
$$V_\epsilon = \{y \in Y : \operatorname{dist}(y, A) < \epsilon\}$$

That is, U_ϵ is an open subset of Y covering almost all of Y, except for avoiding a ϵ-neighbourhood of A, and V is an ϵ-neighbourhood of A. Note that $U_\epsilon \cup V_{2\epsilon} = Y$. (They do overlap!) Then $f(U_\epsilon)$ covers almost all of $\mathbb{S}^1$, and $f(V_{\epsilon/2})$ is a small neighbourhood of 1.

$$p\widetilde{f}(U_\epsilon) = f(U_\epsilon) = \{x \in \mathbb{S}^1 : \operatorname{dist}(x, 1) > \epsilon\}$$
$$p\widetilde{f}(V_{2\epsilon}) = f(V_{2\epsilon}) = \{x \in \mathbb{S}^1 : \operatorname{dist}(x, 1) < 2\epsilon\}$$

This says that $\widetilde{f}(U_\epsilon)$ is an interval of length just a bit smaller than 2ϵ that avoids $f(a) = 2\pi k$, and $\widetilde{f}(V_\epsilon)$ is a small interval containing $f(a) = 2\pi k$. Since U_ϵ and $V_{2\epsilon}$ overlap, the images overlap. Thus the union

$$\widetilde{f}(Y) = \widetilde{f}(U_\epsilon) \cup \widetilde{f}(V_{2\epsilon})$$

is a single open interval of length greater than 2π. So there exist $\alpha, \beta \in \widetilde{f}(Y)$ such that $|\alpha - \beta| = 2\pi$. Since $\widetilde{f}(U_\epsilon)$ and $\widetilde{f}(V_{2\epsilon})$ are both intervals of length less than 2π, α and β can't be in the same one. WLOG assume $\alpha \in \widetilde{f}(U_\epsilon) \setminus \widetilde{f}(V_{2\epsilon})$ and $\beta \in \widetilde{f}(V_{2\epsilon}) \setminus \widetilde{f}(U_\epsilon)$. Then there exist $y_\alpha \in U_\epsilon \setminus V_{2\epsilon}$ and $y_\beta \in V_{2\epsilon} \setminus U_\epsilon$ with $\widetilde{f}(y_\alpha) = \alpha$ and $\widetilde{f}(y_\beta) = \beta$. Then

$$|\alpha - \beta| = 2\pi \Rightarrow p(\alpha) = p(\beta) \Rightarrow p\widetilde{f}(y_\alpha) = p\widetilde{f}(y_\beta) \Rightarrow f(y_\alpha) = f(y_\beta)$$

By construction of U_ϵ, $y_\alpha \in U_\epsilon$ implies that y_α is not in A. Since f is injective except for values in A, this implies that $y_\alpha = y_\beta$, which contradicts the fact that y_α and y_β lie in disjoint neighbourhoods of Y. Therefore, no lift $\widetilde{f}$ exists.

Proposition 1.4.12. *Suppose (X, A) has the HEP.*

(1) (transitivity) *If $X_0 \subset X_1 \subset X_2$ and both pairs (X_2, X_1) and (X_1, X_0) have the HEP, then (X_2, X_0) has the HEP. More generally, if $X = \cup X_k$ has the coherent topology with respect to its subspaces $X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k \subset \cdots$ where each pair of consecutive subspaces has the HEP, then (X, X_0) has the HEP.*

(2) $Y \times (X, A) = (Y \times X, Y \times A)$ has the HEP for all spaces Y.

(3) $(X, A) \times (I, \partial I) = (X \times I, X \times \partial I \cup A \times I)$ has the HEP.

(4) $(Y \operatorname{sup}_\varphi X, Y \cup_\varphi A)$ has the HEP for all spaces Y and all maps $\varpi \colon B \to Y$ defined on a closed subspace B of A. In particular, $(Y \cup_\varphi X, Y)$ has the HEP for any attaching map $\varphi \colon A \to Y$. (See Figure 7.)

(5) The n-cellular extension $(Y \cup_\varphi \coprod \mathbb{D}^n, Y)$ of any space Y has the HEP for any attaching map $\varphi \colon \mathbb{S}^{n-1} \to Y$.

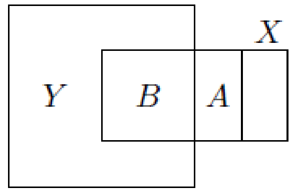


Figure 1.10: The pair $(Y \cup_\varphi X, Y \cup_\varphi A$

Proof. (1): In the first case, there are retractions

$$r_2\colon X_2 \times I \to X_1 \times I \cup X_2 \times \{0\} \quad \text{and}$$
$$r_1\colon X_1 \times I \cup X_2 \times \{0\} \to X_0 \times I \cup X_2 \times \{0\}.$$

Then $r_1 r_2$ is a retraction of $X_2 \times I$ onto $X_0 \times \cup X_2 \times \{0\}$.

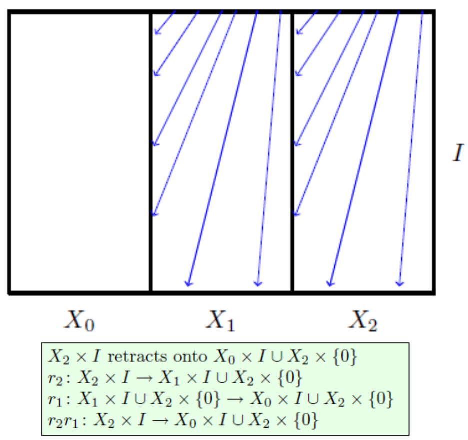


Figure 1.11: Retraction of $X_2 \times I$ onto $X_0 \times I \cup X_2 \times \{0\}$

In the general case, there are retractions

$$r_k\colon X_k \times I \cup X \times \{0\} \to X_{k-1} \times I \cup X \times \{0\}.$$

There is a well-defined retraction

$$r_1 r_2 \cdots r_k \cdots\colon X \times I \to X_0 \times I \cup X \times \{0\}$$

that on $X_k \times I \cup X \times \{0\}$ is

$$X_k \times I \cup X \times \{0\} \xrightarrow{r_k} X_{k-1} \times I \cup X \times \{0\} \xrightarrow{r_{k-1}} \cdots \xrightarrow{r_2}$$
$$X_1 \times I \cup X \times \{0\} \xrightarrow{r_1} X_0 \times I \cup X \times \{0\}$$

This retraction $X \times I \to X_0 \times I \cup X \times \{0\}$ is continuous because the product topology on $X \times I$ is coherent with the filtration $X_k \times I$, $k = 0, 1, \ldots$. (The reader may want to verify this claim!)

(2): We use Proposition 1.4.2. Let $r\colon X \times I \to X \times I$ be a retraction onto $X \times \{0\} \cup A \times I$. Then the product map $\mathrm{id}_Y \times r$ is a retraction of $(Y \times X) \times I$ onto $(Y \times X)\{0\} \cup (Y \times A) \times I$.

(3): See Dugunji,"Topology" Chapter 7 §5 (p. 330).

(4): We use Proposition 1.4.2 again. Let $r\colon X \times I \to X \times I$ be a retraction onto $X \times \{0\} \cup A \times I$. The universal property of quotient maps provides a factorisation, $\overline{\mathrm{id}_{Y \times I} \amalg r}$ of $\mathrm{id}_{Y \times I} \coprod r$

$$\begin{array}{ccc}
(Y \amalg X) \times I & \xrightarrow{\mathrm{id}_{Y \times I} \amalg r} & (Y \amalg X) \times I \\
\Big\downarrow{\scriptstyle q \times \mathrm{id}_I} & & \Big\downarrow{\scriptstyle q \times \mathrm{id}_I} \\
(Y \cup_\varphi X) \times I & \xrightarrow[\overline{\mathrm{id}_{Y \times I} \amalg r}]{} & (Y \cup_\varphi X) \times I
\end{array}$$

that is a retraction of $Y \cup_\varphi X) \times I$ onto $Y \cup_\varphi X \times \{0\} Y \cup_\varphi \cup A \times$. To prove continuity, note that the left vertical map is a quotient map since I is locally compact Hausdorff. This shows that $(Y \cup_\varphi X, Y \cup_\varphi \cup A)$ has the HEP. If the attaching map φ is defined on all of A, we have that $(Y \cup_\varphi X, Y \cup_\varphi \cup A) = (Y \cup_\varphi X, Y)$ so this pair has the HEP.

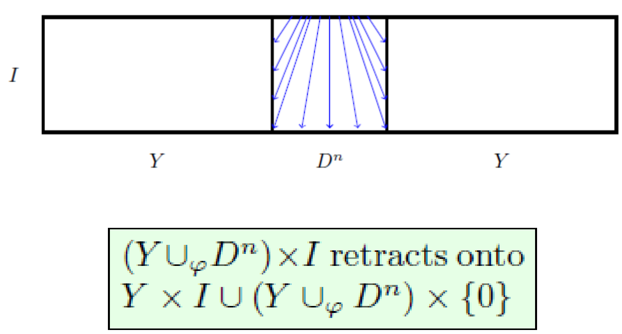


Figure 1.12: Retraction of $(Y \cup \mathbb{D}^n) \times I$ onto $Y \times I \cup (Y \cup \mathbb{D}^n) \times \{0\}$

(5): This is a special case of (4) since $(\coprod \mathbb{D}^n, \coprod \mathbb{S}^{n-1})$ has the HEP (Corollary 1.4.4). □

Corollary 1.4.13. *Any relative CW-complex (X, A) (Definition 1.3.7) has the HEP. In particular, any CW-pair (X, A) has the HEP.*

Proof. There is a filtration of X

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset \cup_X^{n-1} \subset \cup X^n \subset \cdots \subset X$$

where X^n, $n \geq 0$, is obtained from $A \cup X^{n-1}$ by attaching n-cells. Since a cellular extension has the HEP, transitivity (Proposition 1.4.12 (1)) implies that also (X, A) has the HEP. □

Example 1.4.14. ($\mathbb{S}^\infty$ is contractible.) Choose $* = 1$ as the base-point of $\mathbb{R} \supset \mathbb{S}^0 \subset \mathbb{S}^1$. Let $\mathbb{D}_+^{n+1}$ denote the upper half of $\mathbb{S}^{n+1} = \mathbb{D}_-^{n+1} \cup \mathbb{D}_+^{n+1}$. Since the base point $\{*\}$ is a deformation retract of the disc $\mathbb{D}^{n+1}$ there is a homotopy

$$\mathbb{R}^n \colon \mathbb{S}^n \times \left[\frac{n}{n+1}, \frac{n+1}{n+2}\right] \to \mathbb{S}^{n+1}$$

from the inclusion map of $\mathbb{S}^n$ into $\mathbb{S}^{n+1}$ to the constant map $\mathbb{S}^n \to *$ and this homotopy is relative to the base point $\{*\}$. Since (S^1, S^0) has the HEP, the

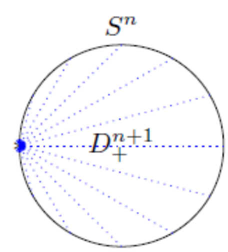


Figure 1.13: Inclusion $\mathbb{S}^n$ to $\mathbb{S}^{n+1}$

partial homotopy $R^0 \colon \mathbb{S}^0 \times [0, 1/2] \to \mathbb{S}^1$ extends to a homotopy $\mathbb{S}^1 \times [0, 1/2] \to \mathbb{S}^1$, relative to the base point, from the identity map of $\mathbb{S}^1$ to some map $f_1 \colon : \mathbb{S}^1 to \mathbb{S}^1$ that sends $\mathbb{S}^0$ to $*$.

Since $(\mathbb{S}^2, \mathbb{S}^1)$ has the HEP, the homotopy

$$\mathbb{S}^1 \times [1/2, 2/3] \to \mathbb{S}^2 \quad (x, t) \mapsto R^1(f_1(x), t),$$

which is constant on $\mathbb{S}^0 \times [1/2, 2/3]$, combined with the already constructed homotopy $\mathbb{S}^1 \times [0, 1/2] \to \mathbb{S}^1$ and the identity on $\mathbb{S}^2 \times \{0\}$, extends to a homotopy $\mathbb{S}^2 \times [0, 2/3] \to \mathbb{S}^2$, constant on $\mathbb{S}^1 \times [1/2, 2/3]$, from the identity map of $\mathbb{S}^2$ to some map $f_2 \colon \mathbb{S}^2 \to \mathbb{S}^2$ that sends $\mathbb{S}^1$ to $*$.

Continue like this and get a homotopy $\mathbb{S}^\infty \times [0, 1] \to \mathbb{S}^\infty$, from the identity to the constant map relative to the base point.

Figure 1.14 shows the beginning of a homotopy $\mathbb{S}^\infty \times I \to \mathbb{S}^\infty$ rel $*$ between the identity map and the constant map. It is continuous because the area where it is constant (indicated by the dotted lines) gets larger and larger as we approach $\mathbb{S}^\infty \times \{1\}$.

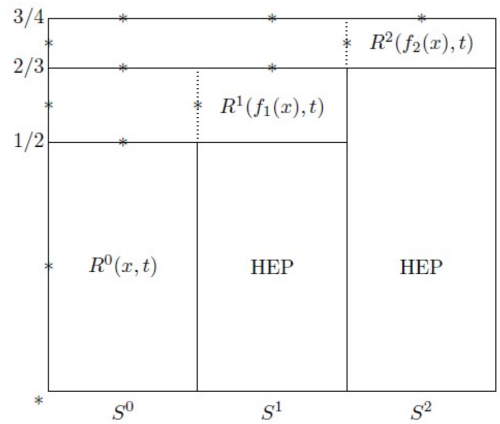


Figure 1.14: $\mathbb{S}^\infty$ is contractible.

In the above example we actually proved the following:

Proposition 1.4.15. *Let X be a CW-complex with skeleta X_n, $n \geq 0$, and base point $* \in X_0$. If all the inclusions $X^n \hookrightarrow X^{n+1}$ are homotopic* rel $*$ *to the constant map $*$, then the identity map of X is homotopic* rel $*$ *to the constant map $*$ and X is contractible.*

Exercise 1.4.16. The Dunce hat is the quotient of the of the 2-simplex by the identifications indicated in Fig 1.15. Then the Dunce hat is contractible, in fact, homotopy equivalent to $\mathbb{D}^2$. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map from $\mathbb{S}^1$ to itself. The cone $C_f = M_f/\mathbb{S}^1 \times \{1\}$ for f is obtained by pinching the top of the mapping cylinder to a point. As M_f is the cylinder $\mathbb{S}^1 \times [0,1]$ with the bottom pasted to $\mathbb{S}^1$ by the map f, C_f is $\mathbb{D}^2$ with $\partial\mathbb{D}^2$ pasted to $\mathbb{S}^1$ by the map f. So the dunce hat is just C_f with $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined as

$$f(e^{2\pi it}) = \begin{cases} e^{2\pi i(3t)}, & 0 \leq t \leq 2/3 \\ e^{2\pi i(2-3t)}, & 2/3 \leq t \leq 1. \end{cases}$$

which is homotopic to the identity by a linear homotopy (note that we make the choice of f for an easy definition of the homotopy)

$$H(e^{2\pi it}, s) = \begin{cases} e^{2\pi i(3t(1-s)+st}, & 0 \leq t \leq 2/3 \\ e^{2\pi i[(2-3t)(1-s)+st]}, & 2/3 \leq t \leq 1. \end{cases}$$

So the dunce hat is homotopic to $C_{\text{id}} \simeq \mathbb{D}^2$ which is contractible.

Example 1.4.17. The unreduced suspension SX and the reduced suspension $\sum X = SX/\{x_0\} \times I$ are homotopy equivalent for all CW-complexes X based at a 0-cell $\{x_0\}$.

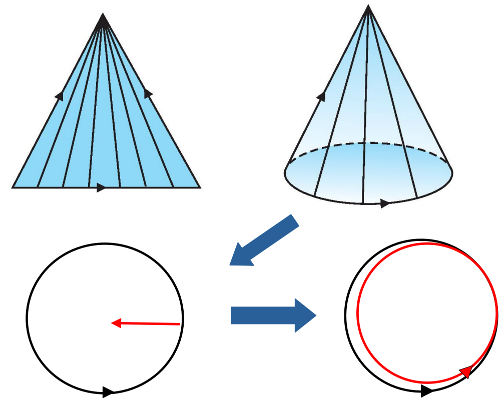


Figure 1.15: Dunce Hat

Example 1.4.18. Homotopic maps have homotopy equivalent mapping cones.

Let $f\colon X \to Y$ be any map. Consider the mapping cone $C_f = Y \cup_f CX$ of f. Since the pair (CX, X) has the HEP (1.4.4) we know that

- (C_f, Y) has the HEP (Proposition 1.4.12.(4))
- The homotopy type of C_f only depends on the homotopy class of f (Theorem 1.4.3.(4))

We claim that the squaring map

$$2\colon \mathbb{S}^1 \to \mathbb{S}^1, \quad z \mapsto z^2 \quad \text{where} \quad \mathbb{S}^1 = z \in \mathbb{C} : |z| = 1$$

is not homotopic to the constant map $0\colon \mathbb{S}^1 \to \mathbb{S}^1$. In fact, $C_2 = \mathbb{S}^1 \cup_2 \mathbb{D}^2 = \mathbb{R}P^2$ and $C_0 = \mathbb{S}^1 \cup_0 \mathbb{D}^2 = \mathbb{S}^1 \vee \mathbb{S}^2$ are not homotopy equivalent.

The complex projective plane $\mathbb{C}P^2 = \mathbb{S}^2 \cup_\varphi \mathbb{D}^4$ is obtained by attaching a 4-cell to the 2-sphere along the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ (Example 1.3.5). If the attaching map were nullhomotopic then $\mathbb{C}P^2$ would be homotopy equivalent to $\mathbb{S}^2 \cup_* \mathbb{D}^4 = \mathbb{S}^2 \vee \mathbb{S}^4$.

There are methods with which we can show that $\mathbb{R}P^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2$ are not homotopy equivalent, and that $\mathbb{C}P^2$ and $\mathbb{S}^2 \vee \mathbb{S}^4$ are not homotopy equivalent, either. Thus the squaring map $2\colon \mathbb{S}^1 \to \mathbb{S}^1$ and the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ are not nullhomotopic.

Example 1.4.19. The homotopy type of the quotient space X/A:
If (X, A) has the HEP so does the pair $(X \cup CA, CA)$ obtained by attaching X to CA (Proposition 1.4.12(4)). Since the cone CA on A is contractible,

$$X \cup CA \to X \cup CA/CA = X/A$$

is a homotopy equivalence (Theorem 1.4.3 (3)) between the cone on the inclusion of A into X and the quotient space X/A. Suppose in addition that the inclusion map $A \hookrightarrow X$ is homotopic to the constant map $0\colon A \to X$, i.e, A is contractible in X. Then there are homotopy equivalences

$$X/A \simeq X \cup CA = X \cup_i CA = C_i \simeq C_0 = X \vee SA$$

as the inclusion map and the constant map have homotopy equivalent mapping cones by Example 1.4.18. For instance, $\mathbb{S}^n/\mathbb{S}^i \simeq \mathbb{S}^n \vee \mathbb{S}^{i+1}$ for all $i \leq 0 < n$. (The inclusion $\mathbb{S}^i \to \mathbb{S}^n$, $0 < i < n$, is nullhomotopic since it factors through the contractible space $\mathbb{S}^n \setminus * = \mathbb{R}^n$.) Hatcher *Algebraic Topology* has an illustration of $\mathbb{S}^2/\mathbb{S}^0 \simeq \mathbb{S}^2 \cup C\mathbb{S}^0 \simeq \mathbb{S}^2 \vee \mathbb{S}^1$. as in Figure 1.16.

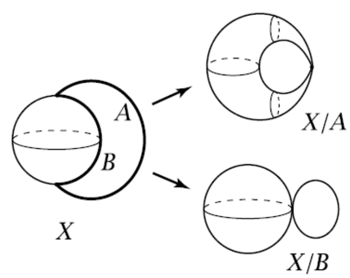


Figure 1.16: $\mathbb{S}^2/\mathbb{S}^0 \simeq \mathbb{S}^2 \vee \mathbb{S}^1$

Example 1.4.20. HEP for mapping cylinders:
Let $f\colon X \to Y$ be a map. We apply Proposition 1.4.12 in connection with the mapping cylinder $M_f = Y \cup_f (X \times I)$ and obtain the following:

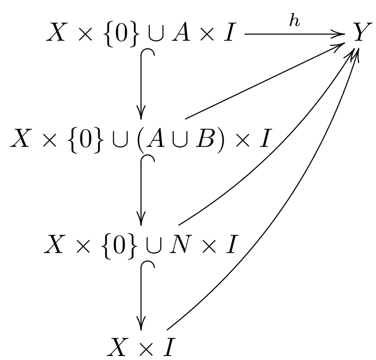
$(I, \partial I)$ has the HEP $\overset{1.4.12(2)}{\Rightarrow}$ $(X \times I, X \times \partial I)$ has the HEP $\overset{1.4.12(4)}{\Rightarrow}$ $(M_f, X \cup Y)$ has the HEP,

$(I, \{0\})$ has the HEP $\overset{1.4.12(2)}{\Rightarrow}$ $(X \times I, X \times \{0\})$ has the HEP $\overset{1.4.12(4)}{\Rightarrow}$ (M_f, Y) has the HEP.

The fact that $(M_f, X \cup Y)$ has the HEP implies that also (M_f, X) has the HEP (simply take a constant homotopy on Y). See Example 1.4.21 below for another application.

Example 1.4.21. HEP for subspaces with mapping cylinder neighbourhoods:
For another application of Proposition 1.4.12, suppose that the subspace $A \subseteq X$ has a mapping cylinder neighbourhood. This means that A has a closed neighbourhood N containing a subspace B (thought of as the boundary of N) such that $N \setminus B$ is an open neighbourhood of A and $(N.A \cup B)$ is homeomorphic to $(M_f, A \cup B)$ for some map $f\colon B \to A$. Then (X, A) has the HEP.

To see this, let $h\colon X \times \{0\} \cup A \times I \to Y$ be a partial homotopy of a map $X \to Y$. Extend it to a partial homotopy on $X \times \{0\} \cup (A \cup B) \times I$ by using the constant homotopy on $B \times I$. Since $(N, A \cup B)$ has the HEP, we can extend further to a partial homotopy defined on $X \times \{0\} \cup N \times I$. Finally, extend to $X \times I$ by using a constant homotopy on $X \setminus (N \setminus B) \times I$. In this way we get extension



The final map is continuous since it restricts to continuous maps on the closed subspaces $X \setminus (N \setminus B) \times I$ and $N \times I$ with union $X \times I$.

1.5 Compact-Open Topology

The compact-open topology is a natural topology on mapping spaces of continuous functions, important because of its role in exhibiting locally compact topological spaces to be exponentiable, as demonstrated below, culminating in Theorem 1.5.6.

Let X be a locally compact Hausdorff space, and Y any Hausdorff space. By Y^X we mean the set of continuous functions $X \to Y$.

Definition 1.5.1. The *compact-open topology on* Y^X is the topology generated by the sets $M(K, U) = \{f \in Y^X | f(K) \subset U\}$, where $K \subset X$ is compact and $U \subset Y$ is open.

Recall that "generated" here means that these sets form a subbasis for the open sets. In what follows, unless otherwise noted, Y^X will always be given the compact-open topology.

Lemma 1.5.2. *Let* $\mathbf{K}$ *be a collection of compact subsets of* X *containing a neighbourhood base at each point of* X*. Let* $\mathbf{B}$ *be a subbasis for the open sets of* Y*. Then the sets* $M(K, U)$*, for* $K \in \mathbf{K}$ *and* $U \in \mathbf{B}$*, form a subbasis for the compact-open topology.*

Proof. Note that $M(K, U) \cap M(K, V) = M(K, U \cap V)$, which implies that it suffices to consider the case in which B is a basis. We need to show that the indicated sets form a neighbourhood basis at each point $f \in Y^X$. Thus it suffices to show that if $K \subset X$ is compact and $U \subset Y$ is open, and $f \in M(K, U)$, then there exist $K_1, \ldots, K_n \in \mathbf{K}$ and $U_1, \ldots, U_n \in \mathbf{B}$ such that $f(x) \in \cup M(K_i, U_i) \subset M(K, U)$.

For each $x \in K$, there is an open set $U_x \in B$ with $f(x) \in U_x \subset U$, and there exists a $K_x \in \mathbf{K}$ which is a neighbourhood of x such that $f(K_x) \subset U_x$. Thus $f \in M(K_x, U_x)$.

By the compactness of K there exist points $x_l, \ldots, x_n$ such that $K \subset K_{x_1} \cup \cdots \cup K_{x_n}$. Then $f \in \cap M(K_{x_i}, U_{x_i}) \subset M(K, U)$. □

Proposition 1.5.3. *For X locally compact Hausdorff, the "evaluation map" $e\colon Y^X \times X \to Y$, defined by $e(f,x) = f(x)$, is continuous.*

Proof. If f and x are given, let U be an open neighbourhood of $f(x)$. Since f is continuous, there is a compact neighbourhood K of x such that $f(K) \subset U$. Thus $f \in M(K,U)$ and $M(K,U) \times K$ is taken into U by the evaluation e. Since $M(K,U) \times K$ is a neighbourhood of (f,x) in $Y^X \times X$, we are done. □

Theorem 1.5.4. *Let X be locally compact Hausdorff and Y and T arbitrary Hausdorff spaces. Given a function $f\colon X \times T \to Y$, define, for each $t \in T$, the function $f_t\colon X \to Y$ by $f_y(x) = f(x,t)$. Then f is continuous $\Leftrightarrow$ both of the following conditions hold:*

(a) each f_t is continuous; and

(b) the function $T \to Y^X$ taking t to f_t is continuous.

Proof. The implication $\Leftarrow$ follows from the fact that f is the composition of the map $X \times T \to Y^X \times X$ taking (x,t) to (f_t, x) with the evaluation $Y \times X \to Y$.

For the implication $\Rightarrow$, (a) follows from the fact that f_t is the composition $X \to X \times T \to Y$ of the inclusion $x \mapsto (x,t)$ with f. To (b), let $t \in T$ be given and let $f_t \in M(K,U)$. It suffices to show that there exists a neighbourhood W of t in T such that $' \in W \Rightarrow f_t \in M(K,U)$. (That is, it suffices to prove the conditions for continuity for a subbasis only.)

For $x \in K$, there are open neighbourhoods $V_x \subset X$ of x and $W_x \subset T$ of t such that $f(V_x \times W_x) \subset U$. By compactness, $K \subset V_{x_1} \cup \cdots \cup V_{x_n} = V$ say. Put $W = W_{X_1} \cap \cdots \cap W_{x_n}$. Then $f(K \times W) \subset f'V \times W) \subset U$, so that $t' \sin W \Rightarrow f_t \in M(K,U)$ as claimed. □

Remark 1.5.5. This theorem implies that *a homotopy $X \times I \to Y$, with X locally compact, is the same thing as a path $I \to Y^X$ in Y^X.*

An often used consequence of Theorem 1.5.4 is that in order to show a function $T \to Y^X$ to be continuous, it suffices to show that the associated function $X \times T \to Y$ is continuous.

Theorem 1.5.6. *(The Exponential Law) Let X and T be locally compact Hausdorff spaces and let Y be an arbitrary Hausdorff space. Then there is the homeomorphism*

$$Y^{X \times T} \xrightarrow{\approx} (Y^X)^T$$

taking f to f^*, where $f^*(t)(x) = f(x,t) = f_t(x)$.

Proof. Theorem 1.5.4 says that the assignment $f \mapsto f^*$ is a bijection. We must show it and its inverse to be continuous. Let $U \subset Y$ be open, and $K \subset X$, $K' \subset T$ compact. Then

$$\begin{aligned} f \in M(K \times K', U) &\Leftrightarrow (t \in K', x \in K \Rightarrow f_t(x) = f(x,t) \in U) \\ &\Leftrightarrow (t \in K' \Rightarrow f_t \in M(K,U)) \\ &\Leftrightarrow f^* \in M(K', M(K,U)). \end{aligned}$$

Now the $K \times K'$ give a neighbourhood basis for $X \times T$. Therefore the $M(K \times K', U)$ form a subbasis for the topology of $Y^{X \times T}$.

Also, the $M(K,U)$ give a subbasis for Y^X and therefore the $M(K', M(K,U))$ give a subbasis for the topology of $(Y^X)^T$.

Since these subbases correspond to one another under the exponential correspondence, the theorem is proved. □

Proposition 1.5.7. *If X is locally compact Hausdorff and Y and W are Hausdorff then there is the homeomorphism*

$$Y^X \times W^X \xrightarrow{\approx} (Y \times W)^X$$

given by $(f,g) \mapsto f \underline{\times} g = (f \times g) \circ \mathrm{diag}$.

Proof. This is clearly a bijection. If $K, K' \subset X$ are compact, and $U \subset Y$ and $V \subset W$ are open then we have

$$\begin{aligned} &(f,g) \in M(K,U) \times M(K',V) \Leftrightarrow \\ &(x \in K \Rightarrow f(x) \in U) \quad \text{and} \quad (x \in K' \Rightarrow g(x) \in V) \Leftrightarrow \\ &(x \in K \Rightarrow (f \underline{\times} g)(x) \in U \times W) \quad \text{and} \quad (x \in K' \to (f \underline{\times} g)(x) \in Y \times V) \Leftrightarrow \\ &(f \underline{\times} g) \in M(K, U \times W) \cap M(K', Y \times V). \end{aligned}$$

Thus $(f,g) \mapsto f \underline{\times} g$ is open.

Also, $(f,g) \in M(K,U) \times M(K,V) \Leftrightarrow (f \underline{\times} g) \in M(K, U \times V)$, which implies that the function in question is continuous. □

Proposition 1.5.8. *If X and T are locally compact Hausdorff spaces and Y is an arbitrary Hausdorff space then there is the homeomorphism*

$$Y^{X+T} \xrightarrow{\approx} Y^X \times Y^T$$

taking f to $(f \circ \mathrm{id}_X, f \circ \mathrm{id}_Y)$.

Proof. This is an easy exercise left to the reader. □

Theorem 1.5.9. *For X locally compact and both X and Y Hausdorff, ^{X} is a covariant functor of Y and a contravariant functor of X.*

Proof. A map $\phi\colon Y \to Z$ induces $\phi^X\colon Y^X \to Z^X$, by $\phi^X(f) = \phi \circ f$. We must show that ϕ^X is continuous. By Theorem 1.5.4 it suffices to show that $Y^X \times X \to Z$, taking (f, x) to $\phi(f(x))$, is continuous. But this is the composition $\phi \circ e$ of ϕ with the evaluation, which is continuous.

Next, for $\psi\colon X \to T$, both spaces locally compact, we must show that $Y^\psi\colon Y^T \to Y^X$, taking ψ to $f \circ \psi$, is continuous. It suffices, by Theorem 1.5.4, to show that $Y^T \times X \to Y$, taking (f, x) to $f(\psi(x))$ is continuous. But this is just the composition $e \circ (\mathrm{id} \times \psi)$, which is continuous. □

Corollary 1.5.10. *For $A \subset X$ both locally compact and X, Y Hausdorff, the restriction $Y^X \to Y^A$ is continuous.*

Theorem 1.5.11. *For X, Y locally compact, and X, Y, Z Hausdorff, the function*

$$Z^Y \times Y^X \to Z^X$$

taking (f, g) to $f \circ g$, is continuous.

Proof. It suffices, by Theorem 1.5.4, to show that the function $Z^Y \times Y^X \times X \to Z$, taking (f, g, x) to $(f \circ g)(x)$, is continuous. But this is the composition $e \circ (\mathrm{id} \times e)$. □

All of these things, and the ones following, have versions in the pointed category, the verification of which is trivial.

We finish this section by showing that, for Y metric, the compact-open topology is identical to a more familiar concept.

Lemma 1.5.12. *Let Y be a metric space, let C be a compact subset of Y, and let $U \supset C$ be open. Then there is an epsilon > 0 such that $B_\epsilon(C) \subset U$.*

Proof. Cover C by a finite number of balls of the form $B_{\epsilon}(x_i)(x_i)$ such that $B_{2\epsilon}(x_i)(x_i) \subset U$. Put $\epsilon = \min(\epsilon(x_i))$. Suppose $x \in B_\epsilon(C)$. Then there is a $c \in C$ with $\mathrm{dist}(x, c) < \epsilon$ and an i such that $\mathrm{dist}(c, x_i) < \epsilon(x_i)$. Thus $x \in B_{2\epsilon}(x_i)(x_i) \subset U$. □

Theorem 1.5.13. *If X is compact Hausdorff and Y is metric then the compact-open topology is induced by the uniform metric on Y^X, i.e., the metric given by* $\mathrm{dist}(f, g) = \sup\{\mathrm{dist}(f(x), g(x)) | x \in X\}$.

Proof. For $f \in Y^X$, it suffices to show that a basic neighbourhood of fin each of these topologies contains a neighbourhood of f in the other topology.

Let $\epsilon > 0$ be given. Let

$$N = B_\epsilon(f) = \{g \in Y^X \,|\, \mathrm{dist}(f(x), g(x)) < \epsilon \quad \text{for all} \quad x \in X\}.$$

Given x, there is a compact neighbourhood N_x of x such that $p \in N_x \to f(p) \in B_{\epsilon/2}(f(x))$. Cover X by $N_{x_1} \cup \cdots \cup N_{x_k}$. We claim that

$$V = M(N_{x_1}, B_{\epsilon/2}(f(x_1))) \cap \cdots \cap M(N_{x_k}, B_{\epsilon/2}(f(x_k))) \subset N.$$

To see this, let $g \in V$, i.e., $x \in N_{x_i} \Rightarrow g(x) \in B_{\epsilon/2}(f(c_i))$. But $f(x) \in B_{\epsilon/2}(f(x_i))$ and so it follows that $g \in V \Rightarrow \text{dist}(f(x), g(x)) < \epsilon$ for all x. That is, $V \subset N$.

Conversely, suppose that $f \in M(K_1, U_1 \cap \cdots \cap M(K_r, U_r)$, i.e, $f(K_i) \subset U_i$ for $i = 1, \ldots, r$. By Lemma 1.5.12, there is an $\epsilon > 0$ such that $B_\epsilon(f(K_i)) \subset U_i$ for all $i = 1, \ldots, r$. If $x \in K_i$ then $B_\epsilon(f(x)) \subset B_\epsilon(F(K_i)) \subset U_i$. Therefore, if $g \in B_\epsilon(f)$ and $x \in K_i$ then $g(x) \in B_\epsilon(f(x)) \subset U_i$. Thus $g \in M(K_i, U_i$ for all i and so $B_\epsilon(f) \subset \cap M(K_i, U_i)$. □

Corollary 1.5.14. *If X is locally compact Hausdorff and Y is metric then the compact-open topology on Y^X is the topology of uniform convergence on compact sets. That is, a net $f_\alpha \in Y^X$ converges to $f \in Y^X$ in the compact-open topology $\Leftrightarrow f_\alpha|_K$ converges uniformly to $f|_K$ for each compact set $K \subset X$.*

Proof. For $\Rightarrow$ recall from Corollary 1.5.10 that $Y^X \to Y^K$ is continuous. Thus $f_\alpha|_L \to f|_K$ in the compact-open topology. But Y^K has the topology of the uniform metric and so $f_\alpha|_K$ converges to $f|_K$ uniformly.

For $\Leftarrow$, suppose that $f_\alpha|_K$ converges uniformly to $f|_K$ for each compact $K \subset X$. Let $f \in M(K, U)$. Then there exists an $\epsilon > 0$ such that $B_\epsilon(f(K)) \subset U$. There is an α such that $\beta > \alpha \Rightarrow \text{dist}(f_\beta(x), f(x)) < \epsilon$ for all $x \in K$. That is, $f_\beta(x) \in B_\epsilon(f(K)) \subset U$. Thus $\beta > \alpha \Rightarrow f_\beta \in M(X, U)$. This implies that f_α converges to f in the compact-open topology. □

Chapter 2

Homotopy Theory II

2.1 Compactly generated spaces

We briefly describe the category of spaces in which algebraic topologists customarily work. The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants. We shall leave the proofs to the reader, but the wise reader will simply take our word for it, at least on a first reading: we do not want to overemphasise this material, the importance of which can only become apparent in retrospect.

The definition of compactly generated spaces

We shall understand compact spaces to be both compact and Hausdorff, following Bourbaki. A space X is said to be *weak Hausdorff* if $g(K)$ is closed in X for every map $g\colon K \to X$ from a compact space K into X. When this holds, the image $g(K)$ is Hausdorff and is therefore a compact subspace of X. This separation property lies between T_1 (points are closed) and Hausdorff, but it is not much weaker than the latter.

A subspace A of X is said to be *compactly closed* if $g^{-1}(A)$ is closed in K for any map $g\colon K \to X$ from a compact space K into X. When X is weak Hausdorff, this holds if and only if the intersection of A with each compact subset of X is closed. A space X is a *k-space* if every compactly closed subspace is closed.

A space X is *compactly generated* if it is a weak Hausdorff k-space. For example, any locally compact space and any weak Hausdorff space that satisfies the first axiom of countability (every point has a countable neighbourhood basis) is compactly generated. We have expressed the definition in a form that should make the following statement clear.

Lemma 2.1.1. *If X is a compactly generated space and Y is any space, then a function $f\colon X \to Y$ is continuous if and only if its restriction to each compact subspace K of X is continuous.*

We can make a space X into a k-space by giving it a new topology in which a space is closed if and only if it is compactly closed in the original topology. We call the resulting space kX. Clearly the identity function $kX \to X$ is continuous. If X is weak Hausdorff, then so is kX, hence kX is compactly generated. Moreover, X and kX then have exactly the same compact subsets.

Write $X \times_c Y$ for the product of X and Y with its usual topology and write $X \times Y = k(X \times_c Y)$. If X and Y are weak Hausdorff, then $X \times Y = kX \times kY$. If X is locally compact and Y is compactly generated, then $X \times Y = X \times_c Y$.

By definition, a space X is Hausdorff if the diagonal subspace $\Delta X = \{(x,x)\}$ is closed in $X \times_c X$. The weak Hausdorff property admits a similar characterisation.

Lemma 2.1.2. *If X is a k-space, then X is weak Hausdorff if and only if ΔX is closed in $X \times X$.*

The category of compactly generated spaces

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

Proposition 2.1.3. *If X is compactly generated and $\pi\colon X \to Y$ is a quotient map, then Y is compactly generated if and only if $(\pi \times \pi)^1(\Delta Y)$ is closed in $X \times X$.*

The interpretation is that a quotient space of a compactly generated space by a "closed equivalence relation" is compactly generated. We are particularly interested in the following consequence.

Proposition 2.1.4. *If X and Y are compactly generated spaces, A is a closed subspace of X, and $f\colon A \to Y$ is any continuous map, then the pushout $Y \cup_f X$ is compactly generated.*

Another source of pathology is passage to colimits over sequences of maps $X_i \to X_{i+1}$. When the given maps are inclusions, the colimit is the union of the sets X_i with the "topology of the union;" a set is closed if and only if its intersection with each X_i is closed.

Proposition 2.1.5. *If $\{X_i$ is a sequence of compactly generated spaces and inclusions $X_i \to X_{i+1}$ with closed images, then the colimit $\varinjlim X_i$ is compactly generated.*

We now adopt a more categorical point of view. We redefine $\mathscr{U}$ to be the category of compactly generated spaces and continuous maps, and we redefine $\mathscr{T}$ to be its subcategory of based spaces and based maps.

Let $w\mathscr{U}$ be the category of weak Hausdorff spaces. We have the functor $k\colon w\mathscr{U} \to \mathscr{U}$, and we have the forgetful functor $j\colon \mathscr{U} \to w\mathscr{U}$, which embeds $\mathscr{U}$ as a full subcategory of $w\mathscr{U}$. Clearly

$$\mathscr{U}(X, kY) \simeq w\mathscr{U}(jX, Y)$$

for $X \in \mathscr{U}$ and $Y \in w\mathscr{U}$ since the identity map $kY \to Y$ is continuous and continuity of maps defined on compactly generated spaces is compactly determined. Thus k is right adjoint to j.

We can construct colimits and limits of spaces by performing these constructions on sets: they inherit topologies that give them the universal properties of colimits and limits in the classical category of spaces. Limits of weak Hausdorff spaces are weak Hausdorff, but limits of k-spaces need not be k-spaces. We construct limits of compactly generated spaces by applying the functor k to their limits as spaces. It is a categorical fact that functors which are right adjoints preserve limits ("RAPL" as coined by Awoedy), so this does give categorical limits in $\mathscr{U}$. This is how we defined $X \times Y$, for example.

Point-set level colimits of weak Hausdorff spaces need not be weak Hausdorff. However, if a point-set level colimit of compactly generated spaces is weak Hausdorff, then it is a k-space and therefore compactly generated. We shall only be interested in colimits in those cases where this holds. The propositions above give examples. In such cases, these constructions give categorical colimits in $\mathscr{U}$.

From here on, we agree that all given spaces are to be compactly generated, and we agree to redefine any construction on spaces by applying the functor k to it. For example, for spaces X and Y in $\mathscr{U}$, we understand the function space $\mathrm{Map}(X, Y) = Y^X$ to mean the set of continuous maps from X to Y with the k-ification of the standard compact-open topology; the latter topology has as basis the finite intersections of the subsets of the form $\{f | f(K) \subset U\}$ for some compact subset K of X and open subset U of Y. This leads to the following adjointness homeomorphism, which holds without restriction when we work in the category of compactly generated spaces.

Proposition 2.1.6. *For spaces X, Y, and Z in $\mathscr{U}$, the canonical bijection*

$$Z^{(X \times Y)} \simeq (Z^Y)^X$$

is a homeomorphism.

Observe in particular that a homotopy $X \times I \to Y$ can equally well be viewed as a map $X \to Y^I$. These adjoint, or "dual," points of view will play an important role in the next two chapters.

2.2 Cofibrations

In this section, we elaborate the fundamental tools and definitions of our study of cofibrations.

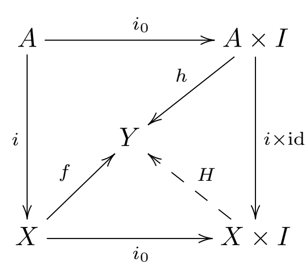
Exact sequences that feature in the study of homotopy, homology, and cohomology groups all can be derived homotopically from the theory of cofibre and fibre sequences that we present in this and the following two chapters. Abstractions of these ideas are at the heart of modern axiomatic treatments of homotopical algebra and of the foundations of algebraic K-theory.

The theories of cofiber and fibre sequences illustrate an important, but informal, duality theory, known as Eckmann-Hilton duality. It is based on the adjunction between Cartesian products and function spaces. Our standing hypothesis that all spaces in sight are compactly generated allows the theory to be developed without further restrictions on the given spaces. We discuss "cofibrations" here and the "dual" notion of "fibrations" in the next chapter.

The definition of cofibrations

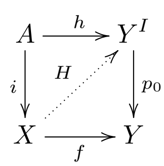
Definition 2.2.1. A map $i\colon A \to X$ is a *cofibration* if it satisfies the homotopy extension property (HEP), i.e, given a map $f\colon X \to Y$ and a homotopy $h\colon A \times I \to Y$ whose restriction to $A \times \{0\}$ is $f \circ i$, there exists an extension H of h to $X \times I$.

This situation is expressed schematically as follows:



where i_0 is the standard inclusion : $i_0(u) = (u, 0)$.

We may write this property in another equivalent (somewhat intricate, as it uses the notion of mapping space) way. $i\colon A \to X$ is a cofibration if there exists a lifting H in the following diagram

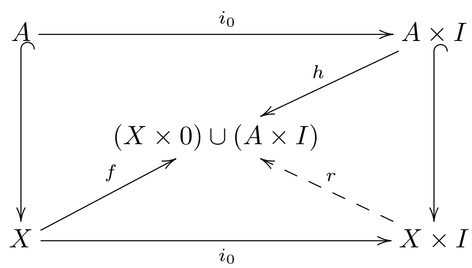


where $p_0(\beta) = \beta(0)$.

Remark 2.2.2. We do not require H to be unique, and it is usually not the case.

Definition 2.2.3. If (X, A) is a pair of topological spaces such that the inclusion map $A \subset X$ is a cofibration, then (X, a) is called a *cofibred pair* or *Bosruk pair* or is said to posses the *absolute homotopy extension property* (AHEP).

A necessary condition for (X, A) to be a cofibred pair is the existence of a retraction $r\colon X \times I \to (X \times 0) \cup (A \times I)$.



(2.2.4)

It is known that this condition is also sufficient.

Theorem 2.2.5. *For an inclusion $A \subset X$ the following are equivalent:*

(1) The inclusion map $A \hookrightarrow X$ is a cofibration.

(2) $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$.

Proof. (1) $\Rightarrow$ (2): Consider the diagram of 2.2.4. The filled-in map r is the desired retraction.

(2) $\Rightarrow$ (1): Composing the retraction of (2) with a map $A \times I \cup X \times 0 \to Y$ gives the homotopy extension property for all Y, which, as mentioned, is equivalent to (1). □

Corollary 2.2.6. *If A is a subcomplex of a CW-complex X, then the inclusion $A \hookrightarrow X$ is a cofibration.*

Proof. This follows from Corollary 1.4.13 which says any CW-pair (X, A) has the HEP. □

The main technical result for proving that particular inclusions are cofibrations is the following. Note that conditions (1) and (2) always hold if X is metric.

Theorem 2.2.7. *Assume that $A \subset X$ is closed and that there exists a neighbourhood U of A and a map $\phi\colon X \to I$, such that:*

(1) $A = \phi^{-1}(0)$;

(2) $\phi(X \setminus U) = \{1\}$; and

(3) U deforms to A through X with A fixed. That is, there is a map $H\colon U \times I \to X$ such that $H(a, t) = a$ for all $a \in A$, $H(u, 0) = u$, and $H(u, 1) \in A$ for all $u \in U$.

Then the inclusion $A \hookrightarrow X$ is a cofibration. The converse also holds.

Proof. We can assume that $\phi = 1$ on a *neighbourhood* of $X \setminus U$, by replacing ϕ with $\min(2\phi, 1)$. It suffices to show that there exists a map

$$\Phi\colon U \times I \to X \times \{0\} \cup A \times I$$

such that $\Phi(x,0) = (x,0)$ for $x \in U$ and $\Phi(a,t) = (a,t)$ for $a \in A$ and all t, since then the map $r(x,t) = \Phi(x, t(1-\phi(x)))$ for $x \in U$ and $r(x,t) = (x,0)$ for $x \notin U$ gives the desired retraction $X \times I \to A \times I \cup X \times \{0\}$.

We define Φ by

$$\Phi(u,t) = \begin{cases} H(u, t/\phi(u)) \times \{0\} & \text{for} \quad \phi(u) > t, \\ H(u,1) \times \{t - \phi(u)\} & \text{for} \quad \phi(u) \le t. \end{cases}$$

We need only show that Φ is continuous at those points $(u,0)$ such that $\phi(u) = 0$, i.e., at points $(a,0)$ for $a \in A$.

Note that $H(a,t) = a$ for all $t \in I$. Thus, for W a neighbourhood of a, there is a neighbourhood $V \subset W$ of a such that $H(V \times I) \subset W$. Therefore, $t < \epsilon$ and $u \in V$ imply that $\Phi(u,t) \in W \times [0,\epsilon]$, and hence that Φ is continuous.

We will now prove the converse.

Let $r\colon X \times I \to A \times I \cup X \times \{0\}$ be a retraction, let $s(x) = r(x,1)$ and put $U = s^{-1}(A \times (0,1])$. Let p_X, p_I be the projections of $X \times I$ to its factors. Then put $H = p_X \circ r\colon U \times I \to X$. This satisfies (3). For (1) and (2), put $\phi(x) = \max_{t \in I} |t - p_I r(x,t)|$ which makes sense since I is compact. That this satisfies (1) and (2) is clear and it remains to show that ϕ is continuous. Let $f(x,t) = |t - p_I r(x,t)|$ and $f_t(x) = f(x,t)$, all of which are continuous. Then

$$\phi^{-1}((-\infty, b]) = \{x | f(x,t) \le b \;\forall\; t\} = \cap_{t \in I} f_t^{-1}((-\infty, b])$$

is an intersection of closed sets and so is closed. Similarly

$$\phi^{-1}([a,\infty)) = \{x | f(x,t) \ge a \text{ for some } t\} = p_X(f^{-1}([a,\infty)))$$

is closed since p_X is a projection. Since the complements of the intervals of the form $[a,\infty)$ and $(-\infty,b]$ give a subbase for the topology of $\mathbb{R}$, the contention follows. □

Let us recall the definition:

Definition 2.2.8. Let X be a topological space; a subspace $A \subset X$ is a *strong deformation retract* of X if there exists a homotopy $H\colon X \times I \to X$ such that

$$\begin{aligned} H(x,0) &= x, \quad x \in X \\ H(x,1) &\in A, \quad x \in X \\ H(a,t) &= a, \quad (a,t) \in A \times I. \end{aligned}$$

The homotopy H is a *strong deformation retraction* of X onto A. The map $r = H(-,1)\colon X \to A$ is a *retraction* and A is a *retract* of X. Thus, a retract A of X with retraction $r\colon X \to A$ is a strong deformation retraction of X if

$$X \xrightarrow{r} A \xhookrightarrow{i} X$$

is homotopic rel $.A$ to id_X.

$$\begin{array}{ccccc}
A\times 0 & \xleftarrow{H(x,0)|_A=\mathrm{id}_A} & A\times I & \xrightarrow{H(x,0)|_A=\mathrm{id}_A} & A\times 1 \\
{\scriptstyle i\times\mathrm{id}_0}\downarrow & & {\scriptstyle i\times\mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle i\times\mathrm{id}_1} \\
X\times 0 & \xleftarrow[H(x,0)=\mathrm{id}_X]{} & X\times I & \xrightarrow[H(x,1)=r(x)]{} & X\times 1
\end{array}$$

It was remarked in Theorem 2.2.5 that if (X, A) is a cofibred pair, then $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$. In fact, we have the following stronger result.

Lemma 2.2.9. *If (X, A) is a cofibred pair, then $(X \times 0) \cup (A \times I)$ is a strong deformation retract of $X \times I$.*

Proof. Let $i\colon X \times \{0\} \cup A \times I \subset X \times I$ be the inclusion map, and let

$$r\colon X \times I \to X \times \{0\} \cup A \times I$$

be a retraction. A homotopy

$$D\colon ir \simeq \mathrm{id}_{X\times I} \,\mathrm{rel}\, X \times \{0\} \cup A \times I$$

is given by

$$D(x, t, s) = (p_X r(x, (1 - s)t), (1 - s)p_I r(x, t) + st).$$

□

Some authours suppose that $i\colon A \to X$ is an inclusion with closed image. That A can be regarded as closed is guaranteed by Proposition 1.4.9. And the following theorem shows that $i\colon A \to X$ can be treated as an inclusion.

Theorem 2.2.10. *If $j\colon A \to X$ is a cofibration, then j is an imbedding, i.e, is a homeomorphism $A \approx j(A)$.*

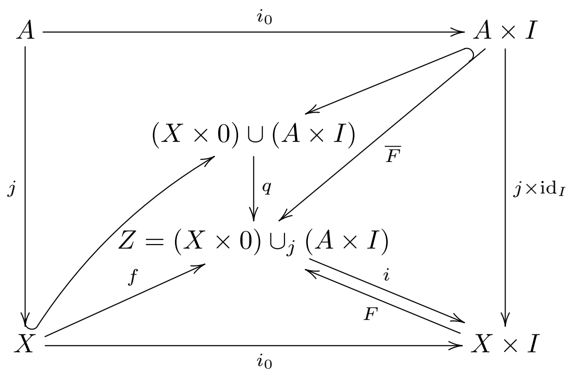
Proof. Let $j\colon A \to X$ be a cofibration and consider the mapping cylinder $Z = (X \times 0) \cup_j (A \times I)$, that is, the quotient space of the topological sum $(X \times 0) \cup (A \times I)$ obtained by identifying $(a, 0) \in A \times I$ with $(j(a), 0) \in X \times 0$ for each $a \in A$. Denote by q the quotient map $(X \times 0) \cup (A \times I) \to Z$. There is a continuous map $i\colon Z \to Z \times I$ defined by

$$iq|_{X\times 0} = \mathrm{id}_{X\times,0},$$
$$iq|_{A\times I} = j \times \mathrm{id}_I\,.$$

Define maps $f\colon X \to Z$ and $\overline{F}\colon A \times I \to Z$ by

$$f(x) = q(x, 0), \quad \overline{F}(a, t) = q(a, t).$$

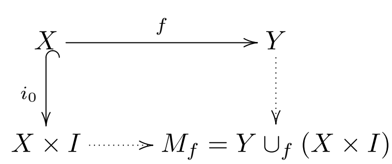
Because j is a cofibration there exists a map $F\colon X\times I \to Z$ such that $F(j(a),t) = q(a,t)$ and $F(x,0) = q(x,0)$ for all $a \in A$, $t \in I$ and $x \in X$. Then $Fi|_{X\times 0}\,\mathrm{id}_{X\times 0}$ and $Fi|_{A\times I} = j \times \mathrm{id}_I$ so $Fi = \mathrm{id}_Z$. i is, therefore, a continuous monomorphism of Z onto $i(Z) = (X \times 0) \cup (j(A) \times I)$. Also, $q|_{A\times 1}$ is a homeomorphism of $A \times 1$ onto $q(A \times 1)$, and consequently $iq|_{A\times 1}$ is a homeomorphism of $A \times 1$ onto $iq(A \times 1) = j(A) \times 1$.



∎

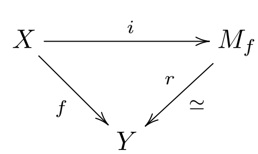
Mapping cylinders and cofibrations

Definition 2.2.11. The *mapping cylinder* of $f\colon X \to Y$ is defined to be the pushout of the maps $f\colon X \to Y$ and $i_0\colon X \hookrightarrow X \times I$, and we note it M_f, so $M_f \equiv Y \cup_f (X \times I)$.



Replacing maps by cofibrations

Remark 2.2.12. The inclusion $i\colon X \hookrightarrow M_f$ clearly satisfies Theorem 2.2.7 and hence is a cofibration. Also, the retraction $r\colon M_f \to Y$ is a homotopy equivalence with homotopy inverse being the inclusion $Y \hookrightarrow M_f$. The diagram



commutes. This shows that *any* map f is a cofibration, up to a homotopy equivalence of spaces.

Also recall the definition of the "mapping cone" of $f\colon X \to F$ as the quotient space

$$C_f = M_f/X \times \{1\} \approx M_f \cup CX.$$

In the case of an inclusion $i\colon A \hookrightarrow X$, we have $C_i = X \cup CA$. There is the map

$$C_i \xrightarrow{h} X/A,$$

defined as the quotient map $X \cup CA \to (X \cup CA)/CA$ composed with the inverse of the homeomorphism $X/A \to (X \cup CA)/CA$. It is natural to ask whether h is a homotopy equivalence. This is not always the case, but the following gives a sufficient condition for it to be so. (cf Theorem 1.4.3 and Example 1.4.19).

Theorem 2.2.13. *If $A \subset X$ is closed and the inclusion $i\colon A \hookrightarrow X$ is a cofibration then $h\colon C_i \to X/A$ is a homotopy equivalence. In fact, it is a homotopy equivalence of pairs*

$$(X/A, *) \simeq (C_i, CA) \simeq (C_i, v),$$

where v is the vertex of the cone.

Proof. The mapping cone $C_i = X \cup CA$ consists of three different types of points, the vertex $v = \{A \times \{1\}\}$, the rest of the cone $\{(a,t) | 0 <\leq\leq 1\}$ where $(a, 0) = a \in A \subset X$, and points in X itself, which we identify with $X \times \{0\}$ to simplify definitions of maps.

Define $f\colon A \times \cup X \times \{0\} \to C_i$, as the collapsing map and extend f to $\overline{f}\colon X \times I \to C_i$ by the definition of cofibration. Then $\overline{f}(a, 1) = v$, $\overline{f}(a, t) = (a, t)$ and$\overline{f}(x, 0) = x$.

Put $\overline{f}_t = \overline{f}|_{X\times\{t\}}$. Since $\overline{f}1(A) = \{v\}$, there is the factorisation $\overline{f}_1 = g \circ j$, where $j\colon X \to X/A$ is the quotient map and $g\colon X/A \to C_i$. (g is continuous by definition of the quotient topology.)

We claim that g is a homotopy equivalence and a homotopy inverse to h.

First we will prove that $hg \simeq \mathrm{id}_{X/A}$. There is the homotopy $h\overline{f}_t\colon X \to X/A$. For all t, this takes A into the point $\{A\}$. Thus it factors to give the homotopy

$$hg \simeq \{h\overline{f}_1\} \simeq \{hf_0\} = \{j\} = \mathrm{id}_{X/A}\,.$$

Next we will show that $gh \simeq \mathrm{id}_{C_i}$. For this, consider $W = (X \times I)/(A \times \{1\})$ and the maps illustrated in Figure 2.1. The map $\overline{f}'$ is induced by $\overline{f}$. The map k is the "top face" map. We see that

$$\begin{aligned} \overline{f}' \circ l &= \mathrm{id}, \\ \pi \circ k &= \mathrm{id} \quad \text{(which we don't need)}, \\ k \circ \pi &\simeq \mathrm{id}, \\ \overline{f}' \circ k &= g \quad \text{(definition of g)}, \\ \pi \circ l &= h. \end{aligned}$$

Hence $g \circ h = \overline{f}' \circ (k \circ \pi) \circ l \simeq \overline{f}' \circ l = \mathrm{id}$, as claimed. □

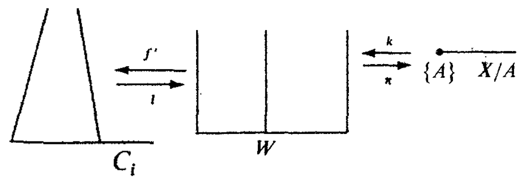


Figure 2.1: A homotopy equivalence and homotopy inverse.

Remark 2.2.14. Theorem 2.2.13 does not hold if $X \times 0 \cup A \times I$ is not a retract of $X \times I$: for example, we repeat Example 1.4.10. Let $A = \{0\} \cup \{l/n | n = 1, 2, \dots\}$, and $X = [0, 1]$. Here C_i is *not* homotopy equivalent to X/A, which is a one-point union of an infinite sequence of circles with radii going to zero. (C_i has homeomorphs of circles joined along edges, but the circles do not tend to a point and so any prospective homotopy equivalence $X/A \to C_i$ would be discontinuous at the image of $\{0\}$ in X/A.)

Let us recall the notion of the pointed category and some notational items. The pointed category has, as objects, spaces with a base point $*$, and, as maps, those maps of spaces preserving the base point. There is also the category of pairs of pointed spaces. There is also the notion of homotopies in this category, those homotopies which preserve the base point.

If $f\colon X \to Y$ is a pointed map then the *reduced mapping cylinder* of f is the quotient space M_f of $(X \times I) \cup Y$ modulo the relations identifying $(x, 0)$ with $f(x)$ and identifying the set $\{*\} \times I$ to the base point of M_f.

The *reduced mapping cone* is the quotient of the reduced mapping cylinder M_f gotten by identifying the image of $X \times \{1\}$ to a point, the base point.

The one-point union of pointed spaces X and Y is the quotient $X \vee Y$ of the disjoint union $X \cup Y$ obtained by identifying the two base points.

The wedge, or smash, product is the pointed space $X \wedge Y = (X \times Y)/(X \vee Y$.

The circle $\mathbb{S}^1$ is defined as $I/\partial I$ with base point $\{\partial I\}$.

The reduced suspension of a pointed space X is $SX = X \wedge \mathbb{S}^1$. It can also be considered as the quotient space $(X \times I)/(X \times \partial I \cup \{*\} \times I)$.

As remarked before, $\mathbb{S}^n \wedge \mathbb{S}^m$ is the one-point compactification of $\mathbb{R}^n \times \mathbb{R}^m$ and hence is homeomorphic to $\mathbb{S}^{n+m}$. Thus we can, and will in this chapter, redefine $\mathbb{S}$ inductively by letting $\mathbb{S}^{n+1} = S\mathbb{S}$. Also note that

$$S(SX) = (SX) \wedge \mathbb{S}^1 = (X \wedge \mathbb{S}^1) \wedge \mathbb{S}^1 = X \wedge \mathbb{S}^2, \quad \text{etc..}$$

The preceding results of this section can all be rephrased in terms of the pointed category. Extending the proofs is elementary, mostly a matter of seeing that the unreduced versions become the reduced versions by taking the quotient of spaces by sets involving the base point. For example, Theorem 2.2.13 would say that if A is a closed, pointed, subspace of the pointed space X and if the inclusion $i\colon A \to X$ is a cofibration (same definition since the base point is automatically taken care of) then $X/A \simeq C_i$, where the latter is now the reduced mapping cone, and the homotopies involved must preserve the base points.

Definition 2.2.15. A base point $x_0 \in X$ is said to be *nondegenerate* if the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration. A pointed Hausdorff space X with nondegenerate base point is said to be *well-pointed*.

Any pointed manifold or CW-complex is clearly well-pointed. A pointed space that is not well-pointed is $\{0\} \cup \{1/n | n \geq 1\}$ with 0 as base point. The reduced suspensions of this also fail to be well-pointed.

If $A \hookrightarrow X$ is a cofibration then X/A, with base point $\{A\}$, is well-pointed as follows easily from Theorem 2.2.7.

If a whisker is appended at the base point of any pointed space X, then changing the base point to the other end of the whisker provides a well-pointed space. (This is, of course, just the mapping cylinder of the inclusion of the base point into X.)

Theorem 2.2.16. *If X is well-pointed then so are the reduced cone CX and the reduced suspension SX. Moreover, the collapsing map $\Sigma X \to SX$, of the unreduced suspension to the reduced suspension, is a homotopy equivalence.*

Proof. Denote the base point of X by $*$. Consider a homeomorphism

$$h\colon (I \times I, I \times \{0\} \cup \partial I \times I) \xrightarrow{\approx} (I \times I, I \times \{0\})$$

which clearly exists. Then the induced homeomorphism

$$\mathrm{id}_X \times h\colon X \times I \times I \xrightarrow{\approx} X \times I \times I$$

carries

$$\begin{aligned} X \times I \times \{0\} \cup X \times \partial I \times I \quad &\text{to} \quad X \times I \times \{0\}, \quad \text{hence} \\ A = X \times I \times \{0\} \cup X \times \partial I \times I \cup \{*\} \times I \times I \quad &\text{to} \quad X \times I \times \{0\} \cup \{*\} \times I \times I. \end{aligned}$$

Therefore,

$$(X \times I \times I, A) \approx I \times (X \times I, X \times \{0\} \cup \{*\} \times I)$$

as pairs. Since $X \times \{0\} \cup \{*\} \times I$ is a retract of $X \times I$ by the definition of “"well-pointed,” it follows that A is a retract of $X \times I \times I$. This implies that the inclusion $X \times \partial I \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration. Therefore, $SX = X \times I/(X \times \partial I \cup * \times I)$ is well-pointed. A similar argument using a homeomorphism

$$(I \times I, I \times \{0\} \cup \{1\} \times I) \xrightarrow{\approx} (I \times I, O \times \{0\})$$

shows that the inclusion $X \times \{1\} \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration and so $CX = X \times I/(X \times \{1\} \cup \{*\} \times)$ is well-pointed.

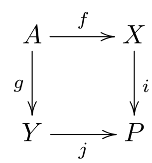
The fact that $X \times \partial I \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration implies that the induced inclusion

$$I \approx \{*\} \times I \hookrightarrow X \times I/(X \times \{0\}, X \times \{1\})$$

is a cofibration by an easy application of Theorem 2.2.7. By Theorem 2.2.13, $\Sigma X \simeq \Sigma X \cup CI \simeq \Sigma X/I = SX$ via the collapsing map. □

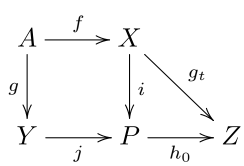
We often construct new spaces and new maps from the given spaces and maps, and one way of such a construction is to take the pushout of two maps. The following proposition states that the class of cofibrations is closed under taking pushouts. Thus we may take pushout of two maps without any restriction.

Proposition 2.2.17. *Let*

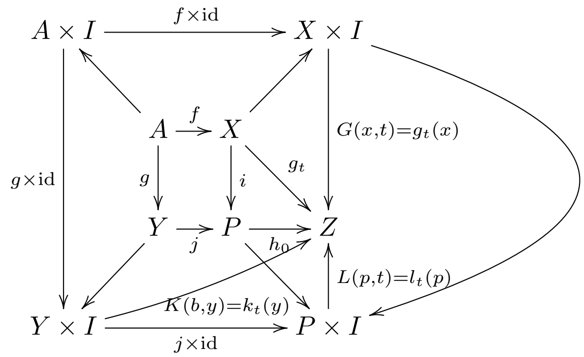


be a pushout square. If g is a cofibration then so is i. In this case, j induces a homeomorphism $j' : Y/g(A) \to P/i(X)$ of cofibres.

Proof. Consider the diagram



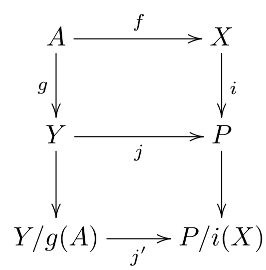
where g_t is a homotopy with $g_0 = h_0 i$. Then $g_0 f = h_0 j g$ and, because g is a cofibration, there is a homotopy $k_t : Y \to Z$ such that $k_t g = g_t f$ and $k_0 = h_0 j$. Then g_t and k_t induce a map $l_t : P \to Z$ such that $l_t i = g_t$ and $l_t j = k_t$. Then l_t is a homotopy since the map $L : P \times I \to Z$ obtained from l_t is continuous. Furthermore, $l_0 i = g_0 = h_0 i$ and $l_0 j = k_0 = h_0 j$. By the uniqueness property of the pushout, $l_0 = h_0$, and so i is a cofibration.



For the second assertion of the proposition, note that j induces $j'\colon Y/g(A) \to P/i(X)$. We regard the pushout P as defined in the proof of Proposition 1.4.7. Then P/iX is obtained from $X \vee Y$ from the following relations:

$$\begin{cases} (f(a),*) \sim (*,g(a)), & \text{for every} \quad a \in A \\ (x,*) \sim *, & \text{for every} \quad x \in X \end{cases}$$

Thus $Y/g(A) \simeq P/i(X)$, and the homeomorphism is j' defined by $j'\langle y\rangle = \langle *, y\rangle$ for $y \in Y$.



□

A criterion for a map to be a cofibration

We want a criterion that allows us to recognise cofibrations when we see them. We shall often consider pairs (X, A) consisting of a space X and a subspace A. Cofibration pairs will be those pairs that "behave homologically" just like the associated quotient spaces X/A.

Definition 2.2.18. A pair (X, A) is an *NDR-pair* (= *neighbourhood deformation retract pair*) if there is a map $u\colon X \to I$ such that $u^{-1}(0) = A$ and a homotopy $h\colon X \times I \to X$ such that $h_0 = \mathrm{id}$, $h(a,t) = a$ for $a \in A$ and $t \in I$, and $h(x,1)$ßA if $u(x) < 1$; (X, A) is a *DR-pair* if $u(x) < 1$ for all $x \in X$, in which case A is a deformation retract of X.

Lemma 2.2.19. *If (h,u) and (j,v) represent (X,A) and (Y,B) as NDR-pairs, then (k,w) represents the "product pair" $(X\times Y, X\times B\cup A\times Y)$ as an NDR-pair, where $w(x,y)=\min(u(x),v(y))$ and*

$$k(x,y,t)=\begin{cases}(h(x,t),j(y,tu(x)/v(y))) & \text{if } \ v(y)\geq u(x)\\ (h(x,tv(y)/u(x)),j(y,t)) & \text{if } \ u(x)\geq v(y).\end{cases}$$

If (X,A) or (Y,B) is a DR-pair, then so is $(X\times Y, X\times B\cup A\times Y)$.

Proof. If $v(y)=0$ and $v(y)\geq u(x)$, then $u(x)=0$ and both $y\in B$ and $x\in A$; therefore we can and must understand $k(x,y,t)$ to be (x,y). It is easy to check from this and the symmetric observation that k is a well defined continuous homotopy as desired. □

Theorem 2.2.20. *Let A be a closed subspace of X. Then the following are equivalent:*

(i) (X,A) *is an NDR-pair.*

(ii) $(X\times I, X\times\{0\}\cup A\times I)$ *is a DR-pair.*

(iii) $X\times\{0\}\cup A\times I$ *is a retract of* $X\times I$.

(iv) The inclusion $i\colon A\to X$ *is a cofibration.*

Proof. The lemma gives that (i) implies (ii), (ii) trivially implies (iii), and we have already seen that (iii) and (iv) are equivalent. Assume given a retraction $r\colon X\times I\to X\times\{0\}\cup A\times I$. Let $\pi_1\colon X\times I\to X$ and $\pi_2\colon X\times I\to I$ be the projections and define $u\colon X\to I$ by

$$u(x)=\sup\{t-\pi_2 r(x,t)|t\in I\}$$

and $h\colon X\times I\to X$ by

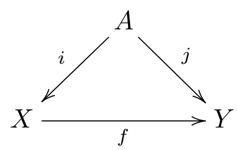
$$h(x,t)=\pi_1 r(x,t).$$

Then (h,u) represents (X,A) as an NDR-pair. Here $u^{-1}(0)=A$ since $u(x)=0$ implies that $r(x,t)\in A\times I$ for $t>0$ and thus also for $t=0$ since $A\times I$ is closed in $X\times I$. □

Cofibre homotopy equivalence

It is often important to work in the category of spaces under a given space A, and we shall later need a basic result about homotopy equivalences in this category. We shall also need a generalisation concerning homotopy equivalences of pairs. The reader is warned that the results of this section, although easy enough to understand, have fairly lengthy and unilluminating proofs.

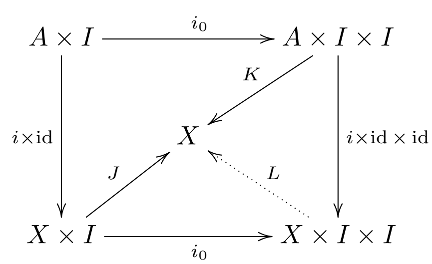
A space *under* A is a map $i\colon A \to X$. A *map of spaces under* A is a commutative diagram



A *homotopy between maps under* A is a homotopy that at each time t is a map under A. We then write $h\colon f \simeq f' \operatorname{rel} A$ and have $h(i(a), t) = j(a)$ for all $a \in A$ and $t \in I$. There results a notion of a *homotopy equivalence under* A. Such an equivalence is called a *cofibre homotopy equivalence.* The name is suggested by the following result, whose proof illustrates a more substantial use of the HEP than we have seen before.

Proposition 2.2.21. *Let $i\colon A \to X$ and $j\colon A \to Y$ be cofibrations and let $f\colon X \to Y$ be a map such that $f \circ i = j$. Suppose that f is a homotopy equivalence. Then f is a cofibre homotopy equivalence.*

Proof. It suffices to find a map $g\colon Y \to X$ under A and a homotopy $g \circ f' \simeq \mathrm{id} \operatorname{rel} A$. Indeed, g will then be a homotopy equivalence, and we can repeat the argument to obtain $f'\colon X \to Y$ such that $f' \circ g \simeq \mathrm{id} \operatorname{rel} A$; it will follow formally that $f' \simeq f \operatorname{rel} A$. By hypothesis, there is a map $g''\colon Y \to X$ that is a homotopy inverse to f. Since $g'' \circ f' \simeq \mathrm{id}$, $g'' \circ j \simeq i$. Since j satisfies the HEP, it follows directly that g'' is homotopic to a map g' such that $g' \circ j = i$. It suffices to prove that $g' \circ f\colon X \to X$ has a left homotopy inverse $e\colon X \to X$ under A, since $g = e \circ g'$ will then satisfy $g \circ f \simeq \mathrm{id} \operatorname{rel} A$. Replacing our original map f with $g' \circ f$, we see that it suffices to obtain a left homotopy inverse under A to a map $f\colon X \to X$ such that $f \circ i = i$ and $f \simeq \mathrm{id}$. Choose a homotopy $h\colon f \simeq \mathrm{id}$. Since $h_0 \circ i = f \circ i = i$ and $h_1 = \mathrm{id}$, we can apply the HEP to $h \circ (i \times \mathrm{id})\colon A \times I \to X$ and the identity map of X to obtain a homotopy $k\colon \mathrm{id} \simeq k_1 \equiv e$ such that $k \circ (i \times \mathrm{id}) = h \circ (i \times \mathrm{id})$. Certainly $e \circ i = i$. Now apply the HEP to the following diagramme:



Here J is the homotopy $e \circ f \simeq \mathrm{id}$ specified by

$$J(x, s) = \begin{cases} k(f(x), 1 - 2s) & \text{if} \quad s \le 1/2 \\ h(x, 2s - 1) & \text{if} \quad s \ge 1/2. \end{cases}$$

The homotopy between homotopies K is specified by

$$K(a,s,t) = \begin{cases} k(i(a), 1-2s(1-t)) & \text{if} \quad s \le 1/2 \\ h(i(a), 1-2(1-s)(1-t) & \text{if} \quad s \ge 1/2. \end{cases}$$

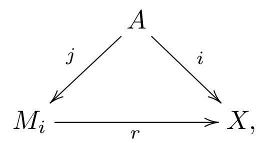
Traversal of L around the three faces of $I \times I$ other than that specified by J gives a homotopy

$$e \circ f = J_0 = L_{0,0} \simeq L_{0,1} \simeq L_{1,1} \simeq L_{1,0} = J_1 = \mathrm{id} \,\mathrm{rel}\, A.$$

□

The proposition applies to the following previously encountered situation.

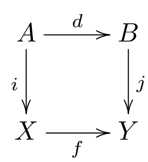
Example 2.2.22. Let $i\colon A \to X$ be a cofibration. We then have the commutative diagramme



where $j(a) = (a,1)$. The obvious homotopy inverse $\iota\colon X \to M_i$ has $\iota(x) = (x,0)$ and is thus very far from being a map under A. The proposition ensures that ι is homotopic to a map under A that is homotopy inverse to r under A.

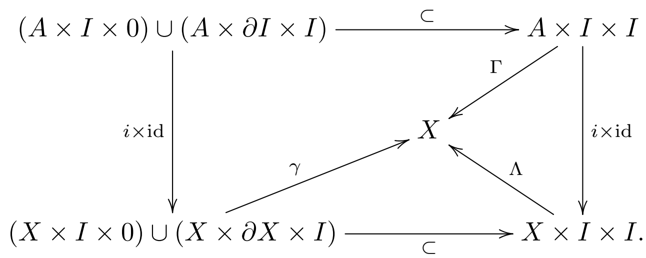
The following generalisation asserts that, for inclusions that are cofibrations, a pair of homotopy equivalences is a homotopy equivalence of pairs. It is often used implicitly in setting up homology and cohomology theories on pairs of spaces.

Proposition 2.2.23. *Assume given a commutative diagram*



in which i and j are cofibrations and d and f are homotopy equivalences. Then $(f,d)\colon (X,A) \to (Y,B)$ is a homotopy equivalence of pairs.

Proof. The statement means that there are homotopy inverses e of d and g of f such that $g \circ j = i \circ e$ together with homotopies $H\colon g \circ f \simeq \mathrm{id}$ and $K\colon f \circ g \simeq \mathrm{id}$ that extend homotopies $h\colon e \circ d \simeq \mathrm{id}$ and $k\colon d \circ e \simeq \mathrm{id}$. Choose any homotopy inverse e to d, together with homotopies $h\colon e \circ d \simeq \mathrm{id}$ and $\ell\colon d \circ e \simeq \mathrm{id}$. By HEP for j, there is a homotopy inverse g' for f such that $g' \circ j = i \circ e$. Then, by HEP for i, there is a homotopy m of $g' \circ f$ such that $m \circ (i \times \mathrm{id}) = i \circ h$. Let $\phi = m_1$. Then $\phi \circ i = i$ and ϕ is a cofibre homotopy equivalence by the previous result. Let $\psi\colon X \to X$ be a homotopy inverse under i and let $n\colon \psi \circ \phi \simeq \mathrm{id}$ be a homotopy under i. Define $g = \psi \circ g'$. Clearly $g \circ j = i \circ e$. Using that the pairs $(I \times I, I \times \{0\})$ and $(I \times I, I \times \{0\} \cup \partial I \times I)$ are homeomorphic, we can construct a homotopy between homotopies Λ by applying HEP to the diagram



Here

$$\gamma(x, s, 0) = \begin{cases} \psi(m(x, 2s)) & \text{if} \quad s \leq 1/2 \\ n(x, 2s-1) & \text{if} \quad s \geq 1/2, \end{cases}$$
$$\gamma(x, 0, t) = (g \circ f)(x) = (\psi \circ g' \circ f)(x),$$
$$\gamma(x, 1, t) = x,$$

while

$$\Gamma(a, s, t) = \begin{cases} i(h(a, 2s/(1+t))) & \text{if} \quad 2s \leq 1+t \\ i(a) & \text{if} \quad 2s \geq 1+t \end{cases}$$

Define $H(x, s) = \Lambda(x, s, 1)$. Then $H\colon g \circ f \simeq \mathrm{id}$ and $H \circ (i \times \mathrm{id}) = i \circ h$. Application of this argument with d and f replaced by e and g gives a left homotopy inverse f' to g and a homotopy $L\colon f' \circ g \simeq \mathrm{id}$ such that $f' \circ i = j \circ d$ and $L \circ (j \times \mathrm{id}) = j \circ \ell$. Adding homotopies by concentrating them on successive fractions of the unit interval and letting the negative of a homotopy be obtained by reversal of direction, define

$$k = (-\ell)(de \times \mathrm{id}) + dh(e \times \mathrm{id}) + \ell$$

and

$$K = (-L)(fg \times \mathrm{id}) + f'H(g \times \mathrm{id}) + L.$$

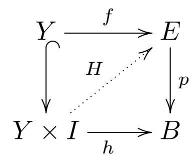
Then $K\colon f \circ g \simeq \mathrm{id}$ and $K \circ (j \times \mathrm{id}) = j \circ k$. □

2.3 Fibrations

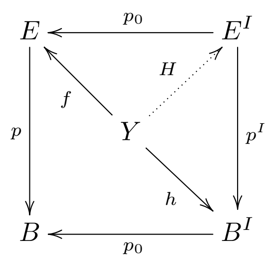
We "dualise" the definitions and theory of the previous section to the study of fibrations, which are "up to homotopy" generalisations of covering spaces.

The definition of fibrations

Definition 2.3.1. A surjective map $p\colon E \to B$ is a *fibration* if it satisfies the covering homotopy property (CHP for short), i.e., given a map $f\colon Y \to E$ and a homotopy $h\colon Y \times I \to B$, there exists a lifting of h to E, whose restriction to $Y \times \{0\}$ is f.



Remark 2.3.2. As we have seen for cofibrations there is an equivalent definition of a fibration in which we can better see the dualisation.



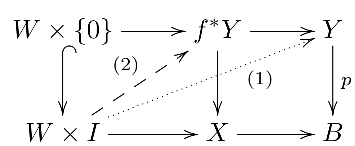
where $p_0(\beta) = \beta(0)$.

Remark 2.3.3. Again we do not require the uniqueness of such a lifting.

The class of fibrations is closed under the base extensions, i.e,

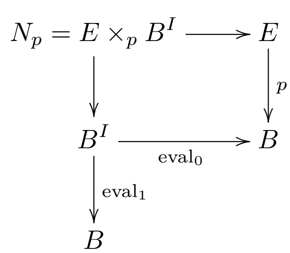
Proposition 2.3.4. *Pullbacks of fibrations are fibrations.*

Proof. Let $p\colon Y \to B$ be a fibration, $f\colon X \to B$ be a map and $p'\colon f^*Y = Y \times_f X \to X$ be the pullback by f.



Since p is a fibration, the prospective map marked (1) exists, maintaining commutativity. Then the map marked (2) exists by the universal property of pullbacks. □

Definition 2.3.5. Let $p\colon E \to B$ be a map. Its *mapping path space* (or *mapping cocylinder* or *mapping path fibration*) is a pullback of p and $\mathrm{eval}_0\colon B^I \to B$. We note it $N_p \equiv E \times_p B^I = \{(e, \beta) | \beta(0) = p(e)\}$.



The mapping path space is the dual of mapping cylinder and we will see that it plays the parallel role for the fibrations.

Definition 2.3.6. Let $p\colon E \to B$ be a map and let N_p be its mapping path space. A map $s\colon N_p \to E^I$ such that $s(e,\beta)(0) = e$ and $p \circ s(e,\beta) = \beta$ is called *path lifting function.*

We have seen that for a map $i\colon A \to X$ to be a cofibration it suffices to admit a homotopy extension for its mapping cylinder. It turns out that for a map $p\colon E \to B$ to be fibration it suffices to have a homotopy lifting for its mapping path space, or equivalently we have

Proposition 2.3.7. *A map $p\colon E \to B$ is a fibration if and only if it admits a path lifting function.*

Proof. Replace Y by N_p in the test diagram of the equivalent definition of fibrations; necessity is then clear. So suppose that we have a path lifting function $s\colon N_p \to E^I$ and maps $f\colon Y \to E$ and $h\colon Y \to B^I$. There is an induced map $g\colon Y \to N_p$, since N_p is a pullback. The composite $s \circ g$ gives the required homotopy lifting. □

As an application of this proposition, we have the following example

Example 2.3.8. If $p\colon E \to B$ is a covering, then p is a fibration with a unique path lifting function.

Example 2.3.9. The evaluation map $p_s\colon B^I \to B$ given by $p_s(\beta) = \beta(s)$ is a fibration.

The relation between fibrations and cofibrations is stated in the following proposition

Proposition 2.3.10. *If $i\colon A \to X$ is a cofibration and B is a space then the induced map $p = B^i\colon B^X \to B^A$ is a fibration.*

Proof. It is an easy task to show that we have the following homeomorphisms

$$B^{M_i} = B^{X\times\{0\}\cup A\times I} \simeq B \times_p (B^A)^I = N_p$$

□

We have seen that every map can be factored as cofibration followed by a homotopy equivalence. We can "dualise" this property and get the following proposition, which will be of great use later.

Proposition 2.3.11. *Any map can be factored as a homotopy equivalence followed by a fibration.*

Proof. □

2.4 Homotopy Exact Sequences

In this section, we elaborate the fundamental tools and definitions of our study of exact homotopy sequences.

2.5 Homotopy Groups

2.6 Homotopy Property of CW Complexes

2.7 The Homotopy Excision And Suspension Theorems