Basic Homotopy Theory

M. M. Postnikov

Preface

A word from the transcriber

This is an attempt to translate the monumental work of M. M. Postnikov "Lectures on algebraic topology - Fundamentals of homotopy theory" into English.

Postnikov is renowned for many good textbooks on mathematics he has written, and most of which have been translated into many languages including English. Unfortunately, as long as the transcriber is aware, no attempt has been made to translate this work into other languages. That is the reason of the devil-may-care behaviour of the transcriber, who hopes his attempt is not a complete waste of time.

The preface by the original authour

This book is a systematic textbook on homotopy theory in the unexpectedly extensive part that can be constructed without involving homological methods.

As is well-known, teaching and studying algebraic topology is extremely complicated by the fact that the theory of homology and cohomology, which occupies a central place in this science- this place (which, by the way, is also true for the modern stage of its development - with an appropriate, more general understanding of homology and cohomology), is extremely cumbersome, requiring for its accurate presentation of entire books. Before reaching at least the simplest applications, the student must cross a vast desert of abstract constructions, the role and meaning of which remain incomprehensible and unknown for him for a long time and which he is forced to study only out of trust in the teacher. The authours of textbooks on algebraic topology have shown a lot of care to make this road easier for students, but they did not achieve significant results. Meanwhile, there is a very simple and elegant way of understanding homology theory, which fits into one or two lectures. Its idea is to define groups of chains of cellular spaces as relative homotopic groups of spans and on this basis to build homology and cohomology groups. Of course, this requires a sufficiently advanced homotopy theory, which therefore needs to be pre-stated. In general, this path may be painfully difficult: instead of crossing a flat, although boring, the desert has to overcome the steep slopes and deep gorges of the mountainous country. But this country is not lifeless, and upon entering it, beautiful views almost immediately open up before the traveller, prompting him to move on. Of course, this path also has its drawbacks, the most important of which is the creation of a false perspective in the student about the meaning of elementary homotopy (= not using homology theory) methods and their underestimation of the power and effectiveness of the methods of homology theory. Another objection to this path is that without homology theory, the proofs of a number of key theorems are significantly complicated and made unnecessarily difficult. However, it should be borne in mind, that the inner nontriviality of the theory can never be circumvented, and if you manage to pull out the beak, then the tail gets bogged down. At the same time, subsequently studying homological proofs of the same theorems and comparing them with elementary homotopy proofs with full force makes it possible to emphasise the power of homology theory and easily correct the false impression that was formed at the beginning. Thus, the advantages of the proposed path significantly outweigh the disadvantages.

The concrete construction of homology theory on the basis of homotopy theory will be carried out in the next issues of these "Lectures".

This book grew out of the summary of a special course that I have repeatedly read to students and postgraduates of the Faculty of Mechanics and Mathematics of Moscow State University. Each lecture of the presentation in the book turned out to be a recording of a real oral lecture, although significantly revised.

Due to the acute shortage of time when reading a special course much more often than in a mandatory course, one has to limit oneself to the idea of evidence, leaving their detailed conduct to the listeners. Auxiliary statements from other departments of mathematics need only be formulated with references to the literature, and examples illustrating the general theory can only be described, also providing their detailed analysis to listeners. When transferring an oral lecture to paper, there is no need to preserve these features and, more moreover, all the evidence should be produced in detail, once the boron of examples is carried out to the end, and the "extraneous" lemmas are proved. This explains the unexpected painful volume of some lectures in the book.

Material that was not presented for one reason or another at lectures, it is taken out in additions. (Thus, the distribution of material by lectures and supplements in the basics was dictated by the requests of the special course and has only very little to do with its internal mathematical value and significance. Nevertheless, when reading the book for the first time, it is recommended to skip the Appendices and return - contact them as needed.)

Lecture 0 has an introductory character and is devoted to the main explanation is based on examples of the subject and method of algebraic topology. In Appendix, the exponential law for mapping spaces is stated.

Lectures 1 and 2 are devoted to cofibrations, fibrations and related issues. In Appendix to these lectures, Dold's theorems on fibrations are presented. (Dual theorems are not mentioned. Their formulation and proof are left to the reader.)

In Lecture 3, general construction methods are outlined homotopically invariant factors, the necessity of transition to pointed spaces is justified, co-H-spaces and H-spaces, products and loop spaces are introduced. In Appendix, it is proved that any connected H-monoid is an H-group.

Lecture 4 contains a general discussion of the category of pointed spaces and its relationship with the category of spaces without base points. In connection with the current issue, the fundamental group is introduced. In Appendix, the mixing of topological spaces and homotopy classes is studied, the Lyusternik-Schnirelman category invariant is introduced, and conditions ensuring the nilpotency or abelicity of homotopy class groups are considered.

Lecture 5 presents the usual material about the absolute-homotopy groups, and in addition, the exact sequences of Puppe are constructed.

Lecture 6 is devoted to covers in general and methods for computing fundamental groups in particular. In Appendix, after presenting the necessary algebraic material, the Seifert-van Kampen theorem is proved.

In Lecture 7 , the concept of degree is introduced and calculated $\pi_m(\mathbb{S}^n)$ groups for $m \leq n$. The Appendix sets out standard geometric consequences of the non-stretchability of the sphere (Borsuk's theorem on an unbounded component, topological invariance of dimension, characterisation of sets that do not dissect the sphere, the theorem of invariance of domain).

In Lecture 8, returning to the general theory of homotopy groups, we introduce relative homotopy groups. In Appendix, the exactness of homotopy sequences of triples and triads is proved.

In the following lecture 9, the following theory of quasi-fibrations (called weak fibrations in the book). In Appendix, Dold's theorem on homotopy fibrations is proved.

Lecture 10 is devoted to James' product sequence theorem. In Appendix, well-known general theorems on homotopy properties of filtrations are proved.

This concludes the first part of the course devoted to the general concepts of homotopy theory and homotopy groups. Only this part of the course is included in this book.

The following lectures (which make up a separate book "Homotopy Theory of cellular Spaces", planned to be published in 1985) mainly concentrate on around the concept of cellular space.

In Lecture 11, the category of cellular spaces is introduced and studied. The more troublesome properties of cellular spaces (local contractibility and paracompactness) are taken out in Appendix.

In Lecture 12, on the basis of the usual smooth approximation technique, the connectivity of pairs (X, X^n) and the cell approximation theorem are proved. As an application, Freudenthal's theorem with the usual consequences is proved. In conclusion, the properties of antipodal maps are considered. In Appendix, after the presentation of the basic concepts of the theory of simplicial spaces, the approximation theorem is proved in the simplicial version.

In Lecture 13, the category of cellular spaces is compared with the category of all spaces (the theorem that any topological space is weakly homotopically equivalent to a cellular space). Whitehead's theorem on homotopy equivalences is also proved here. Appendix to this lecture is devoted to the representability theorems (Brown, Adams and Heller).

At this point, the general theory of cellular spaces is temporarily interrupted and from the next lecture we turn to the theory of homotopy operations.

In Lecture 14, general theorems on homotopy operations (arising from the representability of homotopy groups) are presented, additive operations are characterised, and Whitehead product is introduced.

Lecture 15 discusses generalised Whitehead product and proves its algebraic properties (skew-commutativity, bilinearity, and Jacobi identity). In Appendix to lecture 15 The Hilton-Milnor homotopy theorem is proved - in groups of bouquets of add-ons.

Geometric properties of multiplication The properties of multiplication and Whitehead multiplication are discussed in Lecture 16. The Hopf invariant and its Whitehead generalisations are also introduced there. The Whitehead method computes the Hopf invariant of the Hopf construction and, in particular, the Hopf invariant of the Hopf map. In Addition, Hopf invariants generalised by Hilton are introduced and studied. In particular, the left distributive law for compositional multiplication is discussed.

In Lecture 17, returning for the last time to cellular spaces, we prove the theorem of Blakers and Massey on cutting for triads and, based on it, Freudenthal's theorem for any connected spaces.

In lecture 18, the "difficult part of the Freudenthal's theorem" is proved and the groups $\pi_{n+1}(\mathbb{S}^n)$ and $\pi_{n+2}(\mathbb{S}^n)$ are calculated. In calculating the last group, the key role is played by the fact that the element $\eta_n \circ \eta_{n+1}$ of group $\pi_{n+2}(\mathbb{S}^n)$ is nonzero. The "modern" proof of this fact is based on the theory of cohomology operations. Since this path is not yet available to us, we are forced to present a direct geometric proof proposed at the time by G. W. Whitehead.

The material of the final lecture 19 concentrates around the question of the effect of cell gluing on homotopy groups of π_n. For $n = 1$, we obtain a well-known description of the generators and relations of the fundamental groups of cellular spaces, which, in particular, allows us to prove the Seifert-van Kampen theorem for these spaces in its classical formulation. At $n > 1$ killing spaces are introduced, Eilenberg-Mac Lane spaces are constructed and the group $\pi_n(X^n, X^{n-1})$ is calculated (on this basis and will be built next semester homology theory). In Appendix to this lecture, three-dimensional manifolds and their fundamental groups are briefly considered.

Although algebraic topology has developed mainly before the eyes of our generation, there are already many dark places in its history. This puts the authors of textbooks on topology in front of a number of intractable tasks, for example, when compiling a bibliography, which ideally should be an annotated student's guide to the labyrinth of journal literature. Without prior clarification of all priorities, influences and borrowings, any such bibliography will contain a lot of historical errors and will create an occasion for discussions, accusations and insults. A simple enumeration of all known to the author of articles (or only articles used by him), for educational purposes, is almost useless, for sure, due to the inevitable randomness of their choice, that will give the same result.

Also a very difficult question about the authorship of certain theorems; even the concept of "the author of the theorem" has no clear explication (say, for example, who is the author of the theorem that group $[K, X]$ of homotopy classes of maps from a n-dimensional cellular space K to a $(n-1)$-connected space X is isomorphic to the cohomology group $H^n(K;\pi_n(X))$: Hopf, who first described the group $[K, X]$, but in other terms and only for the case when X is $\mathbb{S}^n$, Whitney, who attracted cohomology, or Whitehead, who introduced cellular spaces and casually remarked that Whitney's formulation is suitable for any cellular spaces?).

Fuchs and Rokhlin in their famous textbook [10] cut the Gordian knot of these problems in one fell swoop: they left theorems nameless, and in the bibliography they limited to an un-commented list of books and articles containing additional information to which there are references in the text.

In these "Lectures" another decision was made: a complete and commented bibliography is given, not only books-with few exceptions-only in Russian the language is considered the most accessible, and the traditional names of theorems are interpreted as simple, easy-to-reference labels that are not necessarily associated with authorship (which is why in indisputable cases is specifically indicated).

M. M. Postnikov

Contents

Lecture 0

In this introductory lecture, we will explain what algebraic topology is and how it is applied to solving specific geometric problems.
Terminology used by Postnikov: *moneomorphism* = a homeomorphism onto its image.

0.1 Extension and retraction tasks

Let X, Y be topological spaces, A be a subspace of the space X, with $f : A \to Y$ and $\overline{f} : X \to Y$ continuous maps. Recall that the map f is called the *restriction* of the map $\overline{f}$ to A, and the map $\overline{f}$ is the *extension* (or *continuation*) of the map f from A to X, if $\overline{f}(a) = f(a)$ for any point $a \in A$. The restriction f of the map $\overline{f}$ to A is indicated by the symbol $\overline{f}|_A$. Equality $f = \overline{f}|_A$ is equivalent to the equality of $f = \overline{f} \circ i$, where $i : A \to X$ is an inclusion (restriction of the identity map $\mathrm{id} : X \to X$).

In the common problem, X, A, Y and f are given and it is required to find out if $\overline{f}$ exists. This problem is represented by a diagramme

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \overline{f}} & \\ Y & & \end{array} \tag{0.1}$$

where the dotted arrow denotes a map whose existence should be proved, and the diagramme is assumed to be commutative (we will keep these conventions throughout the course).

An interesting special case of the extension problem occurs when $Y = A$ and $f = \mathrm{id}$:

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle \mathrm{id}}\downarrow & \swarrow {\scriptstyle r} & \\ A & & \end{array} \tag{0.2}$$

The solution r to this problem is called a *retracting map* (or simply *retraction*). When it exists, the subspace A is called a *retract* of the space X. As a rule, to indicate the retracting the letter r is used for map. It is easy to see that a retracting map is always an *epimorphism* (a surjective map having the property that a subset in A is open if and only if its complete preimage in X is open).

For typographical reasons (to save paper) we will write diagramme (0.2) in the form

$$\mathrm{id} \circlearrowright A \underset{r}{\overset{i}{\rightleftarrows}} X.$$

Generalisation of the extension problem occurs when in diagramme (0.1) A is considered an arbitrary space, and i is an arbitrary map. In this case, the map $\overline{f}$ we will call the extension of the map f *with respect to the map* i. The solution is similar in a logical way to the generalised problem (0.2) we will call the *retraction of the map* i, and the map $i : A \to X$, for which there is a retraction $r : X \to A$, is a *retractible map.* However, this generalisation, in essence, does not give anything new, since, as it is easy to see, any retractible map $i : A \to X$ is a *moneomorphism* (a homeomorphism to its own image) and therefore can be considered as an embedding of A in X.

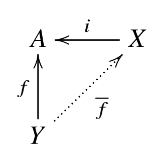
The value of retractions for problem (0.1) is that

Proposition 0.3. *a pair (X, A) ($i : A \to X$) has the property that extension exists for an arbitrary space Y and an arbitrary map $f : A \to Y$ if and only if the subspace A is a retract of the space X (the map i is retractible).*

Proof. Indeed, if the retraction $r : X \to A$ exists, then the map $\overline{f} = f \circ r$ will obviously be, the extension of the map f, and, conversely, if the map $\overline{f} : X \to Y$ exists for any map $f : A \to Y$ for any space Y, then, in particular, it exists for $Y = A$, $f = \mathrm{id}_A$ and is in this case the desired retract r. □

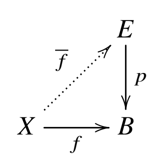
0.2 Lifting and cross-section tasks

Known from general category theory the "trick of turning arrows" translates problem (0.1) into a dual problem



(0.4)

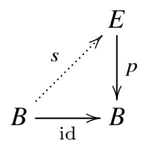
Traditionally, in this problem, it is customary to denote spaces, X, A, Y by E, B, X, the map i by p, and to write the diagramme in an "inverted" form:



(0.5)

The problem (0.5) is called the *lifting problem*, and the map $\overline{f}$ is the lifting of the map f to E. It is also said that the map $\overline{f}$ covers the map f.

Dual to problem (0.2)



(0.6)

whose solution is called the *cross section* of the map p. As a rule, the letter s is used to denote the section. The map p is a retraction of any of its sections (and, conversely, any retractible map is a section of its retraction). Therefore, in particular, every map for which there is a section, is an epiomorphism, and any section of it is a moneomorphism.

Proposition 0.7. *For this map $p : E \to B$ the problem of lifting (0.5) is soluble if and only if for an arbitrary space X and an arbitrary map $f : X \to B$, the map p has a section $s : B \to E$.*

Proof. Indeed, if s exists, then the map $\overline{F} = s \circ f$ covers the map f, and, conversely, if the map $\overline{F} : X \to E$ exists for any map $f : X \to B$ and any space X, then, in particular, it exists for $X = B$, $f = \mathrm{id}_B$, and is in this case the desired section s. □

It is useful to keep in mind that tasks (0.5) and (0.6) (also as generalised problems (0.1) and (0.2)) make sense in an arbitrary category.

0.3 The subject and method of algebraic topology

Algebraic topology can be defined in the first approximation as the science that deals with the solution of problem (0.1), (as well as problems (0.2), (0.5) and (0.6) in the category $\mathcal{T}op$ of topological spaces and continuous maps. Since problem (0.1) appears in mathematics almost everywhere (it suffices to note that any the existence theorem can be considered as a statement about the solubility in the corresponding category of some problem (0.1)) and since most mathematical objects are endowed with the structure of a topological space, this explains why algebraic topology plays one of the key roles in modern mathematics.

Very often there are specific tasks that do not have the form (0.1), but more or less easily reduced to task (0.1). General methods and principles of this kind of information are also usually included in algebraic topology.

Let on the category of $\mathcal{T}op$ (or on some its sufficiently broad subcategory) is given a functor Π that takes values in some category A. By applying the functor Π to diagramme (0.1), we get the diagramme (0.8)

$$\begin{array}{ccc} \Pi A & \xrightarrow{\Pi i} & \Pi X \\ {\scriptstyle \Pi f}\downarrow & \swarrow{\scriptstyle \varphi} & \\ \Pi Y & & \end{array} \tag{0.8}$$

representing by writing the problem of finding a morphism φ in category A that satisfies the relation $\Pi f = \varphi \circ \Pi i$. Each solution of problem (0.1) gives us a solution $\varphi = \Pi\overline{f}$ of the problem (0.8), so

Proposition 0.9. *the solubility of problem (0.8) is a necessary condition for the solubility of problem (0.1).*

Therefore, if problem (0.8) has no solution, then problem (0.1)) is all the more insoluble.

We can say that the whole algebraic topology comes down to the application of this simple consideration. Therefore, it is advisable to choose category A and functor Π only for each specific task. Of course, in order for the transition from problem (0.1) to problem (0.8) to make practical sense, you need to choose category A consisting of simpler objects than topological spaces. In principle, the simplest mathematical objects are considered in algebra. Therefore, one of the "algebraic" (studied in algebra) categories is usually chosen as category A, and thus the geometric problem (0.1) is replaced by the algebraic problem (0.8) (which is usually called the derived algebraic problem).

For problem (0.5), the derived algebraic problem has, of course, the form

$$\begin{array}{ccc} & & \Pi E \\ & \nearrow{\scriptstyle \varphi} & \downarrow{\scriptstyle \Pi p} \\ \Pi X & \xrightarrow[\Pi f]{} & \Pi B \end{array}$$

It goes without saying that for wide application the described method requires a sufficiently large stock of functors Π. Therefore, the first technical task of algebraic topology is the construction and study of such functors. Over time, as more and more difficult problems (0.1) and (0.5) entered science, it was necessary to build more and more complex functors Π, and by now algebraic topology certainly holds the record for the complexity of the specific algebraic objects used.

Since the affirmative solution of problem (0.8), generally speaking, says nothing about the existence of a solution to problem (0.1), the method of algebraic topology, in principle, can only give "negative" answers. It is not a disadvantage, but rather an advantage, since it is the theorems about the non-existence of solutions, as a rule, that are the most difficult and interesting. However, sometimes (not especially often) it is possible to prove that the solubility of a derivative algebraic problem is not only necessary, but also sufficient for the solubility of the original problem. Such results are also included in algebraic topology.

0.4 Example: the drum theorem and Brouwer's fixed point theorem

Let us illustrate these general remarks with a simple but effective example.

Let E^n be a ball $|x| \le 1$ in the space R^n, and $\mathbb{S}^{n-1}$ be an $(n-1)$-dimensional sphere $|x| = 1$ bounding it. In due course we will show that on the category of $\mathcal{Top}$ there exists such a functor Π taking values in the category of abelian groups that $\Pi E^n = 0$ and $\Pi\mathbb{S}^{n-1} \ne 0$. Using this functor, it is immediately shown that

Proposition 0.10. *the sphere $\mathbb{S}^{n-1}$ is not a retract of the ball E^n,*

i.e. that the problem below has no solution.

$$\mathrm{id} \circlearrowright \qquad \mathbb{S}^{n-1} \rightleftarrows \mathbb{E}^n$$

Indeed, the derived algebraic problem below

$$\mathrm{id} \circlearrowright \qquad \Pi\mathbb{S}^{n-1} \rightleftarrows \Pi\mathbb{E}^n$$

with $\Pi\mathbb{E}^n = 0$ and $\Pi\mathbb{S}^{n-1} \ne 0$ is obviously insoluble.

The fact that the circle $\mathbb{S}^1$ is not a retraction of the circle $\mathbb{E}^2$ is a theoretical explanation of why a film can be stretched over the circle, i.e., a drum can be made. Therefore, the proved theorem is sometimes called the *drum theorem*.

An easy consequence of the drum theorem is Brouwer's fixed point theorem, which states that

Proposition 0.11. *for any continuous map $f : E^n \to E^n$ there is at least one fixed point, i.e. such a point $x \in E^n$, that $f(x) = x$.*

Proof. Indeed, if $f(x) \ne x$, then a straight line passing through the points x and $f(x)$ is defined. Let $r(x)$ be the one of the two points of intersection of this line with the sphere $\mathbb{S}^{n-1}$ that is not separated from the point x to the point $f(x)$. If

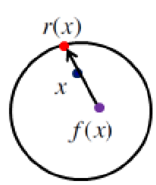


Figure 1:

$f(x) \ne x$ for all points $x \in E^n$, then this construction defines a continuous map $r : E^n \to \mathbb{S}^{n-1}$, which is obviously a retract map ($r(x) = x$ if $x \in \mathbb{S}^{n-1}$). Since the existence of such a map contradicts the drum theorem, the inequality $f(x) \ne x$ for all points $x \in E^n$ cannot be fulfilled. □

Brouwer's theorem (and its generalisations) are a source of innumerable theorems of the existence of solutions to a wide variety of equations in analysis (since any equation can be written in the form $f(x) - x$).

0.5 Example: complex projective plane and Hopf map

Another interesting example will be obtained by considering the complex projective plane $\mathbb{C}P^2$, the points of which are the classes $[z_0 : z_1 : z_2]$ of proportional triples $(z_0, z_1, z_2) \in \mathbb{C}^3$, complex numbers other than triples $0 = (0, 0, 0)$ (and whose topology is the coset topology of the space $\mathbb{C}^3 \setminus 0$). The points of the plane $\mathbb{C}P^2$, for which $z_2 = 0$, constitute an "improper line", which is naturally identified with a complex projective line consisting of classes $[z_0 : z_1]$ of proportional pairs $(z_0, z_1) \in \mathbb{C}P^2 \setminus 0$. Next semester we will construct a contravariant functor $\widetilde{H}^*$ from the category of $\mathscr{T}op$ to the category of $(\mathbb{Z}/2\mathbb{Z})-\mathscr{A}\ell g^*$, graded algebras over the field of $\mathbb{Z}/2\mathbb{Z}$ and calculate its value on the spaces $\mathbb{C}P^1$ and $\mathbb{C}P^n$. (Recall that an algebra A is called a *graded algebra* if it is decomposed into a direct sum of lineals A^n, and $A^nA^m \subset A^{n+m}$ for any n and m; the formula $\deg a = n$ means that $a \in A^n$.) Due to the contravariance of the functor $\widetilde{H}^*$ after applying it to diagramme (0.2) (when $X = \mathbb{C}P^2$ and $A = \mathbb{C}P^1$) the following diagramme is obtained

$$\widetilde{H}^*(\mathbb{C}P^2) \overset{\widetilde{H}^*(i)}{\underset{}{\rightleftarrows}} \widetilde{H}^*(\mathbb{C}P^1) \qquad \circlearrowleft id$$

It turns out that the algebra $\widetilde{H}^*(\mathbb{C}P^1)$ contains a single nontrivial element T_1 (such that $T_1^2 = 0$), and the algebra $\widetilde{H}^*(\mathbb{C}P^2)$ is generated as a linear space by two linearly independent elements T_2 and T_2^2 (such that $T_2^3 = 0$); in this case, $\deg T_1 = \deg T_2 = 2$. In addition, since the homomorphisms φ and $\widetilde{H}^*(i)$, being morphisms of the category $(\mathbb{Z}/2\mathbb{Z}) - \mathscr{A}\ell g^*$, preserve the grading, thus $\varphi T_1 = aT_2$ and $\widetilde{H}^*(i)T_2 = bT_1$, where $a, b \in \mathbb{Z}/2\mathbb{Z}$. But $\widetilde{H}^*(i)T_2 \circ \varphi = \mathrm{id}$ and therefore $a = b = 1$. Therefore, $T_2^2 = (\varphi T_1)^2 = \varphi(T_1^2) = 0$, which is impossible, because $T_2^2 \neq 0$ in $\widetilde{H}^*(i)(\mathbb{C}P^2)$. The resulting contradiction proves that $\mathbb{C}P^2$ is not retractable on $\mathbb{C}P^1$.

Note that in this discussion we essentially used the multiplicative structure of the algebra $\widetilde{H}^*(X)$.

Let $\mathbb{E}^4$ be a unit ball of space $\mathbb{C}^2$ consisting of points (z_0, z_1) such that $|z_0|^2 + |z_1|^2 \leq 1$. The formula

$$h : (z_0, z_1) \mapsto [z_0 : z_1 : 1 - |z_0|^2 - |z_1|^2]$$

defines - obviously, an epiomorphic - map $h : \mathbb{E}^4 \to \mathbb{C}P^2$, so that the plane $\mathbb{C}P^2$ turns out to be the coset space of the ball $\mathbb{E}^4$. On the inside $|z_0|^2 + |z_1|^2 < 1$ of the ball $\mathbb{E}^4$ the map h is a homeomorphism on $\mathbb{C}P^2 \setminus \mathbb{C}^1$, and on its boundary sphere $\mathbb{S}^3 : |z_0|^2 + |z_1|^2 = 1$ - is a continuous map $(z_0, z_1) \mapsto [z_0 : z_1]$ to $\mathbb{C}P^1$. Since the complex projective line $\mathbb{C}P^1$ is naturally identified with the Riemann sphere $\mathbb{C}^+$ (the point $[z_0 : z_1]$ corresponds to the complex number $z = z_0/z_1$ or - when $z_1 = 0$ is the symbol ∞) and, therefore, with the sphere $\mathbb{S}^2$, the last map is called the *Hopf map* - we can consider the map $\mathbb{S}^3 \to \mathbb{S}^2$. We will denote the Hopf map by the symbol $\hbar$.

The group $\mathbb{S}^1$ of complex numbers of the form $e^{i\theta}$ acts on the sphere $\mathbb{S}^3$ according to the formula $e^{i\theta}(z_0, z_1) = (e^{i\theta}z_0, e^{i\theta}z_1)$, and the orbits of this action are large circles of the sphere $\mathbb{S}^3$, representing the preimage of the points of the sphere $\mathbb{S}^2$ when $\hbar$ is selected (so this map induces a homeomorphism $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{S}^2$).

Thus, we can say that the complex projective plane $\mathbb{C}P^2$ is obtained from a ball $\mathbb{E}^4$ by contracting a certain family of large circle into points of its boundary sphere $\mathbb{S}^3$ (just as the real projective space $\mathbb{R}P^n$ is obtained from the ball E^n by identifying pairs of antipodal points of its boundary sphere $\mathbb{S}^{n-1}$).

Suppose that the Hopf map can be extended to $\mathbb{E}^4$. This assumption means that there is a map $g : \mathbb{E}^4 \to \mathbb{C}^1$ satisfying the relation $g|_{\mathbb{S}^3} = \hbar|_{\mathbb{S}^3}$. But then the formula $r = g \circ \hbar^{-1}$ will exactly determine the continuous map $r : \mathbb{C}P^2 \to \mathbb{C}P^1$, identity on $\mathbb{C}P^1$, i.e. being retraction. Since such a map cannot exist, this proves that the Hopf map $\hbar : \mathbb{S}^3 \to \mathbb{S}^2$ does not extend to $\mathbb{E}^4$.

0.6 Reduction of the extension problem to the retraction problem

The relationship between the non-retractibility of $\mathbb{C}P^2$ onto $\mathbb{C}P^1$ and the non-extendibility of Hopf map is quite general.

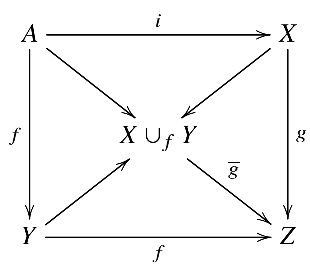
In the situation of diagram (0.1), i.e. for a pair (X, A) and a continuous map $f : A \to Y$, we can consider the space $X \cup_f Y$, which is the coset space of the disjoint union $X \sqcup Y$ by the minimal equivalence relation in which $a \sim f(a)$ for any point $a \in A$. This space is said to be obtained by *gluing* the space X to the space Y by map f. This terminology finds its justification in the fact that the restriction of the factorisation maps $X \sqcup Y \to X \cup_f Y$ on Y is, as can be easily seen, a homeomorphism, so that it is possible, by applying this homeomorphism, to consider the space Y as a subspace of the space $X \cup_f Y$. Similarly, the subspace $X \setminus A$ can also be considered a subspace of space $X \cup_f Y$, and then equality will take place

$$X \cup_f Y = (X \setminus A) \cup Y.$$

The space $X \cup_f Y$ has the property (called the *universal property*) that for any topological space Z and any commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[j]{} & Z \end{array}$$

where j and g are continuous maps, and i is an embedding, the map $\overline{g} : X \cup_f Y \to Z$, coinciding on $X \setminus A$ with g, and on Y with j, is continuous (and has the property that the following diagramme commutes.)



In particular, if the map j is an embedding, and the map g on $X \setminus A$ is a bijective map on $X \setminus Y$, then the map $\overline{g}$ will be a bijective continuous map and, therefore, will be a homeomorphism, if the space $X \cup_f Y$ is compact, and the space Z is Hausdorff. Since the space $X \cup_f Y$ is obviously compact when the spaces X and Y are compact, these conditions are met when $X = \mathbb{E}^4$, $A = \mathbb{S}^3$, $Y = \mathbb{C}P^1$, $Z = \mathbb{C}P^2$ and $f = \hbar$. Thus, we can assume that the complex projective plane $\mathbb{C}P^2$ is obtained as a result of gluing to the sphere $\mathbb{S}^2$ to the ball $\mathbb{E}^4$ by means of the Hopf map:

$$\mathbb{C}P^2 = \mathbb{E}^4 \cup_{\hbar} \mathbb{S}^2$$

If in diagramme (0.1) there is a map $\overline{f} : X \to Y$, then by the universal property (applied to $g = \overline{f}$) there is a map $\overline{g} : X \cup_f Y \to Y$, identical on Y, i.e. being a retraction. Conversely, if there is a retraction $r : X \cup_f Y \to Y$, then the composition with this retraction, restricted to X, factorisation maps $X \sqcup Y \to X \cup_f Y$ will be the map $X \to Y$, which is the extension of the map f. This proves the following proposition.

Proposition 0.12. *In diagramme (0.1) the extension $\overline{f}$ of the map $f : A \to Y$ exists if and only if the space $X \cup_f Y$ retracts to the space Y.*

Thus, the general problem of extension (0.1) is reduced to its particular case (0.2). This reduction is often useful.

0.7 Vector fields on spheres

Let us now give an example of the problem of lifting (0.5) (or, more precisely, its special case (0.6)).

A *vector field* on a sphere $\mathbb{S}^n$ is a continuous map that maps to each point $x \in \mathbb{S}^n$ some vector $v(x)$ touching the sphere at this point. Since the vector v touching the sphere $\mathbb{S}^n$ at point x is characterised by the condition $(x, v) = 0$, vector fields on $\mathbb{S}^n$ can be considered as continuous maps $v : \mathbb{S}^n \to \mathbb{R}^{n+1}$ satisfying the relation $(x, v(x)) = O$, $x \in \mathbb{S}^n$. We will be particularly interested in fields consisting of vectors of unit length, and such sets of them that at each point their vectors are orthogonal. In other words, we will be interested in sets $v_1, \ldots, v_n$ of such selections $v_i : \mathbb{S}^n \to \mathbb{S}^n$, $i = 1, \ldots, m$ that at any point $x \in \mathbb{S}^n$ there are equalities $(x, v_i(x)) = 0$ and $(v_i(x), v_j(x)) = O$, $i = 1, \ldots, m$. We will call such sets *m-frame fields* (or, in short, *m-fields*) on the sphere $\mathbb{S}^n$. Of course, it is assumed here that $0 \le m \le n$.

If $n = 2k - 1$, then we can treat the sphere $\mathbb{S}^n$ as the unit sphere of the complex space $\mathbb{C}^k$, and then the formula $v(x) = ix$, where $i = \sqrt{-1}$, will obviously define some 1-field on $\mathbb{S}^n$. For $n = 4k - 1$ the sphere $\mathbb{S}^n$ is the unit sphere of the quaternion space $\mathbb{H}^k$ and formulae $v_1(x) = ix$, $v_2(x) = jx$, $v_3(x) = kx$ define some 3-field on $\mathbb{S}^n$. Similarly, using the so-called Clifford algebras, the special cases of which are the algebras $\mathbb{C}$ and $\mathbb{H}$, on the sphere $\mathbb{S}^n$ it is possible, as Radon and Hurwitz showed, to construct a (2^a+8b-1)-field where a and b are integers such that $0 \le a \le 3$ and 2^{a+4b} is the highest power of two which divides the number $n + 1$. The question arises whether this result is accurate or whether there are such n that an m-field can be constructed on the sphere $\mathbb{S}^n$ for $m \ge 2^a + 8b$. For example, for $i = 5$, the Radon-Hurwitz method gives us only a 1-field on the $\mathbb{S}^5$ sphere. Is there a 2-field on this sphere?

To answer this question, we must reformulate it in the form of one of the tasks (0.1) - (0.6). To this end, we will introduce into consideration the set $V_{n+1,m+1}$ of all $(m + 1)$-frames of the space $\mathbb{R}^{n+1}$, i.e. $(m + 1)$-member of orthonormal families of vectors $(v_1, \ldots, v_{m+1})$ of this space. Since $(m+1)$-frames are naturally identified with $(m + 1) \times (n + 1)$-matrices whose columns are orthonormal, the set $V_{n+1,m+1}$ turns out to be a subset of the topological space $\mathbb{R}(m + 1, n + 1)$ of all $(m + 1) \times (n + 1)$-matrices, and therefore itself is a topological space (in fact, even a smooth manifold). It is clear that by comparing for each point $(v_1, \ldots, v_{m+1}) \in V_{n+1,m+1}$ with vectors $v_{m+1} \in \mathbb{S}^n$, we get a continuous map

$$p : V_{n+1,m+1} \longrightarrow \mathbb{S}^n. \tag{0.13}$$

Now if $\{v_1, \ldots, v_m\}$ is an arbitrary m-field on $\mathbb{S}^n$, then the formula $s(x) = (v_1(x), \ldots, v_m(x), x)$, $x \in \mathbb{S}^n$, will determine the section $s : \mathbb{S}^n \to V_{n+1,m+1}$ to the map p and, conversely, any cross section of this map will specify some m-field. Thus, the question of the existence of a field of m-frames on the sphere $\mathbb{S}^n$ is reformulated in the form of a question about the existence of at least one section for the map (0.13).

The corresponding derivative algebraic problem (obtained by using the functor $\widetilde{H}^*$) has the form

$$\mathrm{id} \circlearrowright \qquad \widetilde{H}^*(\mathbb{S}^n) \underset{\varphi}{\overset{\widetilde{H}^*(p)}{\rightleftarrows}} \widetilde{H}^*(V_{n+1,m+1}) \tag{0.14}$$

where, as it turns out, $\widetilde{H}^*(\mathbb{S}^n)$ is a graded algebra over $\mathbb{Z}/2\mathbb{Z}$ with one generator σ of degree n, subordinate to the relation $\sigma^2 = 0$. As for the algebra $\widetilde{H}^*(V_{n+1,m+1})$, then, as can be shown, in this algebra all elements of degree $< n$ are equal to zero and there is the only non-zero element α of degree n, which is the image of the element σ mapped by $\widetilde{H}^*(p)$. Therefore, the map $\varphi : \widetilde{H}^*(V_{n+1,m+1}) \to \widetilde{H}^*(\mathbb{S}^n)$, which closes the diagram (0.14), should be determined by the formula

$$\varphi(\xi) = \begin{cases} \sigma, & \text{if} \quad \xi = \alpha, \\ 0, & \text{if} \quad \xi \ne \alpha, \end{cases} \qquad \xi \in \widetilde{H}^*(V_{n+1,m+1})$$

Since this map is obviously a homomorphism of algebras, we do not get any contradiction. This means that for the problem under consideration, the functor $\widetilde{H}^*$ was too weak to give a definite answer.

However, as we will show in the next semester, the $\widetilde{H}^*$ functor actually takes values in the category of graded $\mathbb{Z}/2\mathbb{Z}$-algebras, which are simultaneously modules over some remarkable algebra called the *Steenrod algebra*. Therefore, the map φ in the diagram (0.14) must be not only a homomorphism of algebras, but also a homomorphism of modules, i.e. it must be permutable with actions of elements of the Steenrod algebra. The Steenrod algebra is quite complicated, but it is enough for us now to know that for any $m \geq 1$ it has an element denoted by the symbol Sq^m, under the action of which the degrees of all elements of the algebra $\widetilde{H}^*(X)$ increase by m. Since there are no nonzero elements of degree $\neq n$ in the algebra $\widetilde{H}^*(\mathbb{S}^n)$, it follows that $\mathrm{Sq}^m\,\sigma = 0$ for any $m \geq 1$. At the same time, it turns out (a very non-trivial fact!) that if $m = 2^k$ is the highest power of two dividing the number $n+1$, then in the algebra $\widetilde{H}^*(V_{n+1,m+1})$ there is a relation $\mathrm{Sq}^m\,\alpha \neq 0$. Therefore for $m = 2^k$, the above map φ cannot be a homomorphism of modules (because then the equality $\mathrm{Sq}^m\,\alpha = \mathrm{Sq}^m\,\varphi(\sigma) = \varphi(\mathrm{Sq}^m\,\sigma) = \varphi(0) = 0$ would take place.) This proves that the field of 2^k-frames on the sphere $\mathbb{S}^n$ does not exist. In particular, there is no 2-field on the sphere $\mathbb{S}^5$.

If, as above, we represent k as $a + 4b$, where $0 \leq a \leq 3$, then the equality $2^k = 2^a + 8b$ will take place only when $b = 0$. Therefore, for $b > 0$, the proven result (known, by the way, as the theorem of Steenrod and Whitehead) gives only a partial answer to the above question. The full answer (asserting the accuracy of the Radon-Hurwitz estimate) was obtained about twenty years ago by Adams, who used a very powerful functor $\widetilde{KO}$ (which we will also study the next year).

0.8 Homotopies, cofibrations, and the effectiveness of the algebraic topology method

We see that the solution of a particular problem by the method of algebraic topology is naturally divided into two stages. At the first stage, this problem is reformulated as one of the problems (0.1), (0.2), (0.5) and (0.6), and at the second stage, the corresponding derived algebraic problem is studied. The main difficulty here lies in choosing a suitable algebraic functor, which, on the one hand, must be efficiently computable (at least for the spaces involved in the problem only) and, on the other hand, must take values in the category of which structurally has sufficiently rich objects to give a definite answer.

The examples considered are characteristic of algebro-topological problems also in the sense that these problems are, as a rule, nontrivial and interesting already for the simplest (from the point of view of general topology) spaces-spheres, balls, their 'finite unions' (so-called "polyhedra") etc. Therefore, although we formulated the main problems of algebraic topology without any a priori conditions for the spaces appearing in them, but in practice we will not hesitate to impose on these spaces any general methodological restrictions such as, say, the axioms of separability or certain conditions of local simplicity, which for one reason or another will be convenient for us. Exemption from these conditions will lie beyond the scope of our presentation.

However, even the simplest topological spaces have the cardinality of continuum, and efficiently computable algebraic objects are finite or countable. This means that when moving from task (0.1) to task (0.8), there is a colossal loss of information. Only for this reason, strictly speaking, the method of algebraic topology turns out to be applicable to specific geometric problems. But, the question is, why is it not lost essential information, i.e. why does algebraic topology successfully slip between the Scylla of non-computable informativeness and the Charybdis of computable uninformativeness?
(*Transcriber's note*: Scylla and Charybdis are sea monsters appearing in Greek mythology. "Between Scylla and Charybdis" means "to choose the lesser of two evils".)
The answer to this question turns out to be very interesting.

Definition 0.15. Let X and Y be topological spaces. A *homotopy from X to Y* is an arbitrary continuous map

$$F : X \times I \to Y, \tag{0.16}$$

where $I = [0, 1]$ is a *unit segment.*

Any homotopy (0.16) by the formula

$$f_t(x) = F(x, t), \quad x \in X, 0 \le t \le 1,$$

that is, according to the formula $f_t = F \circ \sigma_t$, $0 \le t \le 1$, where $\sigma_t : X \to X \times I$ - map $x \mapsto (x, t)$, defines a family of continuous maps

$$f_t : X \to Y, \quad 0 \le t \le 1. \tag{0.17}$$

Such a family is also called a homotopy from X to Y. Thus, a family of maps $f_t : X \to Y$ is a homotopy if and only if when the map $F : X \times I \to Y$ is defined by the formula $F(x, t) = f_t(x)$, $x \in X$, $0 \le t \le 1$, continuously. In this case, it is also said that the maps f_t *continuously depend on t.*

Homotopy (0.16) and (0.17) is said to *connect* the map $f_0 : X \to Y$ with the map $f_1 : X \to Y$.

Maps $f, g : X \to Y$ are called *homotopic* if there exists a homotopy $F : X \times I \to Y$ connecting the map f with the map g (i.e. such that $f = f_0$ and $g = f_1$). In this case, write $F : f \sim g$ or just $f \sim g$.

Clearly, the homotopy of two maps means that one of them can be continuously transformed into the other.

The question of homotopy of maps $f, g : X \to Y$ is obviously equivalent to the problem of extension the map $h : (X \times 0) \cup (X \times 1) \to Y$ to $X \times I$, given by the formula

$$h(x, t) = \begin{cases} f(x), & \text{if} \quad t = 0, \\ g(x), & \text{if} \quad t = 1. \end{cases}$$

Thus, homotopy theory (which studies the homotopy relation of maps) is included in algebraic topology.

It is easy to see that

Proposition 0.18. *on the set of* $\mathscr{T}\!op(X,Y)$ *of all continuous maps* $X \to Y$, *the homotopy relation is an equivalence relation.*

Proof. Indeed, first of all, $F : f \sim f$ where $F(x,t) = f(x)$ for any $x \in X$ and $t \in I$. Secondly, if $F : f \sim g$ then $G : g \sim f$, where $G(x,t) = F(x, 1-t)$, $x \in X$, $t \in I$. Finally, if $F : f \sim g$ and $G : g \sim h$, then $H : f \sim h$, where

$$H(x,t) = \begin{cases} F(x,2t), & \text{if } \ 0 \le t = 1/2, \\ G(x,2t-1), & \text{if } \ 1/2 \le t \le 1. \end{cases}$$

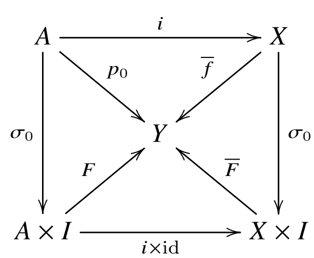
□

Therefore, all continuous maps of $X \to Y$ are divided into classes of homotopy maps to each other. These classes are called *homotopy classes* of maps from X to Y.

The set of all homotopy classes of maps from X to Y is denoted by the symbol $[X,Y]$, and the class containing this map $f : X \to Y$ is denoted by the symbol $[f]$.

The transition to homotopy classes makes everything significantly more efficient, since for "reasonable" spaces X and Y the set $[X,Y]$ turns out to be, as a rule, finite, or countable.

Definition 0.19. It is said that the map $i : A \to X$ is a *cofibration* or that it satisfies the *homotopy extension* axiom (*axiom* **HE**) if for any space Y, any homotopy $f_t : A \to Y$ and any map $\overline{f} : X \to Y$ satisfying the relation $\overline{f} \circ i{=}f_0$, there exists a homotopy $\overline{f}_t : X \to Y$ such that $\overline{f}_0 = \overline{f}$ and $\overline{f}_t \circ i = f_t$ for any $t \in I$:



$$F(a,t) = f_t(a), \quad \overline{F}(x,t) = \overline{f}_t(x). \tag{0.20}$$

A pair (X,A) consisting of a space X and its subspace A is called a *cofibration* (or *Borsuk pair*) if the inclusion $i : A \to X$ satisfies the axiom **HE**.

In particular, for any cofibration (X,A), it follows that

Proposition 0.21. *if the maps* $f, g : A \to Y$ *are homotopic and* f *is extensible to* X, *then* g *is also extensible to* X.

In other words, for the cofibration (X, A), the map property $f : A \to Y$ is extensible to X depends only on its homotopy class $[f]$. This means that in problem (0.1) we can consider their homotopy classes instead of maps, which, according to the above, leads us from the world of continuum powers to the world of countable sets, for an adequate study of which there are no obvious obstacles by means of algebra. This explains, why, for cofibrations, the method of algebraic topology makes it possible to efficiently solve problem (0.1). In the case when (X, A) is not a cofibration, this method does not work. Fortunately, all really occurring on in practice, problems (0.1) either have the property that for them pairs (X, A) are cofibrations, or they are trivially reduced to such problems.

It can be said that the natural object of studying algebraic topology is not so much continuous maps as their homotopy classes. In this sense, algebraic topology is almost completely absorbed by homotopy theory.

Thus, although it is generally impossible to put an equal sign between algebraic topology and homotopy theory, they are in fact intertwined so closely that they become indistinguishable from each other.

0.9 Homotopy category

It is easy to see that

Proposition 0.22. *the homotopy relation is consistent with the composition of maps, i.e. if $f \dim g$, where $f, g : X \to Y$, then $f \circ k \sim g \circ k$ and $h \circ f \sim h \circ g$ for any continuous maps $k : Z \to X$ and $h : Y \to Z$.*

Proof. Indeed, if the maps f and g are connected by the homotopy $f_t : X \to Y$, then the map $f \circ k$ and $g \circ k$ will be connected by the homotopy $f_t \circ : Z \to Y$, and the maps $h \circ f$ and $h \circ g$ are homotopy $h \circ f_t : X \to Z$. □

It follows that the formula

$$[g] \circ [f] = [g \circ f], \quad f : X \to Y, g : Y \to Z$$

well defines the composition of homotopy classes. It is clear that with respect to this composition, the totality of all topological spaces and all homotopy classes of their continuous maps forms a category. This category is called the *homotopy category* and is denoted by the symbol $[\mathscr{T\!op}]$ (or $\mathscr{H\!o\,T\!op}$). It is the natural domain of algebraic topology.

For any category $\mathscr{A}$ and any of its objects A, B the symbol $\mathscr{A}(A, B)$ denotes the set of all morphisms $A \to B$. In these notations

$$[X, Y] = [\mathscr{t\!op}](X, Y)$$

for any topological spaces X and Y.

The transition from the category $\mathscr{T\!op}$ to the category $[\mathscr{T\!op}]$ can be easily axiomatised.

We will call category $\mathscr{A}$ a *category with homotopies*, if for any of its objects A, B in the set $\mathscr{A}(A, B)$ some families of $\{f_t, t \in I\}$ morphisms $f_t : A \to B$, called *homotopies*, are assigned, and the following axioms are satisfied:

(1°) Every family $\{f_t : A \to B\}$ for which $f_1 = f_0$ for all $t \in I$, is a homotopy.

(2°) For any homotopy $\{f_t : A \to B\}$ family $\{f_{1-t} : A \to B\}$ is a homotopy.

(3°) For any homotopy $\{f_t : A \to B\}$ and $\{g_t : A \to B\}$, having the property that $f_1 = g_0$, family $\{h_t : A \to B\}$, defined by the formula

$$h_t = \begin{cases} f_{2t} & \text{if} \quad 0 \leq t \leq 1/2, \\ g_{2t-1} & \text{if} 1/2 \leq t \leq 1, \end{cases}$$

is a homotopy.

(4°) For any homotopy $\{f_t : A \to B\}$ and any morphisms $k : C \to A$ and $h : B \to C$, the family $\{f_t \circ k : C \to B\}$ and $\{h \circ f_t : A \to C\}$ are homotopies.

Homotopy classes are defined in such a category and a transition is possible to the corresponding homotopy categories $\mathscr{H}o - \mathscr{A} = [\mathscr{A}]$.

We will not systematically deal with this abstract nonsense[1], but it will often be useful for us to keep it in mind.

[1]This is not a swear word, but a term proposed by N. Steenrod to denote general-category constructions

Appendix

0.A Basic concepts of general topology

Let us list the basic concepts of general topology that we assume to be known (see, for example, [1] or [8]).

Topologies and topological spaces. Open and closed sets. Neighbourhood. Bases and pre-bases. Local bases (fundamental neighbourhood systems). Axioms of countability (first and second; separability condition).

Continuous maps. Open maps. Monomorphisms, epiomorphisms and homeomorphisms.

Subspaces. The coset space and the coset topology. Direct products and direct sums (disjoint unions) of topological spaces.

Axioms of separability: Hausdorff, regular and normal spaces.

Coverings (open). The refinement of the coverings. Compact, locally compact and paracompact spaces.

We will need only the simplest properties of these concepts. In particular, we will assume that the bijective continuous map of a compact space to a Hausdorff space is a homeomorphism and that any Hausdorff locally compact space is regular.

We will also assume it is known that

Proposition 0.23. *for any normal space X and any closed subspace A of it, every continuous map $A \to \mathbb{R}^n$ can be extended to all X.*

This statement is called the *Tietze theorem* (despite the fact that it was first proved by Uryson).

A special case of the Tietze theorem (necessary, however, to prove) states that for two closed disjoint subspaces A and B of a normal space X there exists a continuous function $\varphi : X \to I$, which is equal to zero on A and one on B. This special case is called Uryson's lemma, and any such function φ is called *Uryson function* for A and B.

We will recall other less well-known (or more special) general methodological results when we need them.

The exception is the so-called exponential law, which it will be convenient for us to state immediately.

0.B Exponential law for sets of maps

Let X and B be arbitrary topological spaces. Although there are quite a few ways to topologise the set of $\mathscr{T\!op}(X, B)$ of all continuous maps $X \to B$, but we will use only the *compact-open topology* of this set (also called the *topology of compact convergence*). In this topology a set is open if and only if it is a union of finite intersections of sets $\mathscr{W}(K, U)$, each of which is given by a compact set of $K \subset X$, an open set of $U \subset B$ and consists of all maps $f : X \to B$ for which $f(K) \subset U$. (In the terminology of the theory of topological spaces, the sets $\mathscr{W}(K, U)$ constitute the prebase of a compactly open topology.)

The set of $\mathscr{T\!op}(X, B))$ provided the compact-open topology, will be denoted by the symbol B^X. This notation is justified by the fact that in the case when X is a discrete space consisting of n points $x_1, \dots, x_n$, the space B^X is naturally homeomorphic to the space $B \times \cdots \times B = B^n$ by homeomorphism $B^X \to B^n$ by the correspondence

$$f \mapsto (f(x_1), \dots, f(x_n)), \quad f \in B^X, (f(x_1), \dots, f(x_n)) \in B^n.$$

(Another justification for this designation is the exponential law $(B^X)^Y = B^{X \times Y}$ proved below.)

For any three spaces X, Y and B and any continuous map $f : X \times Y \to B$, the formula

$$[(\theta f)(y)]x = f(x, y), \quad x \in X, y \in Y,$$

defines some map

$$\theta f : Y \to B^X$$

of the space Y to the space B^X, called the *map associated with* f. It turns out that

Proposition 0.24. *in the compact-open topology, the map θf is continuous.*

Proof. Obviously, it is enough to show that for any compact set $K \subset X$ and any open set $U \subset B$ the preimage $(\theta f)^{-1}\mathscr{W}(K, U)$ of $\mathscr{W}(K, U) \subset B^X$ is open in Y, i.e. that each point y_0 of it is its internal point. But since $(\theta f)(y_0) \in \mathscr{W}(K, U)$, i.e. $[(\theta f)(y_0)]K \subset U$, then $f(x, y_0) \in U$ for any point $x \in K$, i.e. $K \times y_0 \subset f^{-1}(U)$. On the other hand, the set $f^{-1}(U)$ is open in $X \times Y$ and therefore is by combining sets of the form $G \times H$ ("rectangles"), where G is open in X and H is open in K. Due to the compactness of the set K (and, therefore, so is the set $K \times y_0$), there is a finite system

$$G_1 \times H_1 \dots, G_m \times H_m$$

of these rectangles, covering the set $K \times y_0$. Then the open subset $H = H_1 \cap \cdots \cap H_m$ of the space Y will contain a point y_0 and each of its points y will have the property that $K \times y \subset f^{-1}(U)$, and hence $(\theta f)(y) \in \mathscr{W}(K, U)$. In other words, H will be the neighbourhood of the point y_0 in Y contained in the set $(\theta f)^{-1}\mathscr{W}(K, U)$, and, therefore, the point y_0 will be the inner point of the last set. □

So the correspondence $\theta : f \to \theta f$ is a map

$$\theta : \mathcal{T}op(X \times Y, B) \to \mathcal{T}op(Y, B^X)$$

from the set $\mathcal{T}op(X \times Y, B)$ to the set $\mathcal{T}op(Y, B^X)$. This map is called the *association map.*

Equality $\theta f = \theta g$, where $f, g : X \times Y \to B$ means that

$$f(x, y) = [(\theta f)(y)]x = [(\theta g)(y)]x = g(x, y)$$

for any points $x \in X$, $y \in Y$, hence $f = g$. Hence, *the association map is injective.*

To find out when the map θ is bijective, we will introduce the map

$$\omega : X \times B^X \to B,$$

defined by the formula

$$\omega(x, f) = f(x), \quad x \in X, f : X \to B,$$

and called the *evaluation map.*

It turns out that

Proposition 0.25. *the association map $\theta : \mathcal{T}op(X \times Y) \to \mathcal{T}op(Y, B^X)$ is bijective for any space Y, if and only if the evaluation map $\omega : X \times B^X \to B$ is continuous.*

Proof. if the map ω is continuous, then for any map $g : Y \to B^X$, the map $f = \omega \circ (\mathrm{id} \times g) : X \times Y \to B$ (i.e., the map $(x, y) \mapsto \omega(x, g(y)) = [g(y)]x)$ will be continuous, i.e. it will lie in $\mathcal{T}op(X \times Y, B)$, and, obviously, will have the property that $\theta f = g$. Therefore, if the map ω is continuous, then the map θ is surjective, and, therefore, bijective (for, as already noted above, this map is always injective). Conversely, if the map θ is bijective for any Y and, in particular, for $Y = B^X$, then the continuous map $\omega' = \theta^{-1}(\mathrm{id}) \in \mathcal{T}op(X \times B^X, B)$ will be for any point $(x, f) \in X \times B^X$ satisfy the relationship

$$\omega'(x, f) = [\theta(\omega')f](x) = f(x) = \omega(x, f)$$

and, therefore, it will coincide with ω. Therefore, the map ω is continuous. □

On the other hand, it is easily shown that

Proposition 0.26. *if the space X is locally compact and Hausdorff, then for any space B the evaluation map ω is continuous.*

Proof. In fact, let $(x_0, f_0) \in X \times B^X$, and let U arbitrary neighbourhood of the point $\omega(x_0, f_0) = f_0(x_0)$ in the space B. Since the map is $f_0 : X \to B$ is continuous, and the space X is locally compact and Hausdorff (and therefore regular), then in X there exists such a neighbourhood V of a point x_0 with a compact closure $\overline{V}$, with $f_0(\overline{V}) \subset U$. Therefore, an open set $\mathcal{W}(\overline{V}, U)$ containing the point f_0 is defined in B^X, and in $X \times B^X$ - an open set $W = V \times \mathcal{W}(\overline{V}, U)$

containing the point (x_0, f_0). At the same time, for any point $(x, f) \in W$, the inclusion holds

$$\omega(x, f) = f(x) \in f(\overline{V}) \subset (U),$$

showing that $\omega(W) \subset U$. So, for each point $(x_0, f_0 \in \times B^X$ and any neighbourhood of U of the point $\omega(x_0, f_0) \in B$, there is a neighbourhood W of the point (x_0, f_0) such that $\omega(W) \subset U$. □

Thus,

Proposition 0.27. *if the space X is locally compact and Hausdorff, then for any spaces Y and B the association map*

$$\theta : \mathcal{Top}(X \times Y, B) \to \mathcal{Top}((Y, B^X)$$

is bijective.

This statement is called the *exponential law for sets of maps.* The rather strong conditions imposed on the space X (by the way, very close to the necessary ones) are a particular manifestation of the general defectiveness of the category $\mathcal{Top}$ in relation to the direct product. A general way to get rid of such conditions (which we will encounter in other situations later) is to move to a category whose objects are topological spaces, and morphisms are such maps of $f : X \to Y$ topological spaces (called *kaonic maps*)[2] that for any compact space K and any continuous map $\varphi : K \to X$ composite map $f \circ \varphi : K \to Y$ is continuous, or, which is essentially equivalent, to a category whose morphisms are continuous maps, and objects are such topological spaces X (called *kaonic spaces*) that the set $C \subset X$ is closed if and only if, for any compact space K and any continuous map $f : K \to X$, its preimage $\varphi^{-1}C$ is closed in K. However, in this course we will not pursue the formal perfection of the theory and will prefer to stay in the more familiar category of $\mathcal{Top}$.

0.C Exponential law for mapping spaces

In connection with the exponential law, the question also naturally arises about its validity for mapping spaces, i.e., whether the associated map will be a homeomorphism of space $B^{X \times Y}$ of (or at least in B) space $(B^X)^Y$. The answer is, generally speaking, affirmative only for Hausdorff spaces X and Y. Although we will need this fact only in a limited way, we will prove it here in full generality for the sake of completeness,

First of all, we show that

Proposition 0.28. *if the space Y is Hausdorff, then for any spaces X and B the association map*

$$\theta : B^{X \times Y} \to (B^X)^Y$$

is continuous.

[2]The kaon, a special case of meson particle, is made of one quark and one antiquark.

Proof. Indeed, by definition, every open set in $(B^X)^Y$ is a union of finite intersections of sets of the form $\mathscr{W}(L,V)$, where L is a compact subset in Y and V is an open subset in B^X. Therefore, it is sufficient to show that for any sets of the form $\mathscr{W}(L,V)$, its preimage by the map θ is open in $B^{X\times Y}$. Let $f \in \mathscr{W}(L,V)$ and $y \in L$. Since $f(y) \in V$ and since the set V, in turn, is the union of finite intersections of sets of the form $\mathscr{W}(K_i,U_i)$, where K_i are compact sets in X, and U_i are open sets in B, then in X there are such compact sets $K_1^y,\dots,K_{n_y}^y$, and in B there are such open sets $U_1^y,\dots,U_{n_y}^y$ that $f(y) \in V_y \subset V$, where

$$V_y = \mathscr{W}(K_1^y,U_1^y) \cap \cdots \cap \mathscr{W}(K_y^{n_y},U_y^{n_y}).$$

But then, since the map $f : Y \to B^x$ is continuous, in Y there will be such a neighbourhood G_y of the point y that $f(G_y) \subset V_y$. At the same time, since the space L, being Hausdorff (as a subspace of the Hausdorff space Y) and compact, regular, then in L there exists such a neighbourhood H_y of $y \in L$, such that $\overline{H}_y \subset G_y \cap L$.

Let's now choose such a finite system of points, $y_1,\dots,y_m$ in L such that the neighbourhood of $H_{y_1},\dots,H_{y_m}$ covers L (such a system exists due to the compactness of L). Since

$$f(\overline{H}_{y_i}) \subset f(G_{y_i}) \subset V_{y_i} \subset V, \quad i = 1,\dots,m,$$

and

$$V_{y_i} = \mathscr{W}(K_1^i,U_1^i) \cap \cdots \cap \mathscr{W}(K_{n_i}^i,U_{n_i}^i)$$

where for the sake of brevity we set

$$K_j^i = K_j^{y_i}, \quad U_j^i = U_j^{y_i}, \quad \text{and} n_i = n_{y_i}$$

for any $i = 1,\dots,m$ and any $j = 1,\dots,n_i$, thus we have

$$f \in \mathscr{W}(\overline{H}_{y_i},\mathscr{W}(K_j^i,U_j^i)), \quad i = 1,\dots,m, j = 1,\dots,n$$

(sets $\overline{H}_{y_i}$, being closed subsets of a compact set L, are compact) and means $f \in W$, where

$$W = \cap_{i=1}^m \cap_{j=1}^{n_i} \mathscr{W}(\overline{H}_{y_i},\mathscr{W}(K_j^i,U_j^i).$$

On the other hand, if $g \in W$, i.e. if

$$g(\overline{H}_{y_i}) \subset \cap_{j=1}^{n_i}\mathscr{W}(K_j^i,U_j^i) = V_{y_i} \subset V$$

for any $i = 1,\dots,m$, then

$$g(L) = \cup_{i=1}^m g(\overline{H}_{y_i}) \subset V$$

and therefore $g \in \mathscr{W}(L,V)$. Thus $W \subset \mathscr{W}(L,V)$. This proves that any point $f \in \mathscr{W}(L,V)$ has a neighbourhood of the form W in $\mathscr{W}(L,V)$ and, therefore, the set $\mathscr{W}(L,V)$ is a union of sets of the form W, i.e., finite intersections of sets of the form $\mathscr{W}(L,\mathscr{W}(K,U))$, where K is a compact set in X, and U is an open set in B. Therefore, it is enough for us to prove that a complete preimage is open in $B^{X\times Y}$ when θ maps each set of the form $\mathscr{W}(L, \mathscr{W}(K, U))$. But this is obvious, because, as follows directly from the definition of the map θ, this preimage is the set $\mathscr{W}(K \times L, U)$. □

Thus, for Hausdorff Y (and any X and B), the map θ is continuous (and, as we know, injective) map of the space $B^{X\times Y}$ into the space $(B^X)^Y$. We show that

Proposition 0.29. *if X is also a Hausdorff space, then the map θ is an open (and hence homeomorphic) map since* $\operatorname{im}\theta = \theta(B^{X\times Y})$.

Proof. It is clear that it suffices to prove that for any compact set M of $XtimesY$ and any open set U of B the set $\theta\mathscr{W}(M, U)$ is open in $\operatorname{im}\theta$, that is, that any of its points $g_0 = \theta f_0$, where $f_0 \in \mathscr{W}(M, U)$, is its inner point. Let K and L be projections of the set M in X and Y, respectively. Since the map $f_0 : X \times Y \to B$ is continuous, and the spaces K and L, being compact and Hausdorff, are regular (even normal), then for any point $z = (x, y) \in M$, where $x \in K$, $y \in L$, there are such neighbourhoods V_z and W_z of points x and y in the spaces K and L, respectively, that $f_0(\overline{V}_z \times \overline{W}_z) \subset U$ and, therefore, $(g_0, \overline{V}_z)\overline{W}_z \subset U$, i.e.

$$g_0 \in \mathscr{W}(\overline{V}_z, \mathscr{W}(\overline{W}_z, U))$$

(since the sets $\overline{V}_z$ and $\overline{W}_z$, being closed subsets of compact spaces, are compact, then the symbol $\mathscr{W}(\overline{V}_z, \mathscr{W}(\overline{W}_z, U))$ makes sense). On the other hand, since the set M is compact, there exists in it such a finite system of points $z_1, \dots, z_n)$ that the sets

$$V_{z_1} \times W_{z_1}, \dots V_{z_n} \times W_{z_n}$$

cover M. The set

$$W = (\operatorname{im}\theta) \cap [\cap_{i=1}^n \mathscr{W}(\overline{V}_{z_i}, \mathscr{W}(\overline{W}_{z_i}, U))]$$

is open in $\operatorname{im}\theta$ and, according to what has just been said, contains the map g_0.

Let $g = \theta f$ be an arbitrary map from W. Since $g \in \mathscr{W}(\overline{V}_z, \mathscr{W}(\overline{W}_z, U))$ for any $i = 1, \dots, n$, it is that $g\overline{V}_{z_i})\overline{W}_{z_i} \subset U$, i.e., $f(\overline{V}_{z_i} \times \overline{W}_{z_i}) \subset U$, and hence,

$$f(M) \subset f\left(\cup_{i=1}^n (\overline{V}_{z_i} \times \overline{W}_{z_i})\right) \subset U,$$

i.e., $f \in \mathscr{W}(M, U)$. This proves that $\mathscr{W} \subset \theta\mathscr{W}(K, U)$, and, it means that the point g_0 is the inner point of the set $\operatorname{im}\theta$. □

Combining the results obtained with the exponential law, we obtain that

Proposition 0.30. *if*

(a) the space X is Hausdorff and locally compact,

(b) the space Y is Hausdorff,

then for any space B the association map

$$\theta : B^{X\times Y} \to (B^X)^Y$$

is a homeomorphism.

This statement is known as the *exponential law for mapping spaces.*

Lecture 1

1.1 Homotopies and paths

Recall that a *path* in the topological space X is an arbitrary continuous map $u\colon I \to X$. The point $u(0)$ is called the *starting point* of the path, and the point $u(1)$ is its *ending point*. It is also said that the points $u(0)$ and $u(1)$ are *connected* by u.

All paths of space X make up the topological space X^I.

A topological space X is called *connected* (or, more precisely, *path connected*) if any two of its points can be connected by a path[1]. Each space is X is a union of maximal connected subsets called its *components*.

By denoting a topological space consisting of a single point with the symbol pt, we can consider each path $I \to X$ as a map $\mathrm{pt} \times I \to X$, i.e. as a homotopy from pt to X. In accordance with this, the components of the space X are nothing more than homotopy classes of maps $\mathrm{pt} \to X$.

On the other hand, it is clear that the exponential law (for sets of maps) allows, on the contrary, any a homotopy can be interpreted as some path by which a map associated with an arbitrary homotopy $F\colon X \times I \to Y$ is nothing but a path $I \to Y^X$ in the map space Y^X.

Unfortunately, this elegant and visual interpretation of homotopy is quite adequate only for locally compact and Hausdorff spaces X. For arbitrary spaces X, not every path in Y^X will be a homotopy from X to Y. Therefore, in particular, only for of locally compact and Hausdorff spaces X, the homotopy classes of maps from X to Y can be identified with the components of the space Y^X. In general, each such component can consist of several homotopy classes.

A more satisfactory result will be obtained when we will apply the exponential law by rearranging X and I beforehand. Then the map associated with an arbitrary homotopy $F\colon X \times I \to Y$ (which we will allow ourselves to denote with the same letter F) will be given by the formula

$$(F(x))(t) = F(x,t), \quad x \in X, t \in I,$$

[1] Transcriber's note: This conventions is adopted by Fuchs-Rokhlin, as connected but not path-connected spaces are pathological in view of algebraic topology.

and will be a continuous map $X \to Y^I$. Since the space I (which here plays the role of the space X from the formulation of the exponential law) is obviously locally compact and Hausdorff, we see, therefore, that

Proposition 1.1. *for any spaces X and Y, the homotopies $X \times I \to Y$ from X to Y are in natural bijective correspondence with continuous maps $X \to Y^I$.*

Therefore, we will also call the maps $X \to Y^I$ *homotopies from X to Y*.

How with this interpretation homotopies are obtained are the maps f_0 and f_1 related to them? To answer this question, we note that for any spaces X and B, any map $f\colon X \to B$ and any point $x_0 \in X$ formula

$$\omega_{x_0} f = f(x_0)$$

defines some map

$$\omega_{x_0}\colon B^X \to B$$

(slice by x_0 of the ω evaluation map). Although the map ω is generally discontinuous, the map ω_{x_0} for any point $x_0 \in X$ is continuous, since for each open set $U \subset B$ is its preimage $\omega_{x_0}^{-1} U$ when mapped by ω_{x_0} is an open set $\mathscr{W}(\{x_0\}, U)$ (in any topological space, all single-point subsets are compact).

In particular, the maps of the path space

$$\omega_0\colon B^I \to B, \quad \omega_1\colon B^I \to B$$

are continuous, which correspondingly maps each path to its starting and ending points.

1.2 Cofibrations

Now it is clear that (we replace B with Y)

Proposition 1.2. *for each homotopy $F\colon X \to Y^I$ compositions $\omega_0 \circ F\colon X \to Y$ and $\omega_1 \circ F\colon X \to Y$ are maps connected by this homotopy.*

Therefore, for example, the axiom of homotopy extension can be written in the form of a diagramme

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle F}\downarrow & \overset{\widetilde{F}}{\swarrow} & \downarrow{\scriptstyle \bar{f}} \\ Y^I & \xrightarrow[\omega_0]{} & Y \end{array} \tag{1.3}$$

much more visual and convenient than Diagramme (0.20) of the Lecture 0.

Lemma 1.4. *Each cofibration $i\colon A \to X$ is an injective map, and if its image $i(A)$ is closed in X, then it is a moneomorphism.*

Proof. Let

$$CA = A \times I / A \times 0$$

The quotient is obtained from the product space of $A \times I$ by contracting the subspace $A \times 0$ to a point. (The space CA is called a *cone* over A; we will return to it in the next lecture.) Further, let a_0 be the image of the subspace $A \times 0$ in CA (the *vertex* of the cone), and let $j\colon A \to CA$ be the moneomorphism induced by the map $a \mapsto (a, 1)$, $a \in A$. We can consider the factorisation map $A \times I \to CA$ as a homotopy from A to CA, connecting the constant map $\mathrm{const}_A \to CA$, $a \mapsto a_0$, with the map j. Therefore, since $\mathrm{const}_A = \mathrm{const}_X \circ i$, where const_X is a map $X \to CA$, $x \mapsto a_0$, then, according to the axiom **HEP**, there exists a map $g\colon X \to CA$ such that $g \circ i = j$. But the map j is injective. Therefore, the map i is also injective. Also, $iP = g^{-1}(jP) \cap iA$ for any subset $P \subset A$. Therefore, if iA is closed, then for any closed subset of $P \subset A$, the set iP is closed (since by applying the continuity of the map g and the moneomorphism of the map j, the set $g^{-1}(jP)$ closed). Hence, in this case the map i is a moneomorphism. □

The concept of a cofibration allows for a useful and very important dualisation.

1.3 Push-outs

According to the general Proposition 0.12 of the Lecture 0, the axiom of homotopy extension can be formulated as a requirement for the existence of some kind of retraction. The corresponding construction is quite elementary, but with a view to dualising it later, we will present it now in general categorical terms, which will allow us, in particular, to prove the Proposition 0.12 of the lecture 0 anew.

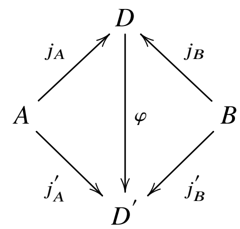
Let $\mathscr{A}$ be an arbitrary category, and let $i_A\colon C \to A$ and $i_B\colon C \to B$ be two morphisms of this category with the same domain of definition of C.

A *straight cone* over a pair of (i_A, i_B) comprises each pair (j_A, j_B) and morphisms $j_A\colon A \to D$, $j_B\colon B \to D$ satisfying the relation $j_A \circ i_A = j_B \circ i_B$ i.e. such that the diagramme

$$\begin{array}{ccc} C & \xrightarrow{i_A} & A \\ {\scriptstyle i_B}\downarrow & & \downarrow{\scriptstyle j_A} \\ B & \xrightarrow[i_B]{} & D \end{array} \tag{1.5}$$

commutes. A *morphism of the cone* $(j_A\colon A \to D, j_B\colon B \to D)$ into the cone $(j_A'\colon A \to D', j_B'\colon B \to D')$ comprises a morphism $\varphi\colon D \to D'$ such that $j_A \circ \varphi =$

j_A' and $j_B \circ \varphi = j_B'$, i.e. that there is a commutative diagramme



It is clear that all straight cones (over a given pair (i_A, i_B)) and all their morphisms make up a category $\underrightarrow{\mathscr{Con}}(i_A, i_B)$.

An initial object of this category, i.e. such a cone $(j_A\colon A \to D, j_B\colon B \to D)$ that for any other cone (j_A', j_B') over (i_A, i_B) in the category $\underrightarrow{\mathscr{Con}}(i_A, i_B)$ there is a single morphism $(j_A, j_B) \to (j_A', j_B')$, called an *push-out* (or *amalgam*) of the diagramme $A \stackrel{A}{\longleftarrow} C \stackrel{B}{\longrightarrow} B$ or, admitting a certain inaccuracy, a push-out of objects A and B with an amalgamated object C. Sometimes the D object itself is called a push-out. Diagramme (1.5), in which D is a push-out, is called a *universal* (or *Cartesian*) *square*.

It is easy to see that

Proposition 1.6. *in the category $\mathscr{Top}$ an amalgam exists for any diagramme $A \stackrel{I_A}{\longleftarrow} C \stackrel{i_B}{\longrightarrow} B$.*

Proof. The space D will be the coset space of the disjoint union $A \sqcup B$ by the minimum equivalence relation in which $a \sim b$ if there exists such a $c \in C$ that $i_A(c) = a$ and $i_B(c) = b$, and the maps $j_A\colon A \to D$ and $j_B\colon In \to D$ are maps induced by inclusions $A \to A \sqcup B$ and $B \to A \sqcup B$. Indeed, by construction $j_A \circ i_A = j_B \circ i_B$ and for any cone $(j_A', j_B' \in \underrightarrow{\mathscr{Con}}(i_A, i_B)$, maps $a \mapsto j_A'$, $b \mapsto J_B'$ induce (obviously the only) map $\varphi\colon D \to D'$, for which $j_A' = \varphi \circ j_A$, $j_B' = \varphi \circ j_B$. □

In the case when $C = A \cup In$, and the maps i_a and I_b are inclusions, the push-out of D is the union of $A \cup in$. This explains the origin of the term "push-out".

1.4 Push-outs and co-induced cofibrations

For the diagramme $X \stackrel{i}{\longleftarrow} A \stackrel{f}{\longrightarrow} Y$, where i is an embedding, the push-out will be the space $X \cup_f Y$ constructed in Lecture 0. Assuming a certain liberty, we will use the notation $X \cup_f Y$ for the pushu-out of the diagramme $X \stackrel{i}{\longleftarrow} A \stackrel{f}{\longrightarrow} Y$ and in the case of an arbitrary map i. The corresponding maps are $X \to X \cup fY$ and $Y \to X \cup_f Y$ (which are constraints of the factorisation map $X \sqcup Y \to X \cup_f Y$) we will denote by the symbols $f_\#$ and i_f. Together with the maps i and f, they make up a co-universal square

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_{\#}} \\ Y & \xrightarrow[\ i_f\ ]{} & X \cup_f Y \end{array} \qquad (1.7)$$

The map i_f is said to be *co-induced* by the map i via the map i_f. If the map i is injective, then the map i_f will be a moneomorphism to a closed subspace. In this case, having identified the points y and $i_f(y)$, we will consider Y to be a closed subspace of the space $X \cup_f Y$ (which is consistent with the conventions from Lecture 0 concerning the case when i is an embedding).

Proposition 0.12 of Lecture 0, i.e. the statement that

Proposition 1.8. *the extension of $\overline{f}\colon X \to Y$ of the map $f\colon A \to Y$ with respect to the map $i\colon A \to X$ exists if and only if the co-induced map $i_f\colon Y \to X \cup_f Y$ is retractible,*

remains true for any map i.

Proof. Indeed, if the map i_f is retractible and $r\colon X \cup_f Y \to Y$ is the corresponding retraction, then the composition is $\overline{f} = r \circ f_{\#}$ will obviously satisfy the relation $\overline{f} \circ = f$. Conversely, if $\overline{f}$ exists, then the pair $(\mathrm{id}\colon Y \to Y, \overline{f}\colon X \to Y)$ will be a straight cone over (i, f), and therefore there will be a morphism $r\colon X \cup_f Y \to Y$ push-outs $X \cup_f Y$ into this cone. But then $r \circ i_f = \mathrm{id}$, so that the morphism r will be a retract of the map i_f. □

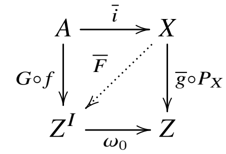
Thus, *the problem of extension of the map f is equivalent to the problem of the existence of a retraction of the co-induced map i_f.* However, this is essentially a general categorical statement, it would be virtually useless for us if, when moving from i to i_f, we left the cofibration class. And in fact, everything is fine in this regard, i.e.

Proposition 1.9. *the map $i_f\colon Y \to X \cup_F Y$, co-induced by the cofibration $i\colon A \to X$, is also a cofibration.*

Proof. Indeed, from any diagramme of the form

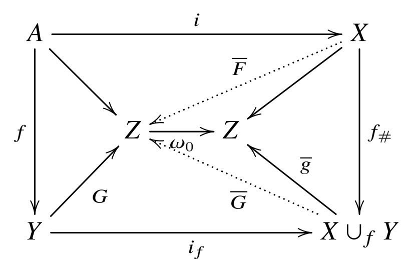
$$\begin{array}{ccc} Y & \xrightarrow{\ i_f\ } & X \cup_f Y \\ {\scriptstyle G}\big\downarrow & \overset{\overline{G}}{\swarrow} & \big\downarrow{\scriptstyle \overline{g}} \\ Z^I & \xrightarrow[\ \omega_0\ ]{} & Z \end{array} \qquad (1.10)$$

by composing its vertical arrows with the vertical arrows of Diagramme 1.7, we get the diagramme



for which the homotopy $\overline{F}$ exists by condition. Obviously, $\overline{F}$ with G together homotopically forms a cone $(\overline{F}, G)$ over the pair (i, f). Therefore, there is a morphism $\overline{G}\colon X \cup_f Y \to Z^I$ from the cone $(f_\#, i_f)$ into the cone $(\overline{F}, G)$, which will be the homotopy closing Diagramme (1.10). Thus, for any diagramme of the form (1.10), there is a closing homotopy $\overline{G}$ and, therefore, the map i_f is a cofibration. □

This proof is clearly depicted by the diagramme



(1.11)

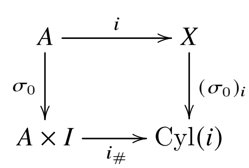
consideration of which makes the proof quite obvious.

1.5 The mapping cylinder

Definition 1.12. The *cylinder* $\mathrm{Cyl}(i)$ of the continuous map $i\colon A \to X$ is the push-out $(A \times I) \cup_i X$ of the diagramme

$$A \times I \xleftarrow{\sigma_0} A \xrightarrow{i} X.$$

This push-out is obtained by gluing the space $A \times I$ to the space X by the map $(a, 0) \mapsto i(a)$. There is a co-universal square for it



(1.13)

the map $(\sigma_0)_i$ of which is a monomorphism (note that we "transposed" the square (3), which, of course, does not interfere with anything). We will assume that the space X is embedded in the cylinder $\mathrm{Cyl}(i)$ by means of the monomorphism $(\sigma_0)_i$.

For any point $(a,t) \in A \times I$ we will denote the point $i_{\#}(a,t) \in \mathrm{Cyl}(i)$ by the symbol $[a,t]$. Thus, each point from $\mathrm{Cyl}(i)$ either has the form $[a,t]$, $a \in A$, $t \in I$, or is a point x from X. In this case, $[a,0] = i(a)$ for any point $a \in A$.

Due to the universality, the formula

$$\begin{aligned} j[a,t] &= (ia,t), \quad (a,t) \in A \times I, \\ j(x) &= (x,0), \quad x \in X, \end{aligned}$$

well determine the continuous map

$$j\colon \mathrm{Cyl}(i) \to X \times I.$$

In the most important special case, when $A \subset X$ and i is an embedding, the map j, as it is easy to see, is an injective map to a subspace $\widetilde{A} = (X \times 0) \cup (A \times I)$ of the product $X \times I$. Generally speaking, it is not a monomorphism. However,

Proposition 1.14. *if A is closed in X, then j is a moneomorphism,*

so in this case, considering j an embedding, we can identify the cylinder $\mathrm{Cyl}(i)$ with the subspace $\widetilde{A}$ of the space $X \times I$.

Proof. Indeed, by definition, the set $C \subset \mathrm{Cyl}(i)$ is closed if its preimage in X and in $A \times I$ are closed. In the case when $A \subset X$, this is equivalent to the fact that the intersections of $jC \cap (X \times 0)$ and $jC \cap (A \times I)$ are closed respectively in $X \times 0$ and $A \times I$. But since $X \times 0$ is closed in $X \times I$, the intersection of $jC \cap (X \times 0)$ is closed in $X \times 0$ if and only if it is closed in $X \times I$. Similarly, if A is closed in X, then $A \times I$ is closed in $X \times I$, and, therefore, the intersection of $jC \cap (A \times I)$ is closed in $A \times I$ if and only if it is closed in $X \times I$. Therefore, if A is closed, then for any set closed in $\mathrm{Cyl}\, i$, C is a set with

$$jC = (jC \cap (X \times 0)) \cup (jC \cap (A \times I))$$

is closed in $X \times I$, and, therefore, in $\widetilde{A} = (X \times 0) \cup (A \times I)$. □

Pairs of (X, A) with closed A are called *closed*.

1.6 Characterisation of cofibrations

Proposition 1.15. *A map $i\colon A \to X$ is a cofibration if and only if the map $j\colon \mathrm{Cyl}(i) \to X \times I$ is retractible.*

Proof. Homotopy of $F\colon A \times I \to Y$ and extension of $\overline{f}\colon X \to Y$ to X of its initial map $f_0 = F \circ \sigma_0$ define a map

$$F \sqcup \overline{f}\colon (A \times I) \sqcup X \to Y,$$

that takes the same values at points $(a,0)$ and $i(a)$, $a \in A$. Therefore, the map $F \sqcup \overline{f}$ induces some map $g\colon \mathrm{Cyl}(t) \to Y$, which has the property that

$g[a,t] = F(a,t)$ and $g(x) = \overline{f}(x)$ for any points $[a,t]$ and x of $\mathrm{Cyl}(i)$. Note that the pair $(F, \overline{f})$ is a cone over (σ_0, i) and g is nothing but a morphism of the cone $(i_\#, (\sigma_0)_i)$ into the cone $(F, \overline{f})$ provided by the co-universality of the push-out $\mathrm{Cyl}(i)$. Therefore, if the retraction is $r\colon X \times I \to \mathrm{Cyl}(i)$ exists, then the map $\overline{F} = g \circ r\colon X \times I \to Y$ will be a homotopy with the initial map f (for this map satisfies the relation $\overline{F} = g$, and therefore $\overline{F}(x,0) = (\overline{F} \circ j)(x) = g(x) = \overline{f}(x)$ for any point $x \in X$), which is an extension (in relation to the map i) of the homotopy F (for $\overline{F}(ia,t) = (\overline{F} \circ j)[a,t] = g[a,t] = F(a,t)$ for any point $(a,t) \in A \times I$). Thus, if the map j is retractible, then the map i is a cofibaration.

Conversely, the map $i_\#$ can be viewed as a homotopy from A to $\mathrm{Cyl}(i)$ with the initial map, being a constraint (with respect to i) maps $(\sigma_0)_i$; therefore, if the map $i\colon A \to X$ is a cofibration, then there is a map $r\colon X \times I \to \mathrm{Cyl}(i)$ satisfying the relations $r(x,0) = x$ and $r(ia,t) = [a,t]$, i.e. being a retraction of the map j. □

Corollary 1.16. *A closed pair (X,A) is a cofibration if and only if the subspace $\widetilde{A} = (X \times 0) \cup (A \times I)$ is a retract of the space $X \times I$.*

Ström showed that the assumption about the closeness of the sub-space A in this corollary is actually superfluous. However, in accordance with our general set-up, we will ignore this statement, since it is only valid for non-Hausdorff spaces X, thus

Proposition 1.17. *if for a pair (X,A) such that the space X is Hausdorff and the subspace $\widetilde{A} = (X \times 0) \cup (A \times I)$ is a retract of the space $X \times I$, the pair (X,A) is closed.*

Proof. Indeed, the composition $j \circ r$ of the reacting map $r\colon X \times I \to \widetilde{A}$ and the embeddings $j\colon \widetilde{A} \to X \times I$ is a continuous map $X \times I \to X \times I$, the sets of fixed points of which is $\widetilde{A}$. By applying the Hausdorff space $X \times I$, it follows that $\widetilde{A}$ is closed in $X \times I$. But then its preimage A is also closed (in X) with a continuous map $\sigma_1 : x \mapsto (x,1)$. □

Note, by the way, that

Proposition 1.18. *For a closed cofibration (X,A) the subspace A is even functionally separable, i.e. there exists a continuous function $\varphi\colon X \to I$, such that $\varphi(a) = 0$ if and only if $a \in A$.*

Proof. Indeed, let $r\colon X \times I \to \widetilde{A}$ be a retraction map and let $\rho(x,t)$ be the projection of the point $r(x,t)$, $(x,t) \in X \times I$ onto I. Consider the function $\varphi\colon X \to I$ defined by the formula

$$\varphi(x) = \max_{t \in I}(t - \rho(x,t)), \quad x \in X.$$

By applying Lemma 1.19 proved below, the function φ is continuous on X. If $a \in A$, then $\rho(a,t) = t$ for any $t \in I$ and therefore $\varphi(a) = 0$. Conversely, if $\varphi(a) = 0$, i.e. $\rho(a,t) \geq t$ for all $t \in T$, then, in particular, $\rho(a,t) > 0$ for $t > 0$

and, therefore, $r(a,t) \in A \times I$. Since the subspace A (and therefore the subspace $A \times I$) is closed by condition, it follows that

$$(a,0) = r(a,0) = \lim_{t \to 0} r(a,t) \in A \times I$$

and thus $a \in A$. □

Lemma 1.19. *For any compact space C, arbitrary topological space X and any continuous function $\psi\colon X \times C \to \mathbb{R}$ the function $\varphi\colon X \to \mathbb{R}$ defined by the formula*

$$\varphi(x) = \max_{c \in C} \psi(x,c), \quad x \in X,$$

is continuous on X.

Proof. From the compactness of the space C it follows, firstly, that for any point $x \in X$ the maximum is reached, i.e. there exists (generally speaking, not the only one) a point $c_x \in C$ such that $\varphi(x) = \psi(x, c_x)$. Secondly, the function ψ is equally continuous at each point x of the space X, i.e. for any $\varepsilon > 0$ and any point $x_0 \in X$, there is a neighbourhood $U \subset X$ of the point x_0, such that $|\psi(x,c) - \psi(x_0,c)|\varepsilon$ for any points $x \in U$ and $c \in C$. Therefore, if $X \in U$ and $\varphi(x) \geq \varphi(x_0)$, then

$$|\varphi(x) - \varphi(x_0)| = \varphi(x) - \varphi(x_0) \leq \psi(x, c_x) - \psi(x, c_{x_0}) = |\psi(x, c_x) - \psi(x, c_{x_0}| < \varepsilon$$

and if $\varphi(x) \leq \varphi(x_0)$, then

$$|\varphi(x) - \varphi(x_0)| = \varphi(x_0) - \varphi(x) \leq \psi(x_0, c_{x_0}) - \psi(x_0, c_{x_0}) = |\psi(x, c_{x_0}) - \psi(x_0, c_{x_0}| < \varepsilon.$$

So $|\varphi(x) - \varphi(x_0)| < \varepsilon$ for any point $x \in U$, i.e. the function φ is continuous. □

1.7 The product of cofibrations

The pair $(X,A) \times (Y,B) := (X \times Y, X \times B \cup A \times Y)$ is called the *product* of pairs (X,A) and (Y,B).

Proposition 1.20 (Theorem of Ström)**.** *The product*

$$(Z,C) = (X \times Y, X \times B \cup A \times Y)$$

of closed cofibrations (X,A) and (Y,B) is a closed cofibration.

Proof. According to the corollary to Proposition 1.15, the subspace $\widetilde{A} = X \times 0 \cup A \times I$ and $\widetilde{B} = Y \times 0 \cup B \times I$ are retracts of the spaces $X \times I$ and $Y \times I$, respectively. Let $r\colon X \times I \to \widetilde{A}$ and $s\colon Y \times I \to \widetilde{B}$ be the corresponding retractions, and let

$$r(x,t) = (\overline{r}(x,t), \rho(x,t)), \quad (x,t) \in X \times I,$$
$$s(x,t) = (\overline{s}(y,t), \sigma(y,t)), \quad (y,t) \in Y \times I,$$

where $\overline{r}(x,t) \in X$, $\overline{s}(y,t) \in Y$ and $\rho(x,t), \sigma(y,t) \in I$, with $\rho(x,t) = 0$ if $\overline{r}(x,t) \notin A$ and $\sigma(y,t) = 0$ if $\overline{s}(y,t) \notin B$. Now, we let

$$\varphi(x) = \max_{t\in I}(t - \rho(x,t)), \quad x \in X,$$
$$\psi(y) = \max_{t\in I}(t - \sigma(y,t)), \quad y \in Y.$$

And let

$$P(x,y,t) = \begin{cases} 0, & \text{if} \quad t \le \min(\varphi(x), \psi(y)), \\ t - \min(\varphi(x), \psi(y)), & \text{if} \quad t \ge \min(\varphi(x), \psi(y)), \end{cases}$$

and

$$\overline{R}(x,y,t) = (\overline{r}(x, \min(t, \psi(y))), \overline{s}(y, \min(t, \varphi(x)))).$$

If $\overline{R}(x,y,t) \notin C = X \times B \cup A \times Y$, then $\overline{s}(y, \min(t, \varphi(x))) \notin B$, $\overline{r}(y, \min(t, \psi(y))) \notin A$, so $\sigma(y, \min(t, \varphi(x))) = 0$, $\rho(x, \min(t, \psi(y))) = 0$. Hence $\min(t, \varphi(x)) \le \psi(y)$ and $\min(t, \psi(y)) \le \varphi(x)$, i.e., $t \le \min(\varphi(x), \psi(y))$, and therefore $P(x,y,t) = 0$. Hence the formula

$$R(x,y,t) = (\overline{R}(x,y,t), P(x,y,t)), \quad (x,y,t) \in Z \times I,$$

defines some map

$$R\colon Z \times I \to \widetilde{C} = Z \times 0 \cup C \times I.$$

At the same time, if $(x,y,t) \in \widetilde{C}$, i.e. either $t = 0$, or $y \in B$ (and, therefore, $\psi(y) = 0$), or $x \in A$ (and, therefore, $\psi(x) = 0$), then $P(x,y,t) = t$ and $\overline{R}(x,y,t) = (\overline{r}(x,0), \overline{s}(y,0)) = (x,y)$, i.e. $R(x,y,t) = (x,y,t)$. Therefore, the map R is a retraction, and therefore the pair (Z,C) is a cofibration. □

Corollary 1.21. *For any closed cofibration* (X,A) *and any* $n > 0$ *the pair* $\overbrace{(X,A) \times \cdots \times (X,A)}^{n}$ *is a cofibration.*

Proof. Obvious induction. □

Corollary 1.22. *For any closed cofibration* (X,A) *and any topological space* Y *the pair* $(X \times Y, A \times Y)$ *is a closed cofibration.*

Proof. This is a special case of Proposition 1.20, obtained when $B = \varnothing$. However, this corollary follows directly from the corollary of Proposition 1.15, since for each retraction $r\colon X \times I \to \widetilde{A}$ the map $r \times \mathrm{id}\colon X \times I \times Y \to \widetilde{A} \times Y$ will be a retraction $X \times I \times Y = X \times Y \times I$ on the subspace $\widetilde{A} \times Y = (X \times Y \times 0) \cup (A \times Y \times I) = \widetilde{A \times Y}$. □

It is clear that

Proposition 1.23. *the composition of two cofibrations is a cofibration, so in particular if* $X \supset A \supset B$ *and pairs* (X,A) *and* (A,B) *are cofibrations, then the pair* (X,B) *will also be a cofibration.*

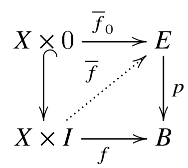
Applying this statement to the pairs $(X \times Y, X \times B)$ and $(X \times B, A \times B)$ and using Corollary 1.22, we will immediately get

Corollary 1.24. *For any two closed cofibrations* (X,A) *and* (Y,B), *the pair* $(X \times Y, A \times B)$ *is also a closed cofibration.*

1.8 Fibrations

According to the general categorical principle of duality, for the concept of a cofibration, there must be a dual concept resulting in "the reversal of all arrows".

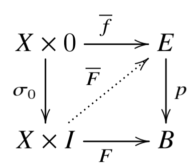
Definition 1.25. A map $p\colon E \to B$ is called a *fibration* (in the sense of Hurevicz) if for any space X, any homotopy $f_t\colon X \to B$ and any map $\overline{f}$ satisfying the relation $p \circ \overline{f} = f_0$, there is such a homotopy $\overline{f}_t\colon X \to E$ that $\overline{f}_0 = \overline{f}$ and $p \circ \overline{f}_t = f_t$ for any $t \in I$.



Thus, the map $p\colon \to B$ is a fibration if, the initial map $f_0\colon X \to B$ of some homotopy $f_t\colon X \to B$ can be lifted to E, it follows that on E it is possible to lift each map f_t, and the lift $\overline{f}_t$ can be chosen so that they form a homotopy.

This requirement is called the *axiom of covering homotopy* (in short, the *axiom* **CH**).

Clearly, the axiom of covering homotopy is represented by the diagramme



(1.26)

the dual of Diagramme (1.3).

In particular, we see that

Proposition 1.27. *if the maps $f, g\colon X \to B$ are homotopic and f can be lifted to E, then g can also be lifted to E, so that for fibrations the property of map $f\colon X \to B$ allowing a lift to E depends only on its homotopy class $[f]$*

Thus, in the lifting problem (problem (0.5) of Lecture 0) we can also move into the homotopy category.

This explains why in algebraic topology the lifting problem is considered only for fibrations. (However, as we will see in Lecture 3, this is not a serious limitation.)

Lemma 1.28. *If the map $p\colon \to B$ is a fibration, then for any path $u\colon I \to B$ and any point $e \in E$ such that $p(e) = u(0)$, there exists a path $v\colon I \to E$ starting at the point e and the covering path u (i.e. such that $p \circ y = u$).*

Proof. Let us define a homotopy $u_t\colon I \to B$ by the formula

$$u_t(s) = u(ts), \quad s, t \in I.$$

Since $u_0(s) = u(0) = p(e)$ for all $s \in I$, then $p \circ 0_e = u_0$, where $0_e : s \mapsto e$ — a constant path at the point e. Therefore, there is a homotopy $v_t : I \to E$ such that $p \circ v_1 = u_t$ for all $t \in I$ (and $v_0 = 0_e$). In particular, $p \circ v_1 = u_1$. Since $v_1(0) = e$ and $u_1 = u$, this proves the lemma (with $v = v_1$). □

Corollary 1.29. *For any fibration $p\colon E \to B$ (with non-empty E), the set $p(E)$ is the union of the components of the space B. In particular, if the space B is connected, then p is an injective map.*

Proof. Let $b \in p(E)$, and let $u\colon I \to B$ be an arbitrary path starting at b. We need to prove that $u(1) \in p(E)$. Let $p(e) = b$, and let $v\colon I \to B$ be a path covering the path u and starting at e. Then $p(v(1)) = u(1)$ and hence $u(1) \in p(E)$. □

The last statement of this corollary is dual to the first statement of Lemma 1.4. The dualisation of the second statement of Lemma 1.4 (the question of epimorphism of fibrations) is of little interest, and we will not deal with it. One can, for example, show that the fibration $E \to B$ will be an epimorphic map in the case when the space B is connected and locally connected, but discussing whether this condition on B can be considered dual to the condition of closure of the subspace iA from Lemma 1.4 would lead us too far to the side. That some conditions on B are necessary is shown by an example of an identical map of a set of rational numbers $\mathbb{Q}$ with the discrete topology to $\mathbb{Q}$ with the conventional topology. While the map is not an epimorphism, as an easy check shows, it is a fibration.

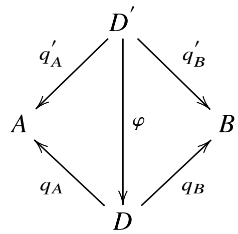
1.9 Pull-backs and induced fibrations

Let $\mathscr{A}$ be an arbitrary category, and let $p_A\colon A \to C$ and $p_B\colon B \to C$ are two morphisms of this category with the same target domain. The *cocone* over the pair (p_A, p_B) is a pair (q_A, q_B) of morphisms $q_A\colon D \to A$ and $q_B\colon D \to B$ satisfying the relation $p_A \circ q_A = p_B \circ q_B$ i.e. such that the diagramme

$$\begin{array}{ccc} D & \xrightarrow{q_A} & A \\ {\scriptstyle q_B}\big\downarrow & & \big\downarrow{\scriptstyle p_A} \\ B & \xrightarrow[p_B]{} & C \end{array} \tag{1.30}$$

commutes. The *morphism* of the cone $(q'_A\colon D' \to A, q'_B\colon D' \to B)$ into the cone $(q_A\colon D \to A, q_B\colon D \to B)$ is a morphism $\varphi\colon D' \to D$ of category $\mathscr{A}$, such that

$q_A \circ \varphi = q'_A$ and $q_B \circ \varphi = q'_B$, i.e. such that there is a commutative diagramme



It is clear that all cocones (over a given pair (p_A, p_B)) and all their morphisms make up a category $\underleftarrow{\mathrm{Cone}}(p_A, p_B)$. The *terminal object* of this category, i.e. such a cone $(q_A\colon D \to A, q_B\colon D \to B)$ that for any other cone (q'_A, q'_B) over (p_A, p_B) in the category $\underleftarrow{\mathrm{Cone}}(p_A, p_B)$ there is a single morphism $(q'_A, q'_B) \to (p_A, p_B)$, called the *pull-back* of the diagramme $A \xrightarrow{p_A} C \xleftarrow{p_B} B$ or, admitting a certain inaccuracy, *pull-back objects A and B over object C*. Sometimes a pull-back is called the object D itself. Diagramme (1.30), in which D is an pull-back, is called a *universal* (or *co-Cartesian*) *square*.

Since in any category a terminal object (when it exists) up to the canonical isomorphism is defined in a unique way, the same is true for pull-back.

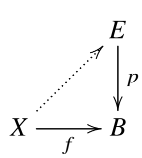
It is easy to see that

Proposition 1.31. *in the category $\mathcal{T}\!op$ the pull-back exists for any diagramme $A \xrightarrow{p_A} C \xleftarrow{p_B} B$.*

Proof. The space D will be a subset of the direct product $A \times B$, consisting of points $(a, B) \in A \times B$ such that $p_A(a) = p_B(b)$, and the maps $q_A\colon D \to A$ and $q_B\colon D \to B$ are the constraints of the natural projections $A \times B \to A$ and $A \times B \to B$. Indeed, by the construction of $p_A \circ q_A = p_B \circ q_B$ and for any cone $(q'_A\colon D' \to A, q'_B\colon D' toB)$ formula $\varphi(x) = (q'_A(x), q'_B(x))$, $x \in D'$, defines the (obviously unique) map $\varphi\colon D' \to D$, for which $q'_A = q_A \circ \varphi$ and $q'_B = q_B \circ \varphi$. □

In the case when $A \subset C$ and $B \subset C$, and the maps p_A and p_B are embeddings, the pull-back D is naturally identified with the intersection of $A \cap B$ of the subspaces A and B.

With each lifting problem



we will associate the pull-back diagramme

$$E \xrightarrow{p} B \xleftarrow{f} X.$$

We will denote this pull-back with the symbol $E \cap_f X$, and the projections $E \cap_f X \to X$ and $E \cap_f X \to E$ with the symbols p_f and $f^\#$. Thus, $E \cap_f X$ is a subspace of the direct product of $E \times X$ consisting of points (e, x) for which $p(e) = f(x)$, and the maps p_f and $f^\#$ act according to the formulae $(e, x) \mapsto x$ and $(e, x) \mapsto e$. Together with the maps f and p these maps make up a universal square

$$\begin{array}{ccc} E \cap_f X & \xrightarrow{f^\#} & E \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

The map $p_f \colon E \cap_f X \to X$ is said to be *induced* by the map $p \colon E \to B$ and the map $f \colon X \to B$.

Naturally, the map p_f has the properties dual to the properties of the co-induced map i_f. Namely,

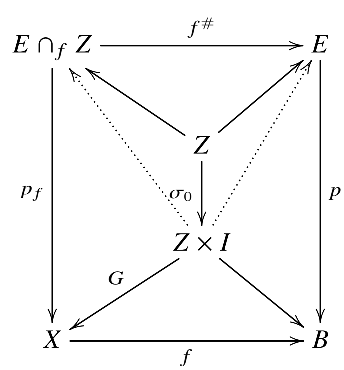
Proposition 1.32. *the map $\overline{f} \colon X \to E$, covering the map $f \colon X \to B$, exists if and only if the induced map $p_f \colon E \cap_f X \to X$ has a section.*

Proof. Indeed, if there is a section $s \colon X \to E \cap_f X$ to the map p_f, then the composition $\overline{f} = f^\# \circ s$ will obviously satisfy the relation $p \circ \overline{f} = f$. Conversely, if the map $\overline{F}$ exists, then the pair $(\mathrm{id} \colon X \to X, \overline{f} \colon X \to E)$ will be the cocone over (f, p), and therefore there will be a morphism $s \colon X \to E \cap_f X$ of this cone into the cone $(p_f, f^\#)$. In particular, there will be equality $p_f \circ s = \mathrm{id}$, so that the morphism s will be section to the map p_f. □

Thus, *the problem of lifting the map f is equivalent to the problem of the existence of the induced cross section p_f*. At the same time, with regard to the applicability of the method of algebraic topology, we lose nothing, because

Proposition 1.33. *if the map $p \colon E \to B$ is a fibration, then for any map $f \colon X \to B$, the induced map $p_f \colon E \cap_f X$ will also be a fibration.*

Proof. For the proof, it is enough to consider the diagramme



the dual diagramme of Diagramme (diag:01-5). □

The map p_f and $f^{\#}$ have certain properties of naturalness (functoriality). For example, if $f = \mathrm{id}$, then the space $E \cap_f X = E \cap_{\mathrm{id}} B$ is naturally identified with E, and the maps $f^{\#}$ and p_f are identified with the maps id and p respectively. Similarly, for any map $g\colon Y \to X$ the space $(E \cap_f X) \cap_g Y$ is naturally identified with the space $E \cap_{f \circ g} Y$, and the maps $f^{\#} \circ g^{\#}$ and $(p_f)_g$ with the maps $(f \circ g)^{\#}$ and $p_{f \circ g}$, respectively. Thus, modulo the above identifications

$$\mathrm{id}^{\#} = \mathrm{id}, \quad p_{\mathrm{id}} = p,$$
$$(f \circ g)^{\#} = f^{\#} \circ g^{\#}, \quad p_{f \circ g} = (pg)f.$$

A note on terminology. The construction of the space $E \cap_f X$ can be considered as a generalisation of the construction of the direct product. Therefore, in English the space (as well as the corresponding fibration $p_f\colon E \cap_f X \to X$) is called *fibre product*. Russian tracing paper “stratified product” is an indisputable solecism that does not adorn the Russian mathematical terminology. The use of this term, which sometimes occurs in a general categorical situation, is already completely meaningless and represents an obvious spoil of language.

It should also be cautioned against using the asymmetric term “induced map” (induced morphism) outside the scope of the lifting problem and, similarly, the term “co-induced morphism” outside the scope of the extension problem.

1.10 The mapping cocylinder and the axiom of the covering path

We now dualise the notion of a mapping cylinder.

Definition 1.34. *Cocylinder* $\mathrm{Cocyl}(p)$ of the continuous map $p\colon E \to B$ is the pull-back $B^I \cap_{\omega_0} E$ of the diagramme

$$B^I \xrightarrow{\omega_0} B \xleftarrow{p} E.$$

By definition, this pull-back is a subspace of the direct product of $B^I \times E$, consisting of such pairs (u, e), where $u\colon I \to B$, and $e \in E$, that $u(0) = p(e)$. For it there is a universal square

$$\begin{array}{ccc} \mathrm{Cocyl}(p) & \xrightarrow{\omega_0^{\#}} & E \\ {\scriptstyle p_{\omega_0}}\downarrow & & \downarrow{\scriptstyle p} \\ B^I & \xrightarrow[\omega_0]{} & B \end{array} \tag{1.35}$$

where $p_{\omega_0}(u, e) = u$ and $\omega 0^{\#}(u, e) = e$. In addition, the formula

$$q(v) = (p \circ v, v(0)). \quad v \in E^I,$$

defines, obviously, a continuous map

$$q\colon E^I \to \mathrm{Cocyl}(p).$$

Proposition 1.36. *The map $p\colon E \to B$ is a fibaration if and only if to the map $q\colon E^I \to \mathrm{Cocyl}(p)$ there is a section $s\colon \mathrm{Cocyl}(p) \to E^I$.*

Before proving this Proposition we will state a few comments that have an independent interest.

Setting Diagramme (1.26) without the dotted arrow is obviously equivalent to setting a commutative diagramme

$$\begin{array}{ccc} X & \xrightarrow{\overline{f}} & E \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle p} \\ B^I & \xrightarrow[\omega_0]{} & B \end{array} \tag{1.37}$$

in which the same letter F is used to denote the map $X \to B^I$, associated with the homotopy $F\colon X \times I \to B$. The commutativity of Diagramme (1.37) means that the pair $(F, \overline{f})$ is an co-cone over the pair (ω_0, p). The corresponding morphism $(F, \overline{f}) \to (p_{\omega_0}, \omega_0^{\#})$ is a map $\varphi\colon\ : X \to \mathrm{Cocyl}(p)$, acting according to the formula $x \mapsto (\overline{f}x, Fx)$.

On the other hand, having Diagramme (1.37), we can allocate a subspace in the cylinder $\mathrm{Cocyl}(\overline{f}) \subset X \times E^I$ the map $\overline{f}$ consisting of such pairs $(x, v), x \in X$, $v\colon I \to E, \overline{f}(x) = *v(0)$, that $F(x) = p \circ v$.

We will denote this subspace by the symbol $\mathrm{Cocyl}(\overline{F}, f)$ and we will call it the *cocylinder of the pair* $(\overline{F}, f)$. We will denote its projection $(x, v) \mapsto x$ onto the space X by the symbol q_X. As the immediate checking whether a commutative diagramme takes place

$$\begin{array}{ccc} \mathrm{Cocyl}(F, \overline{f}) & \longrightarrow & E^i \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[\varphi]{} & \mathrm{Cocyl}(p) \end{array}$$

the upper horizontal arrow of which is a projection of $(x, v) \mapsto v$.

At the same time, it is easy to see that

Proposition 1.38. *for Diagramme (1.37) there is a covering homotopy $\overline{F}\colon X \times I \to E$ if and only if the projection $q_X\colon \mathrm{Cocyl}(F, \overline{f}) \to X$ has the section $s_X\colon X \to \mathrm{Cocyl}(F, \overline{f})$.*

Proof. Indeed, the formula

$$s_X = (x, \overline{F}x), \quad x \in X,$$

establishes a bijective correspondence between sections s_X and homotopies $\overline{F}$ interpreted as maps $X \to E^I$. □

Now Proposition 1.36 becomes almost obvious.

Proof. (of Proposition 1.36) If a section $s\colon \mathrm{Cocyl}(p) \to E^I$ exists, then for any diagramme (1.37) the formula

$$s_X(x) = (x, (s \circ \varphi)(x)), \quad x \in X,$$

defines some section of $s_X\colon X \to \mathrm{Cocyl}(F, f)$ for the map q_X. Therefore, for Diagramme (1.37) there is a covering homotopy $\overline{F}\colon X \times I \to E$. Consequently, the map $p\colon E \to B$ is a fibration.

Conversely, let the map $p\colon E \to B$ is a fibration and, then, for any diagramme (1.37), the corresponding map q_X has a section s_X. In particular, this will be the case for Diagramme (1.35) for $X = \mathrm{Cocyl}(p)$ and $F = p_{\omega_0}$, $\overline{f} = \omega_0^{\#}$. But it is clear that the projection

$$\mathrm{Cocyl}(p_{\omega_0}, \omega_0^{\#}) \to E^I, \quad ((e, u), v) \mapsto v,$$

is a homeomorphism (for $e = v(0)$ and $u = p \circ v$) and with this homeomorphism to the sections $\mathrm{Cocyl}(p) \to \mathrm{Cocyl}(p_{\omega_0}, \omega_0^{\#})$, projections $((e,u), v) \mapsto (e, u)$ correspond to sections $\mathrm{Cocyl}(p) \to E^I$ of the map $q\colon v \mapsto (v(0), p \circ v)$. □

The statement that the map s is a cross section map of q, means that $s(e, u)$ is a path in E, covering this path u in B and starting at this point $e \in E$, projecting to the beginning of the path u. The fact of the existence of such a path is the content of Lemma 1.19. What is new is the statement that, firstly, this choice can be made in a continuous (by u and e) way, and secondly, that the possibility of such a choice is not only necessary, but also sufficient for the map p to be a fibration.

The requirement for the existence of a section s is usually called the *axiom of the covering path* (in short, the *axiom* **CP**).

1.11 Fibrations of mapping spaces

For an arbitrary space Y, each continuous map the map $I\colon A \to X$ determines, by the formula

$$i_Y\colon (f) = f \circ i, \quad f\colon X \to Y$$

a map

$$i_Y\colon \mathscr{Top}(X, Y) \to \mathscr{Top}(A, Y).$$

Since the map i is continuous, then for any compact set $K \subset A$ the set $iK \subset X$ is also compact. Therefore, in $\mathscr{Top}(X, Y)$ for any open the set $U \subset Y$ the set $\mathscr{W}(iK, U)$ is defined, which is obviously a preimage of the set $\mathscr{W}(K, U)$ by the map the i_Y. This proves that

Proposition 1.39. *in the compact open topology the map i_Y is continuous, i.e. it is a map $i_Y\colon Y^X \to Y^A$ of topological spaces.*

Proof. If $i\colon A \to X$ is an embedding, then $i_Y f$ is nothing other than the restriction of the map $f\colon X \to A$ to A. Therefore, in this case we will call i_Y a *restriction map.* □

Proposition 1.40. *If the space X is locally compact and Hausdorff, its subspace A is closed and the pair (X, A) is a cofibration, then for any space Y the constraint map*

$$i_Y : Y^X \to Y^A$$

is a fibration.

Proof. It should be shown that for any diagramme of the form

$$\begin{array}{ccc} Z & \xrightarrow{\overline{f}} & Y^X \\ {\scriptstyle\sigma_0}\downarrow & \overline{F}\nearrow & \downarrow{\scriptstyle i_Y} \\ Z \times I & \xrightarrow[F]{} & Y^A \end{array} \tag{1.41}$$

there is a closing map $\overline{F}$. But since the space X (and, of course, by applying closure, the subspace A) is locally compact and Hausdorff, the exponential law applies to the horizontal map of this diagramme, i.e. these maps are associated with some maps $\overline{g}\colon Z \times X \to Y$ and $G\colon Z \times A \to Y^I$ (we allow ourselves to rearrange multipliers in direct products in a convenient way for us), which are, obviously, the diagramme

$$\begin{array}{ccc} Z \times A & \xrightarrow{\mathrm{id}\times i} & Z \times X \\ {\scriptstyle G}\downarrow & \overline{G}\nearrow & \downarrow{\scriptstyle \overline{g}} \\ Y^I & \xrightarrow[\omega_0]{} & Y \end{array}$$

Since in the terms of the Proposition 1.40 thee pair $(Z\times X, Z\times A)$ is a cofibration (see Corollary 1.22 of Proposition 1.20), for the last diagramme there is a closing map $\overline{G}$. The associated map is $Z \times I \to Y^X$ and will obviously be the map $\overline{F}$ closing Diagramme (1.41). □

Example 1.42. For $X = I$ and $A = \{1\}$, the set $\widetilde{A} = (I \times 0) \cup (1 \times I)$, which is the union of the lower and right sides of the square $X \times I = I \times I$, is obviously a retraction of this square (the retracting map is for example, a projection centred at $(0, 2)$; see fig. 1.11.1). Since in this case the space Y^A is naturally identified with the space Y, and since the the constraint map i_Y passes in this case into the map $\omega_1 : Y^I \to Y$, we get, by applying Proposition 1.40, (denoting Y by X) that for each space X

Proposition 1.43. *the map*

$$\omega_1 : X^I \to X$$

is a fibration.

Note now that

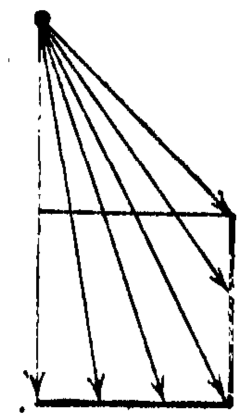


Figure 1.11.1:

Proposition 1.44. *for any fibration $p\colon E \to B$ and any subspace $A \subset B$, the map*

$$p_A = p|_{p^{-1}}(A)\colon p^{-1}(A) \to A$$

is a fibration,

Proof. since this map is naturally identified with the fibration p_i, by the induced fibration p by the embedding map $i\colon A \to B$. □

To apply this general remark to the bundle ω_1, for any subspace $A \subset X$, we will introduce the subspace $X^I(A)$ of the space X^I, consisting of all paths ending in A. This subspace exactly coincides with the preimage $\omega_1^{-1}(A)$ of the subspace A by the map ω_1. Denoting the map constraint ω_1 by $X^I(A)$ again through ω_1, we get, therefore, that

Proposition 1.45. *for any pair (X, A) the map*

$$\omega_1\colon X^I(A) \to A$$

is a fibration.

The last statement is useful to generalise. To do this, returning again to the situation to which Proposition 1.40 refers, suppose that we are given arbitrary families $\{A_\mu\}$ and $\{Y_\mu\}$ subspaces of spaces A and Y with indices from the same set M. Let $[Y^A]m$ is a subspace of the space Y^A consisting of such maps $f\colon A \to Y$ that $f(A_\mu) \subset Y_\mu$ for any $\mu \in M$, $[Y^A]_M$ is its preimage of the restriction map i_Y (lying in the space Y^X). Denoting the restriction of the map i_Y to $[Y^X]_M$ again by i_Y, we get that

Proposition 1.46. *in the conditions of Proposition 1.40 for any families $\{A_\mu\}$ and $\{Y_\mu\}$ the map*

$$i_Y\colon [Y^X]_M \to [Y^A]_M$$

is a fibration.

Example 1.47. Let $X = I$ and $A = \{0,1\}$. It is easy to see that in this case the conditions of Proposition 1.40 are met (the proof that the pair $(I, \{0,1\})$ is a cofibration, i.e. that the square $I \times I$ is retracted into its three sides $I \times 0 \cup \{0,1\} \times I = I \times 0 \cup 0 \times I \cup 1 \times I$ is illustrated in Fig. 1.11.2). Assuming that

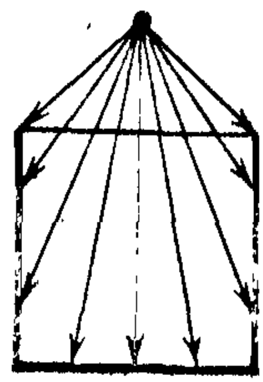


Figure 1.11.2:

the set of indices M consists of only one element μ, and assuming $A_\mu = \{0\}$, $Y_\mu = \{y_0\}$ where y_0 is some point of the space Y, we can obviously identify the space $[Y^A]_M$ with the component of the space Y containing the point y_0, and the space $[Y^X]_M$ with the subspace of all paths of the space Y starting at the point y_0. Replacing again Y by X (and y_0 by x_0), and denoting with the symbol $P(X, x_0)$ the set of all paths of the space X starting at the point $x_0 \in X$, we get,

Proposition 1.48. *for any connected space X and any of its points x_0, the map*

$$\omega_1 \colon P(X, x_0) - \longrightarrow X, \quad u \mapsto u(1),$$

is a fibration.

Proof. For each diagramme of the form

$$\begin{array}{ccc} C & \xrightarrow{\overline{f}} & P(X, x_0) \\ {\scriptstyle \omega_0}\downarrow & \overset{\overline{F}}{\nearrow} & \downarrow{\scriptstyle \omega_1} \\ C \times I & \xrightarrow[F]{} & X \end{array}$$

the covering homotopy $\overline{F}$ can be defined by the formula

$$F(x,s)(t) = \begin{cases} \overline{f}(x)\left(\frac{2t}{2-s}\right), & \text{if} \quad 0 \le t \le \frac{2-s}{2}, \\ f(x, s+2t-2) & \text{if} \quad \frac{2-s}{2} \le t \le 1. \end{cases}$$

□

This fibration is called the *Serre fibration* of paths spaces.

In a more general way, we can consider the space $P(X, A)$ consisting of paths beginning in $A \subset X$, or its subspace $P(X, A, B)$, consisting of paths ending in $B \subset X$. In both cases,

Proposition 1.49. *the maps*

$$\omega_1 \colon P(X, A) \to X, \quad \omega_1 \colon P(X, A, B) \to B$$

are fibrations.

Note in conclusion that the conditions of Proposition 1.40 not only sufficient, but also necessary. More precisely,

Proposition 1.50. *if the space X is locally compact and Hausdorff, the pair (X, A) is closed and for each space Y the restriction map $Y^X \to Y^A$ is a fibration, then the pair (X, A) is a cofibration.*

However, in practice, this fact does not have to be used, therefore we will not prove it.

Appendix

1.A The axiom of weak extension of covering homotopy

By definition, the map $p\colon E \to B$ is a fibration if it satisfies the axiom of covering homotopy, which is expressed by the diagramme

$$\begin{array}{ccc} X & \xrightarrow{\overline{f}} & E \\ {\scriptstyle\sigma_0}\downarrow & \overset{\overline{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B \end{array} \tag{1.51}$$

Now let $A \subset X$, and let the covering homotopy $\overline{F}$ already be built on $A \times I \subset X \times I$. Is it possible to extend it to a covering homotopy on all $X \times I$?

Together with this map $\overline{f}\colon X \to E$ (considered as a map $X \times 0 \to E$), this homotopy $A \times I \to E$ constitutes some map $\overline{A} \to E$, where $\overline{A} = (X \times 0) \cup (A \times I)$, which we will still denote by $\overline{f}$ and which closes the commutative diagramme

$$\begin{array}{ccc} \widetilde{A} = (X \times 0) \cup (A \times I) & \xrightarrow{\overline{f}} & E \\ {\scriptstyle\widetilde{\sigma}_0}\downarrow & \overset{\overline{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B \end{array} \tag{1.52}$$

where $\widetilde{\sigma}_0$ is the inclusion. Our question now boils down to the following question: is there a continuous map $\overline{F}\colon X \times I \to E$, indicated in this diagramme by the dotted arrow?

Of course, here it is assumed that the map $\overline{f}\colon \widetilde{A} \to E$ is continuous (a sufficient condition for which is the closeness of the subspace A).

Surprisingly, the requirements to map $p\colon E \to B$ be a fibration, the answer to the question about the existence of a homotopy $\overline{F}$ turned out to be affirmative for any $\overline{f}$ and F with not overly restrictive conditions on the pair (X, A). For example, as we will show in the next lecture, it is enough for a pair of (X, A) be a closed cofibration.

Now we will consider sufficient conditions of a somewhat similar plan connecting the existence of a homotopy $\overline{F}$ with the possibility of homotopy extension $A \times I \to E$ for at least some neighbourhood of the subspace A.

A subset U of a space X is called a *functional neighbourhood* (or *rim*) of a subset A if $A \subset U$ and there exists such a continuous function $\varphi\colon X \to I$ that $\varphi|_A = 0$, $\varphi|_{X\setminus U} = 1$. Every functional neighbourhood is a (not necessarily open) neighbourhood of the closure $\overline{A}$ of the set A, and if the space X is normal, then by applying Urysohn's lemma (see Lecture 0 Section 0.A) any neighbourhood U of the set $\overline{A}$ is a functional neighbourhood of the set A.

We emphasise that we do not require the functional neighbourhood to be an open set. Therefore, any subset containing a functional neighbourhood U of the set A will also be a functional neighbourhood of the set A.

Note that

Proposition 1.53. *for any continuous function $\varphi\colon X \to I$ and each $t_0 \in I$, $t_0 \neq 0$, the set $V = \varphi^{-1}([0, t_0]) \subset U$ is a functional neighbourhood of the set $A = \varphi^{-1}(0)$.*

Proof. Indeed, the function

$$\psi(x) = \min\left(1, \frac{\varphi(x)}{t_0}\right), \quad x \in X,$$

is continuous and has the property that $\psi = 0$ by A and ψ =1 outside of V □

Since the functional neighbourhood V is closed, it follows, in particular, that any functional neighbourhood U contains a closed functional neighbourhood.

If for the subspace A from Diagramme (1.52) we are given some of its functional neighbourhood U, then we can consider the diagramme

$$\begin{array}{ccc} \widetilde{A}' & \xrightarrow{\overline{f}'} & E \\ {\scriptstyle \widetilde{\sigma}_0'}\big\downarrow & \overset{\overline{F}'}{\nearrow} & \big\downarrow{\scriptstyle p} \\ U \times I & \xrightarrow[F']{} & B \end{array} \tag{1.54}$$

where $\widetilde{A}' = (U \times 0) \cup (A \times I)$, $\overline{f}' = \overline{f}|_{\widetilde{A}'}$, $\widetilde{\sigma}_0' = \widetilde{\sigma}_0|_{\widetilde{A}'}$, and $F' = F|_{U\times I}$. The map $\overline{F}'$ of this diagramme is nothing more than a covering homotopy, which is an extension of the homotopy $A \times I \to E$ over U.

Definition 1.55. We will say that the map $p\colon E \to B$ satisfies the *axiom of weak covering homotopy extension* (in short, the *axiom* **WCHE**), if a map $\overline{F}$ exists for each diagramme (1.52), for which it is possible to find such a functional area U of the subspace A that for the corresponding Diagramme (1.54) there is a map $\overline{F}'$.

It is clear that the functional neighbourhood U we can always assume closed here.

Proposition 1.56. *A map $p\colon E \to B$ if satisfies the axiom of a weak extension of the covering homotopy if and only if it satisfies the axiom of the covering homotopy, i.e. it is a fibration.*

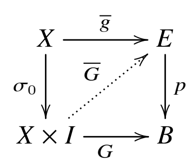
Proof. Since with $A = \varnothing$ the axiom **WCHE** passes into the axiom of CH (it is enough to put $U = \varnothing$), only the statement needs proof that any fibration satisfies the axiom **WCHE**. In other words, we need to show that for Diagramme (1.52) there exists a homotopy $\overline{F}$ if the map $p\colon E \to B$ is a fibration and there is such a closed functional neighbourhood U of the subspace A that for the corresponding Diagramme (1.54) there exists a homotopy $\overline{F}'$.

Let $\varphi\colon X \to I$ be such a continuous function that $\varphi = 0$ on A and $\varphi = 1$ outside U, and let

$$G(x,t) = F(x, \min(1, 1-\varphi(x)+t)), \quad (x,t) \in X \times I$$

$$\overline{g}(x) = \begin{cases} f(x), & \text{if} \quad \varpi(x) = 1 \\ \overline{F}'(x, 1-\varphi(x)), & \text{if} \quad x \in U \end{cases}$$

(the map $\overline{g}\colon X \to E$ is well-defined; it is continuous since the sets U and $\varphi^{-1}(1)$ are closed). Automatic verification shows that the diagramme



commutes. Since the map $p\colon E \to B$ is by condition a fibration, there is a closing homotopy $\overline{G}\colon X \times I \to E$ for this diagramme. Then the formula

$$\overline{F}(x,t) = \begin{cases} \overline{F}'(x,t), & \text{if} \quad x \in U, 0 \le t \le 1-\varphi(x), \\ \overline{G}(x, t-1+\varphi), & \text{if} \quad 1+\varphi(x) \le t \le 1, \end{cases}$$

will well determine the homotopy $\overline{F}\colon X\times \to$, that closes Diagramme (1.52). □

1.B Weak maps

Let $A \subset U \subset X$. A section $s\colon A \to E$ to the map $g\colon E \to X$ over A is called *continued on* U if there exists a section $\overline{s}\colon U \to E$ map g over U such that $\overline{s}|_A = s$.

By analogy with the axiom **WCHE**, we will say that the map $g\colon E \to X$ satisfies the *axiom of weak section extension* (in short, the *axiom* **WSE**) if for any subspace $A \subset X$ each section $s\colon A \to E$ of the map g over A, continued on some functional neighbourhood U of the subspace A, is continued on all X.

Maps satisfying the axiom **WSE**, we will call *weak maps* for brevity.

Applying the axiom **WSE** to $A = \varnothing$, we get, in particular, that

Proposition 1.57. *any weak map has a cross section.*

By applying Proposition 1.36, it follows that

Proposition 1.58. *if for the map $p\colon E \to T$o the natural map*

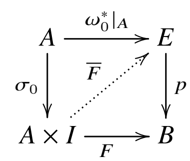
$$g\colon E^I \to \mathrm{Cocyl}(p), \quad v \mapsto (p \circ v, v(0))$$

is weak, then the map $p\colon E \to B$ is a fibration.

The converse is also true, i.e.

Proposition 1.59. *for any fibration $p\colon E \to B$, the map $q\colon E^I \to \mathrm{Cocyl}(p)$ is weak.*

Proof. Indeed, for any subset of $A \subset \mathrm{Cocyl}(p)$ there is a commutative diagramme



the map F which maps an arbitrary point $((u, e), t)$ of the product $A \times I$ where $t \in I$, $u \in B^I$, $e \in E$ and $u(0) = 0$, to the point $u(t) \in B$.

In this case, the plugging map $\overline{F}\colon A \times I \to E$, considered by applying the exponential law,as a map $A \to E^I$, will be nothing more than a section of the map q over A. Therefore the axiom **WSE** for the map q is a consequence of the axiom **WCHE** (See 1.55.) for the map p. □

This explains our interest in weak maps. The theory of these maps was constructed by Dold. We will now present its new results.

1.C Two lemmas about weak maps

Lemma 1.60. *For any weak map $q\colon E \to X$ and any open set $U \subset X$, the complement of which is functionally distinguished, the map*

$$q_U = q|_{q^{-1}(U)}\colon q^{-1}(U) \to U$$

is also weak.

Caveat: the proof is rather involved.

Proof. Let $A \subset U$, and let s be a section of the map q_U over A that, there exists a continuous function $\varphi\colon U \to I$, an open set $V \subset U$ and a section $\overline{s}\colon V \to E$ which maps q_U over V such that $\varphi = 0$ on A, $\varphi = 1$ outside U and $\overline{s}|_A = s$. We need to prove that there exists a section s^* of the map q_U on all U such that $s^*|_A = s$. At the same time, without loss of generality, we can obviously assume that $A = \varphi^{-1}(0)$ and $V = U \setminus \varphi^{-1}(1)$.

By condition there is a continuous function $\omega\colon X \to I$ such that $U = X \setminus \omega^{-1}(1)$. For any $n > 2$ we put

$$W_n = U_{n+1} \cap V_n \quad \text{where}$$
$$U_n = X \setminus \omega^{-1}\left(\left[\frac{n-1}{n}, 1\right]\right), \quad V_n = U \setminus \varphi^{-1}\left(\left[\frac{1}{n}, 1\right]\right).$$

It is clear that $U_n \subset U_{n+1}$, $U_n \subset U$ and $W_n \subset U$. Therefore, in particular, for each $n \geq 2$, the section $\overline{s}$ is defined on the W_n.

It turns out that for any $n \geq 2$ there exists a section $s_n\colon X \to E$ of the map q such that:

a) if $x \in U_n$, then $s_{n+1}(x) = s_n(x)$;

b) if $x \in W_n$, then $s_n(x) = \overline{s}(x)$.

Indeed; a direct check shows that the formulae

$$\psi(x) = \begin{cases} \min(1, \max(0, 1 - 6(1-\varphi(x))(1-\omega(x))), & \text{if } \ x \in U, \ \text{ i. e. } \ 0 \leq \omega(x) \leq 1, \\ 1, & \text{if } \ x \notin U, \ \text{ i. e. } \ \omega(x) = 1, \end{cases}$$

defines a continuous function $\psi\colon X \to I$ such that $X \setminus \psi^{-1}(1) = U \setminus \varphi^{-1}(1) = V$ and $W_2 \subset \psi^{-1}(0)$. Therefore, an open (not only in U, but also in X) set V is, in X, a functional neighbourhood of the set W_2. Therefore due to the weakness of the map q, there is a section s_2, coinciding on W_2 with the section $\overline{s}$. Thus, the existence of the section s_n for $n = 2$ is fully proved (note that the condition a) is meaningless for $n = 2$).

Reasoning by induction, let us now assume that for some $n \geq 2$ the section s_n satisfying the conditions a) and b) has already been constructed. It is easy to see that there are numerical functions $\alpha_n, \beta_n\colon I \to I$ such that

$$\frac{n-1}{n} \leq \alpha_n(t) < \beta_n < 1 \quad \text{for every} \quad t \in I,$$
$$\beta_n(t) \leq \frac{n}{n+1}, \quad \text{if} \quad t \geq \frac{1}{n},$$
$$\alpha_n(t) \geq \frac{n+1}{n+2}, \quad \text{if} \quad t \leq \frac{1}{n+1}.$$

For example, you can put

$$\alpha_n(t) = \begin{cases} \frac{n+1}{n+2} & \text{if } \ 0 \leq t \leq \frac{1}{n+1}, \\ \frac{n+3-2(n+1)t}{n+2} & \text{if } \ \frac{1}{n+1} \leq t \leq \frac{1}{n}, \\ \frac{n-1}{n} & \text{if } \ \frac{1}{n} \leq t \leq 1, \end{cases}$$

$$\beta_n(t) = \begin{cases} \frac{n+2}{n+3} & \text{if } \ 0 \leq t \leq \frac{1}{n+1}, \\ \frac{n^2+5n+2-2(n+1)t}{(n+1)(n+3)} & \text{if } \ \frac{1}{n+1} \leq t \leq \frac{1}{n}, \\ \frac{n}{n+1} & \text{if } \ \frac{1}{n} \leq t \leq 1, \end{cases}$$

Let T_n be a subset of the set U consisting of points $x \in X$ for which $\omega(x) < \beta_n(\varphi(x))$. By condition, if $\varphi(x) \geq \frac{1}{n}$, then $\beta_n(\varphi(x)) \leq \frac{n}{n+1}$, and therefore if, in addition, $x \in T_n$, then $\omega(x) < \frac{n}{n+1}$, i.e. $x \in U_{n+1}$. This proves that $T_n \subset U_{n+1} \cup V_n$. Since $U_n \cap V_n = W_n$ and $s_n = \overline{s}$ on W_n, it follows that the formula

$$t_n(x) = \begin{cases} s_n(x), & \text{if} \quad x \in U_{n+1}, \quad \text{i. e.} \quad \omega(x) < \frac{n}{n+1}, \\ \overline{s}(x), & \text{if} \quad x \in V_n, \quad \text{i. e.} \quad \varphi(x) < \frac{1}{n} \end{cases}$$

On the other hand, the formula

$$\varphi_n(x) = \begin{cases} 0, & \text{if} \quad \omega(x) \leq \alpha_n(\varphi(x)), \\ \frac{\alpha_n(\varphi(x)) - \omega(x)}{\alpha_n(\varphi(x)) - \beta_n(\varphi(x))}, & \text{if} \quad \alpha_n(\varphi(x)) \leq \omega(x) \leq \beta_n(\varphi(x)), \\ 1, & \text{if} \quad \beta_n(\varphi(x)) \leq \omega(x) \quad \text{or} \quad \omega(x) = 1 \quad \text{i. e.} \quad x \notin U \end{cases}$$

defines a continuous function $\varphi_n \colon X \to I$ such that $\varphi_n = 1$ outside T_n and $\varphi_n = 0$ on the set X_n of all points $x \in U$ for which $\omega(x) < \alpha_n(\varphi(x))$. Therefore, T_n is a functional neighbourhood of the set X_n in X, and since the section t_n is defined on T_n, then due to the weakness of the map q, there is a section $s_{n+1} \colon X \to E$, coinciding with the section t_n on X_n.

If $x \in U_n$, i.e. $\omega(x) < \frac{n-1}{n}$, then $\omega(x) < \alpha_n(\varphi(x))$, and this means $x \in X_n$ Moreover, if $x \in U_{n+1}$, then $s_{n+1}(x) = t_n(x) = s_n(x)$. Similarly, if $x \in W_{n+1}$ and thus $\varphi(x) < \frac{1}{n+1}$, $\omega(x) < \frac{n+1}{n+2}$, then $\omega(x) < \alpha_n(\varphi(x))$, i.e., $x \in X_n$. And since in addition, $x \in V_n$, then $s_{n+1}(x) = t_n(x) = \overline{s}(x)$.

Thus, the existence of sections of s_n is proved for $n \geq 2$.

It is clear that

$$\cup_{n=2}^{\infty} U_n = U.$$

Therefore, by applying condition a), the formula

$$s^*(x) = s_n(x), \quad \text{if} \quad x \in U_n,$$

well defines on U a certain section $s^* \colon U \to E$ of the map q (or, equivalently, the map q_U). In addition, since$A \subset V_n$ for any $n \geq 2$, for each point $x \in A$, there exists an n such that $x \in U_n$, and therefore

$$s^*(x) = s_n(x) = \overline{s}(x) = s(x), \quad x \in A.$$

Therefore $s^*|_A = s$. □

By applying Lemma 1.60, if the map $q \colon E \to X$ is weak, then for any open cover of $\{U_\alpha\}$ of the space X all the maps $q_\alpha = q_{U_\alpha} \colon q^{-1}(U_\alpha) \to U_\alpha$ are also weak. It turns out that under very weak general methodological conditions for covering $\{U_\alpha\}$, the converse is also true, i.e. the map q is weak if all the map q_α are weak.

Let X be an arbitrary topological space.

A family $\{\varphi_\alpha\}$ of continuous functions $\varphi_\alpha \colon X \to I$ is called *locally finite* if for any point $x \in X$ there exists a neighbourhood $U \subset X$, in which only a finite number of functions φ_α are nonzero. A locally finite family of functions $\{\varphi_\alpha\}$ is called a *partition of unity* if for any point $x \in X$ the equality

$$\sum_\alpha \varphi_\alpha(x) = 1$$

holds (note that, due to the condition of local finiteness, this sum has a definite meaning). A partition of the unity $\{\varphi_\alpha\}$ is called *subordinate* to the open cover $\}U_\alpha\}$ (with the same set of indices) if $\varphi_\alpha = 0$ outside U_α for any α. (Note that there is another, more restrictive definition in the literature, which requires that U_α contains not only the set, where $\varphi_\alpha \neq 0$, but also its closure.) An open cover $\{U_\alpha\}$ is called *numerable*, if there is a partition of the unity $\{\varphi_\alpha\}$ subordinate to it.

Remark 1.61. It is easy to see that a Hausdorff space is paracompact if and only if each of its open covers is enumerable. Therefore, the condition of the enumerability of coverings is one of the variants of the paracompact condition.

It is obvious that for any continuous map $f : Y \to X$ and any open cover $\{U_\alpha\}$ of the space X, the set $f^{-1}(U_\alpha)$ constitute an open cover of the space Y. At the same time,

Proposition 1.62. *if the coverage is $\{U_\alpha\}$ is numerable, then the coverage of $\{f^{-1}(U_\alpha)\}$ will also be numerable.*

Proof. Indeed, it is clear that for the partitions of unity $\{\varphi_\alpha\}$ subordinate to the covering of $\{U_{alpha}\}$, functions $\varphi_\alpha \circ f : Y \to I$ will constitute a partitions of unity subordinate to the covering $\{f^{-1}(U_\alpha)\}$. □

We will call the covering $\{f^{-1}(U_\alpha)\}$ the covering preimage of $\{U_\alpha\}$ for the map f.

Lemma 1.63. *Let $q : E \to X$ be a map. If there exists a numerable covering $\{U_a\}$ for the space X such that that all maps*

$$q_\alpha : q^{-1}(U_\alpha) \to U_\alpha$$

are weak, then the map $q : E \to X$ is also weak.

The proof is involved.

Proof. Let $A \subset X$ and $s_0 : A \to E$ be a section of the map q over A such that there exists a continuous function $\varphi : X \to I$, an open set $V \subset A$ and a section $\overline{s}_0 : V \to E$ for the map q over V, such that $\varphi = 0$ on A, $\varphi = 1$ outside V and $\overline{s}_0|_A = s_0$.

Consider an arbitrary partition of unity $\{\varphi_\alpha, \alpha \in A\}$, subordinate to the covering $\{U_a\}$. Assuming that the symbol 0 is not contained in the set of indices A, we put

$$\psi_0 = 1 - \varphi, \quad \psi_\alpha = \varphi\psi_\alpha, \quad \alpha \in A.$$

It is clear that the family $\{\psi_\beta\}$, where $\beta \in A \cup \{0\}$ is a partition of unity. Due to the local finiteness of the family $\{\psi_\beta\}$ for any subset of $B \subset A \cup \{0\}$ a continuous function

$$\psi_B : x \mapsto \sum_{\beta \in B} \psi_\beta(x), \quad x \in X$$

taking values in I is defined. Let

$$V_B = X \setminus \psi_B^{-1}(0).$$

Is it clear that $A \subset V_B$ if $0 \in B$, and $A \cap B_B = \varnothing$ if $0 \notin B$. In addition, $A \subset V_0 \subset V$.

Let's now consider the set S of all .pairs of the type (B, s), where B is an arbitrary subset of $A \cup \{0\}$, containing element 0, with s is a section over V_B such that $s|_A = s_0$. Since $(\{0\}, \bar{s}_0|_{V_{\{0\}}}) \in S$, the set S is not empty.

We introduce a partial ordering to the set S, assuming that $(B, s) \leq (B^{'}, s^{'})$, if $B \subset B^{'}$ (and, therefore, $V_B \subset V_{B'}$) and $s^{'}(x) = s(x)$ if $\varphi_{B'}(x) = \varphi'_B(x)$ (i.e. $\psi_B = 0$ for any $\beta \in B^{'} \setminus B$).

Let K be an arbitrary chain in S. Putting,

$$\Gamma = \cup_{(B,s) \in K} B,$$

for any open set $W \subset V_\Gamma$, we denote by Γ_A a subset of the set Γ consisting of all indices $\beta \in \Gamma$ for which the function ψ_B is nonzero at least at one point of W. Let $\{W\}$ be the family of all open sets $W \in V_\Gamma$ for which the set Γ_W is finite. Due to the local finiteness of the partition of unity $\{\psi_B\}$, the family $\{W\}$ covers the set V_Γ.

For each set $W \in \{W\}$ we consider a subset/ K_W of a chain K consisting of pairs (B, s) such that $\Gamma_W \subset B$. Due to the finiteness of the set Γ_W, for any pair $(B, s) \in K$ there exists a pair $(B^{'}, s^{'}) \in K_W$ such that $(B, s) \leq (B^{'}, s^{'})$.

If $(B, s) \in K_W$, then $W \subset V_B$, and if (B, s), $(B^{'}, s^{'}) \in K_W$, then $s = s^{'}$ on W. Therefore , the formula

$$t(x) = s(x), \quad x \in V_\Gamma,$$

where s is an arbitrary section of the q map for which there exists a set of indices $B \subset \Gamma$ and an open set $W \in \{W\}$ containing a point x that $(B, s) \in K_W$, well defines on V_γ the section $t : V_1 \to E$ to the maps q, which obviously has the property that $(\Gamma, t) \in S$.

If $(B, s) \in K$, then by the construction $B \in \Gamma$. Let x be a point of V_B such that $\psi_\gamma = 0$ for all $\gamma \in \Gamma \setminus B$. Choosing an arbitrary neighbourhood W of the point x belonging to the family $\{W\}$, consider in K_W an arbitrary element $(B^{'}, s^{'}) \leq (B, s)$. Since $\psi_\gamma(x) = 0$ at $\gamma \in B^{'} \setminus B$, then follows $s(x) = s^{'}(x)$. But by the construction $s^{'}(x) = t(x)$. Therefore, $s(x) = t(x)$.

This proves that $(B, s) \leq (\Gamma, t)$ for any element $(B, s) \in K$, i.e. that the element (Γ, t) is the upper bound of the chain K.

Thus, in the partially ordered set S, any chain has an upper bound. Therefore, by applying Zorn's Lemma, the set S has a maximum element (B, s).

Let $B \neq A \cup \{0\}$, and let $\alpha \in A \setminus B$. The formula

$$\psi(x) = \max\left(0, \frac{\psi_\alpha(x) - \psi_\beta(x)}{\psi_\alpha(x)}\right), \quad \psi_\alpha(x) \neq 0,$$

defines on the set $V_\alpha = X \setminus \psi_\alpha^{-1}(0)$ a continuous function $\psi \colon V_\alpha to I$, which obviously has the property $V_\alpha \setminus \psi^{-1}(1) \subset V_B$. In particular, we see that the section s is defined on the set $V_\alpha \setminus \psi^{-1}(1)$. Since this set is by definition a functional neighbourhood of the set $\psi^{-1}(0)$, we thus obtain that the section $s|_{\psi^{-1}(0)}$ of the map $q_{V_\alpha} \colon q^{-1}(V_\alpha) \to V_\alpha \setminus \psi^{-1}(1)$ satisfies the conditions of the axiom **WSE**. But it is clear that $V_\alpha \subset U_\alpha$, and therefore $q_{V_\alpha} = q_\alpha|_{q^{-1}(V_\alpha)}$. Since the map q_α is weak by condition, therefore the map q_{V_α} is also weak according to Lemma 1.60. Therefore, for the section $s|_{\psi^{-1}(0)}$ there is a section $s' \colon V_\alpha \to E$ for the map q_{V_α} over V such that $a' = s$ on $\psi^{-1}(0)$.

Now let $\Gamma = B \cup \{\alpha\}$. Putting for any point $x \in V_\Gamma = V_B \cup V_\alpha$

$$t(x) = \begin{cases} s(x), & \text{if } \psi_\alpha(x) \leq \psi_\beta(x), \\ s'(x), & \text{if } \psi_\alpha(x) \geq \psi_\beta(x), \end{cases}$$

we will obviously get a cross section $t \colon V_\Gamma \to E$ of the map q over V_Γ such that $(\Gamma, t) \in S$ and $(B, s) < (\Gamma, t)$. Since this contradicts the maximality of the element $(B, , s)$, it is thus proved that $B = A \cup \{0\}$ and, consequently, that $V_B = X$.

Thus, the section s is defined on all X and coincides on A with the section s_0. Hence, the map q is weak. □

1.D Lemma on coverings of space $X \times I$

To apply the results obtained to fibrations, we need the following technical lemma, which is useful in other matters as well.

Lemma 1.64. *For each numerable covering $\{U_\alpha, \alpha \in A\}$ of the space $X \times I$, we can find a numerable covering $\{V_\beta, \beta \in B\}$ of the space X, which has the property that for any $\beta \in B$ there exists a positive number $\epsilon_\beta > 0$ such that for any segment $J \subset I$ of length $\leq \epsilon_\beta$ there is such a $\alpha \in A$ that $V_\beta \times J \subset U_\alpha$.*

Let us prove the following classical lemma beforehand.

Lemma 1.65 (Lebesgue lemma)**.** *For an arbitrary cover U_α of a compact metric space X, there exists a positive number $\varepsilon > 0$ such that any subset of $K \subset X$ of a diameter smaller than ε is contained in some element of the coverage $\{U_\alpha\}$.*

Proof. If such a number ε does not exist, then for any $n > 0$ in X there will be a subset of K_n of a diameter smaller than $1/n$ that is not contained in any element of the coverage of $\{U_\alpha\}$. Let $x_n \in K_n$. Since the space X is compact by condition, there exists a point $x_0 \in X$, any neighbourhood of which contains infinitely many points x_n. Let $x_0 \in U_{\alpha_0}$, and let d be the distance from x_0 to

$X \setminus U_{\alpha_0}$. If $n > 2/d$ and $\rho(x_0, x_n) < d/2$, then for any point $x \in K_n$ there is an inequality

$$\rho(x_0, x) \leq \rho(x_0, x_n) + \rho(x_n, x) \leq \frac{d}{2} + \frac{1}{n} < d,$$

showing that, contrary to the assumption, $K_n \subset U_{\alpha_0}$. The resulting contradiction proves the lemma. □

The upper bound of the numbers ε is called the *Lebesgue number* of the covering $\{U_\alpha\}$.

Now we can prove Lemma 1.64.

Proof. (of Lemma 1.64) By condition, there is a partition of unity $\{\varphi_\alpha\}$, subordinate to the covering $\{U_\alpha\}$, i.e. such that $X \setminus \varphi_\alpha^{-1}(0) \subset U_\alpha$ for any $\alpha \in A$. In order not to introduce new letters, we assume that $X \setminus \varphi_\alpha^{-1}(0) = U_\alpha$. It is clear that this assumption does not limit generality.

Let B be the set of all finite sequences of elements of the set A. Thus, each element of $\beta \in B$ has the form $(\alpha_1, \ldots, \alpha_r)$, where $r \leq 0$. We will denote the length of the sequence β with the symbol $|\beta|$.

To each element $\beta = (\alpha_1, \ldots c, \alpha_r) \in B$ we will associate the continuous function $\psi_\beta \colon X \to I$ defined by the formula

$$\psi_\beta(x) = \prod_{i=1}^{r} \min_{t_i \in I_i} \varphi_{\alpha_i}(X, t_i), \quad \text{where} \quad i_i = \left[\frac{i-1}{r+1}, \frac{i+1}{r+1}\right].$$

Let $V_\beta = X \setminus \psi_\beta^{-1}(0)$. It is clear that $x \in \beta$ if and only if $\varphi_\alpha \neq 0$ on $\{x\} \times I_i$ for each $i = 1, \ldots, r$. In particular, we see that $V_\beta \times I_i \subset U_{\alpha_i}$, $i = 1, \ldots, r$.

Since any segment $J \subset I$ of length less than $\frac{1}{i+1}$ is contained in one of the segments I_i, $i = 1, \ldots, r$, it follows that for $\varepsilon_\beta = \frac{1}{|\beta|+1}$ the family $\{V_\beta\}$ has the property claimed by Lemma 1.64. Therefore, to prove Lemma 1.64, we only need to show that the family $\{V_\beta\}$ is a numerable covering of the space X.

By definition of the product topology, each point $(x, t) \in X \times I$ has a "rectangular" neighbourhood $U_{x,t} \times V_{x,t}$ contained in some element of the covering $\{U_\alpha\}$. Here $U_{x,t}$ is some neighbourhood in X of the point $x \in X$, and $V_{x,t}$ is some neighbourhood in I of the point $t \in I$. For any point $x \in X$ all neighbourhoods of the form $V_{x,t}$ form an open covering of the segment I. Let r be an integer such that the length $\frac{2}{r+1}$ of the corresponding segments I_i, $i = 1, \ldots, r$, does not exceed the Lebesgue number of this covering. Then every set of the form $\{x\} \times I_i$, $i = l, \ldots, r$, will be contained in some neighbourhood $U_{x,t} \times V_{x,t}$ and therefore in some element $U_{\alpha_i^{(0}}$ of the covering $\{U_\alpha\}$. Therefore, $\varphi_{\alpha_i^{(0)}} \neq 0$ on $\{x\} \times I_i$, and this means $x \in V_{\beta^{(0)}}$ where $\beta^{(0)} = (\alpha_1^{(0)}, \ldots, \alpha_r^{(0)})$. This proves that the family $\{V_\beta, \beta \in B\}$ is a covering of the space X.

In order not to introduce a new notation, we can assume that the neighbourhood $U_{x,t} \times V_{x,t}$ also has the property that each of them intersects only with a finite number of elements of the covering $\{U_\alpha\}$). Then by choosing for each point $x \in X$ from the covering $\{V_{x,t}\}$ of the segment I (which is a compact space) a finite subcovering $\{V_{x,t_1}, \dots, V_{x,t_n}\}$ and putting

$$U = U_{x,t_1} \cap \dots \cap U_{x,t_n},$$

we will get a neighbourhood U of the point x in the space X such that $U_\alpha \cap (U \times I) \neq \varnothing$ only for a finite number of indices $\alpha \in A$. Hence, in sequences $\beta \in B$ for which $U \cap (U \times I) \neq \varnothing$, only a finite number of different indices can participate $\alpha \in A$ and, therefore, for any $r > 0$ such sequences with $|\beta| < r$ will be of only a finite number. Therefore , the formula

$$\lambda_r(x) = \sum_{|\beta|<r} \psi_\beta(x), \quad x \in X,$$

will define on X some continuous non-negative function $\lambda_r \colon X \to \mathbb{R}$ (for $r = 1$ we, by definition, consider $\lambda_1 = 0$). Let

$$\overline{\psi}_\beta(x) = \max(0, \psi_\beta(x) - r\lambda_r(x)), \quad \text{where} \quad r = |\beta|, \beta \in B.$$

It is clear that $\overline{\psi}_\beta(x) = 0$ i$\psi_\beta(x) = 0$. Therefore, the sequences $\beta \in B$ and $|\beta|$ for which $\overline{\psi}_\beta(x) \neq 0$ on U, are also of a finite number.

Having now chosen the sequence for the point $x \in X$ $\beta_0 \in B$ with $x \in V_{\beta_0}$, of length $|\beta_0| = r_0$ which has the smallest possible value, consider an arbitrary number $r > r_0$ for which $\psi_{\beta_0} > \frac{1}{r}$. Then $r\lambda_r > 1$, and therefore $r\lambda_r > 1$ in some neighbourhood of the point x (which we can consider coinciding with the neighbourhood U constructed above). Thus, for any β with $|\beta| = r$ in the neighbourhood of U, the equality $\overline{\psi}_\beta = 0$ holds. Therefore, $|\beta| \neq 0$ on U only for a finite number of sequences of $\beta \in B$. Thus, it is proved that the family $\{\overline{\psi}_\beta\}$ is locally finite, and therefore the formula

$$\overline{\psi}(x) = \sum_{\beta \in B} \overline{\psi}_\beta(x), \quad x \in X,$$

well defines on X some continuous function $\overline{\psi} \colon X \to \mathbb{R}$.

Since by the condition the length of $r_0 = |\beta|$ is the sequence of β_0 for which $x \in V_\beta$, i.e. $\psi_{\beta_0} \neq 0$, has the smallest possible value, then $\psi_\beta = 0$ for $|\beta| < r_0$, and, therefore, $\lambda_{r_0} = 0$. Therefore, $\overline{\psi}_{\beta_0}(x) = \psi_{\beta_0}(x) \neq 0$ and, therefore, $\overline{\psi}(x) \neq O$.

This proves that the functions $\overline{\psi}_\beta / \overline{\psi}$ are defined everywhere on X. Since they obviously constitute a partition of unity subordinate to the covering of $\{V_\beta\}$, Lemma 1.64 is thereby completely proved. □

Corollary 1.66. *For each numerable covering $\{U_\alpha, \alpha \in A\}$ of the space B there can be found a numerable covering $\{V_\beta, \beta \in B$ of the space B^I, which has the property that for any $\beta \in B$ there exists a positive number $\varepsilon_\beta > 0$ such that for an arbitrary segment $J \subset I$ of length $\leq \varepsilon_\beta$, there is an $\alpha = \alpha(\beta, J) \in A$ such that $u(J) \subset U_\alpha$ for each path $u \in V_\beta$.*

Proof. It is enough to apply Lemma 1.64 to the numerable covering of the space $B^I \times I$, which is the preimage of the covering $\{U_\alpha\}$ by the evaluation map

$$\omega\colon B^I \times I \to X, \quad (u,t) \mapsto u(t).$$

□

Using the notation introduced in the Section 0.C of Appendix to Lecture 0 for basic sets of compactly open topology, we can write the properties of sets Vp claimed in this consequence as a formula

$$V_\beta \subset \cap_J \mathscr{W}(J, U_{\alpha(\beta,J)}),$$

where the intersection is taken over all segments $J \subset I$ of length $\leq \varepsilon_\beta$.

Now let n_β be an integer such that $n_\beta \varepsilon_\beta > 1$, and let $I_{\beta,i}$, be a segment $\left[\frac{i-1}{n_\beta}, \frac{i}{n_\beta}\right]$, where $i = 1, 2, \dots, n_\beta$. Then $V_\beta \subset W_\beta$, where

$$W_\beta = \cap_{i=1}^{n_\beta} \mathscr{W}(I_{\beta,i}, U_{\alpha_i(\beta)}), \quad \alpha_i(\beta) = \alpha(\beta, I_{\beta,i}).$$

Sets W_β are open in the space of B^I and make up the numerable covering of this space (the partition of the unity subordinate to the covering of $\{V_\beta\}$, will obviously be subordinate to the covering $\{W_\beta\}$). Denoting them again by V_β, we get

Corollary 1.67. *For each numerable covering $\{U_\alpha, \alpha \in A\}$ of the space B there can be found a numerable covering $\{V_\beta, \beta \in B\}$ of the space B^I, having the the property that for any index β in B there are indices $\alpha_1 = \alpha_1(\beta), \dots, \alpha_{n_\beta} = \alpha_{n_\beta}(\beta)$ in A such that $u \in V_\beta$ if and only if $u(I_{\beta,i}) \subset U_{\alpha_{\alpha_i(\beta)}}$ for each $i = 1, \dots, n_\beta$.*

Here it is convenient to introduce into consideration the parts of the paths $n \in V_\beta$ on the segments I_β, i.e. the paths u_i, $i = 1, \dots, n_\beta$, determined (taking into account the necessary parameter transformation) by the formula

$$u_i(t) = u\left(\frac{t+i-1}{n}\right), \quad 0 \leq t \leq 1.$$

Due to the inclusion of $u(I_{\beta,i}) \subset U_{\alpha_i(\beta)}$, we can consider the path u_i as a path in $U_{\alpha_i(\beta)}$. Hence, by matching the paths $u \in V_\beta$ with the sequence $(u_1, \dots, u_{n_\beta})$ of paths u_i we will get (as it is easy to see a homeomorphic) map of the set V_β to a subset of the product $U^I_{\alpha_{1(\beta)}} \times \cdots \times U^I_{\alpha_{n_\beta}(\beta)}$ consisting of sequences $(u_1, \dots, u_{n_\beta})$, $u_i \in U^I_{\alpha_o(\beta}$ such that $u_i(0) = u_{i1}(1)$ for any $i > 1$. Henceforth, in order not to introduce unnecessary notation, we will identify the paths $u \in V_\beta$ with the corresponding sequences $(u_1, \dots, u_{n_\beta})$.

1.E Dold's theorem

Let's now consider an arbitrary map $p\colon E \to B$, its cocylinder $\mathrm{Cocyl}(p) \subset E \times B^I$ and the map $q\colon E^I \to \mathrm{Cocyl}(p)$, $u \mapsto (u(0), p \circ u)$. Still assuming a given numerable covering $\{U_\alpha, \alpha \in A\}$ of the space B, we denote by W_β the preimages under the natural projection of $p_{\omega_0}\colon \operatorname{Cocyl}(p) \to B^I$, $(u,e) \mapsto u$ provided by Corollary 1.67 of the Lemma 1.64 for the sets of V_β. According to the identifications described above, we can consider as points of each set W_β sets of the form $(e, u_1, \ldots, u_{n_\beta})$, where u_i, $i = 1, \ldots, n_\beta$, are paths in $U_{\alpha_i,(\beta)}$ such that $u_i(0) = u_{i-1}(1)$ for $i > 1$, and e is a point from E (and in fact from $p^{-1}(U_{\alpha_i,(\beta)})$, such that $u_1(0) = p(e)$. Being a preimage of the numerable coverings under a continuous map, the family $\{W_\beta, \beta \in B\}$ is a numerable covering of the space $\operatorname{Cocyl}(p)$.

For every $\beta \in B$ we will introduce the map

$$q_B = q|_{q^{-1}(W_\beta)}\colon q^{-1}(W_\beta) \to W_\beta.$$

Here is the way that $v \in q^{-1}(W_\beta) \subset E^I$ is also convenient to split into parts, i.e. each such path is identified with the sequence $(v_1, \ldots, v_n)$ of paths

$$v_i(t) = v\left(\frac{t+i-1}{n_\beta}\right), \quad 0 \le t \le 1.$$

Because, as it is easy to see that $q^{-1}(W_\beta) = p_*^{-1}(V_\beta$, where $p_*\colon E^I \to B$ map $u \mapsto p \circ u$, the sequence $(v_1, \ldots c, v_{n_\beta})$ paths in E if and only if the path $v \in q^{-1}(W_\beta)$ satisfies $v_i(0) = v_{i-1}(1)$ for $i > 1$ and each path v_i, $i = 1, \ldots, n_\beta$, is a path in $p^{-1}(U_{\alpha_i(\beta)})$. The map of qf_β will be determined by the formula

$$q_\beta(v_1, \ldots, v_{n_\beta}) = (v_1(0), p \circ v_1, \ldots, p \circ v_{n_\beta}).$$

Accordingly, each section of $s\colon W_\beta \to q^{-1}(W_\beta)$ maps to q_β, we can identify the sequence $(s_1, \ldots, s_{n_\beta})$ with the continuous maps $s_i\colon W_\beta \to E^I$, having the following properties:

a) for any point $(e,u) \in W_\beta$ the path $v_i = s_i(e,u)$ is a path in $p^{-1}(U_{\alpha_i)\beta)})$;

b) if $u = (u_1, \ldots, u_{n_\beta})$, then $p \circ v_i = u_i$ for every $i = 1, \ldots, n_\beta$;

c) $v_1(0) = e$ and $v_i(0) = v_{i-1}(1)$ for $i > 1$.

However, the map s_i are more convenient to interpret as the homotopy $W_\beta \times I \to E$, or, more precisely, by applying the condition a), as the homotopy $W_\beta \times I \to p^{-1}(U_{\alpha_i})$, where $\alpha_i = \alpha_i(\beta)$. In this case, the conditions b) and c) will be equivalent to the commutativity of the diagramme

$$\begin{array}{ccc} W_\beta & \xrightarrow{\;s_{i-1}\circ\sigma_i\;} & p^{-1}(U_{\alpha_i}) \\ {\scriptstyle\sigma_0}\downarrow & \overset{s_i}{\nearrow} & \downarrow{\scriptstyle p_i} \\ W_\beta \times I & \xrightarrow[\;\varphi_i\;]{} & U_{\alpha_i} \end{array} \qquad i = 1, \ldots, n_\beta \tag{1.68}$$

where the homotopy $\varphi_i\colon W_\beta \times I \to U_{\alpha_i}$ is defined by the formula

$$\varphi_i((e,u),t) = u_i(t), \quad \text{for} \quad u = (u_1, \ldots, u_{n_\beta}),$$

and p_i represents a map $p_{\alpha_i}: p^{-1}(U_{\alpha_i}) \to U_{\alpha_i}$, induced by the map p. (By the map $s_{i-1} \circ \sigma_i$ for $i = 1$ here, of course, we mean the projection $\omega_0^*: (e,u) \mapsto e$.)

Lemma 1.69. *If all maps p_i, $i = 1, \dots, n_\beta$ satisfy the axiom* **WCHE** *(See 1.55.), then the map q_β is weak.*

Proof. Let $A \subset U \subset W_\beta$, and the set U bes a functional neighbourhood of the set A (in W_β), and let $\overline{s}: u \to q^{-1}(W_\beta)$ be an arbitrary section of the map q_β over U, i.e. there is a sequence of homotopies $\overline{s}_i: U \times I \to p^{-1}(U_{\alpha_i})$, for which the following diagramme commutes

$$\begin{array}{ccc} U & \xrightarrow{\overline{s}_{i-1}\circ\sigma_i} & p^{-1}(U_{\alpha_i}) \\ {\scriptstyle\sigma_0}\downarrow & \overset{\overline{s}_i}{\nearrow} & \downarrow{\scriptstyle p_i} \\ U \times I & \xrightarrow[\varphi_i]{} & U_{\alpha_i} \end{array}$$

We must prove that there exists a section s of the map q_β over the entire set W_β coinciding on A with the section $\overline{s}$, i.e. that there exist homotopies $s_i: W_\beta \times I \to p^{-1}(U_{\alpha_i})$, $i = 1, \dots, n_\beta$, for which Diagramme (1.68) are commutative and which on $A \times I$ coincide with homotopies $\overline{s}_i$.

Let φ be a continuous function such that $\varphi = 0$ on A and $\varphi = 1$ outside U, and let

$$U_i = \varphi^{-1}\left(\left[0, 1 - \frac{i}{n_\beta}\right]\right), \quad i = 1, \dots, n_\beta.$$

Then

$$A \subset U_{n_\beta} \subset \cdots \subset U_{i+1} \subset U_i \subset \cdots \subset u_1 \subset U,$$

moreover, each set U_i, $i < n_\beta$, will be a functional neighbourhood of the set U_{i+1}, and the set U will be a functional neighbourhood of the set U_i.

We will construct a homotopy s_i, by induction on i, for which we additionally require that for each $i = 1, \dots, n_\beta$ the following equality takes place

$$s_i|_{U_i \times I} = \overline{s}_i|_{U_i \times I};$$

in other words, we will replace each Diagramme (1.68) with a diagramme

$$\begin{array}{ccc} \widetilde{U_i} & \xrightarrow{\psi_i} & p^{-1}(U_{\alpha_i}) \\ {\scriptstyle\widetilde{\sigma}_0}\downarrow & \overset{s_i}{\nearrow} & \downarrow{\scriptstyle p_i} \\ W_\beta \times I & \xrightarrow[\varphi_i]{} & U_{\alpha_i} \end{array} \qquad i = 1, \dots, n_\beta, \tag{1.70}$$

where $\widetilde{U_i} = (W_\beta \times 0) \cup (U_i \times I)$, and ψ_i the map defined by the formula

$$\psi_i((e,u),t) = \begin{cases} s_{i-1}((e,u),t), & \text{if} \quad t = 0, \\ \overline{s}_i((e,u),t), & \text{if} \quad (e,u) \in U_i \end{cases}$$

(for $i = 1$, instead of $s_{i-1}((e,u),t)$, you should write e).

But Diagramme (1.70) has the form of Diagramme (1.52) with $X = W_\beta$, $A = U_i$, $\overline{f} = \psi_i$, $F = \varphi_i$, and $p = p_i$. Since in the corresponding Diagramme (1.54) (with $U = U_{i-1}$ for $i > 1$) the closing map $\overline{F}'$ obviously exists (it will be a restriction on $U_{i-1} \times I$ of the homotopy $\overline{s}_i$), then by applying the axiom **WCHE** (See 1.55.), the homotopy s_i in Diagramme (1.70) also exists. Thus, homotopies s, in a step by step fashion, are constructed for all $1 = 1, \ldots, n_\beta$. □

Now we can prove the main theorem of Dold.

Theorem 1.71. *If for the map $p\colon E \to B$ there exists a numerable covering $\{U_\alpha\}$ of the space B such that each map*

$$p_\alpha = p|_{p^{-1}(U_\alpha)}\colon p^{-1}(U_\alpha) \to U_\alpha$$

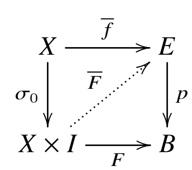
is a fibration, then the map $p\colon E \to B$ will also be a fibration.

Proof. According to Proposition 1.56, the fibration p_α satisfies the axiom **WCHE**, and therefore, according to Lemma 1.69, all maps $q_\beta\colon q^{-1}(W_\beta) \to W_\beta$ are weak, where, recall, W_β is a subset of the space Cocyl(p), which is the preimage of the projection $(u,e) \mapsto u$ of subsets of $V_\beta \subset B^I$ from Corollary 1.67 of Lemmas 1.64. But, as already noted above, the family $\{W_\beta\}$ is a numerable covering of the space Cocyl(p). Therefore, according to Lemma 1.63, the map $q\colon E^I \to \mathrm{Cocyl}(p)$ is weak, and hence the map p is a fibration. □

1.F Locally trivial fibrations

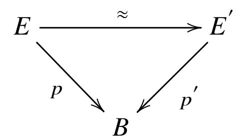
Dold's theorem finds an important application to the so-called locally trivial fibrations.

If in the diagramme below



the space E is the product $B \times Y$ of space B by some space Y (and, therefore, the map $\overline{f}\colon X \to E$ is given by the formula $\overline{f}(x) = (f_1(x), f_2(x))$ where $f_1\colon X \to B$, $f_2\colon X \to Y$), and $p\colon E \to B$ is a projection $(b,y) \mapsto b$, $b \in B$, $y \in Y$, then the homotopy $\overline{F}$ obviously exists (and is given by the formula $\overline{F}(x,t) = (F(x,t), f_2(x)))$. Hence, for any spaces B and Y, the projection $B \times Y \to B$ is a fibration.

We will call two maps $p\colon E \to B$ and $p^{'}\colon E^{'} \to B$ are *isomorphic* if a homeomorphism $E \xrightarrow{\approx}$ exists such that the diagramme



is commutative. It is clear that a map isomorphic to a fibration will itself be a fibration. Therefore, in particular, every map $p\colon E \to B$ isomorphic to the projection $B \times X \to B$ of the direct product will be a fibration. Such fibrations are called *trivial fibrations.*

Accordingly, a map $p\colon E \to B$ is called a *locally trivial fibration* if there exists an open covering $\{U_\alpha\}$ of the space B such that all maps $p_\alpha\colon p^1(U_\alpha) \to U_\alpha$ are trivial fibrations.

This terminology suggests its justification in that, following directly from Theorem 1.71, *if a space is paracompact, then every locally trivial fibration is actually a fibration.*

Remark 1.72. Dold prefers a variant of the definition in which the covering $\{U_\alpha\}$ is assumed to be numerable. Then the assumption about the paracompactness of the space B becomes unnecessary.

Remark 1.73. It should be borne in mind that in mathematics the term "fibration" is used in many different senses, sometimes almost uncorrelated with each other. Therefore, strictly speaking, it is always necessary to clarify what the meaning of this term is meant. In these lectures, a "fibration", unless explicitly stated to the contrary, will always be understood as a fibration in the sense of Definition 1.25 of Lecture 1 (i.e., in the sense of Hurevicz).

Remark 1.74. One of the most important classes of fibrations are the so-called *bundles in the sense of Steenrod*, also called *bundles with a structural group* or *skew products* (in English fibre bundle). These include vector bundles known from differential geometry and numerous bundles constructed with their help (say, bundles into spheres). All bundles in the sense of Steenrod are, by definition, locally trivial and therefore (if their base in paracompact) are fibrations in the sense of Hurevicz.

All this gives us an inexhaustible supply of examples of concrete fibations.

Lecture 2

2.1 Homotopy equivalences

Definition 2.1. A continuous map $f : X \to Y$ is called *homotopy equivalence* if its homotopy class $[f]$ is an isomorphism of the homotopy category $[\mathscr{T}\!op]$. This is equivalent to the fact that there is a continuous map $g : Y \to X$ (called *inverse homotopy equivalence*) such that

$$f \circ g \sim \mathrm{id}_Y, \quad \text{and} \quad g \circ f \sim \mathrm{id}_X . \tag{2.2}$$

Since $[g]$ is nothing but $[f]^{-1}$, and since in any category a morphism, an inverse isomorphism, is defined in a unique way, *the inverse homotopy equivalence up to homotopy is defined in a unique way.*

Spaces that are isomorphic in the category $[\mathscr{T}\!op]$, i.e. connected by homotopy equivalence, are called *homotopically equivalent.* Homotopically equivalent spaces are also said to have the same *homotopy type.*

Similarly, two continuous maps $f : X \to Y$ and $f^{'} : X^{'} \to Y^{'}$ are called *homotopically equivalent* if they are isomorphic in the category of morphisms of the category $[\mathscr{T}\!op]$, i.e. if there exist homotopy equivalences $\varphi : X \to X^{'}$ and $\psi : Y \to Y^{'}$ such that the following diagramme

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X^{'} & \xrightarrow[f^{'}]{} & Y^{'} \end{array}$$

is *homotopically commutative* (commutative in the category $[\mathscr{T}\!op]$), i.e. it has the property that $\psi \circ f \sim f^{'} \circ \varphi$.

A remarkable fact justifying our interest in fibrations and cofibaraiotns from a new and unexpected side is indicated in the following theorem.

Theorem 2.3. *Any continuous map is homotopically equivalent to both a fibration and a cofibration.*

Thus, in homotopy theory, without loss of generality, we can consider all maps to be fibrations or cofibrations if desired!

2.2 Reduction of an arbitrary map to a cofibration

The proof of Theorem 2.3 is based on the concepts of a cylinder and a mapping cylinder introduced in Lecture 1. However, for purely technical reasons (and to tell the truth, it's mostly by tradition) "reversed" cylinders and cocylinders are convenient here, resulting in replacing t with $1-t$. In addition, it will be convenient for us here to denote the map for which the cylinder and cocylinder are being constructed with the symbol $f : X \to Y$, which will free up the letters i and p for other purposes.

Thus, the *reversed cylinder* of the map $f : X \to Y$ is called a push-out $(X \times I) \cup_f Y$ of the diagramme $X \times I \xleftarrow{\sigma_1} X \xrightarrow{f} Y$, obtained by gluing the direct product of $X \times I$ to the space Y by the map $(x, 1) \mapsto f(x)$, that is, the coset space of the disjunct union $(X \times I) \sqcup Y$ by the minimal equivalence relation in which $(x, 1) \sim f(x)$ for any point $x \in X$. We will denote the reversed cylinder with the same symbol $\mathrm{Cyl}(f)$ as the straight cylinder from Lecture 1.

For an inverted cylinder , a co-universal square (1.13) of Lecture 1 has (after transposition) the form

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle (\sigma_1)_f} \\ X \times I & \xrightarrow[f_\#]{} & \mathrm{Cyl}(f). \end{array} \tag{2.4}$$

For any point $(x, t) \in X \times I$, we will denote its image by the map $f_\#$ by $[x, t]$. It is clear that $[x_1, t_1] = [x_2, t_2]$ if and only if either $x_1 = x_2$ and $t_1 = t_2$, or $t_1 = t_2 = 1$ and $f(x_1 = f(x_2)$.

For the sake of simplicity, we denote the co-induced map σ_1 by the symbol j. In addition, we will introduce into consideration the map $i = f_\# \circ \sigma_0 : X \to \mathrm{Cyl}(f)$, that is, the map $x \mapsto [x, 0]$. Obviously, both maps i and j are moneomorphisms.

As a rule, the points are $x \in X$ we will identify x and y with the points ix and jy, respectively, i.e. we will consider the maps i and j as attachments. Thus, by applying this agreement, $X \subset \mathrm{Cyl}(f)$ and $Y \subset \mathrm{Cyl}(f)$.

Each point of $z \in \mathrm{Cyl}(f)$ either has the form $[x, t]$, $x \in X$, $t \in I$, or is some point $y \in Y$. In this case, $[x, 1] = f(x)$ for any point $x \in X$.

Lemma 2.5. *The map i is a cofibration (and, consequently, the pair $(\mathrm{Cyl}(f), X)$ is a cofibration).*

Proof. The formula

$$\bar{r}([x, t], \tau) = \begin{cases} ([x, t\tau + t - \tau], 0), & \text{if} t\tau + t - \tau \geq 0, \\ (x, -t\tau - t\tau), & \text{if} t\tau + t - \tau \leq 0, \end{cases}$$

where $(x, t] \in \mathrm{Cyl}(f)$, and $\tau \in I$ together with the formula

$$\bar{r}(y, \tau) = (y, 0), \quad y \in Y \subset \mathrm{Cyl}(f), \quad 0 \leq \tau \leq 1,$$

well determines the corresponding map

$$\overline{r} : \mathrm{Cyl}(f) \times I \to (\mathrm{Cyl}(f) \times 0) \cup (X \times I).$$

Since X is obviously closed in $\mathrm{Cyl}(f)$, this proves Lemma 2.5 (see Proposition 1.15 of Lecture 1). □

Define the map

$$r : \mathrm{Cyl}(f) \to Y \tag{2.6}$$

by the formulae

$$\begin{aligned} r(x,t]) = [x,1], &\quad [x,t] \in \mathrm{Cyl}(f), \\ r(y) = y, &\quad y \in Y \subset \mathrm{Cyl}(f). \end{aligned}$$

Lemma 2.7. *The map r is a homotopy equivalence.*

Proof. Let, as above, $j : Y \to \mathrm{Cyl}(f)$ an inclusion. It is clear that $r \circ j = \mathrm{id}$. On the other hand, $H : j \circ r \sim \mathrm{id}$, where the homotopy

$$H : \mathrm{Cyl}(f) \times I \to \mathrm{Cyl}(f)$$

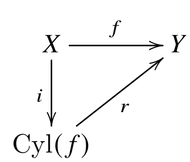
is defined by the formula

$$\begin{aligned} H([x,t],\tau) = [x, \tau t + 1 - \tau], &\quad [x,t] \in \mathrm{Cyl}(f), \\ H(y,\tau) = y, &\quad y \in Y \subset \mathrm{Cyl}(f). \end{aligned} \tag{2.8}$$

□

Corollary 2.9. *Any continuous map $f : X \to Y$ is homotopically equivalent to a cofibration $i : X \to \mathrm{Cyl}(f)$.*

Proof. It suffices to note that $f(x) = [x,1] = r[x,0] = (r \circ i)(x)$ for any point $x \in X$, i.e., the following diagramme is commutative.



□

This corollary proves Theorem 2.3 with respect to cofibrations.

2.3 Deformation retract

The homotopy equivalence of r has the property that $r \circ j = \mathrm{id}$ and, therefore, is a retraction. This type of retraction deserves a separate name.

Let $A \subset X$ and $i : A \to X$ is an inclusion.

Definition 2.10. A map $f : X \to A$ is called *deformation retraction* if $r \circ i = \mathrm{id}\, A$ and $i \circ r \sim \mathrm{id}_X$. A subspace A for which there is a deformation retraction $X \to A$ is called a *deformation retract* of the space X. The homotopy $X \times I \to X$ connecting the maps $i \circ r$ and id_X is called a *retraction deformation.*

A space X is said to be *deformable* into the subspace A if there exists a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{id}$ and $f_1(X) \subset A$, i.e. if there exists a map $\varphi : X \to A$ such that $i \circ \varphi \sim \mathrm{id}_X$. It is easy to see that

Proposition 2.11. *a subspace A is a deformation retract of the space X if and only if it is a retract of the space X and the space X is deformed into the subspace A.*

Proof. Indeed, if $r : X \to A$ is a retraction and $\varphi : X \to A$ is a map such that $i \circ \varphi \sim \mathrm{id}$, then $i \circ r \sim i \circ r \circ i \circ \varphi = i \circ \varphi \sim \mathrm{id}$, and, therefore, r is a deformation retraction. □

Every deformation retraction $r : X \to A$ is, of course, a homotopy equivalence. The inverse homotopy equivalence in this case will be the inclusion $i : A \to X$. Conversely

Proposition 2.12. *if the inclusion $i : A \to X$ is a homotopy equivalence, then in the case where the pair (X, A) is a cofibration, the subspace A will be a deformation retraction of the space X.*

Proof. Indeed, let $j : X \to A$ be the homotopy equivalence inverse to the inclusion $i : A \to X$. By definition, the map $j|_A = j \circ i$ is homotopic to the identity map id_A. Therefore , the map j is homotopic to the map $r : X \to A$, for which $r|_A = \mathrm{id}_A$, i.e. being a retraction $X \to A$. Since $i \circ r \sim i \circ j \sim \mathrm{id}_X$, this retraction is a deformation retraction. □

Interestingly, an arbitrary homotopy equivalence is reduced to deformation retractions and corresponding embeddings. Namely, it turns out that

Proposition 2.13. *for any homotopy equivalence $f : X \to Y$ there exists a space Z containing both spaces X and Y as deformation retracts, such that $f = r \circ i$, where $i : X \to Z$ is an inclusion, and $r : Z \to Y$ is a deformation retraction. Moreover, for this space Z, we can take the (reversed) Cylinder* $\mathrm{Cyl}(f)$ *of the map f.*

All this follows directly from the following proposition.

Proposition 2.14. *A map $f : X \to Y$ is a homotopy equivalence if and only if the space X is a deformation retract of the cylinder* $\mathrm{Cyl}(f)$.

Proof. If X is a deformation retract, then the inclusion $i : X \to \mathrm{Cyl}(f)$ is a homotopy equivalence. Thus, in decompositions $f = r \circ i$, both maps i and r are homotopy equivalences. Therefore, the map f is a homotopy equivalence, too.

Conversely, let the map $f : X \to Y$ be a homotopy equivalence, and let $g : Y \to X$ be the inverse homotopy equivalence. We should construct a retraction $\rho : \mathrm{Cyl}(f) \to X$ and a homotopy $K : \mathrm{Cyl}(f) \times I \to \mathrm{Cyl}(f)$, connecting the identical map $\mathrm{id} : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$ with the map $i \circ \rho : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$. We will compose a homotopy K from four consecutive homotopies. First we will define the homotopy 2.8 on the segment $[0, 1/4]$. This means that at $0 \leq \tau \leq 1/4$, the homotopy K will be determined by the formulae

$$K([x,t],\tau) = [x, t + 4\tau - 4t\tau], \quad [x,t] \in \mathrm{Cyl}(f),$$
$$K(y,\tau) = y, \quad y \in Y \subset \mathrm{Cyl}(f).$$

As a result, from the map $\mathrm{id} : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$ we get the map $j \circ r : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$, which takes values in Y.

On the next segment $[1/4, 1/2]$ we will take as a homotopy K the composition of the map $j \circ r$ and the homotopy $G : Y \times I \to Y$ connecting the identical map $\mathrm{id} : Y \to Y$ with the map $f \circ g : Y \to Y$, i.e., more precisely, homotopy

$$j \circ G \circ (r \times \mathrm{id}) : \mathrm{Cyl}(f) \times I \to \mathrm{Cyl}(f),$$

connecting the map $j \circ r$ with the map $j \circ f \circ g \circ r$, and subjected to a linear transformation of the parameters. Thus, the homotopy K for $\tau \in [1/4, 1/2]$ will be determined by the formulae

$$K([x,t],\tau) = jG(f(x), 4\tau - 1), \quad [x,t] \in \mathrm{Cyl}(f),$$
$$K(y,\tau) = jG(y, 4\tau - 1), \quad y \in Y \subset \mathrm{Cyl}(f).$$

The map $j \circ f \circ g \circ r$ translates the point $[x,t] \in \mathrm{Cyl}(f)$ to the point $f(g(f(x))) = [g(f(x)), 1]$, and the point $y \in \mathrm{Cyl}(f)$ to the point $f(g(y)) = [g(y), 1]$. Therefore, the formulae $([x,t],\tau) \mapsto [g(f(x)), 1 - \tau]$ and $(y,\tau) \mapsto (g(y), 1 - \tau)$ will well determine the homotopy of this map to the map $[x,t] \mapsto g(f(x))$, $y \mapsto g(y)$. We will take this homotopy for the homotopy K on the segment $[1/2, 3/4]$. Given the shift of the parameters, we get, therefore, that the homotopy K is determined for $\tau \in [1/2, 3/4]$ by the formulae

$$K([x,t],\tau) = [g(f(x)), 3 - 4\tau], \quad [x,t] \in \mathrm{Cyl}(f),$$
$$K(y,\tau) = [g(y), 3 - 4\tau] \quad y \in Y \subset \mathrm{Cyl}(f).$$

With $\tau = 3/4$, we actually get a map to X. Therefore, for the points $[x,t]$, the homotopy $F : X \times I \to X$ can be applied to it, connecting the map $g \circ f : X \to X$ with the identity map $\mathrm{id} : X \to X$. This homotopy we will take K as a homotopy on the segment $[3/4, 1]$. At the same time, in order to avoid a gap for $t = 1$, you need to multiply the parameter by $1 - t$. In addition, it is necessary to take into account the shift of the parameter change area. Thus at $\tau \in [3/4, 1]$ we get for the homotopy K of the formula

$$K([x,t],\tau) = iF(x,(1-t)(4\tau-3)), \quad [x,t] \in \mathrm{Cyl}(f),$$
$$K(y,\tau) = ig(y), \quad y \in Y \subset \mathrm{Cyl}(f).$$

The resulting homotopy $K : \mathrm{Cyl}(f)\times I \to \mathrm{Cyl}(f)$ connects the identical map $\mathrm{id} : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$ with the map $i \circ rho : \mathrm{Cyl}(f) \to \mathrm{Cyl}(f)$, where ρ is the map $\mathrm{Cyl}(f) \to X$ defined by the formulae

$$\rho[x,t] = F(x,1-t), \quad [x,t] \in \mathrm{Cyl}(f),$$
$$\rho(y) = g(y), \quad y \in Y \subset \mathrm{Cyl}(f).$$

(due to the relation $F(x,0) = g(f(x))$, the map ρ is well-defined).

To complete the proof of Proposition 2.14, it remains to note that since $\rho[x,0] = F(x,1) = x$, we have $\rho \circ i = \mathrm{id}$. □

2.4 Contractible spaces and cones

A particularly interesting case is when the space Y consists of only one point. We will denote such a space by the symbol pt, and the (unique) map $X \to \mathrm{pt}$ by the symbol const_X (or just const).

Definition 2.15. A space X is called *contractible* if the map const_X is a homotopy equivalence.

Since the inverse homotopy equivalence $g : \mathrm{pt} \to X$ is given by the point $x_0 = g(\mathrm{pt})$ of the space X, and $\mathrm{const}_X \circ g = \mathrm{id}_{\mathrm{pt}}$ and $g \circ \mathrm{const}_{x_0} = \mathrm{const}_{x_0}$, where $\mathrm{const}_{x_0} : X \to X$ is a *constant map* $x \mapsto x_0$, the space X is contractible if there exists such a point $x_0 \in X$, such that $\mathrm{const}_{x_0} \sim \mathrm{id}_X$. It is said that the space X *contracts to the point* x_0.

It is curious that the contractible space shrinks to any point of its own. Indeed, if $x_0, x_1 \in X$ and $\mathrm{const}_{x_0} \sim \mathrm{id}$, then $\mathrm{const}_{x_1} \sim \mathrm{const}_{x_0} \circ \mathrm{const}_{x_0} \sim \mathrm{id}$.

The homotopy connecting the map id_X with the map of const_{x_0} is called the *contraction* of the space X.

Examples of a contractible spaces are the n-dimensional ball $\mathbb{E}^n$ (and, in particular, a segment $I = [0,1]$) and, in general, any convex set or at least stellar set relative to some point of its own body.

The cylinder of the map const_X is called a *cone* over the space X. Two cones are possible: straight and reversed. The first one is obtained from the product $X \times I$ by contracting to the point of the subspace $X \times 0$, and the second by contracting to the point of the subspace $X \times 1$. As a rule, we will consider an inverted cone and denote it with the symbol CX. Thus, by definition

$$CX = (X \times I)/(X \times 1).$$

We will denote the image of the point $(x,t) \in X\times I$ for the coset map $X\times I \to CX$, by the symbol $[x,t]^C$ or simply $[x,t]$. In particular, all symbols of the form $[x,1]$

denote a single point of the cone CX, depending on x. This point is denoted by the symbol p_0 and is called the *vertex* of the cone CX.

The correspondence $[x, t] \mapsto [x, \tau + t - \tau t]$, $\tau \in I$, defines a homotopy from CX to CX connecting an identical map with a constant map $[x, t] \mapsto p_0$. This means that the cone CX shrinks to its vertex (and, therefore, to any point).

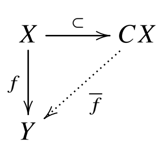
Each continuous map $f : X \to Y$ is determined by the formula

$$(Cf)[x, t] = [f(x), t], \quad x \in X, t \in I,$$

so $Cf : CX \to CY$ is continuous, and it is clear that the correspondences $X \mapsto CX$, $f \mapsto Cf$ constitute a functor from $\mathcal{T}op$ to $\mathcal{T}op$.

If $t_1, t_2 \neq 1$, then $[x_1, t_1] = [x_2, t_2]$ if and only if $x_1 = x_2$ and $t_1 = t_2$. In particular, we see that the correspondence $x \mapsto [x, 0]$ defines a moneomorphic map $X \to CX$. A subspace of the cone CX consisting of points of the form $[x, 0]$, $x \in X$, is called the *base* of the cone. Usually, by means of the moneomorphism $x \mapsto [x, 0]$, it is identified with X. According to Lemma 2.5, the pair (CX, X) is a cofibration, and according to proposition 2.14, the space X is contractible if and only if X is a deformation retract of the cone CX.

A map $f : X \to Y$ is called *null-homotopic* if it is homotopic to the constant map $\mathrm{const}_{y_0} : X \to Y$, $x \mapsto y_0$, where $y_0 \in Y$, i.e. if there exists a homotopy $F : X \times I \to Y$ such that $F(x, 0) = x$ and $F(x, 1) = y_0$ for any point $x \in X$. Since such homotopies are in natural bijective correspondence with the maps $\overline{f} : CX \to X$ that coincide on X with the map f (the homotopy corresponding to the map f is by the composition of the factorisation map $X \times I \to CX$ and the map $\overline{f}$), we obtain that the map $f : X \to Y$ can be extended to CX if and only if is homotopic to zero :



It is also obvious that the space Y is contractible if and only if for each space X any map $X \to Y$ is null-homotopic.

2.5 Relative homotopies and strong deformation retracts

A deformation retraction (2.6) also has the following property that for the corresponding homotopy (2.8) there is an equality $H(y, t) = y$ for any point $y \in Y$ and any $t \in I$

It is also worth introducing the appropriate general terminology here.

Definition 2.16. A homotopy $f_t : X \to Y$ is called *fixed* (or *connected* on the subspace $A \subset X$, if $f_t(a) = f_0(a)$ for any point $a \in A$ and any $t \in I$. A homotopy fixed on A is also called a *homotopy relative to A*.

Of course, a homotopy fixed on A can only be associated with maps $f, g : X \to Y$ that coincide on A, i.e. such that $f|_A = g|_A$. Maps coinciding on A are called (*connected*) *homotopy relative to* A (notation $f \sim g \operatorname{rel} A$ or $f \overset{A}{\sim} g$) if they are connected by a homotopy fixed on A. This relation is an equivalence relation and the corresponding classes $[f] \operatorname{rel} A$ are called *homotopy classes relative to* A.

All maps $X \to Y$ coinciding on A constitute a subspace $\langle Y^X, f_0 \rangle$ of the space Y^X, consisting of extensions to X of some fixed map $f_0 : A \to Y$. In this case, homotopies with respect to A can be considered as paths of this subspace. The converse is generally true only if the space X is Hausdorff and locally compact, and then the classes $[f] \operatorname{rel} A$ are nothing but components of the subspace $\langle Y^X, f_0 \rangle$.

Definition 2.17. A deformation retraction $r : X \to A$ is called *strict* (or *strong*) if $i \circ r \sim \operatorname{id} \operatorname{rel} A$. If such a retraction exists, then the subspace A is called a *strong deformation retraction* of space X. In this case, we will write $X \searrow\!\!\searrow A$.

Proposition 2.18. *If the pair (X, A) is a closed cofibration, then each deformation retraction $r : X \to A$ is a strong deformation retraction.*

To prove this proposition, we need the following lemma.

Lemma 2.19. *For any closed cofibration (X, A), the pair $(X \times I, X_A)$, where $X_A = (X \times 0) \cup (A \times I) \cup (X \times 1)$ is also a cofibration.*

We will prove this lemma below, but for now we will use it is used to prove Proposition 2.18.

Proof. (of Proposition 2.18.) Let $i : A \to X$ be an inclusion and $F : X \times I \to X$ be a free homotopy connecting the map $i \circ r : X \to X$ with the identity map id_X. Since, according to Lemma 2.19, the pair $(X \times I, X_A)$ is a cofibration, so there exists a homotopy from $X \times I$ to X, the initial map of which is the homotopy F and which for every $\tau \in I$ is given by X_A by the formula

$$F_\tau(x, t) = \begin{cases} F((i \circ r)x, \tau), & \text{if} \quad t = 0, \\ F(x, t + (1 - t)\tau), & \text{if} \quad x \in A, 0 \le t \le 1, \\ x, & \text{if} \quad t = 1. \end{cases}$$

The end map F_1 of this homotopy will be the homotopy $X \times I \to X \operatorname{rel} A$ connecting id to $i \circ r$. □

Corollary 2.20. *For a closed cofibration (X, A), the subspace A is a strong deformation retract of the space X if and only if the embedding $i : A \to X$ is a homotopy equivalence.*

Corollary 2.21. *A closed pair (X, A) is a cofibration if and only if the space $\widetilde{A} = (X \times 0) \cup (A \times I)$ is a strong deformation retract of the space $X \times I$.*

Proof. If $X \times I \searrow \widetilde{A}$, then the pair (X, A) is a cofibration by applying the corollary from Proposition 1.15 of Lecture 1. Conversely, since the product $X \times I$ is obviously deformed into the subspace $X \times O$ and, consequently, into the subspace $\widetilde{A}$, then in the case when there is a retraction $r : X \times I \to \widetilde{A}$ (i.e., by applying the same corollary of Proposition 1.15 of Lecture 1, when the pair (X, A) is a cofibration), this retraction will be a deformation retraction and, therefore, by applying Proposition 2.18 (applied to the pair $(X \times I, \widetilde{A})$) a strong deformation retraction. (However, you can do without any references if you notice that putting $r(x,t) = (\overline{r}(x,t), \rho(x,t))$, where $x \in X$, $t \in I$ and $\overline{r}(x,t) \in X$, $\rho(x,t) \in I$, we can define the deformation retraction $g_t : X \times I \to X \times I$ by an explicit formula

$$g_\tau(x,t) = (\overline{r}(x,(1-\tau)t), (1-\tau\rho(x,t)+\tau t), \quad x \in X, \tau \in I.$$

Indeed, it is clear that $g_0 i \circ r$, $g_1 = \mathrm{id}$ and $g_\tau(x,t) = (x,t)$, if $(x,t) \in \widetilde{A}$). □

2.6 Homotopy invariance of the gluing operation

Let (X, A) be a closed cofibration, $F : A \times I \to Y$ be an arbitrary homotopy and $f : A_t o Y$, $a \mapsto F(a,0)$ be the initial homotopy map of F. Since $A \times I \subset X \times I$, the space is $(X \times I) \cup_F Y$. Let us compare it with the space $X \cup_f Y$.

Consider for this purpose the space $\widetilde{A} \cup_F Y$.

Since $A \times I \subset \widetilde{A} \subset X \times I$ and $(X \times I) \searrow \widetilde{A}$, the space $\widetilde{A} \cup_F Y$ is contained in the space $(X \times I) \cup_F Y$ and is its strong deformation retract. On the other hand, since $(A \times I) \cap \widetilde{A} = A \times 0$ and $F(a,0) = f(a)$, $a \in A$, the space $\widetilde{A} \cup_F Y$ is naturally identified with the space $X \cup_f Y$. This proves that

Proposition 2.22. *the space $X \cup_f Y$ is homotopically equivalent to the space $(X \times I) \cup_F Y$.*

A similar statement holds, of course, for the end map $g : A \to Y$, $a \mapsto F(a,1)$, homotopy F. Hence,

Proposition 2.23. *if a closed pair (X, A) is a cofibration, then for any two homotopy maps $f : X \to Y$ and $g : X \to Y$ the spaces $X \cup_f Y$ and $X \cup_g Y$ are homotopically equivalent.*

In this sense, the operation of gluing spaces is homotopically invariant.

Note that in the process of proving, we actually constructed some homotopy $\ell_t : X \sqcup Y \to X \cup_f Y$, the initial map ℓ_0 of which is the factorisation map $q : X \sqcup Y \to X \cup_f Y$ and which has the property, that its restriction $\ell_t|_A$ to A, considered as a homotopy from A to B, is nothing more than a homotopy of $a \mapsto (a,t)$ connecting the map f with the map g. At he same time the map

$$\varphi : X \cup_g Y \to X \cup_f Y.$$

induced by the map ℓ_1, (i.e. such that $\ell_1 = \varphi \circ q'$, where q' is the factorisation map $X \sqcup Y \to X \cup_g Y$)), is a homotopy equivalence.

From this remark it easily follows that

Proposition 2.24. *for any closed cofibration (X, A) and any map $f : A \to Y$ every homotopy equivalence of $h : Y \to Z$ extends to some homotopy equivalence*

$$\overline{h} : X \cup_f Y \to X \cup_{h\circ f} Z.$$

Indeed, it is clear that the map $\overline{h} : X \cup_f Y \to X \cup_{h\circ f} Z$ coinciding on Y with map h and identical on X is uniquely defined and continuous. Therefore, it is only necessary to prove that it is a homotopy equivalence.

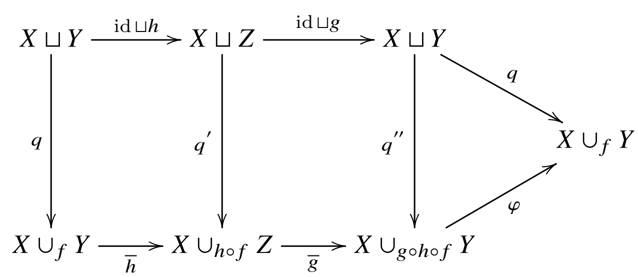
For this purpose, let us consider the inverse homotopy equivalence $g : Z \to Y$ and its extension

$$\overline{g} : X \cup_{h\circ f} Z \to X \cup_{g\circ h\circ f} Y,$$

identical on X. Since $g \circ h \circ f \sim f$, then as has been proved, there is a homotopy equivalence

$$\varphi : X \cup_{g\circ h\circ f} Y \to X \cup_f Y.$$

In the diagramme



the vertical arrows of which are factorisation maps, both squares are commutative by construction, and the right triangle is homotopy commutative by the remark just made. Therefore

$$\varphi \circ \overline{g} \circ \overline{h} \sim q \circ (\mathrm{id} \sqcup (g \circ h)).$$

But it's easy to see that the formula

$$k_t = \begin{cases} q \circ (\mathrm{id} \sqcup s_{2t}), & \text{if} \quad 0 \le t \le 1/2, \\ \ell_{2t-1} \circ (\mathrm{id} \sqcup (g \circ h)), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

where $s_t : Y \to Y$ is the homotopy connecting the identity map of the space Y with the map $g \circ h$, and $\ell_t : X \sqcup Y \to X \cup_f Y$ is the homotopy connecting the map q with map $\varphi \circ q''$, well defines homotopy $k_t : X \sqcup Y \to X \cup_f Y$, connecting the q map to the map

$$(\varphi \circ q'') \circ (\mathrm{id} \sqcup (g \circ h)) = \varphi \circ \overline{g} \circ \overline{h} \circ q$$

and compatible with the projection q, i.e. having the form $k_t = \overline{k}_t \circ q$, where $\overline{k}_r : X \cup_f Y \to X \cup_f Y$. At the same time,

$$\overline{k}_0 = \mathrm{id} \quad \text{and} \quad \overline{k}_1 = \varphi \circ \overline{g} \circ \overline{h}.$$

Thus, $\varphi \circ \overline{g} \circ \overline{h} \sim \mathrm{id}$, where φ is a homotopy equivalence.

Applying the same reasoning $h \circ f$ and g instead of f and k, we get that $\psi \circ \overline{\overline{h}} \circ \overline{g} \sim \mathrm{id}$, where $\overline{\overline{h}}$ is some map, and ψ is a homotopy equivalence.

But if φ' is a homotopy equivalence inverse to the homotopy equivalence φ, then $\overline{g} \circ \overline{h} \sim \varphi'$, and, therefore, $\overline{g} \circ \overline{h} \circ \varphi \sim \varphi' \circ \varphi \sim \mathrm{id}$. Therefore

$$\psi \circ \overline{\overline{h}} \sim \psi \circ \overline{\overline{h}} \circ \overline{g} \circ \overline{h} \circ \varphi \sim \overline{h} \circ \varphi,$$

and therefore $(\overline{h} \circ \varphi) \circ \overline{g} \sim \mathrm{id}$.

This proves that $\overline{g}$ is a homotopy equivalence (with the inverse homotopy equivalence $\overline{h} \circ \varphi$). Therefore, the homotopy equivalence will also be the map $\overline{h}$.

2.7 Neighbourhood deformation retracts and cofibrations

Note that Lemma 2.19 has not yet been proved. Its proof is based on a single, interesting for its own sake, proposition that gives a local characterisation of closed cofirations.

Definition 2.25. A subspace A of a topological space X is called a *neighbourhood strong deformation retract* (abbreviated *NSDR*) if it is a strong deformation retract of some open set $U \supset A$, i.e. if there exists a fixed homotopy $g_t : U \to U$ on A such that $g_0(x) = x$ and $g_1(x) \in A$ for any point $x \in U$.

Similarly, a subspace of A is called an *weak NSDR* if there exists a homotopy $g_t : U \to X$ fixed on A such that $g_0(x) = x$ and $g_1(x) \in A$ for each point $x \in U$.

An NSDR (in the strong or weak sense) is called *functionally distinguished* (abbreviated as *FNSDR*) if there exists a continuous function $f : X \to I$ such that $A = \varphi^{-1}(0)$ and $X \setminus U = \varphi^{-1}(1)$.

Lemma 2.26. *A subspace $A \subset X$ is a FNSDR in the weak sense if and only if there is a continuous function $\psi : X \to I$ and a fixed homotopy $G : X \times I \to X$ on A such that $A = \psi^{-1}(0)$, $G(x,0) = x$ and $G(x,t) \in A$ fort $\psi(x) < t$.*

Proof. If the function ψ and the homotopy G exist, then the homotopy constraint of G on $U \times I$, where $U = X \setminus \varphi^{-1}(1)$, $\varphi = \psi$ will be a homotopy from U to X fixed on $A = \varphi^{-1}(0)$, having the property that $g_0(x) = x$ and $g_1(x) \in A$ for each point $x \in U$.

Conversely, let there be a function $\varphi : X \to I$ such that $A = \varphi^{-1}(0)$, and a homotopy $g_t : U \to X$ fixed on A, where $U = X \setminus \varphi^{-1}(1)$ such that $g_0(x) = x$ and $g_1(x) \in A$ for each point $x \in U$. Then $A = \psi^{-1}(0)$, where $\psi(x) = \min(2\varphi(x), 1)$, and the formula

$$G(x,t) = \begin{cases} x, & \text{if} \quad \varphi(x) \geq t, \\ g_\tau(x), & \text{if} \quad \varphi(x) < t, \end{cases}$$

where

$$\tau = \begin{cases} \frac{t}{\varphi(x)} - 1, & \text{if} \quad 0 < \varphi(x) \leq 2\varphi(x), \\ 1 & \text{if} \quad t \geq 2\varphi(x), \end{cases}$$

well defines a homotopy $G : X \times I \to X$ such that $G(x, 0) = x$ and $G(x, t) \in A$ for $\psi(x) < t$. □

Proposition 2.27 (local characterisation of closed cofibrations)**.** *A closed pair (X, A) is a cofibration if and only if the subspace A is a FNSDR in the weak sense.*

Proof. Let (X, A) be a closed cofibration, and let $f : X \times I \to \widetilde{A}$ be an arbitrary retraction, where, as always, $A = (X \times 0) \cup (A \times I)$. Let, as above

$$r(x, t) = (\widetilde{r}(x, t), \rho(x, t), \quad \text{where} \quad \widetilde{r}(x, t) \in X, \rho(x, t) \in I.$$

As shown in Proposition 1.18, for the function $\psi(x) = \max_{t \in I}(t - \rho(x, t))$ there is an equality $\psi^{-1}(0) = A$. In addition, if $\psi(x) < t$, then $\rho(x, t) > 0$, and therefore $\overline{f}(x, t) \in A$. Thus, the function ψ and the homotopy $G = \overline{r}$ satisfy the conditions of Lemma 2.26, and therefore A is a FNSDR in the weak sense.

Conversely, if A is a FNSDR in the weak sense, then, according to Lemma 2.26, there exists a continuous function $\psi : X \to I$ such that $\psi^{-1}(0) = A$, and a fixed homotopy $G : X \times I \to X$ on A, such that $G(x, 0) = x$ and $G(x, t) \in A$ for $\varphi(x) < t$. Thus the formula

$$r(x, t) = \begin{cases} (G(x, t), 0)), & \text{if} \quad t \le \psi(x), \\ (G(x, t), t - \psi(x)), & \text{if} \quad t \ge \psi(x), \end{cases} \quad (x, t) \in X \times I,$$

will determine the retraction $f : X \times I \to \widetilde{A}$, so the pair (X, A) is a cofibration. □

Example 2.28. Let $f : X \to Y$ be an arbitrary map and $\mathrm{Cyl}(f)$ its reversed cylinder. It is clear that the formulae

$$\varphi[x, \tau] = \tau, \varphi(1)\ 1, \quad x \in X, \tau \in I, y \in Y,$$

define a continuous function $\varphi : \mathrm{Cyl}(f) \to I$, for which $\varphi^{-1}(0) = X$ and $\varphi^{-1}(1) = Y$, and the formula

$$g_t([x, \tau]) = [x, (1 - t)\tau], \quad x \in X, t \in I,$$

defines the homotopy $g_t : U \to \mathrm{Cyl}(f)$, $U = \mathrm{Cyl}(f) \backslash Y$, for which $g_0[x, \tau] = [x, \tau]$ and $g_1[x, \tau] = -[x, 0] \in X$. This shows that the subspace X of the cylinder $\mathrm{Cyl}(f)$ is a FNSDR in the weak sense and, therefore, the pair $(\mathrm{Cyl}(f), X)$ is a cofibration. Thus we have proved Lemma 2.5 anew.

Now we can prove Lemma 2.19 as well.

Proof. (of Lemma 2.19) Since the pair (X, A) is a closed cofibration, then, according to proposition 2.27, there exists a continuous function $f\varphi : X \to I$ and a homotopy $g_t : U \to X \operatorname{rel} A$, such that $U = X \setminus \varphi^{-1}(1)$, $\varphi^{-1}(0) = A$ and $g_0(x) = x$, $g_1 = A$ for any point $x \in U$. But then, as a direct checking shows, the continuous function $\psi : X \times I \to I$ defined by the formula

$$\psi(x, \tau) = 2 \min(2\varphi(x), \tau, 1 - \tau), \quad (x, \tau) \in X \times I,$$

has the property that $\psi^{-1}(0) = X_A$, and the homotopy $h_t : V \to X \times I$, where $V = X \times I \setminus \psi^{-1}(1)$ is a set of points $(x, \tau) \in X \times I$, for which either $\tau \neq \frac{1}{2}$ or $\varphi(x) < \frac{1}{4}$, defined (see Fig. 2.7.1) by the formula

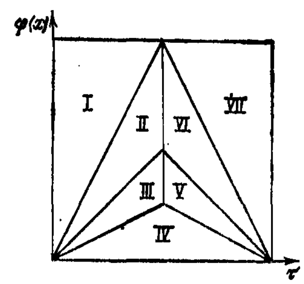


Figure 2.7.1:

$$h_t(x,\tau) = \begin{cases} (x, \tau(1-t)), & \text{if } 2\tau \le \varphi(x), \\ (g(x, (\frac{2\tau}{\varphi(x)} - 1)t, \tau(1-t)), & \text{if } \varphi(x) \le 2\tau \le \min(2\varphi(x), 1), \\ (g(x,t), (\tau - 2\varphi(x))t + \tau), & \text{if } \varphi(x) \le \tau \le \min(2\varphi(x), 1/2), \\ (g(x,t), \tau), & \text{if } 2\varphi(x) \le\le 1 - 2\varphi(x), \\ (g(x,t), \tau + (2\varphi(x) + \tau - 1)t), & \text{if } \max(1 - 2\varphi(x), 1/2) \le 2\tau \le 2 - \varphi(x), \\ (g(x, (\frac{2(1-\tau)}{\varphi(x)} - 1)t), \tau + t - \tau t), & \text{if } \max(2(1-\varphi(x)), 1) \le 2\tau \le 2 - \varphi(x), \\ (x, \tau + t - \tau t), & \text{if } 2 - \varphi(x) \le 2\tau, \end{cases}$$

the properties that $h_0(x, \tau) = (x, \tau)$, $h_1(x, \tau) \in X_A$ for any point $(x, \tau) \in X \times I$ and $h_t(x, \tau) = (x, \tau)$, for any point $(x, \tau) \in X_A$. Thus, EX_A is FNSDR in the weak sense, and therefore the pair $(X \times I, X_A)$ is a cofibration. □

2.8 Strong deformation retracts and homotopy equivalences

Let's now return to the map $f : X \to Y$ and its cylinder Cyl(f). Since, as already noted, homotopy (1.10) is a homotopy relative to Y, we see that for any map $f : X \to Y$ the space Y is a strong deformation retraction of the cylinder Cyl(f).

As for the space X, then, according to proposition 2.14, in order for it to be a strong deformation retraction of the cylinder Cyl(f), it is necessary in any case that the map $f : X \to Y$ be a homotopy equivalence. Moreover, if we remember that the deformation retraction constructed in the proof of proposition 2.14 is obviously not a homotopy with respect to X, it turns out that this necessary condition is also sufficient.

Proposition 2.29. *A map $f : X \to Y$ is a homotopy equivalence if and only if the space X is a strong deformation retract of the cylinder* $\mathrm{Cyl}(f)$.

Proof. This proposition follows directly from Propositions 2.14, 2.18 and Lemma 2.5. □

Corollary 2.30. *A space X is contractible if and only if it is a strong deformation retract of the cone CX.*

2.9 Two more axioms characterising fibrations

The axiom on the covering extension and the axiom on the covering homotopy extension

The concept of strict deformation retraction also allows us to give a new characterisation of maps that are fibrations.

We will say that the map $p : E \to B$ satisfies the *axiom of the covering map extension* (in short, the *axiom* **CME**) if for any pair (X, A) in which the subspace A is functionally separable and is a strong deformation retract of the space X, for each commutative diagramme of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & E \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \overline{f}} & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array} \tag{2.31}$$

there exists a map $\overline{f}$ where i is the inclusion.

Since the subspace $X = X \times 0$ of the product $X \times I$ is obviously functionally separable and is its strong deformation retract,

Proposition 2.32. *every map $p : E \to B$ satisfying the axiom of the extension of the covering map is a fibration.*

The converse is also true, i.e.

Proposition 2.33. *any fibration satisfies the axiom of the extension of the covering map.*

Proof. Indeed, let, in the diagramme (2.31), the map $p : E \to B$ be a fibration, and the subspace A of the space X be functionally separable and represent a strong deformation retract of the space X. Let, further, $r : X \to A$ be a strong deformation retraction, $H : X \times I \to X$ be a homotopy fixing A, connecting maps $i \circ r : X \to X$ with the identity map $\mathrm{id} : X \to X$, and $\varphi : X \to I$ is a continuous function such that $\varphi^{-1}(0) = A$. Then it is clear that the formula

$$\overline{H}(x,t) = \begin{cases} H(x, t/\varphi(x)), & \text{if} \quad t \le \varphi(x) \quad \text{and} \quad x \notin A, \\ x, & \text{if} \quad t \ge \varphi(x) \quad \text{or} \quad x \in A, \end{cases}$$

well defines some continuous map $\overline{H} : X \times I \to X$, which is also a homotopy fixed on A connecting the map $i \circ r$ to the identity map id. At the same time, if we similarly "excite" the map σ_0, i.e. consider the map $\overline{\sigma}_0 : X \to \times I$, defined by the formula

$$\overline{\sigma}_0(x) = (x, \varphi(x)), \quad x \in X,$$

then the equality $\overline{H} \circ \overline{\sigma}_0 = \mathrm{id}$ will take place (whereas $H \circ \sigma_0 = i \circ r$). However, $\overline{\sigma}_0 = \sigma_0$ on A, i.e. $\overline{\sigma}_0 \circ i = \sigma_0 \circ i$.

Let us now consider the diagramme

$$\begin{array}{ccccccc} A & \xrightarrow{i} & X & \xrightarrow{r} & A & \xrightarrow{g} & E \\ {\scriptstyle i}\downarrow & & {\scriptstyle \sigma_0}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\overline{\sigma}_0]{} & X \times I & \xrightarrow[\overline{H}]{} & X & \xrightarrow[f]{} & B \end{array}$$

According to what has just been said, this diagramme is commutative.

By combining the corresponding maps, we can combine the central and right squares of this diagramme into one (also commutative) diagramme

$$\begin{array}{ccc} X & \xrightarrow{g \circ r} & E \\ {\scriptstyle \sigma_0}\downarrow & \overset{\overline{G}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[G]{} & B \end{array}$$

where $G = f \circ \overline{H}$. Since the map $p : E \to B$ is by condition a fibration, there is a covering homotopy $\overline{G} : X \times I \to E$ for this diagramme, and then it is directly verified that the map $\overline{f} = \overline{G} \circ \overline{\sigma}_0$ closes the diagram (2.31). $\square$

The result obtained has an important corollary concerning the problem of extending the covering homotopy:

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\overline{f}} & E \\ {\scriptstyle \widetilde{\sigma}_0}\downarrow & \overset{\overline{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B \end{array} \tag{2.34}$$

where $A \subset X$ and $\widetilde{A} = (X \times 0) \cup (A \times I)$ (see diagram (1.52) from the Appendix to Lecture 1).

We will say that the map $p : E \to B$ satisfies the axiom of *covering homotopy extension* (in short, the axiom **HE**) if for any diagram (2.34) in which (X, A) is a closed cofibration, there exists a covering homotopy $\overline{F}$.

For $A = \varnothing$, the axiom **CHE** turns into the axiom **HE**, so that

Proposition 2.35. *any map satisfying the axiom of the covering homotopy extension is a fibration.*

And conversely, it turns out that

Proposition 2.36. *any fibration satisfies the axiom of the covering homotopy extension.*

Proof. Since fibrations satisfy the axiom **CME**, to prove this statement, it is sufficient to show that the axiom **CME** follows from the axiom **CH**, i.e. that for any closed cofibration (X, A) Diagramme (2.34) is a special case of Diagramme (2.31), or, in other words, that the subspace $\widetilde{A}$ of the space $X \times I$ is its strong deformation retract and is functionally separable. The first property is provided by by the above-proven Corollary 2.30 to Proposition 2.29, and for the proof of the second it is sufficient to show that for a functional distinguished subspace $A \subset X$, the subspace $\widetilde{A} \subset X \times I$ is also functionally allocated (for, as was proved in Proposition 1.18 in Lecture 1, for any closed cofibration (X, A), the subspace A is functionally separable). But if $\varphi : X \to I$ is a continuous function such that $A = \varphi^{-1}(0)$, then the function $\psi : X \times I \to I$ defined by the formula $\psi(x, t) = \min(t, \varphi(x))$, $(x, t) \in X \times I$, will have the property that $\widetilde{A} = \psi^{-1}(0)$. □

Combining the result obtained with the other characterisations of fibrations proved above, we obtain

Theorem 2.37. *For an arbitrary map $p : E \to B$, the following axioms are equivalent:*

CH *Covering homotopy.*

CP *Covering path.*

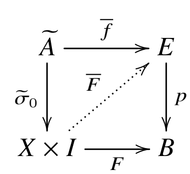
WCHE *Weak covering homotopy extension.*

CHE *Covering homotopy extension.*

CME *Covering map extension.*

Thus, fibrations can be characterised by each of these axioms.

Corollary 2.38. *Let in the diagramme where p is a fibration*



the homotopy F be stationary on a closed subspace of $A \subset X$. Then, if the pair (X, A) is a cofibration, then the homotopy $\overline{F}$ can also be chosen to be fixed on A.

Proof. The condition of immobility of homotopy $\overline{F}$ on A sets this homotopy on $\widetilde{A}$. Therefore, we can apply the axiom on the covering homotopy extension. □

2.10 The preimage of the cofibration

An important proposition follows from this corollary, concerning the preimages of cofibrations.

Proposition 2.39. *For any fibration $p : E \to B$ and any closed subspace $A \subset B$ having the property that the pair (B, A) is a cofibration, the pair $(E, p^{-1}A)$ will also be a cofibration.*

Proof. According to proposition 2.27 and Lemma 2.26, there exists a continuous function $\psi : B \to I$ such that $\psi^{-1}(0) = A$, and a fixed homotopy $g_t : B \to B$ on A, such that $g_0 = \mathrm{id}_B$ and $g_t(x) \in A$ for $t > \psi(x)$. The homotopy $p \circ g_t : E \to B$ is fixed on $p^{-1}A$, and its initial map $p \circ g_0 = p$ is covered by the map id_E. Therefore, according to Corollary 2.20, there exists a fixed homotopy on $p^{-1}A$ and a homotopy $h_t : E \to E$ such that $h_0 = \mathrm{id}_E$ and $h_t \circ p = p \circ g_t$ for any $t \in I$. Consider on E the function $\overline{\psi} = \psi \circ p : E \to I$ and the homotopy $\overline{g}_t : E \to E$ defined by the formula

$$\overline{g}_t(x) = h_\tau(x), \quad \text{where} \quad \tau = \min(t, \overline{\psi}(x)), x \in E.$$

It is clear that $\overline{\psi}^{-1}(0) = p^{-1}A$, $\overline{g}_0(x) = x$, and $\overline{g}_t(x) \in p^{-1}A$ for $t > \overline{\psi}(x)$. Therefore, the pair $(E, p^{-1}A)$ is a cofibration. □

2.11 Reduction of an arbitrary map to a fibration

Let us now return to Theorem 2.3 (which, recall, we so far proved only for cofibrations).

The *reversed cocylinder* of the map $f : X \to Y$ is by definition the push-out of the diagramme $Y^I \xrightarrow{\omega_1} Y \xleftarrow{f} X$, i.e. a subset of the product of $Y^I \times X$ consisting of points (u, x), $u : I \to Y$, $x \in X$, such that $u(1) = f(x)$. We will denote this cylinder with the same symbol $\mathrm{Cocyl}(f)$.

In the notation of Lecture 1 a commutative diagramme takes place for an inverted cocylinder

$$\begin{array}{ccc} \mathrm{Cocyl}(f) & \xrightarrow{\omega_1^\#} & X \\ {\scriptstyle f_{\omega_1}}\downarrow & & \downarrow{\scriptstyle f} \\ Y^I & \xrightarrow[\omega_1]{} & Y \end{array}$$

The map $\omega_1^\# : (u, x) \mapsto x$ we will denote by the symbol R to simplify the formulae. In addition, we will introduce into consideration the map $p = \omega_0 \circ f_{\omega_1} : \mathrm{Cocyl}(f) \to Y$, acting according to the formula $(u, x) \mapsto u(0)$.

Lemma 2.40. *The map $p : \mathrm{Cocyl}(f) \to Y$ is a fibration.*

Proof. Let $\mathrm{Cocyl}(p)$ be the cocylinder (non-inverted) of the map p and $q : \mathrm{Cocyl}(f)^I \to \mathrm{Cocyl}(p)$ be the map $w \mapsto (p \circ w, w(0))$, $w : I \to \mathrm{Cocyl}(f)$.

According to Proposition 1.36 of Lecture 1, to prove Lemma 2.40, it is enough for us to construct a map $s : \mathrm{Cocyl}(p) \to \mathrm{Cocyl}(f)^I$, such that $g \circ s = \mathrm{id}$. The exponential law allows us to consider the map s as a map

$$\mathrm{Cocyl}(p) \times I \to \mathrm{Cocyl}(f)^I,$$

i.e., (we use this law again) as a pair of maps

$$a : \mathrm{Cocyl}(p) \times I \times I \to Y, \quad b : \mathrm{Cocyl}(p) \times I \to X,$$

connected for any points $c \in \mathrm{Cocyl}(p)$, $t \in I$ by the relation

$$a(c, t, 1) = f(b(c, t)).$$

In this interpretation, the identity $g \circ s = \mathrm{id}$ is equivalent, as it is easy to see, to the identities

$$v(t) = a(c, t, 0), \quad u(x) = a(c, 0, \tau), \quad x = b(c, 0),$$

which should be the case for any numbers t, $\tau \in$ and any point

$$c = (v, \overline{c}) \in \mathrm{Cocyl}(p), \quad \text{where} \quad v : I \to Y, \overline{c} = (u, x) \in \mathrm{Cocyl}(f),$$

i.e. for any point $x \in X$ to any paths $v : I \to Y$ and $u : I \to Y$ connected by the relations

$$u(0) = v(0) \quad u(1) = f(x).$$

Assuming a natural liberty, we will denote (v, u, x) by the former symbol c.

We will define the map b by the formula

$$b(x, t) = x, \quad c = (v, u, x) \in \mathrm{Cocyl}(p) \quad \text{and} \quad t \in I.$$

Then the map a will satisfy the relations

$$v(t) = a(c, t, 0), \quad u(\tau) = a(c, 0, t), \quad a(c, t, 1) = f(x)$$

for any points $c = (v, u, x) \in \mathrm{Cocyl}(p)$ and $(t, \tau) \in I \times I$. But it is clear that these relations are satisfied by the map $a : \mathrm{Cocyl}(p) \times I \times I \to Y$, defined by the formula

$$a(c, t, \tau) = (\widetilde{c} \circ \varphi)(t, \tau), \quad c = (v, u, x) \in \mathrm{Cocyl}(p), \quad t, \tau \in I,$$

where φ, as above, is a retraction map $I \times I \to (I \times 0) \cup (0 \times I)$, for which $\varphi(1 \times I) = (1, 0)$, and $\widetilde{c}$ is the map $(I \times 0) \cup (0 \times I) \to Y$, defined by the formulae

$$\widetilde{c}(t, 0) = v(t), \quad \widetilde{c}(0, \tau) = u(\tau), \quad t, \tau \in I.$$

(Question: Why is the map a continuous?)

Thus Lemma 2.40 is fully proved. □

For each point $y \in Y$ with the symbol 0_y, we will denote a constant path at the point y, given by the formula

$$0_y(t) = y \quad \text{for any point} \quad t \in I.$$

In particular, for each point $x \in X$ the symbol $0_{f(x)}$ denotes a constant path at the point $f(x) \in Y$. Because $0_{f(x)}(1) = f(x)$, the pair $0_{f(x)}, x)$ lies in $\mathrm{Cocyl}(f)$. Denoting this pair with the symbol $i(x)$, we therefore get a (obviously continuous) map

$$i : X \to \mathrm{Cocyl}(f).$$

With the map $r : \mathrm{Cocyl}(f) \to X$, $(u, x) \mapsto x$, the map i is related by the formula $r \circ i = \mathrm{id}$. Therefore, i is a moneomorphism, and therefore, identifying each point $x \in X$ with a point $i(x)$, we will embed X in $\mathrm{Cocyl}(f)$. In this case, the subspace X will be a retract of the space $\mathrm{Cocyl}(f)$ with a retraction map r.

Lemma 2.41. *The retract X is a strong deformation retract of the space* $\mathrm{Cocyl}(f)$*, so that, in particular, the map i is a homotopy equivalence.*

Proof. The formula

$$H((u,x),t) = (v_t, x), \quad (u,x) \in \mathrm{Cocyl}(f), \quad t \in I$$

where v_t is the path $I \to Y$ defined by the formula

$$v_t = u(1 - t + \tau t), \quad \tau \in I,$$

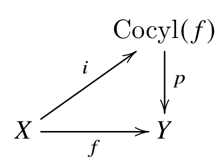
(obviously satisfying the condition $v_t(1) = f(x)$), defines the homotopy

$$H : \mathrm{Cocyl}(f) \times I \to \mathrm{Cocyl}(f),$$

fixed on X and connecting the map $i \circ r : (u,x) \mapsto (0_{f(x)}, x)$ with the identity $\mathrm{id} : (u,x) \mapsto (u,x)$. □

Corollary 2.42. *Any continuous map $f : X \to Y$ is homotopically equivalent to the fibration $p :$* $\mathrm{Cocyl}(f) \to Y$*.*

Proof. It suffices to note that $f(x) = 0_{f(x)}(0) = p(0f(x), x) = (p \circ i)(x)$ for any point $x \in X$, i.e. that the diagramme



commutes, i.e., $f = p \circ i$. □

Thus, Theorem 2.3 is also proved with respect to fibrations.

Appendix

Introduced in an *ad hoc* fashion at the end of the Appendix to Lecture 1 the concept of map isomorphism $p : E \to B$ and $p' : E' \to B$ it can be included in the general categorical framework, which leads to a significant and very important generalisation of all our considerations.

2.A Category $\mathscr{Top}_{B_0}$

Let $\mathscr{A}$ be an arbitrary category and B_0 be some of its fixed object. An object X of the category A, considered together with some morphism $\pi_X : X \to B_0$, is called an *object over* B_0. The morphism π_X is called the *projection* of the object X.

With another - equivalent - point of view, objects over B_0 are considered to be the maps of π_X themselves.

For any two objects X and Y over B_0, a morphism over B_0 of an object X into an object Y is an arbitrary morphism $f : X \to Y$ of the category $\mathscr{A}$, for which the diagramme

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle \pi_X}\searrow \quad \swarrow{\scriptstyle \pi_Y} & \\ & B_0 & \end{array}$$

commutes. It is clear that all objects over B_0 and all their morphisms over B_0 constitute a category. We will denote this category by the symbol $\mathscr{A}_{B_0}$.

In particular, for any topological space B_0, the category $\mathscr{Torus}_{B_0}$, whose isomorphisms are the isomorphisms of maps introduced in the Appendix to Lecture 1.

In the case when the space B_0 consists of only one point (it is a terminal object of the category $\mathscr{Top}$), the category $\mathscr{Top}_{B_0}$ is naturally identified with the category $\mathscr{Top}$. Thus, $\mathscr{Top}_{B_0}$ generalises $\mathscr{Top}$. This generalisation is quite meaningful, since in an arbitrary category $\mathscr{Top}_{B_0}$, an equally advanced homotopy theory is possible, as in the category $\mathscr{Top}$.

However, for obvious reasons, we won't develop in detail the theory of homotopies in categories $\mathscr{Tor}_{B_0}$, and we will limit ourselves to its very initial concepts and results. For us, this theory will have only an auxiliary value, as one of the tools for a more in-depth study of the category $\mathcal{Top}$.

A note on terminology

For a topological space X over B_0, the preimage $\pi_X^{-1}(b)$ of an arbitrary point $b \in B_0$ when projected $\pi_X : X \to B_0$ is usually called the *fibre* of this space over the point B. For any two spaces X and Y over B_0, a continuous map $f : X \to Y$ if and only if is a map over B_0 when for each point $B \in To$ it translates the fibre $\pi_X^{-1}(b)$ to the fibre $\pi_Y^{-1}(b)$. Therefore, the map over B_0 are also called *fibrewise maps.*

2.B Homotopies, cofibrations, and fibrations of the category $\mathcal{Top}_{B_0}$

Homotopy in the category $\mathcal{Top}_{B_0}$ are introduced completely naturally.

Let X and Y be topological spaces over B_0 (objects of the category $\mathcal{Torus}_{B_0}$). Maps $f, g : X \to Y$ over B_0 (morphisms of the category $\mathcal{Top}_{B_0}$) are called *homotopy* in $\mathcal{Top}_{B_0}$ (or over B_0), and also *fibrewise homotopy* if there exists such a homotopy $f_t : X \to Y$ that $f_0 = f$, $f_1 = g$ and for any $t \in I$, the map $f_t : X \to Y$ is a map over B_0. The homotopy f_t is said to be a *homotopy over* B_0 between f and g. The homotopy relation over B_0 is denoted by the symbol $f \underset{B_0}{\sim} g$.

It is clear that fibrewise homotopies satisfy the general axioms (1°) - (4°) from Lecture 0, therefore, in particular, the relation of homotopy over B_0 is an equivalence relation. The corresponding equivalence classes are called *homotopy classes over* B_0 (or *fibrewise homotopy classes*). The class containing the map f is denoted by the symbol $[f]_{B_0}$.

Since the formula

$$[g]_{B_0} \circ [f]_{B_0} = [g \circ f]_{B_0}$$

well defines the composition of any homotopy classes over B_0, then a category arises $[\mathcal{Top}_{B_0}]$, whose objects are spaces over B_0, and morphisms are homotopy classes over B_0.

Remark 2.43. The category $[\mathcal{Top}_{B_0}]$ should be distinguished from the category $[\mathcal{Top}]_{B_0}$. These categories are connected by an obvious functor

$$[\mathcal{Top}_{B_0}] \to [\mathcal{Top}]_{B_0}, \tag{2.44}$$

which is identical on objects, but, generally speaking, not injective and not surjective on morphisms. See page 111 below.

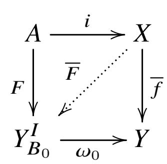
A morphism $f : X \to Y$ in the category $\mathcal{Top}_{B_0}$ is called a *homotopy equivalence over* B_0 (or a *fibrewise homotopy equivalence*) if there exists a morphism $g : Y \to X$ such that $g \circ f \underset{B_0}{\sim} \mathrm{id}$ and $f \circ g \underset{B_0}{\sim} \mathrm{id}$, i.e. if its homotopy class over B_0 is an isomorphism of the category $[\mathcal{Top}_{B_0}]$.

Each subspace A of the space X over B_0 is automatically a space over B_0 (with the projection $\pi_A = \pi_X|A$), and the embedding $i : A \to X$ is a map over B_0. In the case when there is a map $r : X \to A$ over B_0, such that $r \circ i = \mathrm{id}$, the subspace of A is called a *retraction over* B_0 (or a *fibrewise retraction*) of the space X (and the map r is a *retraction over* B_0 or *fibrewise retraction*). If, in addition, $i \circ r \underset{B_0}{\sim} \mathrm{id}$, then the retract A is called a *deformation retract over* B_0, and if $i \circ \underset{B_0}{\sim} \mathrm{id}\,\mathrm{rel}\,A$ (i.e. if the map $i \circ r$ and id are connected by homotopy over B_0, fixed on A), then a *strong deformation retract over* B_0.

For any space X over B_0 we will consider the product of $X \times I$ as the space over B_0 with the projection $(x, t) \mapsto \pi_X(x)$. Then homotopies over B_0 will be nothing more than maps over B_0 of the form $X \times I \to Y$.

Maps $X \to Y^I$ corresponding exponentially to such homotopies will obviously be characterised by the fact that they translate X into the subspace $Y^I_{B_0}$ of the space Y^I consisting of paths $u : I \to Y$ such that $\pi_Y \circ u = \mathrm{const}$. Putting $\pi_Y \circ u = 0_{\pi(u)}$, we get the projection $\pi : Y^I_{B_0} \to B_0$, with respect to which these maps will be maps over B_0. Thus, homotopies over B_0 can also be interpreted as maps over B_0 of the form $X \to Y^I_{B_0}$.

A map of $i : A \to X$ over B_0 is called a *cofibration over* B_0 if for each diagramme of the form



all maps of which are maps over B_0, there is a closing map $\overline{F}$ over B_0.

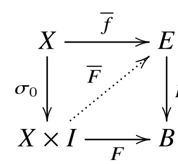
For any map $f : X \to Y$ over B_0, its cylinder $\mathrm{Cyl}(f)$ is a space over B_0 relative to the projection $[x, t] \mapsto \pi_X(x)$, $y \mapsto \pi_Y(y)$, and, as it is easy to see, Lemmas 2.5 and 2.7 of Lecture 2 will remain valid over B_0, i.e.

Proposition 2.45. *the map $i : X \to \mathrm{Cyl}(f)$ will be a cofibration over B_0, and the map $r : \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence over B_0 (and even a strong deformation retraction over B_0).*

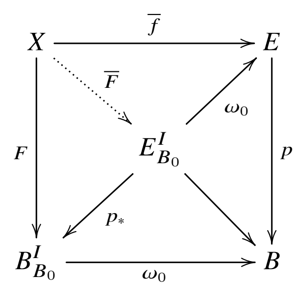
The Proposition 2.14 of Lecture 2 will also remain valid, as well as Lemma 2.26 of lecture 2 (in which, of course, for the pair (X, A) we need to require it to be a cofibration over B_0, i.e. that the embedding $i : A \to X$ be a cofibration over B_0). Therefore, Proposition 2.39 of Lecture 2 will also remain in force. In particular,

Proposition 2.46. *for any homotopy equivalence $f : X \to Y$ over B_0 the subspace X of the cylinder $\mathrm{Cyl}(f)$ will be its strong deformation retract over B_0.*

In a dual way, the map $p : E \to B$ over B_0 is called a *fibration over* B_0 if for each diagramme of the form



or, equivalently, each diagramme of the form



for all maps over B_0, there exists over B_0, a closing map of $\overline{F}$ (a *covering homotopy over* B_0).

The *cocylinder over* B_0 of the map $p : E \to In$ over B_0 is called a subspace $\mathrm{Cocyl}_{B_0}(p) \subset \mathrm{Cocyl}(p)$, consisting of pairs (u, e), $u : I \to B$, $e \in E$, $u(0) = p(e)$, such that $u \in B^I_{B_0}$, i.e. $\pi_B \circ u = \mathrm{const}$. It is a space over B_0 with the projection $(u, e) \mapsto \pi_B(p(e))$. It is clear that the map $q : E^I \to \mathrm{Cocyl}(p)$, $u \mapsto (p \circ u, u(0))$, will move $E^I_{B_0}$ to $\mathrm{Cocyl}_{B_0}(p)$ and therefore induces the map

$$E^I_{B_0} \to \mathrm{Cocyl}_{B_0}(p), \quad u \mapsto (p \circ u, u(0)),$$

which we will also denote by q. Obviously, this map is a map over B_0.

Having now re-examined the proof of Proposition 1.36 of Lecture 1, we will immediately make sure that it is fully preserved for fibrations over B_0, so that

Proposition 2.47. *the map $p : E \to B$ over B_0 will there be a fibration over B_0, if and only if there is a section* $\mathrm{Cocyl}_{B_0}(p) \to E^I_{B_0}$ *to the map* $q : E^I_{B_0} \to \mathrm{Cocyl}_{B_0}(p)$ *(which is automatically being a map over B_0).*

The results of Lecture 2 concerning fibrations are also transferred to the category $\mathcal{T}op_{B_0}$.

Reversed cocylinder over B_0 of the map $f : X \to Y$ over B_0 is the subspace of the the cocylinder $\mathrm{Cocyl}(f)$ consisting of pairs (u, x), $u : I \to Y$, $x \in X$, $u(1) = f(x)$, such that $u \in Y^I_{B_0}$. It is indicated by the symbol $\mathrm{Cocyl}_{B_0}(f)$ and is a space over B_0 with projection $(u, x) \mapsto \pi_X(f(x))$, and the map

$$p : \mathrm{Cocyl}_{B_0}(a) \to Y, \quad (u, x) \mapsto u(0)$$

is a map over B_0. Moreover, the map p will be a fibration over B_0. The proof comprises a verbatim repetition of the proof of Lemma 2.40 of Lecture 2.

Similarly, Lemma 2.41 of Lecture 2 is also preserved in the category $\mathcal{T}op_{B_0}$, i.e. for maps

$$r : \mathrm{Cocyl}_{B_0}(f) \to X, \quad (u, x) \mapsto x,$$
$$i : X \to \mathrm{Cocyl}_{B_0}(f), \quad x \mapsto (0_{f(x)}, x),$$

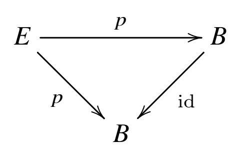
there are relations

$$r \circ i = \mathrm{id}, \quad i \circ r \underset{B_0}{\sim} \mathrm{id}\,.$$

In particular, the map i is a homotopy equivalence over B_0.

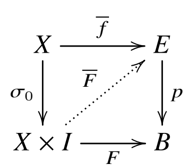
2.C Homotopy fibrations.

Application of these concepts and results to the category $\mathcal{T}op$ is based on the obvious observation that we can consider any map $p : E \to B$ as a map over B, considering the spaces E and B as spaces over B with projections $\pi_E = p$ and $\pi_B = \mathrm{id}$:



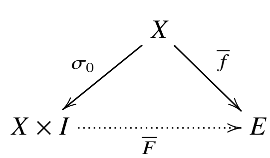
Then the statement that the map $s : B \to E$ is a cross section of the map p, would be equivalent to saying that it is a map over B.

Similarly, in any diagramme of the form



(2.48)

we can consider the spaces X and $X \times I$ as spaces over B with projections $\pi_X = p \circ \overline{f} = \overline{F} \circ \sigma_0$ and $\pi_{X\times I} = \overline{F}$ accordingly. Then the maps σ_0 and $\overline{f}$ will be maps over B, so over the category $\mathcal{T}op_B$ the diagramme will take place



(2.49)

and, therefore, the condition that the map $p : E \to B$ is a fibration will be equivalent to the fact that for any diagramme (2.49) over B, i.e. for any space X over B for which on the product $X \times l$ is given a structure of the space over B (i.e., the projection $\pi_{X\times I} : X \times I \to B$ is given) such that the embedding $\sigma_0 \times X \to X \times I$ is a map over B, and for any map $\overline{f} : X \to E$ it is necessary over

B, there is a map $\overline{F} : X \times I \to E$ over B, closing this diagramme, i.e. such that $\overline{F} \circ \sigma_0 = \overline{f}$.

This reformulation is not completely meaningless, since it allows for immediate generalisation.

Definition 2.50. It is said that the map $p : E \to B$ is a *homotopy fibration* ((or a *fibrarion in the sense of Dold*), if for any diagramme of the form (2.49) (consisting of maps over B) there exists a map $\overline{F} : X \times I \to E$ (over B) such that $\overline{F} \circ \sigma_0 \underset{B}{\sim} \overline{f}$, i.e., in other words, if for each diagram of the form (2.4) there is a map $\overline{F} : X \times I \to E$ such that its upper triangle is homotopically commutative over B (whereas the lower triangle is still commutative).

In other words, the map $p : E \to B$ is a homotopy fibration if, under the conditions of the axiom **HE** (see Definition 0.19), the covering homotopy begins (generally speaking) not with a given map $X \to E$, but with a map that is homotopic to it over B.

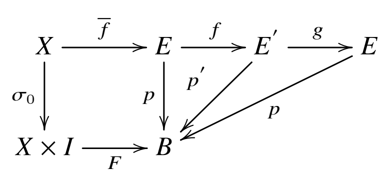
Any fibration is, of course, a homotopy fibration, but as the simplest examples show the converse is not true: (right angle $E = I \times \{0\} \cup \{0\} \times I$ on the plane $\mathbb{R} \times \mathbb{R}$ and its projection $p : (x, y) \mapsto x$ on the segment $B = I$).

Now it will be convenient for us to shift the emphasis somewhat and consider objects of the $\mathcal{T}op_B$ the map $p : E \to B$ themselves.

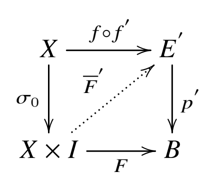
Lemma 2.51. *Any map $p : E \to B$ that is homotopy equivalent over B to a homotopy fibration $p' : E' \to B$, will also be a homotopy fibration.*

In other words, the property of being homotopy is is invariant with respect to fibrewise homotopy equivalences.

Proof. By condition , there are maps $f : E \to E'$, $g : E'\ toE$ over B such that $f \circ g \underset{B}{\sim} \mathrm{id}$ and $g \circ f \underset{B}{\sim} \mathrm{id}$. They allow each diagramme of the form (2.4) to be supplemented to a commutative diagramme



Since the map p' is a homotopy fibration, for a composite diagramme



there is a closing map $\overline{F}' : X \times I \to E'$, for which $p' \circ F' = F$ and $F' \circ \sigma_0 \underset{B}{\sim} f \circ f'$.

But then the map $\overline{F} = g \circ \overline{F}^{'} : X \times I \longrightarrow E$ will satisfy the relations

$$p \circ \overline{F} = p \circ g \circ \overline{F}^{'} = p^{'} \circ \overline{F}^{'} = F \quad \text{and} \quad F \circ \sigma_0 \underset{B}{\sim} g \circ f \circ \overline{f} \underset{B}{\sim} \overline{f}$$

that is, it will close Diagramme (2.49) in $[\mathscr{T}\!op_B]$. □

Remark 2.52. The relation $f \circ \underset{B}{\sim} \text{id}$ is not used in the proof of Lemma 2.51.

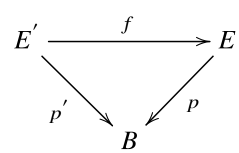
Remark 2.53. The homotopy fibration $(I \times \{0\} \cup (\{0\} \times I) \longrightarrow I$ is fibrewise homotopically equivalent to the identity fibration $I \longrightarrow I$. This shows that for fibrations, the analogue of Lemma 2.51 is incorrect.

The validity of Lemma 2.51 for homotopy fibrations is for us the raison d'être of this concept.

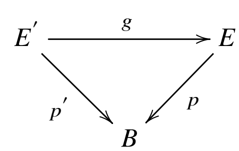
2.D Homotopy equivalences in comparison with fibrewise homotopy equivalences

Important circumstances concerning homotopy (and therefore ordinary) fibrations are revealed in connection with questions about the surjectivity on the morphisms of the functor (2.2) (for $B_0 = B$).

For any two objects $p^{'} : E^{'} \longrightarrow B$ and $p : E \longrightarrow B$ in the category $\mathscr{T}\!op\mathscr{B}$, the statement that this functor is surjective for morphisms from $p^{'}$ to p means that for any homotopy commutative diagramme of the form



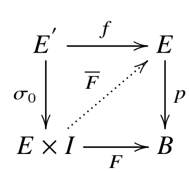
the map f is homotopic to the map $g : E^{'} \longrightarrow E$ over B, i.e. the map for which the diagramme



is commutative. It turns out that

Proposition 2.54. *this is obviously the case if the map $p : E \longrightarrow B$ is a fibration (at least homotopy).*

Proof. Indeed, in this case, in the diagramme



where F is the homotopy connecting the map $p \circ f$ and p', there is a covering homotopy $\overline{F}$, for which $\overline{F} \circ \sigma_0 \underset{B}{\sim} f$, and therefore the map $g = \overline{F} \circ \sigma_1$ will be homotopic to the map f (for $\overline{F} \circ \sigma_1 \overset{\overline{F}}{\sim} \overline{F} \circ \sigma_0$) and will have the property that $p \circ g = p \circ \overline{F} \circ \sigma_1 = F \circ \sigma_1 = p'$. □

It follows, in particular, that

Proposition 2.55. *if the homotopy fibration $p : E \to B$ has a section in the category $[\mathcal{T}op]$, then it will have a section in the category $\mathcal{T}op$, i.e., in other words, if there is a map $t : B \to E$, such that $p \circ t \sim \mathrm{id}$, then there is also a map $s : B \to E$ such that $p \circ s = \mathrm{id}$*

Proof. It is enough to apply the proven statement to the case when $E' = B$, $p' = \mathrm{id}$ and $f = t$. □

The following lemma expresses a much deeper fact.

Lemma 2.56. *If the map $p : E \to B$ is a homotopy fibration, then for any map $f : E \to E$ over B, homotopic (in $\mathcal{T}op$) to the identity map id_E, there exists over B a map $g : E \to E$ such that $f \circ g \underset{B}{\sim} \mathrm{id}$.*

Proof. By condition, there is a homotopy $F : E \times I \to E$ that connects the map f with the map id. Since $p \circ f = p$, the homotopy $p \circ F : E \times I \to B$ will connect the map p to itself. Therefore, the diagramme

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}} & E \\ {\scriptstyle \sigma_0}\downarrow & \overset{\overline{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ E \times I & \xrightarrow[p \circ F]{} & B \end{array}$$

is commutative, and, consequently, for it there will be a homotopy $\overline{F} : E \times I \to E$ covering the homotopy $p \circ F$ and such that $\overline{F} \circ \sigma_0 \underset{B}{\sim} \mathrm{id}$. Since, in view of the equality of $p \circ \overline{F} \circ \sigma_1 = p \circ F \circ \sigma_1 = p$, the map $g = \overline{F} \circ \sigma_1 : E \to E$ is a map over B, Lemma 2.56 will be proved if we show that the relation $f \circ g \underset{B}{\sim} \mathrm{id}$ holds for this map.

To this end, we will introduce into consideration the homotopy $H : E \times I \to E$, defined by the formula

$$H(x,t) = \begin{cases} (f \circ \overline{F})(x, 1 - 4t), & \text{if} \quad 0 \le t \le 1/4, \\ (f \circ G)(x, 4t - 1), & \text{if} \quad 1/4 \le t \le 1/2, \\ f(x), & \text{if} \quad 1/2 \le t \le 3/4, \\ F(x, 4t - 3), & \text{if} \quad 3/4 \le t \le 1, \end{cases}$$

where $G : E \times I \to E$ is a homotopy over B connecting the map $\overline{F} \circ \sigma_0$ to the map id_E. The homotopy H connects the map $f \circ g$ to the map id_E, but is not homotopy over B, since

$$(p \circ H)(x,t) = \begin{cases} (p \circ F)(x, 1-4t), & \text{if} \quad 0 \le t \le 1/4, \\ p(x), & \text{if} \quad 1/4 \le t \le 3/4, \\ (p \circ F)(x, 4t-3), & \text{if} \quad 3/4 \le t \le 1, \end{cases}$$

whereas for homotopy over B equality $(p \circ H)(x,t) = p(x)$ must be the case for all points $(x,t) \in E \times I$.

To fix the case, we'll look at the map $\Phi : E \times I \times I \to B$, defined by the formula

$$\Phi(x,t,\tau) = \begin{cases} (p \circ F)(x, 1-4t(1-\tau)), & \text{if} \quad 0 \le t \le \alpha(\tau), \\ p(x), & \text{if} \quad \alpha(\tau) \le t \le \beta(\tau), \\ (p \circ F)(x, 1-4(1-t)(1-\tau), & \text{if} \quad \beta(\tau) \le t \le 1, \end{cases}$$

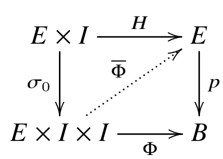
where $x \in E$, $0 \le t, \tau \le 1$, and

$$\alpha(\tau) = \min(1/2, 1/4(1-\tau)), \quad \beta(\tau) = \max(1/2, (3-4\tau)/4(1-\tau)).$$

A direct check shows that the map of Φ is continuous and that

$$\Phi(x,t,0) = (p \circ H), \quad (x,t), (x,\tau) \in E \times I,$$

i.e. that there is a commutative diagramme



Therefore, there is a map $\overline{\Phi} : E \times I \times I \to E$ covering the map Φ such that $H \underset{B}{\sim} \overline{\Phi} \circ \sigma_0$.

Because, as it is easy to see, for any point $x \in E$ and any $t, \tau \in I$ there are equalities

$$\Phi(x,0,\tau) = \Phi(x,t,1) = \Phi(x,1,\tau) = p(x),$$

from the fact that $\overline{\Phi}$ covers Φ, it follows that the correspondence

$$(x,x) \mapsto \overline{\Phi}(x,O,\tau), \quad (x,t) \mapsto \overline{\Phi}(x,t,1), \quad (x,\tau) \mapsto \overline{\Phi}(x,1,\tau)$$

define homotopies over B,the composition of which connects the map

$$\overline{\varphi}_0 : x \mapsto \overline{\Phi}(x,0,0) = (\overline{\Phi} \circ \sigma_0)(x,0)$$

with the map

$$\overline{\varphi}_1 : x \mapsto \overline{\Phi}(x,1,0) = (\overline{\Phi} \circ \sigma_0)(x,1).$$

Thus $\overline{\varphi}_0 \underset{B}{\sim} \overline{\varphi}_1$.

On the other hand, if $\Psi : E\times I\times I \to E$ is a homotopy over B connecting H to $\overline{\Phi}\circ /sigma_0$, then the homotopy $(x,\tau)\mapsto \Psi(x,0,\tau)$ will be a homotopy over B connecting the map $x\mapsto H(x,0) = (f\circ g)(x)$ with the map $x\mapsto \Psi(x,0,1) = \overline{\Phi}(x,0,0) = \overline{\varphi}_0(x)$, and the homotopy $(x,\tau)\mapsto \Psi(x,1,\tau)$ will be a homotopy over B connecting the map $x\mapsto H(x,1) = x$ with the map $x\mapsto \Psi(x,1,1) = \overline{\Phi}(x,1,0) = \overline{\varphi}_1(x)$.

Thus, $f\circ g \underset{B}{\sim} \overline{\varphi}_0$ and $\mathrm{id}_E \underset{B}{\sim} \overline{\varphi}_1$.

Consequently,

$$f\circ g \underset{B}{\sim} \varphi_0 \underset{B}{\sim} \varphi_1 \underset{B}{\sim} \mathrm{id}_E,$$

and Lemma 2.56 is proved. □

An immediate consequence of Lemma 2.56 is the following statement stating that the fibrewise fibration (at least homotopy) is a homotopy equivalence.

Proposition 2.57. *For any homotopy fibration $p' : E' \to B$ and $p : E\to B$, every map $f : R' \to E$ over B, which is a homotopy equivalence, will be a homotopy equivalence over B.*

Proof. By the condition, there is a continuous map $f' : E \to R'$ such that $f\circ f' \sim \mathrm{id}$ and $f'\circ f \sim \mathrm{id}$. At the same time, we can assume (replacing, if necessary, the map f with a homotopic map) that f' is a map over B. Then the map $f\circ f' : E\to E$ will satisfy all the conditions of Lemma 2.56 and, therefore, there will be a map $f'' : E\to E$ over B such that $(f\circ f')\circ f'' \underset{B}{\sim} \mathrm{id}$. Therefore, putting $g = f'\circ f''$, we obtain a map $g : E\to E'$ over B such that $f\circ g \underset{B}{\sim} \mathrm{id}$.

Since the map g is also a homotopy equivalence, the same construction applies to it. Therefore, over B there exists a map $h : E'E$ such that $g\circ h \underset{B}{\sim} \mathrm{id}$. But then

$$h \underset{B}{\sim} (f\circ g)\circ h = f\circ (g\circ h) \underset{B}{\sim} f,$$

and therefore $g\circ f \underset{B}{\sim} \mathrm{id}$.

Thus, f is a homotopy equivalence over B with the inverse equivalence g. □

Corollary 2.58. *Any homotopy fibration $p' : E' \to B$ is homotopically equivalent to an ordinary fibrewise fibration.*

Proof. According to the corollary of Lemma 2.41 of Lecture 2, the map p' decomposes into the composition $p\circ i$ with a homotopy equivalence $i : E' \to E$ and a fibration $p : E\to B$, where E is the cocylinder $\mathrm{Cocyl}(p')$ of the map p'. Since the equality $p' = p\circ i$ means that the homotopy equivalence i is a map over B from the homotopy fibration $p'E' \to B$ to the fibration $p : E\to B$ (considered as objects of the category $\mathcal{T}op_B$), Proposition 2.57 applies to it. Hence, this homotopy equivalence will be a homotopy equivalence over B. □

2.E Collapsing maps

Another important application of Proposition 2.57 relates to to the maps $p : E \to B$, which are, as selected over B, a homotopy equivalences over B. For such maps, the inverse equivalence $i : B \to E$, being a map on B, will automatically be a section of the map p and, therefore, a moneomorphism (homeomorphism onto its image). By embedding B by this moneomorphism into E, we obtain from p a deformation retraction over B. In this sense,

Proposition 2.59. *every map $p : E \to B$, which is a homotopy equivalence over B, will also be a deformation retraction over B.*

Maps $p : E \to B$ that are homotopy equivalences over B, we will call *collapsing maps*. According to what has been said, they are deformation retractions $p : E \to B$, such that with a homotopy connecting the maps id and $i \circ p$, each point $x \in E$ moves to the point $p(x) \in B$ of the set $p^{-1}(p(x))$.

Remark 2.60. The concept of collapsing map is dual to the concept of strong deformation retraction.

By definition, each map $p : E \to B$ is a morphism of the category $\mathscr{T\!op}_B$ of this map itself, considered as an object of the category $\mathscr{T\!op}_B$ of the object $\mathrm{id} : B \to B$. Therefore, the collapsing map $p : E \to B$ is homotopically equivalent over B to the fibration id. So, by applying Lemma 2.51, it is a homotopy fibration. In addition, it will of course be a homotopy equivalence.

Conversely, let $p : E \to B$ be a homotopy fibration, which is a homotopy equivalence. Since this map is also a map over B of homotopy fibrations (of itself of the fibration id), Proposition 2.57 applies to it, according to which it will be a homotopy equivalence over B, i.e. a collapsing map.

Thus, it has been proven that

Proposition 2.61. *the map $p : E \to B$ is a collapsing if and only if it is a homotopy fibration and simultaneously a homotopy equivalence.*

Further, it is easy to see that

Proposition 2.62. *any collapsing map $p : E \to B$ is weak*

(i.e., satisfies the axiom **WCHE** defined in Section 0.A in Appendix to Lecture 1).

Proof. Indeed, if U is a functional neighbourhood in B of a subset of $A \subset B$ and $\overline{s} : U \to E$ is an arbitrary section of the map p over U, then (without loss of generality that the neighbourhood of U is closed) the formula

$$s(x) = \begin{cases} F(\overline{s}(s), \varphi(x)), & \text{if} \quad x \in U, \\ i(x) & \text{if} \quad \varphi(x) = 1, \end{cases}$$

where $i : B \to E$ is the section of the map p, which is the inverse homotopy equivalence over B, F is the homotopy $E \times I \to E$ over B, connecting the map

$i \circ p$ with the map id, and φ is a function $B \to I$ such that $\varphi = 0$ on A and $\varphi = 1$ outside U, well defines the section $s : B \to E$ for the map p, coinciding on A with the section $\overline{s}$. □

2.F Dold's theorem on fibrewise homotopy equivalences

Now we can prove an important theorem of Dold, which asserts the local character of the notion of fibrewise homotopy equivalence.

Let $p' : E' \to B$ and $p : E \to B$ are arbitrary maps (objects of the category $\mathcal{T}op_B$), and let $f : E' \to E$ be a map over B from p' to p:

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

Let, further, $\{U_\alpha, \alpha \in A\}$ be the numerable covering of the space B, and let $\{V'_\alpha\}$ and $\{V_\alpha\}$ be its preimages under the maps p' and p, respectively (which, as we know, are numerable covers of spaces E' and E, respectively).

Then for any $\alpha \in A$ the map f will induce some map

$$f_\alpha : V'_\alpha \to V_\alpha,$$

closing the commutative diagramme

$$\begin{array}{ccc} V'_\alpha & \xrightarrow{f_\alpha} & V_\alpha \\ & {\scriptstyle p'_\alpha}\searrow \quad \swarrow{\scriptstyle p_\alpha} & \\ & U_\alpha & \end{array} \qquad p'_\alpha = p'|_{V'_\alpha}, \quad p_\alpha = p'|_{V_\alpha}$$

that is, being a map over U_α.

If f is a fibrewise homotopy equivalence (over B), then f_α will also obviously be fibrewise homotopy equivalences (over U_α): the inverse equivalences of $g_\alpha : V_\alpha \to V'_\alpha$ will be induced by the inverse equivalence of $g : E \to E'$. It turns out, that the converse statement is also true.

Theorem 2.63. *If for any $\alpha \in A$ the map $f_\alpha : V'_\alpha \to V_\alpha$ is a fibrewise homotopy equivalence, then the map $f : E' \to E$ will also be a fibrewise homotopy equivalence.*

Proof. As noted at the beginning of this Appendix, the map f over B decomposes into a composition of the homotopy equivalence $i : E' \to \mathrm{Cocyl}_B(f)$ over B and the fibration $\overline{f} : \mathrm{Cocyl}_B(f) \to E$ over B:

$$f : E' \xrightarrow{f} \mathrm{Cocyl}_B(f) \xrightarrow{\overline{f}} E$$

(we denote the fibration with the symbol $\overline{f}$, since the former symbol p is currently used for another purpose). A similar decomposition

$$f_\alpha : V'_\alpha \xrightarrow{i_\alpha} \mathrm{Cocyl}_{U_\alpha}(f_\alpha) \xrightarrow{\overline{f}_\alpha} V_\alpha$$

admits, of course, every map f_α. At the same time, the cocylinder $\mathrm{Cocyl}_{U_\alpha}(f)$ is naturally identified with the preimage of the set U_α with the projection $\mathrm{Cocyl}_B(f) \to B$, $(x, u) \mapsto p(x)$, and the map $\overline{f}_\alpha$ is the restriction of the map of $\overline{f}$ on this preimage.

Since by the condition the maps f_α are homotopy equivalences, so the maps $\overline{f}_\alpha$ will also be homotopy equivalences and therefore, being fibrations, they will be collapsing, and hence weak maps. Since the covering $\{V_\alpha\}$ is numerable and $V'_\alpha = f_\alpha^{-1}(V_\alpha)$ for every $\alpha \in A$, hence, by applying Lemma 1.63 of the Appendix to Lecture 1, it follows that the map $\overline{f}$ is also weak and therefore has a section $\overline{f}' : E \to \mathrm{Cocyl}_B(f)$, which is automatically a map over B. Therefore, the map $g = j \circ f'$, where j is the homotopy equivalence of $\mathrm{Cocyl}_B(f) \to E'$, the inverse of the equivalence i, will be a map over B, satisfying the relation

$$f \circ g = \overline{f} \circ i \circ j \circ \overline{f}' \underset{B}{\sim} \overline{f} \circ \overline{f}' = \mathrm{id}\,.$$

In addition, the map g will translate each set V_α into a set V'_α and, therefore, will induce some map $g_\alpha : V_\alpha \to V'_\alpha$, which obviously satisfies the relation $f_\alpha \circ g_\alpha \underset{U_\alpha}{\sim} \mathrm{id}$ and therefore is a homotopy equivalence over B, the inverse of the homotopy equivalence $f_\alpha : V'_\alpha \to V_\alpha$.

We see, therefore, that the map g satisfies the same conditions as the map f (with the permutation E and E'). Therefore, according to what has already been proven, there exists a map $h : E' E$ such that $g \circ h \underset{B}{\sim} \mathrm{id}$. But then $h \underset{B}{\sim} f \circ g \circ h \underset{B}{\sim} f$, and hence $g \circ f \underset{B}{\sim} \mathrm{id}$.

So $f \circ g \underset{B}{\sim} \mathrm{id}$ and $g \circ f \underset{B}{\sim} \mathrm{id}$, so f is homotopy equivalence over B. □

2.G Induced homotopy fibrations

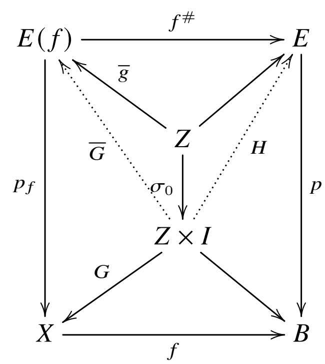
From Lecture 1 we know that for any fibration $p : E \to B$ and any continuous map $f : X \to b$ the induced map

$$p_f : E(f) \to X,$$

where to simplify formulae the pushout $E \cap_f X$ being denoted by $E(f)$, is also a fibration. It is easy to see that this statement also holds for homotopy fibrations, i.e.

Proposition 2.64. *for any homotopy fibration $p : E \to B$, the map $p_f : E(f) \to X$ is also a homotopy fibration.*

Proof. Indeed, with the reasoning as in Lecture 1, let's consider an arbitrary diagramme of the form



Since $p : E \to B$ is a homotopy fibration, there is a map $H : Z \times I \to E$ such that $p \circ H = f \circ G$ and $H \circ \sigma_0 \sim f^{\#} \circ \overline{g}$ over B. The equality $p \circ H = f \circ G$ means that the pair (H, G) is an inverse cone over the pair (p, f). Therefore, there is a unique map $\overline{G} : Z \times I \to E(f)$, for which $f^{\#} \circ \overline{G} = H$ and $p_f \circ \overline{G} = G$. Similarly, if $h_t : Z \to E$ is a homotopy over B connecting the maps $H \circ \sigma_0$ and $f^{\#} \circ g$, then for any $t \in I$ there will be an equality $p \circ h_t = f \circ G \circ \sigma_0$, showing that the pair $(h_t, G \circ \sigma_0)$ is an inverse cone over the pair (p, f). Therefore, there is a unique map $\overline{h}_t : Z \to E(f)$, for which $f^{\#} \circ \overline{h}_t = h_t$ and $p_f \circ \overline{h}_t = G \circ \sigma_0$. The maps h_t constitute a homotopy (why?) from Z to $E(f)$. Since $f^{\#} \circ \overline{h}_0 = p_f \circ \overline{G} \circ \sigma_0$ and $p_f \circ \overline{h}_0 = G \circ h_0 = p_f \circ \overline{G} \circ \sigma_0$, then by applying uniqueness $\overline{h}_0 = \overline{G} \circ \sigma_0$. Similarly, since $f^{\#} \circ \overline{h}_1 = h_1 = f^{\#} \circ \overline{g}$ and $p_f \circ \overline{h}_1 = G \circ \sigma_0 = p_f \circ \overline{h}_0$, then $\overline{h}_1 = \overline{g}$. Finally, since $p_f \overline{h}_t = G \circ \sigma_0 = p_f \circ \overline{h}_0$, the homotopy $\overline{h}_t$ is a homotopy over X. So $p_f \circ \overline{G} = G$ and $\overline{G} \circ \sigma_0 \sim \overline{g}$ over X, as required. □

2.H Fibrations induced by homotopic maps

Now let's compare the fibration $p_f : E(f) \to X$ with the fibration $p_g : E(g) \to X$, where g is the map $X \to B$, homotopic to the map f.

To this end, we will first consider an arbitrary homotopy fibration $p : E \to B_0 \times I$ with a base of the form $B_0 \times I$. Let $E_0 = p^{-1}(B_0 \times 0)$ and $p_0 : E_0 \to B_0$ be the restriction of the map of p to E_0 (we identify $B_0 \times 0$ with B_0). Let, further, $i_0 : E_0 \to E$ be an inclusion.

We will consider E_0 and E to be spaces over B_0 with projections p_0 and $\text{proj}_{B_0} \circ p$, respectively. Then i_0 will obviously be a map over B_0.

Lemma 2.65. *The map i_0 is a homotopy equivalence over B_0.*

Proof. Consider a commutative diagramme

$$\begin{array}{ccc} E & \xrightarrow{\ \mathrm{id}\ } & E \\ {\scriptstyle \sigma_0}\downarrow & \overset{\overline{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ E \times I & \xrightarrow[F]{} & B_0 \times I \end{array}$$

the map F of which is defined by the formula

$$F(e,t) = (b, \tau(1-t)), \quad \text{if} \quad p(e) = (b,\tau), e \in E, b \in B, \tau, t \in I.$$

Since the map p is by convention a homotopy fibration, there is a covering ($\circ F = F$) homotopy for this diagramme $\overline{F} : E \times I \to E$, which has the property that id $\sim \overline{F} \circ \sigma_0$ over $B_0 \times I$, and therefore over B_0. It is clear that the homotopy of $\overline{F}$ is a homotopy over B_0, such that $\overline{F} \circ \sigma_0 \sim \overline{F}\sigma_1$ over B_0. Hence, id $\sim \overline{F} \circ \sigma_i$ over B_0. But since $p \circ \overline{F} \circ \sigma_1 = F \circ \sigma_1 : e \mapsto (b,0)$, if $p(e) = (b,\tau)$, then $(\overline{F} \circ \sigma_1(E) \subset E_0$ and, therefore, $\overline{F} \circ \sigma_1 = i_0 \circ r$ where $r : E \to E_0E$ is some map over B_0.

Thus, we have constructed a map $r : E \to E_0$ such that $i_0 \circ r \sim$ id over B_0. Therefore, to complete the proof of the lemma it remains only to show that $r \circ i_0 \sim$ id over B_0.

Since the homotopy connecting the maps id_B and $\overline{F} \circ \sigma_0$ is a homotopy over $B_0 \times I$, it induces some homotopy from E_0 to E_0 over B_0, connecting the map id_{E_0} with the map $(\overline{F} \circ \sigma_0)_0 : E_0 \to E_0$, induced by the map $\overline{F}\sigma_0$. On the other hand, if $e \in E_0$, i.e. $p(e) = (B,0)$, then by construction $F(e,t) = (b,0)$ and, therefore, $(p \circ F)(e,t) = (b,0)$, i.e. $\overline{F}(e,t) \in E_0$ for any $t \in I$. Therefore , the homotopy $\overline{F}$ also induces some homotopy from E_0 to E_0 over B_0, connecting the map $(\overline{F} \circ \sigma_0)_0$ with the map $(\overline{F} \circ \sigma_1)_0 : E_0 \to E_0$, induced by the map $\overline{F} \circ \sigma_1$. Hence, id $\sim (\overline{F} \circ \sigma_1)_0$ over B_0.

But the fact that the map $(\overline{F} \circ \sigma_1)_0$ is induced by the map $\overline{F} \circ \sigma_1$ means that there is a commutative diagramme

$$\begin{array}{ccc} E_0 & \xrightarrow{(\overline{F}\circ\sigma_1)_0} & E_0 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ E & \xrightarrow[\overline{F}\circ\sigma_1]{} & E \end{array}$$

Hence, $i_0 \circ (\overline{F} \circ \sigma_1)_0 = (\overline{F} \circ \sigma_1) \circ i_0 = i_0 \circ r \circ i_0$, and therefore $(\overline{F} \circ \sigma_1)_0 = r \circ i_0$. So, $r \circ i_0 \sim$ id over B_0. □

Remark 2.66. It is clear that the same statement is true for embedding $i_1 : E_1 \to E$, where $E_1 = p^{-1}(B_0 \times 1)$.

We apply Lemma 2.65 to the homotopy fibration $p_F : E(F) \to X \times I$ induced by some homotopy $F : X \times I \to B$ from the homotopy fibration $p : E \to B$. If F connects the map $f : X \to B$ with the map $g : X \to B$, then, as follows directly from the functorial property of the pushout, the fibration $(p_F)_0 : E(F)_0 \to X$ for the fibration p_F coincides with the fibration $p_f : E(f) \to X$ (so, in particular, $E(F)_0 = E(f)$), and the fibration $(p_F)_1 : E(F)_1 \to X$ coincides with the fibration $p_g : E(g) \to X$. Thus, according to Lemma 2.65, there are homotopy equivalences over X

$$E(f) \xrightarrow{i_0} E(F) \xleftarrow{i_1} E(g)$$

Hence, the homotopy fibrations $p_f : E(g) \to X$ and $p_g : E(g) \to X$ are homotopically equivalent over X.

This proves the following Proposition.

Proposition 2.67. *Homotopy fibrations $p_f : E(f) \to X$ and $p_g : E(g) \to X$ induced from the homotopy fibration $p : E \to B$ by homotopy maps $f, g : X \to B$ are fibrewise homotopy equivalent.*

The fibrewise homotopy equivalence connecting the fibrations p_f and p_g is uniquely determined up to the fibrewise homotopy by the homotopy F connecting the maps f and g. We will denote it with the symbol $p(F)$.

By definition of the induced fibration for each map $f : X \to B$ the map $f^\# : E(f) \to E$ is defined for which the diagramme

$$\begin{array}{ccc} E(f) & \xrightarrow{f^\#} & E \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

is commutative. At the same time, it is clear that for any homotopy $F : X \times I \to B$ that connects the map f with the map g, the constraints $(F^\#)_0 = F^\# \circ i_0$ and $(F^\#)_1 = F^\#) \circ i_1$ of the map $F^\# : E(F) \to E$ on the subspaces $E(F)_0) = E(f)$ and $E(F_1) = E(g)$, respectively, coincide with the maps $f^\#$ and $g^\#$, so that the diagramme

$$\begin{array}{ccccc} E(F) & \xrightarrow{i_0} & E(f) & \xleftarrow{i_1} & E(g) \\ & {\scriptstyle f^\#}\searrow & \downarrow{\scriptstyle F^\#} & \swarrow{\scriptstyle g^\#} & \\ & & E & & \end{array}$$

is commutative. For the homotopy equivalence $p(F)$, it follows that

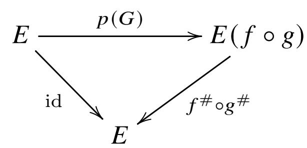
Proposition 2.68. *the diagramme*

$$\begin{array}{ccc} E(F) & \xrightarrow{p(F)} & E(g) \\ & {\scriptstyle f^\#}\searrow \quad \swarrow{\scriptstyle g^\#} & \\ & E & \end{array}$$

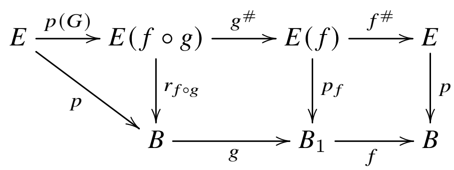
is homotopy commutative.

(The statement about homotopy commutativity is the best possible one, since the map $p(F)$ is defined up to homotopy.)

Now let $f : B_i \to B$ be a homotopy equivalence with inverse homotopy equivalence $g : B \to B_i$ and let $G : B \times I \to B$ be a homotopy such that $G : \mathrm{id} \sim f \circ g$. Because, as we know (see lecture 1), $\mathrm{id}^{\#} = \mathrm{id}$ and $(f \circ g)^{\#} = f^{\#} \circ g^{\#}$, for composition $f^{\#} \circ g^{\#}$ of maps $f^{\#} : E(f) \to E$ and $g^{\#} : E(f \circ g) \to E(f)$ there is a homotopy commutative diagramme



meaning that in the diagramme



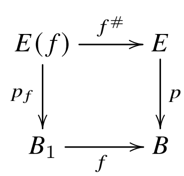
the composition of the arrows of the top line is homotopic to the identity map. So $f^{\#} \circ g^{\#} \sim \widehat{p}(G)$, where $\widehat{p}(G)$ is a homotopy equivalence and the inverse to the equivalence of $p(G)$, and therefore $p(G) \circ f^{\#} \circ g^{\#} \sim p(G) \circ \widehat{p}(G) \sim \mathrm{id}$. This proves that there are maps $\varphi : E(f) \to E(f \circ g)$ and $\psi : E \to E(f)$ (namely, maps $\varphi = p(G) \circ f^{\#}$ and $\psi = g^{\#} \circ p(G)$) such that $\varphi \circ g^{\#} \sim \mathrm{id}$ and $f^{\#} \circ \psi \sim \mathrm{id}$.

Applying this statement to the fibration $p_f : E(f) \to B_1$ and to the map $g \circ f$ (also by the condition homotopy identity map), we, in particular, get, that there is a map $\psi_1 : E(f) \to E(f \circ g)$ such that $g^{\#} \circ \psi \sim \mathrm{id}$. But then $\psi_1 \sim (\varphi \circ g^{\#}) \circ \psi_1 = \varphi \circ (g^{\#} \circ \psi_1) \sim \varphi$ and, therefore, $g^{\#} \circ f \sim \mathrm{id}$. Thus, $g^{\#}$ is a homotopy equivalence with the inverse homotopy equivalence φ.

Since $f^{\#} \circ g^{\#} \circ p(G) \sim \mathrm{id}$, it follows that the map $f^{\#}$ is also a homotopy equivalence (with inverse homotopy equivalence $f^{\#} \circ p(G)$).

Thus we have proven

Corollary 2.69. *If in the diagramme*



the map f is a homotopy equivalence, then the map $f^{\#}$ will also be a homotopy equivalence.

Another interesting case arises when the map f is homotopic to the constant map.

Corollary 2.70. *If the map $f : X \to B$ is homotopic to the constant map* $\mathrm{const} : X \to B$, $x \mapsto b_0$, *then for any homotopy fibration $p : E \to B$, the induced fibration $p_f : E(f) \to X$ is fibrewise homotopically equivalent to the trivial fibration with a fibre $F = p^{-1}(b)$.*

Proof. By definition, the space $E(\mathrm{const})$ consists of pairs $(x, e) \in X \times E$ for which $p(e) = \mathrm{const}(x) = b_0$ and, therefore, is the product $X \times F$. The map p_{const} : $E(\mathrm{const}) \to X$, $(x, e) \mapsto x$, will therefore be a projection of $X \times F \to X$. □

A space B is called *semilocally contractible* if any of its points has a neighbourhood U such that the embedding $U \to B$ is homotopic to the constant map. A fibration $p : E \to B$ is called *homotopically locally trivial* if any point in the space B has a neighbourhood U such that the fibration $p_U : p^{-1}(U) \to U$ is homotopically equivalent to the trivial fibration $U \times F \to U$ (where F, generally speaking, depends on U).

Corollary 2.71. *If the space B is semilocally contractible, then any homotopy fibration $p : E \to B$ is homotopically locally trivial.*

Induced fibrations also have the property of functoriality with respect to fibrewise maps, i.e. for each continuous map $f : X \to B$ any fibrewise map $h : E \to E'$ induces (by the formula $h_f(x, e) = (x, h(e))$) some fibrewise map $h_f : E(f) \to E'(f)$, with $h_{\mathrm{id}} = \mathrm{id}$, $h_{f \circ g} = h_f \circ h_g$ and, similarly, $\mathrm{id}_f = \mathrm{id}$, $(h \circ k)_f = h_f \circ k_f$. In addition, for any homotopy $F : X \times I \to B$ connecting the map f to the map g with the constraint $(h_F)_0 = h_F \circ i_0$ and $(h_F)_1 = h_F \circ i_1$ the maps h_F on the subspaces $E(F)_0 = E(f)$ and $E(F)_1 = E(g)$ correspond to the maps h_f and h_g, respectively, so that the diagramme

$$\begin{array}{ccccc} E(f) & \xrightarrow{i_0} & EF & \xleftarrow{i_1} & E(g) \\ \big\downarrow{\scriptstyle h_f} & & \big\downarrow{\scriptstyle h_F} & & \big\downarrow{\scriptstyle h_g} \\ E'(f) & \xrightarrow[i_0']{} & E' & \xleftarrow[i_1']{} & E'(g) \end{array}$$

is commutative. Therefore

Proposition 2.72. *the diagramme*

$$\begin{array}{ccc} E(f) & \xrightarrow{p(F)} & E(g) \\ \big\downarrow{\scriptstyle h_f} & & \big\downarrow{\scriptstyle h_g} \\ E'(f) & \xrightarrow[p'(f)]{} & E'(g) \end{array}$$

is commutative.

In the special case when g is a constant map $\mathrm{const} : X \to B$, $x \mapsto b_0$, and, therefore, the fibrations $p_g : E(g) \to X$ and $p_g : E'(g) \to X$ have accordingly the form $X \times F \to X$ and $X \times F' \to X$, the map h_g is given by the correspondence $(x, y) \mapsto (x, h(y))$, $y \in F$, and therefore will be a fibrewise homotopy equivalence if the homotopy equivalence is the map $h_{b_0} : F \to F'$, induced by the map h. But then the fibrewise homotopy equivalence will, of course, be the map h_f.

We will formulate this result as a separate corollary.

Corollary 2.73. *If the map $f : X \to B$ is homotopic to the constant map* const $: X \to B$, $x \mapsto b_0$, *then for any fibrewise map $h : E \to E'$ with the homotopy fibration $p : E \to B$ and the homotopy fibration $p' : E' \to B$, having the property that the map $h_{b_0} : p^{-1}(b_0) \to p'^{-1}(b_0)$ is a homotopy equivalence, and the map*

$$h_f : E(f) \to E'(f)$$

of induced fibrations will be a fibrewise homotopy equivalence.

2.I Maps that are homotopy equivalences on fibres

A space B is called *enumerably semilocally contractible* if there exists an enumerable covering $\{U_\alpha\}$ such that any embedding $U_\alpha \to B$ is homotopic to the constant map.

It is said that the layered map $h : E \to E'$ of from homotopy fibration $p : E \to B$ to a homotopy fibration $p' : E' \to B$ is a fibrewise homotopy equivalence if for any point $b \in B$ the map $h_b : p^{-1}(b) \to p'^{-1}(b)$ is a homotopy equivalence. It is clear that any fibrewise homotopy equivalence will be a homotopy equivalence on fibres. For a numerically semi-locally contractible space B, the converse is also true:

Proposition 2.74. *If the space B is enumerably semilocally contractible, then any fibrewise map $h : E \to E'$ form a homotopy fibration $p : E \to B$ to a homotopy fibration $p' : E' \to B$, which is a homotopy equivalence on fibres, will be a fibrewise homotopy equivalence.*

Proof. Let $\{U_\alpha\}$ be a enumberable covering of the space B such that for any α the embedding $U_\alpha \to B$ is homotopic to the constant map. Then according to corollary 2.73 every map

$$h_{U_\alpha} : p^{-1}(U_\alpha) \to p'^{-1}(U_\alpha)$$

will be a fibrewise homotopy equivalence (recall that for any set $U \subset B$ the fibration $_U = p|_{p^{-1}(U)} : p^{-1}(U) \to U$ coincides with the fibration $p_i : E(i) \to U$, induced by the embedding $i : U \to B$). Therefore, according to Theorem 2.63, the map $h : E \to E'$ will also be a fibrewise homotopy equivalence. □

Remark 2.75. For any point $b_0 \in B$ the space $^{-1}(b_0)$ is a pushout of $E(f)$ corresponding to the map $f : \mathrm{pt} \to B$, translating the point pt to the point $b_0 \in B$. On the other hand, each path $U : I \to B$ of the space B we can consider one as a homotopy connecting maps pt $\to B$ corresponding to points $b_0 = u(0)$ and $b_1 = u(1)$. Therefore, for maps $h_{b_0} : p^{-1}(b_0) \to p^{'-1}(b_0)$ and $h_{b_1} : p^{-1}(b_1) \to p^{'-1}(b_1)$ there is a homotopically commutative diagramme

$$\begin{array}{ccc} p^{-1}(b_0) & \longrightarrow & p^{-1}(b_1) \\ {\scriptstyle h_{b_0}}\downarrow & & \downarrow{\scriptstyle h_{b_1}} \\ p^{'-1}(b_0) & \longrightarrow & p^{'-1}(b_1) \end{array}$$

the horizontal arrows of which are homotopy equivalences. Hence, the map h_1 is a homotopy equivalence if and only if the the map h_0 is a homotopy equivalence. Hence the map $h : E \to E^{'}$ is a homotopy equivalence on fibres if and only if in each connected component of the space B there exists a point b such that the map $h_0 : p^{-1}(b) \to p^{'-1}(b)$ represents a homotopy equivalence.

Lecture 3

3.1 Homotopy invariant functors

As explained in introductory Lecture 0, the main task of algebraic topology is to construct and study various algebraic functors Π defined on the category of $\mathcal{Top}$, with the aim of their subsequent application to the processing of geometric problems into derived algebraic problems. However, we don't need any functors. Since we mean to consider only geometric problems whose formulation is invariant with respect to the transition to homotopy maps (according to the "power" arguments explained in Lecture 0, only such problems can count on an effective solution by means of algebra), it is natural to limit ourselves to *homotopy invariant* functors Π, having the property that $\Pi f = \Pi g$ when $f \sim g$, i.e., in other words, being a composition of the factorisation functor $\mathcal{Top} \to [\mathcal{Top}]$ and some functor $\overline{\Pi}$ given on the homotopy category $[\mathcal{Top}]$. (In general, it would be more consistent to leave the category of $\mathcal{Top}$ and finally move into the category of $[\mathcal{Top}]$, but for purely psychological reasons we prefer stay in the more familiar reality of the $\mathcal{Top}$ category.) Therefore, we will focus our attention on methods for constructing only homotopy invariant functors.

3.2 The functor π_0

The simplest example of a homotopy invariant functor is obtained by considering for any topological space X the set $\pi_0 X$ of all its components. Since for each continuous map $f : X \to Y$ component $[f(x)]$ of the space Y containing the point $f(x)$ obviously depends only on the component $[(x)]$ the space X containing the point x, then the correspondence

$$\pi_0 f : [x] \mapsto [f(x)]$$

well defines some map of the set $\pi_0 f : \pi_0 X \to \pi_0 Y$, and it is clear that thereby we get the functor

$$\pi_0 : \mathcal{Top} \to \mathcal{Ens}$$

from the category $\mathcal{Top}$ to the category of sets $\mathcal{Ens}$, which is obviously homotopy invariant.

The corresponding functor $[\mathscr{T\!op}] \to \mathscr{E\!ns}$ is also denoted by the symbol π_0.

As we already noted in Lecture 1, the components of the space X are naturally identified with homotopy classes of maps $\mathrm{pt} \to X$, where pt is a topological space consisting of a single point. Therefore, $\pi_0 X = [\mathrm{pt}, X]$ and, accordingly, $\pi_0[f]$ is nothing more than the map $f_* : [\mathrm{pt}, X] \to [\mathrm{pt}, Y]$, defined by the formula $[x] \mapsto [f \circ x]$, where $x : \mathrm{pt} \to X$.

3.3 Representable functors

This interpretation of the functor π_0 allows for a very far-reaching generalisation to any categories of $\mathscr{A}$. Namely, by arbitrarily selecting some object K in the category $\mathscr{A}$, we can construct a functor K_* from $\mathscr{A}$ to $\mathscr{E\!ns}$, matching any object X from $\mathscr{A}$ the set

$$K_*(X) = \mathscr{A}(K, X)$$

for all morphisms of $K \to X$, and any morphism $f : X \to Y$ from $\mathscr{A}$ to the map

$$K_*(f) : K_*(X) \to K_*(Y)$$

(also denoted by the symbol f_K or f_*), defined by the formula

$$K_*(f)(\alpha) = f \circ \alpha, \quad \alpha : K \to K.$$

Similarly, putting for any object X of $\mathscr{A}$

$$K^*(X) = \mathscr{A}(X, k)$$

and by matching each morphism $f : X \to Y$ of $\mathscr{A}$ with the map

$$K^*(f) : K^*(Y) \to K^*(X)$$

(also denoted by the symbol f^* or f^K), defined by the formula

$$K^*(f)(\beta) = \beta \circ f, \quad \text{for any morphism} \quad \beta : Y \to K,$$

we get a contravariant functor K^* from $\mathscr{A}$ to $\mathscr{E\!ns}$ (i.e., a functor from the opposite category $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{E\!ns}$).

The functors obtained by this construction are called *representable*. The object K is said to *represent* the functor K_* (or the functor K^*).

As applied to the homotopy category, this general technique allows us to construct two homotopy invariant functors for each topological space K

$$K_* : \mathscr{T\!op} \to \mathscr{E\!ns}, \quad K^* : \mathscr{T\!op}^{\mathrm{op}} \to \mathscr{E\!ns}$$

representable by this space (as functors on $[\mathscr{T\!op}]$). By definition

$$K_*(X) = [K, X], \quad K^*(X) = [X, K] \tag{3.1}$$

for any topological space X and

$$\begin{aligned} f_*(\alpha) &= [f \circ \alpha], \quad \alpha : K \to X, \\ f^*(\beta) &= [\beta \circ f], \quad \beta : Y \to X, \end{aligned} \tag{3.2}$$

for any continuous map $f : X \to Y$.

When $K = \text{pt}$ we get the functor π_0.

In some cases, with a special choice of the space K in the set $[K, X]$ or $[X, K]$, it is possible to introduce an algebraic structure (groups, rings, etc.) so that a functor is obtained in the corresponding algebraic category. Such functors are also said to be *represented by* the space K.

It turns out that many homotopy invariant functions of algebraic topology (including almost all contravariant functors) are representable. (This is partly explained by Brown's representability theorem, which we will prove in the next semester.) Therefore, at first it is reasonable to limit ourselves to studying only representable functors defined by formulae (3.1) and (3.2).

3.4 Category of groups

There is an obvious case when the set $K^*(X) = [X, K]$ has a natural (with respect to X) group structure: this is the case when K is a topological group. Indeed, for any topological group K and any topological space X, the set of $\mathcal{T\!op}(X, K)$ of all continuous maps $X \to K$ is obviously a group with respect to the multiplication operation $(f, g) \mapsto fg$, defined by the formula

$$(fg)(x) = f(x)g(x), \quad x \in X, \quad f, g : X \to K. \tag{3.3}$$

The unit of this group is a constant map $\text{const} : X \to K$ which maps to the unit e of the group K, and the element f^{-1}, the inverse of the element $f : X \to K$, is determined by the formula

$$f^{-1}(x) = f(x)^{-1} \tag{3.4}$$

(don't confuse f^{-1} with the inverse map!). If $\{f_t : f_0 \sim f_1$ and $\{g_t : g_0 \sim g_1$, then $\{f_t g_t : f_0 g_0 \sim f_1 g_1$ and, therefore, the formula $[f][g] = [fg]$ well defines multiplication in the set $[X, K]$, with respect to which this set is a group. At the same time, for any continuous map $\varphi : X \to Y$ the map $\varphi^* : [Y, K] \to [X, K]$, $[f] \mapsto [\varphi \circ f]$, will obviously be a homomorphism.

This design is definitely too tight for our purposes (which is evident, for example, from the fact that it gives too much - a group structure on the set of $\mathcal{T\!op}(X, K)$, completely unnecessary for us), but it will serve as a starting point for broader generalisations. To do this, it is necessary to comprehend the concept of a group from the general standpoint of category theory.

Multiplication in a group K is nothing more than some map

$$m : K \times K \to K, \quad (x, y) \mapsto xy, \tag{3.5}$$

form the direct product $K\times K$ into K, and the operation $x \mapsto x^{-1}$ of the transition to the inverse element is some map

$$\mu : K \to K.$$

At the same time, the fact that the element $e \in K$ is a unit of the group K means that for a constant map $\mathrm{const}_e : K \to K$, $x \mapsto e$, which is convenient for us to denote by the symbol ε, composite maps

$$K \xrightarrow{\mathrm{id}\times\varepsilon} K\times K \xrightarrow{m} K, \quad x \mapsto (x,e) \mapsto xe,$$
$$K \xrightarrow{\varepsilon\times\mathrm{id}} K\times K \xrightarrow{m} K, \quad x \mapsto (e,x) \mapsto ex,$$

represent identical maps, i.e. that the diagrammes

$$\begin{array}{ccc} & K\times K & \\ {\scriptstyle \mathrm{id}\times\varepsilon}\nearrow & & \searrow{\scriptstyle m} \\ K & \xrightarrow[\mathrm{id}]{\qquad} & K \end{array} \qquad \begin{array}{ccc} & K\times K & \\ {\scriptstyle \varepsilon\times\mathrm{id}}\nearrow & & \searrow{\scriptstyle m} \\ K & \xrightarrow[\mathrm{id}]{\qquad} & K \end{array} \tag{3.6}$$

are commutative. (Here the symbol is $\alpha\times\beta$, where $\alpha : X \to A$ and $\beta : X \to B$, we denote the map $X \to A\times B$, defined by the formula $(\alpha\times\beta)(x) = (\alpha(x), \beta(x))$. In the case when $\alpha : X \to A$ and $\beta : y \to B$, the same symbol will mean the expression $X\times Y \to A\times B$, defined by the formula $(\alpha\times\beta)(x,y) = (\alpha(x), \beta(y))$. Thus, when $X = Y$ there is ambiguity. However, with some care, it does not lead to misunderstandings.)

Similarly, the associativity of multiplication (3.5) means the commutativity of $(m\circ(\mathrm{id}\times m) = m\circ(m\times\mathrm{id}))$ in the diagramme

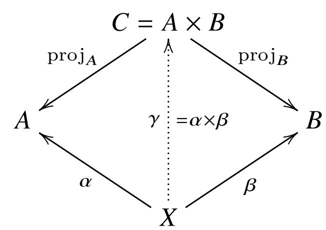
$$\begin{array}{ccc} K\times K\times K & \xrightarrow{\mathrm{id}\times m} & K\times K \\ {\scriptstyle m\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle m} \\ K\times K & \xrightarrow[m]{\qquad} & K \end{array} \tag{3.7}$$

and the identities $xx^{-1} = e$, $x^{-1}x = e$ are the commutativity of diagrammes

$$\begin{array}{ccc} & K\times K & \\ {\scriptstyle \mathrm{id}\times\mu}\nearrow & & \searrow{\scriptstyle m} \\ K & \xrightarrow[\varepsilon]{\qquad} & K \end{array} \qquad \begin{array}{ccc} & K\times K & \\ {\scriptstyle \mu\times\mathrm{id}}\nearrow & & \searrow{\scriptstyle m} \\ K & \xrightarrow[\varepsilon]{\qquad} & K \end{array} \tag{3.8}$$

Now let $\mathscr{A}$ be an arbitrary category. Recall that the (*direct*) *product* of objects A and B of the category $\mathscr{A}$ is an object C, such that considered together with two morphisms $\mathrm{proj}_A : C \to A$, $\mathrm{proj}_B : C \to B$, for any object $X \in \mathscr{A}$ and any morphisms $\alpha : X \to A$, $\beta : X \to B$ there is a single morphism $\gamma : X \to C$

having the property that $\mathrm{proj}_A \circ \gamma = \alpha$ and $\mathrm{proj}_B \circ \gamma = \beta$. The object C, if it exists, is unique up to canonical isomorphism. It is denoted by the symbol $A \times B$, and the morphism γ is denoted by the symbol $\alpha \times \beta$.



Similarly, the product of any (including empty!) families of objects of the category $\mathscr{A}$ is defined. In this case, the product of an empty family of objects will be an object 0, which has the property that for any object $A \in \mathscr{A}$ there is one and only one morphism $A \to 0$. Such an object is called a *terminal object* of the category $\mathscr{A}$. (We have already encountered this term in Lecture 1 in connection with the concept of pullback.)

Category $\mathscr{A}$ is called *finitely multiplicatively closed* (*fm-closed* for short) if for any finite family of its objects, there exists their product. In particular, the fm-closed category has a terminal object.

A morphism $A \to B$ in an fm-closed category is called *constant* if it is passed through a terminal object, i.e. it is a composit morphism of the form $A \to 0 \to B$.

Examples of fm-closed categories are the categories $\mathscr{E}\mathit{ns}$, $\mathscr{T}\mathit{op}$, $[\mathscr{T}\mathit{op}]$ and $\mathscr{G}\mathit{rp}$. In each of these categories, products are ordinary direct products of sets, topological spaces or groups, and terminal objects are singleton sets, spaces or groups. The constant morphisms of these categories are constant maps (for the category of groups, trivial homomorphisms).

Now it is clear that the concept of a group can be transferred to an arbitrary fm-closed category $\mathscr{A}$. Namely, an object K of such a category is naturally called a *$\mathscr{A}$ category of group* (or $\mathscr{A}$-group), if it is given morphisms $m : K \times K \to K$, $\mu : K \to K$, $\varepsilon : K \to K$, such that:

(a) the morphism ε is constant;

(b) diagrammes (3.6), (3.7) and (3.8) are commutative.

Groups of the $\mathscr{E}\mathit{ns}$ category of sets are ordinary abstract groups, groups of the $\mathscr{T}\mathit{op}$ category are topological groups (and, say, groups of the $\mathscr{D}\mathit{iff}$ category smooth manifolds are Lie groups).

Question: What are the groups of the $\mathscr{G}\mathit{rp}$ category?
Answer: See Lemma 4.93 from the Appendix to Lecture 4.

It is obvious that other algebraic structures allow a similar "categorisation". For example, a monoid of the category $\mathscr{A}$ (or an $\mathscr{A}$-monoid), where $\mathscr{A}$ is an arbitrary fm-closed category, is an object K of the category $\mathscr{A}$, for which the following morphism $m : K \times K \to K$ and a constant morphism $\varepsilon : K \to K$ such that Diagrammes (3.6), (3.7) are commutative are given.

In the situations we are interested in, the associativity condition (commutativity of diagram (3.7) will, oddly enough, often be superfluous. Unfortunately, there is still no generally accepted name for the corresponding algebraic structure (multiplication with one) We will call the set in which multiplication with one is given *unitoid*.
(Transcrier's note: that is, a unitoid is a magma with an identity element. Thus, an H-space is a homotopy unitoid.)
Accordingly for any fm-closed category $\mathscr{A}$, its object K will we call a *unitoid of the category* $\mathscr{A}$ (or a $\mathscr{A}$*-unitoid*) for which a morphism $m : K \times K \to K$ and a constant morphism $\varepsilon : K \to K$ are given such that Diagrammes (3.6) are commutative.

Formula (3.3) in categorical notation has the form

$$fg = m \circ (f \times g), \quad f, g : X \to K,$$

and formula (3.4) is the form

$$f^{-1} = \mu \circ f, \quad f : X \to K.$$

These formulae do make sense for any $\mathscr{A}$-group K and any object $X \in \mathscr{A}$ and, as is automatically checked, define the structure of the group in the set $\mathscr{A}(X, K)$ of all morphisms $X \to K$. The unit of this group is a constant morphism $X \to K$, which is a composition of an arbitrary morphism $\alpha : X \to K$ and the morphism $\varepsilon : K \to K$ (and obviously independent of the choice of morphism α).

A trivial check shows that for any morphism $f : X \to Y$ of the category $\mathscr{A}$ the map $f^* : \beta \mapsto \beta \circ f$ from $\mathscr{A}(Y, K)$ to $\mathscr{A}(X, K)$ is a homomorphism of groups and that the correspondences of $X \mapsto \mathscr{A}(X, K)$, $f \mapsto f^*$ define a functor from the category $\mathscr{A}$ to the category of groups, i.e., in other words, that the structure of the group in $\mathscr{A}(X, K)$ is natural in X

Conversely, suppose that for some object K of the category $\mathscr{A}$ all the sets $\mathscr{A}(X, K)$, $X \in \mathscr{A}$, are equipped with the natural structure of the group in X. Then, in particular, the set $\mathscr{A}(K \times K, K)$ of all morphisms $K \times K \to K$ will be a group. By definition, this group contains two morphisms $\mathrm{proj}_1 : K \times K \to K$ and $\mathrm{proj}_2 : K \times K \to K$. The product of these morphisms (as elements of the group $\mathscr{A}(K \times K, K)$) we denote by m. Similarly, the set $\mathscr{A}(K, K)$ is also a group. We will denote the unit of this group by e, and the element inverse to the element $\mathrm{id} : K \to K$ by μ. The equality $\mathrm{id} \cdot \varepsilon = \varepsilon \cdot \mathrm{id} = \mathrm{id}$ in the group $\mathscr{A}(K, K)$ is exactly equivalent to the commutativity of Diagrammes (3.6), and the equality $\mathrm{id} \cdot \mu = \mu \cdot \mathrm{id} = \varepsilon$ is equivalent to the commutativity of Diagrammes (3.8). As for Diagramme (3.7), its commutativity is equivalent to the relation $\mathrm{proj}_1 \cdot (\mathrm{proj}_2 \cdot \mathrm{proj}_3) = (\mathrm{proj}_1 \cdot \mathrm{proj}_2) \cdot \mathrm{proj}_3$ for the elements $\mathrm{proj}_1, \mathrm{proj}_2, \mathrm{proj}_3 : K \times K \times K \to K$ of the group $\mathscr{A}(K \times K \times K \to K, K)$. Moreover, from the fact that the map f^* for any morphism $f : X \to Y$ of the category $\mathscr{A}$ is a homomorphism from the group $\mathscr{A}(Y, K)$ to the group $\mathscr{A}(X, K)$ and therefore translates the unit of the first group into the unit of the second group, it immediately follows that for any morphism $f : K \to K$, the equality $\varepsilon = \varepsilon \circ f$ holds. In particular, this equality holds for every constant morphism $f : K \to K$ (existing because the group $\mathscr{A}(0,K)$ is not empty), which is possible only if the morphism ε itself is constant. Thus, the morphism m, μ, and ε satisfy all the conditions for defining a $\mathscr{A}$-group, so that with respect to them the object K is a group of the category $\mathscr{A}$. Finally, as the obvious automatic check shows, the structure of the group in each set $\mathscr{A}(X,K)$, defined by the formulae (3.3) and (3.4), coincides with the given one.

This proves that

Proposition 3.9. *the sets $\mathscr{A}(X,K)$ have a group structure natural in X if and only if the object K is a group of category $\mathscr{A}$.*

It is clear that the same statement is true with respect to monoids and unitoids.

3.5 H-groups, H-monoids and H-unitoids (H-spaces)

With respect to the category $[\mathscr{T}\!op]$, we obtain, in particular, that

Proposition 3.10. *sets $[X,K]$ have a group, monoid, or unitoid structure natural in X, if and only if the topological space K is a group, a monoid, or a unitoid of the category $[\mathscr{T}\!op]$.*

For groups of the category $[\mathscr{T}\!op]$ morphisms m, μ and ε are by definition homotopy classes of continuous maps. However, in practice it turns out to be convenient to consider some of their representatives instead of these classes (choosing, of course, in the class ε a constant map const_e, where e is some point). Thus, we come to the concept of a topological space K, for which continuous maps

$$m : K \times K \to K, \quad \mu : K \to K$$

and a point $e \in K$ such that Diagrammes (3.6), (3.7) and (3.8) (with $\varepsilon = \mathrm{const}_e$) are homotopically commutative, are given. This kind of topological space we will call an *H-group* the natural term "homotopy group", unfortunately, is already occupied).
(*Transcriber's note*: i.e., by the group $[\mathbb{S}^n, X]$.

Similarly, H-*monoids* and H-*unitoids* are defined. However, in topology it is customary to call H-unitoids H-*spaces*. Despite the colourlessness of this term (established only due to the lack of a good name in algebra for sets with a multiplication), we, in order not to break with tradition, will also use it.
(*Transcriber'snote*: J. P. Serre coined the term "magma" for a set with a multiplication.)

For any two elements x, y of arbitrary H-space K, element $m(x,y)$ is usually denoted by the symbol xy and is called the *product* of these elements. Similarly (in the case when K is an H-group), the element $\mu(x)$ is denoted by the symbol x^{-1} and is called the *inverse element* (or, more commonly, *homotopy inverse*) to the element x. The element e is called a *homotopy unit*.

According to the above general statement

Proposition 3.11. *for each H-space K and any topological space X, the set $[X,K]$ has a natural X structure of a unitoid (the structure of a monoid if K is a H-monoid, and the structure of a group if K is H-group).*

In this case, the multiplication in $[X,K]$ is determined by the formula

$$[f][g] = [h], \quad f,g : X \to K,$$

where h is the map $X \to K$ defined by formula (3.3), i.e. formula

$$h(x) = f(x)g(x), \quad x \in X.$$

The unit of this multiplication is the homotopy class $[\mathrm{const}_e]$ of the constant map $\mathrm{const}_e : X \to K$, and the operation $[f] \mapsto [f]^{-1}$ defined (in the case where K is H-group) by the formula $[f]^{-1} = [f^{-1}]$, where $f^{-1} : x \mapsto f(x)^{-1}$ for any point $x \in X$.

It is clear that by replacing the map m (and in the case of H-groups and the map μ) by an arbitrary homotopy map to it, and the element e is an arbitrary element lying in the same connected component, from a given H-space (H-monoid or H-group) we get again an H-space (H-monoid or H-group). This H-space (H-monoid or H-group) is called *equivalent* to the original H-space (H-monoid or H-group). In other words, two H-spaces (H-monoids or H-groups) are equivalent if they define the same unitoid (respectively, the same monoid or the same group) of the category $[\mathscr{Top}]$.

From the point of view of general algebra, the *morphisms* of H-spaces (H-monoids and H-groups) should be called continuous maps $f : K \to L$ for which the following diagrammes are commutative

$$\begin{CD} K\times K @>{f\times f}>> L\times L \\ @V{m}VV @VV{m}V \\ K @>>{f}> L \end{CD} \qquad \begin{CD} K @>{f}>> L \\ @V{\varepsilon}VV @VV{\varepsilon}V \\ K @>>{f}> L \end{CD} \tag{3.12}$$

and also in the case of H-group,the diagramme

$$\begin{CD} K @>{f}>> L \\ @V{\mu}VV @VV{\mu}V \\ K @>>{f}> L \end{CD} \tag{3.13}$$

We will call this kind of maps *algebraic morphisms.*

However, since H-spaces are for us only representatives of the unitoids of the category $[\mathscr{Top}]$, of much greater interest to us will be the maps $f : K \to L$, for which Diagrammes (3.12) and (3.13) are only homotopy commutative. We will call such maps *homotopy morphisms.*

Remark 3.14. In the case when K and L are H-groups, a continuous map $f : K \to L$ will be their homotopy morphism only if the first diagram in (3.12) is homotopically commutative (for f). Indeed, passing into the homotopy category $[\mathscr{T}\!op]$, we can reformulate this statement as a statement that the map of groups of the category $[\mathscr{T}\!op]$, preserving multiplication, translates one into one and the inverse elements in the inverse. For groups of the category $\mathscr{E}ns$, this is a well-known statement with a trivial proof (since $e^2 = e$, then $f(e^2) = f(e)$, and therefore $f(e) = f(e)^2 f(e)^{-1} = f(e)f(e)^{-1} = E$; since $xx^{-1} = e$, then $f(x)f(x^{-1}) = f(e) = e$, and thanks to this $f(x)^{-1} = f(x)^{-1}f(x)f(x^{-1}) = f(x^{-1})$). But, reformulating this proof in the language of diagrammes, we immediately discover that it remains valid in an arbitrary category, and therefore in the category $[\mathscr{T}\!op]$.

It is clear that all H-spaces (H-monoids or H-groups) and all their homotopy (or only algebraic) morphisms make up a category. At the same time, according to Remark 3.14, the category of H-groups and their homotopy morphisms will be a complete subcategory of H-monoids or H-spaces,

Remark 3.15. The category of H-groups and their algebraic morphisms is not a complete subcategory of the corresponding category of H-monoids or H-spaces.

Homotopy morphisms, which are homotopy equivalences, we will call *homotopy isomorphisms*.

Remark 3.16. Homotopy isomorphisms are not isomorphisms of any category.

Equivalences of H-spaces are their homotopy isomorphisms, which are identical maps.

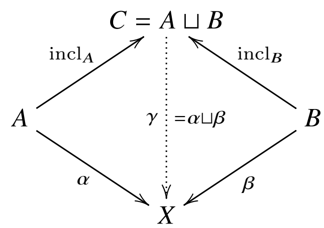
If $f : K \to L$ is a homotopy equivalence between an H-space K with an arbitrary space L, then you can obviously introduce in L the structure of an H-space to which the map f will be a homotopy isomorphism. Up to equivalence, this structure is unique.

At the same time, it is clear that an H-space homotopically isomorphic to a H-monoid or H-group will itself be an H-monoid or, respectively, an H-group.

3.6 Category of cogroups

Let us now consider the dual situation.

Let again $\mathscr{A}$ be an arbitrary category. An object C of the category $\mathscr{A}$, considered together with two morphisms $\mathrm{incl}_A : A \to C$, $\mathrm{incl}_B : B \to C$, is called the *sum* (or *coproduct*) of objects A and B if for any object $X \in \mathscr{A}$ and any morphisms $\alpha : A \to X$, $\beta : B \to X$ there is a single morphism $\gamma : C \to X$ that has the property that $\gamma \circ \mathrm{incl}_A = \alpha$, $\gamma \circ \mathrm{incl}_B = \beta$. The object C, if it exists, is unique up to canonical isomorphism. It is denoted by the symbol $A \sqcup B$, and the morphism γ is by the symbol $\alpha \sqcup \beta$.



Similarly , the sum of any (including the empty one!) families of objects of category $\mathscr{A}$ is defined. In this case, the sum of the empty family of objects will be the object $0_{\mathscr{A}}$, which has the property that for any object $A \in \mathscr{A}$ there is one and only one morphism $0_{\mathscr{A}} \to A$. Such an object is called an *initial object* of the category $\mathscr{A}$.

A category $\mathscr{A}$ is called *finitely additively closed* (in short, *fa-closed*) if for any finite family of its objects there exists their sum. In particular, the fa-closed category has an initial object.

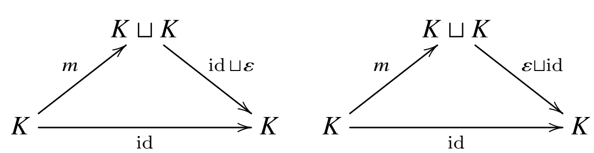
A morphism $A \to B$ in an fa-closed category is called *constant* if it passes through an initial object, i.e. it is a compound morphism of the form $A \to 0_{\mathscr{A}} \to B$.

Examples of fa-closed categories are all the same categories $\mathscr{E}ns$, $\mathscr{T}op$, $[\mathscr{T}op]$ and $\mathscr{G}rp$. The sum of objects in the first three categories is their disjunct union, and in the last one is their free product. The initial object of the categories $\mathscr{E}ns$, $\mathscr{T}op$ and $[\mathscr{T}op]$ is an empty set of $\varnothing$ (empty space), and the initial object of the category $\mathscr{G}rp$ is a trivial group (which is also a terminal object). The only constant morphisms The categories $\mathscr{E}ns$, $\mathscr{T}op$ and $[\mathscr{T}op]$ are "maps" of the form $\varnothing \to X$, and in the category $\mathscr{G}rp$ the constant morphisms coincide with the constant ones (i.e. they are trivial homomorphisms).

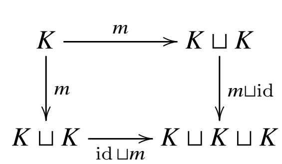
An object K of an arbitrary fa-closed category $\mathscr{A}$ is called a *cogroup of the category* $\mathscr{A}$ (in short, a $\mathscr{A}$-cohroup), if morphisms $m : K \to K \sqcup K$, $\mu : K \to K$, $\varepsilon : K \to K$ such that:

(a) the morphism ε is constant;

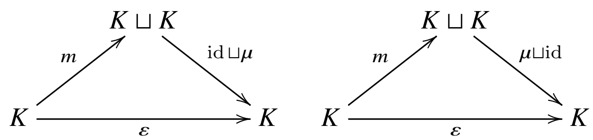
(b) there are commutative diagrams dual to Diagramems (3.6), (3.7) and (3.8) accordingly.



(3.17)



(3.18)



(3.19)

If only the morphisms m and ε are given, and only Diagrammes (3.17) are commutative, then the object K is called a *counitoid in the category* $\mathscr{A}$ (or a *$\mathscr{A}$-counitoid*). A category of counitoid for which Diagramme (3.18) is also commutative is called a *comonoid in the category* $\mathscr{A}$ (or *$\mathscr{A}$-comonoid*).

For any $\mathscr{A}$-cogroup K and any object $X \in \mathscr{A}$, formula

$$f + g = (f \sqcup g) \circ m, \quad f, g : K \to X \tag{3.20}$$

defines in the set $\mathscr{A}(K, X)$ all morphisms $K \to X$ is an addition operation, the zero of which is - as follows directly from the commutativity of Diagrammes (3.17) - a constant map $K \to X$ of the form $\beta \circ \varepsilon$, where β is an arbitrary morphism $K \to X$ (it is easy to see that the morphism $\beta \circ \varepsilon$ does not depend on the choice of morphism β). From the commutativity of diagram (3.18) follows the associativity of addition (3.20), and from the commutativity of diagram (3.19), that the element $-f = f \circ \mu$ is the inverse to the element f with respect to addition (3.20). Thus, with respect to the operation (3.20) the set $\mathscr{A}(K, X)$ is a group, and, as it is easy to see, this group structure is natural with X.

Conversely, let for an object K of the category $\mathscr{A}$ all the sets $\mathscr{A}(K, X)$, $X \in \mathscr{A}$, are provided with a natural X-structure of the (additive) group. Then in the group $\mathscr{A}(K, K \sqcup K)$ the element m will be defined - the sum of the morphisms incl_1 and incl_2, and in the group $\mathscr{A}(K, K)$ - the zero element ε and the element μ, inverse to the element id A more or less automatic check shows that for these morphisms diagrams (3.17),(3.18) and (3.19) are commutative and that operation (3.20) in the set $\mathscr{A}(K, X)$ coincides with the addition given in $\mathscr{A}(K, X)$. In addition, it follows from the naturalness property that the morphism ε is constant. Thus, in full duality to the situation for groups $\mathscr{A}(X, K)$, we obtain that

Proposition 3.21. *sets $\mathscr{A}(K, X)$ have a group structure natural in X if and only if the object K is a subgroup of the category $\mathscr{A}$.*

It is clear that the same statement is true with respect to comonoids and counitoids.

3.7 Category $\mathcal{Top}^\bullet$

For the case that interests us first of all categories $\mathcal{Top}$ and $[\mathcal{Top}]$ the results obtained are of catastrophic character. Indeed, it is as a permanent map $K \to X$ in the category $\mathcal{Top}$ exists only when $K = \varnothing$, then *in the categories $\mathcal{Top}$ and $[\mathcal{Top}]$ there are no nonempty groups*, and therefore for any nonempty K in the sets $\mathcal{Top}(K, X)$ and $[\mathcal{Top}](K, X) = [K, X]$ it is impossible to introduce a group structure natural for X!

To remedy the situation, you can offer a few different ways. The simplest- and, apparently, the most important- is based on the following definition.

Definition 3.22. A *pointed space* is a pair (X, x_0), consisting of a topological space X and some of its point x_0 (called the *base point* of the pointed space). Very often, the base point is only implied (= not explicitly specified) and the pointed space is simply denoted by X.

Often, the base point is also indicated by the symbol pt (or $*$), the same for all spaces.

Pointed map $f : (X, x_0) \to (Y, y_0)$ from the pointed space (X, x_0) into the pointed space (Y, y_0) is a continuous map $X \to Y$ such that $f(x_0) = y_0$.

It is clear that pointed spaces and their pointed maps constitute a category. We will denote this category with the symbol $\mathcal{T}\!op^\bullet$. It is related to the category $\mathcal{T}\!op$ by the functor ignoring base points

$$\mathcal{T}\!op^\bullet \to \mathcal{T}\!op$$

that translates (X, x_0) into X.

In the $\mathcal{T}\!op^\bullet$ there is a product of any family of objects: it will be their product as topological spaces, in which a base point is marked, each coordinate of which is a base point of the corresponding factor. In particular, for two factors

$$(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0)).$$

The terminal object of the category $\mathcal{T}\!op^\bullet$ is a singleton pointed space $(\{\mathrm{pt}\}, \mathrm{pt})$.

Thus, up to the based points this reasoning works and terminal objects of the category $\mathcal{T}\!op^\bullet$ are the same as in the category $\mathcal{T}\!op$.

The situation is different with sums and initial objects (which is the raison d'être of the category $\mathcal{T}\!op^\bullet$).

Definition 3.23. A *bouquet sum* (or, in short, a *bouquet*) of a family of pointed spaces X_α is the coset space of their disjoint union resulting in the identification of the base points. For the base point of the bouquet, the image of the base points of the components is taken.

The bouquet of spaces X_α is indicated by the symbol $\vee X_\alpha$. To denote a bouquet of a finite family of pointed spaces, the following notation is also used

$$X_1 \vee X_2 \vee \cdots \vee X_n.$$

The bouquet $X_1 \vee X_2 \vee \cdots \vee X_n$ is naturally homeomorphic to the coordinate cross of the product $X_1 \times X_2 \vee \cdots \times X_n$ consisting of points, all of whose coordinates, except perhaps one, are base points. For example, for two terms

$$X \vee Y = (X \times y_0) \cup (x_0 \times Y).$$

We will also denote the point (x, y_0) of the bouquet $X \vee Y$ by the symbol x_I and the point (x_0, y) by the symbol y_{II}.

Each family of pointed maps $f_\alpha : X_\alpha \to Z$ obviously defines a pointed map

$$f = \vee f_\alpha : \vee X_\alpha \to Z,$$

having the property that $f \circ \mathrm{incl}_\alpha = f_\alpha$ for any α, where

$$\mathrm{incl}_\alpha : X_\alpha \to \vee X_\alpha$$

are the canonical inclusions. Since the latter property obviously characterises the map f in the unique way, therefore,

Proposition 3.24. *the bouquet sum is the sum in the category of $\mathscr{Top}^\bullet$.*

For a finite family of $f_1, \ldots, f_n$ of the maps $\vee f_\alpha$ the map $\vee f_\alpha$ is also denoted by the symbol $f_1 \vee \cdots \vee f_n$. In particular, for two maps $f : X \to Z$ and $g : Y \to Z$ the map $f \vee g : X \vee Y \to Z$ is defined, acting according to the formulae

$$(f \vee g)(x_I) = f(x), \quad (f \vee g)(y_{II}) = g(y), \quad x \in X, y \in Y.$$

Any family of maps $f_\alpha : X_\alpha \to Y_\alpha$ naturally also defines the map $\vee X_\alpha \to \vee Y_\alpha$, which we will denote with the same symbol $\vee f_\alpha$ (or, respectively, $f_1 \vee \cdots \vee f_\alpha$).

Note that the map $\vee f : \vee X_\alpha \to Z$ (map $\vee f_\alpha : \vee X_\alpha \to \vee Y_\alpha$) is a restriction of the map $\prod f : \prod X \to Z$ (the map $\prod f_\alpha : \prod X_\alpha \to \prod Y_\alpha$).

Initial objects of the category $\mathscr{Top}$ coincide with terminal objects $(\{\mathrm{pt}\}, \mathrm{pt})$ and, therefore, co-constant maps are constant.

Note that, as in the category of groups, in the category of $\mathscr{Top}^\bullet$ for any two objects X, Y there is a single constant (aka co-constant) map $\mathrm{const} : X \to Y$.

By analogy with the category $\mathscr{Top}^\bullet$, we can introduce the category $\mathscr{Ens}^\bullet$ of pointed sets, the objects of which are sets with a base point marked in them, and morphisms are maps that translate base points into based ones. It is convenient, however, to consider this category a complete subcategory of the category $\mathscr{Top}^\bullet$, considering every set is like a topological space with a discrete topology.

In the language of algebra, a pointed set is nothing more than a set with a single zero operation. Thus, the structure of the pointed set is the weakest of all possible algebraic structures. Unfortunately, in many situations there is no other structure on the sets we need, and we have to limit ourselves to what we have.

For example, the π_0 functor on the category $\mathscr{Top}^\bullet$. Naturally, it is a functor with values in $\mathscr{Ens}^\bullet$ (the component containing the base point is marked), but in general, no richer algebraic structure can be introduced into the sets of $\pi_0 X$.

3.8 Category $[\mathscr{Top}^\bullet]$

Having introduced the category $\mathscr{Top}^\bullet$, we must now construct the corresponding homotopy category $[\mathscr{Top}^\bullet]$. To do this, we need to define the concept of homotopy for pointed maps. We will do this in the most natural way, taking the pointed maps as homotopy $(X, x_0) \to (Y, y_0)$ (or, in short, a pointed homotopy) an arbitrary homotopy $f_t : X \to Y$, having the property that for any $t \in T$ the map f_t is a pointed map $X \to Y$, that is, it satisfies the relation $f_t(x_0) = y_0$. In other words, pointed homotopies are exactly homotopies rel$\{x_0\}$.

In the interpretation of homotopies as maps $F : X \times I \to Y$ the condition $f_t(x_0) = y_0$ means that $F(x_0, t) = y_0$ for all $t \in I$, i.e. that $F(x_0 \times I) = y_0$. In this regard, in the category of $\mathcal{T\!op}^\bullet$ it is advisable to consider, instead of space $X \times I$, the space

$$X \dot{\times} I = (X \times I)/(x_0 \times I),$$

resulting from the product of $X \times I$ by collapsing the segment $x_0 \times I$ to a point. This space is naturally pointed - its base point is the image of the segment $x_0 \times I$, and the homotopy of the pointed maps are nothing more than the pointed map $X \dot{\times} I \to Y$.

Sometimes(very rarely) we will have to consider ordinary homotopy pointed maps that do not satisfy the condition $f_t(x_0) = y_0$. We will call such homotopies *free*.

For homotopy classes of pointed maps $[f]\,\mathrm{rel}\{x_0\}$, we will use the more expressive notation $[f]^\bullet$, and the set of all homotopy classes of pointed maps $[f]^\bullet : (X, x_0) \to (Y, y_0)$ we will denote either by the symbol $[(X, x_0), (Y, y_0)]$ or by the symbol $[X, Y]^\bullet$. We will sometimes denote the pointed homotopy relation by the symbol $\dot{\sim}$. Thus, the formulae $f \dot{\sim} g$,, $f \sim g\,\mathrm{rel}\{x_0\}$ and $[f]^\bullet = [g]^\bullet$ mean the same thing.

It is clear that the homotopy class of the composition of pointed maps depends only on the homotopy classes of these maps, which makes it possible to determine the composition of classes $[f]^\bullet$ with respect to which the totality of all pointed spaces and all homotopy classes of their pointed maps is a category. We will denote this category by the symbol $[\mathcal{T\!op}]^\bullet$ (or the symbol $[\mathcal{T\!op}^\bullet]$).

For any pointed homotopy $f_t : X \to Z$ and $g_t : Y \to Z$ the map $f_t \times g_t$ and the restriction $f_t \vee g_t$ are obviously homotopy. Therefore, the products (sums) of pointed spaces in the category $\mathcal{T\!op}^\bullet$ will be products (sums) in the category $[\mathcal{T\!op}]^\bullet$. The same is, of course, true with respect to initial (= terminal) objects. As for the constant (= co-constant) morphisms of the category $[\mathcal{T\!op}]^\bullet$, they will be homotopy classes of constant maps. Therefore, in the category $[\mathcal{T\!op}]^\bullet$, as in the category $\mathcal{T\!op}^\bullet$, for any two objects X and Y, there is one and only one constant (= co-constant) morphism $X \to Y$.

3.9 H-cogroups, H-comonoids, H-counitoids

Thus, we see that formal obstacles to the existence in the category of $[\mathcal{T\!op}]^\bullet$ there are no cogroups, comonoids and counitoids. However, as in a similar situation for $[\mathcal{T\!op}]$-groups, we prefer instead of homotopy classes m, μ, ε consider their arbitrary representatives. Thus we come to the following definitions.

The pointed space (K, e) is called *H-counitoid* (or, more traditionally, a *co-*

H-space) if a pointed map is given

$$m : K \to K \vee K.$$

that Diagrammes (3.17) (in which $K \sqcup K$ is replaced by $K \vee K$ and ε is a constant map $\mathrm{const}_\varepsilon : K \to K$) are homotopy commutative (of course, pointed), i.e. if the space K is a $[\mathcal{Top}]^\bullet$ counitoid with respect to homotopy classes $[m]^\bullet$ and $[\varepsilon]^\bullet$.

If, in addition, the diagramme (3.18) is homotopy commutative, then the H-counitoid K is called the *H-comonoid*, and if the map $\mu : K \to K$ is given for which the Diagrammes (3.19) are homotopy commutative, then the H-comonoid is called a H-*cogroup*.

The map m is called a *co-multiplication*, and the base point ε is called a *homotopy unit*.

Morphisms of H-counitoids, H-comonoids and H-cogroups are determined in an obvious way. As in the case of H-spaces, they can be algebraic or more generally homotopic.

Two H-counitoids (two H-comonoids or two H-cogroups) are called *equivalent* if, as pointed spaces, they coincide, and their multiplications (and in the case of cogroups and the map μ) are pointed homotopies, i.e. if they define the same counitoide (the same comonoid or the same cogroup) of the category $[\mathcal{Top}]^\bullet$.

According to the general-category (theory), results obtained above

Proposition 3.25. *for each H-cogroup (each H-comonoid or H-counitoide) K and any pointed space X the set $[K, X]$ has a group structure (monoid structure or unitoid structure) that is natural in X.*

Of course, in order for all this to make meaningful sense, we need to have a fairly large stock of examples of specific H co-groups. We will now outline one way of constructing them, although not the most general, but sufficient for our purposes.

3.10 Suspensions

Let X be an arbitrary (non-pointed) topological space.

Definition 3.26. The *suspension* SX over the space X is the coset space of the product $X \times I$ with respect to equivalence, in which $(x_1, t_1) \sim (x_2, t_2)$ if and only if either $x_1 = x_2$, $t_1 = t_2$, or $t_1 = t_2 = 0$, or$t_0 = t_1 = 1$. Thus, the factorisation map $X \times I \to SX$ is a homeomorphism on the subspace $X \times (0, 1)$ and maps each of the subspaces $X \times 0$ and $X \times 1$ to one point respectively (its own for each subspace - see Fig. 3.10.1). We will call the point $_0 \in SX$, which is the image of the subspace $X \times O$, the *south pole* of the suspension SX, and the point $_{p1} \in SX$, which is the image of the subspace $X \times 1$ its *north pole*. We will call the images of the segments $X \times I$ the *meridians* of the suspension, and the image of the subspace $X \times 1/2$ its equator. The equator is naturally homeomorphic to the space X, and we will usually identify it with this space.

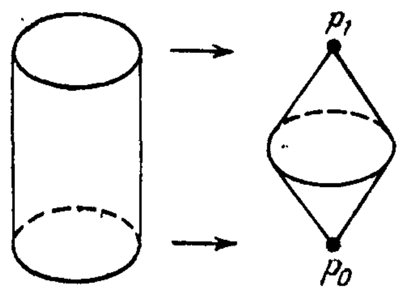


Figure 3.10.1:

We will denote the point of the suspension SX which is the image of the point $(x, t) \in X \times I$ by the symbol $[x, t]^S$ or simply $[x, t]$.

Thus,

$$[x, 0] = p_0, [x, 1] = p_1 \quad \text{for any point} \quad x \in X,$$

and for $0 < t_1, t_2 < 1$, the equality $[x_1, t_1] = [x_2, t_2]$ holds if and only if $x_1 = x_2$ and $t_1 = t_2$.

It is clear that the correspondence $[x, t]^C \to [x, t]^S$ induces homeomorphism of the space CX/X to the space SX. Thus, we can assume that $SX = CX/X$. (See Section 2.4.)

In the case when X is a unit sphere $\mathbb{S}^{n-1}$ of the space $\mathbb{R}^n$, by setting up $S\mathbb{S}^{n-1}$ by means of the homeomorphism

$$[x, t] \mapsto (-\cos \pi t, \sin \pi i x), \quad x \in \mathbb{S}^{n-1}, t \in I, \tag{3.27}$$

it is naturally identified with a single sphere $\mathbb{S}^n$ in the space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. At the same time, the poles, meridians and equator acquire the usual elementary geometric meaning.

In particular, $\mathbb{S}^1 = S\mathbb{S}^0$.

Thus, although the sphere $\mathbb{S}^0$ is disconnected, its suspension $\mathbb{S}^1$ is connected. In general, the setting of SX over any space X is connected, since any point of it is connected by a meridian segment with each pole.

For any continuous map $f : X \to Y$, the correspondence

$$[x, t] \mapsto [f(x), t], \quad x \in X, t \in I,$$

defines a continuous map $Sf : SX \to SY$, and it is clear that the correspondences $X \mapsto SX$, $f \mapsto Sf$ make up some functor

$$S : \mathcal{T}op \to \mathcal{T}op.$$

We will call this functor the *suspension functor*.

When the equator X is pulled into a point in the suspension SX, a space SX/X appears, homeomorphic, obviously, to the bouquet $SX \vee SX$ of two copies

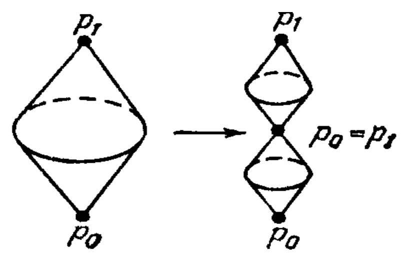


Figure 3.10.2:

of the suspension SX, resulting in the identification, say, of the south pole of one instance with the north pole of the other (see Fig. 3.10.2). (A homeomorphism $SX/X \to SX \vee SX$ is induced, for example, by the map

$$m : SX \to SX \vee SX,$$

defined by the formula

$$m[x,t] = \begin{cases} [x, 2t]_I, & \text{if} \quad 0 < t < 1/2, \\ [x, 2t-1]_{II}, & \text{if} \quad 1/2 < t < 1, \end{cases} \tag{3.28}$$

where $x \in X$, $t \in I$.)

The map (3.28) is, by definition, a kind of multiplication in the space SX, and, as we will show below, if we arbitrarily select a base point in SX, then

Proposition 3.29. *with respect to this multiplication, the suspension SX will be an H-cogroup.*

Thus we have obtained a large stock of various H-cogroups.

3.11 Reduced suspensions

However, a careful analysis of the presented construction reveals certain roughness in it, due to the fact that although, due to the connectivity of the space SX, the choice of the base point in it has no real meaning, but for the functoriality of the whole construction, this choice should be made in some "natural" way. Unfortunately, we have two natural candidates for the role of the base point - the north pole and the south pole,the choice between which introduces an unpleasant element of arbitrariness. (This, in particular, was reflected in the fact that although, according to the general theory, when constructing a bouquet $SX \vee SX$ we need for both copies of the space SX to remove the same base point, but above, for symmetry, we chose the north pole p_0 in one instance, and the south pole p_1 in the other.) Moreover, since we actually mean to work in the category of $\mathscr{T}\!op^\bullet$, we must also consider the initial space X to be pointed, and then in order for its identification $x \mapsto [x, 1/2]$ with the equator of the suspension SX to be an identification in the category of $\mathscr{T}\!op^\bullet$, it is necessary for the base point of the suspension SX to select the point $[x_0, 1/2]$, where x_0 is the base point of the space X.

Definition 3.30. The *reduced suspension* $S^\bullet X$ over the pointed space X is the coset space of the product $X \times I$, resulting in the collapsing the subspace $X_{\mathrm{pt}} = (X \times 0) \cup (x_0 \times I) \cup (X \times 1)$ to a point:

$$S^\bullet X = (X \times I)/X_{\mathrm{pt}}$$

(which is taken as the base point of the above suspension). In other words,

$$S^\bullet X = SX/Sx_0,$$

where Sx_0 is the meridian passing through the point x_0.

Instead of $S^\bullet X$, we will also write $S(X, x_0)$.

The points of the above suspension will be denoted by the former symbols $[x, t]$ or, when necessary, $[x, t]^S$. So now $[x_1, t_1] = [x_2, t_2]$ if and only if either $x_1 = x_2$ and $t_1 = t_2$, or $x_1 = x_2 = \mathrm{pt}$, or $t_1 = 0, 1$, $t_2 = 0, 1$ (and then $[x_1, t_1] = [x_2, t_2] = \mathrm{pt}$).

The correspondence $x \mapsto [x, 1/2]$ will still imbed X into $S^\bullet X$, but now it will already be a pointed map. Assuming that $X \subset S^\bullet X$, we will thereby identify the point x_0 with the base point of the suspension $S^\bullet X$.

Similarly, the correspondence $[x, t] \mapsto [f(x), t]$ for any pointed map $f : X \to Y$ will determine the pointed map $S^\bullet f : S^\bullet X \to S^\bullet Y$, and the correspondences $X \mapsto S^\bullet X$, $f \mapsto S^\bullet f$ will constitute some functor

$$S^\bullet : \mathscr{T}\!op^\bullet \to \mathscr{T}\!op^\bullet.$$

We will call this the *reduced suspension functor.*

Finally, the same formula (3.28) will give us a multiplication $m : S^\bullet X \to S^\bullet X \vee S^\bullet X$, with respect to which the space $S^\bullet X$ is also, as we will show below, an H-cogroup (with x_0 as the unit).

Thus, the transition to the above suspension removes all the difficulties noted above and at the same time nothing, in fact, spoils. This, however, should not be surprising, since, as we will prove in the next lecture, the spaces SX and $S^\bullet X$ are homotopically equivalent under very broad general methodological conditions.

Remark 3.31. Moreover, under slightly more restricted conditions, the spaces SX and $S^\bullet X$ are even homeomorphic. We will not investigate this question in full generality and will limit ourselves to proving that

Proposition 3.32. *for any $n > 0$ the reduced suspension $S^\bullet \mathbb{S}^{n-1}$ over the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ is homeomorphic to the n-dimensional sphere $\mathbb{S}^n = S\mathbb{S}^{n-1}$.*

Proof. Indeed, by identifying the space $\mathbb{R}^n$ with the product $\mathbb{R}\times\mathbb{R}^{n-1}$, and the space $\mathbb{R}^{n+1}$ with the product $\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$, the homeomorphism $d : S^\bullet\mathbb{S}^{n-1} \to \mathbb{S}^n$ can be defined, for example, by the formula

$$d[(x,\mathbf{x},t] = \begin{cases}(1-2t+2tx, 2t\mathbf{x}, 2\sqrt{t(1-2t)(1-x)}, & \text{if} \quad 0 \le t \le 1/2,\\ (2t-1+2(1-t)x, 2(1-t)\mathbf{x}, 2\sqrt{(1-t)(2t-1)(1-x)}, & \text{if} \quad 1/2 \le t \le 1,\end{cases} \tag{3.33}$$

where $x^2 + |\mathbf{x}|^2 = 1$, $x \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^{n-1}$ and $t \in I$ (this homeomorphism maps each meridian of the suspension to the circumference of the sphere $\mathbb{S}^n$, carved by the hyperplane of the space $\mathbb{R}^{n+1}$ passing through the points $(x,\mathbf{x},0)$ and $s_0 = (1,\mathbf{0},0)$ parallel to the axis x_{n+1}). □

3.12 Loop spaces

The conceptual advantage of the above suspension is also manifested in the fact that it allows a dual construction.

Let (X,x_0) be an arbitrary pointed space.

Definition 3.34. The path $u : I \to X$ of the pointed space (X,x_0) is called a *loop* if $u(0) = u(1) = x_0$. The set of all loops of the space (X,x_0), being a subset of the topological space X^I, is a topological space. We will denote it with the symbol $\Omega(X,x_0)$ or simply ΩX.

We will consider the space ΩX to be a pointed space, taking the constant loop $0_{x_0} : I \to X$, $t \mapsto x_0$ as its base point.

Each pointed map $f : X \to Y$ is defined by the formula

$$(\Omega f)u = f \circ u \tag{3.35}$$

defines the pointed map $\Omega f : \Omega X \to \Omega Y$, and it is clear that the correspondences $X \mapsto \Omega X$, $f \mapsto \Omega f$ make up a functor

$$\Omega : \mathcal{Top}^\bullet \to \mathcal{Top}^\bullet.$$

We will call this functor the *loop functor.*

We will introduce multiplication into ΩX by defining the product of uv of two loops $u, v \in \Omega X$ by the formula

$$(uv)(t) = \begin{cases} u(2t) & \text{if} \quad 0 \le t \le 1/2,\\ v(2t-1) & \text{if} \quad 1/2 \le t \le 1.\end{cases} \tag{3.36}$$

It is clear that this definition is correct and the resulting map $(u,v) \mapsto uv$ from the product $\Omega X \times \Omega X$ to the space ΩX is continuous.

Below we show that

Proposition 3.37. *with respect to this multiplication, the space ΩX is an H-group.*

The (homotopy) unit of this H-group is the constant loop 0_{x_0}, and the map $u \mapsto u^{-1}$ is given by the formula $u^{-1}(t) = u(1-t)$.

Note that $\Omega X = \Omega X_0$, where X_0 is the component of the space X containing the base point. Therefore, considering the space ΩX, we will, as a rule, consider the space X to be connected.

3.13 Adjoint functors S and Ω

In general category theory, two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are called *adjoint* (more precisely, the functor F is called *adjoint on the left* with the functor G, and the functor G is called *adjoint on the right* with the functor F) if for any $X \in \mathscr{C}$, $Y \in \mathscr{D}$ a natural (with respect to X and Y) bijective map is given

$$\varphi : \mathscr{D}(FX, y) \xrightarrow{\approx} \mathscr{C}(X, GY)$$

(called *adjoint isomorphism*).

Example 3.38. When $\mathscr{C} = \mathscr{E}ns$, $\mathscr{D} = \mathscr{G}rp$ the functor $\mathscr{G}rp \to \mathscr{E}ns$ ignoring the group structure is adjoint on the right with the $\mathscr{E}ns \to \mathscr{G}rp$ functor, which maps an arbitrary set X to a free group with the set X as free generators.

Adjoint functors have a number of important properties, but they are of interest to us now only in connection with the theory of groups and cogroups.

Let the functor F have the property that for any object $X \in \mathscr{C}$, the object FX is a subgroup of the category $\mathscr{D}$ and, therefore, for any object $Y \in \mathscr{D}$ the set $\mathscr{D}(FX, Y)$ is a cogroup. Then the set $\mathscr{C}(X, GY)$ will also be a group, and the group structure on this set will be natural for X (and, of course, for Y, but we don't care about that at the moment). Therefore, the object $GY\mathscr{C}$ will be a category group $\mathscr{C}$. Since this reasoning is completely reversible, it is proved that

Proposition 3.39. *the functor F takes values in the category of $\mathscr{D}$-cogroups if and only if the functor G takes values in the category of $\mathscr{C}$-groups.*

Let's return now to the category $\mathscr{T}op^\bullet$.

Proposition 3.40. *The functor $S^\bullet : \mathscr{T}op^\bullet \to \mathscr{T}op^\bullet$ is adjoint on the left with the functor $\Omega : \mathscr{T}op^\bullet \to \mathscr{T}op^\bullet$*

Proof. By composing an arbitrary pointed map $f : S^\bullet X \to X$ with the factorisation map $XxI \to S^\bullet X$ and moving to the associated map, we get the map $\varphi f : X \to Y^I$, explicitly defined by the formula

$$(\varphi f)(x)(t) = f([x,t]), \quad x \in X, t \in I$$

Since by the condition $f([x,0]) = f([x,1]) = x_0$ map φf is actually a map in ΩX, so that the correspondence $f \mapsto \varphi f$ gives us some (obviously natural with respect to X and Y) map

$$\varphi : \mathscr{T}op^\bullet(S^{\bullet X,Y}) \to \mathscr{T}op^\bullet(X, \Omega Y),$$

and a direct check shows that this map is bijective. □

A functor $T : \mathscr{Top}^\bullet \to \mathscr{Top}^\bullet$ is called *homotopy invariant* if, for any (pointed) homotopy $f_t : X \to Y$, the maps $T(f_t) : TX \to TY$ constitute a homotopy from TX to TY (automatically pointed). For such a functor, the correspondence $[f]^\bullet \mapsto [Tf]^\bullet$ is well defined and together with the correspondence $X \mapsto TX$ makes up the functor $[\mathscr{Top}^\bullet] \to [\mathscr{Top}^\bullet]$, which is called a homotopisation of the functor T and is usually denoted by the same symbol T.

Examples of homotopy functors are, as can be easily seen, the functors $S^\bullet$ and Ω. We will also call their homotopisations a *suspension functor* and a *loop functor* (from $[\mathscr{Top}^\bullet] \to [\mathscr{Top}^\bullet]$).

Remark 3.41. For the homotopy class $\alpha = [f]^\bullet \in [X, Y]^\bullet$ the homotopy class $S^\bullet\alpha = [S^\bullet f]^\bullet$ is also denoted by the symbol $E\alpha$.

It is clear that the adjoint isomorphism for the homotopy functors $S^\bullet$ and Ω translates into homotopy. Therefore, the functors $S^\bullet$ and Ω from $[\mathscr{Top}^\bullet] \to [\mathscr{Top}^\bullet]$ are *also paired (= adjoint).*

Therefore, the statement that the above suspension $S^\bullet X$ is an H-cogroup is equivalent to the statement that the loop space is an H-group. So, only one of them needs a proof.

3.14 Topological monoids of Moore loops

Definition 3.42. The *Moore loop* of the pointed space (X, x_0) is a continuous map

$$u : [0, a] \to X$$

from the segment $[0, a]$, where $a \geq 0$, into the space X, such that $u(0) = u(a) = x_0$.

The number a is called the *length* of the Moore loop u.

Moore loops of length 1 are ordinary loops of space X (i.e. loops in the sense of Definition 3.34).

Similarly, the *Moore path of length* $a \geq 0$ of the topological space X is an arbitrary map $u : [0, a] \to X$. The point $u(0)$ is called the *beginning* of the path u, and the point $u(a)$ is its *end*.

For each Moore path $u : [0, a] \to X$ the formula $u^\#(t) = u(at)$ defines an ordinary path $u^\#$ of the space X, and by matching the paths u with a pair $(u^\#, a)$, we get a map of the set of all Moore paths of the space X into the product $X^I \times R^+$, where R^+ is the semiaxis of all non-negative real numbers. It is clear that this map is injective. Considering it as an embedding, we will introduce the topology of the subspace of the product $X^I \times R^+$ into the set of Moore loops $\Omega^M X$. Obviously, the space ΩX will then be a subspace of the space $\Omega^M X$, and the map $u \mapsto u^\#$ will be a retraction $\Omega^M X \to \Omega X$.

Moreover, it is easy to see that

Proposition 3.43. *this retraction is a strong deformation retraction and, hence, a homotopy equivalence.*

Proof. Appropriate corresponding deformation $f_\tau : \Omega^M X \to \Omega^M X$, $0 \le \tau \le 1$, can be obtained by matching each Moore loop $u : [0, a] \to X$ with the Moore loop $f_\tau u$ of length $a + \tau - \tau a$, given by the formula

$$(f_\tau u)(t) = \begin{cases} u(t_0), & \text{if} \quad a \le 1 \text{ and } 0 \le t \le a, \\ x_0, & \text{if} \quad a \le 1 \text{ and } a \le t \le a + \tau - \tau a, \\ u(\frac{at}{a+\tau-\tau}), & \text{if} \quad a \ge 1, \end{cases}$$

where $0 \le t \le a + \tau - a\tau$. □

Let u and v be two Moore paths of lengths a and B, respectively, having the property that the end of $u(a)$ of the path u coincides with the beginning of $v(0)$ of the path v. Then the formula

$$(uv)(t) = \begin{cases} u(t), & \text{if} \quad 0 \le t \le a, \\ v(t - a), & \text{if} \quad a \le t \le a + b, \end{cases}$$

defines the Moore path uv of length $a + b$, called the *product* of paths u and v.

In particular, the product uv is defined for any two Moore loops $u, v \in \Omega^M x$, and the corresponding map

$$\Omega^M X \times \Omega^M X \to \Omega^M X, \quad (u, v) \mapsto uv, \tag{3.44}$$

as it is easy to see, is continuous. In addition, it is clear that it is associative and the loop const : $[0, 0] \to X$ (which we will also denote by the symbol e) is its unit. This means that

Proposition 3.45. *with respect to multiplication (3.44), the space $\Omega^M X$ is a topological monoid.*

On the other hand, the formula

$$m_\tau(u, v)(t) = \begin{cases} u(\frac{2(a+b)t}{(a+b)+(a-b)\tau} a), & \text{if} \quad 0 \le\le t \le \frac{(a+b)+(a-b)\tau}{2(a+b)}, \\ v(\frac{2(a+b)t-(a+b)-(a-b)\tau}{(a+b)+(a-b)\tau} b), & \text{if} \quad \frac{(a+b)+(a-b)\tau}{2(a+b)} \le t \le 1, \end{cases}$$

(where a and b are the lengths of the Moore paths u and v) defines - as directly verified - the homotopy $m_\tau : \Omega^M X \times \Omega^M X \to \Omega X$, connecting the map $m_0 : (u, v) \mapsto u^\# v^\#$ with the map $m_1 : (u, v) \mapsto (uv)^\#$. Hence, the map $u \mapsto u^\#$ is a homotopy morphism and, therefore, being a homotopy equivalence, a homotopy isomorphism.

This proves that

Proposition 3.46. *the space with multiplication ΩX is homotopically isomorphic to the topological monoid $\Omega^M X$.*

3.15 The loop space is an H-group.

Hence, the space QX is a H-monoid (and therefore the space $S^\bullet X$ is an H-comonoid).

Since $e^\# = 0_{x_0}$, the constant loop 0_{x_0} is the homotopy unit of the H-monoid ΩX (and the base point of the reduced suspension $S^\bullet X$ is its homotopy unit).

Remark 3.47. Of course, the latter statements are easily proved directly. Homotopy from $\Omega X \times \Omega X \times \Omega X$ to ΩX, the binding map $(u, v, w) \mapsto (uv)w$ and $(u, v, w) \mapsto u(vw)$, can be defined by the formula

$$[f_\tau(u,v,w)](t) = \begin{cases} u(\frac{ut}{1+\tau}), & \text{if } \ 0 \le t \le \frac{1+\tau}{4}, \\ v(4t-\tau-1), & \text{if } \ \frac{1+\tau}{4} \le t \le \frac{1+\tau}{4}, \\ w(\frac{4t-\tau-1}{2-\tau}), & \text{if } \ \frac{2+\tau}{4} \le t \le\le 1, \end{cases}$$

where $u, v, w \in \Omega X$, and the homotopies from ΩX to ΩX connecting the maps $u \mapsto u0_{x_0}$ and $u \mapsto 0_{x_0}u$ with the identical map are formulae

$$[f_\tau(u)](t) = \begin{cases} u(\frac{2t}{1+\tau}), & \text{if } \ 0 \le t \le \frac{1+\tau}{2}, \\ x_0, & \text{if } \ \frac{1+\tau}{2} \le t \le 1, \end{cases}$$

$$[g_\tau(u)](t) = \begin{cases} x_0, & \text{if } \ 0 \le t \le \frac{1-\tau}{2}, \\ u(\frac{2t+\tau-1}{1+\tau}), & \text{if } \ \frac{1-\tau}{2} \le t \le 1, \end{cases}$$

for $u \in \Omega X$.

The corresponding homotopies for the space $S^\bullet X$ have a similar ("adjoint") form.

Conventionally constructed homotopies are shown in Figures 3.15.1 and 3.15.2.

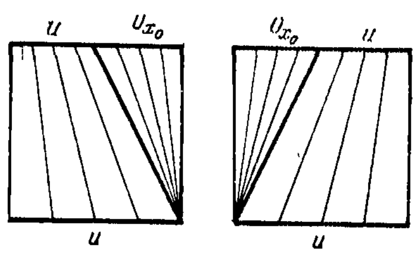


Figure 3.15.1:

Proposition 3.48. *The space ΩX is an H-group (and the space $S^\bullet X$ is therefore an H-cogroup).*

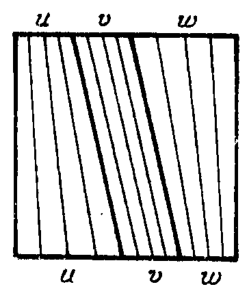


Figure 3.15.2:

Proof. In light of all the above, we only need to prove that the maps $u \mapsto uu^{-1}$ and $u \mapsto u^{-1}u$ are homotopic to the constant map $\Omega X \to \Omega X$. To the map $u \mapsto uu^{-1}$, the corresponding homotopy is defined by the formula

$$[f_\tau(u)](t) = \begin{cases} u(2t), & \text{if } \ 0 \le t \le \frac{1-\tau}{2}, \\ u(1-\tau), & \text{if } \ \frac{1-\tau}{2} \le t \le \frac{1+\tau}{2}, u \in \Omega X, \\ u(2-2t), & \text{if } \ \frac{1+\tau}{2} \le t \le 1, \end{cases}$$

(see Fig. 3.15.3), and to the map $u \mapsto u^{-1}u$ is obtained from this homotopy by

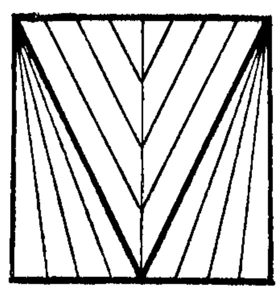


Figure 3.15.3:

replacing u with u^{-1} (and u^{-1} by u). □

It is clear that the H-group will also be a topological monoid of the Moore loops $\Omega^M X$.

Appendix

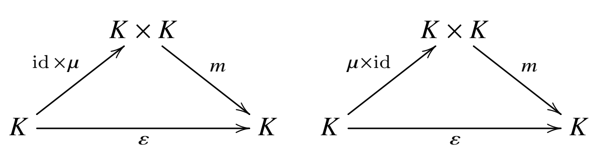
3.A H-monoids that are H-groups

In algebra, we are used to the fact that the structure of a monoid is significantly weaker than the structure of a group, which is expressed in the existence of the most diverse monoids that are not groups. It turns out that in the category $[\mathscr{Top}]$ the situation is quite different, and with very weak general methodological assumptions, *every connected H-monoid is an H-group.* Here the connectedness condition aims to exclude monoids considered in algebra. However, the same can be achieved under less restrictive conditions.

To formulate these conditions, we recall that for any H-monoid K its component the set $\pi_0 K$ is naturally a monoid, and for H-group K this monoid will be a group. Thus, in order for the H-monoid K to be a H-group, it is necessary that the monoid $\pi_0 K$ be a group. It turns out that with appropriate general methodological constraints, the necessary condition is also sufficient. Exactly, the following Proposition is true.

Proposition 3.49. *Any numerable semi-locally contractible H-monoid K for which the monoid $\pi_0 K$ is a group is an H-group.*

Proof. This sentence states, therefore, that for H-monoid K there is a continuous map $\mu K \to K$, $x \mapsto x^{-1}$, for which the diagrammes



$\varepsilon = \mathrm{const}_e$, (3.50)

are homotopically commutative. At the same time, it is easy to see that the map μ is characterised uniquely by these conditions (of course, up to homotopy). Indeed, according to Remark 3.14 of Lecture 3, the identical map $K \to K$ should translate one map μ into another. □

We assume the proof of Proposition 3.49 with a few general remarks.

3.B Left and right shifts in H-monoids

For any fixed point $a \in K$ we will introduce in consideration, the *left shift* L_a by a and the *right shift* R_a by a, which are maps $K \to K$, defined respectively by formulae

$$L_a x = ax, \quad R_a x = xa, \quad x \in K.$$

The fact that the point e is a homotopy unit, means that the maps L_e and R_e are homotopy to the identical map id. In addition, it follows from the homotopy associativity of multiplication in K that for any two points $a, b \in K$ the map $L_a \circ L_b$ and $R_a \circ R_b$ are homotopic to the map L_{ab} and R_{ba}, respectively.

Each path $u : I \to K$ connecting the point a to the point B determines by the formula $t \mapsto L_{u(t)}$ (by the formula $t \mapsto R_{u(t)}$) the homotopy connecting the map L_b (map R_b) with the map L_b (the map R_b). On the other hand, if for the H-monoid K the set $\pi_0 K$ is a group, then for any element $a \in K$ there exists an element $a^{'} \in K$ such that for each point $x \in K$ points $a^{'}(ah)x$ and $a(a^{'}x)$ (or, equivalently, the points $(a^{'}a)x$ and $(aa^{'})x)$ lie in the same component as the point x, i.e. they can be connected to this point in some way. In particular, the points $(a^{'}a)e$ and $(aa^{'})e$, which means that the points $a^{'}a$ and $aa^{'}$ lie in the same component as the point e. Therefore, the maps $L_{a^{'}a} : x \mapsto (a^{'}a)x$ and $L_{aa^{'}} : x \mapsto (aa^{'})x$ are homotopic to the map L_e, and hence to the identical map id. Similarly, it is shown that the maps $R_{a^{'}a}$ and $R_{aa^{'}}$ are also homotopic to the identical map id. Since $L_{aa^{'}} \sim L_a \circ L_{a^{'}}$, and $L_{a^{'}a} \sim L_{a^{'}} \circ L_a$ (and similarly $R_{aa^{'}} \sim R_a \circ R_{a^{'}}$, and $R_{a^{'}a} \sim R_{a^{'}} \circ R_a$), this proves the following lemma.

Lemma 3.51. *If for a H-monoid K the set $\pi_0 K$ is a group, then for any element $a \in K$ maps L_a and R_a are homotopy equivalences.*

Conversely, let for an H-monoid K, say, the map L_a be homotopy equivalences. Then the map $\ell : K \times K \to K \times K$, defined by the formula

$$\ell(a, x) = (a, ax), \quad a, x \in K,$$

and being, obviously, a map over K, with respect to the projection $\mathrm{proj}_1 : K \times K \to K$, $(a, x) \mapsto a$, will be a homotopy equivalence on fibres. Therefore, if the H-monoid K is numerable and semi-locally contractible, then, according to Proposition 2.74 of the Appendix 2.11 to Lecture 2, the map ℓ will be a fibrewise homotopy equivalence.

Let $\ell^{'} : K \times K \to K \times K$ be an inverse fibrewise homotopy equivalence. Since $\mathrm{proj}_1 \circ \ell = \mathrm{proj}_1$ for any point $(a, x) \in K \times K$ we will have the equality $\ell^{'}(a, x) = (a, \lambda(a, x))$, where $\lambda : K \times K \to K$ is a map such that the maps

$$(a, x) \mapsto \lambda(a, ax) \quad \text{and} \quad (a, x) \mapsto a\lambda(a, x), \quad (a, x) \in K \times K,$$

from $K \times K$ to K are homotopic projections of $(a, x) \mapsto x$. Putting $x = e$ and $\mu^{\ell}(a) = \lambda(a, e)$, we obtain, in particular, a map $\mu^{\ell} : K \to K$, for which the map $a \mapsto a\mu^{\ell}(a)$ is homotopic to the map $\mathrm{const}_e : K \to K$, $a \mapsto e$, i.e. for which the first diagramme of (3.50) is homotopy commutative.

Similarly, it is proved that if the homotopy equivalences are the maps $R_a, a \in K$, then there exists a map $\mu^r : K \to K$ for which the second diagramme of (3.50) is homotopy commutative.

But if there are two maps μ^ℓ and $j\mu^r$, then the map $\mu^r : x \mapsto \mu^r(x)$ will be homotopic to the map $x \mapsto \mu^r(x)x\mu^\ell(x)$, and the map $\mu^\ell : x \mapsto \mu^\ell(x)$ will be homotopic to the map $x \mapsto \mu^r(x)x\mu^\ell(x)$. Therefore, due to the homotopy associativity of multiplication in the H-monoid K, the maps μ^r and μ^ℓ turn out to be homotopic. Therefore, we can assume that $\mu^r = \mu^\ell$.

Thus, the following lemma is proved.

Lemma 3.52. *If for any element $a \in K$ of a numerable semilocally contractible H-monoid K, the maps L_a and R_a are homotopy equivalences, then the H-monoid K will be an H-group.*

The proof of Proposition 3.49 is immediately obtained by comparing Lemmas 3.51 and 3.52.

Remark 3.53. By applying Lemma 3.51, the condition of Lemma 3.52 is not only sufficient, but also necessary, i.e.

Proposition 3.54. *a numerable semilocally contractible H-monoid K is an H-group if and only if for any element $a \in K$ the maps L_a and R_a are homotopy equivalences.*

Lecture 4

The introduction to the consideration of pointed spaces forces us to raise the question of the price that we have to pay for it, i.e. the question of how much the constructions and results of previous lectures are modified and complicated when switching to pointed spaces.

Fortunately, it turns out that in most cases this transition is carried out almost painlessly.

For example, since for any pointed spaces (X, x_0) and (Y, y_0) the set of $\mathscr{T\!op}^\bullet((X, x_0), (Y, y_0))$ lies in the $\mathscr{T\!op}(X, Y)$, it is automatically provided with an induced topology. The resulting topological space is denoted by the symbol $(Y, y_0)^(X, x_0)$. It is naturally pointed: its base point is the constant map $\mathrm{const} : X \to Y$. When there is no possibility of misunderstandings, we will denote the space $(Y, y_0)^(X, x_0)$ with the former symbol Y^X.

4.1 Exponential law for pointed maps

However, the situation with the exponential law for the category $\mathscr{T\!op}^\bullet$ turns out to be more complicated because the map $\theta f : Y \to B^X$ associated with the pointed map $f : (X \times Y, (x_0, y_0)) \to (B, b_0)$, will not, generally speaking, be a map to $(B, b_0)^{(X, x_0)}$ (unless $f(x_0, y) = b_0$ for any point $y \in Y$, i.e. $f(x_0 \times Y) = b_0$). Moreover, even if the last requirement is met, the map θf will not, generally speaking, be a pointed map $(Y, y_0) \mapsto (B, b_0)^{(X, x_0)}$ (unless, $f(x, y_0) = b_0$ for any point $x \in X$, i.e. $f(X \times y_0) = b_0$). This shows that the role of the space $X \times Y$ in the exponential law for pointed maps should be played by the coset space

$$(X, x_0) \wedge (Y, y_0) = (X \times Y)/((X \times y_0) \cup (x_0 \times Y)),$$

resulting from the product of $X \times Y$ by pulling the coordinate cross into one point

$$(X, x_0) \vee (Y, y_0) = (X \times y_0) \cup (x_0 \times Y)$$

(which is considered to be the base point of this space).

Definition 4.1. The space $(X, x_0) \wedge (Y, y_0)$ is called a *smash* (or a *smash product*) of the pointed spaces (X, x_0) and (Y, y_0).

To denote a smash product, instead of the sign $\wedge$, the sign $\otimes$ is also used (Fuchs and Rokhlin [10].) Instead of $(X, x_0) \wedge (Y, y_0)$, they often write simply $X \wedge Y$ (or $X \otimes Y$).

The image of the point $(x, y) \in X \times Y$ in the space $X \wedge Y$ is denoted by the symbol $x \wedge y$. In these notations, the base point of the space $X \wedge Y$ is the point $x_0 \wedge y_0$.

Similarly, the smash product $X_1 \wedge \cdots \wedge X_n$ is defined for any number of pointed spaces $X_1, \ldots, X_n$. They will be the coset space of the product $X_1 \times \cdots \times X_n$, in which all points, at least one coordinate of which is the base point of the corresponding multiplier, are identified.

In order to avoid misunderstandings, we emphasise that this product *is not* a product in the category $\mathscr{T}\!op^\bullet$ in the sense of general category theory: it will be, as we already know, a direct product

$$(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0)).$$

It should also be borne in mind that, generally speaking, mixed multiplication does not have the associativity property, i.e. there are such pointed spaces X, Y and Z that the spaces $(X \wedge Y) \wedge Z$ and $X \wedge Y \wedge Z$ are not homeomorphic. (An example is the spaces $X = \mathbb{Q}$, $Y = \mathbb{Q}$ and $Z = \mathbb{Z}$, where $\mathbb{Q}$ is the set of rational numbers in the usual topology.) We will consider the question of the associativity of mixed multiplication in the Appendix 4.16 to this lecture.

At the same time, the mixed multiplication is obviously commutative, i.e. *for any pointed spaces X and Y, the spaces $X \wedge Y$ and $Y \wedge X$ are naturally homeomorphic.*

Identifying the circle $\mathbb{S}^1$ with the space $I/\{0, 1\}$, resulting in the identification of the end points of the segment I, we immediately get that for any pointed space (X, x_0) the smash $\mathbb{S}^{-1} \wedge X$ is the same as the suspension $S^\bullet X$.

Pointed maps $f : (X \times Y, (x_0, y_0)) \to (B, b_0)$ for which the associated map θf is a map $(Y, y_0) \to (B, b_0)^{(X, x_0)}$, i.e. for which $f((X \times y_0) \cup x_0 \times Y)) = b_0$, are found obviously, in natural bijective correspondence with pointed maps $\overline{f} : X \wedge Y \to B$ (matching is done by combining the maps $\overline{f}$ with the factorisation map $X \times Y \to X \wedge Y$). This allows you to embed the set $\mathscr{T}\!op^\bullet(X \wedge Y, B)$ into the set $\mathscr{T}\!op^\bullet(X \times Y, B) \subset \mathscr{T}\!op(X \times Y, B)$ and thereby assume that the map θ is defined on $\mathscr{T}\!op^\bullet(X \wedge Y, B)$. Thus, the association map θ now turns out to be the map

$$\mathscr{T}\!op^\bullet(X \wedge Y, B) \to \mathscr{T}\!op^\bullet(Y, B^X) \tag{4.2}$$

It is clear that it is still always (for any X, Y, B) injective, and bijective if and only if the corresponding map $\mathscr{T}\!op^\bullet(X \times Y, B) \to \mathscr{T}\!op^\bullet(Y, B^X)$ is bijective for the category $\mathscr{T}\!op$. In particular,

Proposition 4.3. *the map (4.2) is bijective if the space X is locally compact and Hausdorff.*

If, in addition, the space Y is also Hausdorff, then considered as a map $B^{X \wedge Y} \to (B^X)^Y$, it is a homeomorphism.

This exponential law for pointed maps, unlike the usual exponential law, is not directly applicable to pointed homotopies, which are maps of the form $X \times I \to Y$, and not $X \wedge I \to Y$ (note that the space $X \wedge I$ does not even make sense, since the segment I is not pointed). Therefore, homotopy requires a separate parallel discussion.

As it was said in Lecture 1, each (free) homotopy $X \times I \to Y$ admits two interpretations - as a path in the space Y^X or as a map $X \to Y^I$ into the path space Y^I. It is clear that the first interpretation also holds for pointed homotopies - each pointed homotopy $X \dot{\times} I \to Y$ is naturally identified with some path in the space $(Y, y_0)^{(X,x_0)}$, but as for the non-pointed case, this interpretation is adequate only under strong constraints on the space X (for example, if this space is Hausdorff and locally compact).

In order to transfer the second (formally more important) interpretation to the case of pointed homotopies, for any pointed space (Y, y_0), we will agree once and for all to consider the path space Y^I as a pointed space, taking the constant path $0_{y_0} : I \to Y$, $t \mapsto y_0$ as its base point, at the point y_0. Then the pointed homotopies $X \dot{\times} I \to Y$ will naturally be identified with the pointed maps $(X, x_0) \to (Y^I, 0_{y_0})$.

4.2 Fibrations and cofibrations in the category $\mathcal{T}op^\bullet$

In particular, since all maps $\omega_t : Y^I \to Y$ will now obviously be pointed, it follows that the axiom of the homotopy extension property in the form of Diagramme (1.3) of Lecture 1 will remain valid for pointed maps as well.

The concepts of cofibrations and fibrations are transferred to the case of the category $\mathcal{T}op^\bullet$ in an obvious way: it is only necessary to assume all the maps pointed in the corresponding diagrammes. At the same time, as was just noted, both variants of the definition of the fibration (one using the interpretation of homotopies as maps of $X \times I \to Y$, and the other as maps of $X \to Y^I$) are still equivalent to each other. Characterisation of fibrations using the axiom of the covering path also, of course, is literally transferred to the category $\mathcal{T}op^\bullet$.

Nevertheless, the transition to pointed maps is reflected differently on the cofibrations and fibrations. Since the extension of the pointed map is automatically a pointed map, the pointed map $i : (A, a_0) \to (X, x_0)$, being a cofibration (as a map $i : A \to X$ of the category $\mathcal{T}op^\bullet$), will be a cofibration in the category $\mathcal{T}op^\bullet$. In particular, taking an arbitrary cofibration (X, A) of the category $\mathcal{T}op$ and arbitrarily selecting the base point in A, we get a cofibration of the category $\mathcal{T}op^\bullet$.

On the contrary, from the fact that the map $p : E \to B$ is a fibration of the category $\mathcal{T}op$, it does not follow that the pointed map $p : (E, e_0) \to (B, b_0)$ will be a fibration of the category $\mathcal{T}op^\bullet$, since the pointed covering homotopy $\overline{F} : X \times I \to E$ for the homotopy $F : X \times I \to B$, may not be pointed.

To at least partially correct the case, we will introduce the following definition.

Definition 4.4. A point x_0 of a topological space X is called *nondegenerate* if it is closed in X and the pair (X, x_0) is a cofibration, i.e. - which by applying Proposition 2.27 of Lecture 2 is equivalent - if a one-point set $\{x_0\}$ is a FNSDR in a weak sense: (FNSDR = Functionally Distinguished Neighbourhood Strong Deformation Retract. See Definition 2.25). The pointed space (X, x_0) is called *well-pointed* if its base point x_0 is non-degenerate.

It follows directly from the Corollary 2.38 of Theorem 2.37 of Lecture 2 that if the pointed map $p : (E, e_0) \to (B, b_0)$ is a fibration in the category $\mathscr{T}\!op$, then in the category $\mathscr{T}\!op^\bullet$ the axiom of covering homotopy will hold for any well-pointed space (X, x_0). Assuming freedom of speech, we can thus say that with respect to well-pointed spaces, any fibration of the category $\mathscr{T}\!op$ is a fibration of the category $\mathscr{T}\!op^\bullet$.

This is quite a lot, because, as we once did we will see that almost all the pointed spaces that actually occur are well-pointed.

Remark 4.5. The pointed map, which is a (co)-fibration of the category $\mathscr{T}\!op^\bullet$, a priori may not be a (co)-fibration of the category $\mathscr{T}\!op$. However, in practice, such pathological situations do not occur.

4.3 The lemma about gluing a whisker

In connection with the notion of a well-pointed space, it is useful to keep in mind the following simple lemma.

Lemma 4.6. *Any topological space X is homotopically equivalent to a topological space X' having a non-degenerate point x_0. In this case, we can assume that the space X is contained in the space X' as its strong deformation retract. Moreover, the space X' can be chosen so that there exists a path connecting x_0' with any pre-defined point $x_0 \in X$.*

Proof. Noting the point 1 in the segment I, and in the space X this point x_0, we construct a bouquet $X' = X \vee I$. Each point of X' is either a point x from X, or the number $\tau \in I$, with $x_0 = 1$. It is clear that $X \subset X'$ is a strong deformation retract of the space X' (the corresponding homotopy $f_\tau : X' \subset X'$, fixed on X, translates each point $\tau \in I \subset X'$ to the point $t + (1 - t)\tau \in I$). Let x_0' be the point $0 \in I \subset X'$. Putting $\varphi(x) = 1$ and $\varphi(\tau) = \tau$, we get on X' a continuous function $\varphi : X' \to I$, for which $\varphi^{-1}(0) = \{x_0\}$ and $\varphi^{-1}(1) = X$, and putting $g_\tau(\tau) = (1 - t)\tau$, we get a homotopy $g_t : U \to X'$, where $U = X' \setminus X = [0, 1)$, such that $g_0(\tau) = \tau$ and $g_1(\tau) = x_0'$ for any $\tau \in U$. Therefore, the point x_0' is non-degenerate. □

It is said that the space X' is obtained from the space by *gluing a whisker*.

Thus, gluing a whisker, practically without changing the space X, turns it into a space with a non-degenerate base point.

4.4 Cylinders and co-cylinders in the category $\mathscr{T\!op}^\bullet$

To bring pushouts and pullbacks into the category $\mathscr{T\!op}^\bullet$, it is enough to construct for the corresponding diagrammes their pushouts and pullbacks in the category $\mathscr{T\!op}$ and then naturally mark the points in them (in the pushout diagramme $A \leftarrow C \rightarrow B$, the base point will be the image of the base point of the space C, and in the pullback diagramme $A \rightarrow C \leftarrow B$ is the base point of the direct product of $A \times B$). Thus, up to the base points of pushuout and pullback in the category $\mathscr{T\!op}^\bullet$ are the same as in the category $\mathscr{T\!op}$.

In particular, if you ignore the base points, then the cocylinder of pointed map $f : X \rightarrow Y$ will be its cocylinder as an non-pointed map.

At the same time, the corresponding statement is incorrect for the cylinder, since in its construction the space $X \times I$ is replaced by the space $X \dot{\times} I$, which means that the cylinder $\operatorname{Cyl}^\bullet(f)$ of the pointed map $f : X \rightarrow Y$ is obtained from its non-pointed mapping cylinder $\operatorname{Cyl}(f)$ by shrinking the segment $x_0 \times I$ to a point:

$$\operatorname{Cyl}^\bullet(f) = \operatorname{Cyl}(f)/(x_0 \times I).$$

In the literature, the cylinder $\operatorname{Cyl}^\bullet(f)$ is usually called the *reduced mapping cylinder* (and the cylinder $\operatorname{Cyl}(f)$, respectively, is called *non-reduced*).

Despite this modification, the reduction of the extension problem to the retraction problem carried out with the help of cylinders is completely preserved for pointed maps. Of course, the reduction of the lifting problem to the cross-section problem carried out with the help of cylinders is also preserved.

4.5 Contractible pointed spaces

A pointed space (X, x_0) is called *contractible* (notation $X \searrow \mathrm{pt}$) if the constant map $\mathrm{const} : X \rightarrow X$, $x \mapsto x$ is pointed homotopic to the identity map $\mathrm{id} : X \rightarrow X$, $x \mapsto x$, i.e. if the point x_0 is a strong deformation retract of the spaces X.

It follows directly from Proposition 2.18 of Lecture 2 that a well-pointed space (X, x_0) is contractible if and only if the non-pointed space X is contractible. The corresponding pointed homotopy $X \dot{\times} X$ is called a *contraction*.

This means that the contractibility property is essentially indifferent to which category ($\mathscr{T\!op}$ or $\mathscr{T\!op}^\bullet$) we are working.

4.6 Reduced cones

The reduced (reversed) cylinder of the map const is called the *reduced cone* over the space X and is denoted by the symbol $C^\bullet X$. Thus, $C^\bullet X = CX/Cx_0$.

We will denote the points of the cone $C^\bullet X$ with the former symbols $[x, t]$, as before identifying the points $[x, 1]$ and x, i.e. identifying the space X with the base the cone $C^\bullet X$. In particular, according to this embedding, the base point of the cone CX will be the point $x_0 \in X$.

It is clear that $S^\bullet X = C^\bullet X/X$.

Each pointed map $f : X \to Y$ defines by the formula

$$(C^\bullet f)[x,t] = [f(x),t], \quad x \in X, t \in I,$$

the pointed map $C^\bullet f : C^\bullet X \to C^\bullet Y$, and it is clear that the correspondences $X \to C^\bullet X$, $f \mapsto C^\bullet f$ constitute a functor from $\mathcal{T}op^\bullet$ to $\mathcal{T}op^\bullet$.

The cone $C^\bullet X$ is obviously contractible (as a pointed space), and

Proposition 4.7. *the space X is contractible if and only if it is a retract of the cone $C^\bullet X$.*

Moreover, it turns out that - as for non - pointed spaces - the words "is a retract" can be replaced here with the words "is a strong retract".

Indeed, we can consider any retraction $r : C^\bullet X \to X$ as a retraction $CX \to X$, and it is easy to see that the deformation retract F, constructed from this retraction in the way described in Lecture 2, will be stationary at C_{x_0}, subject to obvious precautions, and therefore, will induce a deformation retraction of the cone $C^\bullet X$.

Explicitly, the deformation retraction F can be specified, for example, by the formula

$$F([x,t],s) = \begin{cases} [r[r[x, \frac{t-2s}{1-2s}], \frac{2(1-3s)}{1-2s}], 0], & \text{if } 2s \le t \le 1, 1/4 \le s \le 1/3, \\ [r[x, \frac{t-2s}{1-2s}], \frac{2s}{1-2s}]], & \text{if } 2s \le t \le 1, 0 \le s \le 1/4, \\ [r[x, \frac{2(1-t-s)}{1-2s}], 0], & \text{if } 1/2 \le t \le 2/3, t/2 \le s \le 1-t, \\ [x, \frac{2(t-s)}{1-2s}], & \text{if } 0 \le t \le 1/3, t/2 \le s \le t, \\ [r[x, \frac{2(1-s-t}{1-2t}], 0], & \text{if } 1/2 \le t \le 2/3, 1-t \le s \le 1/2, \\ [x, \frac{2(s-t)}{1-2t}], & \text{if } 1/2 \le t \le 2/3, 1/2 \le s \le t, \\ [r[x, 9t+2s-2st-8], 0] & \text{if } 2/3 \le t \le 1, 1/3 \le s \le 1/2, \\ [r[x, 10-9t+2s-2st], 0] & \text{if } 2/3 \le t \le 1, 1/2 \le s \le 2/3, \\ [x, \frac{t+2s-2)}{2s-1}], & \text{if } 2-2s \le t \le 1, 2/3 \le s \le 1, \\ [x, 0], & \text{if } 0 \le t \le 2/3, t \le s \le (2-t)/2, \end{cases}$$

which determines this deformation on the square (t,s) separately on each of its ten parts, shown in Fig. 4.6.1.

The pointed map $f : X \to Y$ is called *nul homotopic* if it is pointed homotopic to the constant map. Just as in the case of non-pointed spaces, the map $f : X \to Y$ is nul homotopic if and only if it can be extended to $C^\bullet X$:

$$\begin{array}{ccc} X & \xrightarrow{\subset} & C^\bullet X \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \overline{f}} & \\ Y & & \end{array}$$

In addition, a pointed space Y is contractible if and only if, for every pointed space X, any pointed map $X \to Y$ is null homotopic.

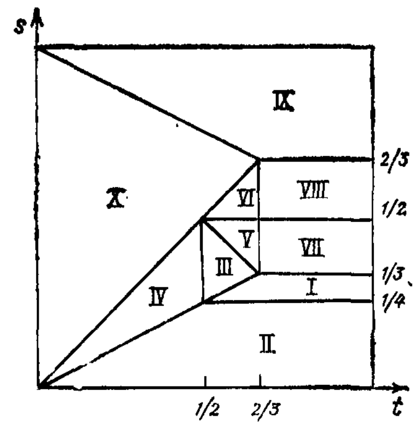


Figure 4.6.1:

4.7 Serre fibrations

In contrast to the case of the category $\mathscr{T}op$, the concept of a cone in the category $\mathscr{T}op^\bullet$ can be detailed. The corresponding dual object is the space $PX = P(X, x_0)$, already familiar to us from Lecture 1, of all paths of the space X starting at the base point x_0 (formally, it would be more convenient here to either go to the straight cone or assume that PX consists of paths, ending at the point x"; due to the fact that we take an inverted cone for $C'X$, $1 - t$ appears in the formulas below instead of t).

We will consider PX to be a pointed space, taking the constant path 0_{x_0} : $t \mapsto x_0$ as its base point.

Like the cone $C^\bullet X$, the space PX is contractible (the corresponding contraction is given by the correspondence $u \mapsto u_\tau$, $u \in PX$, $\tau \in I$, where u_τ the path is defined by the formula $u_\tau(t) = u((1 - \tau)t)$, $t \in I$).

It is clear that any pointed map $f : X \to Y$ defines by the formula,

$$((Pf)u)(t) = f(u(t)), \quad u \in PX, t \in I,$$

the continuous pointed map $Pf : PX \to PY$, and the correspondences $X \mapsto PX$, $f \mapsto Pf$ constitute a functor from $\mathscr{T}op^\bullet$ to $\mathscr{T}op^\bullet$.

It is easy to see that, like the functor $S^\bullet$ and Ω, the functors $C^\bullet$ and P are adjoint, i.e. for any pointed spaces X and Y, there is a natural (with respect toy X and Y) bijective map

$$\mathscr{T}op^\bullet(C^\bullet X, Y) \xrightarrow{\approx} \mathscr{T}op^\bullet(X, PY)$$

(each map $f : C^\bullet X \to Y$ corresponds to a map $g : X \to PY$, defined by the formula $g(x)(t) = f[x, 1 - t]$, $x \in X$, $t \in I$). However, unlike the case of the

$S^\bullet$ and Ω functors, the transition to the homotopy category $[\mathscr{T}\!op^\bullet]$ makes little sense for the functors $C^\bullet$ and P, since for any pointed spaces X and Y sets $[C^\bullet X, Y]^\bullet$ and $[X, PY]^\bullet$ are singleton sets due to the contractibility of the spaces X and PY.

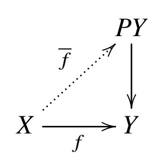
If the space X is contractible, then each contraction $X \times I \to X$ interpreted as a map $X \to X^I$, will be (after replacing t with $1-t$) a map in PX, which has the property that its composition with the Serre fibration $\omega_1 : PX \to X$, $u \mapsto u(1)$ is an identical map of id_X, i.e. it will be a section of the fibration ω_1. Conversely, it is clear that any section $X \to PX$ of the fibration ω_1, considered as a map in X^I, will be a pointed homotopy connecting the map const_X to the map id_X, i.e. after replacing t with $1-t$ will be a contraction of the space X. Thus,

Proposition 4.8. *the space X is contractible if and only if the Serre fibration $\omega_1 : PX \to X$ has the section $s : X \to PX$.*

(It can be shown that this section also has the property that $\mathrm{id} \sim s \circ p$, and the corresponding homotopy $f_t : PXPX$ can be chosen so that for any $t \in I$ equality $p \circ f_t = p$ takes place. In the terminology introduced in Appendix 2.11 to Lecture 2, this means that the space X is contractible if and only if the Serre fibration $\omega_1 : PXX$ collapses. This refinement is dual to the statement for cones obtained by replacing retracts with strong deformation retracts, and is proved in a dual way.)

Now it is clear that

Proposition 4.9. *the pointed map $f : X \to Y$ is null homotopic if and only if it can be lifted to PY:*



Remark 4.10. (A note on terminology) G. W. Whitehead [19] warns against using the term *cocone* for the space PX. Although we do not see this term as a special crime, but we will refrain from using it. (Perhaps the space PX should be called "nus"?)
Transcriber's note: 'nus" (Russian) = "nous" (Greek), meaning the God-given capacity of each person to think (reason); the mind; mental capacity to exercise reflective thinking.

4.8 Pointed homotopy equivalences

A pointed map $f : X \to Y$ is called a *pointed homotopy equivalence* (or *homotopy equivalence of the category $\mathscr{T}\!op^\bullet$*) if there is a pointed map $g : Y \to X$ (inverse pointed homotopy equivalence) such that $f \circ g \sim \mathrm{id}$ and $g \circ f \sim id$.

Of course, any pointed homotopy equivalence will also be an ordinary homotopy equivalence (homotopy equivalence of the category $\mathcal{Top}^\bullet$). For well-pointed spaces, the converse is also true:

Proposition 4.11. *If the pointed spaces (X, x_0) and (Y, y_0) are well-pointed, then each pointed map $f : (X, x_0) \to (Y, y_0)$, which is a homotopy equivalence of the category $\mathcal{Top}$, will also be a pointed homotopy equivalence.*

Thus, the seemingly ambiguous term "pointed homotopy equivalence" is not really such (at least for well-pointed spaces).

Remark 4.12. If the space is (X', x_0') is obtained from the space (X, x_0) by gluing a whisker, then in the case when the space (X, x_0) is well-pointed, the deformation retraction $X' \to X$ will, by applying Proposition 4.11, be a pointed homotopy equivalence. Thus, as expected,

Proposition 4.13. *gluing a whisker to a well-pointed space does not change its pointed homotopy type.*

(On the contrary, it is easy to see that if the space (X, x_0) is not well-pointed, then the space (X', x_0') cannot be pointed homotopy equivalent to it).

We will preface two lemmas to the proof of Proposition 4.11.

Lemma 4.14. *If the pointed space (X, x_0) is well-pointed, then for any pointed space (Y, y_0) each map $f : X \to Y$, for whose point $f(x_0)$ lies in the same component of the space Y as the point y_0, is homotopic to some pointed map $(X, x_0) \to (Y, y_0)$.*

Proof. By condition, the point $f(x_0)$ can be connected in some way $u : I \to Y$ with the point y_0 We can consider this path as a homotopy of the map $f|_{x_0}$. Therefore, according to the axiom of homotopy extension (applicable by condition to the pair (X, x_0)), there exists a homotopy $f_t : X \to Y$ such that $f_t(x_0) = u(t)$ for any $t \in l$. Since $f_1(x_0) = u(1)y_0$, the map f_1 is a pointed map $(X, x_0) \to (Y, y_0)$, homotopic to the map f. □

Lemma 4.15. *If the pointed space (X, x_0) is well-pointed, then for any pointed map $f : (X, x_0) \to (X, x_0)$, freely homotopic to the identical map* id*, there exists a pointed map $f' : (X, x_0) \to (X, x_0)$, such that $f' \circ f \sim$* id.

Proof. Let $F : X \times I \to X$ be a free homotopy connecting the map f to the identical maps id. Since the pair (X, x_0) is a cofibration, there exists a homotopy $f_t' : X \to X$ such that $f_0' =$ id and $f_t'(x_0) = F(x_0, t)$ for any $t \in I$. Then the formula

$$G(x,t) = \begin{cases} f_{1-2t}'(f(x)), & \text{if} \quad 0 \le t \le 1/2, \\ F(x, 2t-1), & \text{if} 1/2 \le t \le 1, \end{cases}$$

defines a homotopy $G : X \times I \to X$, connecting the map $f' \circ f$, where $f' = f_1'$, with the map id. Since the pair is $(X \times I, x_0 \times I)$ is also a cofibration, there exists a homotopy $\Phi : (X \times I) \times I \to X$ with the initial map G such that

$$\Phi(x_0, t, \tau) = \begin{cases} F(x_0, 1 - 2t(1-\tau)) & \text{if} \quad 0 \le t \le 1/2, \\ F(x_0, 1 - 2(1-t)(1-\tau)), & \text{if} 1/2 \le t \le 1, \end{cases}$$

Then the map

$$(x, \tau) \mapsto \Phi(x, 0, \tau), \quad (x, t) \mapsto \Phi(x, t, 1), \quad (x, \tau) \mapsto \Phi(x, 1, 1 - \tau)$$

will be pointed homotopies sequentially connecting the map $f' \circ f : x \mapsto \Phi(x, 0, 0)$ with the map $\mathrm{id} : x \mapsto \Phi(x, 1, 0)$. □

Remark 4.16. Lemma 4.15 and Proposition 2.18 of Lecture 2 are special cases of a general statement dual to Lemma 2.56 from the Appendix 2.11 to Lecture 2.

Proof. (of Proposition 4.11) Let $\overline{g} : Y \to X$ be the homotopy equivalence inverse to the homotopy equivalence $f : X \to Y$. Then the map $\overline{g} \circ f : X \to X$ is homotopic to the identical map id, which implies that the point $\overline{g}(y_0) = (\overline{g} \circ f)(x_0)$ lies in the same component of the space X as the point x_0. Therefore, according to Lemma 4.6, the map $\overline{g}$ is homotopic to some pointed map $(Y, y_0) \to (X, x_0)$. Hence, without loss of generality, we can assume the map $\overline{g}$ to be pointed.

But then, homotopic to the identity map, the map $\overline{g} \circ f : X \to X$ is also pointed, and therefore, according to Lemma 4.14, there exists a pointed map $h = (\overline{g} \circ f)' : X \to X$ such that $h \circ (\overline{g} \circ f) \sim \mathrm{id}$. This proves that there exists a pointed map $g : Y \to X$ (namely, the map $g = h \circ \overline{g}$) such that $g \circ f \sim \mathrm{id}$.

The map g is of course also a homotopy equivalence (the inverse of the homotopy equivalence of f). Therefore, for the same reasons, a pointed map $\overline{f} : X \to Y$ exists for it in turn, such that $\overline{f} \circ g \sim \mathrm{id}$. But then $\overline{f} \sim \overline{f} \circ g \circ f \sim f$ and, therefore, $f \circ g \sim \mathrm{id}$. Therefore, f is a pointed homotopy equivalence (with g as the inverse pointed homotopy equivalence). □

4.9 Maps ignoring base points

Proposition 4.11 suggests that the relation of pointed homotopy should coincide with the relation of ordinary homotopy (at least for smooth pointed spaces), i.e. that the map

$$[X, Y] \to [X, Y], \tag{4.17}$$

that occurs when the base points are ignored, is bijective. However, this assumption is false and the situation here is actually more complicated.

Indeed, in order that the map $f : X \to Y$ be homotopic to the pointed map $(X, x_0) \to (Y, y_0)$ in any case, it is necessary that the map $\pi_0 f : \pi_0 X \to \pi_0 Y$ translates the component of the space X containing the point x_0 into the component of the space Y containing the point y_0. This means that the image of the map (4.17) is obviously contained in the subset $[X, Y]_0$ of the set $[X, Y]$, consisting of homotopy classes of maps $X \to Y$ satisfying the last condition. Since for a disjoint subset Y $[X,Y]_0$, is necessarily a proper subset of the set $[X,Y]$, we see, therefore, that even for the objectivity of the map (4.17) it is necessary to assume the space Y connected. For a disconnected space Y, the question of surjectivity can only stand in relation to the map

$$[X,Y]^\bullet \to [X,Y]_0, \tag{4.18}$$

induced by the map (4.17) (and, note, with a connected Y coinciding with the map (4.17)).

But the answer to this question is exactly given by Lemma 4.6. Thus, according to this lemma, *if the space X is well-pointed, then for any pointed space Y the map (4.18) is surjective.*

In particular,

Proposition 4.19. *if the pointed space X is well-pointed, and the pointed space Y is connected, then the map (4.17) is surjective.*

The question of the injectivity of the map (4.17) (or, equivalently, the map (4.18)) requires some adjustments, because even for the "smoothest" spaces, the map (4.17) may not be injective, and the study of the question of the structure of the preimages of the elements of the set $[X,Y]$ with this map is a meaningful geometric problem.

4.10 The fundamental group of pointed spaces

Definition 4.20. Loops $u, v \in \Omega X$ of a pointed space (X, x_0) are called *homotopic* if they are homotopic with respect to a two-element set $\{0,1\} \subset I$. The set $[I,X]\,\mathrm{rel}\{0,1\}$ of all classes of $[u]$ homotopy loops is denoted by the symbol $\pi_1(X,x_0)$ (or simply $\pi_1 X$). If the set $\pi_1(X,x_0)$ consists of only one element, i.e. if any two loops of the space X are homotopic, then the pointed space X is called *simply connected*.

By applying the exponential homotopy law I in $X\,\mathrm{rel}\{0,1\}$ are identified with the paths of the space ΩX, and thus the set $\pi_1 X$ is identified with the sets $\pi_0\Omega X$ of all components of this space:

$$\pi_1 X = \pi_0 \Omega X.$$

But since $\pi_0 K = [pt, K]$, then

Proposition 4.21. *for each H-space (each H-monoid or each H-group) K multiplication is transferred to its coset $\pi_0 K$ and with respect to this multiplication, the set $\pi_0 K$ is a unitoid (respectively, a monoid or a group).*

With respect to $K = \Omega X$, we obtain, therefore, that

Proposition 4.22. *the formula*

$$[u][v] = [uv] \tag{4.23}$$

well defines the multiplication with respect to which this set is a group.

The group $\pi_1 X$ is called the *fundamental group* of the pointed space X.

The unit of this group is the homotopy class $[0_{x_0}]$ of the constant loop $0_{x_0} : I \to X$, $t \mapsto x_0$, and the element α^{-1} inverse to the element $\alpha = [u]$ is the homotopy class α^{-1} of the inverse loop $u^{-1} : t \mapsto u(1-t)$.

Each pointed map $f : (X, x_0) \to (Y, y_0)$ determines, by the formula $f_* : [u] \mapsto [f \circ u]$, the map $f_* : \pi_1 X \to \pi_1 Y$, which is a homomorphism, and it is clear that the correspondences $X \mapsto \pi_1 X$, $f \mapsto f_*$ constitute a functor from the category $\mathscr{T}\!\mathit{op}^\bullet$ to the category $\mathscr{G}\mathit{rp}$. In other words, multiplication (4.23) is natural with respect to X.

Remark 4.24. Interestingly, as we will show in Lecture 6, multiplication (4.23), as well as the inverse multiplication $[u][v] = [vu]$ are the only natural group multiplications on X that can be defined in the set $\pi_1 X$.

4.11 The action of the group $\pi_1 Y$ on the set $[X, Y]^\bullet$

Recall that the action of the group *Pi* on an object A of some category $\mathscr{A}$ is an arbitrary homomorphism $R : \Pi \to \operatorname{Aut} A$, $\xi \mapsto R_\xi$, of this group into the group of automorphisms $\operatorname{Aut} A$ of the object A. If A is a set (generally speaking, equipped with an additional structure), then the image of the element $\alpha \in A$ with the automorphism R_ξ, $\xi \in \Pi$, is usually denoted by the symbol $\xi\alpha$. The resulting map

$$\Pi \times A \to A, \quad (\xi, \alpha) \mapsto \xi\alpha,$$

is also called an *action*. In order for the map $\Pi \times A \to A$ to be an action, it is necessary and sufficient that for any elements $\xi, \eta \in \Pi$, $\alpha \in A$ the following diagramme commutes

$$\begin{array}{ccc} \Pi \times \Pi \times A & \longrightarrow & \Pi \times A \\ \downarrow & & \downarrow \\ \Pi \times A & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} (\xi, \eta, \alpha) & \longmapsto & (\xi\eta, \alpha) \\ \downarrow & & \downarrow \\ (\xi, \eta\alpha) & \longmapsto & \xi(\eta\alpha) = (\xi\eta)\alpha \end{array}$$

and so that for any element $\xi \in \Pi$ the map $\alpha \mapsto \xi\alpha$, $\alpha \in A$, is a morphism of the object A on itself. With $\mathscr{A} = \mathscr{E}\mathit{ns}$, the last condition is automatically fulfilled, and, for example, with $\mathscr{A} = \mathscr{A}\mathit{b}\mathscr{G}\mathit{rp}$, it means that for any elements $\alpha, \beta \in A$ the following equality must be met

$$\xi(\alpha + \beta) = \xi\alpha + \xi\beta.$$

An example of an action is the action $(\xi, \alpha) \mapsto \xi\alpha\xi^{-1}$ of a group Π on itself by means of internal automorphisms.

The *orbit* of the action of $\Pi \times A \to A$ defined by the element $\alpha \in A$, is the set $\Pi\alpha$ of all elements of the form $\xi\alpha$, $\xi \in \Pi$. Orbits are the equivalence classes with respect to the equivalence relation in which $\alpha \sim \beta$ if $\beta = \xi\alpha$ for some $\xi \in \Pi$. The corresponding coset is denoted by the symbol A/Π. We emphasise that, in general in other words, it does not inherit the structure of the object A. For example, if A is a group, then A/Π will, generally speaking, be only a pointed set (the base point of which is the orbit of unity, which, by the way, is a singleton set).

We will apply these general definitions to the case when the group Π is the fundamental group $\pi_1 Y$, and the set A is the set $[X,Y]^\bullet = [(X,x_0),(Y,y_0)]$ homotopy classes of maps of a *well-pointed* space (X,x_0) to the space (Y,y_0).

Let $\xi = [u] \in \pi_1 Y$ and $\alpha = [f]^\bullet \in [X,Y]^\bullet$ Since the space (X,x_0) is well-pointed, there exists a homotopy $f_t : X \to Y$ such that $f_0 = f$ and $f_t(x_0) = u(1-t)$ for any $t \in I$ (see above proof of Lemma 4.6). Let $[v] = [u]$ and $[g]^\bullet = [f]^\bullet$, and let $g_t : X \to Y$ be a homotopy such that $g_0 = g$ and $g_t(x_0) = v(1-t)$ for any $t \in I$. This homotopy together with the homotopy f_t and the homotopy $\omega_\tau : I \to Y \operatorname{rel}\{0,1\}$, connecting u and v, defines by the formula

$$h_t(x,\tau) = \begin{cases} f_t(x), & \text{if} \quad \tau = 0, \\ \omega_\tau(1-\tau), & \text{if} \quad x = x_0, \\ g_t(x), & \text{if} \quad \tau = 1, \end{cases}$$

some homotopy $h_\tau : X_{\text{pt}} \to Y$, where, according to the notation introduced in Lecture 2, $X_{\text{pt}} = (X \times 0) \cup (x_0 \times I) \cup (X \times 1)$. Since the map h_0 is a restriction on the X_{pt} homotopy $\operatorname{rel}\{x_0\}$ connecting the maps f and g, and since, according to Lemma 2.19 of Lecture 2, the pair $(X \times I, X_{\text{pt}})$ is a cofibration, the homotopy h_t extends to some homotopy $\overline{h}_t : X \times I \to Y$. The end map $\overline{h}_1$ of this homotopy will be the homotopy $\operatorname{rel}\{x_0\}$ connecting the maps f_1 and g_1. This proves that the class $[f_1]^\bullet$ of the map f_1 depends only on the classes $\xi = [u]$ and $\alpha = [f]^\bullet$, and not on their representatives u and f. Therefore, denoting it by $\xi\alpha$, we will well define some map $(\xi,\alpha) \mapsto \xi\alpha$ from $\pi_1 Y \times [X,Y]^\bullet$ to $[X,Y]^\bullet$ as the automatic check shows, by the action of the group $\pi_1 Y$ on the set $[X,Y]^\bullet$.

We will call the constructed action the *canonical action* of the group $\pi_1 Y$ on the set $[X,Y]^\bullet$.

Proposition 4.25. *Orbits of the canonical action*

$$\pi_1 Y \times [X,Y]^\bullet \to [X,Y]^\bullet$$

exactly coincide with the preimages for the map (4.17) homotopy classes of $[X,Y]$.

Proof. By definition, if $\alpha = [f]^\bullet$ and $\xi\alpha = [f_1]^\bullet$, then $f \sim f_1$. Conversely, if $f \sim f_1$ and $f_t : X \to Y$ be a homotopy connecting f to f_1 then $[f_1]^\bullet = [u][f]^\bullet$, where $u : I \to Y$ be the loop $t \mapsto f_{1-t}(x_0)$. □

Proposition 4.25 means that there is equality

$$[X,Y]_0 = [X,Y]^\bullet/\pi_1 Y, \tag{4.26}$$

where $[X,Y]^\bullet/\pi_1 Y$ is the coset of the set $[X,Y]^\bullet$ by the action of the group $\pi_1 Y$. In particular, we see that

Proposition 4.27. *if the space Y is connected and simply connected, then* $[X,Y] = [X,Y]^\bullet$.

Remark 4.28. If X_α and Y_β are components of the spaces X and Y respectively, then the set $[X,Y]$ is naturally represented as a disjoint union of sets $[X_\alpha, Y_\beta]$. Similarly, if X_{α_0} and Y_{β_0} are components of spaces X and Y containing base points, then the set $[X,Y]^\bullet$ is a disjoint union of the set $[X_{\alpha_0}, Y_{\beta_0}]^\bullet$ and sets $[X_\alpha, Y_\beta]$ for $(\alpha,\beta) \neq (\alpha_0,\beta_0)$. At the same time, on the last sets, the map (4.17) is an identity map (in particular, the group $\pi_1 Y$ acts identically on them), so only the following map is of interest

$$[X_{\alpha_0}, Y_{\beta_0}]^\bullet \to [X_{\alpha_0}, Y_{\beta_0}].$$

Since the group $\pi_1 Y$ is naturally identified with the group $\pi_1 Y_{\beta_0}$, we see, therefore, that everything is judged to the case of connected spaces X and Y. Namely, being able to calculate the sets $[X,Y]^\bullet$ for connected pointed spaces X and Y, we will know for the same spaces the sets $[X,Y]$, and therefore we will know these sets (together with the sets $[X,Y]^\bullet$) and for any spaces X and Y. This explains why, in homotopy theory, the restriction to connected spaces is usually considered quite sufficient and does not, in fact, reduce generality. When, in practice, there is a need to apply the results of homotopy theory to disconnected spaces, they are easily modified in the necessary way.

Similarly, although the main geometric interest is, of course, the theory of homotopies in the category of $\mathcal{T}op$ and the transition to the category of $\mathcal{T}op^\bullet$ is caused only by a rigid algebraic necessity, in the light of all the above, we can now focus all attention on the category of $\mathcal{T}op^\bullet$, keeping "in mind" the possible ignoration of the base points.

Remark 4.29. For any pointed space X and any well-pointed (i.e. having a non-degenerate unit) H-cogroup K, the canonical action of the group $\pi_1 X$ is an action on the set of elements of the group $[K,X]^\bullet$. It turns out that

Proposition 4.30. *this action is consistent with the structure of the group on* $[K,X]^\bullet$*, i.e. for any element $\xi = [u]$ of the group $\pi_1 X$ the map $\alpha \to \xi\alpha$, $\alpha \in [K,X]^\bullet$, is an automorphism of the group* $[K,X]^\bullet$.

Proof. In fact, let $f : K \to X$, $g : K \to X$ be arbitrary pointed maps, and $\alpha = [f]^\bullet$ and $\beta = [g]^\bullet$ be their homotopy classes. By definition $\xi\alpha = [f_1]^\bullet$ and $\xi\beta = [g_1]^\bullet$ where f_1 and g_1 are terminal maps of homotopies $f_t, g_t : K \to X$ such that $f_0 = f$, $g_0 = g$ and $f_t(e) = g_t(e) = u(1-t)$, where e is the co-unit of the H-cogroup K. But then the map

$$f_t + g_t = m \circ (f_t \vee g_t) : K \to X, \quad 0 \leq t \leq 1,$$

will obviously constitute a homotopy connecting the map $f+g = m\circ(f\vee g)$ with the map $f_1 + g_1 = m \circ (f_1 \vee g_1)$ and satisfying the relation $(f_t + g_t)(e) = u(1-t)$. Since, by definition, $[f+g]^\bullet = \alpha + \beta$ and $[f_1 + g_1]^\bullet = \xi\alpha + \xi\beta$, this proves that $\xi\alpha + \xi\beta = \xi(\alpha+\beta)$. Consequently, the map $\alpha \to \xi\alpha$ is an automorphism. □

Remark 4.31. The construction of the canonical action of the group $\pi_1 Y$ on the set $[X,Y]^\bullet$ admits a small generalisation, which is often useful.

Let two points y_0 and y_1 lying in the same connected component be selected in the topological space Y. By specifying the point y_0 as the base point, we get the pointed space (Y, y_0), and by specifying the point y_1, we get the pointed space (Y, y_1). Accordingly, for any pointed space (X, x_0), two sets of pointed homotopy classes will be defined

$$[(X,x_0),(Y,y_0)] \quad \text{and} \quad [(X,x_0),(Y,y_1)].$$

Now let $u : I \to Y$ be an arbitrary path in Y connecting the point y_0 with the point y_1. If the space (X, x_0) is well-pointed, then for any map $f : (X, x_0) \to (Y, y_0)$ there exists a free homotopy $f_t : X \to Y$ such that $f_0 = f$ and $f_t(x_0) = u(1-t)$ (and, therefore, $f_1(x_0) = y_0$, i.e, $f_1 : (X, x_0) \to (Y, y_0)$). Arguments that differ only trivially from those used above now show that the class $[f_1]^\bullet \in [(X,x_0),(Y,y_0)]$ depends only on the class $[f]^\bullet \in [(X,x_0),(Y,y_1)]$ (and on the class $[u]\,\mathrm{rel}\{0,1\}$ of the path) and that the resulting map

$$[(X,x_0),(Y,y_1)] \to [(X,x_0),(Y,y_0)] \tag{4.32}$$

is a bijective map of the set $[(X,x_0),(Y,y_1)]$ on the set $[(X,x_0),(Y,y_0)]$.

Transferring to this case the values entered above, we will denote class $[f_1]$ with the symbol $\xi\alpha$, where $\alpha = [f]$ and $\xi = [u]\,\mathrm{rel}\{0,1\}$.

If the path u connects the point y_0 with the point y_1 and the path v connects the point y_1 with the point y_2, then the formula (3.36) of Lecture 3 will well determine their product uv, which is the path connecting the point y_0 with the point y_2, and it is clear that the class $\mathrm{rel}\{0,1\}$ of the path uv depends only on the classes ξ and η of the paths u and v. This class is called the *product* of the classes ξ and η and is denoted by the symbol $\xi\eta$.

If now $\alpha \in [(X,x_0),(Y,y_0)]$, then as shown by an automatic check, equality $(\xi\eta)\alpha = \xi(\eta\alpha)$ will take place.

In the case when (X, x_0) is an H-group (or at least H is a counitoid), the same reasoning as above shows that

Proposition 4.33. *the map (4.32) is an isomorphism of groups (of unitoids).*

We emphasise that the map (4.32) depends on the path and (or, more precisely, on its class $\xi = [u]\,\mathrm{rel}\{0,1\}$) and replacing the path with another (non-homotopic) path, it can be changed.

Remark 4.34. Recall that a *groupoid* is a category whose morphisms are all isomorphisms. For any category $\mathscr{A}$, an *ensemble* (or *local system*) of objects of category $\mathscr{A}$ over a groupoid Π is an arbitrary functor from Π to $\mathscr{A}$. Thus, the ensemble R maps each object $x \in \Pi$ to some object $R_x \in \mathscr{A}$ and each morphism $\xi : x \to y$ to a morphism $R_\xi : R_x \to R_y$ (automatically being an isomorphism), with $R_{\xi\eta} = R_\xi \circ R_\eta$ and $R_{\mathrm{id}} = \mathrm{id}$.

An example of a groupoid is the *fundamental groupoid* ΠY of an arbitrary topological space Y, whose objects are points of space Y, and whose morphisms are homotopy classes rel$\{0,1\}$ of paths $u : I \to Y$. In this groupoid, the relation $\xi : y \to x$ means that the paths of the class ξ connect the point x with the point y, and the composition of morphisms η and ξ is the product $\xi\eta$ of the classes ξ and η.

Ensembles over the groupoid ΠY are called ensembles (local systems) *over the space* Y.

In this terminology, the statements of Remark 4.31 mean that

Proposition 4.35. *for any well-pointed space* (X, x_0) *and any topological space* Y *the sets* $R_y = [(X, x_0), (Y, y)]$ *together with the maps* $R_\xi : \alpha \mapsto \xi\alpha$ *constitute an ensemble of sets over the space* Y, *which is an ensemble of groups (unitoids) when the space* (X, x_0) *is an H-cogroup (H-counitoid).*

For any groupoid Π and any of its objects y_0, the set $\Pi(y_0, y_0)$ of all morphisms $\xi : y_0 \to y_0$ is a group, and for any ensemble R of sets over Π, this group acts on the set R_{y_0}. For the case $\Pi = \Pi Y$ the group $\Pi(y_0, y_0)$ is nothing other than the fundamental group $\pi_1(Y, y_0)$ and its action on the set $R_{y_0} = [(X, x_0), (Y, y_0)]$ is a canonical action from Proposition 4.25.

4.12 Pointed H-spaces

The transition to the category $\mathcal{Top}^\bullet$ requires, of course, a corresponding modification in the concepts of H-groups, H-monoids and H-unitoids. Now we must consider every H-unitoid K (and, in particular, every H-monoid and every H-group) to be a pointed space with a base point and, accordingly, assume multiplication $m : K \times K \to K$ as a pointed map (which is equivalent to equality $e^2 = e$), and homotopies connecting maps $x \mapsto xe$ and $x \mapsto ex$ with identical map, pointed homotopies. In addition, for H-monoids and H-groups, diagrams)3.7) and (3.8) of Lecture 3 should now be commutative up to pointed homotopy. Then for any pointed space (X, x_0), the set $[X, K]^\bullet$ will be an unitoid (respectively a monoid or a group) with unit $[0_e]^\bullet$.

According to Proposition 4.25, if the space (X, x_0) is well-pointed, then the group $\pi_1 K$ acts on the set $[X, K]^\bullet$. It turns out that

Proposition 4.36. *this action is trivial, i.e.* $\xi\alpha = \alpha$ *for any elements* $\alpha = [f]^\bullet \in [X, K]^\bullet$ *and* $\xi = [u] \in \pi_1 K$.

Proof. Indeed, since by the condition the map $e \mapsto xe$, $x \in K$, is a pointed homotopy to the constant map, the map f is pointed homotopy to the map $g : x \mapsto f(x)e$, and the loop u is homotopic to the loop $v : t \to u(t)e$ i.e. $\alpha = [g]^\bullet$ and $\xi = [v]$. On the other hand, the formula

$$g_t(x) = f(x)u(1-t)$$

defines a homotopy $g_t : X \to K$ for which $g_0 = g_1 = g$ and hence $g_t(x_0) = u(1-t)$. Hence, $\xi\alpha = [g_1]^\bullet = [g]^\bullet = \alpha$. □

Since every H-unitoid (H-monoid or H-group) of the category $\mathscr{Top}^\bullet$ is, of course, an H-unitoid (respectively, an H-monoid or H-group) of the category $\mathscr{Top}$, the set of free homotopy classes $[X, K]$ of maps $X \to K$ will also be a unitoid (respectively, a monoid or group), and the natural map $[X, K]^\bullet \to [X, K]$ will be a homomorphism. Since, according to what has just been proved, this map is bijective (if the space X is well-pointed), we finally get that

Proposition 4.37. *for any well-pointed space X and any pointed H-unitoid (any H-monoid, any H-group) K unitoids (respectively monoids and groups) $[X, K]^\bullet$ and $[X, K]$ are naturally isomorphic.*

4.13 H-spaces with real units

The statement that the point $e \in K$ is a homotopy unit of the H-space K means that the constraint $m|_{K\vee K}$ of multiplications $m : K \times L \to K$ on the bouquet $K \vee K = (K \times e) \cup (e \times K)$ is homotopic to the *collapsing map* $\nabla = \mathrm{id} \vee \mathrm{id} : K \vee K \to K$, which translates the points $x_I = (x, e)$ and $x_{II} = (e, x)$ of this bouquet to the point $x \in K$, and the statement that the point e is a real (strict) unit, means that $m|_{K\vee K} = \nabla$. Therefore, if the point e is non-degenerate and, therefore, according to Proposition 1.36 of Lecture 1, the pair $(K \times K, K \vee K) = (K, e)^2$ is a cofibration, then multiplication $m : K \times K \to K$ with homotopy unit e is homotopic to multiplication $m' : K \times K \to K$ for which this unit is a real unit.

At the same time, if, with respect to the multiplication of m, the space K was an H-monoid, i.e. if the map $m \circ (m \times \mathrm{id})$ and $m \circ (\mathrm{id} \times m)$ from $K \times K \times K$ to K were homotopic, then the pointed maps $m' \circ (m' \times \mathrm{id})$ and $m' \circ (\mathrm{id} \times m')$ will also be homotopic (freely). But due to the non-degeneracy of the base point of the space $K \times K \times K$ (proved by a twofold application of Lemma 1.28 of Lecture 1), the statement about the coincidence of the sets $[X, K]^\bullet$ and $[X, K]$ applies to this space. Therefore, pointed maps $m' \circ (m' \times \mathrm{id})$ and $m' \circ (\mathrm{id} \times m')$ will be homotopic and pointed. Thus, with respect to the multiplication of m', the space K will be a pointed H-monoid.

Finally, if the H-space K is an H-group, then, for similar reasons, the map μ will be homotopic to the pointed map $\mu' : (K, e) \to (K, e)$, and for this map, diagrams (3.8) of Lecture 3, in which m and μ are replaced by m' and μ', will be pointed homotopy, i.e. the pointed H-monoid K with multiplication m'' will be a pintted H-group.

This proves that

Proposition 4.38. *every H-space (H-monoid or H-group) whose homotopy unit is non-degenerate is equivalent to a pointed H-space (H-monoid, H-group) with a real unit.*

Without the assumption of non-degeneracy of the unit, one can only assert that the H-space K is homotopically isomorphic to the pointed H-space K' with a real unit. To prove it, it is enough to stick to the K' whiskers and apply the previous statement.

We see, therefore, that in the transition from the category of $\mathscr{T}op$ to the category of $\mathscr{T}op^\bullet$, we lose nothing (and gain nothing) with respect to the K^* functions, since the stock of H-spaces K in both categories is practically the same, and each of them gives the same algebraic objects to $K^*(X)$.

4.14 Cofibrations and relative homeomorphisms

In addition to the question of the relationship between the categories of pointed and non-pointed spaces, we have one more small debt left from the previous lecture — homotopy equivalence of the reduced and on the reduced suspensions. Although the latter question is quite elementary in itself, we will take this opportunity to present in connection with it some general results that have an independent interest.

For any pair (X, A), the space X/A is called the *cofibre*. For example, the cofibre of the pair $(X \times Y, X \vee Y)$ is the smash product $X \wedge Y$, and the cofibre of the pair (CX, X) is the suspension SX.

Note that the cofibre X/A of an arbitrary pair (X, A) is naturally a pointed space, the base point of which is the image of the subspace A for the factorisation map $X \to X/A$.

A continuous map $\varphi : XtoY$ is called a *map* $(X, A) \to (Y, B)$ *from a pair* (X, A) *to a pair* (Y, B) if $\varphi(A) \subset B$. The composition of the mappings of the pairs $(X, A) \to (Y, B)$ and $(Y, B) \to (Z, C)$ is obviously a map of $(X, A) \to (Z, C)$. Therefore, pairs and their maps make up a category. We will denote this category by the symbol $\mathscr{T}op_2$.

It is clear that any map of pairs $\varphi : (X, A) \to (Y, B)$ induces a pointed map $\varphi^\bullet : X/A \to Y/B$ of their cofibres with correspondences $(X, A) \mapsto X/A$, $\varphi \mapsto \varphi'$ constitute a functor from the category $\mathscr{T}op_2$ to the category $\mathscr{T}op^\bullet$.

The map of pairs $f : (X, A) \to (Y, B)$ is called a *relative homeomorphism* if as the map $X \to Y$ it is an epimorphism on $f(X)$, It is easily verified that

Proposition 4.39. *the map $X/A \to Y/B$ induced by the relative homeomorphism $(X, A) \to (Y, B)$ is a homeomorphism.*

For any pair (X, A) and any surjective map $g : A \to B$ factorisation map

$$f : (X, A) \to (X \cup_g B, B)$$

is obviously an surjective relative homeomorphism. It's easy to see that the opposite is also true.

Lemma 4.40. *For any surjective relative homeomorphism $f : (X, A) \to (Y, B)$ the space Y is homeomorphic to the space $X \cup_g B$, where $g = f|_A$.*

Proof. Consider the diagramme

$$\begin{array}{ccc} X \sqcup B & \xrightarrow{f \sqcup i} & Y \\ \Big\downarrow{\scriptstyle\varphi} & \nearrow_{\widehat{f}} & \\ X \cup_g B & & \end{array}$$

where $i : B \to Y$ is an embedding, φ is a factorisation map, and $\widehat{f}$ is a map that matches on $X \setminus A = \varphi(X \setminus A)$ with the map f, and on B with the map i. Since this diagramme is obviously commutative, then, as the map $\widehat{f}$ is epimorphic, it is continuous. In addition, it is obviously bijective, and for the inverse map there is a relation $\varphi = \widehat{f}^{-1} \circ (f \sqcup i)$. But the surjective relative homeomorphism f is, by definition, an epimorphism, from which it immediately follows that the map $f \sqcup i$ is also epiomorphic. Hence, by applying the continuity of the map φ, the map $\widehat{f}^{-1}$ is continuous, and, therefore, the map $\widehat{f}$ is a homeomorphism. □

Relative homeomorphisms are mostly interesting to us because of their following property.

Lemma 4.41. *If the pair (X, A) is a cofibration, then for any relative homeomorphism $\varphi : (X, A) \to (Y, B)$, the pair (Y, B) will also be a cofibration.*

Proof. The statement that the pair (Y, B) is a cofibration means that for each diagramme of the form (where j is the inclusion)

$$\begin{array}{ccc} B & \xrightarrow{j} & Y \\ {\scriptstyle F}\Big\downarrow & \overline{F} & \Big\downarrow{\scriptstyle \overline{f}} \\ Z^I & \xrightarrow[\omega_0^\bullet]{} & Z \end{array} \tag{4.42}$$

there is a closing homotopy of $\overline{F}$. But by superimposing this diagramme with the following diagramme

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle\varphi}\Big\downarrow & & \Big\downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[j]{} & Y \end{array}$$

we will get a similar diagramme (where i is the inclusion)

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle F\circ\varphi}\Big\downarrow & \overline{G} & \Big\downarrow{\scriptstyle \overline{f}\circ\omega} \\ Z^I & \xrightarrow[\omega_0^\bullet]{} & Z \end{array}$$

for a pair (X, A) for which the homotopy $\overline{G}$ by the condition exists. Since the map φ outside A is injective, the formula

$$\overline{F}(g) = \begin{cases} F(y), & \text{if} \quad y \in B, \\ \overline{G}(x), & \text{if} \quad y \notin B \text{ and } y = \varphi(x), \end{cases}$$

well defines some map $\overline{F} : X \to Z^I$, satisfying the relation $\overline{G} = \overline{F} \circ \varpi$ and therefore, by applying the epimorphic map φ, continuous. To complete the proof, it remains to note that the map $\overline{F}$ obviously closes the diagram (4.42). □

Corollary 4.43. *The cofibre X/A of an arbitrary cofibration (X, A) is a well-pointed space.*

Corollary 4.44. *For any well-pointed spaces X and Y, the space*

$$X \wedge Y = (X \times Y)/X \vee Y, \quad X \vee Y = (X \times y_0) \cup (x_0 \times Y),$$

is also well-pointed.

Proof. According to the Proposition 1.20 Lecture 1, the pair $(X \times Y, X \wedge Y) = (X, x_0) \times (Y, y_0)$ is a cofibration. □

Corollary 4.45. *For any well-pointed space X, the pair (CX, X) is a cofibration.*

Proof. Identification map $(X \times I, X_{\text{pt}}) \to (C^\bullet X, X)$ is a relative homeomorphism, and the pair $(X \times I, X_{\text{pt}}$ is a cofibration (see Lemma 2.19 of Lecture 1). □

4.15 Cofibrations of contractible subspaces

Let us now prove the following important lemma.

Lemma 4.46. *If for a cofibration (X, A) the subspace A is contractible, then the coset expression*

$$\xi : X \to X/A$$

is a homotopy equivalence.

Proof. Let $f_t : A \to A$ be a homotopy connecting the identity map $\text{id}_A : A \to A$ with a constant map $\text{const}_{a_0} : A \to A$. Since the pair (X, A) is a cofibration, there exists a homotopy $\overline{f}_t : X \to X$ such that $\overline{f}_0 = \text{id}_X$ and $\overline{f}_t \circ i = i \circ f_t$ for any $i \in I$, where $i : A \to X$ is an embedding. In particular, $\overline{f}_1 \circ i = i \circ \text{const} = \text{const}$, i.e. $\overline{f}_1|_A = \text{const}$. Therefore, the map $\overline{f}_1$ induces some map $h : X/A \to X$ for which $h \circ \xi = \overline{f}_1$. Hence $\text{id} \sim h \circ \xi$. Similarly, since $\overline{f}_t(a) \in A$ for any $a \in A$ and any $t \in I$, the homotopy $\overline{f}_t$ induces the homotopy $\overline{g_t} : X/A \to X/A$, for which $\overline{g}_1 = \text{id}$ and $\overline{g}_1 = \xi \circ h$. Hence, $\text{id} \sim h \circ \varphi$, so ξ and h are mutually inverse homotopy equivalences. □

4.16 Concluding remarks on suspensions and loop spaces

Let's apply the obtained general results to the suspension SX and its meridian S_{x_0}. If the space X is well-pointed, and therefore, by applying Lemma 4.15, the pair $(X \times I, X_{\mathrm{pt}})$, where $X_{\mathrm{pt}} = (X \times 0) \cup (x_0 \times I) \cup (X \times 1)$, is a cofibration, then, according to Lemma 4.41, the pair (SX, S_{x_0}) with cofibre $S^\bullet X = SX/S_{x_0}$ will also be a cofibration (the factorisation map $(X \times I, X_{\mathrm{pt}}) \to (SX, S_{x_0})$ is a relative homeomorphism). Hence, since the meridian S_{x_0}, being homeomorphic to the segment I, is contractible,

Proposition 4.47. *for any well-pointed space X, the factorisation map $SX \to S^\bullet X$ is a homotopy equivalence.*

In addition, by applying Corollary 4.43 of Lemma 4.41 to the pair $(X\times I, X_{\mathrm{pt}})$, we get that

Proposition 4.48. *if a space X is well-pointed, then its reduced suspension $S^\bullet X$ is also well-pointed.*

A similar statement holds for the loop space, i.e.

Proposition 4.49. *for any well-pointed space (X, x_0), the loop space $(\Omega X, 0_{x_0})$ is also well-pointed.*

Proof. Indeed, if $\varphi : X \to I$ is a continuous function such that $\varphi^{-1}(0) = \{x_0\}$, then the formula

$$\widehat{\varphi}(u) = \max_{\tau} \varphi(u(\tau)), \quad 0 \le t \le 1,$$

will determine the continuous function $\widehat{\varphi} : \Omega X \to I$, for which $\widehat{\varphi}^{-1}(0) = \{0_{x_0}\}$. In this case, the set $\widehat{U} = \Omega X \setminus \widehat{\varphi}(1)$ is naturally identified with the space ΩU, where $U = X \setminus \varphi^{-1}(1)$, and therefore for any homotopy $g_t : U \to X \,\mathrm{rel}\{x_0\}$ the formula

$$\widehat{g}_t(u) = g_t \circ u, \quad u \in \widehat{U}, \quad 0 \le t \le 1,$$

will be correct to define some homotopy $\widehat{g}_t : \widehat{U} \to \Omega X$. Also, if $g_0(x) = x$ and $g_1(x) = x_0$ for any point $x \in U$, then $\widehat{g}_0(u) = u$ and $\widehat{g}_1(u) = 0_{x_0}$ for any path $u \in \widehat{U}$. □

It is clear that this reasoning holds for the space of Moore loops $\Omega^M X$ as well.

Appendix

In this Appendix we will consider two unrelated questions: the question of the properties of composite multiplication and the question of the conditions ensuring the nilpotency or abelicity of the group $[X, K]^{\bullet}$.

4.A The lemma on the product of epiomorphisms

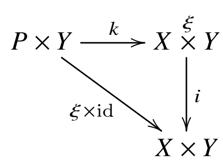
The key to the properties of composite multiplication is the following lemma, which we will need in many other questions.

Lemma 4.50. *If the space Y is Hausdorff and locally compact, then for any epiomorphism $\xi : P \to X$, the map*

$$\xi \times \mathrm{id} : P \times Y \to X \times Y, \quad (p, y) \mapsto (\xi(p), y),$$

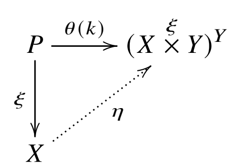
is also an epimorphism.

Proof. Introducing into consideration the coset space $X \overset{\xi}{\times} Y$ of the space $P \times Y$ with respect to the equivalence $(p_1, y_1) \sim (p_2, y_2)$ if and only if $\xi(p_1) = \xi(p_2)$ and $y_1 = y_2$, we will represent the map $\xi \times \mathrm{id}$ as a composition of the factorisation map $k : P \times Y \to X \overset{\xi}{\times} Y$ and some continuous bijective map $i : X \overset{\xi}{\times} Y \to X \times Y$:



The map $\xi \times \mathrm{id}$ is an epimorphism if and only if the map i is a homeomorphism.

With this in mind, we will consider the map $\theta(k) : P \to (XxY)^Y$ associated with the map k. By definition $[\theta(k)p](y) = k(p, y)$ for any points $p \in P$, $y \in Y$, and, therefore, $\theta(k)p_1 = \theta(k)p_2$ if and only if $\xi(p_1) = \xi(p_2)$. Therefore, in the diagramme



there is a closing map $\eta : X \to (X \overset{\xi}{\times} Y)^Y$. It is continuous, since the map $\theta(k)$ is continuous, and the map ξ is an epimorphism. Since the space Y is Hausdorff and locally compact, and, therefore, the association map θ is bijective, then there is a map $j : X \times Y \to (X \overset{\xi}{\times} Y$ such that $\theta j = \eta$, and since $\eta \circ \xi = \theta(k)$, $j \circ (\xi \times \mathrm{id}) = k$. Therefore $(i \circ j) \circ (\xi \times \mathrm{id}) = i \circ k = \xi \times \mathrm{id}$, and therefore $i \circ j = \mathrm{id}$, the map $\xi \times \mathrm{id}$ is subjective. Hence, the map i is a homeomorphism (with inverse homeomorphism j), and the map $\xi \times \mathrm{id}$ is an epiomorphism. □

Remark 4.51. We needed Hausdorffness and local compactness of the space Y in Lemma 4.50 only in order to ensure the bijectivity of the map θ. Therefore,

Proposition 4.52. *if the exponential law is true for the spaces X and Y, i.e. if for the space B the association map*

$$\theta : B^{X \times Y} \to (B^Y)^X$$

is bijective, then for any epimorphism $\xi : P \to X$ the map $\xi \times \mathrm{id} : P \times Y \to X \times Y$ is also an epiomorphism.

Corollary 4.53. *If the spaces X and Y are Hausdorff and locally compact, then for any epiomorphisms $\xi : P \to X$ and $\eta : Q \to Y$ the map*

$$\xi \times \eta : P \times Q \to X \times Y, \quad (p, q) \mapsto (\xi(p), \eta(q)),$$

is also an epiomorphism.

Proof. Is it enough to notice that $\xi \times \eta = (\xi \times \mathrm{id}) \circ (\mathrm{id} \times \eta)$, and take into account that the composition of two epiomorphisms is an epiomorphism. □

4.B The smash product of homotopy classes

It is clear that for any pointed spaces A, B, X, Y and any pointed maps $f : A \to X$, $g : B \to Y$ the formula $a \wedge b \mapsto f(a) \wedge f(b)$, $a \in A$, $b \in B$, well defines the pointed map

$$f \wedge g : A \wedge B \to X \wedge Y,$$

moreover, the correspondences $(X, Y) \mapsto X\wedge$, $(f, g) \mapsto f \wedge g$ constitute a two-argument functor from $\mathcal{Top}^\bullet$ to $\mathcal{Top}^\bullet$, covariant on both arguments.

Proposition 4.54. *For any homotopy $f_t : A \to X$ and $g_t : B \to Y$ the map $f_t \wedge g_t : A \wedge B \to X \wedge Y$ also constitutes a homotopy.*

Proof. You need to show that the map

$$H : (A \wedge B) \times I \to X \wedge I, \quad (a \wedge b, t) \mapsto f_t(a) \wedge g_t(b),$$

is continuous. But if $j' : A \times B \to A \wedge B$ and $j : X \times Y \to X \wedge Y$ are canonical factorisation maps, then there is a commutative diagramme

$$\begin{array}{ccc}
(A \times B) \times I & \xrightarrow{\overline{H}} & X \times Y \\
{\scriptstyle j' \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle j} \\
(A \wedge B) \times I & \xrightarrow[H]{} & X \wedge Y
\end{array}$$

where the map $\overline{H}$ is defined by the formula $\overline{H}((a,b),t) = (f_t(a), g_t(b))$ and is therefore continuous. Hence, the map $H \circ (j \times \mathrm{id}) = j \circ \overline{H}$ is continuous, and since, according to Lemma 4.50, the map $j \times \mathrm{id}$ is an epiomorphism, then the map H is continuous. □

It follows directly from Proposition 4.54 that for any homotopy classes

$$\alpha = [f]^\bullet \in [A, X]^\bullet, \quad f : A \to X,$$
$$\beta = [g]^\bullet \in [B, Y]^\bullet, \quad g : B \to Y,$$

the formula

$$\alpha \wedge \beta = [f \wedge g]^\bullet$$

well defines a homotopy class

$$\alpha \wedge \beta \in [A \wedge B, X \wedge Y]^\bullet,$$

which is called the *smash product* of the classes α and β.

Obviously, the correspondences $(X, Y) \mapsto X \wedge Y$, $(\alpha, \beta) \mapsto \alpha \wedge \beta$ make up a double functor from $[\pi^\bullet]$ to $[\pi^\bullet]$ (homotopisation of the functor $\wedge$ from $\pi^\bullet$ to $\pi^\bullet$). In particular, this means that for any spaces A, X, X', B, Y, Y' and any homotopy classes $\alpha \in [A, X]^\bullet$, $\beta \in [B, Y]^\bullet$, $\xi \in [X, X']^\bullet$, $\eta \in [Y, Y']^\bullet$ there is equality

$$(\alpha \wedge \beta) \circ (\xi \wedge \eta) = (\alpha \circ \xi) \wedge (\beta \circ \eta).$$

It is clear that for any three pointed spaces X, Y and Z the correspondences

$$x_I \wedge z \mapsto (x \wedge z)_I, \quad y_{II} \wedge Z \mapsto (y \wedge Z)_{II}$$

define a canonical homomorphism

$$(X \wedge Y) \wedge Z \mapsto (X \wedge Z) \vee (Y \wedge Z),$$

having the naturality, i.e. such that for any maps $f : A \to X$, $g : B \to Y$, $h : C \to Z$ there is a commutative diagramme

$$\begin{array}{ccc}
(A \vee B) \wedge C & \longrightarrow & (A \wedge C) \vee (B \wedge C) \\
{\scriptstyle (f \vee g) \wedge h} \downarrow & & \downarrow {\scriptstyle (f \wedge h) \vee (g \wedge h)} \\
(X \vee B) \wedge Z & \longrightarrow & (X \wedge Z) \vee (Y \wedge Z)
\end{array}$$

We will always identify the spaces $(X \vee Y) \wedge Z$ and $(X \wedge Z) \vee (Y \wedge Z)$ in the future by means of this homeomorphism.

In particular, this allows us for any H cogroup K with the multiplication $m : K \to K \vee K$ and any pointed space C,treating $m \wedge \mathrm{id}$ as a map $K \wedge C \to (K \wedge C) \vee (K \wedge C)$: and an automatic, albeit somewhat tedious, checking shows that with respect to this map,

Proposition 4.55. *the space $K \wedge C$ is an H-cogroup.*

Therefore, for any elements $\alpha, \beta \in [K, X]^\bullet$ and $\gamma \in [C, Z]^\bullet$, the element $\alpha \wedge \gamma + \beta \wedge \gamma \in [K \wedge C, X \wedge Z]$ will be defined, On the other hand, in the group $[K, X]^\bullet$ the element $\alpha + \beta$ will be defined, and therefore in the group $[K \wedge C, X \wedge Z]^\bullet$ the element $(\alpha + \beta) \wedge \gamma$ will be defined. A direct calculation using definitions shows that these elements are the same:

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma.$$

Similarly, it is proved that for any pointed space A and any H-cogroup L with multiplication m, the space $A \wedge L$ is an H-cogroup with multiplication $\mathrm{id} \wedge m$, and for any elements $\alpha \in [A, X]^\bullet$, and $\beta, \gamma \in [L, Y]^\bullet$ there is the equality

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma.$$

In this sense,

Proposition 4.56. *the smash product of homotopy classes is distributive with respect to addition.*

As already noted in Lecture 4, the operation of smash product of spaces is commutative, i.e. the formula $x \wedge y \mapsto y \wedge x$ defines a canonical homomorphism $X \wedge Y \to Y \wedge X$. Identifying by means of this homeomorphism of the spaces $X \wedge Y$ and $Y \wedge X$ (as well as by means of a similar homeomorphism $a \wedge b \mapsto b \wedge a$ of the space $A \wedge B$ and $B \wedge A$), for any elements $\alpha \in [A, X]^\bullet$, $\beta \in [B, Y]^\bullet$ we can consider the homotopy class $\beta \wedge \alpha$ as an element of the set $[A \wedge B, X \wedge Y]^\bullet$. It is clear that then this class coincides with the class $\alpha \wedge \beta$. In this sense,

Proposition 4.57. *the smash product of homotopy classes is commutative.*

Similarly, if for the spaces X, Y, Z the map

$$(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z), \quad (x \wedge y) \wedge z \mapsto x \wedge (y \wedge z), \tag{4.58}$$

is a homeomorphism (or at least a homotopy equivalence) and if the spaces A, B, C have the same property, then after the corresponding identifications for any elements $\alpha \in [A, X]^\bullet$, $\beta \in [B, Y]'^\bullet$ and $\gamma \in [C, Z]^\bullet$ there will be the equality $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$. In this sense,

Proposition 4.59. *the smash products of homotopy classes are associative.*

4.C Associativity of the smash product of locally compact Hausdorff spaces

For which spaces X, Y, Z is the map (4.58) a homeomorphism? To answer this question, we first compare the space $(X \wedge Y) \wedge Z$ with the space $X \wedge (Y \wedge Z)$.

By definition, the space $X \wedge Y$ is the coset space $(X \times Y)/(X \vee Y)$ of the space $X \times Y$, the space $(X \wedge Y) \wedge Z$ is the coset space $[(X \wedge Y) \times Z]/[(X \wedge Y) \vee Z]$ of the space $(X \wedge Y) \times Z$ and the space $X \wedge Y \wedge Z$ is the coset space $(X \times Y \times Z)/(X \vee Y \vee Z)$ of the space $X \times Y \times Z$. Let

$$\begin{aligned} j :& X \times Y \to X \wedge Y, \\ j' :& X \times Y \times Z \to X \wedge Y \wedge Z, \\ j'' :& (X \wedge Y) \times Z \to (X \wedge Y) \wedge Z \end{aligned}$$

be the corresponding factorisation maps, and let

$$\xi : X \wedge Y \wedge Z \to (X \wedge Y) \wedge Z$$

be the map $x \wedge y \wedge z \mapsto (x \wedge y) \wedge z$ induced by the homeomorphism

$$\bar{\xi} : X \times Y \times Z \to (X \times Y) \times Z, \quad (x, y, z) \mapsto ((x, y), z).$$

Then there is a commutative diagramme

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{\bar{\xi}} & (X \times Y) \times Z \\ & & \big\downarrow {\scriptstyle j \times \mathrm{id}} \\ {\scriptstyle j'} \big\downarrow & & (X \wedge Y) \times Z \\ & & \big\downarrow {\scriptstyle j''} \\ X \wedge Y \wedge Z & \xrightarrow[\xi]{} & (X \wedge Y) \wedge Z \end{array} \tag{4.60}$$

from which it follows that the map $\xi \circ j'$ is continuous. Since the map j' is an epiomorphism,

Proposition 4.61. *the map ξ is continuous.*

Our goal is to find conditions under which the map ξ is a homeomorphism. It is clear that this map is bijective. Therefore, it all comes down to the question of the continuity of the inverse map ξ^{-1}. Due to the commutativity of diagramme (4.60) the map $\xi^{-1} \circ j'' \circ (j \times \mathrm{id})$ is continuous (and even epiomorphic). Since the map j'' is an epiomorphism, it follows that the map ξ^{-1} is continuous if and only if the map $j \times \mathrm{id}$ is epiomorphic, for which, by applying Lemma 4.50, is enough for the space Z which is Hausdorff and locally compact. Thus, we have proved the following proposition.

Proposition 4.62. *If the space Z is locally compact and Hausdorff, then for any spaces X and Y the canonical map*

$$X \wedge Y \wedge Z \to (X \wedge Y) \wedge Z, \quad x \wedge y \wedge z \mapsto (x \wedge y) \wedge z,$$

is a homeomorphism.

It is clear that the analogue of Proposition 4.62 holds for the canonical map

$$X \wedge Y \wedge Z \to X \wedge (Y \wedge Z), \quad x \wedge y \wedge z \mapsto x \wedge (y \wedge z),$$

i.e. this map is homeomorphic if the space X is locally compact and Hausdorff.

Corollary 4.63. *If the spaces X and Z are locally compact and Hausdorff, then for any space Y the canonical map*

$$(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z), \quad (x \wedge y) \wedge z \mapsto x \wedge (y \wedge z),$$

is a homeomorphism.

Thus, for locally compact and Hausdorff spaces X and Z, all three spaces $(X \wedge Y) \wedge Z$, $X \wedge (Y \wedge Z)$ and $X \wedge Y \wedge Z$ are canonically homeomorphic and therefore can be identified with each other. In this sense,

Proposition 4.64. *the smash product of spaces is associative.*

Corollary 4.65. *If the space Y is locally compact and Hausdorff, then for any space X the canonical maps*

$$\begin{aligned} S^\bullet(X \wedge Y) \to S^\bullet X \wedge Y, &\quad [x \wedge y, t] \mapsto [x, t] \wedge y, \\ S^\bullet X \wedge Y \to S^\bullet(X \wedge Y), &\quad [x, t] \wedge y \mapsto [x \wedge y, t], \end{aligned}$$

are mutually inverse homeomorphisms.

Similarly, if the space X is locally compact and Hausdorff, then for any space Y, the canonical maps will be the inverse of each other homeomorphisms

$$\begin{aligned} S^\bullet(X \wedge Y) \to X \wedge S^\bullet Y, &\quad [x \wedge y, t] \mapsto x \wedge [y, t] \\ X \wedge S^\bullet Y \to S^\bullet(X \wedge Y), &\quad x \wedge [y, t] \mapsto [x \wedge y, t]. \end{aligned}$$

Proof. Suffice it to recall that $S^\bullet X = \mathbb{S}^1 \wedge X$. □

Considering these homeomorphisms as identifications, we get that for any homotopy classes $\alpha \in [L, X]^\bullet$, $\beta \in [B, Y]'^\bullet$, $\gamma \in [C, Z]^\bullet$ the identity

$$E(\alpha \wedge \beta) = E\alpha \wedge \beta = \alpha \wedge E\beta. \tag{4.66}$$

(See Remark 3.41 of Lecture 3.)

4.D Homotopy associativity of the smash product of well-pointed spaces

Unfortunately, in homotopy theory we cannot limit ourselves to Proposition 4.62, since many spaces naturally arising in this theory (for example, path spaces) are not locally compact. Therefore, we are forced to look for other less restrictive conditions that ensure the associativity of the smash product (at least up to homotopy equivalence).

Proposition 4.67. *For any well-pointed spaces X, Y, Z, the canonical bijective map*

$$\xi : X \wedge Y \wedge Z \to (X \wedge Y) \wedge Z, \quad x \wedge y \wedge z \mapsto (x \wedge y) \wedge z,$$

is a homotopy equivalence.

Proof. According to Corollary 4.44 of Lemma 4.41 Lecture 4 the space $X \wedge Y$ is well-pointed, i.e. (Lemma 2.26 and Proposition 2.27 of Lecture 2) there exists a function $\varphi : X \wedge Y \to I$, such that $\varphi^{-1}(0) = x_0 \wedge y_0$ and a homotopy $g_t : X \wedge Y \to X \wedge Y$, such that $g_0 = \mathrm{id}$ and $g_t(x \wedge y) = x_0 \wedge y_0$ for $t > \varphi(x \wedge y)$. In this case, the homotopy g_t is induced by some homotopy with similar properties $\overline{g}_t : X \times Y \to X \times Y$, i.e. for any $t \in I$ there is a commutative diagramme

$$\begin{array}{ccc} X \times Y & \xrightarrow{\overline{g}_t} & X \times Y \\ \downarrow j & & \downarrow j \\ X \wedge Y & \xrightarrow[g_t]{} & X \wedge Y \end{array}$$

where, as above, $j : X \times Y \to X \wedge Y$ is the factorisation map (a similar statement is true, of course, for the function φ, but we won't need it).

It is easy to see that the formulae

$$\overline{h}_t(x \wedge y \wedge z) = j'(\overline{g}_t(x, y), y),$$
$$h_t((x \wedge y) \wedge z) = g_t(x \wedge y) \wedge z,$$

where $j' : X \times Y \times Z \to X \wedge Y \wedge Z$ is the factorisation map, well determine homotopies

$$\overline{h}_t : X \wedge Y \wedge Z \to X \wedge Y \wedge Z, \quad h_t : (X \wedge Y) \wedge Z \to (X \wedge Y) \wedge Z,$$

for which the commutative diagramme

$$\begin{array}{ccc} X \wedge Y \wedge Z & \xrightarrow{\xi} & (X \wedge Y) \wedge Z \\ \downarrow \overline{h}_t & & \downarrow h_t \\ X \wedge Y \wedge Z & \xrightarrow[\xi]{} & (X \wedge Y) \wedge Z \end{array}$$

takes place and which have the property that $\overline{h}_0 = \mathrm{id}$, $h_0 = \mathrm{id}$.

Let

$$\eta = \xi^{-1} \circ h_1 : (X \wedge Y) \wedge Z \to X \wedge Y \wedge Z.$$

Assuming that the map η is continuous and considering the relation $\eta = \overline{h}_1 \circ \xi^{-1}$, we immediately get that

$$\xi \circ \eta = h_1 \sim h_0 = \mathrm{id}, \quad \eta \circ \xi = \overline{h}_1 \sim \overline{h}_0 = \mathrm{id},$$

i.e. that ξ and η are mutually inverse homotopy equivalences.

Thus, to complete the proof of proposition 4.67, we only need to prove the continuity of the map η.

To this end, we will consider a commutative diagramme

$$\begin{array}{ccc} X \wedge Y \wedge Z & \xrightarrow{j'''} & (X \wedge Y) \wedge Z \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j\wedge\mathrm{id}} \\ X \wedge Y \wedge Z & \xrightarrow[\xi]{} & (X \wedge Y) \wedge Z \end{array}$$

where j, j' and j''' are the factorisation maps, and its boundary on the closed subspace $A = \varphi^{-1} \wedge Z$ of the space $(X \wedge Y) \wedge Z$, i.e. the diagramme

$$\begin{array}{ccc} B' & \xrightarrow{j'''|_{B'}} & A' \\ {\scriptstyle j'|_{B'}}\downarrow & & \downarrow{\scriptstyle (j\wedge\mathrm{id})|_{A'})} \\ B & \xrightarrow[\xi|_B]{} & A \end{array}$$

where $A' = (j \wedge \mathrm{id})^{-1}A$, $B = \xi^{-1}A$, and $B' = (j')^{-1}B$. The maps $j'|_{B'}$ and $j'''|_{B'}$ of the last diagramme, being restrictions of epiomorphisms, are themselves epiomorphisms. As for the map $(\xi' \wedge \mathrm{id})|_{A'}$, it will obviously be a homeomorphism. Therefore, the closing map $\xi|_B$ of this diagramme is an epiomorphism, and, being moreover a bijective continuous map, it is a homeomorphism. This proves that the map ξ^{-1} on a closed set A is continuous. Therefore, the map $\eta = \overline{h}_1 \circ \xi^{-1}$ is continuous on A.

Let, then, $U = (X \wedge Y) \setminus \varphi^{-1}(1)$ and $B = \overline{U} \wedge Z \subset (X \wedge Y) \wedge Z$. By the condition $g_1(U) = x_0 \wedge y_0$ and, therefore, by continuity, $g_1(\overline{U}) = x_0 \wedge y_0$ (the point $x_0 \wedge y_0$ being non-degenerate, closed). Therefore, $h_1(B) = (x_0 \wedge y_0) \wedge z_0$, and therefore $\eta(B) = x_0 \wedge y_0 \wedge z_0$, i.e. $\eta|_B = \mathrm{const}$. Hence, the map η is continuous also on B. Since $A \cup B = (X \wedge Y) \wedge Z$, the continuity of the map η is fully proved, □

Corollary 4.68. *For any well-pointed spaces X and Y, the map*

$$S^\bullet(X \wedge Y) \to S^\bullet X \wedge Y, \quad [x \wedge y, t] \mapsto [x, t] \wedge y,$$

is a homotopy equivalence.

Proof. The map in question is a composition of the map

$$S^\bullet(X \wedge Y) \to X \wedge Y \wedge \mathbb{S}^1, \quad [x \wedge y, t] \mapsto x \wedge y \wedge e^{2\pi i t},$$

and, being by applying Proposition 4.62 (and identifications $S^\bullet(X \wedge Y) = \mathbb{S}^1 \wedge (X \wedge Y) = (X \wedge Y) \wedge \mathbb{S}^{-1}$) a homeomorphism, and the map

$$X \wedge Y \wedge \mathbb{S}^1 \mapsto S^\bullet X \wedge Y, \quad x \wedge y \wedge e^{2\pi i t} \mapsto [x, t] \wedge y,$$

being by applying Proposition 4.67 (and identifications $X \wedge Y \wedge \mathbb{S}^1 = \mathbb{S}^1 \wedge X \wedge Y$, $\mathbb{S}^1 \wedge X = S^\bullet X$) a homotopy equivalence. □

Remark 4.69. We emphasise that the map $[x, t] \wedge y \mapsto [x \wedge y, t]$, generally speaking, is not a continuous map.

It is clear that the analogue of Proposition 4.67 is also valid for the space $X \wedge (Y \wedge Z)$, i.e. (assuming that the spaces X, Y, Z are well-pointed) the canonical map

$$X \wedge Y \wedge Z \to X \wedge (Y \wedge Z), \quad x \wedge y \wedge z \mapsto x \wedge (y \wedge z),$$

is a homotopy equivalence. Hence,

Proposition 4.70. *for well-pointed spaces X, Y and Z, the spaces $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ are homotopically equivalent*

(although there is no canonical homotopy equivalence between them, generally speaking).

Remark 4.71. It immediately follows from the results obtained that the associativity property of smash product of homotopy classes are valid for any well-pointed spaces. The same is true for the identities of (4.66).

4.E The invariant cat X

Let us now proceed to the study of the algebraic structure of groups $[X, K]^\bullet$.

Let X be an arbitrary pointed space with a base point x_0.

Definition 4.72. We say that $\operatorname{cat} X < n$ if

$$X = A_1 \cup \cdots \cup A_n,$$

where $A_1, \ldots, A_n$ are sets such that for any $k = 1, \ldots, n$ there is a homotopy $f_t^{(k)} : X \to X$, having the property that $f_0^k = \mathrm{id}$ and $f_1^{(k)}(A_k) = x_0$.

If there is no n for which $\operatorname{cat} X < n$, then we write $\operatorname{cat} X = \infty$. Otherwise, the smallest n for which $\operatorname{cat} X < n + 1$ is denoted by $\operatorname{cat} X$.

Remark 4.73. There are many different variants of Definition 4.72 in the literature. In the very first variant proposed by Lusternik and Schnirelman, it was only required that for any $k = 1, \ldots, n$ the embedding $A_k \to X$ was homotopic to the constant map. However, in "reasonable" situations, all these options are equivalent. For example, for a connected space X, the definition of Lusternik and Schnirelman will become Definition 4.72 if it is additionally required that all pairs (X, A_k) are closed cofibrations.

Remark 4.74. In addition, from the time with Lusternik and Schnirelman to the most recent time, the symbol $\operatorname{cat} X$ denoted a number one more than ours.

It is clear that

Proposition 4.75. *if* $\operatorname{cat} X < \infty$*, then the space* X *is connected, and* $\operatorname{cat} X = 0$ *if and only if the space* X *is contractible.*

Let X_k^n be a subspace of the space $X^n = \underbrace{X \times \cdots \times X}_{n}$, consisting of points with at least $n-k$ coordinates equal to x_0. For example, $X_0^n = \{(x_0, \dots, x_0)\}$, $X_1^n = \underbrace{X \vee \cdots \vee X}$ and $X_n^n = X^n$.

The subspace X_{n-1}^n will be of particular importance for us. It is clear that

$$(X^n, X_{n-1}^n) = (X, x_0)^n,$$

so in particular (see Corollary 1.21 of Proposition 1.20 Lecture 1),

Proposition 4.76. *if the point* x_0 *is nondegenerate, then the pair* (X^n, X_{n-1}^n) *is a cofibration.*

It is also useful to keep in mind that

$$X_{n-1}^n = \operatorname{proj}_1^{-1}(x_0) \cup \cdots \cup \operatorname{proj}_n^{-1}(x_0),$$

where, as always, $\operatorname{proj}_k : X^n \to X$ is the projection on the k-th factor.

The map $f : X \to Y$ is said to be *contractible to the subspace* $B \subset Y$ if a homotopy $f_t : X \to Y$, exists such that $f_0 = f$ and $f_1(X) \subset B$.

Lemma 4.77. *The inequality* $\operatorname{cat} X < n$ *holds if and only if the diagonal map*

$$\Delta_X^{(n)} = \Delta : X \to X^n, \quad x \mapsto (x, \dots, x), \tag{4.78}$$

is contractible on X_{n-1}^n.

Proof. Let $\operatorname{cat} X < n$, and let $f_t^{(k)} : X \to X$, $k = 1, \dots, n$, are homotopies such that $f_0^{(k)} = \mathrm{id}$ and $f_1^{(k)}(A_k) = x_0$, where $A_1 \cup \cdots \cup A_n = X$. Then the formula

$$f_t(x) = (f_t^{1)}(x), \dots, f_t^{(n)}(x)), \quad x \in X,$$

will obviously define a homotopy $f_t : X \to X$, such that $f_0 = \Delta$ and $f_1(X) \subset X_{n-1}^n$.

Conversely, let there be a homotopy $f_t : X \to X^n$ such that $f_0 = \Delta$ and $f_1(X) \subset X_{n-1}^n$. Then for the homotopy $f_t^{(k)} = \operatorname{proj}_k \circ f_t : X \to X$, $k = 1, \dots, n$, the relations will take place $f_0^{(k)} = \mathrm{id}$ and $f_1^{(k)}(A^k) = x_0$, where $A^k = (f_1^{(k)})^{-1}(x_0)$. But since $X_{n-1}^n = \operatorname{proj}_1^{-1}(x_0) \cup \cdots \cup \operatorname{proj}_n^{-1}(x_0)$, $X = A_1 \cup \cdots \cup A_n$. Therefore, $\operatorname{cat} X < n$. □

Corollary 4.79. *The inequality* $\operatorname{cat} X < 2$ *holds if and only if the space* X *is an H-counitoid.*

Proof. By definition, a space X is an H-counitoid if there exists a map $m : X \to X \vee X \subset X \times X$ such that both maps $\mathrm{proj}_1 \circ m, \mathrm{proj}_2 \circ m : X \to X$ are homotopic to the identity map. The corresponding homotopies obviously constitute a homotopy from X to $X \times X$ connecting the map m, considered as a map $X \to X \times X$, with the diagonal map $\Delta : X \to X \times X$. This shows that the map m exists if and only if the map Δ is contractible on $X_1^2 = X \vee X$. □

In particular, $\operatorname{cat} S^\bullet X < 2$ for any space X (and $\operatorname{cat} S^\bullet X = 1$, if $\operatorname{cat} X > 0$). However, this also follows directly from the definition (since the space X is the union of two cones, and each cone is contractible).

In addition, we see that every H-counitoid is a connected space.

The calculation of $\operatorname{cat} X$ (for $\operatorname{cat} X \geq 2$) is, in general in short, quite a difficult task.

Corollary 4.80 (Bass' theorem). *For any spaces X and Y , there is an inequality*

$$\operatorname{cat}(X \times Y) \leq \operatorname{cat} X + \operatorname{cat} Y.$$

Proof. If $f_t : X \to X^n$ is a homotopy such that $f_0 = \Delta_X^{(n)}$ and $f_1(X) \subset X_{n-1}^n$, and $g_t : Y \to Y^m$ is a homotopy such that $f_0 = \Delta_Y^{(m)}$ and $f_1(Y) \subset Y_{m-1}^m$, then the formula

$$h_t(x, y) = (f_t(x), \Delta_X^m(x), g_t(y), \Delta_Y^n(y)), \quad x \in X, y \in Y, t \in I,$$

defines a homotopy from $X \times Y$ to $X^n \times X^m \times Y^m \times Y^n = (X \times Y)^{n+m}$ such that

$$h_0 = \Delta_{X \times Y}^{(n+m)},$$
$$h_1(X \times Y) \subset X_{n-1}^n \times X^m \times Y_{m-1}^m \times Y^n \subset (X \times Y)_{n+m-1}^{n+m}$$

Therefore, if $\operatorname{cat} X < n$ and $\operatorname{cat} Y < m$, then $\operatorname{cat}(X \times Y) < n + m$. □

4.F The nilpotence of the group $[X, K]^\bullet$

Recall that the *commutator* of elements x, y of the group G is the element

$$[x, y] = xyx^{-1}y^{-1}.$$

In a more general way, you can define a commutator (more precisely, a *right-hand commutator*) $[x_1, \ldots, x_n]$ of elements $x_1, \ldots, x_n$ of the group G by the inductive formula

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

For $n = 1$, it is convenient to assume that $[x_1] = x_1$ for any element $x_1 \in G$.

If the commutator of any n elements of the group G is equal to e, then we write $\operatorname{nil} G < n$. A group G is called *nilpotent* if there exists an n such that $\operatorname{nil} G < n$. The smallest n for which $\operatorname{nil} G < n + 1$ will be denoted by the symbol $\operatorname{nil} G$. If the group G is not nilpotent, then we write $\operatorname{nil} G = \infty$.

Remark 4.81. There are many equivalent definitions of the number $\operatorname{nil} G$. We have chosen the most convenient one for us.

It is clear that $\operatorname{nil} G = 0$ if and only if $G = \{e\}$, and $\operatorname{nil} G = 1$ if and only if the group G is Abelian.

Instead of commutators, it is convenient to consider the corresponding commutator maps

$$\varphi_n : G^n \to G,$$

defined by formulae

$$\varphi_1 = \operatorname{id}, \quad \varphi_2 = \varphi : (x, y) \mapsto xyx^{-1}y^{-1}, \tag{4.82}$$

$$\varphi_n = \varphi \circ (\operatorname{id} \times \varphi_{n-1}), \quad n \geq 2. \tag{4.83}$$

It is clear that $\operatorname{nil} G < n$ if and only if $\varphi_n = \text{const}$.

Obviously, $[e, x] = [x, e] = x$ for each element $x \in G$, i.e. $\varphi_2|_{G \vee G} = \text{const}$. By trivial induction it follows that $\varphi_n|_{G^n_{n-1}} = \text{const}$ for any $n \geq 1$.

All this, of course, is transferred to any H-groups K, and the commutator maps $\varphi_n : K^n \to K$ are determined by the same formulas (4.82) (only the formula for φ needs to be written with brackets due to the lack of associativity: $f : (x, y) \mapsto ((xy)x^{-1})y^{-1})$. At the same time,

Proposition 4.84. *if the unit of the H-group K is non-degenerate, then*

$$\varphi_n|_{K^n_{n-1}} \sim \text{const} \quad \textit{for any} \quad n \geq 1.$$

Proof. Indeed, for $n = 1$ and for $n = 2$, this is obvious (and true without assuming the non-degeneracy of unity). Let's make an inductive transition from n to $n+1$, assuming for simplification of formulae $\varphi_n|_{K^n_{n-1}} = \varphi'_n$.

Since $(K^n, /K^n_{n-1}) = (K, e)^n$, and the pair (K, e) is by convention a cofibration, then (K^n, K^n_{n-1}) will also be a cofibration. Therefore, if $\varphi'_n \sim \text{const}$, then there is a homotopy $f_t : K^n \to K$, such that $f_0 = \varphi_n$ and $f_1(K^n_{n-1}) = e$. Therefore, the map $\varphi_{n+1} = \varphi \circ (\operatorname{id} \times \varphi_n) = \varphi \circ (\operatorname{id} \times f_0)$ will be homotopic to the map $\varphi \circ (\operatorname{id} \times f_1)$, and, therefore, the map φ'_{n+1} is homotopic the map $\varphi \circ (\operatorname{id} \times f_1)|_{K^{n+1}_n}$. But since $K^{n+1}_n = (K \times K^n_{n-1}) \cup (e \times K^n)$, then

$$(\operatorname{id} \times f_1)(K^{n+1}_n) \subset (K \times e) \cup (e \times K) = K \vee K,$$

and therefore

$$\varphi \circ (\operatorname{id} \times f_1)|_{K^{n+1}_n} = (\varphi|_{K \vee K}) \circ ((\operatorname{id} \times f_1)|_{K^{n+1}_n}).$$

Hence, $\varphi'_n \sim \text{const}$, for $\varphi_{K \vee K} \sim \text{const}$. □

Proposition 4.85. *For any connected H-group K with a non-degenerate unit e and any pointed space X, there is an inequality*

$$\operatorname{nil}[X, L] \leq \operatorname{cat} X. \tag{4.86}$$

In particular, if $\operatorname{cat} X < \infty$, then the group $[X, K]$ is nilpotent.

Proof. For arbitrary maps $g_k : X \to K$, $k = 1, \ldots, n$, the commutator of their homotopy classes are set, obviously, by the map

$$X \xrightarrow{\Delta} X^n \xrightarrow{g_1 \times \cdots \times g_n} K^n \xrightarrow{\varphi_n} K. \tag{4.87}$$

In view of the fact that the H-group K is connected, according to the general results of Lecture 4, we can consider the maps g_k pointed without loss of generality. But then the map is $g_1 \times \cdots \times g_K$ will translate X^n_{n-1} into K^n_{n-1} and, therefore, there will be a commutative diagramme of the form

$$\begin{array}{ccc} X^n & \xrightarrow{g_1 \times \cdots \times g_n} & K^n \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_K} \\ X^n_{n-1} & \xrightarrow[g]{} & K^n_{n-1} \end{array}$$

the vertical arrows of which are inclusions. Therefore, if $\operatorname{cat} X < n$, i.e. if the map Δ is homotopic to a map of the form $i_X \circ f$, where $f : X \to X^n_{n-1}$ and then the map (4.87) will be homotopic to the map

$$\varphi_n \circ (g_1 \times \cdots \times g_n) \circ i_X \circ f = \varphi_n \circ i_K \circ g \circ f = \varphi_n' \circ g \circ f$$

and, therefore (since $\varphi_n' \sim \mathrm{const}$), will be homotopic to the constant map, i.e. its homotopy class, will be the unit of the group $[X, K]$. Thus, if $\operatorname{cat} X < n$, then $\operatorname{nil}[X, K] < n$, which is equivalent to inequality (4.86). □

Remark 4.88. The inequality (4.86) is meaningful only when $\operatorname{cat} X < \infty$. Therefore, $[X, K]$ can be replaced in it by $[X, K]^\bullet$ and in this form it will also be true for non-closed H-groups K.

Corollary 4.89. *For any H-group K and any H-counitoid L, the group $[L, K]^\bullet$ is Abelian.*

4.G The abelicity of the group $[L, K]^\bullet$

The latter result can be generalised and simultaneously dualised.

Let L be an arbitrary H-counitoid and K be an arbitrary H-space (H-unitoid). Then, in the set $[L, K]^\bullet$, two structures of a unitoid will be defined - one arising from the fact that L is an H-counitoid, and the second - from the fact that K is an H-unitoid.

Proposition 4.90. *For any H-counitoid L and any H-unitoid K, these two unitoide structures on the set $[L, K]$ coincide.*

This unitoid is an abelian (commutative) monoid.

In particular, if K is an H-cogroup or L is an H-group, then $[L, K]^\bullet$ will be an Abelian group.

According to the general definitions of Lecture 3, the concept of a unitoid makes sense over an arbitrary fm-closed category $\mathscr{C}$. In particular, since the category of unitoids is an fm-closed category (the product of $G_1 \times G_2$ of unitoids G_1 and G_2 is their direct product as sets with a component-wise multiplication operation: $(\alpha_1, \alpha_2)(\beta_1, \beta_2) = (\alpha_1\beta_1, \alpha_2\beta_2)$), we can talk about the unitoids of the unitoid category. By definition, a unitoid G (the operation in which we will now write additively and, accordingly, call its unit *nul*) is a *unitoid of the unitoid category* if a homomorphism of unitoids is given

$$G \times G \to G, \tag{4.91}$$

i.e. a multiplication $(\alpha, \beta) \mapsto \alpha\beta$ such that

$$\alpha_1\beta_1 + \alpha_2\beta_2 = (\alpha_1 + \alpha_2)(\beta_1 + \beta_2), \tag{4.92}$$

for any elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G$, and the element $0 \in G$ is the same with respect to this multiplication, i.e. $0\alpha = \alpha 0 = \alpha$ for any element $\alpha \in G$.

Lemma 4.93. *Every unitoid G of the category of unitoids is an Abelian monoid. For any elements $\alpha, \beta \in G$ has the equality*

$$\alpha + \beta = \alpha\beta.$$

Proof. We have

$$\alpha + \beta = \alpha 0 + 0\beta = (\alpha + 0)(0 + \beta) = \alpha\beta.$$

Similarly,

$$\alpha + \beta = 0\alpha + \beta 0 = (0 + \beta)(\alpha + 0) = \beta\alpha.$$

Finally,

$$(\alpha\beta)\gamma = \alpha\beta + 0\gamma = (\alpha + 0) + (\beta + \gamma) = \alpha(\beta + \gamma) = \alpha(\beta\gamma).$$

□

Proof. (of 4.90) According to Lemma 4.93, it is sufficient to prove that the unitoid $G = [L, K]^\bullet$ (with the operation induced by multiplication $m_1 : L \to L \vee L$) is a unitoid of the unitoid category (with operation (4.91) induced by multiplication $m : K \times K \to K$), i.e. that for any elements $\alpha_1, \beta_1, \alpha_2, \beta_2 \in G$ the equality (4.92) holds (identities $0\alpha = \alpha 0 = \alpha$ will take place by definition). But for any two pointed maps $f, g : L \to K$, the map $f \times g : L \to K \times K$ is defined by the formula

$$(f \times g)(x) = (f(x), g(x)), \quad x \in L,$$

and the map $f \vee g : L \vee L \to K$ - by the formula

$$(f \vee g)(z) = \begin{cases} f(x), & \text{if} \quad z = x_I, \\ g(x), & \text{if} \quad z = x_{II}, \end{cases} \quad z \in L \vee L,$$

(we identify $L \vee L$ with $(L \times e) \cup (e \times L)$ and denote (x, e) by x_I and (e, x) by x_{II}). Therefore, for any four pointed maps $f_1, g_1, f_2, g_2 : L \to K$, the map $h = (f_1 \times g_2) \vee (f_2 \times g_2) : L \vee L \to K \times K$ is given by the formula

$$h(z) = \begin{cases} (f_1(x), g_1(x)), & \text{if} \quad z = x_I, \\ (f_2(x), g_2(x)), & \text{if} \quad z = x_{II}, \end{cases}$$

and the map $h' = (f_1 \vee f_2) \times (g_1 \vee g_2)$ by the formula

$$h'(z) = (f_1 \vee f_2) \times (g_1 \vee g_2)(z) = \begin{cases} (f_1(x), g_1(x)), & \text{if} \quad z = x_I, \\ (f_2(x), g_2(x)), & \text{if} \quad z = x_{II}, \end{cases}$$

We see, therefore, that $h = h'$.

On the other hand, by definition

$$[f]^\bullet + g^\bullet = [(f \vee g) \circ m_1]^\bullet,$$
$$[f]^\bullet \cdot g^\bullet = [m \circ (f \vee g)]^\bullet,$$

and therefore, if $\alpha_1 = [f_1]^\bullet$, $\beta_1 = [g_1]^\bullet$, $\alpha_2 = [f_2]^\bullet$, $\beta_2 = [g_2]^\bullet$, then

$$\alpha_1\beta_1 + \alpha_2\beta_2 = [(m \circ h \circ m_1],$$
$$(\alpha_1 + \beta_1)(\alpha_2\beta_2) = [(m \circ h' \circ m_1].$$

Since $h = h'$, this proves that $\alpha_1\beta_1 + \alpha_2\beta_2 = (\alpha_1 + \beta_1)(\alpha_2\beta_2)$. □

Corollary 4.94. *For any two pointed spaces X and Y, the group $[S^\bullet X, \Omega Y]'^\bullet$ is Abelian.*

4.H Groups $[S^nX, \Omega^mY]^\bullet$

Since the functor $S^\bullet$ acts from the category of $\mathcal{Top}^\bullet$ to the same category $\mathcal{Top}^\bullet$, it can be iterated indefinitely. We will put

$$S^0X = X, \quad S^{n+1}X = S^\bullet(S^nX) \quad \text{for} \quad n \geq 0,$$

(so $S^1X = S^\bullet X$).

Iterations of the loop functor are defined similarly

$$\Omega^0X = X, \quad \Omega^{n+1}X = \Omega(\Omega^nX) \quad \text{for} \quad n \geq 0.$$

In this case, due to the adjointness of the functors $S^\bullet$ and Ω, for any pointed spaces X, Y and any numbers $n \geq 0$, $m \geq 0$, there are equalities

$$\begin{aligned} [X, \Omega^{n+m}Y]^\bullet &= [S^1X, \Omega^{n+m-1}Y]^\bullet = \cdots \\ &= [S^nX, \Omega^mY]^\bullet = \cdots = [S^{n+m}X, Y]^\bullet. \end{aligned}$$

Corollary 4.95. *For $n + m \geq 1$, the set $[S^nX, \Omega^mY]^\bullet$ has a group structure natural in X and Y. For $n + m \geq 2$, this group is Abelian.*

Lecture 5

5.1 Homotopy groups

The simplest concrete H-cogroup is certainly the suspension $S^\bullet\mathbb{S}^{n-1}$ over the unit $(n-1)$-dimensional $(n > 0)$ sphere $\mathbb{S}^{n-1}$ (in which, say, a base point is $\boldsymbol{s}_0 = (-1, 0, \dots, 0)$). Since this suspension is homeomorphic to the n-dimensional sphere $\mathbb{S}^n$ (see Remark 3.31 of Lecture 3), we get, therefore, that

Proposition 5.1. *for any pointed space* (X, x_0) *the set*

$$[(\mathbb{S}^n, \boldsymbol{s}_0), (X, x_0)] = [\mathbb{S}^n, X]^\bullet \tag{5.2}$$

is a group.

Definition 5.3. The group (5.2) is denoted by the symbol $\pi_n(X, x_0)$ (or simply $\pi_n X$) and is called the *n-dimensional* (or *n-th*) *homotopy group* of the space X.

Of course, the multiplication in $\mathbb{S}^n$ depends on the choice of the homeomorphism $S^\bullet\mathbb{S}^{n-1} \to \mathbb{S}^n$. Therefore, in order to fix this multiplication (and hence the structure of the group in $\pi_n X$), it is necessary to fix this homeomorphism once and for all (at least up to homotopy).

We will choose for the homeomorphism $S^\bullet\mathbb{S}^{n-1} \to \mathbb{S}^n$ the homeomorphism d defined by formula (3.27) of Lecture 3. In the future we will always assume that for $n > 0$ the sphere $\mathbb{S}^n$ is identified with the suspension $S^\bullet\mathbb{S}^{n-1} \to \mathbb{S}^n$ by means of this homeomorphism and in accordance with this point $d[\boldsymbol{x}, t] \in \mathbb{S}^n$, $\boldsymbol{x} \in \mathbf{S}^{n-1}$, $t \in I$, we will simply denote $[\boldsymbol{x}, t]$.

Note that $[\boldsymbol{s}_0, t] = \boldsymbol{s}_0$ for any $t \in I$.

For $n = 0$, we get not a group, but only a pointed set $[\mathbb{S}^0, X]^\bullet$ (the base point of which is the homotopy class of the constant map const : $\mathbb{S}^0 \to X$.) It is clear that $[\mathbb{S}^0, X]^\bullet = [\mathrm{pt}, X]$, i.e. that the set $[\mathbb{S}^0, X]^\bullet$ is the set of $\pi_0 X$ components of the space X introduced in Lecture 3 (which, by the way, explains the designation $\pi_0 X$ for this set).

Although the set $\pi_0 X$, generally speaking, is not a group, we will still allow ourselves to speak without reservations about homotopy groups $\pi_n X$ for all $n \geq 0$.

The operation in the group $\pi_n X$, $n \geq 1$, we will denote the + sign and call it *addition* (with the possible exception of the group $\pi_1 X$; see below). In this regard, the fact that the group consists of only one element, we will write with the formula $\pi_n X = 0$ (even if $n = 0$ or $n = 1$).

If elements α and β of the group $\pi_n X$ are represented by the maps $a : (\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$ and $b : (\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$, then their sum $\alpha + \beta$ will be represented by the map $c = (a \vee b) \circ m : (\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$, where m is the co-multiplication $\mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$, defined by the formula (3.28) of lecture 3 (for $X = \mathbb{S}^{n-1}$). Explicitly , the map with is given by the formula

$$c[\boldsymbol{x}, t] = \begin{cases} a[\boldsymbol{x}, 2t], & \text{if } 0 \leq t \leq 1/2, \\ b[\boldsymbol{x}, 2t-1], & \text{if } 1/2 \leq t \leq 1, \end{cases} \tag{5.4}$$

where, in accordance with the above, $[\boldsymbol{x}, t]$ denotes $d[\boldsymbol{x}, t]$, $x \in \mathbb{S}^{n-1}$, $t \in I$.

According to the general theory of functors of the form $[K, X]^\bullet$ every pointed map $f : X \to y$ defines by the formula

$$f_*[a]^\bullet = [f \circ a], \quad a : (\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0),$$

the homomorphism $f_* : \pi_n X \to \pi_n Y$ is also denoted by the symbol π_n (or f_n), and the correspondences $X \mapsto \pi_n X$, $f \mapsto f_*$ represent a functor from the category of $\mathcal{top}^\bullet$ to the category of groups $\mathcal{Grp}$:

$$\mathcal{Top}^\bullet \to \mathcal{Grp}$$

(for $n = 0$ - in the category of pointed sets $\mathcal{Ens}^\bullet$). We will call this functor the *functor of n-dimensional homotopy groups.*

In particular, the embedding $X_0 \to X$ of the component X_0 into the point x_0 induces the homomorphism $\pi_n X_0 \to \pi_n X$. It is clear that when $n > 0$ this homomorphism is an isomorphism.

Thus, in the theory of homotopy groups, without loss of generality, we can (for $n > 0$) limit ourselves only to the connected spaces X.

5.2 An alternative definition of homotopy groups

Of course, when constructing the group $\pi_n X$, the sphere $\mathbb{S}^n$ can be replaced by any space that is homeomorphic to it (i.e., in generally accepted terminology, any topological sphere S^n). It is only necessary to fix a certain homeomorphism $S^n \to \mathbb{S}^n$ once and for all.

Remark 5.5. Naturally, it is sufficient to specify the homeomorphisms $S^n \to \mathbb{S}^n$ only up to homotopy. In Lecture 12[1] we will show that there are only two distinct homotopy classes of such homeomorphisms that can be identified with the orientations of the sphere S^n. Hence, if the pointed topological sphere $\mathbb{S}^n$ is oriented, then each pointed map $S^n \to X$ uniquely defines some element of the

[1]The transcriber guesses that Postnikov refers to Lecture 2 of "Cellular Homotopy".

group $\pi_n X$. Therefore, the elements of the group $\pi_n X$ can be defined - which was done fifty years ago - as homotopy classes of pointed maps to the space X of all possible oriented spheres of dimension n. However, this conceptually more complex definition does not, as experience has shown, have any real advantages and currently no one uses it.

Let $\mathbb{E}^n$ be a unit n-dimensional ball consisting of $\boldsymbol{x} \in \mathbb{R}^n$, for which $|x| \leq 1$. The boundary of this ball (for $n > 0$) is an $(n-1)$ dimensional sphere $\mathbb{S}^{n-1}$, and the coset space $\mathbb{E}^n/\mathbb{S}^{n-1}$ is homeomorphic to the sphere $\mathbb{S}^n$. Therefore, by choosing a certain homeomorphism $\mathbb{E}^n/\mathbb{S}^{n-1} \to \mathbb{S}^n$, we can consider the homotopy classes of maps $\mathbb{E}^n/\mathbb{S}^{n-1} \to X$ as elements of the group $\pi_n X$. (As always, it means that a point is based in the coset space $\mathbb{E}^n/\mathbb{S}^{n-1}$, which is the image of the sphere $\mathbb{S}^{n-1}$ with the canonical map of the identification $\mathbb{E}^n \to \mathbb{E}^n/\mathbb{S}^{n-1}$). But, as it is easy to see, composing with the factorisation map $\mathbb{E}^n \to \mathbb{E}^n/\mathbb{S}^{n-1}$ establishes a bijective correspondence between the pointed maps $\mathbb{E}^n/\mathbb{S}^{n-1} \to X$ and the maps $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, x_0)$. Since this correspondence obviously translates pointed homotopies into homotopies of pairs (i.e., in this case homotopies with respect to $\mathbb{S}^{n-1}$), we obtain, therefore, that the homotopy classes of maps $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, x_0)$ can be considered as elements of the group $\pi_n X = \pi_n(X, x_0)$:

$$\pi_n(X, x_0) = [(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, x_0)].$$

This identification of homotopy classes of maps $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, x_0)$ with elements of the group $\pi_n X$ depends, of course, on the choice of the homeomorphism $\mathbb{E}^n/\mathbb{S}^{n-1} \to \mathbb{S}^n$, or, equivalently, the relative homeomorphism

$$\chi^{(n)} : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (\mathbb{S}^n, \boldsymbol{s}_0), \tag{5.6}$$

and at the map level is set by the correspondence $a \mapsto a \circ \chi^{(n)}$, where $a : (\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$. We will agree once and for all to take for homeomorphism (5.6) a relative homeomorphism defined (by applying the identification $\mathbb{R}^{n+1}) = \mathbb{R} \times \mathbb{R}^n$) by the formula

$$\chi^{(n)}(\boldsymbol{x}) = \left(\cos \pi|\boldsymbol{x}|, \frac{\sin \pi|\boldsymbol{x}|}{|\boldsymbol{x}|}\right), \quad |\boldsymbol{x}| \in \mathbb{E}^n \subset \mathbb{R}^n.$$

Instead of a unit ball, one can, of course, take a *unit cube* I^n homeomorphic to it, consisting of points $t = (t_1, \dots, t_n)$ of the space $\mathbb{R}^n$, for which $0 \leq t_i \leq 1$ for any $i = l, \dots, n$. By denoting the boundary of this cube with the symbol $\dot{I}^n$, we obtain, in a way such that

Proposition 5.7. *the elements of the group $\pi_n X$ can be interpreted as homotopy classes* rel $\dot{I}^n$ *of maps of $a : (I^n, \dot{I}^n) \to (X, x_0)$,*

i.e. functions

$$a : (t_1, \dots, t_n) \mapsto a(t_1, \dots, t_n) \in X$$

variables $t_1, \dots, t_n \in I$ such that $a(t_1, \dots, t_n) = x_0$ if at least one of the arguments $t_1, \dots, t_n$ is zero or one.

Here, of course, it is also necessary to fix a certain relative homeomorphism

$$\chi = \chi_n : (I^n, \dot{I}^n) \to (\mathbb{S}^n, \boldsymbol{s}_0). \tag{5.8}$$

For $n = 1$, we define this homeomorphism by the formula

$$\chi_1(t) = \begin{cases} (1-4t, 2\sqrt{2t(1-2t)}), & \text{if} \quad 0 \le t \le 1/2, \\ (4t-3, -2\sqrt{2(1-t)(2t-1)}), & \text{if} \quad 1/2 \le t \le 1 \end{cases}$$

(cf. with formula (3.33) of Lectures 3 for$x = -1$ and $\boldsymbol{x}$ absent), and for $n > 1$, identifying the cube I^n with the product $I \times I^n$ by the formula

$$\chi_n(t, \boldsymbol{t}) = [\chi_{n-1}(t), \boldsymbol{t}], \quad t \in I, \quad \boldsymbol{t} \in I^{n-1}.$$

The addition operation (5.4) transferred using relative homeomorphism (5.8) to the maps $a : (I^n, \dot{I}^n) \to (X, x_0)$, will, as it is easy to see, be given by the formula

$$(a+b)(t, \boldsymbol{t}) = \begin{cases} a(2t, \boldsymbol{t}), & \text{if} \quad 0 \le t \le 1/2, \\ b(2t-1, \boldsymbol{t}), & \text{if} \quad 1/2 \le t \le 1 \end{cases} \tag{5.9}$$

(for $n = 1$, there is no argument $\boldsymbol{t}$). Thus, for $n \ge 1$ we come to an alternative definition of the group$\pi_n X$, in which any recollection of the spheres is completely eliminated. In this definition, the elements of the group $\pi_n X$ are the homotopy classes rel $\dot{I}^n$ of maps $a : (I^n, \dot{I}^n) \to (X, x_0)$, and addition is induced by the operation of addition of maps given by the formula (5.9).

This definition of group $\pi_n X$ is the most convenient in practice, and therefore it is usually considered the main one. Of course, at the same time it is necessary to re-check the axioms of the group for $\pi_n X$.

In particular, for $n = 1$, comparing the definitions, we get that

Proposition 5.10. *the group $\pi_1 X$ coincides with the fundamental group introduced in Lecture 4*

(which was not in vain indicated there by the symbol $\pi_1 X$). The only difference is that in Lecture 4, the operation in the group $\pi_1 X$ was called multiplication. Thus, for the group $\pi_1 X$ we have two competing notation systems - additive and multiplicative. We will consider them completely equal and in each case we will use the one that is more convenient. (In some cases, we will even allow ourselves to use both notations in the same formula!)

Remark 5.11. Similarly, the sum of any number of maps can be determined $(I^n, \dot{I}^n) \to (X, x_0)$. For example, the sum $a + b + c$ of three maps a, b, c : $(I^n, \dot{I}^n) \to (X, x_0)$ is defined by the formula

$$(a+b+c)(t, \boldsymbol{t}) = \begin{cases} a(3t, \boldsymbol{t}), & \text{if} \quad 0 \le t \le 1/3, \\ b(3t-1, \boldsymbol{t}), & \text{if} \quad 1/3 \le t \le 2/3, \\ c(3t-2, \boldsymbol{t}), & \text{if} \quad 2/3 \le t \le 1. \end{cases}$$

Of course, the map $a + b + c$ is different from the map $(a + b) + c$ (and from the map $a + (b + c)$), and not homotopic with respect to $\dot{I}^n$.

For each map $a : (I^n, \dot{I}^n) \to (X, x_0)$ we will use the symbol $-a$ to denote the map $(I^n, \dot{I}^n) \to (X, x_0)$ defined by the formula

$$(-a)(t, \boldsymbol{t}) = a(1-t, \boldsymbol{t}), \quad (t, \boldsymbol{t}) \in I \times I^{n-1} = I^{n+1}.$$

It is clear that in the group $\pi_n X$, the homotopy classes of maps a and $-a$ define mutually opposite elements.

In appropriate situations, we will naturally reduce the entered designations. For example, instead of $a + (-b) + c$, we will simply write $a - b + c$. (Note that the map $a - b + c$ is thus determined by the formula

$$(a-b+c)(t, \boldsymbol{t}) = \begin{cases} a(3t, \boldsymbol{t}), & \text{if} \quad 0 \leq t \leq 0, \\ b(2-3t, \boldsymbol{t}), & \text{if} \quad 1/3 \leq t \leq 2/3, \\ c(3t-2, \boldsymbol{t}), & \text{if} \quad 2/3 \leq t \leq 1, \end{cases}$$

where $(t, \boldsymbol{t}) \in I \times I^{n-1} = I^n$.)

5.3 Inductive definition of homotopy groups

The last definition of the groups $\pi_n X$ suggests considering the space $\Omega^n X$ of all maps $(I^n, \dot{I}^n) \to (X, x_0)$ (equipped with the subspace topology of the space X^{I^n}). Regarding the operation (5.9) this space is an H-space (even an H-group). The homotopy unit (which is better now called *null homotopy*) of this space is the constant map $I^n \to X$, $\boldsymbol{t} \mapsto x_0$. For $n = 1$, it coincides with the space ΩX.

By applying the exponential law, homotopies with respect to $\dot{I}^n$ are nothing but paths of the space $\Omega^n X$. Consequently, we can identify the group $\pi_n X$ with the group

$$\pi_n X = \pi_0 \Omega^n X$$

(cf. with a similar formula for the group $\pi_1 X$ in Lecture 4).

For $n > 1$, each map $a : (t, \boldsymbol{t}) \mapsto a(t, \boldsymbol{t})$ is identified with the path $t \mapsto a^{\#}(t)$ of the space $\Omega^{n-1} X$, where $a^{\#}(t)$ is a point $\boldsymbol{t} \mapsto a(t, \boldsymbol{t})$ of this space. Thus, for $n > 1$, the equality takes place

$$\Omega^n X = \Omega(\Omega^{n-1} X) \tag{5.12}$$

(if we put $\Omega^0 X = X$, then for $n = 1$ the formula (5.12) will turn into the equation $\Omega^1 X = \Omega X$ already known to us).

Therefore

$$\pi_n X = \pi_0 \Omega^n X = \pi_0 \Omega(\Omega^{n-1} X) = \pi_1 \Omega^{n-1} X$$

and, in general, as the obvious induction shows,

$$\pi_n X = \pi_k \Omega^{n-k} X \tag{5.13}$$

for any $k = 0, 1, \ldots, n$.

The formula (5.13) for $k = 1$ can, as was originally done by Hurevicz, be used as the basis for another, inductive, definition of the groups $\pi_n X$. It has the advantage that it does not require verification of the axioms of the group (provided that for the group $\pi_1 X$ these axioms have already been verified).

Remark 5.14. A number of special names can be found in the literature for maps $(\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$ and $(I^n, \dot{I}^n) \to (X, x_0)$; for example, Fuchs and Rokhlin (see [10]) adhere to the ancient tradition of calling maps $(\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$ *spheroids* of the space X. By analogy of the map $(I^n, \dot{I}^n) \to (X, x_0)$ could be called *cuboids* of the space X, and bearing in mind the formula (5.12), *n-dimensional* (or *n-fold*) loops of this space. We will not give preference to any of these terms.

5.4 The action of the group $\pi_1 X$ on the groups $\pi_n X$

According to Remark 4.29 of Lecture 4, *for $n > 0$, the group $\pi_1 X$ acts on the group $\pi_n X$*, i.e. a map is defined

$$\pi_1 X \times \pi_n X \to \pi_n X, \quad (\xi, \alpha) \mapsto \xi\alpha, \quad \xi \in \pi_1 X, \quad \alpha \in \pi_n X, \tag{5.15}$$

such that:

a) for each element $\xi \in \pi_1 X$, the map $R_\xi : \alpha \mapsto \xi\alpha$, $\alpha \in \pi_n X$, is an automorphism of the group $\pi_n X$, i.e. it is bijective and for any elements $\alpha, \beta \in \pi_n X$ there is equality
$$\xi(\alpha + \beta) = \xi\alpha + \xi\beta;$$

b) the map $R : \xi \to R_\xi$ is a homomorphism from the group $\pi_1 X$ into the group $\operatorname{Aut} \pi_1 X$ of automorphisms of the group$\pi_1 X$, i.e.
$$(\xi\eta)\alpha = \xi(\eta\alpha)$$
for any elements $\xi, \eta \in \pi_1 X$, $\alpha \in \pi_n X$.

At the same time, comparing the definitions, we immediately find that the maps $a : (I^n, \dot{I}^n) \to (X, x_0)$ and $b : (I^n, \dot{I}^n) \to (X, x_0)$ then and only then set the elements α and $\xi\alpha$ of the group $\pi_n X$, when there is a map $F : I^{n+1} \to X$ from the cube $I^{n+1} = I^n \times I$ into the space X such that

$$F(\boldsymbol{t}, 0) = a(\boldsymbol{t}), \quad F(\boldsymbol{t}, 1) = b(\boldsymbol{t}) \tag{5.16}$$

for any point $\boldsymbol{t} \in I^n$ and
$$F(\boldsymbol{t}, t) = u(1 - t) \tag{5.17}$$
for any point $\boldsymbol{t} \in \dot{I}^n$ and for any $t \in I$, where $u : I \to X$ is the loop defining the element ξ of the group $\pi_1 X$. If desired, you can take this as a definition of action (5.15), but then statements a) and b) will need proof (which, however, is carried out quite automatically).

Sometimes, when it is necessary to emphasise the dependence of R on n, we will write $R_\xi^{(n)}$ instead of R_ξ.

It is obvious that

Proposition 5.18. *for any pointed map $f : X \to Y$ and any elements $\alpha \in \pi_n X$ and $\xi \in \pi_1 X$ there is an equality*

$$f_*(\xi\alpha) = f_*(\xi) f_*(\alpha)$$

where, when applied to $\xi\alpha$ and α, the symbol f_* means the homomorphism $\pi_n f : \pi_n X \to \pi_n Y$, and when applied to ξ, the homomorphism $\pi_1 f : \pi_1 X \to \pi_1 Y$ (the functoriality property of the action $R : \pi_1 X \to \operatorname{Aut} \pi_n X$). In the technical language of group theory with operators, this property means that

Proposition 5.19. *the homomorphism $f_* := \pi_n f : \pi_n X \to \pi_n Y$ is a $\pi_1 X$-homomorphism*

with respect to the action of $\pi_1 X \to \operatorname{Aut} \pi_n Y$ induced by the homomorphism $f_* = \pi_1 f : \pi_1 X \to \pi_1 Y$ (i.e. - in other terminology - *is a $\pi_1 f$-homomorphism*).

For $n = 1$, conditions (5.16) and (5.17) mean that with two possible movements on the sides of the square I^2 from the point $(0,0)$ to the point $(1,1)$ in one case we run through the path au^{-1}, and in the other -the path $u^{-1}b$. Therefore, by composing the map F with a piecewise linear map of the square

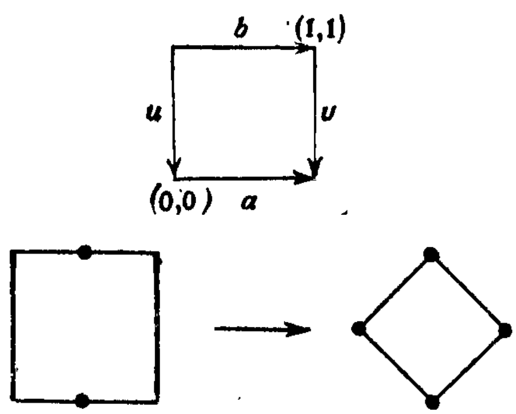


Figure 5.4.1:

I^2 onto itself, map the segments $0 \times I$ and $1 \times I$, respectively, to points $(0,0)$ and $(1,1)$, and the segments $I \times 0$ and $I \times 1$ by breaking them in the middle, into the polyline $0 \times I \cup I \times 1$ and $I \times 0 \cup 1 \times I$, respectively, we get a map of the square $I^2 = I \times I$ to the space X, which is a homotopy from I to X with respect to $I = \{0,1\}$, connecting the path au^{-1} with the path $u^{-1}b$. This means that by going to the multiplicative notation in the group $\pi_1 X$ and denoting the element $\xi\alpha$, in order to distinguish it from the product, with the symbol $\xi(\alpha)$, we will have equality $\alpha\xi^{-1} = \xi^{-1} \cdot \xi(\alpha)$. Thus it is proved that

$$\xi(\alpha) = \xi\alpha\xi^{-1} \quad \text{for any elements} \quad \alpha, \xi \in \pi_1 X, \tag{5.20}$$

i.e.

Proposition 5.21. *the automorphism $R_{\xi'} : \alpha \mapsto \xi(\alpha)$ is an inner automorphism of the group $\pi_1 X$ defined by the element ξ.*

5.5 Abelian spaces

Definition 5.22. A space X is called *homotopically simple in dimension n* if the group $\pi_1 X$ acts trivially on the group $\pi_n X$, i.e. if $\xi\alpha = \alpha$ for any elements $\xi \in \pi_1 X$, $\alpha \in \pi_n X$. A space that is homotopically simple in all dimensions is called *abelian*.

As was noted in Lecture 4, the fundamental group of an arbitrary H-space acts trivially on the group of homotopy classes of maps into this space from any pointed space. In particular, this means that

Proposition 5.23. *any H-space is abelian.*

However, this fact is easily proved directly, since for every H-space X with a real unit (which, as we know from Lecture 4, does not limit generality in essence) the formula

$$F(\boldsymbol{t}, t) = a(\boldsymbol{t})u(1-t), \quad \boldsymbol{t} \in I^n, \quad t \in I,$$

defines for any elements $a \in \Omega^n X$, $u \in \Omega X$ a map $F : I^{n+1} \to X$ satisfying conditions (5.16) and (5.17) with $b = a$.

Of course, the class of abelian spaces is wider than the class of H-spaces. For example, it is clear that any simply connected space is abelian. In its place, we will give examples of non-connected abelian spaces that are not H-spaces.

According to formula (5.20)

Proposition 5.24. *a space X is homotopically simple in dimension 1 if and only if the group $\pi_1 X$ is abelian.*

In particular,

Proposition 5.25. *for any abelian space (and, for example, for any H-space) X, the group $\pi_1 X$ is abelian.*

5.6 Abelicity of homotopy groups for $n \geq 2$

As for the groups $\pi_n X$ for $n > 1$, the following remarkable proposition holds.

Proposition 5.26. *For $n > 1$, the homotopy group $\pi_n X$ of an arbitrary pointed space X is abelian.*

Thus, for $n > 1$, we can assume that the functor π_n takes values in the category of abelian groups $\mathscr{A}\mathscr{b}\mathscr{G}\mathscr{r}\mathscr{p}$.

Since $\mathbb{S}^n = S^2\mathbb{S}^{n-2}$ for $n > 1$, Proposition 5.26 is a special case of Corollary 4.65 of Proposition 4.62 of Appendix to Lecture 4. We will prove again here this proposition (and we'll even give it two proofs).

Proof. (The first proof) By applying the formula (5.13) (for $k = 2$) is sufficient to prove Proposition 5.26 for the group $\pi_2 X$.

Let $a : (I^2, \dot{I}^2) \to (X, x_0)$, $b : (I^2, \dot{I}^2) \to (X, x_0)$, and let c be the map $(I^2, \dot{I}^2) \to (X, x_0)$, resulting from the map $a + b$by "adding on the second coordinate" of the permanent map. Schematically , the c map can be represented by the drawing 5.6.1

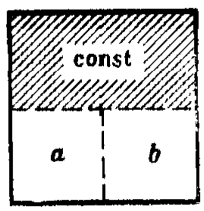


Figure 5.6.1:

and algebraically it is given by formulae

$$c(t,s) = \begin{cases} a(2t, 2s), & \text{if} \quad 0 \leq t, s \leq 1/2, \\ b(2t-1, 2s), & \text{if} \quad 0 \leq s \leq 1/2 \leq t \leq 1, \\ x_0, & \text{if} \quad 1/2 \leq s \leq 1. \end{cases}$$

Let us produce a deformation (homotopy with respect to $\dot{I}^2$) over this map, schematically represented by the Figure 5.6.2

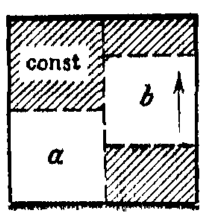


Figure 5.6.2:

and algebraically given by formulae

$$c_\tau(t,s) = \begin{cases} a(2t, 2s), & \text{if} \quad 0 \leq t, s \leq 1/2, \\ b(2t-1, 2s-\tau), & \text{if} \quad \frac{1}{2} \leq t \leq 1, \frac{\tau}{2} \leq s \leq \frac{1+\tau}{2}, 0 \leq \tau \leq 1, \\ x_0, & \text{in other cases.} \end{cases}$$

As a result, we will get a map schematically represented by the Figure 5.6.3

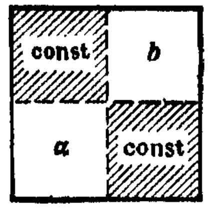


Figure 5.6.3:

Having then done the first deform on this map first as in the Figure 5.6.4

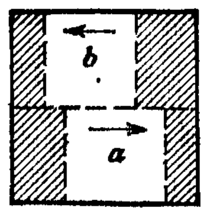


Figure 5.6.4:

and then the deformation as in the Figure 5.6.5

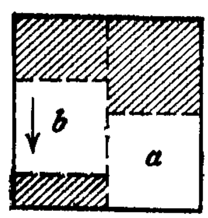


Figure 5.6.5:

we will get the map as in the Figure 5.6.6

To complete the proof, it remains to note that the maps (5.6.1) and (5.6.6) are homotopic with respect to $\dot{I}^2$, respectively, to the maps $a + b$ and $b + a$. □

Proof. (The second proof) According to the formula (5.13) (for $k = 1$)

$$\pi_n X = \pi_1 \Omega^{n-1} X,$$

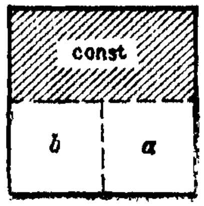


Figure 5.6.6:

and the group $\pi_1\Omega^{n-1}X$, being the fundamental group of the H space $\Omega^{n-1}X = \Omega(\Omega^{n-2}X)$, is abelian. □

An abelian additive group in which the multiplicative group Π acts is called a Π-*module*. Thus, we can say that

Proposition 5.27. *for $n \geq 2$, the group $\pi_n X$ is a $\pi_1 X$ module.*

5.7 Ensemble of homotopy groups

In the case when a given space X is not pointed, we have to consider the groups $\pi_n(X,x)$ for all points $x \in X$ at the same time, without giving any of them any advantage. At the same time, in according to Remark 4.28 of Lecture 4, any path $u \in X^I$ will determine some isomorphism $u_\#$ of the group $\pi_n(X,u(1))$ to the group $\pi_n(X,u(0))$, depending only on the homotopy class $\xi = [u]\,\mathrm{rel}\{0,1\}$ of the path u. In the case when $u(0) = u(1) = x_0$, i.e. when the path u is a loop, this is the automorphism already known to us $R_\xi^{(n)} : \alpha \mapsto \xi\alpha$, $\alpha \in \pi_n(X,x_0)$. For $u(0) \neq u(1)$ it is constructed in exactly the same way (using maps F satisfying conditions (5.16) and (5.17), and we will keep the same notation $R_\xi^{(n)}$ for it (or just R_ξ).

At the same time, according to Remark 4.29 of Lecture 4, the correspondences $x \mapsto \pi_n(X,x)$, $\xi \mapsto R_\xi$ will constitute an ensemble of groups over the space X.

Assuming a certain liberty, we will denote this ensemble with the symbol $\{\pi_n(X,x)\}$.

Definition 5.28. The ensemble $\{\pi_n(X,x)\}$ is called the *ensemble of n-dimensional homotopy groups* of a topological space X.

For $n > 1$, it is an ensemble of abelian groups.

Each morphism $\xi : y \to x$ of an arbitrary groupoid Π defines by the formula

$$\overline{R}_\xi(\eta) = \xi\eta\xi^{-1}, \quad \eta \in \Pi(y,y),$$

some isomorphism $\overline{R}_\xi : \Pi(y,y) \to \Pi(x,x)$ of the group $N(y,y)$ and the morphism $y \to y$ with the group $\Pi(x,x)$ and the morphism $x \to x$, and the correspondences $x \mapsto \Pi(x,x)$, $\xi : \xi \mapsto \overline{R}_\xi$ make up an ensemble of groups over Π. It turns out that,

Proposition 5.29. *in the case when the groupoid Π is the fundamental groupoid ΠX of the topological space X, this ensemble coincides with the ensemble $\{_p i_1(X,x)\}$ of fundamental groups, i.e. $\overline{R}_\xi = R^{(1)}_\xi$.*

Proof. Indeed, for $y = x (= x_0)$, this is exactly the statement expressed by formula (5.20), the proof of which is fully preserved for $y \neq x$. □

On the other hand, it is easy to see that

Proposition 5.30. *for any ensemble sets (or groups) R over a groupoid Π each isomorphism $R_\xi : R_y \to R_x$, $\xi : y \to x$, is an $\overline{R}_\xi$-isomorphism (with respect to the natural actions of the groups $\Pi(y,y)$ and $\Pi(x,x)$ on the sets R_y and R_x), i.e. for any elements $\alpha \in R_y$ and $\eta \in \Pi(y,y)$ there is an equality*

$$R_\xi(\eta\alpha) = \overline{R}_\xi(\eta) R_\xi(\alpha).$$

Proof. Indeed, since, by definition, $\eta\alpha = R_\eta \alpha$, the latter equality is equivalent to the formula

$$R_\xi \circ R_\eta = R_{\overline{R}_\xi(\eta)} \circ R_\xi.$$

On the other hand, $R_\xi \circ R_\eta = R_{\xi\eta}$, so

$$R_{\overline{R}_\xi(\eta)} \circ R_\xi = R_{\xi\eta\xi^{-1}} \circ R_\xi = R_{\xi\eta}.$$

□

For the ensemble $\{\pi_n(X,x)\}$ from here (and from equality $\overline{R}_\xi = R^{(1)}_\xi$) it follows that for any points $x_0, x_1 \in X$, any elements $\eta \in \pi_1(X, x_1)$, $\alpha \in \pi_n(X,x)$ and any class ξ of paths connecting the point x_0 with the point x_1, there is an equality

$$R^{(n)}_\xi(\eta\alpha) = R^{(1)}_\xi(\eta) R^{(n)}_\xi(\alpha), \tag{5.31}$$

i.e., that for every $n \geq 1$ the isomorphism $R^{(n)}_\xi : \pi_n(X,x_1) \to \pi_n(X,x_0)$ is an $R^{(1)}_\xi$-isomorphism.

If the space X is connected, then for any two points $x_0, x_1 \in X$ in the groupoid ΠX there exists at least one morphism $\xi : x_1 \to x_0$. Therefore, as follows directly from formula (5.31), if, with some choice of the base point $x_0 \in X$, the connected space X turned out to be homotopically simple in some dimension $n \geq 1$, then it will be homotopically simple in dimension n and with any other choice of this point. A connected space X having this property is naturally called *homotopy simple in dimension n* (and a space homotopy simple in all dimensions is an *abelian* space).

According to Proposition 4.25 of Lecture 4, for every connected space X homotopically simple in dimension n with any choice of the base point $x_0 \in X$ ignoring base points, the map

$$\pi_n(X, x_0) \to [\mathbb{S}^n, X]$$

is bijective. By transferring the group structure from $\pi_n(X, x_0)$ to $[\mathbb{S}^n, X]$, we thereby define the set $[\mathbb{S}^n, X]$, $n \geq 1$, as a group (it is necessary to be abelian even when $n = 1$). This group is called the *n-dimensional homotopy group of the homotopically simple space* X and is denoted by the former symbol $\pi_n X$.

We emphasize that, in contrast to the case of pointed spaces, the group $\pi_n X$ is defined only for spaces X connected and homotopically simple in dimension n (for example, simply connected.) At the same time, on the subcategory of the category of $\mathscr{T}\!\mathit{op}$ consisting of such spaces, the correspondence $\pi_n : X \mapsto \pi_n X$ is obviously a functor.

5.8 Homotopy groups of abelian spaces

It is clear that for each ensemble R and with a connected space X, all objects R_x are isomorphic with each other. In particular, we see that

Proposition 5.32. *if the space X is connected, then for any $n \geq 1$ all groups $\pi_n(X, x)$, $x \in X$, are isomorphic with each other.*

This means that for a connected non-pointed space X, we can also talk about its homotopy group $\pi_n X$, but only as an abstract group (given up to isomorphism). Therefore, in particular, for connected spaces, the condition $\pi_n X = 0$ makes sense (meaning that for one, and therefore for any choice of the base point x_0, the equality $\pi_n(X, x_0)$ takes place).

5.9 Aspherical spaces

Definition 5.33. A pointed (or non-pointed, but connected) space X is called *aspherical in dimension n* if $\pi_n X = 0$. A space that is aspherical in all dimensions $\leq n$ is called *n-connected.*

Thus, pointed spaces aspherical in dimension 0 (or, equivalently, 0-connected) are nothing but connected spaces, and aspherical in dimension 1 are nothing but simply connected spaces. 1-connected spaces are connected and simply connected spaces.

It is clear that the space X is 1-connected if and only if for any points $x_0, x?1 \in X$ in the groupoid ΠX there is a unique morphism $x_0 \to x_1$.

An example of a space that is aspherical in all dimensions (i.e., ∞-*connected*) is the single-point space pt. Therefore, any space that is homotopically equivalent to the space pt, i.e. any contractible space, will also be an aspherical space in all dimensions. The converse, generally speaking, is true only with some additional assumptions of a general methodological nature.

By definition, the map $(\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$ specifies the zero element of the group $\pi_n X$ if it is pointed null homotopic (homotopic to the constant map). But since the group $\pi_1 X$ acts on the group $\pi_n X$ by automorphisms, the $\pi_1 X$-orbit of zero consists only of zero. Therefore, by applying Proposition 4.25 of Lecture 4, the map $(\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$ will set the zero element of the group $\pi_n X$ even when it is freely null homotopic.

Since, for a connected space X, each map $\mathbb{S}^n \to X$ is homotopic to some pointed map $(\mathbb{S}^n, \boldsymbol{s}_0) \to (X, x_0)$, it follows in particular that

Proposition 5.34. *a connected space X is aspherical in dimension n if and only if each map $\mathbb{S}^n \to X$ from an arbitrary n-dimensional sphere $\mathbb{S}^n$ to the space X is null homotopic.*

As we know (see Lecture 1), a null homotopic map $X \to Y$ is equivalent to its extension to the cone CX. On the other hand, the correspondence $[\boldsymbol{x}, t] \mapsto t\boldsymbol{x}$, $t \in I$, $x \in \mathbb{S}^n$, defines, obviously fixed on $\mathbb{Z}^n$, the homeomorphism of the straight cone $C\mathbb{S}^n$ with the unit ball $\mathbb{E}^{n+1}$. Therefore, the space X is aspherical in dimension n if and only if any map $f : \mathbb{S}^n \to X$ can be extended to $\mathbb{E}^{n+1}$:

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\ t\ } & \mathbb{E}^{n+1} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \overline{f}} & \\ X & & \end{array}$$

We will constantly use this simple asphericity criterion in the future.

5.10 Homotopy sequence of fibrations

Let $p : (E, e_0) \to (B, b_0)$ be an arbitrary pointed fibration and $F = p^{-1}(b_0)$ - its fibre. Due to the decomposition $I^{n+1} = I^n \times I$ each map $a : (I^{n+1}, \dot{I}^{n+1}) \to (B, b_0)$ we can consider the homotopy from I^n to B, connecting the constant map const : $I^n \to B$, $\boldsymbol{t} \to b_0$, with itself and (for $n \geq 1$) fixed on $\dot{I}^n$. Since the constant map $I^n \to B$ is covered by the constant map $I^n \to E$, it therefore follows from the axiom **WCHE** that there is a homotopy $\varphi : I^n \times I \to E$ covering the homotopy a and (for $n \geq 1$) fixed on $\dot{I}^n$. In particular, $(p \circ \varphi)(\boldsymbol{t}, 1) = a(\boldsymbol{t}, 1) = b_0$ for any point $\boldsymbol{t} \in I^n$, i.e. $\varphi(\boldsymbol{t}, 1) \in F$. Therefore, putting $b(t) = \varphi(\boldsymbol{t}, 1)$, $t \in I$, we get a map $b : I^n \to F$, translating (for $n \geq 1$) the boundary $\dot{I}^n$ of the cube I^n to the point $e_0 \in F$, i.e. a map $(I^n, \dot{I}^n) \to (F, e_0)$. Conventionally, the transition from a to b is shown in Fig. 5.10.1.

Now let a' be another map $(I^{n+1}, \dot{I}^{n+1}) \to (B, b_0)$, and let b' be a map $(I^{n+1}, \dot{I}^{n+1}) \to (F, e_0)$ resulting from a' by the described construction (using the covering homotopy φ'). Then it is easy to see that

Proposition 5.35. *if $a \sim a' \operatorname{rel} \dot{I}^{n+1}$, then $b \sim b' \operatorname{rel} \dot{I}^n$.*

Proof. Indeed, by identifying the cube I^{n+1} with the product $I^n \times I$ i.e. by denoting its points with symbols $(\boldsymbol{t}, t)$, where $\boldsymbol{t} \in I$, $t \in I$, we can each homotopy

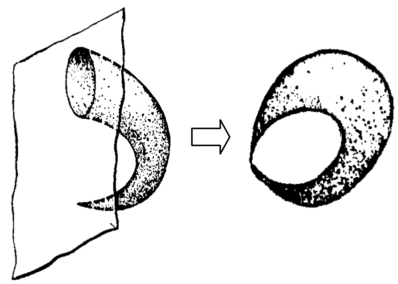


Figure 5.10.1:

$H : I^{n+1} \times I \to B$ relative to $\dot{I}^{n+1}$, connecting the maps a and a', to be interpreted as a homotopy $I^{n+1} \times I \to B$ and in another way, taking the argument t as a deformation parameter. Then the homotopy H will bind the constant expression $I^{n+1} \to B$ with itself, it will be stationary on $\dot{I}^{n+1} \times I \subset \dot{I}^{n+1}$, and on $I^n \times 0 \subset \dot{I}^{n+1}$ and $I^n \times 1 \subset \dot{I}^{n+1}$ it will - after identifying $I^n \times 0$ and $I^n \times 1$ with I^n - coincide, respectively, with maps a and a' interpreted as homotopies. Therefore, according to the axiom **WCHE**, there is a homotopy $\overline{H}$ from $I^n \times I$ to E, covering the homotopy H and on $I^n \times 0$ and $I^n \times 1$ coinciding, respectively, with the homotopy φ and φ'. The terminal map of this homotopy, considered as a map in F and will obviously be a homotopy from I^n to F relative to $\dot{I}^n$, connecting the map b with the map b'. □

The proven statement means that although the correspondence $a \mapsto b$ is constructed with a certain arbitrariness, the homotopy class (rel $\dot{I}^n$) $\beta \in \pi_n F$ of the map b is uniquely determined by the homotopy class (rel $\dot{I}^{n+1}$) $\alpha \in \pi_n B$ of the map a, so we get some well-defined map $\alpha \mapsto \beta$ of the group $\pi_n B$ to the group $\pi_n F$. It is entirely obvious (since the maps a we add "at the first coordinate", and raise to E "at the last coordinate"), that

Proposition 5.36. *for $n \geq 1$ the map $\alpha \mapsto \beta$ is a homomorphism.*

We will denote this homomorphism by the symbol ∂_n or simply ∂.

Together with the homomorphisms $i_* : \pi_n F \to \pi_n E$ and $p_* : \pi_n E \to \pi_n B$ the homomorphism $\partial : \pi_{n+1} B \to \pi_n F$ allows you to write a left-right-free sequence of abelian groups

$$\cdots \xrightarrow{p_*} \pi_{n+1} B \xrightarrow{\partial} \pi_n F \xrightarrow{i_*} \pi_n E \xrightarrow{p_*} \pi_n B \xrightarrow{\partial} \cdots, \quad (5.37)$$

ending with three, generally speaking, non-abelian groups and three pointed sets:

$$\cdots \xrightarrow{\partial} \pi_1 F \xrightarrow{i_*} \pi_1 E \xrightarrow{p_*} \pi_1 B \xrightarrow{\partial} \pi_0 F \xrightarrow{i_*} \pi_0 E \xrightarrow{p_*} \pi_0 B.$$

This sequence is called the *homotopy sequence of the fibration $p : E \to B$.*

Homomorphisms ∂ link together the "obvious" segments $\pi_n F \to \pi_n E \to \pi_n B$ of the sequences (5.37). On this basis, they are called *connecting homomorphisms.*

Finite or infinite sequence

$$\cdots \to A \to B \to C \to D \to \cdots, \tag{5.38}$$

of groups connected by homomorphisms (or, more generally, pointed sets) are called *exact in the term B* if the kernel of the "outgoing" homomorphism $B \to C$ (the preimage of the base point in the case when C is only a pointed set) coincides with the image the "incoming" homomorphism $A \to B$ (which, therefore, must necessarily be an normal subgroup, i.e.- in the old terminology - a normal divisor). A sequence is called *exact* if it is exact in every term (except for the extreme terms when they exist).

Proposition 5.39. *The homotopy sequence (5.37) of an arbitrary pointed fibration $p : E \to B$ is an exact sequence.*

Proof. Let's prove the exactness of the sequence (5.37) in all its members in turn.

Exactness in the term $\pi_n E$. Since $i \circ p = \text{const}$, then $i_* \circ p_* = 0$ and hence $\operatorname{im} i_* \subset \ker p_*$.
Conversely, turning on $\alpha \in \ker p_*$, means that to the map $a : (I^n, \dot{I}^n) \to (E, e_0)$ defining element α, there exists a homotopy $G : I^n \times I \to B$ fixed on $\dot{I}^n$ such that

$$G(t, 0) = (p \circ a)(\boldsymbol{t}) \quad \text{and} \quad G(t, 1) = b_0 \quad \text{for any point} \quad t \in I.$$

According to the axiom **WCHE**, this homotopy can be covered by a homotopy $\overline{G} : I^n \times I \to E$, also fixed on $\dot{I}^n$, such that $\overline{G}(\boldsymbol{t}, 0) = a(\boldsymbol{t})$ for any point $\boldsymbol{t} \in I^n$. The map

$$b : (I^n, \dot{I}^n) \mapsto (E, e_0), \quad \boldsymbol{t} \mapsto \overline{G}(\boldsymbol{t}, 1),$$

gives the same element α of the group $\pi_n f$ as the map a, but has the properties that $b(\boldsymbol{t}) \in F$ for any point $\boldsymbol{t} \in I^n$, and therefore considered as a map in F gives such an element β of the group $\pi_n F$ that $i_* \beta = \alpha$. Hence, $\ker p_* \subset \operatorname{im} i_*$.

Exactness in the term $\pi_n F$. By definition, each element $\alpha \in im\partial \subset \pi_n F$ is set by the map $a : (I^n, \dot{I}^n) \mapsto (F, e_0)$, for which there exists a homotopy $\overline{a} : I^n \times I \to E$ fixed on $\dot{I}^n$, connecting the constant map $I^n \to E$, $\boldsymbol{t} \mapsto e_0$, with the map a, considered as a map in E (in this case $\alpha = \partial\beta$, where $\beta \in \pi_{n+1} B$ is the homotopy class of the map $b = p \circ \overline{a} : (I^{n+1}, \dot{I}^{n+1}) \to (B, b_0)$). Hence, $i{*}\alpha = 0$, i.e. $\operatorname{im}\partial \subset \ker i_*$.
Conversely, enabling $\alpha \in \ker i_* \subset \pi_n F$ means that there exists a homotopy $\overline{a} : I^n \times I \to E$ fixed on $/\dot{i}^n$, such that $\overline{a}(\boldsymbol{t}, 0) = e_0$, $\boldsymbol{t} \in I^n$, and a map $\boldsymbol{t} \mapsto \overline{a}(\boldsymbol{t}, 1)$, considered as a map in F, gives the element α. Then the map $b = p \circ \overline{\alpha}$ will be the map $(I^{n+1}, \dot{I}^{n+1}) \to (B, b_0)$, for the homotopy class $\beta \in \pi_{n+1} B$ of which the equality $\partial\beta = \alpha$ holds. Hence, $\ker i_* \subset \operatorname{im}\partial$.

Exactness in the term $\pi_{n+1} B$ By definition, the element $\alpha \in \operatorname{im} p_* \subset \pi_{n+1} B$ is given by the map

$$a = p \circ \overline{a} : (I^{n+1}, \dot{I}^{n+1}) \to (B, b_0), \quad \text{where} \quad \overline{a} : (I^{n+1}, \dot{I}^{n+1}) \to (E, e_0).$$

Therefore, the class $\partial\alpha \in \pi_n F$ will have a constant map $\boldsymbol{t} \mapsto \overline{a}(\boldsymbol{t}, 1)$, $\boldsymbol{t} \in l^n$, and, therefore, this class will be zero. Hence, $\operatorname{im} p_* \subset \ker \partial$.
Conversely, the inclusion $\alpha \in \ker \partial \subset \pi_{n+1} B$ means that for the map $a : (I^{n+1}, \dot{I}^{n+1}) \to (B, b_0)$ of class α, there is a covering map $\overline{a} : I^{n+1} \to E$ such that $\overline{a}(\boldsymbol{t}, 0) = e_0$, for any point $\boldsymbol{t} \in I^n$, and the map $\overline{a}_1 : \boldsymbol{t} \mapsto \overline{a}(\boldsymbol{t}, 1)$, $\boldsymbol{t} \in I^n$, considered as a map in F, is homotopic relative to $\dot{I}^n$ to the constant map. Then the map $\overline{b} : I^n \times I \to E$, defined by the formula

$$\overline{b}(\boldsymbol{t}, t) = \begin{cases} \overline{a}(\boldsymbol{t}, 2t), & \text{if} \quad 0 \le t \le 1/2, \\ (i \circ G)(\boldsymbol{t}, 2t - 1), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

where $G : I^n \times I \to F$ is a homotopy fixed on $\dot{I}^n$ connecting the map $\overline{a}_1$ with the constant map $I^n \to F$, $\boldsymbol{t} \mapsto e_0$, will obviously be the map $(I^{n+1}, \dot{I}^{n+1}) \to (E, e_0)$ and, therefore, it will set some element β of the group $\pi_{n+1} E$. By definition, the image $p_*\beta$ of this element with the homomorphism $p_* : \pi_{n+1} E \to \pi_{n+1} B$ will be given by by the map $p \circ \overline{b}$. But it is clear that

$$(p \circ \overline{b})(\boldsymbol{t}, t) = \begin{cases} a(\boldsymbol{t}, 2t), & \text{if} \quad 0 \le t \le 1/2, \\ b_0, & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

that is, up to the transposition of the first and last coordinates - the map $p \circ b$ is the sum of (in the sense of formula (5.9)) the map a and a constant map. Therefore, the map $p \circ \overline{b}$ will specify the same element α of the group $\pi_{n+1} B$ as the map a. This means that $p_*\beta = \alpha$. Hence, $\ker \partial \subset \operatorname{im} p_*$. □

5.11 Algebraic properties of exact sequences

Let us explain the algebraic consequences of the exactness property of a homotopy sequence.

The statement that the homomorphism $C \to D$ from the exact sequence (5.38) is a monomorphism (i.e. that there is equality $\ker(C \to D) = 0$), is equivalent in view of the exactness of this sequence in the term C to the statement that the previous homomorphism $B \to C$ is null (i.e. that there is equality $\operatorname{im}(B \to C) = 0$). Similarly, the statement that the homomorphism $A \to B$ is an epimorphism ($\operatorname{im}(A \to B) = B$) is equivalent in view of the exactness of the sequence (5.38) in the term B to the equality $\ker(B \to C) = B$, i.e. the statement that the following homomorphism $B \to C$ is null. Schematically:

$$\bullet \xrightarrow{epi} \bullet \xrightarrow{null} \bullet \xrightarrow{mono} \bullet$$

Thus,

Proposition 5.40. *in the exact sequence of groups, each monomorphism is preceded by an epimorphism through an arrow, and, conversely, each epimorphism is followed by a monomorphism through an arrow.*

Similarly,

Proposition 5.41. *the equality to zero of any member of the exact sequence is equivalent to the fact that the preceding (through the arrow) homomorphism is an epimorphism, and the subsequent one is a monomorphism:*

$$\bullet \xrightarrow{epi} \bullet \to 0 \to \bullet \xrightarrow{mono} \bullet$$

In particular, the statement that the homomorphism $A \to B$ is a monomorphism (epimorphism), is equivalent to the statement that the sequence $0 \to A \to B$ (the sequence $A \to B \to 0$) is exact. Therefore, the statement that the homomorphism $A \to B$ is an isomorphism is equivalent to the statement about the exactness of the sequence $0 \to A \to B \to 0$.

Sequences of the form deserve special attention

$$0 \to A \to B \to C \to 0$$

called *short sequences.* The statement about the exactness of a short sequence is equivalent to the statement that the group B is an *extension* of the group A by means of the group C.

Another interesting case arises when in sequence (5.38) every third group is zero. For convenience of formulation, denote the sequence (5.38) by

$$\cdots \to C_{n+1} \to A_n \to B_n \to C_n \to A_{n-1} \to \cdots , \tag{5.42}$$

we can state that in the exact sequence (5.42) all homomorphisms $A_n \to B_n$ are isomorphisms if and only if all groups C_n are zero. More generally,

Proposition 5.43. *in the sequence (5.42), the groups C_n are equal to zero for all $n \leq r$ (for all $n > r$), if and only if the homomorphisms A_r toB_r for $n < r$ (for $n > r$) are isomorphisms, and the homomorphism $A_r \to B_r$ is an epimorphism (monomorphism) thus, in the sequence (5.42), the groups C_n are equal to zero for all $n < r$ (for all $n > r$), if and only if the homomorphisms $A_r \to B_r$ for $n < r$ (for $n > r$) are isomorphisms if and only if the homomorphisms $A_r \to B_r$ are an epimorphisms (monomorphisms).*

5.12 Homotopy groups of covering spaces

A surjective map $p : \widetilde{X} \to X$ is called a *covering* if the space X is connected and each point of the space X (also obviously connected) has a neighbourhood U such that the set $p^{-1}(U)$ is a disjoint union of open sets, each of which the map p maps homeomorphically to U. The last condition obviously means that the map $P|_U : p^{-1}(U) \to U$ induced by the map p is isomorphic (as an object of the category $\mathcal{T}\!op_U$) to the projection of the direct product $U \times F \to U$, where F is some discrete space. Thus, covers are exactly locally trivial fibrations $\widetilde{X} \to X$ with discrete fibres (and a connected space X). Therefore, if the space X is paracompact, then any covering $\widetilde{X} \to X$ is a fibration. (See Appendix 1.11 to Lecture 1.)

Remark 5.44. With the help of the properties of coverings proved in the next lecture, it can be easily shown that the latter statement is true even without the assumption of paracompact space X. However, due to our general attitudes, we will ignore this circumstance.

In particular, we see that if the space X is paracompact, then for any pointed covering $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ there is an exact homotopy sequence

$$\cdots \to \pi_n F \to \pi_n \widetilde{X} \xrightarrow{p_*} \pi_n X \to \pi_{n-1} F \to \cdots \to \pi_1 X \xrightarrow{\partial} \pi_0 F \to 0$$

(due to the connectivity $\pi_0 \widetilde{X} = \pi_0 X = 0$). But since the fibre F is discrete, then $\pi_n F = 0$ for $n > 0$, from which it immediately follows that

Proposition 5.45. *for $n > 1$ the covering $p : \widetilde{X} \to X$ induces isomorphisms*

$$p_* : \pi_n \widetilde{X} \to \pi_n X$$

of homotopy groups.

Thus, when passing to the covering space X, the higher homotopy groups $\pi_n X$, $n > 1$, remain the same.

Example 5.46. The circle $\mathbb{S}^1$ is covered by a straight line $\mathbb{R}$ (if the points of the circle are considered complex numbers z with $|z| = 1$, then the covering $\mathbb{R} \to \mathbb{S}^1$ can be given by the formula $t \mapsto e^{it}$). Hence, $\pi_n \mathbb{S}^1 = \pi_n \mathbb{R}$ for any $n > 1$. But since the line $\mathbb{R}$ is contractible, $\pi_n \mathbb{R} = 0$ for all $n \geq 0$. This proves that

$$\pi_n \mathbb{S}^1 = 0 \quad \text{for any} \quad n > 1. \tag{5.47}$$

Example 5.48. Having mapped to each point of the unit sphere $\mathbb{S}^m$, $m \geq 1$, the space $\mathbb{R}^{m+1}$ has a one-dimensional subspace passing through it (a point of the m-dimensional projective space $\mathbb{R}P^m$) we will obviously get a covering $\mathbb{S}^m \to \mathbb{R}P^m$. Therefore,

$$\pi_n \mathbb{S}^m \approx \pi_n \mathbb{R}P^m \quad \text{for any} \quad n > 1 \quad \text{and} \quad m \geq 1. \tag{5.49}$$

The last example can be complexified.

5.13 Hopf fibrations

Let $\mathbb{S}^{2m+1}$ be a unit sphere of $m+1$-dimensional complex space $\mathbb{C}^{m+1}$ (given by the equation $|z_0|^2 + \cdots + |z_m|^2 = 1$), and $\mathbb{C}P^m$ be a complex m-dimensional projective space (the set of all one-dimensional subspaces of the space $\mathbb{C}^{m+1}$ or, equivalently, the set of all proportional classes $(z_0 : z_1 : \cdots : z_m)$ of $(m+1)$-tuples $(z_0, z_1, \ldots, z_m) \neq (0, \ldots, 0)$). By matching each point $(z_0, z_1, \ldots, z_m) \in \mathbb{S}^{2m+1}$ one-dimensional subspace passing through it, i.e. the class $(z_0 : z_1 : \cdots : z_m)$, we get some map

$$\hslash : \mathbb{S}^{2m+1} \to \mathbb{C}P^m, \quad (z_0, z_1, \ldots, z_m) \mapsto (z_0 : z_1 : \cdots : z_m).$$

Let Ui be an open set in $\mathbb{C}P^m$ consisting of all points $(z_0 : z_1 : \cdots : z_m)$ for which $z_i \neq 0$, and let $\hbar^{-1}U_i$ be its preimage in the sphere $\mathbb{S}^{m+1}$ (consisting of points $(z_0, z_1, \ldots, z_m) \in \mathbb{S}^{2m+1}$, for which $z_i \neq 0$). Then it is easy to see that the formula

$$(z, (z_0 : z_1 : \cdots : z_m)) \mapsto \frac{|z_i|z}{z_i\sqrt{|z_0|^2 + \cdots + |z_m|^2}}(z_0, z_1, \ldots, z_m),$$
$$z \in \mathbb{S}^1, \quad (z_0 : z_1 : \cdots : z_m) \in U_i,$$

well defines the map $\mathbb{S}^1 \times U_i \to \hbar^{-1}U_i$, which is a homeomorphism translating the projection $\mathbb{S}^1 \times U_i \to U_i$ into the map $\hbar^{-1}U_i \to U_i$ induced by the map $\hbar$, This proves that the map $\hbar$ is a locally trivial fibration. It is called a *Hopf fibration* (or *map*). Its fibre is the circle $\mathbb{S}^1$.

Since $\pi_n\mathbb{S}^1 = 0$ for $n > 1$, it follows directly from the exactness of the homotopy sequence of the Hopf fibrations that

Proposition 5.50. *for any $n \geq 3$ the homomorphism*

$$\hbar_* : \pi_n\mathbb{S}^{2m+1} \to \pi_n\mathbb{C}P^m, \tag{5.51}$$

induced by the fibration $\hbar$, is an isomorphism.

For $m = 1$ the complex projective line $\mathbb{C}P^1$ is homeomorphic to the sphere $\mathbb{S}^2$ (Riemann sphere). Therefore in this case , the Hopf fibration has the form

$$\hbar : \mathbb{S}^3 \to \mathbb{S}^2$$

(and is given by the formula $(z_1, z_2) \mapsto \frac{z_2}{z_1}$; see Lecture 0) and the isomorphism (5.51) turns into an isomorphism

$$\hbar_* : \pi_n\mathbb{S}^3 \to \pi_n\mathbb{S}^2, \quad n \geq 3, \tag{5.52}$$

commonly called *Hopf isomorphism.*

In particular, we see that

Proposition 5.53. *the group $\pi_3\mathbb{S}^2$ is isomorphic to the group $\pi_3\mathbb{S}^3$.*

Note that the isomorphism (5.52) is given by the correspondence $f \mapsto \hbar \circ f$, where $f : \mathbb{S}^n \to \mathbb{S}^3$, and $\hbar : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map. For $n = 3$, this means that the element $\iota_3 = [\mathrm{id}]$ of the group $\pi_3\mathbb{S}^3$ corresponds to the group $\pi_3\mathbb{S}^2$ of the homotopy class $\eta_3 = [\hbar]$ of the Hopf map.

5.14 Functoriality of the homotopy sequence of fibrations

In addition to the property of accuracy, the homotopy sequence of the fibration also has the property of functoriality with respect to the maps of the fibration $p : E \to B$ to an arbitrary other fibration $p_1 : E_1 \to B_1$, i.e. maps $f : E \to E_1$

such that for some (obviously uniquely defined) map $B \to B_1$ (which we will also denote by f) there is a commutative diagramme

$$\begin{array}{ccc} E & \xrightarrow{f} & E_1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ B & \xrightarrow[f]{} & B_1 \end{array}$$

If the fibrations p and p_1 are pointed, then the map $f : E \to E_1$ (and therefore the map $f : B \to B_1$) is also assumed to be pointed. In this case, the map f induces some map $F \to F_1$ of fibres of fibrations p and p_1 (which we will denote with the same symbol f). Therefore, each map $f : E \to E_1$ of pointed fibrations generates a diagramme

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_{n+1}B & \longrightarrow & \pi_n F & \longrightarrow & \pi_n E & \longrightarrow & \pi_n B & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \pi_{n+1}B_1 & \longrightarrow & \pi_n F_1 & \longrightarrow & \pi_n E_1 & \longrightarrow & \pi_n B_1 & \longrightarrow \cdots \end{array} \tag{5.54}$$

the horizontals of which are homotopy sequences of fibrations p and r_1, and the vertical arrows are induced by maps f. The functorial property is that

Proposition 5.55. *this diagramme is commutative, i.e. that its vertical arrows constitute a homomorphism of the top row to the bottom.*

Proof. For the proof, it is sufficient to note that after applying the map f, all constructions related to the homotopy sequence of the fibration $p : E \to B$ pass into the corresponding constructions for the homotopy sequence of the fibration $p_1 : E_1 \to B_1$. □

By introducing into consideration the category of pointed fibrations and the category of exact sequences, we can say that the correspondence between the fibration and its homotopy sequence, is a functor from the first category to the second.

5.15 Axiomatic description of homotopy groups

Note that the commutativity of all squares of diagram (5.54), except for squares containing homomorphisms ∂, is a consequence of the fact that the correspondence $\pi_n : X \to \pi_n X$ is a functor, and the commutativity of each square

$$\begin{array}{ccc} \pi_{n+1}B & \xrightarrow{\partial} & \pi_n F \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \pi_{n+1}B_1 & \xrightarrow[\partial]{} & \pi_n F_1 \end{array}$$

means that the homomorphism $\partial : \pi_{n+1}B \longrightarrow \pi_n F$ is a natural transformation (morphism) of the functor $\pi_{n+1} \circ \beta$ in the functor $\pi_n \circ \varphi$, where β and φ are functors from the category of pointed fibrations to the category $\mathcal{T}op^\bullet$, matching each fibrations $p : E \longrightarrow B$, respectively, its base B and its fibre F.

Suppose now that for any $n \geq 0$ on the category $\mathcal{T}op^\bullet$, some homotopy invariant functor π_n^a is given, taking values for $n \geq 2$ in the category $\mathcal{A}b\mathcal{G}rp$, for $n = 1$ in the category $\mathcal{G}rp$ and for $n = 0$ in the category $\mathcal{E}ns$, and for the category of pointed fibrations, some natural transformation ∂ of the functor $\pi_n^a \circ \beta$ to the functor $\pi_n^a \circ \varphi$. Then for any fibration $p : E \longrightarrow B$ we can write the following

$$\cdots \longrightarrow \pi_{n+1}^a B \xrightarrow{\partial} \pi_n^a F \xrightarrow{i_*} \pi_n^a E \xrightarrow{p_*} \pi_n^a B \longrightarrow \cdots \tag{5.56}$$

similar to sequence (5.37).

We will say that $\{\pi_n^a, \partial\}$ is a *system of axiomatically defined homotopy groups* if

1 (the exactness axiom) For any pointed fibration $p : E \longrightarrow B$, the sequence (5.56) is exact.

2 (the dimension axiom) For a single-point space pt, for any $i \geq 1$, the equality holds

$$\pi_n^a(\mathrm{pt}) = 0.$$

3 (the initial condition) For any pointed space X, the set $\pi_0^a X$ is in natural (i.e., functorial) bijective correspondence with the pointed set $\pi_0 X$ of the components of the space X.

(We will discuss the reasons for calling the axiom 2 the axiom of dimension in the next semester.)

By applying the homotopy invariance of the functors π_n^a it follows from the axiom of dimension that $\pi_n X = 0$ for $n \geq 1$ for any contractible space X. In particular, $\pi_n^a(PX) = 0$, where PX, as always, is the space of paths in the space X starting at the base point x_0, Therefore, applying the exactness axiom to the Serre fibration $PX \longrightarrow X$, we get that

Proposition 5.57. *for any axiomatically given homotopy groups π_n^a, there is an isomorphism*

$$\pi_n^a X \approx \pi_{n-1}^a \Omega X.$$

Cf. with the formula (5.13) for $k = n - 1$

Hence, by induction on n (the initial step of which is provided by the initial condition (3), it immediately follows that for any $n \geq 0$ there is a natural (with respect to X) bijective map $\pi_n^a X \longrightarrow \pi_n X$. At the same time, it follows directly from Remark 4.24 of Lecture 4 that for $n = 1$ this map is either an isomorphism or an anti-isomorphism (a bijective map that changes the order of multipliers to the opposite), from which it follows by the same induction that this map will be an isomorphism for any $n \geq 2$ (since for $n \geq 2$ the group $\pi_n X$ is abelian, then every anti-isomorphism $\pi_n^a X \longrightarrow \pi_n X$ is an isomorphism). Thus, we see that

Proposition 5.58. *the axioms 1, 2 and the initial condition 3 unambiguously up to isomorphism (or for $n = 1$ up to anti-isomorphism) characterise homotopy groups π_n.*

Used in the proof of this statement Remark 4.24 of lecture 4 we will prove in the next lecture.

Appendix

The technique by which groups are defined in an arbitrary category can also be applied to exact sequences. For the sake of certainty, we will limit ourselves to the category $[\mathscr{T}op^\bullet]$, although almost everything is automatically transferred - with obvious and self-evident changes - to arbitrary categories.

5.A Exact sequences of pointed spaces

Definition 5.59. A sequence

$$\cdots \to A_{n-1} \xrightarrow{a_n} A_n \xrightarrow{a_{n+1}} A_{n+1} \to \cdots \tag{5.60}$$

of pointed spaces and their maps are called *exact in the term* A_n if, for any pointed space X, the sequence of pointed sets

$$\cdots \to [X, A_{n-1}]^\bullet \xrightarrow{(a_n)_X} [X, A_n]^\bullet \xrightarrow{(a_{n+1})_X} [X, A_{n+1}]^\bullet \to \cdots \tag{5.61}$$

is exact in the term $[X, A_n]^\bullet$. Sequence (5.60) is called *exact* if it is exact in every term (except, of course, the extreme members, if there are any).

Example 5.62. The pointed map $p : (E, e_0) \to (B, b_0)$ is called a *quasi-fibration* if for any diagramme

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ {\scriptstyle\sigma_0}\downarrow & \overset{\overline{G}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[G]{} & B \end{array} \qquad G \circ \sigma_1 = \text{const}, \tag{5.63}$$

over the category $\mathscr{T}op^\bullet$, in which the terminal map $G \circ \sigma_1$ of the homotopy G is a constant map, there is a covering homotopy $\overline{G} : X \times I \to E$. (Recall that $\sigma_0 : x \mapsto (x, 0)$ and $\sigma_1 : x \mapsto (x, 1)$.) Since the fact of the existence of a homotopy G means that $p \circ g \sim \text{const}$, and the fact of the existence of a homotopy $\overline{G}$ means that $g \sim i \circ f$, where i is an embedding $F \to E$, $F = p^{-1}(b_0)$, and f is the terminal map $\overline{G} \circ \sigma_1$ of the homotopy $\overline{G}$, considered as a map $X \to F$, then

Proposition 5.64. *the map $p : E \to B$ is a quasi-fibration when and only when the three-term sequence*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

is exact in the term E.

Note that any (pointed) fibration is, of course, a quasi-fibration.

The *map* of the sequence (5.60) into a similar sequence $\{B_n\}$ is a sequence $\{f_n\}$ of pointed maps $f_n : A_n \to B_n$, which is for any n the diagramme

$$\begin{array}{ccc} A_n & \longrightarrow & A_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ B_n & \longrightarrow & B_{n+1} \end{array}$$

is commutative. A (family of) maps $\{f_n\}$ is called *homotopy equivalence* if all the maps f_n are pointed homotopy equivalences. It is clear that

Proposition 5.65. *a sequence of pointed spaces that is homotopically equivalent to an exact sequence is also an exact sequence.*

5.B Short exact sequences of H-groups

Of particular importance are the exact sequences of spaces ending with a one-point space pt (or starting with such a space).

It is easy to see that

Proposition 5.66. *the sequence*

$$\cdots \to E \xrightarrow{p} B \to \text{pt} \tag{5.67}$$

is exact in term B if and only if there exists a map $s : B \to E$ that $p \circ s \sim \text{id}$ (the cross section of the map p in the category $[\mathscr{T}\!op^\bullet]$).

Proof. Indeed, the exactness of the sequence (5.67) in the term B means that for any space X, the map $p_X : [X, E]^\bullet \to [X, B]^\bullet$, $[f]^\bullet \mapsto [f \circ p]^\bullet$, $f : X \to E$, is surjective. In particular, it is surjective for $X = B$, and, therefore, there is a map $s : B \to E$ such that $p_B[s]^\bullet = [\text{id}]^\bullet$, i.e. $s \circ p \sim \text{id}$.
Conversely, if such a map s exists, then $p_X[s \circ g]^\bullet = [g]^\bullet$ for any map $g : X \to B$. □

Remark 5.68. If the map $p : E \to B$ is a (at least homotopy) fibration , then this condition is equivalent to the existence of a map $s : B \to E$ (sections of the map p in the category $\mathscr{T}\!op^\bullet$) that $p \circ s = \text{id}$. See the Appendix 2.11 to Lecture 2, where, however, this fact is proved for non-pointed fibrations.

By analogy with the case of groups, sequences of spaces having the form

$$\text{pt} \to F \xrightarrow{i} E \xrightarrow{p} B \to \text{pt}, \tag{5.69}$$

are called *short sequences.*

Remark 5.70. Below we will show (see remark 5.97) that in the case when in the sequence (5.69) the map p is a fibration and the map i is an embedding of its fibre, *this sequence is exact in the term* B, i.e. (see remark 5.68) if and only if there is a section s for the fibration p.

In particular, we see that for any pointed spaces F and B there is a short exact sequence

$$\mathrm{pt} \to F \xrightarrow{\mathrm{incl}_F} F \times B \xrightarrow{\mathrm{proj}_B} B \to \mathrm{pt}, \tag{5.71}$$

where $\mathrm{incl}_F : a \mapsto (a, b_0)$ and $\mathrm{proj}_B : (a, b) \mapsto b$, $a \in F$, $b \in B$. (Of course, the exactness of this sequence is easily and directly proved.)

A short sequence (5.69), whose members F, E, B are H-groups, and the maps i and p are morphisms (homotopy) of H-groups, we will call the *sequence of H-groups*. For any exact sequence of H-groups and any pointed space X, the sequence

$$1 \to [X, F] \xrightarrow{i_X} [X, E] \xrightarrow{p_X} [X, B] \to 1 \tag{5.72}$$

is the usual short exact sequence of groups.

A short exact sequence of groups

$$1 \to F \xrightarrow{i} E \xrightarrow{p} B \to 1 \tag{5.73}$$

is called a *splitting sequence* if the homomorphism $p : E \to B$ has (in the category of groups) the section $s : B \to E$. This section is called a *splitting homomorphism*, and the group E is called a *semidirect product* of groups F and B. An example of a splitting sequence is a sequence whose splitting homomorphism is the sequence

$$1 \to F \xrightarrow{\mathrm{incl}_F} F \times B \xrightarrow{\mathrm{proj}_B} B \to 1,$$

the splitting homomorphism of which is the homomorphism $\mathrm{incl}_B : b \mapsto (1, b)$, $b \in B$.

Proposition 5.74. *For any splitting sequence of groups (5.73), the formula* $\theta(a, b) = i(a) \cdot s(B)$, $(a, b)? \in F \times B$, *defines a bijective map* $\theta : F \times B \to E$.

Proof. Indeed, let t and q be maps $E \to E$ defined by the formulae $t(x) = (s \circ p)(x)^{-1}$ and $q(x) = x \cdot t(x)$, $x \in E$. Then

$$(p \circ q)(x) = p(x) \cdot (p \circ t)(x) = p(x) \cdot (p \circ s \circ p)(x)^{-1} = p(x) \cdot p(x)^{-1} = 1,$$

and therefore, by applying exactness, there exists an element $r(x) \in F$ such that $i(r(x)) = q(x)$. Consider the map $r \times p : E \to F \times B$ defined by the formula $(r \times p)(x) = (r(x), p(x))$, $x \in B$. By definition

$$(\theta \circ (r \times p))(x) = \theta(r(x), p(x)) = (i \circ r)(x) \cdot (s \circ p)(x) = q(x) \cdot t(x)^{-1} = x$$

for any element $x \in E$. In addition, $(p \circ \theta)(a, b) = (p \circ i)(a) \cdot (p \circ s)(b) = 1 \cdot b = b$, and therefore

$$\begin{aligned} i((r \circ \theta)(a, b)) &= (q \circ \theta)(a, b) = \theta(a, b) \cdot (t \circ \theta)(a, b) = \theta(a, b) \cdot (s \circ p \circ \theta)(a, b)^{-1} \\ &= i(a) \cdot s(b) \cdot s(b)^{-1} = i(a) \end{aligned}$$

for any element $(a, b) \in F \times B$. So $(r \circ \theta)(a, b) = a$ and hence $((r \times p) \circ \theta)(a, b) = (a, b)$. Thus, θ and $r \times p$ are mutually inverse bijective maps. □

If the groups F, E and B are abelian, then the maps θ and $r \times p$ are homomorphisms and, therefore, isomorphisms. Thus (we move on to the additive notation and change the notation somewhat),

Proposition 5.75. *in any splitting sequence of abelian groups*

$$0 \longrightarrow A \xrightarrow{i} C \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

the group C is isomorphic to the direct sum $A \oplus B$ of groups A and B.

The corresponding injections of $A \to C$ and $B \to C$ will be homomorphisms of i and s, and projections of $C \to B$ and $C \to A$ are homomorphisms p and r (and the last homomorphisms will be be determined from the relation $i(r(x)) = x - s(p(x))$, $x \in C$).

Following Eilenberg and Steenrod [11], we call for abelian groups $A_1, \dots, A_n$

injections of natural monomorphisms $\mathrm{incl}_\alpha : A_\alpha \to \oplus_\beta A_\beta$, and

projections of natural epimorphisms $\mathrm{proj}_\alpha : \oplus_\beta A_\beta \to A_\alpha$.

Recall that $\mathrm{proj}_1 \circ \mathrm{incl}_1 + \cdots \mathrm{proj}_n \circ \mathrm{incl}\, n = \mathrm{id}$.

Similarly, a short exact sequence of H-groups (5.69) is called *splitting* if an H-group morphism $s : B \to E$ exists such that $p \circ s \sim \mathrm{id}$ holds. In this case, each sequence (5.69) will be a splitting sequence of groups with a splitting homomorphism $s_X : [X, B] \to [X, E]$. For any H-groups F and B, the sequence (5.71) is a splitting short exact sequence of the H-group (with respect to multiplication in $F \times B$, defined in the usual coordinate way). Moreover,

Proposition 5.76. *for any splitting short exact sequence of H-groups (5.69), the space E is homotopically equivalent to the product $F \times B$.*

Proof. Indeed (we actually repeat the group-theoretic reasoning stated above), if t and q are maps of $E \to E$ such that $[t]^\bullet = [s \circ p]^{-1}$ and $[q]^\bullet = [\mathrm{id}]^\bullet \cdot [t]^\bullet$ in the group $[E, E]^\bullet$ then

$$p_F[q]^\bullet = [p \circ q]^\bullet = [p]^\bullet \cdot [p \circ t]^\bullet = [p]^\bullet \cdot [p \circ s \circ p]^{\bullet -1}$$

in the group $[E, B]^\bullet$, and therefore there is a map $r : E \to F$, such that $i_E[r]^\bullet = [q]^\bullet$ i.e. $[i \circ r]^\bullet = [q]^\bullet$. Therefore, to the map $\theta : F \times B \to E$, given by the formula $\theta = m \circ (i \times s)$, where $i \times s : F \times B \to E \times E$, and $m : E \times E \to E$ is a multiplication in E, i.e. being with respect to m by the product of the maps $i \circ \mathrm{proj}_F : F \times B \to E$ and $s \circ \mathrm{proj}_B : F \times B \to E$, in the group $[E, E]^\bullet$ equality will take place

$$\begin{aligned}[\theta \circ (r \times p)]^\bullet &= [m \circ (i \times s) \circ (r \times p)]^\bullet = [m \circ ((i \circ r) \times (s \circ p))]^\bullet \\ &= [i \circ r]^\bullet \cdot [s \circ p]^\bullet = [q]^\bullet \cdot [t]^{-1} = [\mathrm{id}]^\bullet.\end{aligned}$$

On the other hand, since the map p is a homotopy morphism of H-groups, i.e. $p \circ m \sim m' \circ (p \times p)$, where $m' : B \times B \to B$ is the multiplication in B, then in the group $[F \times B, B]^\bullet$ we will have the equality

$$\begin{aligned}[p \circ \theta]^\bullet &= [p \circ m \circ (i \times s)]^\bullet = [m' \circ ((p \circ i) \times (p \circ s))]^\bullet \\ &= [p \circ i \circ \mathrm{proj}_F]^\bullet \cdot [p \circ s \circ \mathrm{proj}_B]^\bullet = [\mathrm{const}]^\bullet \cdot [\mathrm{proj}_B]^\bullet = [\mathrm{proj}_B]^\bullet,\end{aligned}$$

and, therefore, in the group $[F \times B, E]^\bullet$ the equality

$$\begin{aligned}I_{F\times B}[r \circ \theta]^\bullet &= [i \circ r \circ \theta]^\bullet = [q \circ \theta]^\bullet = ([\mathrm{id}]^\bullet \cdot [t]^\bullet) \circ [\theta]^\bullet \\ &= [\theta]^\bullet \cdot [t \circ \theta]^\bullet = [\theta]^\bullet \cdot [s \circ p \circ \theta]^{\bullet -1} \\ &= [\theta]^\bullet \cdot [a \circ \mathrm{proj}_B]^{\bullet -1} = [i \circ \mathrm{proj}_F]^\bullet = I_{F\times B}[\mathrm{proj}_F]^\bullet.\end{aligned}$$

Since the homomorphism $i_{F\times B}$ is by condition a monomorphism, this proves that $[r \circ \theta]^\bullet = [\mathrm{proj}_F]^\bullet$ in the group $[F \times B, F]^\bullet$. Hence,

$$[(r \times p) \circ \theta]^\bullet = [(r \times \theta) \times (p \circ \theta)]^\bullet = [\mathrm{proj}_F \times \mathrm{proj}_B] = [\mathrm{id}]$$

in the group $[F \times B, F \times B]^\bullet$. So $\theta \circ (r \times p) \sim \mathrm{id}$ and $(r \times p) \circ \theta \sim \mathrm{id}$, i.e. θ and $r \times p$ are mutually inverse homotopy equivalences. □

If the H-groups F, E and B are abelian (i.e., each exact sequence (5.72) consists of abelian groups), then the homotopy equivalences θ and $r \times p$ will be, as the automatic calculation shows, morphisms of H-groups, so that in this case the H-group E is homotopically isomorphic to the H-group $F \times B$.

5.C Short exact sequences of H-cogroups

All these results are immediately dualised.

Definition 5.77. A sequence

$$\cdots \to A_{n+1} \xrightarrow{a_n} A_n \xrightarrow{a_{n-1}} A_{n-1} \to \cdots$$

of pointed spaces and their maps are called *coexact in the term A_n* if, for any pointed space X, the sequence of pointed sets

$$\cdots \to [A_{n+1}, X]^\bullet \xrightarrow{a_n^X} [A_n, X]^\bullet \xrightarrow{a_{n-1}^X} [A_{n-1}, X]^\bullet \to \cdots$$

is exact in the term $[A_n, X]^\bullet$. A sequence is called *coexact* if it is coexact in every term (except, of course, the extreme terms, if there are any).

It is clear that a sequence homotopically isomorphic to a(nother) coexact sequence is coexact. An example of a coexact sequence is a three-term sequence

$$A \xrightarrow{i} X \xrightarrow{j} X/A, \tag{5.78}$$

where $i : A \to X$ is an arbitrary cofibration (considered as an embedding), and $j : X \to X/A$ is the factorisation map.

Sequence

$$\mathrm{pt} \to A \xrightarrow{i} X \to \cdots$$

is co-exact in the term A if and only if when there exists a map $r : X \to A$ such that $r \circ i \sim \mathrm{id}$ (in the case when the map i is a cofibration, such that $r \circ i = \mathrm{id}$), and a short sequence of the form

$$\mathrm{pt} \to A \xrightarrow{i} X \xrightarrow{j} X/A \to \mathrm{pt},$$

where i is a cofibration, and j is a factorisation map if and only if it is coexact in the term A, i.e. when $X \searrow A$.

A sequence consisting of H-cogroups and their (homotopy) morphisms is called a *sequence of H-cogroups*. Short exact sequence of H-cogroups

$$\mathrm{pt} \to A \xrightarrow{i} X \xrightarrow{j} B \to \mathrm{pt},$$

is called *co-splitting* if there exists a morphism of H-cogroups $r : X \to A$ such that $r \circ i \sim \mathrm{id}$. For any co-splitting short co-exact sequence of H-cogroups, the space X is homotopically equivalent to the bouquet $A \vee B$, and the homotopy equivalence $X \to A \vee B$ will be a coproduct (with respect to the multiplication in X) of the maps $\mathrm{incl}_A \circ r$ and $\mathrm{incl}_B \circ j$, where $\mathrm{incl}_A : A \to A \vee B$ and $\mathrm{incl}_B : B \to A \vee B$ are canonical inclusions.

The proofs of all these statements are obtained by an obvious dualisation of the proofs of the corresponding statements for exact sequences, and we will leave them to the reader.

5.D Homotopy fibres of pointed maps

Let us now turn from these general — essentially purely category-theoretic — considerations to more meaningful constructions.

Let $p : (E, e_0) \to (B, b_0)$ be an arbitrary pointed map (it is convenient for us to move away from the standard notation a little now), and let $\mathrm{Cocyl}(p)$ be its reversed cylinder. Recall (see Lecture 2) that the points of the the cylinder $\mathrm{Cocyl}(p)$ are pairs of (u, e), where $u : I \to B$ and $e \in E$, with $u(1) = p(e)$, and that the formula $q(u, e) = u(0)$ defines some fibration $q : \mathrm{Cocyl}(p) \to B$.

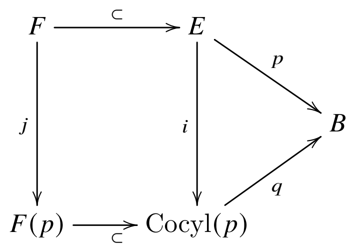
Definition 5.79. The fibre of the fibration $q : \mathrm{Cocyl}(p) \to B$ is called the *homotopy fibre* of p and is denoted by the symbol $F(p)$.

It is clear that the correspondence $p \mapsto F(p)$ is a functor.

It is easy to see that

Proposition 5.80. *if the map $p : E \to B$ is a pointed homotopy fibration, then its homotopy fibre $F(p)$ is pointed homotopically equivalent to its ordinary fibre $F = p^{-1}(b_0)$.*

Proof. Indeed, it is clear that the inclusion $i : E \to \mathrm{Cocyl}(p)$, $\mapsto (e, 0_{p(e)})$, translates the fibre F into the fibre $F(p)$ and therefore induces some map $j : F \to F(p)$. In the corresponding commutative diagramme



(5.81)

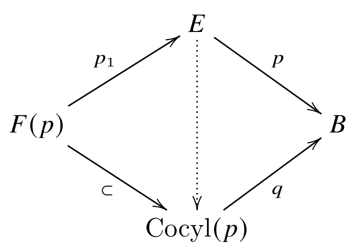
the map i, being a homotopy equivalence and at the same time a map over B (with respect to the projections p and q), will, according to Proposition 2.57 of the Appendix to Lecture 2 (in its version for the category $\mathscr{T\!op}^\bullet$), be a fibre homotopy equivalence and, therefore, the map j will be homotopy equivalence. □

Since diagram (5.81) is commutative and its vertical arrows are pointed homotopy equivalences, the upper row of this diagram is homotopically isomorphic to its lower exact row and, therefore, is an exact sequence. This proves that

Proposition 5.82. *any homotopy fibration is a quasi-fibration.*

A very unexpected result!

In the general case of an arbitrary pointed map $p : E \to B$, we can consider the map $p_1 : F(p) \to E$, which is a restriction of the projection $(u, e) \mapsto e$. This map has the property that the diagramme



the dotted line is homotopy commutative, which means that the upper line of this diagramme is homotopically isomorphic to its lower exact line and therefore is also an exact sequence. This proves that

Proposition 5.83. *for any pointed map $p : E \to B$ there is a three-term exact sequence*

$$F(p) \xrightarrow{p_1} E \xrightarrow{p} B. \tag{5.84}$$

Iterating this construction and assuming $p_n = (p_{n-1})_1$, $p_0 = p$, we get an exact sequence infinite to the left

$$\cdots \to F(p_n) \xrightarrow{p_n} F(p_{n-1}) \to \cdots \to F(p) \xrightarrow{p_1} E \xrightarrow{p} B \tag{5.85}$$

of pointed spaces, the terms $F(p_n)$, $n \geq 0$, which are called *iterated homotopy fibres* of the pointed map p.

Further, comparing the definitions, we immediately get, that the map $p_1 : F(p) \to E$ is induced by the Serre fibration $\omega_1 : PB \to B$ by the map p (i.e., in the notation introduced in Lecture 1, is the map $(\omega_1)_p$). Hence, the map $p_1 : F(p) \to E$ is a fibration. Therefore, all maps p_n, $n \geq 1$, from the sequence (5.85) will also be fibrations.

5.E The Puppe exact sequence

Applying the functor Ω to the sequence (5.85), we get the sequence

$$\cdots \to \Omega F(p_n) \xrightarrow{\Omega p_n} \Omega F(p_{n-1}) \to \cdots \to \Omega F(p) \xrightarrow{\Omega p_1} \Omega E \xrightarrow{\Omega p} \Omega B \tag{5.86}$$

On the other hand, it is easy to see that

Proposition 5.87. *the space ΩB is homotopically equivalent to the space $F(p_1)$, the space ΩE is homotopically equivalent to the space $F(p_2)$, the space $\Omega F(p) = \Omega F(p_0)$ is homotopically equivalent to the space $F(p_3)$ and, in general, the space $\Omega F(p_n)$ is homotopically equivalent to the space $F(p_{n+2})$ (it is convenient to conditionally assume that $F(p_{-1}) = E$, $F(p_{-2}) = B$).*

Proof. Indeed, since the map p_{n+i}, $n \geq 0$, is a fibration, its homotopy fibre $F(p_{n+i})$ is homotopically equivalent to its ordinary fibre F_{n+i}. But since the points of the space $F(p_n)$ are pairs (v, e), where $v \in PF(p_{n-2})$, $e \in F(p_{n-1})$ and $v(1) = p_n(e)$, and the map p_{n+1} is a restricted projection $(v, e) \mapsto e$, the fibre F_{n+1} consists of points of the form (v, e_0), where e is the base point, and $v \in \Omega F(p_{n-2})$, and therefore is homeomorphic to the space $\Omega F(p_{n-2})$. □

To write the homotopy equivalences $j_n : \Omega F(p_{n-3}) \to F(p_n)$, $n \geq 1$, explicitly, we note that for $n \geq 1$ each point $e \in F(p_{n-1})$ in turn has the form (u, b), where $u \in PF(p_{n-3})$, $b \in F(p_{n-1})$ and $u(1) = p_{n-1}(b)$, and $p_n(e) = b$. Therefore, the points of the space $F(p_n)$ can be considered as pairs (v, u), where $v \in PF(p_{n-1})$, $u \in PF(p_{n-3})$, $u(1) = p_{n-1}(v(1))$, and then the map j_n will be determined by the formula $j_n(u) = (0, u)$ (and the map p_{n+1} - by the formula $p_{n+1}(v, u) = (u, v(1))$).

It is natural to expect that the homotopy equivalences of j_n will constitute a homotopy isomorphism of sequence (5.86) to sequence (5.85) without the last three terms, i.e. that the corresponding squares will be homotopy commutative. However, this is not the case, and here we are faced with one of those rare cases when seemingly completely natural constructions lead to noncommutative diagrammes.

To achieve commutativity, each homotopy equivalence j_n with an odd number n must be combined with the homotopy equivalence $\Omega F(p_{n-3}) \to \Omega F(p_{n-3})$, $u \mapsto u^{-1}$, where, as always, $u^{-1} : t \mapsto u(1-t)$, $t \in I$. In accordance with this, we will define homotopy equivalences

$$k_n : \Omega F(p_{n-3}) \to F(p_n), \quad n \geq 1,$$

by the formula

$$k_n(u) = \begin{cases} (0, u^{-1}), & \text{if } n \text{ odd}, \\ (0, u), & \text{if } n \text{ even}, \end{cases} \qquad u \in \Omega F(p_{n-3}).$$

These homotopy equivalences already constitute a homotopic isomorphism, i.e.

Proposition 5.88. *the diagramme (5.89) is homotopy commutative.*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega F(p_{n-2}) & \xrightarrow{\Omega p_{n-2}} & \Omega F(p_{n-3}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle k_{n+1}} & & \downarrow{\scriptstyle k_n} & & \\ \cdots & \longrightarrow & F(p_{n+1}) & \xrightarrow[p_{n+1}]{} & F(p_n) & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & \Omega F(p) & \xrightarrow{\Omega p_1} & \Omega E & \xrightarrow{\Omega p} & \Omega B & & & & & & \\ & & \downarrow{\scriptstyle k_3} & & \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle k_1} & & & & & & \\ \cdots & \longrightarrow & F(p_3) & \xrightarrow[p_4]{} & F(p_2) & \xrightarrow[p_3]{} & F(p_1) & \xrightarrow[p_2]{} & F(p) & \xrightarrow[p_1]{} & E & \xrightarrow[p]{} & B \end{array} \tag{5.89}$$

Proof. Indeed, the map $p_{n+2} \circ k_{n+1}$ translates each loop $\omega^{\varepsilon}, 0$ to the point $(\omega^{\varepsilon}, 0)$, where $\varepsilon = (-1)^{n+1}$, and the map $k_n \circ \Omega p_{n-1}$ - to the point $(0, (p_{n-1} \circ \omega)^{-\varepsilon}) = (0, p_{n-1} \circ \omega^{-\varepsilon})$. Therefore, the homotopy $(\omega, \tau) \mapsto (v_{\omega}, u_{\omega,\tau})$, $(\omega, \tau) \in \Omega F(p_{n-2}) \times I$, where the paths $v_{\omega,\tau}, \in PF(p_{n-2})$, $u_{\omega,\tau} \in PF(p_{n-3})$ are defined by formulae

$$v_{\omega,\tau} = \omega^{\varepsilon}((1-\tau)t), \quad u_{\omega,\tau} = (p_{n-1} \circ \omega^{\varepsilon})(1 - \tau t)$$

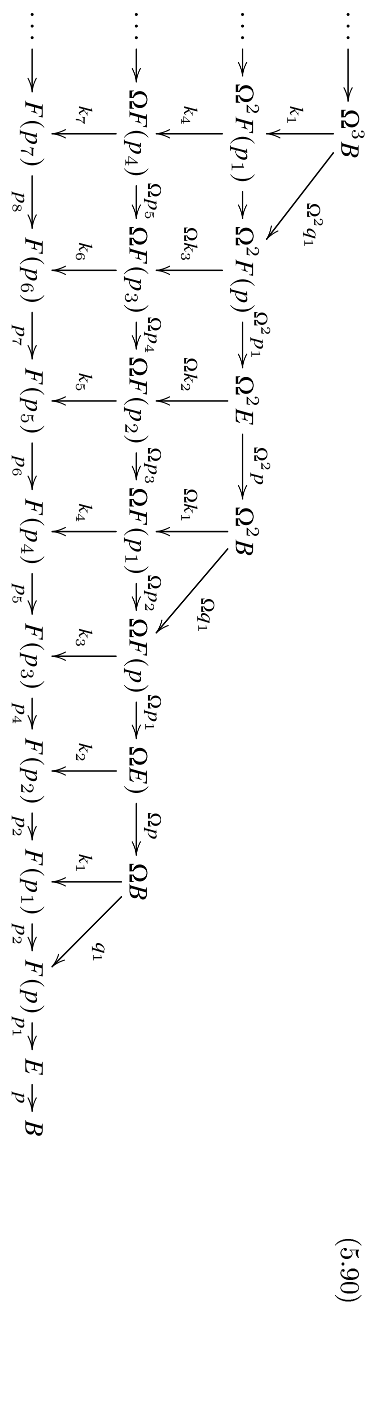
(it is clear that these paths satisfy the relation $u_{\omega,\tau}(1) = p_{n-1}(v_{\omega,\tau}(1))$, connects the first map with the second. $\square$

Now applying the functor Ω to Diagramme (5.89), we obtain a homotopy commutative diagramme whose vertical arrows are also homotopy equivalences. Since the bottom line of this diagramme coincides with the top line of diagram (5.89), they can be stitched into one three-line homotopy commutative diagram.

Iterating this construction, we get an infinite to the left and up homotopy commutative diagram, all vertical arrows of which are homotopy equivalences, the bottom line is a sequence (5.85), and each next line is obtained from the previous one by applying the functor Ω and shifting three terms to the left.

In Diagramme 5.90, the maps

$$q_1 = p_2 \circ k_1, \Omega q_1, \Omega^2 q_1 \ldots,$$



(5.90)

are attached to it, stitching the "terminal" three-term sequences of each line into a single sequence

$$\begin{aligned}&\cdots \to \Omega^{n+1}B \xrightarrow{\Omega^n q_1} \Omega^n F(p) \xrightarrow{\Omega^n p_1} \Omega^n E \xrightarrow{\Omega^n p} \Omega^n B \to \\ &\cdots \to \Omega B \xrightarrow{q_1} F(p) \xrightarrow{p_1} E \xrightarrow{p} B\end{aligned} \tag{5.91}$$

Definition 5.92. The sequence (5.91) of pointed spaces is called the *Puppe sequence* (or *resolvent*) of the map $p : E \to B$.

This sequence Ω is periodic in the sense that under the influence of the Ω functor, it shifts three terms to the left as a whole, and therefore each of its three consecutive terms after applying this functor pass into the next three terms.

It is clear that the compositions of the vertical arrows of Diagramme 5.90 constitute a homotopy isomorphism of the Pupe sequence to the sequence (5.85) of iterated homotopy fibres of the map p. Hence,

Proposition 5.93. *the Puppe sequence is an exact sequence, i.e., for any pointed space X, there is an exact sequence*

$$\cdots \to [X, \Omega^{n+1}B]^\bullet \to [X, \Omega^n F]^\bullet \to [X, \Omega^n E]^\bullet \to [X, \Omega^n B]^\bullet \to \cdots . \tag{5.94}$$

At the same time, according to the results of the Appendix to Lecture 4, all members of the sequence (5.94) are abelian groups (and the maps connecting them are homomorphisms), except for the last six

$$\cdots \to [X, \Omega F]^\bullet \to [X, \Omega E]^\bullet \to [X, \Omega B]^\bullet \to [X, F]^\bullet \to [X, E]^\bullet \to [X, B]^\bullet,$$

of which the first three are groups and the last three are pointed sets.

In the special case when the map $p : E \to B$ is a fibration (at least homotopy), the homotopy fibrer $F(p)$ in the Puppe sequence can be replaced by an ordinary fibre F of the fibration p (while, of course, replacing the map p_1 with the embedding $i : F \to E$, and the map q_1 is the corresponding map $q : \Omega B \to F$). The resulting sequence

$$\cdots \to \Omega^{n+1}B \xrightarrow{\Omega^n q} \Omega^n F \xrightarrow{\Omega^n i} \Omega^n E \xrightarrow{\Omega^n p} \Omega^n B \to \cdots \to \Omega B \xrightarrow{q} F \xrightarrow{i} E \xrightarrow{p} B \tag{5.95}$$

is called the *Puppe sequence of the fibration p*. It is, of course, also exact.

Remark 5.96. The map $q : \Omega B \to F$ in the sequence (5.95) depends on the choice of the section $s : \mathrm{Cocyl}(p) \to E^I$ by the axiom **CP** and maps to the loop $u \in \Omega B$ the end of the path $s(e_0, u)$ starting at the point $e_0 \in F$ and covering this loop:

$$q(u) = s(e_0, u)(1), \quad u \in \Omega B.$$

Therefore, the statement about the exactness of the sequence (5.95) in the term F (which is the main stumbling block when trying to directly - not using Proposition 2.57 from the Appendix to Lecture 2 - prove the exactness of this sequence) boils down to the statement that the map $E^I \to R^I$ defined by the formula $v \mapsto s(v(0), p \circ v)$, $v \in E^I$, can be related to the identity map by a homotopy $f_t : E^I \to E^I$, such that for any path $v \in E^I$ the path $p \circ f_t v \in B$ does not depend on t. It will be a very good exercise for the reader to try to prove the last statement and deduce from it the exactness of the sequence (5.95) in the term F.

Remark 5.97. If there is a cross section $s : B \to E$ to the map $p : E \to B$, then the map $\Omega s : \Omega B \to \Omega E$ will be a cross section of the map $\Omega p : \Omega E \to \Omega B$, from which it directly follows that for any space X the homomorphism $[X, \Omega E]^\bullet \to [X, \Omega B]^\bullet$ is an epimorphism. Therefore, the sequence (5.94) crumbles into a short exact sequences

$$0 \to [X, \Omega^n F(p)]^\bullet \to [X, \Omega^n E]^\bullet \to [X, \Omega B]^\bullet \to 0, \quad n \geq 0,$$

and, therefore, the sequence (5.91) is a short exact sequences

$$\mathrm{pt} \to \Omega^n F(p) \to \Omega^n E \to \Omega^n B \to 0, \quad n \geq 0.$$

In particular, we see that if in sequence (5.69) the map p is a fibration having a section (and the map i is an embedding of its fibre), then this sequence is exact. (See Remark 5.70 above.)

Using the adjointness between the functors Ω and $S^\bullet$, the sequence (5.94) can also be rewritten in the following form:

$$\cdots \to [S^{n+1}X, B]^\bullet \to [S^n X, F]^\bullet \to [S^n X, E]^\bullet \to [S^n X, B]^\bullet \to \cdots. \tag{5.98}$$

A special case of the sequence (5.98) (obtained for $X = \mathbb{S}^0$) is the homotopy sequence of the fibration $p : E \to B$.

Remark 5.99. An analogue of the sequence (5.98) (obtained by replacing F with $F(p)$) holds, of course, for any map $p : E \to B$; in particular, - for $E \subset B$ - for embedding $E \to B$. At the same time, it is easy to see that in the latter case, the space $F(p)$ is nothing more than the path space $P(B, \mathrm{pt}, E)$ of the space B starting at the base point and ending in the subspace E.

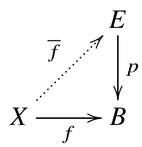
5.F The extended Puppe sequence of a classifying fibration

We will call the pointed fibration $p : E \to B$ *classified* if it is induced (by some map $\varphi : B \to B_0$, called a *classifying map*) fibration $p_0 : E_0 \to B_0$ with a contractible space E_0, i.e. if there exists a universal square

$$\begin{array}{ccc} E & \xrightarrow{\varphi^{\#}} & E_0 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow[\varphi]{} & B_0 \end{array} \tag{5.100}$$

in which the map p_0 is a fibration, and the space E_0 is contractible.

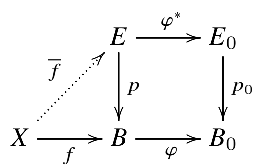
For classified fibrations $p : E \to B$, a simple necessary and sufficient condition for the solubility of an arbitrary lifting problem can be specified



(5.101)

Proposition 5.102. *If the fibration $p : E \to B$ in problem (5.101) is classified and if $\varphi : B \to B_0$ is a classifying map, then the covering map $\overline{f} : X \to E$ exists if and only if the map $\varphi \circ f : X \to B_0$ is null homotopic.*

Proof. Diagrammes (5.100) and (5.101) can be stitched into one diagramme



If the map $\overline{f}$ exists, then $\varphi \circ f = p_0 \circ \varphi^\# \circ \overline{f}$. Since the space E_0 is contractible,

$$\varphi^\# \circ \overline{f} = \mathrm{id} \circ \varphi^\# \circ \overline{f} \sim \mathrm{const} \circ \varphi^\# \circ \overline{f} = \mathrm{const}$$

and therefore $p_0 \circ \varphi^\# \circ \overline{f} \sim \mathrm{const}$. Hence, $\varphi \circ f \sim \mathrm{const}$. (Note that we did not use the universality of square (5.100) in this argument.)

Conversely, let $\varphi \circ f \sim \mathrm{const}$. Since the constant map is covered, then by applying the axiom **CP** (applied to the fibration p_0) there is a map $g : X \to E_0$ covering the map $\varphi \circ f$, i.e. such that the pair (f, g) is a cone over the pair (φ, p_0). Consequently, due to the universality of the square (5.100), there is a morphism $\overline{f} : X \to E$ of this pair into the pair $(p, \varphi^\#)$. This morphism will be the map covering the map f. □

Remark 5.103. In the last argument, we did not use the contractibility of the space E_0. Therefore, if the fibration $p : E \to B$ is induced from some (other) fibration $p_0 : E_0 \to B_0$ by a map $\varphi : B \to B_0$, then for any diagram (5.101) in which the map f has the property that $\varphi \circ f \sim \mathrm{const}$, there is a covering map $\overline{f}$.

This simple sufficient condition for the solubility of the lifting problem is surprisingly often useful.

The existence of the map $\overline{f}$ is equivalent to the fact that the element $[f]$ of the pointed set $[X, B]^\bullet$ belongs to the image $\operatorname{im} p_*$ of the map $p_* : [X, E]'^\bullet \to [X, B]^\bullet$, and the condition $\varphi \circ f \sim \mathrm{const}$ is that this element belongs to the kernel $\ker \varphi_*$ of the map $\varphi_* : [X, B]^\bullet \to [X, B_0]^\bullet$. Therefore, the statement of

Proposition 5.102 is equivalent to the equality $\operatorname{im} p_* = \ker \varphi_*$, i.e., the statement about the exactness of the sequence

$$[X, E]^\bullet \xrightarrow{p_*} [X, B]^\bullet \xrightarrow{\varphi_*} [X, B_0]^\bullet,$$

or, in other words, about the exactness of the sequence

$$E \xrightarrow{p} B \xrightarrow{\varphi} B_0.$$

Thus, we see that without loss of exactness, the Puppe sequence of the classified map $p : E \to B$ can be extended on the right by one term. The resulting exact sequence

$$\begin{gathered} \cdots \to \Omega^n F \xrightarrow{\Omega^n i} \Omega^n E \xrightarrow{\Omega^n p} \Omega^n B \xrightarrow{\Omega^n \varphi} \Omega^{n-1} F \to \cdots \\ \cdots \to F \xrightarrow{i} E \xrightarrow{p} B \xrightarrow{\varphi_0} B_0 \end{gathered} \tag{5.104}$$

is called the *extended sequence* of a Puppe classified fibration.

It should be borne in mind that for the exactness of the sequence (5.104), the classification of the fibration p is only sufficient, but not necessary. For example, according to Remark 5.97, this sequence will remain exact if, in the universal square (5.100), the conditions for the contractibility of the space E_0 are replaced by a weaker condition such that the map $p \circ \varphi : E \to B_0$ is null homotopic.

5.G Mapping cones and Puppe exact sequences

Let us now consider the dual situation.

Let $i : A \to X$ be an arbitrary pointed map, $\mathrm{Cyl}^\bullet(i)$ be its reversed reduced cylinder and $j : A \to \mathrm{Cyl}^\bullet(i)$ be the cofibration of $a \mapsto [a, 0]$, $a \in A$,

Definition 5.105. The cofibre

$$C^\bullet(i) = \mathrm{Cyl}^\bullet(i)/jA \tag{5.106}$$

of the cofibrationi j is called the (*reduced*) *cone* of the map i. It is obtained from the cone $C^\bullet A$ over the space A by gluing the space X to it by the map $[a, 1] \mapsto ia$.

In particular, if $A \subset X$ and i is an inclusion, then

$$C^\bullet(i) = CA \cup X$$

(see Fig. 5.G.1).

If, in addition, the pair (X, A) is a cofibration, then

Proposition 5.107. *the pointed cone $C^\bullet(i)$ is homotopically equivalent to the cofibre X/A of the pair (X, A)*

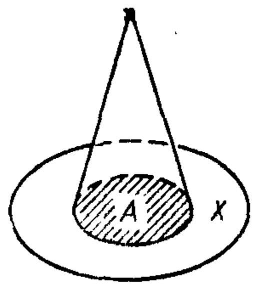


Figure 5.G.1:

Proof. since this cofibre is naturally homeomorphic to the cofibre $C^\bullet(i)/CA$, which, according to Lemma 4.46 of Lecture 4, is homotopically equivalent to the space $C^\bullet(i)$ (the conditions of this lemma are fulfilled, since the cone CA is contractible, and the pair $(C^\bullet(i), CA)$ is relatively homeomorphic to the cofibration (X, A) and therefore is also a cofibration). □

Replacing in the coexat sequence (10) with a cofibre X/A on the cone $WC^\bullet(i)$, we get the sequence

$$A \xrightarrow{i} X \xrightarrow{i_1} C^\bullet(i), \tag{5.108}$$

where i_1 is the natural inclusion (which is obviously a cofibration). The *sequence* (5.108) *is coexact*, being homotopically isomorphic to the coexact sequence

$$A \to \mathrm{Cyl}^\bullet(i) \to C^\bullet(i).$$

Putting $i_n = (i_{n-1})_1$, $i_0 = i$, we therefore get an infinite right-hand coexact sequence

$$A \xrightarrow{i} X \xrightarrow{i_1} C^\bullet(i_1) \to \cdots \to C^\bullet(i_{n-1}) \xrightarrow{i_{n+1}} C^\bullet(i_n) \to \cdots \tag{5.109}$$

all maps of which are cofibrations, except, maybe the first one. The members of $C^\bullet(i_n)$ of this sequence are called *iterated cones of the map i.*

It is obvious that the cone $C^\bullet(i_1)$ is homotopically equivalent to the suspension $S^\bullet A$ (which is the fibre $C^\bullet(i)/X = (CA \cup X)/X$ of the cofibration i_1), the cone $C^\bullet(i_2)$ is homotopically equivalent to the suspension $S^\bullet X$ (which is a cofibre of the cofibration i_2) and, in general, the cone $C^\bullet(i_n)$ is homotopically equivalent to the suspension $S^\bullet C(i_{n-3})$ over the cone $C^\bullet(i_{n-3})$ (it is convenient to assume that $C^\bullet(i_{-2}) = A$, $C^\bullet(i_{-1}) = X$).

By definition, each point of the cone $C^\bullet(i_n)$, $n \geq 1$, either has the form $[x, t]$, where $x \in C^\bullet(i_{n-2})$, $t \in I$, or a point from $C^\bullet(i_{n-1})$, which has the form $[a, t]$, where $a \in C^\bullet(i_{n-3})$, $t \in I$. In other words, $C^\bullet(i_n) = C^\bullet C^\bullet(i_{n-2}) \cup C^\bullet C^\bullet(i_{n-3})$.

map i_{n+2} of the cone

$$C^\bullet(i_n) = C^\bullet C^\bullet(i_{n-2}) \cup C^\bullet C^\bullet(i_{n-3})$$

to the cone

$$C^\bullet(i_{n+1}) = C^\bullet C^\bullet(i_{n-1}) \cup C^\bullet C^\bullet(i_{n-2})$$

is on $C^\bullet C^\bullet(i_{n-2})$ an identical map, and on $C^\bullet C^\bullet(i_{n-3})$

$$C^\bullet(i_{n-1}) = C^\bullet C^\bullet(i_{n-3}) \cup C^\bullet C^\bullet(i_{n-4})$$

is an embedding of the cone $C^\bullet C^\bullet(i_{n-1})$. Thus

$$\begin{aligned} i_{n+2}[x,t] &= [x,t], \quad x \in C^\bullet(i_{n-2}), \quad t \in I, \\ i_{n+2}[a,t] &= [[a,t],0], \quad a \in C^\bullet(i_{n-3}), \quad t \in I, \end{aligned}$$

Homotopy equivalence $j_n : C^\bullet(i_n) \to S^\bullet C^\bullet(i_{n-3})$ for points $[x,t] \in C^\bullet C^\bullet(i_{n-2})$ translates to the base point pt, and the points $[a,t] \in C^\bullet C^\bullet(i_{n-1})$ - to the points of the suspension $S^\bullet C^\bullet(i_{n-3})$ denoted by the same symbol $[a,t]$. We define the homotopy equivalence $k_n : C^\bullet(i_n) \to S^\bullet C^\bullet(i_{n-3})$, assuming that the points $[x,t] \in C^\bullet C^\bullet(i_{n-2})$ still translate to the point pt, and on the points $[a,t] \in C^\bullet C^\bullet(i_{n-3})$, $a \in C^\bullet(i_{n-3})$, $t \in I$, defined by the formula

$$k_n([a,t]) = \begin{cases} [a, 1-t], & \text{if } n \text{ is odd}, \\ [a,t], & \text{if } n \text{ is even}. \end{cases}$$

Then it is easy to see that

Proposition 5.110. *the diagramme*

$$\begin{array}{ccccccccccc} A & \xrightarrow{i} & X & \xrightarrow{i_1} & C^\bullet(i) & \xrightarrow{i_2} & C^\bullet(i_1) & \xrightarrow{i_3} & C^\bullet(i_2) & \longrightarrow & \cdots \\ & & & & & & \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle k_2} & & \\ & & & & & & S^\bullet(A) & \xrightarrow[S^\bullet i]{} & S^\bullet(X) & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C^\bullet(i_n) & \xrightarrow{i_{n+2}} & C^\bullet(i_{n+1}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle k_n} & & \downarrow{\scriptstyle k_{n+1}} & & \\ \cdots & \longrightarrow & S^\bullet C^\bullet(i_{n-3}) & \xrightarrow[S^\bullet i_{n-1}]{} & S^\bullet C^\bullet(i_{n-2}) & \longrightarrow & \cdots \end{array}$$

is homotopically commutative.

Proof. Indeed, every point from $C^\bullet(i_n)$ of the form $[a,t]$, $a \in C^\bullet(i_{n-2})$, $t \in I$, the map $S^\bullet i_{n-1} \circ k_n$ translates, depending on the parity of n, either to the point $[i_{n-1}a, 1-t]$, or to the point $[i_{n-1}a, t]$, and the map $k_{n+1} \circ i_{n+2}$ - to the base point pt. On the contrary, every point from $C^\bullet(i_n)$ of the form $[x,t]$, $x \in C^\bullet(i_{n-2})$, $t \in I$, the map $S^\bullet i_{n-1} \circ k$ translates to the point pt, and the map $k_{n+1} \circ i_{n+2}$ -

either to the point $[i_{n+2}x, t]$, or to the point $[i_{n+2}x, 1-t]$. Thus, (for $n \geq 4$)

$$(S^\bullet \circ k_n)[a,t] = \begin{cases} [a, 1-t] \quad \text{or} \quad [a,t] & \text{if} \quad a \in C^\bullet C^\bullet(i_{n-5}), \\ [[a,1-t],0] \quad \text{or} \quad [a,t,0] & \text{if} \quad a \in C^\bullet C^\bullet(i_{n-6}), \end{cases}$$

$$(k_{n+1} \circ i_{n-2})[a,t] = \text{pt},$$

$$(S^\bullet i_{n-1} \circ k_n)[x,t] = \text{pt},$$

$$(k_{n+1} \circ i_{n+2})[a,t] = \begin{cases} [x,t] \quad \text{or} \quad [x,1-t] & \text{if} \quad a \in C^\bullet C^\bullet(i_{n-4}), \\ [[x,t],0] \quad \text{or} \quad [[x,1-t],0] & \text{if} \quad a \in C^\bullet C^\bullet(i_{n-5}). \end{cases}$$

With $n \leq 4$, the formulae are only simplified. For example, when $n = 1$ the map $S^\bullet i_0 \circ k_1 = S^\bullet i \circ k_1$ and $k_2 \circ ki_3$ from $C^\bullet(i_1) = CX \cup CA$ (see Fig. 5.G.2). to

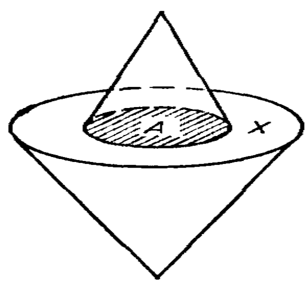


Figure 5.G.2:

$C^\bullet(i_{-1}) = S^\bullet X$ are given by formulae

$$(S^\bullet \circ k_1)[a,t] = [ia, 1-t] \quad [a,t] \in CA,$$
$$(k_2 \circ i_3)[a,t] = \text{pt} \quad [a,t] \in CA,$$
$$(S^\bullet i \circ k_1)[x,t] = \text{pt} \quad [x,t] \in CX,$$
$$(k_2 \circ i_3)[x,t] = [x,t] \quad [x,t] \in CX.$$

Putting

$$g_\tau[a,t] = [ia, 1 - \tau + \tau t], \quad [a,t] \in CA,$$
$$g_\tau[x,t] = [x, \tau t], \quad [x,t] \in CX,$$

we will obviously get a homotopy $g_\tau : C^\bullet(i_1) \to S^\bullet X$, connecting $S^{bullet} i \circ k$ with the map $k_2 \circ i_3$. For $n > 1$, the homotopy g_τ is constructed, but essentially the same way. □

It follows from this that, without violating the exactness, we can replace in sequence (5.109) the part starting with the term $C^\bullet(i_1)$ with the result of applying the $S^\bullet$ functor to sequence (5.109). Iterating this construction, we get an infinite right $S^\bullet$-periodic co-exact sequence

$$A \xrightarrow{i} X \xrightarrow{i_1} C^\bullet(i) \xrightarrow{k} S^1 A \to \cdots \to S^n A \xrightarrow{S^n i} S^n X \xrightarrow{S^n i_1} S^n C^\bullet(i) \xrightarrow{S^n k} S^{n+1} A \to \cdots, \tag{5.111}$$

where $k = k_1 \circ i_2$. This sequence is called the *Puppe coexact sequence* (or *co-resolvent*) of the map $i : A \to X$.

In the case when the map i is also a cofibration (an embedding corresponding to the cofibration (X, A)), the sequence (5.111) is homotopically isomorphic to the coexact sequence

$$A \xrightarrow{i} X \to jX/A \xrightarrow{k} S^1 A \to \cdots \to S^n A \xrightarrow{S^n j} S^n(X/A) \xrightarrow{S^n k} \cdots, \tag{5.112}$$

where $j : X \to X/A$ is the factorisation map. The sequence (5.112) is called the *Puppe coexact sequence of the cofibration* (X, A) (or the *cofibration* $i : A \to X$).

By definition, the sequence (5.112) is exact means that

Proposition 5.113. *for any pointed space Y there is an exact sequence*

$$\cdots \to [S^n(X/A), Y]^\bullet \to [S^n X, Y]^\bullet \to [S^n A, Y]^\bullet \to \cdots, \tag{5.114}$$

all members of which are abelian groups, except the last six:

$$\cdots \to [S(X/A), Y]^\bullet \to [SX, Y]^\bullet \to [SA, Y]^\bullet \to [X/A, Y]^\bullet \to [X, Y]^\bullet \to [A, Y]^\bullet,$$

of which the first three are groups, and the last three are pointed sets.

If the cofibration (X, A) is classified, i.e. $X = A \cup X_0$, where the space X_0 is contractible, and the pair (X_0, A_0), $A_0 = X_0 \cap A$, is a cofibration, then the Puppe sequence of ccofibration can be extended by one term to the left while preserving the exactness:

$$A_0 \xrightarrow{\varphi} A \xrightarrow{i} X \to \cdots$$

where $\varphi : A_0 \to A$ is an embedding (the same will remain true if the condition for the contractibility of the space X_0 is replaced by a weaker condition such that the embedding $i \circ \varphi_0 : A_0 \to X$ was null homotopic).

In the theory of functors dual to the functors of homotopic groups (we will deal with these functors in the next semester), Puppe coexact sequences play the same role as exact Puppe sequences (more precisely, their special cases are homotopy sequences of fibrations) in the theory of homotopy groups.

Lecture 6

In this lecture, using the identification of coverings with fibrations having discrete fibres, we will state the basic properties of covering spaces and on this basis we will obtain practical methods of calculation of fundamental groups. “Direct” methods that are not related to covers will be described in the Appendix.

6.1 The lifting problem for coverings

A remarkable property of coverings (determining success their application in concrete calculations) consists in the fact that it is possible for them (under very broad general methodological assumptions) to specify a necessary and sufficient condition for the solubility of an arbitrary lifting problem

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow p \\ Y & \xrightarrow[f]{} & X \end{array} \tag{6.1}$$

We will consider this problem in the category of $\mathscr{T}\!\mathscr{o}\mathscr{p}^\bullet$, i.e. assuming that p and the given map f are pointed maps and requiring that the desired map $\widetilde{f}$ is also pointed. By applying this assumption, we can, following the general method of algebraic topology described in Lecture 0, apply the functor π_1 to Diagramme (6.1). A necessary condition for the solubility of the corresponding algebraic problem

$$\begin{array}{ccc} & & \pi_1\widetilde{X} \\ & \overset{\varphi}{\nearrow} & \downarrow p_* \\ \pi_1 Y & \xrightarrow[f_*]{} & \pi_1 X \end{array}$$

is the inclusion of

$$\operatorname{im} f_* \subset \operatorname{im} p_*, \tag{6.2}$$

since if φ exists, then $\operatorname{im} f_* = \operatorname{im}(p_* \circ \varphi) = p_*(\operatorname{im} \varphi) \subset \operatorname{im} p_*$. Therefore, inclusion (6.2) is necessary for the solubility of problem (6.1).

The question of the sufficiency of this condition is no longer subject to algebraic methods and should be solved by direct geometric construction. At the same time, of course, we can consider the pointed space Y connected without loss of generality.

First of all, we note that

Proposition 6.3. *for a connected space Y, the covering map $\widetilde{f}$, when it exists, is unique.*

Proof. Indeed, let $\overline{f}$ be another covering map, and let B be a subset of the space Y consisting of all points $b \in Y$, for which $\overline{f}(b) = \widetilde{f}(b)$. This subset contains the base point and therefore is non empty. Let $b \in Y$ and $x = f(b)$. By the condition, there exists a neighbourhood U of the point x such that the set $p^{-1}U$ is a disjoint union of open sets $\widetilde{U}_\alpha$, each of which p maps homeomorphically onto U. If $b \in B$, then let $\widetilde{U}$ be one of these sets that contains the point $\overline{f}(b) = \widetilde{f}(b)$. Then the set $\overline{f}^{-1}(\widetilde{U}) \cap \widetilde{f}^{-1}(\widetilde{U})$ is open in Y, contains the point b and is contained in B. Therefore, the set B is open in Y. If $b \notin B$ and, therefore, $\overline{f}(b) \neq \widetilde{f}(b)$, then the points $\overline{f}(b)$ and $\widetilde{f}(b)$ belong to two different sets $\widetilde{U}_\alpha$, say, $\widetilde{U}_1$ and $\widetilde{U}_2$. Then the set $\overline{f}^{-1}(\widetilde{U}_1) \cap \widetilde{f}^{-1}(\widetilde{U}_2)$ is open in Y, contains the point b and is contained in $Y \setminus B$. Hence, the set B is closed. Being a non-empty, open and closed subset of a connected space Y, it coincides with all Y. Therefore $\overline{f} = \widetilde{f}$. □

It follows that

Proposition 6.4. *the maps $\widetilde{f}, \widetilde{g} : Y \to X$ are homotopic if the maps $f = p \circ \widetilde{f}$, $g = p \circ \widetilde{g}$ are homotopic.*

Proof. Indeed, according to the axiom **CP**, the homotopy G connecting the maps f and g can be covered by the homotopy $\widetilde{G}$ connecting the map $\widetilde{f}$ with some map, covering the map g and therefore coinciding with the map $\widetilde{g}$. □

At the same time, if the homotopy G is stationary on some subspace $B \subset Y$, then for any point $b \in B$, the path $t \mapsto G(b, t)$ will lie in the fibre above the point b and therefore, due to the discreteness of this fibre, it will be a constant map, i.e. the homotopy $\overline{G}$ will also be fixed on B. Therefore, in particular, the maps $\widetilde{f}$ and $\widetilde{g}$ will coincide on B. Thus,

Proposition 6.5. *in order to prove that the maps $\widetilde{f}, \widetilde{g} : Y \to \widetilde{X}$ coinciding on the subspace $B \subset Y$, it is sufficient to prove that the maps $p \circ \widetilde{f}, p \circ \widetilde{g} : Y \to X$ coincide on B and are homotopic relative to B.*

This way of proving a match on B of maps $Y \to \widetilde{X}$ is often useful.

Recall now (see [9]) that the space Y is called *locally (linearly) connected* if any component of each of its open sets is an open set (or, equivalently, each neighbourhood of an arbitrary point contains a connected neighbourhood).

Lemma 6.6. *For any connected and locally connected pointed space Y, the Serre fibration*

$$\omega_1 : PY \to Y, \quad u \mapsto u(1),$$

is an epiomorphism.

Proof. Since the space Y is connected, the map ω_1 is surjective. Therefore, we only need to prove that every set $V \subset Y$ whose complete preimage $\omega_1^{-1}V$ is open in PY is open in Y. But by the definition of topology in PY, the openness of the set $\omega_1^{-1}V$ means that it is a union of finite intersections of sets of the form $\mathscr{W}(K,U) \cap PY$, where, recall, K is a compact (=closed) subset of the segment I, U is an open subset of the space Y, and $\mathscr{W}(K,U)$ is the set of all paths $u : I \to Y$ having the property that $u(K) \subset U$. Therefore, for any path $u_0 \in \omega^{-1}V$, there are compact subsets $K_1, \dots, K_n \subset I$ and open subsets $U_1, \dots, U_n \subset Y$ such that $u_0 \in W_0 \subset \omega^{-1}V$ where

$$W_0 = \mathscr{W}(K_1, U_1) \cap \cdots \cap \mathscr{W}(K_n, U_n) \cap PY.$$

At the same time, renumbering, if necessary, the sets $K_1, \dots, K_n$ we can assume that for some $m = 0, 1, \dots, n$ (the case of $n = 0$ and $m = n$ are not excluded)

$$1 \in K_1 \cap \cdots \cap K_m \quad \text{and} \quad 1 \notin K_{m+1} \cup \cdots \cup K_n.$$

Then $\omega_1(u_0) = u_0(1) \in U_1 \cap \cdots \cap U_m$, and therefore in Y there exists a connected neighbourhood V_0 of the point $y_0 = \omega_1(u_0)$ such that $V_0 = U_1 \cap \cdots \cap U_m$.

Since the neighbourhood V_0 is open (and the set $K = K_{m+1} \cup \cdots \cup K_n$ is closed), there exists a point $t_0 < 1$ of the segment I such that $u_0([t_0, 1]) \subset V_0$, and since the neighbourhood V_0 is connected, for any point $y \in V_0$, there exists a path v lying entirely in V_0 and connecting the points $u_0(t_0)$ and y (see Fig. 6.1.1). Then the path $u = u_0'v$, where $u0'$ is the path $t \mapsto u_0(t_0 t)$, $t \in I$,

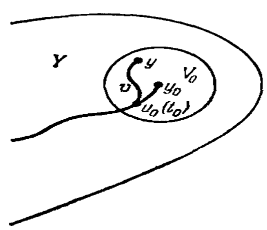


Figure 6.1.1:

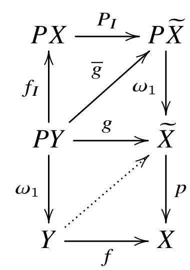
will have the property that $u(K_i) \subset U_i$ for any $i = 1, \dots, n$ (for $i > m$ this is obvious, because $u(K_i) = u_0(K_i) >$, and for $i \leq m$ follows from the fact that $K_i = (L_i \cap [0, t_0]) \cup (K_i \cap [t_0, 1J)$, and both $u(K_i \cap [0, t_0] \subset u_0(K_i) \subset U_i$, and $u(K_i \cap [t_0, 1] \subset v(Y) \subset W_0 \subset U_i$, i.e., will lie in $W_0 \subset \omega_1^{-1}V$. Therefore, $y = \omega_1(u) \in V$. This proves that $V_0 \subset V$, i.e. that the point $y_0 = \omega_1(u) \in V$ is an internal point of the set V. Since u_0 is an arbitrary path from $\omega_1^{-1}V$ and, therefore, y_0 is an arbitrary point from V, Lemma 6.6 is thus fully proved. □

Now we have everything ready to prove the following main theorem.

Theorem 6.7. *If the space Y is connected and locally connected, then the map $\widetilde{f}$ in Diagramme (6.1) exists if and only if the inclusion (6.2) takes place.*

Proof. Since the necessity of this condition has already been proved above, we only need to prove its sufficiency.

To this end, we note that since the space PY is contractible, the map $f \circ \omega_1 : PY \to X$ is homotopic to the constant map (the corresponding homotopy is obtained by combining with the map $f \circ \omega_1$ of the deformation $u \mapsto u_t$, $0 \le t \le 1$, the space PY, where $u_t : \tau \mapsto u(t\tau)$, $0 \le \tau \le 1$), and therefore there is a covering map for it $g : PY \to \widetilde{X}$. For the same reasons (applied to the Serre fibration $\omega_1 : P\widetilde{X} \to \widetilde{X}$) for the map g, there is also a covering map $\overline{g} : PY \to P\widetilde{X}$, explicitly given by the formula $\overline{g}(u) : t \mapsto g(u_t)$, $u \in PY$, $0 \le t \le 1$ and therefore satisfying the relation $p_I \circ \overline{g} = f_I$, where $f_I : PY \to PX$ and $p_I : P\widetilde{X} \to PX$ are maps $u \mapsto f \circ u$, $u \in PY$ and $a \mapsto p \circ a$, $a \in P\widetilde{X}$ (for $(p_I \circ \overline{g})(u)(t) = p(g(u)(t)) = p(g(u_t)) = f(\omega_1(u_t)) = f(u(t)) = f_I(t)$ for any path $u \in PY$). Thus, we have a commutative diagramme



It turns out that

Proposition 6.8. *the map $\overline{g}$ preserves fibres of fibrations $\omega_1 : PY \to Y$ and $\omega_1 : P\widetilde{X} \to \widetilde{X}$, i.e. for any paths $u, v \in PY$ satisfying the relation $\omega_1 u = \omega_1 v$, i.e. the relation $u(1) = v(1)$, the paths $\overline{g}u, \overline{g}v \in P\widetilde{X}$, satisfy the relation $\omega_1(\overline{g}u) = \omega_1(\overline{g}v)$, i.e. the relation $(\overline{g}u)(1) = (\overline{g}v)(1)$.*

Proof. Indeed, if $u(1) = v(1)$, then the loop $uv^{-1} \in \Omega Y$ is defined, and hence the loop $f_I(uv^{-1}) \in \Omega X$, whose homotopy class belongs to the subgroup $\operatorname{im} f_*$ of the group $\pi_1 X$, and therefore, by applying condition (6.2), to the subgroup $\operatorname{im} p_*$. This means that in the space X there is a loop $a \in \Omega\widetilde{X}$, such that the loop $p_I a \in \Omega X$ is homotopic $\operatorname{rel}\{0,1\}$ to the loop $f_I(uv^{-1}) = f_I u \cdot (f_I v)^{-1} = p_I \overline{g} u \cdot (p_I \overline{g} v)^{-1}$, and therefore the path $u' = p_I \overline{g} u = p \circ \overline{g} u$ is homotopic to the path $v' = p_I a \cdot p_I \overline{g} v = p_I(a \cdot \overline{g} v) = p \circ (a \cdot \overline{g} v)$. Hence, the paths $\overline{g}u$ and $a \cdot \overline{g}v$ coincide on $\{0,1\}$. In particular, $(\overline{g}u)(1) = (a \cdot \overline{g}v)(1)(\overline{g}v)(1)$. □

It follows that the formula

$$\widetilde{f}(y) = (\overline{g}u)(1), \quad \text{if} \quad y = u(1), \quad \text{where} \quad y \in Y, u \in PY,$$

well defines the map $\widetilde{f} : Y \to \widetilde{X}$, for which $\widetilde{f} \circ \omega_1 = \omega_1 \circ \overline{g} = g$, and therefore $p \circ \widetilde{f} \circ \omega_1 = f \circ \omega_1$. Since the map ω_1 is surjective, it follows that $p \circ \widetilde{f} = f$, i.e. that the map $\widetilde{f}$ closes Diagramme (6.1).

Thus, Theorem 6.7 is fully proved. □

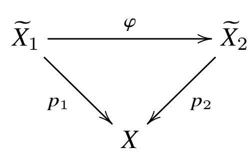
Corollary 6.9. *If the pointed space Y is connected, locally connected, and simply connected, then for each pointed covering $p : \widetilde{X} \to X$, any potined map $f : Y \to$ $-X$ is uniquely covered by some pointed map $\widetilde{f} : Y \to \widetilde{X}$.*

6.2 Coverings and subgroups of the fundamental group

The subgroup $\operatorname{im} p_*$ appearing in Theorem 6.7, of the group $\pi_1 X$ consists of homotopy classes of those loops of the space X that are covered by loops of the space $\widetilde{X}$. Since, according to the remarks made above, the homotopy of the loop space $\widetilde{X}$ is equivalent to the homotopy of their projections in X and, therefore, the homomorphism $p_* : \pi_1\widetilde{X} \to \pi_1 X$ is a monomorphism (which, however, also directly follows - due to the exactness of the homotopy sequence of the covering $p : \widetilde{X} \to X$ - from the discreteness of its fibre), it follows that

Proposition 6.10. *the subgroup $\operatorname{im} p_*$ of the group $\pi_1 X$ is isomorphic to the group $\pi_1\widetilde{X}$.*

The pointed covers of $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of a given pointed space X constitute the category $\mathscr{C}ov^\bullet X$ (which is a complete subcategory of the category $\mathscr{T}op^\bullet_X$), whose morphisms are their maps over X, i.e. pointed maps $\varphi : \widetilde{X}_1 \to \widetilde{X}_2$, such that the diagramme



is commutative.

Proposition 6.11. *If the space X is locally connected, then for covers $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ the morphism $\varphi : \widetilde{X}_1 \to \widetilde{X}_2$ exists if and only if*

$$\operatorname{im} p_{1*} \subset \operatorname{im} p_{2*}.$$

Proof. The morphism φ is a lifting of the map p_1 with respect to the map p_2. Therefore Proposition 6.11 is a special case of Theorem 6.7. (Note that the space $\widetilde{X}_1$ is locally connected if and only if the space X is locally connected.) □

By applying the Corollary 6.9 of Theorem 6.7, if the morphism φ there then it is only one. Hence, and from proposition 6.11, the following corollary immediately follows.

Corollary 6.12. *The pointed covers $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ of a connected and locally connected space X are isomorphic if and only if when*

$$\operatorname{im} p_{1*} = \operatorname{im} p_{2*}.$$

In the case where the morphism φ exists, we will write $p_1 \geq p_2$. The relation $\geq$ is an order relation on classes of isomorphic covers, and for a locally connected space X the correspondence

$$p \mapsto \operatorname{im} p_*$$

is an anti-isomorphic embedding of the set of all classes of isomorphic covers of this space into the set of all subgroups of the group $\pi_1 X$ ordered by inclusion. In this sense, the subgroups $\operatorname{im} p_*$ of the group $\pi_1 X$ classify the coverings $p : \widetilde{X} \to X$.

It is clear that for any element $\xi \in \pi_1 X$ and any point $x \in F$ the end ξx of a path starting at point x and covering an arbitrary loop of class ξ^{-1} depends only on ξ and x and that thereby we get some action of groups $\pi_1 X$ on the fibre F covering $p : \widetilde{X} \to X$. It is also clear that

Proposition 6.13. *this action is transitive (the orbit of any point is the entire fibre F) and that the isotropy subgroup of the point $\widetilde{x}_0 \in F$ is the subgroup* $\operatorname{im} p_*$

In particular, it follows that

Proposition 6.14. *the correspondence $\xi \mapsto \xi\widetilde{x}_0$ defines a bijective map of the coset*

$$\operatorname{coker} p_* = \pi_1 X / \operatorname{im} p_*$$

over the fibre F.

The covering $p : \widetilde{X} \to X$ is called *finite-leaved* if its fibres F are finite, and in this case the cardinality of the fibres is called the *number of covering sheets* $p : \widetilde{X} \to X$.

From the existence of a bijective map $\operatorname{coker} p_* \to F$ it follows directly that

Proposition 6.15. *the cover $p : \widetilde{X} \to X$ is finite if and only if the index* $\operatorname{card}(\operatorname{coker} p_*)$ *of the subgroup* $\operatorname{im} p_*$, *in the group $\pi_1 X$ is finite, and in this case the number of sheets of the cover $p : \widetilde{X} \to X$ is equal to this index.*

The statement that the number of sheets of the cover $p : \widetilde{X} \to X$ is equal to 1 is equivalent to the statement that this cover is bijective and hence homeomorphic map $\widetilde{X} \to X$. Calling the homeomorphisms $\widetilde{X} \to X$ *trivial covers*, we get, therefore, that

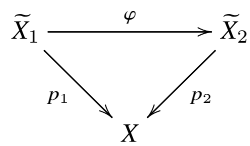
Proposition 6.16. *the cover $p : \widetilde{X} \to X$ is trivial if and only if when* $\operatorname{im} p_* = \pi_1 X$, *i.e. when the map $p_* : \pi_1 \widetilde{X} \to \pi_1 X$ is an isomorphism.*

This seems to be the only case where a simple homotopy condition ensures the map is homeomorphic.

6.3 Automorphisms of coverings

A homeomorphic (generally speaking, not preserving the base point!) map $\varphi : \widetilde{X} \to \widetilde{X}$ is called an *automorphism* (or *transformation*) of the covering of

$p : \widetilde{X} \to X$ if it is an isomorphism on itself as an object of the category $\mathscr{C}ov X$, i.e. if the diagramme



is commutative. Each such automorphism can be considered as a lifting of the map p, and therefore, by applying the uniqueness theorem of lifts, two automorphisms coincide if they act identically on the base point $x_0 \in X$ (recall that the space $\widetilde{X}$ is, by definition, connected). Hence,

Proposition 6.17. *the formula $\varphi \mapsto \varphi(\widetilde{x}_0)$ defines an injective map of the group* $\operatorname{Aut}\widetilde{X} = \operatorname{Aut}(p : \widetilde{X} \to X)$ *of the automorphisms of the covering $p : \widetilde{X} \to X$ into its fibre $F = p^{-1}(x_0)$.*

However, it is more convenient to consider the map $\varphi \mapsto \varphi^{-1}(\widetilde{X}_0)$, which is a composition of the bijective map $\varphi \mapsto \varphi^{-1}$ and the map $\varphi \mapsto \varphi(\widetilde{X}_0)$.

A cover $p : \widetilde{X} \to X$ is called *transitive* if this injective map is bijective, i.e. if the group $\operatorname{Aut}\widetilde{X}$ acts transitively on the fibre F.

Composing the map $\varphi \mapsto \varphi^{-1}(x_0)$ with the bijective map $F \to \operatorname{coker} p_*$, the inverse of the map constructed above $\operatorname{coker} p_* \to F$, we get some injective map

$$\alpha : \operatorname{Aut}\widetilde{X} \to \operatorname{coker} p_*.$$

To the automorphism φ, this map maps a residue class by the subgroup $\operatorname{im} p_*$, an element of the group $\pi_1 X$, defined by the loop of the space X, which is an image when map p of the path of the space X connecting the point x_0 with the point $f(x_0)$.

It is clear that

Proposition 6.18. *the cover $p : \widetilde{X} \to X$ is transitive if and only if when the map α is bijective.*

Let $N_{\pi_1 X}(\operatorname{im} p_*)$ be the normaliser of the subgroup $\operatorname{im} p_*$, i.e. the largest subgroup of the group $\pi_1 X$ in which the subgroup $\operatorname{im} p_*$ is normal. Then the coset group defined by

$$\operatorname{Weyl}(\widetilde{X}) = N_{\pi_1 X}(\operatorname{im} p_*)/\operatorname{im} p_*,$$

is called the *Weyl group* of the covering $p : \widetilde{X} \to X$. As a set, it is a subset of the set $\operatorname{coker} p_*$.

Proposition 6.19. *The map α is a monomorphic map of the automorphism group* $\operatorname{Aut}\widetilde{X}$ *to the Weyl group*

$$\alpha : \operatorname{Aut}\widetilde{X} \to \operatorname{Weyl}(\widetilde{X}). \tag{6.20}$$

Proof. The statement that α maps the group $\operatorname{Aut}\widetilde{X}$ to the group $\operatorname{Weyl}(\widetilde{X})$ is equivalent to the statement that for any automorphism $\varphi \in \operatorname{Aut}\widetilde{X}$ and any path $\widetilde{u}$ connecting in $\widetilde{X}$ the point x_0 with the point $\varphi(\widetilde{x}_0)$, the homotopy class $[p \circ \widetilde{u}]$ of the loop $u = p \circ \widetilde{u} \in \Omega X$ lies in the normaliser $N_{\pi_1(X)}(\operatorname{im} p_*)$ of the subgroup $\operatorname{im} p_*$, i.e. that for any loop $\widetilde{v} \in \Omega(\widetilde{X}, x_0)$ the loop $uvu^{-1} \in \Omega X$, where $v = p \circ \widetilde{v}$, also has the form $p \circ \widetilde{w}$, where $\widetilde{w} \in \Omega(\widetilde{X}, \widetilde{x}_0)$. But $uvu^{-1} = p \circ \widetilde{w}_1$, where $\widetilde{w}_1 = \widetilde{u}\widetilde{v}\widetilde{u}^{-1}$ - a loop at the point $\varphi(\widetilde{x}_0)$. Hence, $uvu^{-1} = (p \circ \varphi) \circ (\varphi^{-1} \circ \widetilde{w}_1) = p \circ \widetilde{w}$ where $\widetilde{w} = \varphi^{-1} \circ \widetilde{w}_1$ is a loop at the point $\widetilde{x}_0$, which is what is required.

Suppose now $\varphi, \psi \in \operatorname{Aut}\widetilde{X}$, and let $\widetilde{u}, \widetilde{v}$ be paths in $\widetilde{X}$, connecting the point $\widetilde{x}_0$ with the points $\varphi(\widetilde{x}_0)$ and $\psi(\widetilde{x}_0)$, respectively. Then let $\widetilde{u}\widetilde{w}$, where $\widetilde{w} = \varphi \circ \widetilde{v}$, connect the point $\widetilde{x}_0$ with the point $(\varphi \circ \psi)(\widetilde{x}_0)$ and there will be the following equality

$$p \circ \widetilde{u}\widetilde{w} = (p \circ \widetilde{u})(p \circ \widetilde{w}) = (p \circ \widetilde{u})(p \circ \widetilde{v}).$$

Therefore $\alpha(\varphi \circ \psi) = \alpha(\varphi)\alpha(\psi)$, i.e. the map α is a homomorphism (and therefore - by applying injectivity - *a fortiori* a monomorphism). □

A covering $p : \widetilde{X} \to X$ is called a *regular covering* (or *Galois covering*) if the subgroup $\operatorname{im} p_*$ is a normal subgroup of the group $\pi_1 X$, i.e. if $\operatorname{Weyl}(\widetilde{X}) = \operatorname{coker} p_*$.

Corollary 6.21. *Any transitive covering of $p : \widetilde{X} \to X$ is regular.*

Proof. For a transitive cover $\alpha(\operatorname{Aut}\widetilde{X}) = \operatorname{coker} p_*$, and, according to Proposition 6.19, $\alpha(\operatorname{Aut}\widetilde{X}) \subset \operatorname{Weyl}\widetilde{X}$. At the same time $\operatorname{Weyl}\widetilde{X} \subset \operatorname{coker} p_*$. Therefore, $\operatorname{coker} p_* = \operatorname{Weyl}(\widetilde{X})$. □

Proposition 6.22. *If the space X is locally connected, then, conversely, any regular cover is transitive.*

Proof. It is enough, obviously, to show that *if the space X is locally connected, then the monomorphism (6.20) is an isomorphism*, i.e. that for any loop $u \in \Omega X$ of the space X, the homotopy class $[u]$ which belongs to the normaliser $N_{\pi_X}(\operatorname{im} p_*)$ as the subgroup $\operatorname{im} p_*$ there is an automorphism $\varphi \in \operatorname{Aut}\widetilde{X}$ such that the point $\varphi(\widetilde{x}_0)$ is the end of the path $\widetilde{u} : I \to \widetilde{X}$ covering the loop u and starting at the point $\widetilde{x}_0$. But since in the commutative diagramme

$$\begin{array}{ccc} \pi_1(\widetilde{X}, \widetilde{x}_1) & \xrightarrow{\widetilde{u}^{\#}} & \pi_1(\widetilde{X}, \widetilde{x}_0) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \pi_1(X, x_0) & \xrightarrow[u^{\#}]{} & \pi_1(X, x_0) \end{array}$$

where $\widetilde{x}_1 = \widetilde{u}_1$, the horizontal arrows are isomorphisms, then

$$p_*\pi_1(\widetilde{X}, \widetilde{x}_0) = (p_* \circ \widetilde{u}^{\#})\pi_1(\widetilde{X}, \widetilde{x}_1) = (u^{\#} \circ p_*)\pi_1(\widetilde{X}, \widetilde{x}_1) = p_*\pi_1(\widetilde{X}, \widetilde{x}_1),$$

and therefore, according to Corollary 6.12 of Proposition 6.11, there is an isomorphism φ of the covering $(\widetilde{X}, \widetilde{X}_0) \to (X, x_0)$ to the covering $(\widetilde{X}, \widetilde{X}_1) \to (X, x_0)$. This isomorphism will be an automorphism $\varphi \in \operatorname{Aut}\widetilde{X}$, for which $\varphi(\widetilde{x}_0) = u(1)$. □

6.4 Completely discontinuous actions of groups

The action of the group G on the topological space X is called *completely discontinuous* if for each point $x \in X$ there exists a neighbourhood U of it such that $g_1U \cap g_2U = \varnothing$ for any two distinct elements $g_1, g_2 \in G$, i.e., which is obviously equivalent the statement if $U \cap gU = \varnothing$ for any element $g \neq e$ of the group G. It is clear that

Proposition 6.23. *all orbits of G_X of a completely discontinuous action are discrete and it is an action without fixed points (i.e. for $g \neq e$ each transformation $x \mapsto gx$ has no fixed points).*

It is also clear that

Proposition 6.24. *every finite group acting without fixed points on a Hausdorff space acts completely discontinuously.*

Proposition 6.25. *For any cover of $p : \widetilde{X} \to X$, the group $\operatorname{Aut}\widetilde{X}$ acts completely discontinuously on the space $\widetilde{X}$. If the cover $p : \widetilde{X} \to X$ is transitive, then the space $\widetilde{X}/\operatorname{Aut}\widetilde{X}$ of orbits of this group is homeomorphic to the space X.*

Conversely, for any connected space $\widetilde{X}$ on which some group G acts completely discontinuously, the natural map $p : \widetilde{X} \to \widetilde{X}/G$ is a transitive cover, the group $\operatorname{Aut}\widetilde{X}$ whose automorphisms are naturally isomorphic to the group G.

In short, transitive covering spaces are exactly spaces in which a certain group acts quite discontinuously.

Proof. Let $\widetilde{x} \in \widetilde{X}$, and let U be a neighbourhood of the point $x = p(\widetilde{x})$ such that the set $p^{-1}(U)$ is a disjoint union of open sets $\widetilde{U}_i$, each of which covers p homeomorphically maps on U. Let, in addition, $\widetilde{U}_0$ be one of the sets $\widetilde{U}_i$ which contains the point $\widetilde{x}$. It is clear that an arbitrary automorphism $\varphi \in \operatorname{Aut}\widetilde{X}$ translates the neighbourhood $\widetilde{U}_0$ of the point $\widetilde{x}$ into that of the sets $\widetilde{U}_i$ which contains the point $\varphi(\widetilde{x})$. Therefore, if $\varphi \neq \mathrm{id}$, then $\widetilde{U}_0 \cap \varphi\widetilde{U}_0 = \varnothing$. This proves the first assertion of proposition 6.25.

Since the orbits of the group $\operatorname{Aut} X$ lie in the layers of the covering $p : \widetilde{X} \to X$, this covering induces a continuous map $\widetilde{X}/\operatorname{Aut}\widetilde{X} \xrightarrow{\overline{p}} X$, which is, as it is not difficult to show, an epiomorphism. If the fibration p is transitive, then the map $\overline{p}$ is obviously injective and, therefore, represents a homeomorphism. This proves the second statement.

Now let $\widetilde{X}$ be a connected space on which the group G acts completely discontinuously, and let $p : \widetilde{X} \to \widetilde{X}/G$ be a natural epiomorphism of $\widetilde{x} \mapsto G\widetilde{x}$. Let $x \in X = \widetilde{X}/G$, $\widetilde{x} \in p^{-1}(x)$, and let $U = p(\widetilde{U})$, where $\widetilde{U}$ is a neighbourhood of the point $\widetilde{x}$, such that $\widetilde{U} \cap g\widetilde{U} = \varnothing$ for any element $g \neq e$ of the group G. It is clear that $p^{-1}(U)$ is a disjoint union of all possible open sets of the form $g\widetilde{U}$, $g \in G$, and on each of these sets the map p is epiomorphic. In addition, if $p(g\widetilde{x}_1) = p(g\widetilde{x}_2)$, i.e. if there exists an element $h \in G$ such that $g\widetilde{U} \cap hg\widetilde{U} = \varnothing$, then $g = hg$, i.e. $h = e$, and therefore $g\widetilde{x}_1 = g\widetilde{x}_2$. Hence, the map p on $g\widetilde{U}$ is injective and therefore, being an epimorphism, is a homeomorphism. Thus, the map $p : \widetilde{X} \to X$ is a covering.

Finally, since $p(g\widetilde{x}) = p(\widetilde{x})$ for any point $\widetilde{x} \in \widetilde{X}$ and any element $g \in G$, then the group G is contained in the group $\operatorname{Aut}\widetilde{X}$ (we identify the elements of the group G with the transformations they produce). Since the group G acts transitively on each fibre (which is - the reader should recall - an orbit), for any automorphism $\varphi \in \operatorname{Aut}\widetilde{X}$ and any point $\widetilde{x}_0 \in \widetilde{X}$ there is an element g of a group G such that $\varphi(\widetilde{x}_0) = gx_0$. Since the map $\varphi \mapsto \varphi(x_0)$, as we know, is injective, it follows that $\varphi = g$. Hence, $G = \operatorname{Aut}\widetilde{X}$. □

Corollary 6.26. *If the space X is the space of orbits $\widetilde{X}/G$ of some group G, which is completely equivalent on a simply connected space $\widetilde{X}$, then*

$$\pi_1 X \approx G. \tag{6.27}$$

Proof. Consider the transitive covering of $p : \widetilde{X} \to X = \widetilde{X}/G$. By applying the simple connectedness of the space $\widetilde{X}$ for this cover, there is an equality $\operatorname{im} p_* = 0$, i.e. the equality $\operatorname{coker} p_* = \pi_1 X$. Therefore $G = \operatorname{Aut}\widetilde{X} \approx \operatorname{coker} p_* = \pi_1 X$. □

This corollary provides us with a powerful way to compute fundamental groups.

Example 6.28. The formula

$$\alpha x = x + 1, \quad x \in \mathbb{R},$$

where α is the generator of an infinite cyclic group F_1 defines a completely discontinuous action of this group on the space $\mathbb{R}$. The corresponding orbits are nothing more than adjacent classes of the additive group $\mathbb{R}$ by the subgroup $\mathbb{Z}$ of integers (isomorphic to the group F_1) and, therefore, the space of orbits $\mathbb{R}/F_1$ by the coset group $\mathbb{R}/\mathbb{Z}$, i.e. the circle $\mathbb{S}^1$. Since the space $\mathbb{R}$, being contractible, is simply connected, it is proved that

$$\pi_1 \mathbb{S}^1 = F_1, \tag{6.29}$$

or, in additive notation, $\pi_1 \mathbb{S}^1 = \mathbb{Z}$.

It is easy to see that with the isomorphism (6.29) to the element $\iota = [id]^\bullet$, $\mathrm{id} : (\mathbb{S}^1, s_0) \to (\mathbb{S}^1, s_0)$ the group $\pi_1 \mathbb{S}^1$ corresponds to the generator α of the group F_1. This means that

Proposition 6.30. *the element ι_1 is the generator of the infinite cyclic group $\pi_1 \mathbb{S}^1$,*

i.e. - in additive notation - that any element of the group $\pi_1 \mathbb{S}^1$ is uniquely represented as $n\iota_1$ where $n \in \mathbb{Z}$.

Thus, each loop $(\mathbb{S}^1, s_0) \to \mathbb{S}^1, s_0)$ corresponds to some integer $n \in \mathbb{Z}$, depending only on the homotopy class of this loop. Visually, it is nothing more than the number of revolutions of this loop. This number can be analytically expressed by a known integral and on this basis obtain a direct proof of equality (6.29). (See Lecture 7 below.)

Remark 6.31. It is important to keep in mind that

Proposition 6.32. *the isomorphism (6.27) has the property of naturality,*

i.e. for any groups G and H acting completely discontinuously on simply connected spaces $\widetilde{X}$ and $\widetilde{Y}$, any homomorphism $\varphi : G \to H$ and any continuous map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ having the property that $\widetilde{f}(g\widetilde{x}) = \varphi(g)\widetilde{f}(\widetilde{x})$, $\widetilde{x} \in \widetilde{X}$, $g \in G$, and therefore inducing a continuous map $f : X \to Y$, $X = \widetilde{X}/G$, $Y = \widetilde{Y}/H$, a commutative diagramme takes place

$$\begin{array}{ccc} \pi_1 X & \xrightarrow{f_*} & \pi_1 Y \\ \approx\big\downarrow & & \big\downarrow\approx \\ G & \xrightarrow[\varphi]{} & H \end{array}$$

In other words, isomorphisms $G \approx \pi_1 X$ and $H \approx \pi_1 Y$ transform the homomorphism φ into the homomorphism f_*.

This remark allows us to compute not only fundamental groups, but also their homomorphisms induced by continuous maps.

Example 6.33. Let f be the *antipodal map* of $\mathbb{S}^1 \to \mathbb{S}^1$, given by the formula $f(\boldsymbol{x}) = -\boldsymbol{x}$. It is easy to see that when identifying $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}_1$, this map is induced by the map $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, acting according to the formula $\widetilde{f}(x) = x + 1/2$, $x \in \mathbb{R}$. Since $\alpha\widetilde{f}(x) = \widetilde{f}(\alpha x)$, where $\alpha : x \mapsto x + 1$, then the corresponding homomorphism $\varphi : F_1 \to F_1$ is an identity map. This proves that

Proposition 6.34. *the antipodal map* $\mathbb{S}^1 \to \mathbb{S}^1$ *induces the identical map* $\mathrm{id} : \pi_1\mathbb{S}^1 \to \pi_1\mathbb{S}^1$ *of the fundamental group of the circle* $\mathbb{S}^1$.

This example will be useful to us in lecture 13[1].

6.5 The fundamental group of a bouquet of circles

As a more complex example of applying the Corollary 6.26 from Proposition 6.25, we calculate the fundamental group $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ of the bouquet $\mathbb{S}^1 \vee \mathbb{S}^1$ of two circles (the figure "eight").

For each integer $r > 0$, we note on the circumference of the plane $\mathbb{R}^2$ of the radius r with the centre $(0, 0)$ the points that are the vertices of the regular $4 \cdot 3^{r-1}$-gon, so that among these points turned out to be the point $(r, 0)$. Let's call these points vertices of rank $r > 0$. We will consider the point $(0, 0)$ to be the vertex of the rank 0. We will connect the vertex $(0, 0)$ with rectilinear segments with all four vertices of rank 1 and each vertex of rank $r > 0$ with the three vertices of rank $r + 1$ closest to it (see Fig. 6.5.1).

[1] The transcriber guesses Postnikov refers to Lecture 3 of "Cellular Homotopy".

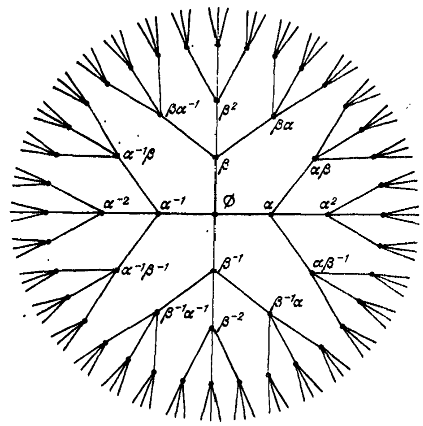


Figure 6.5.1:

We will denote the union of all constructed segments by T, and the intersection of T with a circle of radius r and the centre $(0,0)$ by T_r. Obviously, T_r is contracted to T_{r-1}, from which it follows directly by induction that T_r is contractible. Since any compact subset of the plane is contained in a circle of sufficiently large radius, an arbitrary map $\mathbb{S}^1 \to T$ is mapped into some space T_r and therefore - due to the contractibility of this space, it is homotopic to the constant map. This proves that

Proposition 6.35. *the space T is simply connected.*

(In fact, the space T is contractible; see Proposition 6.91 of the Appendix to this lecture.)

The vertices of the space T are conveniently described using the free group F_2 with two generators α and β. By definition, the elements of this group are *group words* of the form

$$\alpha^{a_1}\beta^{b_1}\cdots\alpha^{a_1}\beta^{b_1}$$

where is $a_1, b_1, \ldots, a_n, b_n \in \mathbb{Z}$, and it is assumed that $a_2, \ldots, a_n$ and $b_1, \ldots, b_{n-1}$ are nonzero. The multiplication of these words consists in their attribution to each other, followed, when required, by the reduction of "similar terms". (Because of this reduction, checking the associativity of multiplication is somewhat difficult; see the Appendix to this lecture.) The unit of the group F_2 is the empty word $\varnothing$.

To the vertex $(0,0)$ we will match the empty word $\varnothing$, and to the four vertices of rank 1 the words α, β, α^{-1}, β^{-1} (in any order). Then the construction continues by induction: if the vertices of the rank r have already been matched with the elements of the group F_2, the three vertices of the rank $r+1$, closest to the top of the rank r, which corresponds to the word σ, we match (in any order) those three of the four words $\sigma\alpha$, $\sigma\beta$, $\sigma\alpha^{-1}$, $\sigma\beta^{-1}$, in which the reduction of such terms is not feasible (for example, if the word σ ends with β in a positive degree, then we take the words $\sigma\alpha$, $\sigma\alpha^{-1}$, $\sigma\beta$; see Fig. 6.5.1). It is obvious that in this way a one-to-one correspondence is established between the vertices of the space T and the elements of the group F_2, and the vertex to which the element $\sigma \in F_2$ corresponds is connected by segments with the vertices corresponding to the elements $\sigma\alpha$, $\sigma\beta$, $\sigma\alpha^{-1}$, $\sigma\beta^{-1}$ and only with these vertices.

We also define the action of the group F_2 in T, assuming that the element $\pi \in F_2$ translates the vertex corresponding to the element $\sigma \in F_2$ into the vertex corresponding to the element $\tau\sigma$, and linearly maps each segment connecting two vertices is divided into a segment connecting their images. It is clear that this is quite a discontinuous action. *The fundamental domain* of this action, i.e. the set of representatives of its orbit, is the union of two segments connecting the point $(0,0)$ with the vertices $\overline{\alpha}$ and $\overline{\beta}$ corresponding to the words α and β, and the vertices themselves $\overline{\alpha}$ and $\overline{\beta}$, are equivalent to the point $(0,0)$. This means that the orbit space T/F_2 is a bouquet $\mathbb{S}^1 \vee \mathbb{S}^1$ of two circles obtained by identifying the points $\overline{\alpha}$ and $\overline{\beta}$ with the point $(0,0)$.

Since the space T, as we have seen, is simply connected, it is proved that

$$\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = F_2. \tag{6.36}$$

It is easy to see at the same time that in the isomorphism (6.36) the elements α and β correspond to the elements ι' and ι'' of the group $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$, which are homotopy classes of canonical embeddings

$$\iota' : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1, x \mapsto x_I, \quad \text{and} \quad \iota'' : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1, x \mapsto x_{II}.$$

Thus,

Proposition 6.37. *the group $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ is a free group with two generators ι' and ι''.*

Similarly, it can be proved that

Proposition 6.38. *the fundamental group of a bouquet of n circles is a free group with n generators,*

but we prefer to prove this by another method in Appendix to this lecture.

6.6 Uniqueness of multiplication in the group $\pi_1 X$

Based on the formula (6.36) (and on its analogue when $n = 3$) it is possible, in particular, to prove the statement made in Remark 4.12 of Lecture 4 (and thus complete the proof of categoricity of axioms for homotopy groups from the previous lecture).

Proposition 6.39. *The multiplication $([u],[v]) \mapsto [uv]$ and the inverse multiplication $([u],[v]) \mapsto [vu]$ are the only natural X multiplications in the set $\pi_1 X = [\mathbb{S}^1, X]^\bullet$.*

For any group G, each element $s(\alpha,\beta) = \alpha^{a_1}\beta^{b_1}\cdots\alpha^{a_n}\beta^{b_n}$ of the group F_2 defines the map $s : G\times G \to G$, translating the element $(g,h) \in G\times G$ and the element $s(g,h) = g^{a_1}h^{b_1}\cdots g^{a_n}h^{b_n}$ of the group G. For example, for $G = F_2$, the elements $s(\alpha,\varnothing) = \alpha^{a_1+\ldots+a_n}$ and $s(\varnothing,\beta) = \beta^{b_1+\ldots+b_n}$ and for $G = F_3$, where F_3 is a free group with three generators α, β, γ, the elements $s(s(\alpha,\beta),\gamma)$ and $s(\alpha,s(\beta,\gamma))$.

The key to Proposition 6.39 is the following combinatorial lemma:

Lemma 6.40. *If the word $s(\alpha,\beta) \in F_2$ satisfies the relations*

$$s(\alpha,\varnothing) = \alpha, \quad s(\varnothing,\beta) = \beta \tag{6.41}$$

and

$$s(s(\alpha,\beta),\gamma) = s(\alpha,s(\beta,\gamma)), \tag{6.42}$$

then either $s(\alpha,\beta) = \alpha\beta$ or $s(\alpha,\beta) = \beta\alpha$.

Proof. The condition $s(\alpha,\varnothing) = \alpha$ means that $a_1+\cdots+a_n = 1$, and the condition $(\varnothing,\beta) = \beta$ - that $b_1+\cdots+b_n = 1$. Therefore, $n \geq 1$, and if $n = 1$, then $a_1 = 1$, $B_1 = 1$ (i.e. $s(\alpha,\beta) = \alpha\beta$), and if $n = 2$ and $a_1 = 0$, then $a_2 = 1$, $b_1 = 1$ (i.e. $s(\alpha,\beta) = \beta\alpha$). Therefore, it is enough for us to prove that the case of $n > 2$ is impossible, and for $n = 2$, $a_1 = 0$ and $b_2 = 0$ are required.

Let first $a_1 > 0$ and $b_n \neq 0$ ($n \geq 2$). Then

$$s(\alpha\beta)^{a_1} = \underbrace{\alpha^{a_1}\beta^{b_1}\alpha^{a_2}\cdots\alpha^{a_n}\beta^{b_n}\cdots\alpha^{a_1}\beta^{b_1}\alpha^{a_2}\cdots\alpha^{a_n}\beta^{b_n}}_{a_1\text{ times}},$$

whence it follows that the word $s(s(\alpha,\beta),\gamma)$ has the form $\alpha^{a_1}\beta^{b_1}\alpha^{a_2}\cdots$. At the same time, the word $s(\alpha,s(\beta,\gamma))$ for $b_1 > 0$ has the form $\alpha^{a_1}\beta^{a_1}\gamma^{b_1}\cdots$ and for $b_1 < 0$ - the form $\alpha^{a_1}\gamma^{-b_n}\beta^{-a_n}\cdots$ and therefore obviously different from the word $s(s(\alpha,\beta),\gamma)$. Since this contradicts condition (6.42), the case $a_1 > 0$ and $b_n \neq 0$ is therefore impossible.

If $a_1 > 0$, but $b_n = 0$, then in the word $s(\alpha,\beta)^{a_1}$ it is necessary to bring similar terms, i.e. for $a_n + a_1 \neq 0$ replace $\alpha^{a_n}\beta^{b_n}\alpha^{a_1}$ by $\alpha^{a_n+a_1}$, and for $a_n + a_1 = 0$ and $b_{n-1}+b_1 \neq 0$ replace $\beta^{b_{n-1}}\alpha^{a_n}\beta^{b_n}\alpha^{a_1}\beta^{b_1}$ with $\beta^{b_{n-1}+b_1}$, etc. Since $a_1+\ldots b+a_n = 1$ and $B_1+\cdots+b_n = 1$, a complete reduction cannot occur in this case, which implies that the word $s(s(\alpha,\beta),\gamma)$ will still have the form $\alpha^{a_1}\beta^{b_1}\alpha^{a_2}\cdots$ (for $n = 2$ - the form is $\alpha^{a_1}\beta\alpha^{a_2}$ with $a_1 + a_2 = 1$), and therefore condition (6.42) cannot be fulfilled again.

Now let $a_1 = -a$, where $a > 0$. Then

$$s(\alpha\beta)^{a_1} = \underbrace{\beta^{-b_n}\alpha^{-a_n}\cdots\beta^{-b_2}\alpha^{-a_1}\cdots\beta^{-b_2}\alpha^{-a_1}\cdots\beta^{-b_n}\alpha^{-a_n}}_{a\text{ times}},$$

whence, when $b_n \neq 0$, it follows that the word $s(s(\alpha,\beta),\gamma)$ has the form $\beta^{b_n}\alpha^{-a_n}\cdots$ and therefore cannot match the word $s(\alpha,s(\beta,\gamma)) = \alpha^{a_1}\cdots$. If $b_n \neq 0$, then - again due to the impossibility of a complete reduction - the word $s(\alpha, s(\beta,\gamma))$ will have the form $\alpha^{-a_n}\beta^{-b_{n-1}}\alpha^{a_{n-1}}\cdots$ (with $n = 2$ - the form $\alpha^{-a_2}\beta^{-1}\alpha^{-a_1}$ with $a_1 + a_2 = 1$) and therefore again cannot coincide with the word $s(\alpha, s(\beta,\gamma))$ (having fort $a_1 > 0$ the form $\alpha^{a_1}\beta^{a_1}\gamma^{b_1}\cdots$, and with $a_1 < 0$ - the form $\alpha^{a_1}\beta^{a_n}\beta^{-b_{n-1}}\cdots$).

Finally, let $a_1 = 0$. If $n = 2$ (i.e. if $s(\alpha,\beta) = \beta^b\alpha\beta^{1-b}$, then $s(s(\alpha,\beta),\gamma) = \gamma^b\beta^b\alpha\beta^{1-b}\gamma^{1-b}$ at that time as $s(\alpha, s(\beta,\gamma)) = (\gamma^b\beta\gamma^{1-b})^b\alpha(\gamma^b\beta\gamma^{1-b})^{1-b}$, i.e., $s(\alpha, s(\beta,\gamma)) = \gamma^b\beta\cdots$, if $b > 0$, and $s(\alpha, s(\beta,\gamma)) = \gamma^{b-1}\beta^{-1}\cdots$, if $b < 0$. Therefore, equality (6.42) is possible only when $b = 0$ or 1, i.e. for $s(\alpha,\beta) = \alpha\beta$ or $\beta\alpha$.

If $n > 2$, then

$$s(s(\alpha,\beta),\gamma) = \begin{cases} \gamma^{b_1}\beta^{b_1}\alpha^{a_2}\cdots, & \text{if} \quad a_2 > 0, \\ \gamma^{b_1}\beta^{-b_n}\alpha^{-a_n}\cdots, & \text{if} \quad a_2 < 0, \end{cases}$$

$$s(\alpha, s(\beta),\gamma) = \begin{cases} \gamma^{b_1}\beta^{a_2}\gamma^{b_2}\cdots, & \text{if} \quad b_1 > 0, \\ \gamma^{b}_{n}\beta^{-a_n}\gamma^{-b_{n-1}}\cdots, & \text{if} \quad b_1 < 0, \end{cases}$$

and in all cases equality (6.42) is impossible. Thus Lemma 6.40 is fully proved. □

Proof. (of Proposition 6.39) Let $(\alpha,\beta) \mapsto a \circ \beta$ be an arbitrary natural X multiplication in $\pi_1 X$. Then for $X = \mathbb{S}^1 \vee \mathbb{S}^1$ is the free group $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ with the generators ι' and ι'', in particular, the word $s(\iota',\iota'') = \iota' \circ \iota''$ will be defined. But it is clear that for any space X and any elements $\alpha,\beta \in \pi_1 X$ there is a map $f : \mathbb{S}^1 \vee \mathbb{S}^1 \to X$ such that $f_*\iota' = \alpha$ and $f_*\iota'') = \beta$. Since $f_*s(\iota',\iota'' = s(f_*\iota', f_*\iota'')$ and due to the naturality of $f_*\iota' \circ f_*\iota'' = f_*(\iota',\iota'')$, then in the group $\pi_1 X$ the following equality holds

$$\alpha \circ \beta = s(\alpha,\beta). \tag{6.43}$$

Therefore, to prove Proposition 6.39, it is sufficient to prove that the word $s(\alpha,\beta)$ satisfies the conditions (6.41) and (6.42) of Lemma 6.40 (and therefore is equal to either $\alpha\beta$ or $\beta\alpha$).

With this in mind, we will first show that

Proposition 6.44. *for any pointed space X, the unit of the group $\pi_1 X$ with respect to the multiplication of $\circ$ is the class 1 of the constant map $\mathbb{S}^1 \to X$.*

Indeed, for any pointed map $f : Y \to X$ the map $f_* : \pi_1 Y \to \pi_1 X$, being by applying naturality a homomorphism with respect to the multiplication $\circ$, translates the unit of this multiplication into one. But with $Y = \text{pt}$, the set of $\pi_1 Y$ is a singleton, and therefore the image of $f_*(1)$ being its only element 1 with the map $f_* :_p i_1(\text{pt}) \to \pi_1 X$, induced by the map $f : \text{pt} \to X$, will be a multiplication unit of $\circ$. This proves everything, since $f_*(1) = 1$.

Hence, by applying formula (6.43), it immediately follows that $s(\alpha, 1) = \alpha$ and $s(1,\beta) = \beta$ for any elements $\alpha,\beta \in \pi_1 X$. For $X = \mathbb{S}^1 \vee \mathbb{S}^1$, this gives (6.41).

Finally, since the multiplication $\circ$ is associative, then for any elements α,β,γ there is an equality $s(s(\alpha,\beta),\gamma) = s(\alpha, s(\beta,\gamma))$, which for $X = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$ gives (6.42). □

6.7 Universal coverings

Let us now return to the coverings in order to round out their theory to a certain extent.

The covering $p_0 : \widetilde{X}_0 \to X$ is called *simply connected* if the space $\widetilde{X}_0$ is simply connected (and, therefore, $\operatorname{im} p_{0*} = 0$).

The space X is called *covered* if there is a simply connected covering for it.

The automorphism group $\operatorname{Aut} \widetilde{X}_0$ of a simply connected covering $p_0 : \widetilde{X}_0 \to X$ is naturally embedded in the fundamental group $\pi 1 X$ and coincides with this group:

$$\operatorname{Aut} \widetilde{X}_0 = \pi_1 X,$$

if the space X is locally connected.

For any subgroup $G \subset \operatorname{Aut} \widetilde{X}_0$, the space $\widetilde{X} = \widetilde{X}_0/G$ is defined, for which, firstly, $\pi_1 \widetilde{X} = G$ and, secondly, the covering $p_0 : X_0 \to X$ induces the map $p : \widetilde{X} \to X$, which is obviously a covering. This, in particular, proves that

Proposition 6.45. *if a connected space X is locally connected and covered, then for any subgroup G of the group $\pi_1 X$ there is a (unique up to isomorphism) covering $p : \widetilde{X} \to X$ such that* $\operatorname{im} p_* = G$.

Thus, for every connected locally connected and covered space X, the correspondence

$$p \mapsto \operatorname{im} p_*$$

is an anti-isomorphism of a partially ordered set of classes of isomorphic covers of the space X to a partially ordered by inclusion set of all subgroups of the group $\pi_1 X$.

In this correspondence (sometimes called *Galois correspondence*), normal subgroups correspond to regular (transitive) coverings, and the corresponding coset groups will be isomorphic to the automorphism groups of these coverings.

A covering $p_0 : \widetilde{X}_0 \to X$ of a space X is called *universal* if $p_0 \geq p$ for any covering $p : \widetilde{X} \to X$ of a space X.

Since the map constructed above $p_0 : \widetilde{X}_0 \to \widetilde{X}_0/G$ is obviously a map over X, we see that $p_0 \geq p$ for any covering $p : \widetilde{X} \to X$, i.e. that

Proposition 6.46. *a simply connected covering of a locally connected covered space X is universal.*

Remark 6.47. A non-closed locally connected space can have a universal covering (of course, it is not connected).

In [9], a topological space X is called simply connected if the identity $\mathrm{id} : X \to X$ is its universal covering. Thus, this concept is different from the one introduced here, but coincides with it for locally connected spaces.

A topological space X is called *semilocally simply connected* if there exists its open covering $\{U_a\}$ having the property that every map $\mathbb{S}^1 \to X$ whose image is contained in one of the elements of this cover is homotopic to a constant map.

It is easy to see that

Proposition 6.48. *any covered space X is semilocally simply connected.*

Proof. Indeed, if $p_0 : \widetilde{X}_0 \to X$ is a simply connected covering and U is an open set in X; over which the fibration p_0 is trivial, then any map $\mathbb{S}^1 \to U$ lifts to $\widetilde{X}_0$, and since $\widetilde{X}_0$ is simply connected, then the lifted map, and therefore the original map, is homotopic to the constant. □

At the same time, it can be shown (see, for example, [9], p. 180) that

Proposition 6.49. *if a connected semi-locally simply connected space is locally connected, then it is covered.*

Since this fact has no direct relation to homotopy theory, we will not prove it here.

6.8 Fundamental groups of topological groups and their coset spaces

In [9] it was also proved that if the topological the group G, considered as a topological space, is covered, and if $p : \widetilde{G} \to G$ is the corresponding simply connected pointed covering, then in G it is possible to introduce in the unique way a group structure with respect to which the covering p will be a homomorphism. The kernel $\ker p$ of this homomorphism is (see [9], p. 202) a discrete Abelian normal subgroup of the group $\widetilde{G}$. Its action by left shifts on $\widetilde{G}$ is quite discontinuous and the corresponding coset space is nothing other than the coset group $\widetilde{G}/\ker p_*$ homeomorphic to the group G. Therefore

$$\ker p_* = \pi_1 G.$$

This equality, like equality (6.27), is one of the most important tools for calculating fundamental groups. In [9] it is taken as the definition of the group $\pi_1 G$.

For any Lie group G and any of its closed subgroup H, the natural map

$$G \to G/H$$

is, as it is easy to see, a locally trivial fibration with H a fibre. (Note that any quotient of a Lie group is obviously paracompact.) Therefore, there is an exact sequence for it

$$\cdots \to \pi_2(G/H) \to \pi_1 H \to \pi_1 G \to \pi_1(G/H) \to \pi_0 H \to \cdots$$

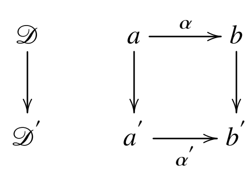
whence it follows that if the subgroup H is connected, then the group $\pi_1(G/H)$ is the quotient group of the group $\pi_1 G$, and if the quotient space G/H is simply connected, then the group $\pi_1 G$ is the quotient group of the group $\pi_1 H$ (see [9], Propositions 8 and 9 of Lecture 12).

Appendix

6.A Limits of diagrammes over an arbitrary category

A *diagramme scheme* is a set divided into two subsets and equipped with two maps of the second subset into the first. The elements of the first subset are called the *vertices* of the scheme, and the second - its *arrows*. The case of an empty set of arrows is not excluded. By the condition, two vertices a and B correspond to each arrow α. It is said that the arrow α is an arrow *from a to b* and write $\alpha : a \to b$.

A *morphism* of a diagramme scheme $\mathscr{D}$ into a diagramme scheme $\mathscr{D}'$ is an arbitrary map $\mathscr{D} \to \mathscr{D}'$ that translates vertices into vertices, arrows into arrows and such that if $a \mapsto a'$, $b \mapsto b'$, $\alpha \mapsto \alpha'$ and $\alpha : a \to b$, then $\alpha' : a' \to b'$.



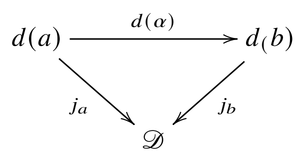
It is clear that each category is a diagramme scheme (we ignore here the differences between "small" and "large" categories). The morphism $\mathscr{D} \to \mathscr{A}$ of the diagramme scheme $\mathscr{D}$ into the category $\mathscr{A}$ (considered as a diagramme scheme) is called a *diagramme of the type $\mathscr{D}$ over the category $\mathscr{A}$.*

This formal definition is an explication of the intuitive concept of a diagramme, which has been quite enough for us so far.

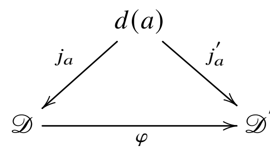
If the diagram scheme $\mathscr{D}$ does not contain arrows, then diagrams of the type $\mathscr{D}$ are nothing more than a family of objects of the category $\mathscr{A}$, whose indices are the vertices of the scheme $\mathscr{D}$.

A (straight) *cone* over the diagramme $d : \mathscr{D} \to \mathscr{A}$ with *vertex* $\mathscr{D}$ is a family of morphisms $j_a : d(a) \to \mathscr{D}$, $a \in \mathscr{D}$, and a category $\mathscr{A}$, such that for any arrow

$\alpha : a \to b$ from $\mathscr{D}$, the diagramme



is commutative ($j_a = j_b \circ d(\alpha)$). The *morphism* of the cone $j_a : d(a) \to \mathscr{D}$ into the cone $j_a^{'} : d(a) \to \mathscr{D}^{'}$ is a morphism $\varphi : \mathscr{D} \to \mathscr{D}^{'}$ of the category $\mathscr{A}$ such that for any vertex $a \in \mathscr{D}$ the diagramme

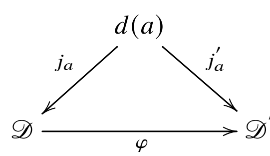


is commutative. All cones above the diagramme d and their morphisms form, obviously, a category. We will denote this category with the symbol $\underrightarrow{\mathrm{Cone}}(d)$.

In the case when the scheme $\mathscr{D}$ has the form $\cdot \leftarrow \cdot \rightarrow \cdot$, diagrammes of the type $\mathscr{D}$ are pairs of morphisms $i_A : C \to A$, $i_b : C \to B$ in the category $\mathscr{A}$, and we get cones over pairs (i_A, i_B) in the sense of Lecture 1.

In the case when the scheme $\mathscr{D}$ has no arrows and, therefore, diagrammes of the type $\mathscr{D}$ are families of $\{A_a, a \in \mathscr{D}\}$ then objects of the category $\mathscr{A}$, cones over $\{A_a, a \in \mathscr{D}\}$ are families of morphisms $j_a : A_a \to D$, and their morphisms are ordinary morphisms of families.

Definition 6.50. The *limit* of the *diagramme* $d : \mathscr{D} \to \mathscr{A}$ is the vertex D of the initial object of the category $\underrightarrow{\mathrm{Cone}}(d)$, i.e. a cone $\{j_a : d(a) \to D\}$ such that for any cone $\{j_a^{'} : d(a) \to D^{'}\} \in \underrightarrow{\mathrm{Cone}}(d)$ there is a single morphism $\varphi : D \to D^{'}$ from $\{j_a\}$ to $\{j_a^{'}\}$. It is also said that the object D is the *limit objects $d(a)$ with respect to morphisms j_a*.



It is clear that the limit of the diagramme (when it exists) is uniquely defined up to canonical isomorphism.

The limit of the diagram $A \leftarrow C \rightarrow B$ is its pushout, and the limit of the family $\{A_a\}$ is the coproduct (direct sum) $\sqcup_a A_a$ of objects A_a.

Proposition 6.51. *A limit exists for an arbitrary diagramme over each of the categories $\mathscr{E}ns$, $\mathscr{E}ns^{\bullet}$, $\mathscr{T}op$ and $\mathscr{T}op^{\bullet}$.*

Proof. For diagrammes $d : \mathscr{D} \to \mathscr{Ens}$ over the category of sets $\mathscr{Ens}$, they will be the coset of the disjoint union set $\sqcup d(a)$ of sets $d(a)$, $a \in \mathscr{D}$, by the smallest equivalence relation in which the elements $x \in d(a)$ and $y \in d(b)$ are equivalent if in $\mathscr{D}$ there exists an arrow $\alpha : a \to b$such that $y = d(\alpha)x$. The same coset will be the limit of the diagramme $d : \mathscr{D} \to \mathscr{Ens}^\bullet$ over the category of pointed sets $\mathscr{Ens}^\bullet$ (due to the pointed nature of all maps $d(a)$ based points of all sets $d(a)$ turn out to be equivalent and their equivalence class is taken beyond the based limit point). Equipped with a coset topology, the same coset set will be the limit of the diagramme d over the categories $\mathscr{Top}$ and $\mathscr{Top}^\bullet$. □

Remark 6.52. It should be borne in mind that for many diagrammes, although there are limits, they are of no interest. An example is the diagramme $A \to C \leftarrow B$, the limit of which, as it is easy to see, is the object C (so in this case the limit does not depend on the homomorphisms $A \to C$ and $B \to C$). For such diagrammes, the dual notion of the *inverse limit* is meaningful, which we will not need yet.

6.B The limits of diagrammes over the category of groups

We will need the above general concepts for the case of the category of groups $\mathscr{Grp}$. At the same time, in order not to stray far from the traditional notation, for an arbitrary diagram $d : \mathscr{D} \to \mathscr{Grp}$, we will denote the group $d(a)$ with the symbol G_a.

Proposition 6.53. *For any diagramme $d : \mathscr{D} \to \mathscr{Grp}$, $\varinjlim d$ exists.*

Proof. Recall that a *word* over the set Y is an arbitrary expression of the form $y_1 y_2 \cdots y_k$ where $k \geq 0$ and $y_1, y_2, \dots, y_k \in Y$. The *product* of the words $y_1, y_2, \dots, y_k$ and $y_1', y_2', \dots, y_\ell'$ is called the word $y_1, y_2, \dots, y_k y_1', y_2', \dots, y_\ell'$. With respect to the multiplication of words, the set $M(Y)$ of all words over Y is a monoid (associative unitoid). The unit of this monoid is the empty word $\varnothing$.

Proposition 6.54. *The monoid $M(Y)$ is a free monoid generated by the set Y,*

i.e., for any monoid M and any map $\varphi : Y \to M$ there exists only one monoid morphism $\overline{\varphi} : M(Y) \to M$, extending the map φ (this morphism is given by the formula $\overline{\varphi}(y_1, y_2, \dots, y_k) = \varphi(y_1)\varphi(y_2)\cdots\varphi(y_k)$).

We will apply this construction to the construction of the limit $\varinjlim d$ for diagrammes $d : \mathscr{D} \to \mathscr{Grp}$, taking for Y the disjunct union of all groups G_a, $a \in \mathscr{D}$, and considering in $M(Y)$ the smallest *congruence* (equivalence relation consistent with the multiplication) for which

(a) $x \sim d(a)x$ for any element $x \in G_a$ and any arrow $\alpha : a \to b$;

(b) $xy \sim z$ for any elements $x, y, z \in G_a$ for which $xy = z$ in the group G_a;

(c) $x \sim \varnothing$ if the element is $x \in G_a$ is a unit of the group.

Let $G^{(0)}$ be the coset monoid of the monoid $M(Y)$ by this congruence. Since for each word $\xi = y_1 \cdots y_k \in M(Y)$ the word $\xi^{-1} = y_k^{-1} \ldots by_1^{-1}$ has, by applying conditions (b) and (c), the property that $\xi\xi^{-1} \sim \varnothing$ and $\xi^{-1}\xi \sim \varnothing$, then the monoid $G^{(0)}$ is a group. Let $j_a^{(0)} : G_a \to G^{(0)}$ be the composition of the embedding $G_a \to Y \to M(Y)$ and the factorisation map $M(Y) \to G^{(0)}$. By applying conditions (b) and (c) each map j_a is a homomorphism of groups, and by applying condition (a), the family of homomorphisms $j_a^{(0)}$ is a cone over the diagramme d. In addition, for any cone $\{j_a : G_a \to G\}$ over d, the homomorphism $\overline{\varphi} : M(Y) \to G$, continuing the map $\sqcup j_a : Y \to G$, obviously has the property that $\overline{\varphi}(\xi) = \overline{\varphi}(\eta)$ for any equivalent words $\xi, \eta \in M(Y)$, and therefore induces a homomorphism $\varphi : G^{(0)} \to G$, being a morphism of the cone $\{j_a^{(0)}\}$ to the cone $\{j_a\}$. Since there is no other morphism $\{j_a^{(0)}\} \to \{j_a\}$ obviously can exist, *this proves that the constructed group $G^{(0)}$ is the limit of diagramme d.* □

To describe the group $G^{(0)}$ more explicitly, we can use the method of generators and relations.

6.C Co-presentations of limits

Recall that a group F with a subset of X allocated in it is called a *free group over X* (and X is the *set of free generators of the group F*) if for any group G each map $f\varphi : X \to G$ extends in a unique way to some homomorphism $\overline{\varphi} : F \to G$. It follows directly from this definition that *free groups with sets of equal-cardinality of free generators are isomorphic.*

At the same time, it turns out that

Proposition 6.55. *for any set X there exists a free group $F(X)$ over X.*

The elements of this group are *group words* over X, i.e. expressions of the form

$$x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}, \tag{6.56}$$

where $a_1, a_2, \ldots, a_k$ (the case of $k = 0$ is not excluded) are nonzero integers, and $x_1, x_2, \ldots, x_k$ are elements of X such that $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, k-1$. In this case, a word of the form x^{+1} is identified with the element $x \in X$ (to ensure the inclusion of $X \subset F(X)$). The multiplication of words consists in their attribution to each other, accompanied by the "reduction of similar terms", so that, in particular, the word (6.56) turns out to be a product of elements $x_1, x_2, \ldots, x_k$ of degrees $x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}$. All axioms of the group, with the exception of associativity, are checked without difficulty (by the unit of the group $F(X)$ is the empty word $\varnothing$, and the word inverse to the word (6.56) is the word $x_k^{-a_k}, \ldots, x_2^{-a_2} x_1^{-a_1}$). As for associativity, its direct proof is somewhat difficult, and we will postpone it for now.

Proof. Having thus temporarily taken on faith that $F(X)$ is a group, we can already quite automatically check that this group is free. Indeed, if the homomorphism $\overline{\varphi} : F(X) \to G$ extending the map $\varphi : X \to G$ exists, then

$$\overline{\varphi}(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}) = \overline{\varphi}(x_1)^{a_1} \overline{\varphi}(x_2)^{a_2} \cdots \overline{\varphi}(x_k)^{a_k} = \varphi(x_1)^{a_1} \varphi(x_2)^{a_2} \cdots \varphi(x_k)^{a_k}$$

for any word (6.56), and hence this homomorphism is defined in a unique way. To prove its existence, it is necessary, as always in similar situations, to take the resulting formula as a definition, i.e. for any word (6.56) put

$$\overline{\varphi}(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}) = \varphi(x_1)^{a_1} \varphi(x_2)^{a_2} \cdots \varphi(x_k)^{a_k}$$

It is obvious that in this way we get a homomorphism $\overline{\varphi} F(X) \to G$, extending the map $\varphi : X \to G$. □

The assignment of an arbitrary group G by the generators and determining relations (called, according to the proposal of A. M. Vinogradov, the "copresentation" of the group; however, recently Yu. I. Merzlyakov proposed a more expressive term - the *genetic code*) consists in the assignment of some a set X, whose elements are called *generators*, and some subset R of a free group $F(X)$, whose elements are called *defining relations*. Let $F_R(X)$ be the coset group of the group $F(X)$ by the smallest normal subgroup $[R]_F$ of the group $F(X)$ containing the set R. They say that the pair $\langle X; R\rangle >$ is a copresentation of the group G, and write $G = \langle X; R\rangle$ if some isomorphism is given $\overline{\varphi} : F_R(X) \approx G$, or, equivalently, some epimorphism $\varphi : F(X) \to G$ with kernel $[R]_F$. This epimorphism is uniquely determined by the elements $\overline{x} = \varphi(x)$ of the group G. Admitting a certain liberty, the elements $\overline{x}$ are usually denoted simply by x (this is generally accepted, although not very successful, since it may well happen that for various elements $x_1, x_2 \in X$ in the group G will take place equality $\overline{x}_1 = \overline{x}_2$). Each element of the group G will then be represented (generally speaking, not in only one way) as a product of the powers of the elements $x \in X$. This explains the use of the term "generators" in relation to these elements.

For clarity, the equality $G = \langle X; R\rangle$ is often written as $G = \langle X; r = 1, r \in R\rangle$ and just $G = \langle X; r_1 = 1, \ldots, r_m = 1\rangle$ if $R = \{r_1 = 1, \ldots, r_m = 1\}$. Moreover, if some word $r \in R$ has the form $r = a^{-1}b$, where $a, b \in F(X)$, then $a = b$ is also written instead of $r = 1$.

If now for some diagramme $d : \mathscr{D} \to \mathscr{G}rp$ over the category $\mathscr{G}rp$ the following presentations are given

$$G_a = \langle X_a; R_a\rangle, \quad a \in \mathscr{D},$$

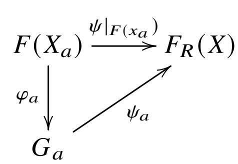
of its groups G_a, then we can easily write a co-presentation of the limit $G = \varinjlim d$ of this diagramme. For any arrow $\alpha : a \to b$ of the diagramme scheme $\mathscr{D}$ and any generator $x_a \in X_a$, we will represent the element $d(a)\overline{x}_a$ of the group G_b (i.e., more precisely, the element $d(a)\overline{x}_a$) in the form of some (generally speaking, not unique) group word $\xi(x_\alpha, \alpha)$ from the generators of X_b. By entering for any $\alpha \in \mathscr{D}$ the epimorphism $\psi_\alpha : F(X_a) \to G_a$ with the kernel $[R_a]_F$, we can more

formally define the word $\xi(x_a, \alpha)$ as an element of the group $F(X_b)$ satisfying the relation
$$(d(\alpha) \circ \varphi_a)x_a = \varphi_b(\xi(x_a, \alpha)).$$
Let
$$X = \sqcup_{a \in \mathscr{D}} X_a$$
be the disjoint union of all sets X_a, $a \in \mathscr{D}$. Denoting for any arrow $\alpha : a \to b$ from $\mathscr{D}$ with the symbol R_a the set of all words over X of the form $x_a^{-1}\xi(x_a, \alpha)$, $x_a \in X_a$, we will put
$$R = (\cup_{a \in \mathscr{D}} R_a) \cup (\cup_{\alpha \in \mathscr{D}} R_\alpha).$$

Proposition 6.57. *The pair $\langle X; R\rangle$ is a co-presentation of the limit of diagramme d:*
$$\varinjlim d = \langle X; R\rangle.$$

Proof. We should construct an epimorphism $F(X) \to G^{(0)})$ from the free group $F(X)$ to the group $G^{(0)} = \varinjlim d$, the kernel of which is the normal subgroup $[R]_F$. We show that this epimorphism can be taken as an extension of φ the map $X \to G^{(0)}$, which is on each term X_a a restriction of the map $j_a \circ \varphi_a : F(X_a) \to G^{(0)}$. It is clear that the homomorphism φ is an epimorphism and that $\varphi(r) = 1$ for any word $r \in R$, i.e. that $[R]_F \subset \ker \varphi$ (we consider all groups $F(X_a)$ naturally embedded in the group $F(X)$; by applying this agreement $\varphi|_{F(X_a} = j_a \circ \varphi_a$ and therefore $\varphi(r_a) = j_a(\varphi_a(r_a)) = 1$ forR any word $r_a \in R_a$ and
$$\begin{aligned}\varphi(x_a^{-1}\xi(x_a, \alpha)) &= (j_a \circ \varphi_a)(x_a)^{-1}(j_b \circ \varphi_b)(\xi(x_a, \alpha))\\ &= (j_a \circ \varphi_a)(x_a)^{-1}(j_b \circ d(\alpha)\varphi_a)(x_a)\\ &= (j_a \circ \varphi_a)(x_a)^{-1}(j_a \circ \varphi_a)(x_a) = 1\end{aligned}$$
for any word $x_a^{-1}\xi(x_a, \alpha) \in R_a$). Thus, only the reverse inclusion needs an additional proof.

With this in mind, consider the canonical epimorphism $\psi : F(X) \to F_R(X)$ with kernel $[R]_F$. Since for any vertex $a \in \mathscr{D}$ there is an inclusion $[R_a]_F \subset [R]_F \cap F(X_a)$, then there is a homomorphism $\psi_a : G_a \to F_R(X)$, closing the commutative diagramme



Since for any element $x_a \in X_a$ and any arrow $\alpha : a \to b$, the word $x_a^{-1}\xi(x_a, \alpha)$ lies in the kernel of the epimorphism ψ, then
$$\begin{aligned}(\psi_b \circ d(\alpha)(\varphi_a(x_a) &= (\psi_b \circ \varphi_b)(\xi(x_a, \alpha)) = \psi(\xi(x_a, \alpha))\\ &= \psi(x_a) = \psi_a(\varphi_a(x_a)),\end{aligned}$$

and, therefore, $\psi_b \circ d(a) = \psi_a$ (because the elements $\varphi_a(x_a) = \overline{x}_a$ generate the group G_a). Thus, the map $\psi_a : G_a \to F_R(X)$ make up a cone over the diagramme d, and therefore there is a homomorphism $\lambda : G^{(0)} \to F_R(X)$ such that $\lambda \circ j_a = \psi_a$ for any $a \in \mathscr{D}$. But then

$$\lambda \circ \varphi|_{F(X_a} = \lambda \circ j_a \circ \varphi_a = \psi_a \circ \varphi_a = \psi|_{F(X_a},$$

and, therefore, $\lambda \circ \varphi = \psi$. Therefore, $\ker \varphi \subset \ker \psi \subset R_F$. □

Example 6.58. The coproduct $\sqcup_a G_a$ of a family of groups (the limit of the diagram without arrows) is called the *free product* of these groups. According to Proposition 6.57, the *co-presentation of the free product is obtained by combining the co-presentations of the multipliers: if* $G_a = \langle X_a; R_a \rangle$, *then*

$$\sqcup_a G_a = \langle \sqcup_a X_a; \sqcup_a R_a \rangle.$$

Generally speaking, the co-presentation from Proposition 6.57 is not the most economical and can often be simplified.

Let $G = \langle X; R \rangle$, and let S be an arbitrary subset of $[R]_F$. Then it is clear that $G = \langle X; R \cup S \rangle$. Similarly, $G = \langle X \cup \{y\}; R \cup \{y^{-1}\xi\} \rangle$, where y is an arbitrary character not contained in X, and ξ is an arbitrary word from $F(X)$. Transformations of the co-presentation $\langle X; R \rangle$ into the co-presentations $\langle X; R \cup S \rangle$ and $\langle X \cup \{y\}; R \cup \{y^{-1}\xi\} \rangle$ (as well as inverse transformations) are called *Tietze transformations*.

Remark 6.59. It can be shown (this statement is known as Tietze's theorem) that two "finite co-presentations set the same (= isomorphic) groups if and only if when they can be translated into each other by Tietze transformations. However, this theorem does not indicate any way to find these transformations, and, moreover, it can be shown that there can be no algorithm that allows this to be done. In this respect, defining a group by generators and defining relations is highly ineffective.

Nevertheless, in many situations, the Tietze transformations make it possible to significantly simplify presentations.

Example 6.60. Let the scheme $\mathscr{D}$ have the form $\cdot \leftarrow \cdot \rightarrow \cdot$, that is, we are dealing with a push-out of the G diagramme $A \xleftarrow{i_A} C \xrightarrow{i_B} B$. Let, further, define $A = \langle X; R \rangle$, $B = \langle Y; Q \rangle$ and $C = \langle Z; P \rangle$. For any generator $z \in Z$, we denote by ξ_z, and η_z the group words over X and Y, which are expressions through the generating elements $i_A(z) \in A$ and $i_B(z) \in B$. Then, according to Proposition 6.57, the push-out G will have a co-presentation of the form $\langle X \sqcup Y \sqcup Z; R \cup Q \cup P \cup U \cup V \rangle$, where U is the set of all words of the form $z^1\xi_z$, $z \in Z$, and V is the set of all words of the form $z^{-1}\eta_z$, $z \in Z$. But since the map i_A is a homomorphism, an arbitrary word from P after substitution instead of each element $z \in Z$ of the corresponding word ξ_z will be a consequence of the relations from R (i.e. it will belong to the normal subgroup $[R]_F$). This means that all the relations from P are a consequence of the relations from R and U (as well as, of course, the relations from S and V), and therefore they can be painlessly removed from the presentation of the group G (the first Tietze transformation). Further, it is clear that any pair of relations of the form $z^{-1}\xi_z = 1$ and $z^{-1}\eta_z = 1$ is equivalent to the relations $z^{-1}\xi_z = 1$ and $\xi_z = \eta_z$. Therefore, by preserving U (as well as R and Q), we can replace V with a set of relations W of the form $\xi_z = \eta_z$. But after this replacement (and removing the relations from P), each $z \in Z$ will appear only in one relation $z^{-1}\xi_z = 1$, and therefore, by throwing out z and this relation, we will not change the group G (the second Tietze transformation). This proves that

Proposition 6.61. *for $A = \langle X; R\rangle$,$B = \langle Y; Q\rangle$ and $C = \langle Z; P\rangle$ the push-out G diagramme $A \leftarrow C \rightarrow B$ has a co-presentation of the form*

$$G = \langle X \cup Y; R \cup Q \cup W\rangle,$$

where W is the set of all relations of the form $\xi_z = \eta_z$, $z \in Z$.

Example 6.62. If in the previous example the group C is a unit group, i.e. if we are talking about a diagramme $A \leftarrow 1 \rightarrow B$, then the push-out G will obviously be a free product of the groups A and B. If the unit group is the group B, then the push-out G will have a presentation of the form

$$G = \langle X; R \cup W\rangle,$$

where W is the set of all relations of the form $\xi_z = 1$, and therefore will be a coset group of the group $A = \langle X; R\rangle$ by a normal subgroup generated by all elements ξ_z or, more precisely, elements $i(z)$, $z \in Z$, where $i = i_A$. Since the last subgroup is obviously nothing more than the smallest normal subgroup containing the image $\operatorname{im} i = iC$ of the homomorphism i, we get, therefore, that

Proposition 6.63. *the push-out of the diagram $A \leftarrow C \rightarrow 1$ is the coset group A' of the group A by the smallest normal subgroup containing the subgroup iC (the corresponding initial cone will be the natural epimorphism $A \rightarrow A'$).*

Of course, this fact is easily established and directly from the definition of push-out.

6.D The structure of the free product of groups

In Example 6.62, we managed to eliminate - at least from the formulation, if not in essence - the generators and defining relations, which should be considered as a certain achievement in the direction of the efficiency of the description of the limit. Interestingly, a similar (but, perhaps, much more difficult) elimination can be carried out in the situations of Examples 6.58 and 6.60.

For any diagramme $d : \mathscr{D} \rightarrow \mathscr{G}rp$, the group $G^{(0)} = \varinjlim d$ is generated, of course, by images of groups G_a with homomorphisms $j_a^{(0)} : G_a \rightarrow G^{(0)}$, i.e. any element g of the group $G^{(0)}$ can be represented as a product $g_1 g_2 \cdots g_k$ each multiplier of which belongs to one of the groups G_a (or, more precisely, is the image of some element of the group G_a with the homomorphism $j_a^{(0)}$). For $g \neq 1$, we can assume that this presentation is reduced, i.e. that none of the factors $g_1, g_2, \ldots, g_k$ is a unit of the corresponding group and any two neighbouring factors belong to different groups. Naturally, in the general case| even the reduced presentation is not unique (at least due to the fact that the homomorphisms $j_a^{(0)}$ are not, generally speaking, monomorphisms). However, it turns out that in the situation of Example 6.58, i.e. in the case when $G^{(0)} = \sqcup_a G_a$, *the reduced presentation of the elements of the group* $G^{0)}$, i.e. from an equality

$$g_1 \cdots g_k = g_1' \cdots g_\ell'$$

of only the two reduced products, it follows that $k = \ell$ and $g_i = g_i'$ for any $i = 1, \ldots, k$. In particular, this means that

Proposition 6.64. *the natural homomorphisms $j_a^{(0)} : G_a \to \sqcup G_a$ of the groups G_a in their free product $\sqcup G_a$ are monomorphisms, and therefore the groups G_a can be considered subgroups of the group $\sqcup G_a$ (which, by the way, justifies our notation a posteriori).*

The natural way to prove this statement (avoiding, by the way, the trouble with presentations) is to consider for these groups G_a the set $\overline{G}$ of all *reduced words*, i.e. words $g_1, \cdots g_k$ over a disjoint union of all groups G_a in which all elements $g_1, \ldots, g_k$ are not units of the corresponding groups and any two neighbouring elements belong to different groups, and prove that:

1) with respect to the natural multiplication operation, which consists in attributing words to each other and then reducing the resulting word, the set $\overline{G}$ is a group;

2) natural embeddings $\overline{j}_a : G_a \to \overline{G}_a$ (matching the same element to each element of $g \neq 1$ of the group G_a, but considered as a reduced word longer than 1) constitute an initial cone (and, therefore, by applying the unity of the initial cone, the group $\overline{G}$ is isomorphic to the group $G^{(0)}$).

Proof. Point 2) of this program does not cause any difficulties: for any family of homomorphisms $j_a : G_a \to G$ the formula

$$\varphi(g_1 \cdots g_k) = j_{a_1}(g_1) \cdots j_{a_k}(g_k),$$

where $a_1, \ldots, a_k$ are indexes such that $g_1 \in G_{a_1}, \ldots, g_k \in G_{a_k}$, well defines a homomorphism $\varphi : \overline{G} \to G$ satisfying the relations $j_a = \varphi \circ \overline{j}_a$ and no other homomorphism $\overline{G} \to G$ satisfying these relations obviously can exist.

However, point 1) encounters - again with regard to proving the associativity of multiplication - serious combinatorial difficulties. To get around these difficulties, van der Waerden proposed the following artificial technique.

For each index $a \in \mathscr{D}$, we define the action of the group G_a on the set $\underline{G}$, assuming for any element $g \in G_a$ and any reduced word $\xi = g_1 g_2 \cdots g_k \in \overline{G}$

$$g\xi = \begin{cases} g_1g_2\cdots g_k, & \text{if} \quad g = 1, \\ gg_1g_2\cdots g_k, & \text{if} \quad g \neq 1, \text{ and } g_1 \notin G_a, \\ g_1^{'}g_2\cdots g_k, & \text{if} \quad g_1 \in G_a \text{ and } g^{'} = gg_1 \neq 1, \\ g_2\cdots g_k, & \text{if} \quad g_1 \in G_a \text{ and } gg_1 = 1. \end{cases} \tag{6.65}$$

An automatic verification shows that these formulas really define the action of the group G_a on the set $\overline{G}$, i.e. $1\xi = \xi$ and $(g^{'}g)\xi = g^{'}(g\xi)$ for any word $\xi \in \overline{G}$ and any elements $g^{'}, g \in G_a$. It is clear that this action is *effective*, i.e. considered as a homomorphism of the group G_a to the group $\operatorname{Aut}\overline{G}$ of all permutations (bijective maps to itself) of the set $\overline{G}$, it is a monomorphism. Therefore, we can consider the group G_a as a subgroup of the group $\operatorname{Aut}\overline{G}$. Let $\widetilde{G}$ be a subgroup of the group $\operatorname{Aut}\overline{G}$ generated by all groups G_a. Each element of the group $\widetilde{G}$ other than one admits a reduced presentation of the form $g_1\cdots g_k$, where $g_1 \in G_{a_1}, \dots g_k \in G_{a_k}$, all elements $g_1, \dots, g_k$ are different from one and no two neighbouring elements belong to the same group. We will denote this element with the symbol $\widetilde{\xi}$, where ξ is the reduced word $g_1\cdots g_k$ from $\overline{G}$ (the difference between $\widetilde{\xi}$ and ξ is that ξ is obtained by formally attributing the elements $g_1, \dots c, g_k$ to each other, and $\widetilde{\xi}$ is a permutation consisting of sequentially performing permutations $g_1, \dots, g_k$). It is clear at the same time that by the map $\xi \to \widetilde{\xi}$ of the set $\overline{G}$ to the group $\widetilde{G}$, described in paragraph 1) a multiplication in $\overline{G}$ translates into a multiplication in the group $\widetilde{G}$ (i.e., is a homomorphism). On the other hand, it follows directly from formulae (6.65) that $\widetilde{\xi}(\varnothing) = \xi$ for any word $\xi \in \overline{G}$ (where $\varnothing$ is an empty word from $\overline{G}$). Therefore, $\widetilde{\xi} = \widetilde{\eta}$ if and only if $\xi = \eta$, i.e. the map $\xi \to \widetilde{\xi}$ is bijective and therefore is an isomorphism. Thus, the unitoid $\overline{G}$ is isomorphic to the group $\widetilde{G}$ and, therefore, is itself a group. □

Thus, we have obtained a completely satisfactory description of the algebraic structure of the free product $\sqcup G_a$.

In particular, we can now prove that the multiplication of words (6.56) is associative, i.e. that the unitoid $F(X)$ is a group. Indeed, if the set X consists of only one element, then this fact is obvious, and the group $F(X)$ will in this case be an infinite cyclic group. Comparing the definitions now, we immediately find that in the general case, the unitoid $F(X)$ is nothing more than the unitoid $\overline{G}$ for infinite cyclic groups $G_a = F(\{x_a\})$, where x_a runs through it a set of X. Hence, $F(X)$ is a group.

In addition, we have obtained that

Proposition 6.66. *any free group is a free product of infinite cyclic groups.*

Similar results can be obtained in the situation of Example 6.58 if we additionally assume that both homomorphisms i_A and i_B are monomorphisms, i.e. if in fact we are dealing with two groups A and B in which isomorphic subgroups of $A^{'} = \operatorname{im} i_A$, $B^{'} = \operatorname{im} i_B$ and an isomorphism is given $i = i_B \circ i_A^{-1} : A^{'} \to B^{'}$. In this case, the push-out of the $G^{(0)}$ diagramme $A \xleftarrow{i_A} C \xrightarrow{i_B} B$ is called the *free product of the groups A and B with the joined subgroup $A^{'} = B^{'}$.*

It turns out that for this free product, the homomorphisms $j_A : A \to G^{(0)}$ and $j_B : B \to G^{(0)}$ are also monomorphisms, i.e. the groups A and B are naturally embedded in the group $G^{(0)}$. Let $A'{}'$ be a system of representatives of the right coset classes of the group A by the subgroup $A^{'}$ other than the subgroup $A^{'}$, i.e. such a subset in A that any element of $a \in A \setminus A^{'}$ is uniquely presented as $a^{'}a^{''}$, where $a^{'} \in A^{'}$, $a^{''} \in A'{}'$, and let similarly $B^{''}$ be a system of representatives of the right coset classes of the group B by the subgroup $B^{'}$ other than the subgroup $B^{'}$. Then it turns out that

Proposition 6.67. *any element of the group $G^{(0)}$ is presented in the unique way as a product of $a^{'}c_1 \cdots c_k$, where $a^{'} \in A^{'}$, and $c_1, \dots, c_k \in A^{''} \cup B^{''}$, and if $c_i \in A^{''}$, then $c_{i+1} \in B^{''}$, and vice versa.*

We need this fact only for a clearer presentation of the structure of concrete examples, and therefore we will leave it without proof (which even after the improvement of van der Waerden's software remains very painstaking).

6.E The Seifert-van Kampen theorem

Now let's return to topology.

Let X be a connected topological space, $\{X_a\}$ be its covering consisting of connected subspaces, and $\mathscr{D}$ be a diagram scheme whose vertices are the indices a of the cover $\{X_a\}$, and the arrows are pairs (a, b) such that $X_a \subset X_b$ (meaning that the pair (a, b) is an arrow $a \to B$). Thus, the covering $\{X_a\}$ naturally turns out to be a diagramme of the type $\mathscr{D}$ over the category $\mathscr{T\!op}$.

If the space X is pointed and all subspaces X_a contain a base point, then the covering $\{X_a\}$ will be a diagramme of type $\mathscr{D}$ over the category $\mathscr{T\!op}$, and the groups $G_a = \pi_i X_a$ and homomorphisms $G_a \to G_b$ induced by $X_a \subset X_b$ with attachments $X_a \to X_B$, will form a diagramme of groups of the type $\mathscr{D}$. We will denote this diagramme with the symbol $\{\pi_1 X_a\}$.

It is clear that the homomorphisms $i_a : \pi_1 X_a \to \pi_1 X$ induced by the embeddings of $X_a \to X$ make up the cone $\{i_a : \pi_1 X_a \to \pi_1 X\}$ over the diagramme $\{\pi_1 X_a\}$ with vertex $\pi_1 X$.

A covering $\{X_a\}$ is called *saturated* if the intersection of any two of its elements is also a covering element. From any covering, you can move to a saturated covering by adding all possible finite intersections of its elements (but the condition of connectivity of the covering elements may be violated).

Theorem 6.68. *For any pointed connected space X and any of its saturated covering $\{U_a\}$ consisting of connected open sets U_a containing the base point, the fundamental group $\pi_1 X$ of a space X is the limit of groups $\pi_1 U_a$,*

$$\pi_1 X = \varinjlim \{\pi_1 U_a\},$$

with respect to homomorphisms $i_a : \pi_1 U_a \to \pi_1 X$, induced by attachments $U_a \to X$.

In a very special special case, this theorem was first proved by Seifert and almost simultaneously by van Kampen. For a long time it was called the van Kampen theorem, but now it is more often called the Seifert-van Kampen theorem. In fact, Crowell seems to have proved it for the first time.

We will break the proof of Theorem 6.68 into a series of lemmas.

Lemma 6.69. *Under the conditions of Theorem 6.68, the group $\pi_1 X$ is generated by images of groups $\pi_1 U_a$ with homomorphisms i_a.*

Proof. Let $\alpha = [u :: (I, \dot{I}) \to (X, x_0)]$ be an arbitrary element of the group $\pi_1 X$. Then if $1/n$ is less than the Lebesgue number (see Appendix 1.11 to Lecture 1) of the preimage of the covering $\{U_a\}$ by the map u, then for any $k = 1, \dots, n$ there exists an index $a_k \in \mathcal{D}$ that the segment $I_k = [(k-1)/n, k/n]$ passes when u is mapped to the open set U_{a_k}, i.e. the path $u_k : I \to X$ defined by the formula

$$u_k(t) = u\left(\frac{t+k-1}{n}\right), \quad i \in I,$$

is actually a path in U_{a_k}. This path connects the point $x_{k-1} = u\left(\frac{k-1}{n}\right)$ (being at $k = 1$ the base point x_0) with the point $x_k = u\left(\frac{k}{n}\right)$ (at $k = n$, also being the base point x_0). For $k < n$, the point x_k belongs to the set $U_{a_k} \cap U_{a_{k+1}}$, which is by condition an element of the covering $\{U_a\}$, and therefore connected and containing the point x_0. Therefore, in $U_{a_k} \cap U_{a_{k+1}}$ there is a path w_k connecting the point x_0 with the point x_k. For $k = 0$, we will assume that this path is a constant path. For $k = n$, we will also take a constant path for v_n. Then for any $k = 1, \dots, m$ in U_{a_k}, the loop $v_k = w_{k-1} u_k w_k^{-1}$ will be defined and in X the product $v_1 v_2 \cdots v_n$ of all these loops will be homotopic rel $\dot{I}$ to the product of $u_1 u_2 \cdots u_n$ of paths u_k, i.e. it will be homotopic to the loop u. This proves that the elements $\beta_k = [v_k]$ of the groups $\pi_1 U_{a_k}$ have the property that the element α is the product of the elements $i_{a_k}(\beta_k)$. Therefore, the group $\pi_1 X$ is generated by the subgroups $i_a(\pi_1 U_a)$. □

We will call a loop u *elementary* if it is contained in one of the elements of the cover $\{U_a\}$, and a loop *subordinate to the cover* $\{U_a\}$ if it is a product of elementary loops. In this terminology, Lemma 6.69 states that

Proposition 6.70. *any loop of the space X is homotopic to* rel $\dot{I}$ *the loop subordinate to the covering $\{U_a\}$.*

Loops u and v subordinate to the covering $\{U_a\}$, we will call *elementary homotopy* if $u = u_1 u' u_2$ and $v = u_1 v' u_2$ where u_1, u_2 are loops subordinate to the covering $\{U_a\}$, and u' and v' are elementary loops, contained in the same element U_a of the coverings $\{U_a\}$ and homotopic rel $\dot{I}$ in U_a.

Lemma 6.71. *Loops u and v subordinate to the covering $\{U_a\}$ are homotopic if and only if when they are subordinately homotopic* rel $\dot{I}$.

Proof. It is clear that subordinate homotopic loops are homotopic. Conversely, let the loops u and v subordinate to the covering $\{U_a\}$ be homotopic, and let $F : I \times I \to X$ be the homotopy connecting them $\operatorname{rel} \dot{I}$.

For any $N > 0$, consider the squares into which the square $I \times I$ is divided by the lines $s = k/N$ and $t = \ell/N$, where $k, \ell = 0, 1, \ldots, N$. Each such square is given by two numbers $k, \ell = 1, \ldots, N$ and consists of all points $(s, t) \in I \times I$ for which

$$\frac{k-1}{N} \leq s \leq \frac{k}{N}, \quad \frac{\ell-1}{N} \leq t \leq \frac{\ell}{N}.$$

We will denote this square with the symbol $I^2_{k\ell}$.

The reasoning already known to us from the proof of Lemma 6.69, using the Lebesgue number to the covering $\{F^{-1}U_a\}$, shows that for a sufficiently large N, each of the sets $F(I^2_{k\ell})$ is contained at least in one element of the coverings $\{U_a\}$. Assuming that N has this property, we denote for any k and ℓ by $a(k, \ell)$ one of the indexes of a for which $F(I^2_{k\ell}) \subset U_a$. Then the map F, bounded to the upper side of the square $I^2_{k,\ell}$, will be some way in $U_{a(k,\ell)}$. We will denote this path with the symbol $\overline{u}_{k\ell}$. Taking into account the transformation of the parameter, it is determined by the formula

$$\overline{u}_{k\ell}(t) = F\left(\frac{t+k-1}{N}, \frac{\ell}{N}\right), \quad t \in I,$$

We will also introduce the path $\overline{v}_{k\ell} : I \to U_{a(k,\ell)}$, defined by the map F on the right side of the square $I^2_{k\ell}$:

$$\overline{v}_{k\ell}(t) = F\left(\frac{k}{N}, \frac{t+\ell-1}{N}\right), \quad t \in I,$$

Then on the lower side of the square $I^2_{k,\ell}$ the map F will define the path $\overline{u}_{k,\ell-1}$ and on the left - the path $\overline{v}_{k-1,\ell}$. Therefore, on the entire square $I^2_{k,\ell}$, the map F will define the homotopy $\operatorname{rel} \dot{I}$ connecting the path $\overline{u}_{k,\ell-1}\overline{v}_{k\ell}$ with the path $\overline{v}_{k-1,\ell}\overline{u}_{k\ell}$. Thus

$$\overline{u}_{k,\ell-1}\overline{v}_{k\ell} \sim \overline{v}_{k-1,\ell}\overline{u}_{k\ell} \quad \operatorname{rel} \dot{I} \quad \text{in} \quad U_{a(k,\ell)}$$

Each point of the form $(i/N, j/N)$ is the vertex of four (or fewer) squares $I^2_{k,\ell}$. Let $U_{b(i,j)}$ be the intersection of the corresponding sets $U_{a(k,\ell)}$. Thus, in particular, $U_{a(i,j)} \subset U_{a(k,\ell)}$ at $(i, j) = ((k-1, \ell), (k, \ell)$ and $(k, \ell-1)$.

Let $k, \ell = 1, \ldots, N$. Consider the set $U_{b(k,\ell)}$. This set is connected and contains the points x_0 and $x_{k\ell} = F(k/N, \ell/N)$. Let $w_{k\ell}$ be an arbitrary path in $U_{b(k,\ell)}$ connecting the points x_0 and $x_{k\ell}$ (and being a constant path if $x_0 = x_{k\ell}$). Then in $U_{a(k,\ell)}$ loops will be defined

$$u_{k\ell} = w_{k-1,\ell}\overline{u}_{k\ell}w^{-1}_{k\ell} \quad \text{and} \quad v_{k\ell} = w_{k,\ell-1}\overline{v}_{k\ell}w^{-1}_{k\ell},$$

and homotopies will take place for these loops

$$u_{k,\ell-1}v_{k\ell} \sim v_{k-1,\ell}u_{k\ell} \quad \operatorname{rel} \dot{I} \quad \text{in} \quad U_{a(k,\ell)}. \tag{6.72}$$

Let us now focus our attention on the polylines in the square $I \times I$ connecting its lower left vertex $(0,0)$ with its upper right vertex $(1,1)$ and consisting of the sides of the squares $I^2_{k,\ell}$. For each such polyline, we will match a loop that is the product of elementary loops $u_{k\ell}$ and $v_{k\ell}$ corresponding to the sides of this polyline. Since we can move from any polyline to any other polyline a sequence of elementary steps, at each of which two sides of one of the squares $I^2_{k\ell}$ are replaced by its other two sides, it follows directly from the formula (6.72) that

Proposition 6.73. *the loops corresponding to any two polylines are subordinately homotopic.*

Therefore, in particular, the loops are subordinately homotopic

$$u^{'} = u_{10}u_{20}\cdots u_{N0}v_{N1}v_{N2}\cdots v_{NN}$$

and

$$v^{'} = v_{00}v_{01}\cdots v_{0N}u_{N1}u_{N2}\cdots u_{NN}$$

corresponding to polylines consisting of (subdivided) sides of the square $I \times I$. But by construction, all loops $v_{N1}v_{N2}\cdots v_{NN}$ and $v_{00}v_{01}\cdots v_{0N}t$ are constant loops (at the point x_0) and, therefore, the subordinated loops $u^{'}$ and $v^{'}$ are homotopic to the loops $u^{''} = u_{10}u_{20}\cdots u_{N0}$ and $v^{''} = u_{N1}u_{N2}\cdots u_{NN}$, corresponding to the lower and upper sides of the square $I \times I$.

Let us now recall that, by the condition, the loop u is, firstly, the restriction of the map F on the lower side of the square $I \times I$, and secondly, it is subordinated to the covering $\{U_a\}$, i.e. is the product $u_1u_2\cdots u_n$ of the element loops $u_1, u_2 \dots, u_n$. Since all of the above is true for any sufficiently large N, we can additionally require that the number N be divisible by n, i.e. have the form $N = nM$, and, consequently, that each loop $u_i, i = 1, \dots, n$, is the product of the paths $\overline{u}_{k0}$ with $(i-1)M \le k \le iM$. At the same time, by the condition for any $i = 1, \dots, n$, there is an index a_i such that the loop is, and therefore every path $\overline{u}_{k0}$ with $(i-1)M \le k \le iM$, is contained in U_{a_i}. It is clear that, without loss of generality, we can assume (by increasing, if necessary, M) that U_{a_i} contains not only the path $\overline{u}_{k0}$ (i.e., more precisely, the image when map F to the lower side of the square $I^{(2)}_{k0}$), but the image $F(I^{(2)}_{k0})$ of the total square $I^{(2)}_{k0}$. This means that for $(i-l)M \le k \le iM$ we can assume that there is an equality $a(k,0) = a_i$ and, consequently, that each loop $U_{k0} = w_{k-1,0}\overline{u}_{k0}w^{-1}_{k0}$ with $(i-l)M \le k \le iM$ is a loop in U_{a_i}. Therefore, the product $u^{'}_i$ of these loops will be in U_a homotopic to the product of the paths $\overline{U}_{k0}$, i.e. it will be homotopic to the loop u_i. This proves that the loop

$$u^{''} = u_{10}u_{20}\cdots u_{N0} = u^{'}_1u^{'}_2\cdots u^{'}_n$$

is subordinately homotopic to the loop $u = u_1u_2\cdots u_n$.

Similarly, it is proved that (for possibly even larger N) the loop $v^{''}$ is subordinately homotopic to the loop v.

Thus, there are subordinate homotopies

$$u \sim u^{''} \sim u^{'} \sim v^{'} \sim v^{''} \sim v,$$

and, therefore, the loops u and v are subordinately homotopic. □

Now we have everything ready to prove Theorem 6.68.

Proof. (of Theorem 6.68) We need to show that for any cone $\{j_a : \pi_1 U_a \to G\}$ over the diagramme $\{\pi_1 U_a\}$ there exists a morphism $\varphi : \pi_1 X \to G$ of the cone $\{i_a : \pi_1 U_a \to \pi_1 X\}$ to the cone $\{j_a : \pi_1 U_a \to G\}$ and that this morphism is unique. But if such a morphism exists and if $\alpha = i_{a_1}\beta_1 \cdot i_{a_2}\beta_2 \cdot \cdots \cdot i_{a_n}\beta_n$ - presentation of the element $\alpha \in \pi_1 X$ as a product of elements of the form $i_a\beta$, $a \in \mathcal{D}$, $\beta \in \pi_1 U_a$ (existing by Lemma 6.69), then

$$\varphi(\alpha) = j_{a_1}\beta_1 \cdot j_{a_2}\beta_2 \cdot \cdots \cdot j_{a_n}\beta_n. \tag{6.74}$$

Hence, the morphism φ is unique. Therefore, everything will be proved if we show that formula (6.74) well defines some map $\varphi : \pi_1 X \to G$ (which will automatically be a homomorphism of groups and a morphism of cones), i.e. that equality

$$i_{a_1}\beta_1 \cdot i_{a_2}\beta_2 \cdot \cdots \cdot i_{a_n}\beta_m = i_{b_1}\gamma_1 \cdot i_{b_2}\gamma_2 \cdot \cdots \cdot i_{b_n}\gamma_m \tag{6.75}$$

follows the equality

$$j_{a_1}\beta_1 \cdot j_{a_2}\beta_2 \cdot \cdots \cdot j_{a_n}\beta_m = j_{a_1}\gamma_1 \cdot j_{a_2}\gamma_2 \cdot \cdots \cdot j_{a_n}\gamma_m. \tag{6.76}$$

Let $u_1, u_2, \dots, u_n$ be arbitrary loops of classes $\beta_1, \beta_2, \dots, \beta_n$, and $v_1, v_2, \dots, v_m$ be arbitrary loops of classes $\gamma_1, \gamma 2, \dots, \gamma_m$. Consider subordinate coverings $\{U_a\}$ for loops $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_m$. Equality (6.75) means that these loops are homotopic. Therefore, according to Lemma 6.71, they are also subordinately homotopic. Thus, it is sufficient for us to prove Equality (6.76) only in the case when the loops u and v are subordinately homotopic. Moreover, for obvious inductive reasons, it is sufficient to prove this equality only for elementary homotopy loops u and v, i.e., in other words, for the case when$n = m$ and $\beta_1 = \gamma_1, \cdots \beta_n = \gamma_m$. In turn, it is enough, obviously, to prove that if the elementary loop $u : (I, \dot{I}) \to (X, x_0)$ is contained in the elements U_a and U_b of the covering $\{U_a\}$, then for the elements $[u]_a$ and $[u]_b$ of the groups $\pi_1 U_a$ and $\pi_1 U_b$ defined by this loop, considered as a loop in U_a and in U_b, respectively, the equality takes place in the group G

$$i_a[u]_a = j_b[u]_b. \tag{6.77}$$

Let $U_c = U_a \cap U_b$, and let $[u]$ be an element of the group $\pi_1 U_c$ defined by the loop and, considered as a loop in U_c. Then $[u] \mapsto [u]_a$ and $[u] \mapsto [u]_b$ with homomorphisms induced by the corresponding embeddings. Therefore, by applying the defining property of the cone in the group G, the equality $j_c[u] = j_a[u]_a$ and $j_c[u] = j_b[u]_b$, and hence Equality (6.77).

Thus, Theorem 6.68 is fully proved, □

6.F Consequences of the Seifert-van Kampen theorem

Using Theorem 6.68, the fundamental group of the bouquet of any well-pointed spaces is easily calculated.

Proposition 6.78. *The fundamental group $\pi_1(X \vee Y)$ of the bouquet $X \vee Y$ of two connected well-pointed spaces X and Y is the free product of the fundamental groups of these spaces:*

$$\pi_1(X \vee Y) = \pi_1 X \sqcup \pi_1 Y.$$

Proof. By the condition, the base point x_0 has in the spaces X and Y (now automatically linked) neighbourhoods U and V such that $U \searrow\!\!\searrow x_0$ and $V \searrow\!\!\searrow x_0$. Consider in the bouquet $X \vee Y$ open sets

$$U^{'} = X \cup V, \quad V^{'} = U \cup Y$$

and their intersection $U^{'} \cap V^{'} = U \vee V$. According to Theorem 6.68, the group $\pi_1(X \vee Y)$ is a push-out of the diagramme

$$\pi_1 U^{'} \leftarrow \pi_1(U^{'} \cap V^{'}) \rightarrow \pi_1 V^{'}.$$

But it is clear that $U^{'} \searrow\!\!\searrow X$, $V^{'} \searrow\!\!\searrow Y$ and $U^{'} \cap V^{'} \searrow\!\!\searrow x_0$. Therefore, this push-out is isomorphic to the free product $\pi_1 X \sqcup \pi_1 Y$ □

Of course, the analogue of proposition 6.53 also holds for a bouquet of any number of well-pointed spaces, i.e.

$$\pi_1(\vee_a X_a) = \sqcup_a \pi_1 X_a$$

for any well-pointed spaces X_a.

Corollary 6.79. *As a consequence, the fundamental group of the bouquet of circles is a free group:*

$$\pi_1(\vee_a \mathbb{S}^1_a) = F(A). \tag{6.80}$$

Here $F(A)$ is a free group, the set of free generators of which is the set of indices of the number the circles of the bouquet $\vee_a \mathbb{S}^1_a$.

Remark 6.81. The reasoning used in the proof of Proposition 6.78 is of a very general nature. It is applicable to any family of subspaces $\{X_a\}$ of the space X (generally speaking, not even a covering) for which there exists a saturated open covering $\{U_a\}$ of the space X such that:

1) for any index a, there is an inclusion of $X_a \subset U_a$ and the corresponding embedding $X_a \rightarrow U_a$ induces a isomorphism $\pi_1 X_a \approx \pi_1 U_a$ (so it will be for example, in the case when $U_a \searrow\!\!\searrow X_a$);,
2) for any pair of indices a and b, the inclusion $X_a \subset X_b$ takes place if and only if $U_a \subset U_b$ holds.

It is clear that

Proposition 6.82. *for any such family*

$$\varinjlim\{\pi_1 X_a\} = \pi_1 X.$$

6.G Graphs

We will verify the Seifert-van Kampen theorem in Lecture 19[2], but for now we will consider one simple but important class of topological spaces whose fundamental groups can be easily calculated using the previous corollary.

Definition 6.83. A Hausdorff space X is called a *graph* if a nonempty discrete subspace X^0 is allocated in it (the points of which are called *vertices* of the graph X) such that

1) the complement $X \setminus X^0$ is a disjoint union of open sets, each of which is homeomorphic to the open interval $(0, 1)$ (these sets - as well as their closures - are called *edges* of the graph X);

2) for any edge $e \subset X \setminus X^0$, there exists a continuous map $u : I \to X$ (called the characteristic path of this edge), the image of which is the closure $\overline{e}$ of the edge e and which is on the inside of $(0, 1)$ of the segment I is homeomorphic to the edge e;

3) any set $C \subset X$ is closed (open) if and only if, for any edge e, the intersection of $C \cap \overline{e}$ is closed (open) in $\overline{e}$.

A graph is called *finite* if it has only a finite number of vertices and edges. For any finite graph, condition 3) is automatically fulfilled.

Since the space X is Hausdorff by condition, it follows from condition 2) that the *boundary* $\dot{e} = \overline{e} \setminus e$ of each edge consists of one or two vertices, and in the first case the set $\overline{e}$ is homeomorphic to a circle, and in the second case to the segment I.

If $\overline{e}$ is homeomorphic to the segment I, then the edge e is called a *simple edge*, and the points from $\dot{e}$ are called its *vertices*. Otherwise, the edge e is called *loop-like edge*. Such an edge, by definition, has only one vertex (however, for the unity of formulations, it is often convenient to assume that a loop-like edge has two, but coinciding vertices).

Every discrete space is a graph without edges. In a sense, the opposite example of a graph is any bouquet of circles. This graph has only one vertex and has no simple edges.

Remark 6.84. In graph theory (which, by the way, has undergone rapid development in recent years due to a number of important applied studies), it is customary to define a graph more abstractly as a collection of two sets (vertices and edges) connected by some map that maps an unordered pair of vertices (possibly coincident) to each edge. Graphs in our sense are then called *geometric realisations* of this kind of "abstract" graphs. This point of view (usually called "combinatorial") is completely equivalent to ours (which can be called "topological"), because, as it is easy to see, abstract graphs are isomorphic if and only if (in an understandable way) their geometric realisations are homeomorphic by means of homeomorphisms that translate vertices into vertices.

[2]The transcriber guesses that Postnikov refers to Lecture 9 of "Cellular Homotopy".

From the equivalence of the combinatorial and topological points of view, it follows that each topological property of graphs is equivalent to some of their combinatorial property, i.e. a property formulated only in terms of vertices and edges.

For example, it is clear that a graph is a bouquet of circles (topological property) if and only if when it has only one vertex (combinatorial property).

To obtain similar statements regarding the properties of connectivity and simple connectedness, we need to introduce a combinatorial analogue of the concept of path.

We will call an arbitrary symbol of the form e^{ε} an *oriented edge* of the graph X, where e is some edge of the graph X, and $\varepsilon = \pm 1$. To each oriented edge e^{ε}, we will arbitrarily match one of the vertices of the edge e, requiring only - in the case when the edge e is simple - that the edge e^{ε} be matched by another vertex of the edge e. The vertex mapped to the edge e^{ε}, we will call it the *initial vertex*, and the other vertex of the edge e - in the case when the edge e is simple - the *final vertex* of the oriented edge e^{ε}. For a loop-like oriented edge, its only vertex will, by definition, be both the initial and the final one.

A graph whose edges are all oriented and whose initial and final vertices are specified for them is called *oriented*.

Remark 6.85. Formally, the concept of an oriented graph is identical to the concept of a circuit diagramme.

A word of the form $v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n}$ consisting of oriented edges of a graph X is called a *route* in X if for any $i = l, \dots, n-1$ the final vertex of the edge $v_i^{\varepsilon_i}$ coincides with the initial vertex of the edge $v_{i+1}^{\varepsilon_{i+1}}$. We also consider the empty word $\varnothing$ (for which $n = 0$) to be a route.

A route $w = v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n}$ is said to *connect* the initial vertex x_0 of the edge $v_1^{\varepsilon_1}$ with the final vertex y_n of the edge $v_n^{\varepsilon_n}$. When $x_0 = y_n$, the route w is called a *closed route with the pole* x_0. An empty route, by definition, is closed. Its pole is considered to be an arbitrary vertex of graph X.

The concept of a route is the combinatorial equivalent of the topological concept of a path. For routes, it is possible to construct a purely combinatorial analogue of homotopy theory of path theory, and thus for any pointed graph X - its "combinatorial" fundamental group, which turns out to be isomorphic to its usual ("topological") fundamental group $\pi_1 X$. We will not do this (although we strongly recommend that the reader independently conduct all the necessary reasoning) and we will freely use topological and geometric considerations for the study of combinatorial constructions.

For each edge e of the graph X, any two of its characteristic paths $I \to X$ differ by some monotone map $I \to I$. Since any two monotone maps $I \to I$ of the same character (i.e. both increasing or both decreasing), as it is easy to see, are homotopic rel $\dot{I}$, we see, therefore, that all characteristic paths $I \to X$ for edge e fall into two classes: paths of the same class (considered as paths in $\overline{e}$) are homotopic rel $\dot{I}$, and maps of different classes are not homotopic. These classes are naturally identified with oriented edges $e^{\pm 1}$, which thus gives a geometric interpretation of the formally combinatorial concept of an oriented edge.

We see, therefore, that each oriented edge, and therefore any route, we, assuming an insignificant arbitrariness that disappears after the transition to homotopy classes, can be considered as a path in X. (This is the reason why routes are an adequate combinatorial analogue of paths. Of course, in order to fully substantiate this thesis, it is necessary and, conversely, to interpret any path in X - with permissible arbitrariness - as a route, but formally we will not need it, and therefore in connection with our common, with the installation explained above, we will leave this to the reader.)

In particular, we see that

Proposition 6.86. *any closed route with* x_0 *as a pole well defines some element of the group* $\pi_1(X, x_0)$.

A route in a graph X is called *simple* if it contains neither matching edges nor edges that differ only in orientations.

We will call a graph X *connected* if any two of its vertices can be connected by a simple route. (The fact that this combinatorial notion of connectivity coincides with the topological one will follow from our final results. Therefore, we will not prove it here, although we strongly recommend the reader to prove it now.)

A graph A is called a *subgraph* of the graph X if every edge (and every vertex) of the graph A is an edge (vertex) of the graph X. It is easy to see that any two vertices of a connected graph are contained in a finite connected subgraph.

6.H Trees

A connected graph T is called a *tree* if for any two of its vertices the simple route connecting them is unique, i.e., which is obviously equivalent if there are no non-empty simple closed routes in it. It is clear that

Proposition 6.87. *no tree contains loop-like edges and that any connected subgraph of a tree is a tree.*

A tree consisting of only one vertex and having no edges is called a *trivial tree*.

A simple edge e of a connected graph X is called *branch* if one (and only one) of its vertices is not the vertex of any other edge. Its other vertex y is said to have a branch e *attached* to the graph X' obtained from the graph X by removing the edge e and the vertex x.

It is clear that

Proposition 6.88. *for any graph* X' *and any of its vertices* y *it is possible to construct a graph* X *resulting from the graph* X' *by attaching a branch* e *at the vertex* y. *At the same time, if* X' *is a tree, then* X *will also be a tree.*

On the other hand, it is easy to see that

Proposition 6.89. *any nontrivial finite tree* T *contains a branch.*

Proof. Indeed, by applying finiteness in T, there is a simple route of maximum length, and it is clear that the initial and final edges of this route are (by applying maximality) branches. □

By obvious induction, it follows that

Proposition 6.90. *any finite tree is obtained from the trivial by sequentially attaching branches.*

Now we can prove the main theorem of the topological theory of trees.

Proposition 6.91. *Every tree T is contractible to any of its vertices x_0 (and therefore simply connected).*

Proof. It is clear that if the graph X is obtained from the graph X' by attaching a branch, then $X \searrow X'$. Therefore, for a finite tree T, Proposition 6.91 is proved by an obvious induction on the number of edges.

Let the tree T be infinite. For any vertex x of a tree T in T, there exists a finite connected subgraph T_x (and, therefore, a tree) containing vertices x_0 and x, and in this subgraph, the path u_x connecting the point x_0 with the point x (for example, we can assume that their path is obtained in the way described above from a simple route connecting the vertices x_0 and x in T_x). Let e be an arbitrary edge of the tree T. By choosing for e the characteristic path $u : I \to T$ (which, due to the simplicity of the edge e, is a homeomorphism) and putting $x = u(0)$, $y = u(1)$, we construct a path $v = u_x u u_y^{-1}$. This path is given by the formula

$$v(s) = \begin{cases} u_x(3s), & \text{if } 0 \le s \le 1/3, \\ u(3s-1), & \text{if } 1/3 \le s \le 2/3, \\ u_y(3-3s), & \text{if } 2/3 \le s \le 1, \end{cases}$$

and is a loop at the point x_0, entirely contained in the finite subgraph $T_e = T_x \cup T_y \cup e$ of the tree T. Since the subgraph T_e is obviously a finite tree and therefore, according to what has already been proved, is simply connected, the loop v is homotopic to zero in T_e, and therefore in T. This means that there is homotopy $G : I \times l \to T$ such that $G(0,t) = G(1,t) = G(s,1) = x_0$, and $G(s,0) = v(s)$ for any $t, s \in I$. But a simple elementary geometric construction (which we will present to the reader) shows that there exists a continuous map $\varphi : I \times I \to I \times I$, which is a homeomorphism on the inside of the square $I \times I$ such that

$$\varphi(s,0) = \begin{cases} (0, 1-3s), & \text{if } 0 \le s \le 1/3, \\ (3s-1, 0), & \text{if } 1/3 \le s \le 2/3, \\ (1, 3s-2), & \text{if } 2/3 \le s \le 1, \end{cases}$$

and

$$\varphi(0,t) = (0,1), \quad \varphi(s,1) = (s,1), \quad \varphi(1,t) = (1,1)$$

for any $s, t \in I$. It is clear that the formula $H = G \circ \varphi-^{-1}$ well defines a continuous map $H : I \times I \to I$ having the property that

$$\begin{aligned} H(0,t) = u_x(1-t), \quad H(1,t) = u_y(1-t), \quad t \in I, \\ H(s,0) = u_s, \quad H(s,1) = x_0, \quad s \in I. \end{aligned} \tag{6.92}$$

Using the fact that the map u is a homeomorphism, we will match any point $z \in e$ and any number $t \in I$ to the point $F(z,t) = H(s,t)$ of the tree T, where s is a number from I such that $z = u(s)$. It is clear that the resulting map $F : e \times I \to T$ is continuous.

On the other hand, it follows directly from formulae (6.92) that this construction is consistent at each vertex x of the tree T, i.e., that the point $F(x,t)$ does not depend on the choice of the edge e whose vertex is x. This means that we have well defined the map $F : T \times I \to T$, which has the property that on any subset of the form $\overline{e} \times I$ it is continuous. But then this map will, by applying condition 3) of Definition 6.83, be continuous and on the entire product $T \times I$, i.e. it will be a homotopy.

To complete the proof, it remains to note that, as directly follows from formulae (6.92), the homotopy F connects the identity map $T \to T$ with the constant map $T \to T$. □

6.I Calculation of the fundamental group of graphs

Now let X be an arbitrary connected graph. Consider the set $\mathfrak{T}$ of all possible *subtrees* (subgraphs that are trees) of the graph X. It is clear that for any family of trees linearly ordered by embedding $T_\lambda \in \mathfrak{T}$ their union $\cup T_\lambda$ is also a tree, i.e., lies in $\mathfrak{T}$. This means that the (obviously non-empty) set $\mathfrak{T}$ satisfies the conditions of Zorn's lemma and therefore contains a maximal element. This proves that

Proposition 6.93. *in any connected graph X there exists a maximal subtree T.*

It is easy to see that

Proposition 6.94. *a tree T is maximal if and only when it contains all vertices of the graph X.*

Proof. ($\Rightarrow$): Indeed, if there are vertices in X that do not belong to the tree T, then in X there is such a simple edge e that one of its vertices lies in T, and the other does not. Then $T \cup \overline{e}$ will be under a tree in X containing a tree T and different from T, which is impossible due to the maximality of T.
($\Leftarrow$): Conversely, let the tree T contain all vertices of the graph X. Then for any vertex $x \in X^0$ in the tree T there will be a single simple route w_x connecting to the vertex x some fixed vertex x_0. If the tree T is not maximal, then in X there exists a simple edge e such that the subgraph $T' = T \cup \overline{e}$ is a tree. Let x and y be the vertices of the oriented edge e^{+1}. Then the routes w_y and $w_x e^{+1}$ will be two different simple routes in T' connecting the vertex x_0 with the vertex y, which is impossible. Hence, the tree T is maximal. □

Assuming the maximum tree $T \subset X$ and the vertex $x_0 \in X^0$ fixed, we can consider a closed route $w_x e^{+1} w_y^{-1}$ for any edge $e \in X \setminus T$, where x, y are the vertices of the oriented edge e^{+1}, and w_x and w_y, as above, are simple routes in T connecting the vertex x_0 with vertices x and y, respectively. According to the above, this route uniquely defines some element of the fundamental group $\pi_1 X$. Assuming a certain liberty, we will designate this element with the symbol $[e]$.

Theorem 6.95. *The fundamental group $\pi_1 X$ of an arbitrary connected graph X is a free group. Elements of the form $[e]$, $e \in X \setminus T$, are (for any choice of the maximum subtree T) its free generators.*

Proof. Consider the coset space $X \setminus T$. It is clear that this coset space is a bouquet of circles, each of which is obtained from some (simple or loop-like) edge $e \in X \setminus T$, i.e., it has the form $\varphi\bar{e}$ where φ is the factorisation map $X \to X/T$. In this case, the image of the element $[e] \in \pi_1 X$, $e \in X \setminus T$, with the homomorphism $\varphi_* : \pi_1 X \to \pi_1(X/T)$ induced by the map φ, is obviously the generator of the fundamental group $\pi_1(\varphi\bar{e})$ of the circle $\varphi\bar{e}$ (in a standard way nested in the free product $pi_1(X/T)$ of the group $\pi_1(\varphi\bar{e})$). This means that the homomorphism φ_* is a homomorphism of the group $\pi_1 X$ on a free group $\pi_1(X/T)$, translating the elements of $[e]$ into free generators of the group $\pi_1(X/T)$.

On the other hand, it is easy to see that

Proposition 6.96. *the pair (X, T) is a cofibration*

(we leave this fact to the reader to prove; it is a trivial special case of Borsuk's general theorem, which we will prove in Lecture 11[3]; see Theorem 1 of Lecture 11). Therefore, by applying Lemma 4.46 of Lecture 4 and Proposition 6.91, the map φ is a homotopy equivalence and, therefore, the homomorphism φ_* is an isomorphism. Hence, the group $\pi_1 X$ is free and the elements $[e]$, $e \in X \setminus T$, are its free generators. □

Remark 6.97. It is easy to see that every graph X is covered, and that for any of its covering $\widetilde{X} \to X$, the space X is also a graph (see in Lecture 19[4], proposition 7). Therefore, by applying Theorem 6.95, for any covering $\widetilde{X} \to X$ of a graph X, the group $\pi_1 X$ is free. Taking for X a bouquet of circles, and for $\widetilde{X} \to X$ a covering corresponding to an arbitrary subgroup G of the group $\pi_1 X$ (and therefore having the property that $\pi_1 \widetilde{X} \approx G$), we immediately get from here that

Proposition 6.98. *any subgroup of a free group is free.*

This is the famous Nielsen-Schreier theorem, the algebraic proof of which is extremely difficult.

[3]The transcriber guesses Postnikov refers to Lecture 1 of "Cellular Homotopy"
[4]The transcriber guesses Postnikov refers to Lecture 9 of "Cellular Homotopy"

Lecture 7

To apply homotopy groups to individual geometric problems, it is necessary to be able to calculate these groups for specific spaces. However, in general, this task turns out to be extremely difficult, and so far, despite all the efforts and sophisticated techniques developed (which we will gradually become familiar with), there is not a single space (except for trivial cases like the circle $\mathbb{S}^1$) about which we could say that all its homotopy groups are known to us. This applies even to such simple spaces as spheres $\mathbb{S}^n$ for $n > 1$.

In this lecture we will calculate the homotopy groups $\pi_m \mathbb{S}^n$ for $m \leq n$. This requires some approximation methods, in principle alien to homotopy theory (but the use of which, apparently, cannot be avoided; interestingly, after $m \leq n$ the groups $\pi_m \mathbb{S}^n$ are calculated, their calculation at $m > n$ - as far as it can be done - can already be produced by purely homotopy, or, better to say, by algebro-topological means). The necessary approximations can be made by means of either smooth or piecewise linear (= simplicial) maps. We will use smooth approximations, because, firstly, the basic concepts of the theory of smooth manifolds are certainly known to the reader from the compulsory course, and secondly, there are many excellent expositions of this theory in Russian; see, for example, [14]. However, we will also present all the necessary information from the theory of simplicial approximations in our place; see the Appendix to Lecture 12[1].

In Lecture 12, we will recalculate the groups $\pi_m \mathbb{S}^n$, $m \leq n$, and based on some more general considerations. However, the more geometric methods of this lecture have their advantages and familiarity with them is by no means useless.

7.1 Smooth maps and smooth homotopies

In this lecture, we will always understand by a *manifold* a smooth (of some prescribed class C^r, $r \geq 2$) compact Hausdorff (and therefore metrisable) manifold, generally speaking, with boundary. Manifolds without boundary we will call *closed manifolds*. The guiding example of a manifold without boundary will be

[1] The transcriber guesses Postnkov refers to Lecture 2 of "Cellular Homotopy".

a sphere $\mathbb{S}^n$, and a manifold with boundary will be its product $\mathbb{S}^n \times I$ with a segment.

On an arbitrary manifold N, it is always possible to introduce a Riemannian metric, moreover, as proved in the theory of Riemann spaces, for every Riemannian metric ρ on N there exists a $\varepsilon_0 > 0$ (usually called *Morse number* of a Riemannian manifold N), such that for any two points $y_0, y_1 \in N$ with $\rho(y_0, y_1) < \varepsilon_0$ in N there is a unique geodesic $t \mapsto \gamma(t; y_0, y_1)$ of length $< \varepsilon_0$, connecting the point $_0$ with the point y_1. For example, for the sphere $\mathbb{S}^n$ (in its natural metric), the Morse number is π.

Assuming that the geodesic $t \mapsto \gamma(t; y_0, y_1)$ is related to the parameter t, proportional to the length of the arc and varying from 0 to 1, consider the map $(y_0, y_1, t) \to \gamma(t; y_0, y_1)$ into the manifold N of the open subset $\{(y_0, y_1, t) | \rho(y_0, y_1) < \varepsilon_0\}$ of the manifold $N \times N \times I$. It follows directly from the theorem on the smooth dependence of solutions of differential equations on the initial data that this map is smooth and, in particular, continuous. Therefore, for any topological space X and any two maps $f, g : X \to M$ satisfying the condition $\rho(f(x), g(x)) < \varepsilon_0$, $x \in X$, the formula

$$F(X, t) = \gamma(t; f(x), g(x))$$

well defines the homotopy $F : X \times I \to N$ connecting the map f with the map g. Thus,

Proposition 7.1. *maps to a manifold are homotopic if they are close enough.*

Recall now that a continuous map $g : M \to N$ of a manifold M to a manifold N is called a *smooth map* if in local coordinates it is written by smooth functions. In the theory of smooth manifolds, it is proved that

Proposition 7.2. *any continuous map $f : M \to N$ can be arbitrarily approximated by smooth maps,*

i.e. that for any $\varepsilon > 0$ there exists a smooth map $g : M \to N$ such that $\rho(f(x), g(x)) < \varepsilon$ for any point $x \in M$. (In each coordinate neighbourhood, the functions defining the map f are approximated - according to the classical Weierstrass polynomial approximation theorem - by means of smooth functions, and then these local approximations are stitched into a single map g.) In combination with the previous result, this gives the following statement (we do not call it a theorem, since it is actually given by us without proof).

Proposition 7.3 (Statement 1)**.** *Any continuous map $f : M \to N$ of a smooth manifold M to a smooth manifold N is homotopic to some smooth map $g : M \to N$.*

For manifolds with boundary, this statement admits correction.

Proposition 7.4 (Statement 1')**.** *If a continuous map $f : M \to N$ is smooth on the boundary ∂M of a manifold M, then there exists a smooth map $g : M \to N$ coinciding with f on ∂M and homotopic f with respect to ∂M.*

The proof of the statement 7.4 is based on the so-called collar theorem, which states that the boundary ∂M has a neighbourhood in M diffeomorphic to the product of ∂M with the half-open interval $[0, 1)$. In addition, some refinement of the approximation theorem is required, which guarantees the coincidence of the approximating and approximating function on the set, where the approximated function is already smooth. However, we will need the statement 7.4 only for the case of maps of the form MN (and in fact even $\mathbb{S}^n \times I \to N$), when most of these technical difficulties are absent (or trivially overcome).

For a closed manifold M, the boundary of the manifold $M \times I$ consists of the manifolds $M \times 0$ and $M \times 1$, and the condition that the map $M \times I \to N$ is smooth on the boundary means that this map, considered as a homotopy of M in N, connects smooth maps. Therefore, with respect to homotopies from M to N, the statement 7.4 gives us that smooth maps of a closed manifold M to an arbitrary manifold N are homotopic if and only if they are smoothly homotopic (i.e. connected by a homotopy from M to N, which is a smooth map of $M \times I \to N$).

Thus, in the study of homotopy classes of maps $M \to N$ (and, in particular, groups $\pi_m \mathbb{S}^n = [\mathbb{S}^m, \mathbb{S}^n]$) without loss of generality, we can limit ourselves only to smooth maps and their smooth homotopies.

7.2 Sard's theorem

For closed manifolds M and N, every smooth map $f : M \to N$ at any point $x \in M$ induces a linear map $(df)_x$ (called the *differential* of the map f at point x) of the tangent space $T_x M$ of the manifold M at the point x to the tangent space $T_y N$ of the manifold N at the point $y = f(x)$. In local coordinates (or, more precisely, in the corresponding bases of tangent spaces), the matrix of the linear map $(df)_x$ is the Jacobi matrix $\partial f / \partial x$ of the functions $f^1, \dots, f^n$ that define the map f in local coordinates; the elements of this matrix are the values at the point $(\xi^1(x), \dots, \xi^m(x))$ of partial derivatives $\partial f^i / \partial \xi^j$ of functions $f^1, \dots, f^n$ at the local coordinates $\xi^1, \dots, \xi^m$ on the manifold M near the point x (here, as everywhere below, the symbol m we denote the dimension of the manifold M, and the symbol n is the dimension of the manifold N).

A point x is called the *critical point* of a smooth map $f : M \to N$ if the linear map $(df)_x$ is not adjective, i.e., if the rank of the Jacobi matrix $\partial f / \partial x$ at point x is less than n. A point $y \in N$ is called a *regular value* of the map f if no point $x \in f^{-1}(y)$ is a critical point.

The famous Sard's theorem (which, however, was already proved by Brown a few years before Sard, and subsequently significantly improved by Dubovitsky) states that for any smooth map $f : M \to N$, the set of all its regular values is dense everywhere (recall that we consider the variety M to be compact; for a non-compact M the statement of Sard's theorem is somewhat weakened). In particular, according to this theorem,

Proposition 7.5. *for any smooth map $f : M \to N$ there are regular values.*

For $m < n$, each point $x \in M$ is a critical point, and therefore regular values are exactly points that do not belong to the image $f(M)$ of the manifold M. Thus, we see that for $m < n$, Sard's theorem boils down to the statement that

Proposition 7.6. *a smooth map of a manifold to a manifold of greater dimension is necessarily not surjective (using somewhat fuzzy, but expressive terminology, we can say that* a smooth map does not increase dimension*).*

7.3 The group $\pi_m \mathbb{S}^n$ for $m < n$

In particular, we see that for $m < n$ for any smooth map $f : \mathbb{S}^m \to \mathbb{S}^n$ there exists a point $y_0 \in \mathbb{S}^n$ such that $f(\mathbb{S}^m) \subset \mathbb{S}^n \setminus y_0$. This means that $f = i \circ f'$, where f' is the map $\mathbb{S}^m \to \mathbb{S}^n \setminus y_0$, and i is the inclusion $\mathbb{S}^n \setminus y_0 \to \mathbb{S}^n$. But the set $\mathbb{S}^n \setminus y_0$ is homeomorphic (by, say, a stereographic projection) to the space $\mathbb{R}^n$ and therefore contractible. Hence, the map f', and hence the map f, is homotopic to the constant map. Since any continuous map $\mathbb{S}^m \to \mathbb{S}^n$ is homotopic to a smooth map, it is proved that for $m < n$ the group $\pi_m \mathbb{S}^n$ consists only of zero:

$$\pi_m \mathbb{S}^n = 0, \quad \text{if} \quad m < n. \tag{7.7}$$

Assuming $m = 1$, we get, thus, that

Proposition 7.8. *for $n > 1$ the sphere $\mathbb{S}^n$ is simply connected.*

7.4 The fundamental group of the space $\mathbb{S}^n/G$

By applying the general isomorphism (6.27) of Lecture 6, it follows that for any group G acting completely discontinuously on the sphere $\mathbb{S}^n$, $n > 1$, the fundamental group of the space $\mathbb{S}^n/G$ is isomorphic to the group G:

$$\pi_1(\mathbb{S}^n/G) = G. \tag{7.9}$$

Example 7.10. A group of order 2 with α as a generator acts completely discontinuously on the sphere $\mathbb{S}^n$ according to the formula $\alpha(x) = -x$, and the corresponding coset space $\mathbb{S}^n/G$ is an n-dimensional projective space $\mathbb{R}P^n$. Therefore,

$$\pi_1 \mathbb{R}P^n = \mathbb{Z}/2\mathbb{Z} \quad \text{for} \quad n > 1$$

(the space $\mathbb{R}P^1$ is a circle, and therefore $\pi_1 \mathbb{R}P^1 = \mathbb{Z}$.)

Example 7.11. A cyclic group of order h with a generator α can act quite discontinuously (= without fixed points) on an odd-dimensional sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ in many ways. In particular, each set of integers mutually prime with h numbers $a_0, a_1, \dots c, a_n$ (given up to terms that are multiples of h) defines this kind of action according to the formula

$$\alpha(z_0, z_1, \dots, z_n) = (\zeta^{a_0} z_0, \zeta^{a_1}, z_1, \dots \zeta^{a_n} z_n), \quad (z_0, \dots, z_n) \in \mathbb{S}^{2n+1}, \tag{7.12}$$

where ζ is the primitive root of 1 of degree h. (In fact, any completely discontinuous action of the group $\mathbb{Z}/h\mathbb{Z}$ on the sphere $\mathbb{S}^{2n+1}$ is equivalent to an action of the form (7.12), but this fact lies beyond the scope of our presentation.) When replacing ζ with another primitive root η associated with ζ by the equality $\zeta = \eta^a$, where a is mutually prime to h, the numbers $a_0, a_1, \dots, a_n$ are replaced by the numbers $aa_0, aa_1, \dots, aa_n$. With this in mind, we will call the quotient space of the sphere $\mathbb{S}^{2n+1}$ by the action (7.12) of the group $\mathbb{Z}/h\mathbb{Z}$ a *lens space of type* $(a_0 : a_1 : \dots : a_n : h)$ and we will denote it with the symbol $L(a_0 : a_1 : \dots : a_n : h)$. According to the general formula (7.9)

$$\pi_1 L(a_0 : a_1 : \dots : a_n : h) = \mathbb{Z}/h\mathbb{Z}.$$

Remark 7.13. A lot of effort has been put into establishing the homeomorphism and homotopy equivalence of lens spaces. However, the question of homotopy equivalence is solved without much difficulty and, as it turns out, two lens spaces $L(a_0 : a_1 : \dots : a_n : h)$ and $L(b_0 : b_1 : \dots : b_n : h)$ are homotopically equivalent if and only i when there exists a number c such that

$$b_0 b_1 \cdots b_n \equiv \pm c^{n+1} a_0 a_1 \cdots a_n \bmod h.$$

The only real difficulty is the question of the homeomorphism of lens spaces. It was solved (and then only partially) only relatively recently on the basis of a very deep and complex technique.

For three-dimensional (case $n = 1$) lens spaces $L(a_0 : a_1 : h)$, the asymmetric notation $L[a, h]$ is accepted, where a is a number such that $a_1 \equiv aa_0 \bmod h$. Two spaces $L[a, h]$ and $L[b, h]$ are homotopically equivalent if and only if there exists a number c such that $b \equiv \pm c^2 a \bmod h$, and are homeomorphic if and only if $c \equiv 1 \bmod h$ or $c \equiv b \bmod h$ (in the latter statement, only "only if" is nontrivial).

7.5 The degree of smooth maps.

Recall that an atlas of a smooth manifold M is called an *oriented atlas* if for any two of its charts (with non-empty intersection) the Jacobian of the transition from the local coordinates of one chart to the local coordinates of the other chart is positive. A manifold having an oriented atlas is called *orientable*. Maximal oriented atlases are called *orientations* of the orientable manifold M. A manifold with a fixed orientation is called *oriented*. Charts of this orientation are called *positive charts*. A connected orientable manifold has exactly two orientations, which are called *opposites*. Each chart of a connected orientable manifold defines some orientation of the manifold with respect to which this chart is positive.

If now M and N are two smooth oriented closed manifolds of the same dimension n, then for any smooth map $f : M \to N$ and any of its non-critical points $x \in M$ we can consider in M and N positive charts containing the points x and $y = f(x)$ and the Jacobian of the functions defining the map f in these charts. It is clear that the sign of this Jacobian does not depend on the choice of chart, i.e. it is well determined by the point x and the map f (given the orientations of the manifolds M and N). We will call it the *map index* of the map f at the point x and we will denote it with the symbol $\varepsilon_f(x)$.

It follows directly from the elementary analytic inverse map theorem that a smooth map $f : M \to N$ of closed manifolds on some neighbourhood of each non-critical point $x \in M$ is a diffeomorphism of this neighbourhood to some neighbourhood of the point $y = f(x)$. Therefore, for any regular value $y \in N$, its preimage $f^{-1}(y)$ consists of a finite number of (non-critical) points. (Note that the compactness of the manifold M is essentially used here.) We will put

$$\deg_y f = \sum_{x \in f^{-1}(y)} \varepsilon_f(x),$$

where the summation is extended to all points $x \in f^{-1}(y)$. Conventionally, the number $\deg_y f$ can be called the "algebraic number" of points from $f^{-1}(y)$.

It is clear that every point $x \in f^{-1}(y)$ has a neighbourhood, all points of which are also non-critical (this will be a neighbourhood in which the map f is a diffeomorphism), and $\varepsilon_{x_1}(f) = \varepsilon_x(f)$ for any point x_1 of this neighbourhood. Therefore,

Proposition 7.14. *each regular value of $y \in N$ the smooth map f has a neighbourhood consisting of regular values, and for any point y_1 of this neighbourhood*

$$\deg_{y_1} f = \deg_y f. \tag{7.15}$$

(In the language of analysis, this means that the function $y \mapsto \deg_y f$ is locally constant.)

It turns out that

Proposition 7.16. *if the manifold N is connected, then equality (7.15) holds for any regular values $y, y_1 \in N$ of the map f, i.e. in this case the number $\deg_y f$ does not depend on the choice of the regular value y (the function $y \mapsto \deg_y f$ is constant).*

It would seem that this directly follows from the local constantness of the function $y \mapsto \deg_y f$, but in fact the situation here is much more subtle, because the set of regular values of the map f even with a connected N is, generally speaking, disconnected To better understand why the function $y \mapsto \deg_y f$ is constant, we will give this two proofs, one of which can be described as "analytical" and the other as "geometric".

Proof. (**Analytical proof**). This proof is based on the following facts, which we will assume to be known,:

1. On any oriented compact smooth n-dimensional manifold N there exists a differential form ω_0 of degree n for which the integral

$$\omega_0[N] = \int_N \omega_0$$

is nonzero.

If a Riemannian metric is introduced on N, then the shape of the volume has this property. For a sphere $\mathbb{S}^n$ assigned to ordinary spherical coordinates with $\theta_1, \theta_2, \ldots, \theta_n$, the shape of the volume is equal to

$$\cos\theta_2 \cos^2\theta_3 \cdots \cos^{n-1}\theta_n (d\theta_1 \wedge d\theta_2 \wedge \cdots \wedge d\theta_n).$$

2. If the manifold N is connected, then, each differential form of the degree of n by N is expressed by the formula

$$\omega = r\omega_0 + d\alpha,$$

where $r \in \mathbb{R}$; and α is some form of degree $n-1$.

3. For any open set $U \subset M$, there exists a form ω of degree n such that $\omega = 0$ outside U and $\omega[M] \neq 0$.

It follows that for any smooth map $f : M \to N$ of closed manifolds and any form ω on M for which $\omega[M] \neq 0$, the number

$$d(f) = \frac{f^*\omega[M]}{\omega[N]} \tag{7.17}$$

does not depend on the choice of the form ω (assuming that the manifold N is connected). Indeed, if $\omega = r\omega_0 + d\alpha$, then by applying Stokes' theorem $\omega[N] = r \inf_N \omega_0 = r\omega_0[M]$ and similarly $(f^*\omega)[M] = r(f^*\omega_0)[M]$. Thus

$$\frac{f^*\omega[M]}{\omega[N]} = \frac{f^*\omega_0[M]}{\omega_0[N]}$$

On the other hand, if $y \in N$ is a regular value of the smooth map $f : M \to N$, $x_1, \ldots, x_s \in M$ are all its preimages and $V_1, \ldots, V_s \subset M$ are disjoint neighbourhoods (which we can assume to be coordinate) of points $x_1, \ldots, x_s$, each of which the map f diffeomorphically maps to the (coordinate) neighbourhood U of the point $y \in N$, then for any form ω, equal to zero outside U,

$$\omega N = \int_N \omega = \int_U \omega, \quad (f^*\omega)M = \int_M f^*\omega = \int_{V_1} f^*\omega + \ldots + \int_{V_s} f^*\omega,$$

and according to the classical rule of replacing variables in multiple integrals, for any $i = 1, \ldots, s$, the equality takes place

$$\int_{V_i} f^*\omega = \varepsilon_i \int_U \omega,$$

where ε_i is the sign of the Jacobian of the map f at the point x_i (or, equivalently, in the neighbourhood of V_i). Thus

$$(f^*\omega)M = \varepsilon_1\omega[N] + \ldots + \varepsilon_s\omega[N] = (\varepsilon_1 + \ldots + \varepsilon_s)[N]$$

and then $d(f) = \varepsilon_1 + \ldots + \varepsilon_s = \deg_y f$.

Since the number $d(f)$ does not depend on the choice of the point y, therefore, the number $\deg_y f$ also has this property.

Note that we not only proved the independence of the number $\deg_y f$ from the point y, but also found an explicit formula for it in the form of a ratio of two integrals. □

Proof. (**Geometric proof**). In this proof, as above, we will assume that a number of statements of the theory of smooth manifolds are known, most of which are clearly obvious, but it is quite troublesome to prove them accurately. Bearing in mind what follows, we will formulate them in a somewhat larger volume than is directly necessary.

Let M and N be smooth manifolds (generally speaking, of different dimensions), P be a submanifold of a manifold N and $f : M \to N$ be a smooth map. If for any point $x \in f^{-1}P$ the space $T_{f(x)}N$ is a smm (not necessarily direct) of its subspaces $(df)_x T_x M$ and $T_{f(x)}P$, then the map f is called *transversal* to P (or *t-regular along* P). If $P = \{y\}$, then this condition reduces to the requirement that the point y be a regular value of the map f. If the submanifold P has a boundary ∂P, then it is additionally required that f be transversal to ∂P, and if, in addition, M has a boundary (the case when the boundary has N we do not need), then it is required that the transversality condition (with respect to both P and ∂P) was also applied to the restriction of the map f on this boundary.

Proposition 7.18 (Statement 2)**.** *If a smooth map $f : M \to N$ is transversal to P, then the preimage $f^{-1}P$ of the submanifold P is a submanifold of the manifold M of dimension* $\dim M + \dim P - \dim N$*, the boundary of which is a preimage of the boundary of the manifold*

In particular,

Proposition 7.19. *the preimage $f^{-1}(y)$ of any regular value $y \in N$ is a submanifold of the manifold M of dimension* $\dim M - \dim N$.

Of course, in accordance with the general definition of t-regularity, in the case when the manifold M has a boundary ∂M, it is assumed here that y is also a regular value of the map $f|_{\partial M}$.

In the latter case, we can additionally assert that the intersection of $f^{-1}(y) \cap \partial M$ of the manifold $f^{-1}(y)$ with the boundary ∂M of the manifold M is its boundary $\partial f^{-1}(y)$, and at no point $x \in \partial f^{-1}(y)$ is the manifold $f^{-1}(y)$ tangent to the manifold ∂M (i.e. $T_x(f^{-1}(y)) \not\subset T_x(\partial M)$).

For any submanifold $P \subset N$ and any smooth map $f : M \to N$ transversal to P, it is also possible to describe fairly accurately the behaviour of the map f near each component Q of the submanifold ^{-1}P. In general, we will not need a description of this, and therefore we will limit ourselves to the three simplest cases.

Case 1. (actually already analysed above). Let P be a point $y \in N$ (the regular value of the map f) and $\dim M = \dim N = n$. Then the preimage

$f^{-1}P = f^{-1}(y)$ consists of individual points and each point is $x \in f(y)$ has a neighbourhood U, which f diffeomorphically maps to some neighbourhood V of the point y. By reducing, if necessary, the neighbourhoods of U and V, we can assume that the neighbourhood of V is diffeomorphic to the open ball $\mathring{\mathbb{E}}^n$ of the Euclidean space $\mathbb{R}^n$. By combining this diffeomorphism with the diffeomorphism $f|_y$, we get the diffeomorphism $U \to \mathbb{E}^n$, for which a commutative diagramme takes place

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ \mathring{\mathbb{E}}^n & \xrightarrow[\mathrm{id}]{} & \mathring{\mathbb{E}} \end{array}$$

Case 2. Let it continue as before $P = \{y\}$ and $\dim M = n+1$. Then $f^{-1}P$ is a compact one-dimensional manifold, and therefore (of course, this "because" needs justification) each of its components is diffeomorphic to either a circle or a segment (which is, as they say, a *embedded arc*). In this case, components diffeomorphic to the segment (arc) can exist only if the manifold M has a boundary ∂M. It turns out that every diffeomorphic segment of the component Q the manifolds $f^{-1}P$ have (in M) a neighbourhood U, and the point y_0 in N has a neighbourhood V such that:

(i) there is a commutative diagramme

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ \mathring{\mathbb{E}}^n \times I & \xrightarrow[\mathrm{proj}]{} & \mathring{\mathbb{E}} \end{array}$$

where the vertical arrows are diffeomorphisms, and the lower horizontal arrow is the projection $\mathrm{proj} : (\boldsymbol{x}, t) \mapsto \boldsymbol{x}$, $\boldsymbol{x} \in \mathring{\mathbb{E}}^n$, $t \in I$.

(ii) the diffeomorphism $U \to \mathring{\mathbb{E}}^n \times I$ maps the arc Q to the segment $0 \times I$, and the intersection of $U \cap \partial M$ to the union $(\mathring{\mathbb{E}}^n \times 0) \cup (\mathring{\mathbb{E}}^n \times 1)$ (see 7.5.1).

Case 3. Let P be a submanifold of a diffeomorphic segment (i.e. an embedded arc), and $\dim = M = \dim N = n$. Then the submanifold $f^{-1}P$ is again one-dimensional and we will again consider its component Q, diffeomorphic to the segment (which is an embedded arc). In this case, the submanifolds Q and P have such neighbourhoods U and V that

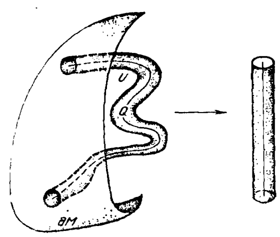


Figure 7.5.1:

(i) there is a commutative diagramme

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ \mathring{\mathbb{E}}^n \times I_\varepsilon & \xrightarrow[\mathrm{id} \times \varepsilon]{} & \mathring{\mathbb{E}} \times I_\varepsilon \end{array}$$

the vertical arrows of which are diffeomorphisms, and the lower horizontal arrow is given by the correspondence

$$((x), t) \mapsto \boldsymbol{x}, \varphi(t)), \quad x \in \mathring{\mathbb{E}}^{n-1}, \ t \in I_\varepsilon,$$

where $I_\varepsilon = (-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, and φ is a smooth function $I_\varepsilon \to I_\varepsilon$ such that $\varphi(0) = 0$, and $\varphi'(0) \neq 0$, $\varphi'(1) \neq 1$;

(ii) the diffeomorphisms $U \to \mathring{\mathbb{E}}^n \times I_\varepsilon \mapsto \mathring{\mathbb{E}}^n \times I_\varepsilon$ represent the arcs Q and P on the segment $0 \times I$.

Note that from these conditions (and the requirement of transversality) it follows, firstly, that $\varphi(1)$ is equal to either 0 or 1, and secondly, that for $0 < t < 1$ the number $\varphi(t)$ is not equal to either 0 or 1.

In all three cases, it is customary to call the lower arrow of the constructed diagrams *normal form* of the map f. Thus, in case 1, the normal form is the identical map id, in case 2, the projection proj, and in case 3, the map of the form id $\times\varphi$ (the map φ can also be further normalised, but we won't need it).

In the language of local coordinates, the existence of these normal forms means that in the vicinity of U and V, one can choose such local coordinates $\xi_1, \dots, \xi_n$ (and $\xi_{n+1} = t$ in case 2) and $\eta_1, \dots \eta_n$, that the map f will be in these coordinates is written by the formulae

$$\eta_1 = \xi_1, \dots, \eta_n = \xi_n \quad \text{in cases} \quad 1 \text{ and } 2,$$
$$\eta_1 = \xi_1, \dots, \eta_{n-1} = \xi_{n-1}, \eta_n = \varphi(\xi_n) \quad \text{in case} \quad 3.$$

Sard's theorem can also be generalised to the case of arbitrary submanifolds $P \subset N$.

Let P and $P^{'}$ be diffeomorphic submanifolds of a manifold N. This means that there exists a smooth manifold P_0 and smooth embeddings $i : P_0 \to N$ and $i^{'} : P_0 \to N$ such that $iP_0 = P$ and $i^{'}P_0 = P^{'}$. Assuming the embeddings i and $i^{'}$ are fixed, we will call the submanifolds P and $P^{'}$ *ε-close* (with respect to a certain metric ρ on N) if $\rho(i(x), i^{'}(x)) < \varepsilon$ for any point $x \in P_0$. The generalised Sard theorem states that

Proposition 7.20. *for any smooth map $f : M \to N$ of smooth compact manifolds, any submanifold $P \subset N$, and any $\varepsilon > 0$, there exists a submanifold $P^{'}$ ε-close to P such that the map f is transversal to $P^{'}$.*

At the same time, if the submanifold P has a boundary ∂P and the map f is transversal to ∂P, then you can additionally require that $\partial P^{'} = \partial P$.

Now we can proceed directly to the proof of the independence of the number $\deg_y f$ from y.

Let M and N be closed smooth oriented manifolds of the same dimension n, and let $f : M \to N$ be a smooth map, and $y_0, y_1 \in N$ be its two regular values. Let, moreover, the manifold N be connected. Then (of course, for $N \neq \mathbb{S}^n$ it needs to be proved) the points y_0, y_1 can be connected by a embedded arc P, and by applying the generalised Sard theorem, it can be assumed that the map f is transversal to P. According to statement 7.18, the preimage $f^{-1}P$ will be a one-dimensional manifold with the boundary $\partial f^{-1}P$, which is the preimage of $f^{-1}(y_0) \cup f^{-1}(y_1)$ of the ends of the arc P. In this case, each point $x_0 \in \partial f^{-1}P$ will be the end of some embedded arc Q - components of the manifold $f^{-1}P$. Let $x_1 \in \partial f^{-1}P$ be the other end of this arc.

According to the above, we can assume that in the manifolds M and N there are positive maps $(U; \xi_1, \dots, \xi_n)$ and $(V; \eta_1, \dots, \eta_n)$ such that:

1) $Q \subset U$ and a point $x \in U$ belongs to Q if and only if when

$$\xi_1(x) = 0, \dots, \xi_{n-1}(x) = 0, \quad 0 \leq \xi_n(x) \leq 1;$$

2) $P \subset V$ and a point $y \in V$ belongs to P if and only if when

$$\eta_1(x) = 0, \dots, \eta_{n-1}(y) = 0, \quad 0 \leq \eta_n(y) \leq 1;$$

3) there is an inclusion $fU \subset V$, and the map f is given in local coordinates $\xi_1, \dots c, \xi_n$ and $\eta_1, \dots c, \eta_n$ by formulae

$$\eta_1 = \xi_1, \dots, \eta_{n-1} = \xi_{n-1}, \eta_n = \varphi(\xi_n),$$

where φ is a smooth function such that $\varphi(0) = 0$ and $\varphi^{'}(0) > 0$, $\varphi^{'}(1) \neq 0$, with $\varphi(1)$ is equal to either 0 or 1, and $0 < \varphi(\xi) < 1$ for $0 < \xi < 1$.

At the same time, without loss of generality, we can assume that $\xi_n(x_0) = 0$, $\xi_n(x_1) = 1$ and similarly that $\eta_n(y_0) = 0$, $\eta_n(y_1) = 1$ (so $x_0 \in f^{-1}y_0$, and $x_1 \in f^{-1}y_0$ if $\varphi(1) = 0$, and $x_1 \in f^{-1}y_1$ if $\varphi(1) = 1$).

The Jacobian of the map f at point x_0 is obviously $\varphi^{'}(0)$, so the contribution of point x_0 to $\deg_{y_0} f$ is $\varepsilon_{x_0} = \operatorname{sgn} \varphi^{'}(0)$. Similarly, the point x_1 gives a contribution to $\deg_{y_0} f$ (for $\varphi(1) = 0$) or to $\deg_{y_1} f$) (for $\varphi(1) = 1$) equal to $\varepsilon_{x_1} = \operatorname{sgn}\varphi'(1)$. On the other hand, it follows directly from the conditions imposed on the φ function (see Fig. 7.5.2) that

$$\operatorname{sgn}\varphi'(1) = \begin{cases} \operatorname{sgn}\varphi'(0), & \text{if} \quad \varphi(1) = 1, \\ -\operatorname{sgn}\varphi'(0), & \text{if} \quad \varphi(1) = 0. \end{cases}$$

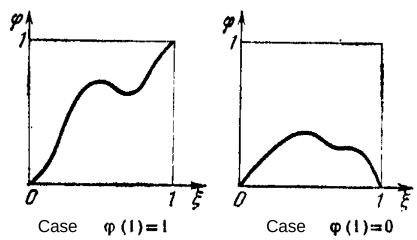


Figure 7.5.2:

Therefore, for $\varphi(1) = 1$, the points x_0 and x_1 give equal contributions of the numbers $\deg_{y_0} f$ and $\deg_{y_1} f$, and for $\varphi(1) = 0$ - mutually decreasing contributions to the number $\deg_{y_0} f$.

Thus, calling the point $x_0 \in f^{-1}y_0$ (the point $x_1 \in f^{-1}y_1$) *essential* if $x_1 \in f^{-1}y_1$ (if $x_0 \in f^{-1}y_0$), and *inessntial* if $x_1 \in f^{-1}y_0$ (respectively $x_0 \in f^{-1}y_1$), we get that the inessential points are included in $f^{-1}y_0$ (and also, of course, in $f^{-1}y_1$) in pairs with different signs (and therefore when counting degrees are mutually annihilated), and each essential of $f^{-1}y_0$ corresponds to each other uniquely in $f^{-1}y_1$, an essential point with the same sign. Therefore, $\deg_{y_0} f = \deg_{y_1} f$. Cf.Fig 7.5.3. □

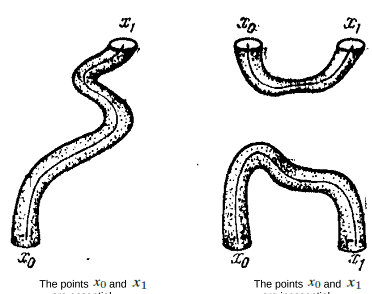


Figure 7.5.3:

Since $\deg_y f$ does not depend on y, then by putting

$$\deg f = \deg_y f,$$

where y is an arbitrary regular value of the map f, we well assign to each smooth map $f : M \to N$ some integer $\deg f$ (coinciding, note, with the number $d(f)$ defined by the integral formula (7.17)). This number is called the *degree of the map* f.

We emphasize that the manifolds M and N are assumed here to be smooth, closed, oriented manifolds of the same dimension, and the manifold N is also assumed to be connected.

7.6 Homotopy invariance of degree

It turns out that

Proposition 7.21. *if two smooth maps $f, g : M \to N$ are smoothly homotopic, then their degrees coincide:*

$$\deg f = \deg g. \tag{7.22}$$

Proof. This is especially easy to prove using formula (7.17), and for this it must even be required that the homotopy $f_t : M \to N$ connecting the map data be smooth. Indeed, according to the analysis theorem on the dependence of integrals on a parameter, the number

$$(f_t^*\omega)[M] = \int_M f_t^*\omega$$

continuously depends on t. Therefore the number

$$\deg f_t = \frac{(f_t^*\omega)[M]}{\omega[M]}$$

also continuously depends on t. Therefore, being an integer, this number is constant. □

With the "geometric" approach, it is advisable to prove a more general statement, fee of extraneous details, the trivial consequence of which is the equality (7.22). To do this, we will need some general results of the theory of smooth manifolds related to orientations, which we will still accept without proof.

For any $(n+1)$-dimensional manifold W with boundary with its boundary ∂W is characterised by the fact that they have charts $(U; \xi_0, \xi_1, \dots \xi_n)$ for which the diffeomorphism $x \mapsto (\xi_0(x), \xi_1(x), \dots \xi_n(x))$ is a map to the half-space $\xi_0 \geq 0$ of the space $\mathbb{R}^{n+1}$. At the same time

$$(U \cap \partial W; \xi_1|_{U\cap\partial W}, \dots, \xi_n|_{U\cap\partial W}) \tag{7.23}$$

will be a chart of the manifold ∂W.

Proposition 7.24 (Statement 3)**.** *If a manifold W is orientable, then its boundary ∂W is also orientable. Any orientation of the manifold W produces some orientation of the manifold ∂W.*

Namely, if the chart $(U;\xi_0,\xi_1,\dots,\xi_n)$ set a given orientation of the manifold W, then, by definition, map (7.23) will set the induced orientation of the manifold ∂W. (Of course, this definition needs - not at all obvious - a correctness check,)

In the case when $W = M\times I$ and, therefore, $\partial W = (M\times 0)\cup(M\times 1)$, the manifold W is orientable if and only if ∂W, i.e., the manifold M is orientable. At the same time, for any map $(U;\xi_1,\dots,\xi_n)$ of the manifold M that sets its orientation o, the chart $(U\times I;\xi_0^*,\xi_1^*,\dots,\xi_n^*)$, where

$$\xi_0^*(x,t)=t, \xi_1^*(x,t)=\xi_1(x),\dots\xi_n^*(x,t)=\xi_n(x), \quad x\in U,\ t\in I,$$

will set the orientation $o\times I$ of the manifold $M\times I$, inducing the orientation o on $M\times 0$, i.e., more precisely, passing into the orientation o when identifying $(x,0)\mapsto x$. On the contrary, on $M\times 1$, the orientation of $o\times I$ will induce the opposite orientation of $-o$, since the orientation of o is obviously induced by the orientation given by the chart $(U\times I;1-\xi_0^*,\xi_1^*,\dots,\xi_n^*)$ which is opposite to the orientation of $o\times I$. Conditionally , this situation can be written with the formula

$$\partial(M\times I)=M\times 0-M\times 1.$$

It follows, in particular, that for any smooth map $h:\partial(M\times I)\to N$, the following formula holds

$$\deg=\deg f-\deg g,$$

where f and g are maps $M\to N$ defined by formulae

$$f(x)=h(x,0), \quad g(x)=h(x,1), \quad x\in M.$$

On the other hand, the fact that the maps f and g are homotopic is equivalent to the statement that the map h is a restriction of some map $F:M\times I\to N$ on $\partial(M\times I)$. Therefore, the formula (7.22) is an immediate consequence of the following general lemma.

Lemma 7.25. *For any smooth map $F:W\to N$ and oriented compact $(n+1)$-dimensional manifold W with boundary ∂W in oriented connected closed n-dimensional manifold N the degree of map $h=F|_{\partial W}:\partial W\to N$ is zero:*

$$\deg(F|_\partial W)=0.$$

Proof. Let y be the regular value of the map F (and hence the map h), and x_0 is an arbitrary point of the prototype $h^{-1}(y)$. In $F^{-1}(y)$, the point x_0 is the end point of some arc-component Q. Let x_1 be the other end of this arc (also lying in $h^{-1}(y)$). Obviously, it is sufficient to prove that the signs $\varepsilon 0$, and ε_1 of the Jacobian of the map h at points x_0 and x_1 are opposite, i.e. that the contributions of these points to the power of $\deg h$ are mutually reduced.

But since this statement has a local character, when proving it, we can go to arbitrary neighbourhoods of the arc Q and the point y. Having chosen the local coordinates in these neighbourhoods accordingly, we can therefore assume (see Case 2 above) that

$$W = \mathring{\mathbb{E}}^n \times I, \quad N = \mathring{\mathbb{E}}^n, \quad F = \text{proj},$$
$$x_0 = (\mathbf{0}, 0), \quad x_1 = (\mathbf{0}, 1), \quad y = 0.$$

To complete the proof, it remains to note that in this situation the equality $\varepsilon_1 = -\varepsilon_0$ is obvious, since on both components $\mathring{\mathbb{E}}^n \times 0$ and $\mathring{\mathbb{E}}^n \times 1$ are boundaries $\partial(\mathring{\mathbb{E}}^n \times I)$ and the map proj is identical, and the orientations of these components are different.

Thus, Lemma 7.25, and hence the formula (7.22), is fully proved. □

7.7 The degree of the homotopy class

Since, as already noted above, any continuous map $M \to N$ is homotopic to a smooth map and two smooth maps are homotopic if and only if they are smoothly homotopic, it follows from the equality (7.22) that for every homotopy class $\alpha \in [M, N]$ the formula

$$\deg \alpha = \deg f,$$

where f is an arbitrary smooth map of the class α, well defines some integer $\deg \alpha$ - the *degree of the class* α.

Here M and N are smooth closed oriented manifolds of the same dimension $n \geq 1$, and the manifold N is connected. Thus, we have defined some map

$$\deg : [M, N] \to \mathbb{Z}.$$

In particular, for $M = N = \mathbb{S}^n$ (and, therefore, for $[M, N] = \pi_n \mathbb{S}^n$) we get the map

$$\deg : \pi_n \mathbb{S}^n \to \mathbb{Z}, \tag{7.26}$$

which, as it is easy to see, is a homomorphism.

Theorem 7.27. *The map (7.26) is an isomorphism.*

It turns out that in Theorem 7.27, the specificity of the sphere $\mathbb{S}^n$ plays a role only when this sphere acts as a manifold N. As for the manifold M, it is enough to require only connectivity from it. In other words,

Proposition 7.28. *for any smooth, closed connected n-dimensional manifold M, the map*

$$\deg : [M, \mathbb{S}^n] \to \mathbb{Z} \tag{7.29}$$

is bijective.

We will prove this statement.

The injectivity of the map (7.29), i.e. the fact that for any $k \in \mathbb{Z}$ there exists a map $f : M \to \mathbb{S}^n$ of degree k, is proved without any difficulty (and even without the assumption that the manifold M connected).

Proof. Indeed, choosing a system of $|k|$ disjoint open balls in the manifold M, we will set the map $M \to \mathbb{S}^n$, requiring that the complement to these balls it translates to the point $s_0 \in \mathbb{S}^n$, and each ball is diffeomorphically mapped to a cell $e^n = \mathbb{S}^n \setminus s_0$, preserving orientation at $k > 0$ and reversing orientation at $k < 0$. It is clear that the degree of this map is k.

For $M = \mathbb{S}^n$, this fact can be proved even more simply by noting that, due to the homeomorphism of the map (7.26), its image is a subgroup of the group $\mathbb{Z}$ containing the degree 1 of the identical map, and therefore coincides with the entire group $\mathbb{Z}$. □

7.8 The injectivity of the map deg

Thus, we only need to prove the injectivity of the map (7.29), i.e. the fact that

Proposition 7.30. *two maps $f, g : M \to S^n$ are homotopic if and only if the degrees of these maps are equal:*

$$\deg f = \deg g.$$

This statement is the simplest example of so-called homotopy classification theorems that establish necessary and sufficient conditions for the homotopy of maps in certain situations.

Due to the general connection between homotopy and extension problems (see lecture 0) this theorem follows directly from the following extension theorem (applied to the manifold $W = M \times I$).

Proposition 7.31. *A smooth map*

$$f : \partial W \to \mathbb{S}^n$$

form the boundary ∂W of a smooth compact oriented $(n+1)$-dimensional connected manifold W into the sphere $\mathbb{S}^n$ can be extended to the entire manifold W if and only if when its degree is zero:

$$\deg f = 0. \tag{7.32}$$

Proof. The necessity of condition (7.32) is the content of Lemma 7.25. Therefore, only its sufficiency needs a proof.

Condition (7.32) means that for some point $y \in \mathbb{S}^n$, its preimage $f^{-1}y$ consists of an even number of points, in one half of which the Jacobian of the map f is positive, and in the other half is negative. Let's construct a family of disjoint nested arcs Q_i connecting in W each point of the first type with some point of the second type, entirely - with the exception of the ends - lying in the interior $W \setminus \partial W$ of the manifold W, and at the ends not touching the boundary ∂W. Obviously, this can always be done (only when $n = 1$, you should choose pairs of connected points with some care). We will prove proposition 7.31 by constructing a smooth map $F : W \to \mathbb{S}^n$ such that:

a) $F|_{\partial W} = h$;

b) the point y is a regular value of the map F;

c) its preimage $F^{-1}y$ consists of arcs Q_i.

We will build such a map in two stages.

Stage 1 Let Q be one of the arcs Q_i It is obvious that this arc has a neighbourhood U ("tube along Q") such that there is a diffeomorphism

$$\varphi : R^n \times I \to U,$$

translating the segment $\mathbf{0} \times I$ into the arc Q and map the boundary $\mathbb{R}^n \times 0 \cup \mathbb{R}^n \times 1$ of the product $\mathbb{R}^n \times I$ into the intersection $U \cap \partial W$. At the same time, the neighbourhoods of U corresponding to all possible arcs of Q_i can be considered disjoint. In addition, by reducing, if necessary, the neighbourhood of U, we can assume that on the components $V_0 = \varphi(\mathbb{R}^n \times 0)$ and $V_1 = \varphi(\mathbb{R}^n \times 0)$ of the intersection $U \cap \partial W$ the map h is a diffeomorphism to some neighbourhood V of a point y.

In Stage 11, we will construct a map F on each neighbourhood U separately. It is clear that it is enough to build smooth map for this

$$\Phi : \mathbb{R}^n \times I \to \mathbb{R}^n,$$

such that

a') $\Phi|_{\mathbb{R}^n \times 0} = \mathrm{id}$ and $\Phi|_{\mathbb{R}^n \times 1} = k$, where

$$k = (\varphi|_{\mathbb{R}^n \times 0})^{-1} \circ (h|_{V_1})^0 \circ (h|_{V_1}) \circ (\varphi|_{\mathbb{R}^n \times 1})$$

(we identify $\mathbb{R}^n \times 0$ with $\mathbb{R}^n$;

b') the point $0 \in \mathbb{R}^n$ is the regular value of the map Φ;

c') its preimage Φ^{-1} is the segment $\mathbf{0} \times I$. (Indeed, if such a map Φ is constructed, then the map F on U can be set by the formula

$$F = (h|_{V_0}) \circ (\varphi|_{\mathbb{R}^n \times 0}) \circ \Phi \circ \varphi^{-1}.)$$

Let $ell : \mathbb{R}^n \to \mathbb{R}^n$ be the differential (at the point $\mathbf{0}$) of the diffeomorphism k (considered as the diffeomorphism of $ell : \mathbb{R}^n \to \mathbb{R}^n$. It is clear that if the neighbourhoods of U and V are chosen small enough, then for any point $x \in \mathbb{R}^n \setminus \mathbf{0}$ the point $\mathbf{0}$ will not belong to the segment with the ends $\ell(\boldsymbol{x})$ and $k(\boldsymbol{x})$. Therefore, the map $\Phi_1 : \mathbb{R}^n \times I \to \mathbb{R}^n$ defined by the formula

$$\Phi_1(\boldsymbol{x}, t) = (1-t)\ell(\boldsymbol{x}) + tk(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n, \ t \in I,$$

will have the property that $\Phi_1^{-1}\mathbf{0} = \mathbf{0} \times I$. In this case, the point $\mathbf{0}$ will obviously be the regular value of the map Φ_1 and there will be equalities

$$\Phi_1|_{\mathbb{R}^n \times 0} = \ell, \quad \Phi_1|_{\mathbb{R}^n \times 1} = k.$$

Thus, the homotopy Φ_1 has all the properties 1a') - 1c'), with the exception of the first of the properties 1a').

To correct the case, we will use the fact that, by the condition, the Jacobians of the diffeomorphisms $h|_{V_0}$ and $h|_{V_1}$ have opposite signs. Since the diffeomorphisms $\varphi|_{\mathbb{R}^n \times 0}$ and $\varphi|_{\mathbb{R}^n \times 1}$ have the same property, then, consequently, the Jacobian of the diffeomorphism k is positive, i.e. the linear operator ℓ has a positive determinant, and, therefore, belongs to the unit component of the general linear group $\mathrm{GL}(n)$. Therefore, in $\mathrm{GL}(n)$ there is a smooth path $t \mapsto \ell_t$ connecting the identical operator $\mathrm{id} = \ell_0$ with the operator $\ell = \ell_1$. We define the (obviously smooth) homotopy $\Phi_0 : \mathbb{R}^n \times 0 \to \mathbb{R}^n$ by putting

$$\Phi_0(\boldsymbol{x}, t) = (\ell_t \boldsymbol{x}, t) \quad \text{for any point} \quad (\boldsymbol{x}, t) \in \mathbb{R}^n \times I.$$

Obviously, this homotopy also has the properties 1b') and 1c'), and the property 1a') for it has the form

$$\Phi_{01}|_{\mathbb{R}^n \times 0} = \mathrm{id}, \quad \Phi_0|_{\mathbb{R}^n \times 1} = \ell.$$

Now it is clear that the homotopy Φ obtained when we first produce the homotopy Φ_0 and then the homotopy Φ_1 has (after appropriate smoothing near the point $t = 1/2$) all the required properties.

Stage 2 As a result of Stage 1 (performed for all arcs Q_i, simultaneously), we will get on some neighbourhood of the union $\cup_i Q_i$, all arcs Q_i, (denote this neighbourhood again with the symbol U) the map F, satisfying all the conditions a) - c) (with the only caveat, that in condition a) the boundary of ∂W should be replaced by the intersection of $U \cap \partial W$).

Since every smooth manifold is - by applying local compactness and Hausdorffness - a regular space, the submanifold $\cup Q_i$ has in W a neighbourhood V such that $\overline{V} \subset U$.

It is clear that at the intersection of the sets $\overline{V} \setminus V$ and $\partial W \setminus V$ (closed in $W \setminus V$), the maps F and h coincide. Therefore, the formula

$$\widehat{F}(x) = \begin{cases} F(x), & \text{if} \quad x \in \overline{V} \setminus V, \\ h(x), & \text{if} \quad x \in \partial W \setminus V, \end{cases}$$

well defines a continuous map $\widehat{F}$ of the union $(\overline{V} \cup \partial W) \setminus V$ of these sets into the sphere $\mathbb{S}^n$ that does not touch the point $\boldsymbol{y}$, i.e. being a map in $\mathbb{S}^n \setminus \boldsymbol{y}$. Since $\mathbb{S}^n \setminus \boldsymbol{y} \approx \mathbb{R}^n$, we can therefore consider $\widehat{F}$ as a map in $\mathbb{R}^n$ and apply Tietze's theorem to it (see Appendix to Lecture 0). According to this theorem, there is a continuous map $\overline{F} : W \setminus V \to \mathbb{S}^n \setminus \boldsymbol{y}$ coinciding with $(\overline{V} \cup \partial W) \setminus V$ with the map $\widehat{F}$, i.e. matching on $\overline{V} \setminus V$ with the map F, and on $\partial W \setminus V$ with the map h. We will define a continuous map $F_1 : W \to \mathbb{S}^n$, assuming that on $\overline{V}$ it coincides with F, and on $W \setminus V$ it coincides with $\overline{F}$.

This continuous map is smooth on ∂W and on M. Therefore (see "Statement 1" = Proposition 7.3 above) it can be smoothed without changing it on ∂W (as well as to some neighbourhood of the submanifold $\cup Q_i$ contained in V). The resulting map will be a smooth extension of the map h from ∂W to all W.

Thus Proposition 7.31 is fully proved. Together with this, Theorem 7.27 is fully proved. □

Remark 7.33. Similarly (by introducing the so-called *degree* mod 2), it can be proved that if a smooth closed connected n-dimensional manifold M is unorientable, then the set $[M, \mathbb{S}^n]$ consists of two elements. As for manifolds with boundary, it is easily proved that any map $M \to \mathbb{S}^n$ is null homotopic.

Remark 7.34. Consideration of the prototypes of regular values is also useful when studying the maps of spheres of various dimensions. On this basis, L. S. Pontryagin constructed his famous method for calculating homotopy groups $\pi_m \mathbb{S}^n$, identifying them with the so-called framed cobordism groups of manifolds. Unfortunately, we do not have time to present this method (an interested reader can refer to Pontryagin's book [7] or to a more accessible, beginner-friendly presentation Milnor [5]),

Remark 7.35. The notion of degree can also be introduced for maps of manifolds with boundary (to manifolds with boundary). We will not develop the corresponding general theory here and only write a few special cases where an ad hoc reduction to the case of maps of spheres is possible.

Case1 A continuos map

$$f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (\mathbb{S}^n, \boldsymbol{s}_0) \tag{7.36}$$

Using the standard relative homeomorphism

$$\chi : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (\mathbb{S}^n, \boldsymbol{s}_0) \tag{7.37}$$

(see lecture 3) any such map we can represent as $f = \widehat{f} \circ \chi$, where $\widehat{f} : (\mathbb{S}^n, \boldsymbol{s}_0) \to (\mathbb{S}^n, \boldsymbol{s}_0)$. Assuming, by definition, that $\deg f = \deg \widehat{f}$, we immediately get that maps (7.36) are homotopic if and only if when their degrees coincide $(\operatorname{rel} \mathbb{S}^{n-1})$.

Case2 Continuous mapa

$$(I^n, \dot{I}^n) \to (\mathbb{S}^n, s_0) \quad \text{and} \quad (\dot{I}^{n+1}, 0) \to (\mathbb{S}^n, s_0).$$

This case is very similar to the previous one, only instead of homeomorphism (7.37), it is necessary to use relative homeomorphism

$$\chi : (I^n, \dot{I}^n) \to (\mathbb{S}^n, s_0) \tag{7.38}$$

or respectively pointed homeomorphism

$$\omega : (\dot{I}^{n+1}, 0) \to (\mathbb{S}^n, s_0) \tag{7.39}$$

We emphasise that here we are dealing with maps of spaces that are not smooth manifolds (even with boundary).

Case3

$$\chi : (I^n, \dot{I}^n) \to (\dot{I}^{n+1}, 0) \tag{7.40}$$

A similar trick is also applicable here, but both homeomorphisms (7.38) and (7.39) are necessary.

It is useful to keep in mind that in these constructions there is no need to assume that maps (7.38) and (7.39) are homeomorphisms - it is enough that they are maps of degree 1.

Of course, in each of the cases 1 - 3 it is possible to give a direct definition of the degree. For example, if the map (7.40) is smooth on the preimage of the interior of one of the faces of the cube I^{n+1}, then its degree is equal to the "algebraic number" of preimages of an arbitrary regular value belonging to this face.

Appendix

7.A The simplest consequences of the fact that $\pi_n\mathbb{S}^n \neq 0$

Surprisingly, the fact that the group $\pi_n\mathbb{S}^n$ (and the groups $\pi_m\mathbb{S}^n$ for $m < n$ are zero) is nonzero has important geometric consequences. For example, since $\pi_n\mathbb{E}^{n+1} = 0$, the functor π_n has the properties that we required from the functor Π in the proof of the drum theorem in Lecture 0. Thus, we are now (and only now!) we can consider this theorem proven. At the same time, Brouwer's fixed point theorem is also proved (see lecture 0).

Further, we can now prove that

Proposition 7.41. *for $m \neq n$ the spheres $\mathbb{S}^m$ and $\mathbb{S}^n$ are not homotopically equivalent.*

Proof. Indeed, if, say, $m < n$, then $\pi_m\mathbb{S}^n = 0$, whereas $\pi_m\mathbb{S}^m \neq 0$. □

It follows that

Proposition 7.42. *for $m \neq n$ spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

Proof. Indeed, any homeomorphism of these spaces would define a homeomorphism - and hence a homotopy equivalence - of their one-point compactifications $\mathbb{S}^m$ and $\mathbb{S}^n$. □

7.B Degrees of maps into spheres

To get more in-depth results, we need a few simple remarks about maps to spheres.

First of all, we note that

Proposition 7.43. *if the space X is normal in a closed pair (X, A), then any map $f : A \to \mathbb{S}^n$ can be extended to some neighbourhood U of the set A.*

Proof. Indeed, due to the embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the map f can be considered as the map $A \to \mathbb{R}^{n+1}$. By Tietze's theorem, this map extends to some map

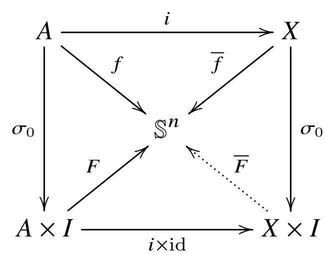
$g : X \to \mathbb{R}^{n+1}$. Let $U = X \setminus g^{-1}(0)$. It is clear that $A \subset V$ and the formula

$$\overline{f}(x) = \frac{g(x)}{|g(x)|}, \quad x \in U,$$

defines a continuous map $\overline{f} : U \to \mathbb{S}^n$ that coincides on A with the map f. □

It follows from this that

Proposition 7.44. *if for a closed pair (X, A) the space X is normally stable (i.e. it is itself normal and the product $X \times I$ is normal), then for each commutative diagram of the form*



(7.45)

there is a closing homotopy $\overline{F} : X \times I \to \mathbb{S}^n$ (i.e., as they say, the pair (X, A) satisfies the axiom **HE** *- cf. Definition0.19 - with respect to maps to spheres).*

Proof. Indeed, applying the previous proposition to the pair $(X \times I, \widetilde{A})$, where, as always,$\widetilde{A} = (X \times 0) \cup (A \times I)$, and to the map $\widetilde{f} : A \to \mathbb{S}^n$, given by the formula

$$\widetilde{f}(x, t) = \begin{cases} f(x), & \text{if} \quad t = 0, \\ F(x, t), & \text{if} \quad x \in A. \end{cases}$$

we will be able to extend this map to some the neighbourhood U of the subspace $\widetilde{A}$. It follows directly from the compactness of the segment I that in the space X there exists a neighbourhood V of the subspace A such that $V \times I \subset U$. According to Urysohn's lemma, there is a function $\varphi : X \to I$ equal to zero on $X \setminus V$ and one on A. Then $(x, \varphi(x)t) \in U$ for any point $(x, t) \in X \times I$, and therefore the formula

$$\overline{F}(x, t) = g(x, \varphi(x)t), \quad x \in X, \ t \in I,$$

where $g : U \to \mathbb{S}^n$ is the extension to U of the map $\widetilde{f}$, well defines the homotopy $\overline{F} : X \times I \to \mathbb{S}^n$, obviously, closing the diagram (7.45). □

Maps $X \to {}^n$ that are homotopic to zero are called also *non-essential maps*. All other maps $X \to {}^n$ are called *essential maps*.

In the case where X is an n-dimensional manifold, essential maps are exactly maps whose degree is nonzero.

Note that

Proposition 7.46. *any essential map is surjective.*

From the proposition proved above, it immediately follows that

Proposition 7.47. *if for a closed pair (X, A) the space X is normally stable, then any non-essential map $f : A \to \mathbb{S}^n$ admits an extension $\overline{f} : X \to \mathbb{S}^n$, which is also a non-essential map.*

7.C Borsuk's theorem on an unbounded component

Now let X be an arbitrary compact subset of the Euclidean space $\mathbb{R}^{n+1}$. Then for any point $\boldsymbol{x}_0 \in \mathbb{R}^{n+1} \setminus X$ the formula

$$p_{\boldsymbol{x}_0} : \boldsymbol{x} \mapsto \frac{\boldsymbol{x} - \boldsymbol{x}_0}{|\boldsymbol{x} - \boldsymbol{x}_0|}, \quad \boldsymbol{x} \in X,$$

defines some continuous map $p_{\boldsymbol{x}_0} : X \to \mathbb{S}^n$. If the points $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ lie in the same component of the complement $\mathbb{R}^{n+1} \setminus X$, then the map $p_{\boldsymbol{u}(t)}$ where $\boldsymbol{u} : t \mapsto \boldsymbol{u}(t)$ is an arbitrary path connecting the points $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ in $\mathbb{R}^{n+1} \setminus X$ constitute, obviously, a homotopy connecting the maps $p_{\boldsymbol{x}_0}$ and $p_{\boldsymbol{x}_1}$. Thus,

Proposition 7.48. *the homotopy class of the map $p_{\boldsymbol{x}_0}$ depends only on the component C of the complement $\mathbb{R}^{n+1} \setminus X$ containing the point $\boldsymbol{x}_0$.*

With respect to the component C, two cases are possible: either this component is bounded, or it is not limited. Let's first have a component C bounded. Since the set X is compact, it is contained in some closed ball E. It is clear that the component C, and therefore its closure $\overline{C}$ are also contained in E. By subjecting the space $\mathbb{R}^{n+1}$ to some parallel transfer and some homotopy, we can ensure that the point $\boldsymbol{x}_0$ coincides with the point 0 (and therefore the map $p = p_{\boldsymbol{x}_0}$, is given by the formula $\boldsymbol{x} \mapsto \boldsymbol{x}/|x|$), and the ball E was a single ball E^{n+1}. With this in mind, assume that the map $p : X \to \mathbb{S}^n$, $\boldsymbol{x} \mapsto \boldsymbol{x}/|x|$), is non-essential. Then, according to the remark made above, it extends to some (also non-essential) map $\overline{p} : X \cup C \to \mathbb{S}^n$. It is clear that the formula

$$r(\boldsymbol{x}) = \begin{cases} \overline{p}(\boldsymbol{x}), & \text{if} \quad \boldsymbol{x} \in X \cup C, \\ \boldsymbol{x}/|\boldsymbol{x}| & \text{if} \quad \boldsymbol{x} \in \mathbb{E}^{n+1} \setminus C, \end{cases}$$

well defines a continuous map $r : \mathbb{E}^{n+1} \to \mathbb{S}^n$, identity on $\mathbb{S}^n$, i.e. being a retraction $\mathbb{E}^{n+1} \to \mathbb{S}^n$. Since such a retraction cannot exist, the assumption about the non-essentiality of the map p is false, i.e. this map is essential.

Now let the C component be unlimited. Without loss of generality, we can assume that the point $\boldsymbol{x}_0$ lies outside the ball E and is therefore separated from X by some hyperplane. Then the directions of the vectors $\boldsymbol{x} - \boldsymbol{x}_0$ for points $\boldsymbol{x} \in E$, and hence for points $\boldsymbol{x} \in X$, are contained in some hemisphere of the sphere $\mathbb{S}^n$ and therefore obviously do not fill this entire sphere. Therefore, the p map is non-essential.

This proves the following

Proposition 7.49. *The point $\boldsymbol{x}_0 \in \mathbb{R}^{n+1} \setminus X$ lies in an unbounded component of the set $\mathbb{R}^{n+1} \setminus X$ if and only if when the map $p_{\boldsymbol{x}_0} : X \to \mathbb{S}^n$ is non-essential.*

This proposition is known as *Borsuk's unbounded component theorem.*

As applied to closed subsets of spheres, it follows from Borsuk's theorem that

Proposition 7.50. *if a closed subset X of a sphere S^{n+1} has the property that each map $X \to \mathbb{S}^n$ is non-essential, then it does not dissect the sphere (the complement $\mathbb{S}^{n+1} \setminus X$ is connected).*

Proof. Indeed, if the complement $\mathbb{S}^{n+1} \setminus X$ is disconnected and $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathbb{S}^{n+1} \setminus X$ are points lying in its various components, then by identifying the punctured sphere $\mathbb{S}^{n+1} \setminus \boldsymbol{x}_1$ with the space $\mathbb{R}^{n+1}$, we we get that the point $\boldsymbol{x}_0$ will belong to a bounded component. Therefore, there will be an essential map $X \to \mathbb{S}^n$ (namely, the map $p_{\boldsymbol{x}_0}$). □

7.D Topological invariance of the dimension of cubable sets

The converse statement is proved by a direct geometric construction that does not use algebro-topological considerations and is based on one property of the sphere $\mathbb{S}^{n+1}$, which it is advisable to discuss in a general way beforehand.

This property is related to the problem of topological invariance of the elementary geometric concept of dimension. To clearly formulate this problem, it is necessary first of all to clearly describe the class of spaces for which the "elementary geometric" dimension makes sense.

Let $N > 0$ be a positive integer. Hyperplanes of the space $\mathbb{R}^n$ with equations of the form $t_i = k2^{-N}$, where $i = 1, \dots, n$, and $k \in \mathbb{Z}$, split this space into a union of closed cubes with sides of length 2^{-N}. These cubes, as well as all their faces (of any dimension), we will call *cubes of fineness N* in the spaces $\mathbb{R}^n$. We will call a subset of the space $\mathbb{R}^n$ a *cubable set* if there exists $N > 0$ such that this subset is represented (obviously, in the unique way) as a union of some family of N fineness cubes containing together with each cube and all its faces. We will call this family the *cubilage of the fineness N* of the cubable set. It is clear that the existence of a cubilage of fineness N implies the existence of a cubilage of any greater fineness. Thus, each cubable set has cubilages of arbitrarily large fineness. Any cubable set is closed. It is compact (= bounded) if and only if all its cubilages are finite. According to Lebesgue's lemma (see Appendix to Lecture 1) for any open cover of a compact cubable set there exists $N > 0$ such that each element of the cubilage of fineness N of this set is contained in some element of the cover.

We will call the *dimension* $\dim X$ of the cubable set X the largest dimension of cubes of its arbitrary cubilage (it is clear that this dimension does not depend on the choice of cubilage).

The problem we are interested in can now be formulated as follows:

Proposition 7.51. *Do the dimensions of two homeomorphic cubable sets coincide?*

The affirmative answer to this question directly follows from the following proposition.

Proposition 7.52. *For a cubable set X, the inequality* $\dim X \leq n$ *holds if and only if, for each closed subset $A \subset X$, any continuous map $f : A \to \mathbb{S}^n$ can be extended to all X.*

Proof. The sufficiency of this condition easily follows from the drum theorem. Indeed, if $\dim X > n$, then in X there is a closed subset B homeomorphic to $(n+1)$-dimensional ball $\mathbb{E}^{n+1}$ (for example, every $(n+1)$-dimensional cube of any cubilage of the space X). Let $g : B \to \mathbb{E}^{n+1}$ be an arbitrary homeomorphism, $A = g^{-1}(\mathbb{S}^n)$ and $f = g|_A$. By the condition, the map $f : A \to \mathbb{S}^n$ can be extended to some map $\overline{f} : X \to \mathbb{S}^n$. But then the map $(\overline{f}|_B) \circ g^{-1}$ will obviously be a retraction of $\mathbb{E}^{n+1} \to \mathbb{S}^n$. Therefore $\dim X \leq n$.

The necessity of the condition is proved by a direct construction. Since the space X is obviously normally stable, then, according to the above, the map f can be extended to some neighbourhood U of the set A. On the other hand, considering a sufficiently small cubilage of the set X, we get that all cubes of this cubilage intersecting A are contained in U. The union of A_0 of all these cubes is a cubable set containing A, to which the map f is extended. Therefore, without loss of generality, we can assume from the very beginning that the set A is a cubable set (and consists of cubes from some cubilage of the set X). In this case, we will extend the map f cube by cube to all cubes of X that do not belong to A. At each step of this extension, we will encounter a situation where the map f is given on the boundary of some cube of dimension $m \leq n$, and the task will be to extend this map to the entire cube. But since the pair (cube, its boundary) is homeomorphic to the pair $(\mathbb{E}^m, \mathbb{S}^{m-1})$ and since, as we already know, $\pi_{m-1}\mathbb{S}^n = 0$ for $m \leq n$, this problem is always soluble. Therefore, moving from cube to cube, as a result we will extend the map f to all X. □

Remark 7.53. Proposition 7.52 suggests a way to define the notion of dimension $\dim X$ for any topological space X. Namely, we can assume that $\dim X \leq n$ if for every closed subset $A \subset X$ any continuous map $A \to \mathbb{S}^n$ can be extended to all X, and $\dim X = n$ if $\dim X \leq n$, but it is not true that $\dim X \leq n-1$. Then Proposition 7.52 will state that for any cubable set X, the dimension in this sense coincides with its elementary-geometric dimension.

If $\dim X > n$, then the map $f : A \to \mathbb{S}^n$ can, generally speaking, be extended only to $X^n \cup A_0$, where X^n is the union of all cubes of dimension $\leq n$ of the considered cubilage of the set X. On the other hand, if $\dim X = n+1$ and if in each $(n+1)$-dimensional cubes from $X \setminus A_0$ are selected by point, then the set $X \setminus K$, where K is the set of all selected points, will obviously be retracted to $X^n \cup A_0$, and therefore the map f can be extended from $X^n \cup A_0$ to $X \setminus K$.

This proves that for any continuous map $f : A \to \mathbb{S}^n$ to the sphere $\mathbb{S}^n$ of a closed subset A of a cubable $(n+1)$-dimensional set X there exists a finite set $K \subset X \setminus A$, such that f is extended to $X \setminus K$.

7.E Sets that do not dissect the spheres

Let us now return to the sets that do not dissect the sphere.

Proposition 7.54. *A closed set $X \subset \mathbb{S}^{n+1}$ does not dissect the sphere $\mathbb{S}^{n+1}$ if and only if when each map $X \to \mathbb{S}^{n+1}$ is non-essential.*

Proof. The sufficiency of this condition was proved above. Therefore, we only need to prove its necessity.

Let the complement $\mathbb{S}^{n+1} \setminus X$ be connected, and let f be an arbitrary continuous map $X \to \mathbb{S}^n$. Since the sphere $\mathbb{S}^{n+1}$ is homeomorphic to the cubable set $\dot{I}^{n+2}$, then according to what has just been proved, we can assume that the map f is extended to some map of the space $\mathbb{S}^{n+1} \setminus K$ to the sphere $\mathbb{S}^n$, where K is a finite subset of the complement $\mathbb{S}^{n+1} \setminus K$. We will call the points from K *singular points of the map f*.

Assuming that the sphere $\mathbb{S}^{n+1}$ is equipped with a Riemannian metric (for example, the induced Euclidean metric of the enclosing space $\mathbb{R}^{n+2}$), we call an arbitrary open ball $e \subset \mathbb{S}^{n+1} \setminus X$ the *cell* of $\mathbb{S}^{n+1} \setminus X$. It is clear that any point $\boldsymbol{x}_0 \in \mathbb{S}^{n+1} \setminus X$ is contained in some cell, the boundary $\dot{e} = \overline{e} \setminus e$ which does not contain points from K and, moreover, any two points $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathbb{S}^{n+1} \setminus X$ it is possible to connect a chain of such cells in which any two consecutive cells intersect.

With this in mind, consider an arbitrary point $\boldsymbol{x}_1 \in K$ and a cell e containing a point $\boldsymbol{x}_1$ and such that its boundary $\dot{e}$ does not intersect with the set K. Let $\boldsymbol{x}_2$ be an arbitrary point of e. Since the set $\overline{e} \setminus \boldsymbol{x}_2$ is obviously retractible to $\dot{e}$, the map f can be extended to $e \setminus \boldsymbol{x}_2$.

The resulting map has the same singular points outside e as the map f, and inside e there is only one singular point $\boldsymbol{x}_2$. We will say that the singular point $\boldsymbol{x}_1$ is *shifted* to position $\boldsymbol{x}_2$. (Note that in this case, all other singular points, if any, contained in e, are also shifted in $\boldsymbol{x}_2$.) Since, as already noted, any two points from $\mathbb{S}^{n+1} \setminus X$ can be connected by a chain of cells, any two neighbouring cells of which intersect, it follows that all points from K can be shifted to some fixed point $\boldsymbol{x}_0 \in \mathbb{S}^{n+1} \setminus X$. In other words, without loss of generality, we can assume that the map f has only one singular point $\boldsymbol{x}_0$, i.e. that it is a map $\mathbb{S}^{n+1} \setminus \boldsymbol{x}_0 \to \mathbb{S}^n$ and, therefore, is non-essential (because the punctured sphere $\mathbb{S}^{n+1} \setminus \boldsymbol{x}_0$ is contractible). But then its restriction to X, i.e. the original map f, is also non-essential. □

7.F The theorem of the invariance of domain

Corollary 7.55. *If $A \subset X \subset \mathbb{S}^{n+1}$ and the pair (X, A) is homeomorphic to the pair $(\mathbb{E}^{n+1}, \mathbb{S}^n)$, then the set $\mathbb{S}^{n+1} \setminus A$ is divided into two components, which are the sets $\mathbb{S}^{n+1} \setminus X$ and $X \setminus A$. In particular, the set $X \setminus A$ is open in the sphere $\mathbb{S}^{n+1}$.*

Proof. Since any map $\mathbb{E}^{n+1} \to \mathbb{S}^n$ is non-essential, the set $\mathbb{S}^{n+1} \setminus X$ is connected. The set $X \setminus A$ is connected because it is homeomorphic to an open ball. This means that the set

$$\mathbb{S}^{n+1} \setminus A = (\mathbb{S}^{n+1} \setminus X) \cup (X \setminus A)$$

is either connected or consists of two components $\mathbb{S}^{n+1} \setminus X$ and $X \setminus A$. But it cannot be connected, because there are essential maps $A \to \mathbb{S}^n$ (for example, the homeomorphism $A \to \mathbb{S}^n$). □

Corollary 7.56 (The theorem of the invariance of domain)**.** *If subsets U and V of the sphere $\mathbb{S}^n$ are homeomorphic and U is open in $\mathbb{S}^n$, then V is also open.*

Proof. Let $f : U \to V$ be a given homeomorphism, and let $\boldsymbol{x} \in U$ and O be a spherical neighbourhood of a point $\boldsymbol{x}$ such that $\overline{O} \subset U$. Just as the pair $(\overline{O}, \dot{O})$ is homeomorphic to the pair $(\mathbb{E}^{n+1}, \mathbb{S}^n)$, the pair $(f\overline{O}, f\dot{O})$ is also homeomorphic to the pair $(\mathbb{E}^{n+1}, \mathbb{S}^n)$. Therefore, according to Corollary 7.55, the set $f\overline{O} \setminus f\dot{O} = fO$ containing the point $f(\boldsymbol{x}) \in V$, is open. Since this set is contained in $V = fU$, this proves that the point $f(\boldsymbol{x})$ is an internal point of the set V, and since any point from V can be represented as $f(\boldsymbol{x})$, the set V is open. □

7.G Topological invariance of the dimension of manifolds

Recall that a topological space X is called a *topological n-dimensional manifold* if any of its points has a neighbourhood homeomorphic to the space $\mathbb{R}^n$.

Corollary 7.57. *If subsets U and V of topological n-dimensional manifolds X and Y are homeomorphic and U is open in X, then V is open in Y.*

Proof. Let $f : U \to V$ be a given homeomorphism, and let $x \in U$. There are spaces homeomorphic to $\mathbb{R}^n$, which means that the space $\mathbb{S}^n \setminus \boldsymbol{s}_0$ is a neighbourhood P and Q of points x and $y = f(x)$ respectively in the manifolds X and Y, such that $P \subset U$ and $fP \subset Q$. Having chosen the homeomorphisms $\varphi : P \to \mathbb{S}^n \setminus \boldsymbol{s}_0$ and $\psi : Q \to \mathbb{S}^n \setminus \boldsymbol{s}_0$, consider the composition $\psi \circ f \circ \varphi^{-1}$, which is a homeomorphism of an open set $\mathbb{S}^n \setminus \boldsymbol{s}_0$ on its subset $(\psi \circ f)(P)$. According to Corollary 7.56, the subset $(\psi \circ f)(P)$ is open in $\mathbb{S}^n$, and therefore in $\mathbb{S}^n \setminus \boldsymbol{s}_0$. So, the set $f(P)$ is open in $Q = \psi^{-1}(\mathbb{S}^n \setminus \boldsymbol{s}_0)$, and therefore and in all the manifold Y. Thus, each point $f(x)$ of the set $V = fU$ has an open neighbourhood fP in Y, contained in V. Hence the set V is open. □

Remark 7.58. Without the assumption that X and Y are manifolds, Corollary 7.57 is incorrect. To get the corresponding counterexample, it is enough to take the abscissa axis as U, V and X, and the union of the abscissa axis and the ordinate axis as Y.

Corollary 7.59. *Topological manifolds X and Y of different dimensions are not homeomorphic.*

Proof. Let $\dim X - \dim Y = n > 0$, and let, contrary to the statement, the manifolds X and Y are homomorphic. Then the manifold is $Y \times \mathbb{R}^n$ will have the same dimension as the manifold X, and at the same time will contain a non-open subset $Y \times \mathbf{0}$, homeomorphic to the manifold X. Since X is open in X, this contradicts Corollary 7.57. □

Remark 7.60. Another, more instructive way of proving corollary 7.59 is that for manifolds an analogue of Proposition 7.18 is proved. We will leave the detailed conduct of the relevant arguments to the reader's initiative.

Lecture 8

8.1 Exact Π sequences

Let us now return to the general theory of homotopy groups.

A homotopy sequence of an arbitrary fibration belongs to the class of left-infinite exact sequences of the form

$$\cdots \xrightarrow{p_{n+1}} B_{n+1} \xrightarrow{\partial_n} F_n \xrightarrow{i_n} E_n \xrightarrow{p_n} B_n \to \cdots, \tag{8.1}$$

all members of which, with the exception of the last six right-hand members, are abelian groups, and which end on the right with three non-abelian (multiplicatively written) groups and three pointed sets:

$$\cdots \to \underbrace{F_2 \xrightarrow{i_2} E_2 \xrightarrow{p_2} B_2}_{\text{abelian groups}} \xrightarrow{\partial_1} \underbrace{F_1 \xrightarrow{i_1} E_1 \xrightarrow{p_1} B_1}_{\text{non-abelian groups}} \xrightarrow{\partial_0} \underbrace{F_0 \xrightarrow{i_0} E_0 \xrightarrow{p_0} B_0}_{\text{pointed sets}},$$

moreover, for any $n \geq 1$, the multiplicatively written groups F_1, E_1, B_1 act respectively on the groups F_n, E_n, B_n (for $n = 1$ - by means of internal automorphisms).

Following Fuchs and Rokhlin we will say that such a sequence (8.1) is a Π-*sequence* if, in addition, the group E_1 acts on the group F_n, and the group B_1 acts on the set F_0 (generally speaking, non-pointed maps), and:

a) the homomorphism p_n for $n \geq 2$ is a p_1-homomorphism;

b) the homomorphism i_n for $n \geq 1$ is a E_1-homomorphism;

c) the homomorphism ∂_n for $n = 0$ is a B_1-homomorphism; (in relation to the action of the group B_1 in itself by means of left translations), and for $n \geq 1$ is an E_1-homomorphism with respect to the action of the group E_1 on the group B_{n+1}, induced by the homomorphism p_n, given by the action of the group B_1;

d) the action of the group F_1 on the group F_n is induced by the homomorphism i_1 by the given action of the group E_1.

For such a sequence, the exactness in the term F_0 will be understood in a stronger sense. Namely, we will require that the preimages of the elements of the set E_0 by the map i_0 coincide with the orbits of the action of the group B_1 in the set F_0.

Proposition 8.2. *The homotopy sequence*

$$\cdots \longrightarrow \pi_{n+1}B \xrightarrow{\partial} \pi_n F \xrightarrow{i_*} \pi_n E \xrightarrow{p_*} \pi_n B \longrightarrow \cdots$$

of an arbitrary pointed fibration $p : E \longrightarrow B$ is an exact Π-sequence.

Since for a homotopy sequence the condition a) is nothing more than a property of the functoriality of the action $R : \pi_1 X \longrightarrow \operatorname{Aut} \pi_n X$ in relation to the map $p : E \longrightarrow B$, to prove proposition 8.2 we need to construct the action of the group $\pi_1 E$ on the group $\pi_n F$ (and the group $\pi_1 B$ on the set $\pi_0 F$) and check the conditions b), c) and d) (as well as the exactness in the term $\pi_0 F$).

In the construction of this action (as well as, by the way, in the construction of the action $R : \pi_1 X \longrightarrow \operatorname{Aut} \pi_n X$), only the non-degeneracy of the point s_0 is used from the properties of the pair $(\mathbb{S}^n, s_0)$. Therefore, instead of the group $\pi_n F$, we consider the general case of a set of the form $[X, F]^\bullet$, where X is an arbitrary well-pointed space, and construct the action of the group $\pi_1 E$ on this set, preserving the group structure when X is an H-cogroup, and such that

1) induced by the embedding $i : F \longrightarrow E$ the map

$$[X, F]^\bullet \longrightarrow [X, E]^\bullet$$

is a $\pi_1 E$-homomorphism (for $X = \mathbb{S}^n$ this will give us the property b);

2) induced by this action by the homomorphism $i_* : \pi_1 F \longrightarrow \pi_1 E$ the action of the group $\pi_1 F$ on the set $[X, F]^\bullet$ coincides with the canonical action from Proposition 4.25 of Lecture 4 (for $X = \mathbb{S}^n$ this will give us the property d)).

To construct this action, we notice that any loop $u \in \Omega E$ is determined by the formula

$$G(x, t) = (p \circ u)(t), \quad x \in X, \ t \in I,$$

a homotopy $G : X \times I \longrightarrow B$ such that for each map $f : X \longrightarrow F$ there is a commutative diagram

$$\begin{array}{ccc} \widetilde{\{x_0\}} & \xrightarrow{\ \overline{g}\ } & E \\ {\scriptstyle \widetilde{\sigma_0}}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[G]{} & B \end{array}$$

where $\widetilde{\{x_0\}} = (X \times 0) \cup (\{x_0\} \times I)$, and $\overline{g}$ is the map given by the formula

$$\overline{g}(x, t) = \begin{cases} (i \circ f)(x), & \text{if} \quad t = 0, \\ u(t), & \text{if} \quad x = x_0, \end{cases} \qquad (x, t) \in \widetilde{\{x_0\}}.$$

The corresponding covering homotopy $\overline{G} : X \times I \to E$ (existing by the axiom **CHE**) has the property that $\overline{G}(x, 1) \in F$ for any point $x \in X$ (and $G(x_0, 1) = e_0$).

Therefore, putting

$$\widehat{f}(x) = \overline{G}(x, 1), \quad x \in X,$$

we get some pointed map $\widehat{f} : X \to F$, and it is obvious that if the formula $\xi\alpha = [\widehat{f}]^\bullet$, where $\xi = [u]^\bullet$, and $\alpha = [f]^\bullet$, well defines the map $(\xi, \alpha) \mapsto \xi\alpha$, then this map will be the action of the group $\pi_1 E$ on the group $[X, F]^\bullet$, having properties 1) and 2).

Therefore, we only need to check the correctness of this construction, i.e. show that

Proposition 8.3. *if* $u \sim u_1 \operatorname{rel}\{0, 1\}$ *and* $f \sim f_1$, *then* $\widehat{f} \sim \widehat{f_1}$.

Proof. To this end, by introducing into consideration the corresponding homotopies $u_t : I \to E \operatorname{rel}\{0, 1\}$ and $f_\tau : (X, x_0) \to (E, e_0)$, we will define the map $\overline{h} : \widetilde{X}_{\mathrm{pt}} \to E$ to the space E from the subspace

$$\widetilde{X}_{\mathrm{pt}} = (X \times I) \times 0 \cup ((X \times 0) \cup (\{x_0\} \times I) \cup (X \times 1)) \times I$$

of the space $(X \times I) \times I$ by putting

$$\overline{h}(x, \tau, t) = \begin{cases} f_\tau(x), & \text{if } \ t = 0, \\ \overline{G}(x, t), & \text{if } \ \tau = 0, \\ \overline{G}_1(x, t), & \text{if } \ \tau = 1, \\ u_\tau(t), & \text{if } \ x = x_0, \end{cases}$$

where $\overline{G}_1$ is the homotopy of $\overline{G}$, built on the map f_1, and the path u_1. It is clear that the map $\overline{h}$ covers over $\widetilde{X}_{\mathrm{pt}}$ the homotopy $H : X \times I \times I \to B$ defined by the formula

$$H(x, \tau, t) = (p \circ u_\tau)(t), \quad (x, \tau, t) \in X \times I \times I.$$

Hence, this map extends to the covering homotopy $\overline{H} : X \times I \times I \to E$, and the map

$$\overline{H} \circ \sigma_1 : (x, \tau) \mapsto \overline{H}(x, \tau, 1), \quad (x, \tau) \in X \times I,$$

will obviously be a pointed homotopy connecting the map $\widehat{f}$ the map $\widehat{g}$. □

Remark 8.4. It is clear that a similar construction holds for any paths in E (and not just loops). As a result, an ensemble arises on the space E, whose group (or pointed set) at the point $e \in E$ is the group $[(X, x_0), (F_b, e)]$, where $B = p(e)$, and $F_b = p^{-1}(b)$ is the fibre above the point b of the fibration $p : E \to B$ (which now there is no need to assume pointed). In particular, we thereby obtain for any fibration $p : E \to B$ (and each $n \geq 0$) on the space E an *ensemble of homotopy groups of fibres* $\{\pi_n(F_b, e)\}$ (for $n = 0$ being an ensemble of pointed sets).

Remark 8.5. When interpreting the elements of the group $\pi_n F$ as homotopy classes of maps $(I^n, \dot{I}^n) \to (F, e_0)$ the element $\xi\alpha \in \pi_n F$ is given by by map $B : (I^n, \dot{I}^n) \to (F, e_0)$ related to the map $a : (I^n, \dot{I}^n) \to (F, e_0)$, specifying the element $\alpha \in \pi_n F$, and a loop $u : (I^n, \dot{I}^n) \to (F, e_0)$, specifying element $\xi \in \pi_1 E$, by the map $G : I^{n+1} \to E$ such that

$$(p \circ G)(\boldsymbol{t}, t) = (p \circ u)(t) \quad \text{for any point} \quad (\boldsymbol{t}, t) \in I^{n+1},$$
$$G(\boldsymbol{t}, 0) = a(\boldsymbol{t}), \quad G(\boldsymbol{t}, 1) = b(\boldsymbol{t}) \quad \text{for any point} \quad \boldsymbol{t} \in I^n.$$

A similar remark is true, of course, with respect to the action of morphisms of the groupoid ΠE on the groups $\pi_n(F_b, e)$ of the ensemble of homotopy groups of fibres of the fibration $p : E \to B$.

Now we can proceed directly to the proof of Proposition 8.2.

Proof. (of Proposition 8.2) It remains for us to check the condition c) and construct the action of the group $\pi_1 B$ on the set $\pi_0 F$ (and also prove the exactness in the term $\pi_0 F$).

First of all, we will check the condition c) for $n \geq 1$, i.e. we show that for any elements $\alpha \in \pi_{n+1} B$ and $\xi \in \pi_1 E$ there is the equality

$$\xi(\partial_n \alpha) = \partial_n(p_*(\xi)\alpha).$$

Let the map $a : (I^{n+1}, \dot{I}^{n+1}) \to (B, b_0)$ define the element $\alpha \in \pi_n B$, and the loop $u : (I, \dot{I}) \to (E, e_0)$ the element ξ. Let, further, H be a map $I^{n+1} \times I \to B$ such that

$$H(\boldsymbol{t}, 0) = a(\boldsymbol{t}) \quad \text{for any point} \quad \boldsymbol{t} \in I^{n+1},$$
$$H(\boldsymbol{t}, t) = (p \circ u)(t) \quad \text{for} \quad \boldsymbol{t} \in \dot{I}^{n+1} \quad \text{and} \quad t \in I,$$

(so the map $b : \boldsymbol{t} \mapsto H(\boldsymbol{t}, 1)$ sets the element $p_*(\xi)\alpha \in \pi_{n+1} B$). By applying the decomposition of $I^{n+1} = I^n \times I$, $\boldsymbol{t} = (\boldsymbol{s}, s)$, we can consider the map H as a homotopy $I^{n+1} \times I \to B$ with the parameter s. The initial map $(\boldsymbol{s}, t) \mapsto H(\boldsymbol{s}, 0, t)$ of this homotopy is given by the formula $(\boldsymbol{s}, t) \mapsto (p \circ u)(t)$ and, therefore, is covered by the map $(\boldsymbol{s}, t) \mapsto ut$. Also, if $\boldsymbol{s} \in \dot{I}^n$ or $s = 0$, and $t = 0$ or $t = 1$, then the homotopy H is covered by the constant map const_{e_0}. Therefore, according to the axiom **CHE**, there is a homotopy

$$\overline{H} : I^{n+1} \times I \to E, \quad (\boldsymbol{s}, s, t) \mapsto \overline{H}(\boldsymbol{s}, s, t),$$

such that

$$\overline{H}(\boldsymbol{s}, s, t) = e_0, \quad \boldsymbol{s} \in \dot{I}^n \quad \text{or} \quad s = 0, \quad \text{and} \quad t = 0 \quad \text{or} \quad t = 1,$$
$$H(\boldsymbol{s}, 0, t) = u(t) \quad \text{for any points} \quad \boldsymbol{s} \in I^n, \quad t \in I,$$
$$(p \circ \overline{H})(\boldsymbol{s}, s, t) = H(\boldsymbol{t}, t) \quad \text{for any points} \quad (\boldsymbol{s}, s, t) \in I^n \times I \times I, \quad \text{where} \quad \boldsymbol{t} = (\boldsymbol{s}, s).$$

Putting $\overline{a}(\boldsymbol{s}, s) = \overline{H}(\boldsymbol{s}, s, 0)$, we get a homotopy $\overline{a} : I^n \times I \to E$, fixed on $\dot{I}^n$, such that $\overline{a}(\boldsymbol{s}, 0) = e_0$ and $p \circ \overline{a} = a$. Hence, the map $c : \boldsymbol{s} \mapsto \overline{a}(\boldsymbol{s}, 1) = \overline{H}(\boldsymbol{s}, 1, 0)$, considered as a map $(I^n, \dot{I}^n) \to (E, e_0)$, sets the element $\partial_n \alpha \in \pi_n F$.

Similarly, putting $\overline{b}(s,s) = \overline{H}(s,s,1)$, we get a homotopy $b : I^n \times I \to E$ fixed on I^n, such that $\overline{b}(s,0) = e_0$ and $p \circ \overline{b} = b$.

On the other hand, the map $G : (s,t) \mapsto \overline{H}(s,1,t)$ obviously satisfies the relations

$$(p \circ G)(t,0) = (p \circ u)(t) \quad \text{for any point} \quad (s,t) \in I^n \times I = I^{n+1},$$
$$G(s,0) = c(s), \quad G(s,1) = c_1(s) \quad \text{for any point} \quad S \in I^n.$$

Therefore, the map $c_1 : (I^n, \dot{I}^n) \to (F, e_0)$ sets the element $\xi(\partial_n \alpha)$.

Thus

$$\xi(\partial_n \alpha) = \partial_n(p_*(\xi)\alpha)$$

Let us now construct the action of the group $\pi_1 B$ on the set $\pi_0 F$.

Let $\xi \in pi_1 B$ and $\alpha \in \pi_0 F$. By selecting for an arbitrary loop $u : (I, \dot{I}) \to (B, b_0)$ of the class ξ the path covering this loop $\overline{u} : I \to E$, starting at some point e_α of the component α, consider the point $u(1) \in F$. An automatic verification shows that the component $\xi\alpha$ of this point depends only on ξ and α (the easiest way to see this is if you notice that $\xi\alpha$ is nothing but the image of the element ξ by the map $\partial_0 : \pi_1(B, b_0) \to \pi_0(F, e_\alpha)$ and that the map $(\xi, \alpha) \mapsto \xi\alpha$ is an action of the group $\pi_1 B$ on the set $\pi_0 F$. At the same time, it is clear that for any elements of $\xi, \eta \in \pi_1 B$ there is equality

$$\partial_0(\xi\eta) = \xi(\partial_0 \eta),$$

meaning that condition c) is also met when $n = 0$.

Thus, to complete the proof of Proposition 8.2, we only need to check the exactness in the term $\pi_0 F$, i.e., to show that for elements $\alpha, \beta \in \pi_0 F$ the equality $i_*\alpha = i_*\beta$ holds if and only if there exists an element $\xi \in \pi_1 B$ such that $\xi\alpha = \beta$. But the equality $\xi\alpha = \beta$ means that the loop defining the element ξ is covered by a path starting in the component α and ending in the component β, and the equality $i_*\alpha = i_*\beta$ means that the components α and β of the fibre F lie in the same component of the space E, i.e. that in E there is a path starting in the component α and ending in the component β. Since any such path is the cover of some loop (namely, its projection into B), the equalities $\xi\alpha = \beta$ and $i_*\alpha = i_*\beta$ are indeed equivalent.

Thus Proposition 8.2 is fully proved. □

Remark 8.6. Similarly, it is possible for any $n \geq 1$ to construct an action of the group $\pi_1 B$ on the set $[\mathbb{S}^n, F]$ of the free homotopy mapping classes $\mathbb{S}^n \to F$. Thus, *in the case when the fibre F is homotopically simple in dimension n* and, therefore, the set $[\mathbb{S}^n, F]$ is a group $\pi_n F$, we get the action of the group $\pi_1 B$ on the group $\pi_n F$ (and for a non-pointed fibration $p : E \to B$ the ensemble $\{\pi_n F_b\}$ of groups $\pi_n F_b$ on the space B).

8.2 Category of pointed pairs $\mathcal{T}op_2^\bullet$

In Lecture 4, we already had the opportunity to introduce the category of pairs $\mathcal{T}op_2$, the objects of which are pairs (X, A) of topological spaces, and morphisms $f : (X, A) \to (Y, B)$ are maps $f : X \to Y$ such that $f(A) \subset B$.

Similarly, for any $n \geq 2$ the *category of n-ples* $\mathcal{T}op_n$ is introduced, whose objects have the form $(X, A_1, \dots, A_{n-1})$, where $X \supset A_1 \supset \cdots \supset A_{n-1}$, but these categories, unlike the category $\mathcal{T}op$, will play a purely official role with us.

The category $\mathcal{T}op^\bullet$ of pointed spaces is obviously a complete subcategory of the category $\mathcal{T}op_2$.

A *pointed pair* is a triple of the form (X, A, x_0), where x_0 is a point of the subspace of A (called the *pair marked with a point*). The complete subcategory of the category $\mathcal{T}op_2$ generated by pointed pairs is denoted by the symbol $\mathcal{T}op_2^\bullet$, and its morphisms are called *pointed maps of pointed pairs.*

There are two obvious functors from the category $\mathcal{T}op_2$, (or the category $\mathcal{T}op_2^\bullet$) to the category $\mathcal{T}op$ (respectively to the category $\mathcal{T}op^\bullet$). The first functor of each pair (X, A) maps the space X and each map $f : (X, A) \to (Y, B)$ is its own, but considered simply as a map $X \to Y$, and the second functor to the pair (X, A) maps the space A, and to the map $f : (X, A) \to (Y, B)$ is the map $A \to B$ induced by it (which we will often denote with the same symbol f).

A homotopy $f_t : X \to Y$ is called a *homotopy of maps of pairs* (or a *homotopy in* $\mathcal{T}op_2$) if $f_t : (X, A) \to (Y, B)$ for any $t \in I$. (Homotopy maps of pairs should not be confused with the narrower concept of homotopy with respect to A.) Similarly, *pointed homotopy maps* of pairs are defined.

It is clear that thus the category $\mathcal{T}op_2$, (category $\mathcal{T}op_2^\bullet$) it turns out to be a category with homotopies in the sense introduced in Lecture 0. Therefore, all the usual homotopy concepts make sense in it: homotopy equivalences, deformation retractions, homotopy invariant functors, etc.

8.3 Relative homotopy groups

An example of a pointed pair is the pair $(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0)$ consisting of an n-dimensional $(n \geq 1)$ ball $\mathbb{E}^n$, its boundary sphere $\mathbb{S}^{n-1}$ and a point $\boldsymbol{s}_0 \in \mathbb{S}^{n-1}$.

Definition 8.7. For any pointed pair (X, A, x_0), the symbol $\pi_n(X, A, x_0)$ or simply $\pi_n(X, A)$ denotes the set

$$[(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0), (X, A, x_0)] = [(\mathbb{E}^n, \mathbb{S}^{n-1}), (X, A)]^\bullet$$

of the homotopy classes of pointed maps $(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0) \to (X, A, x_0)$.

We will consider this set to be a pointed set, the base point of which is the class of constant map const : $\mathbb{E}^n \to X$, $a \mapsto x_0$ (obviously representing the map $(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0) \to (X, A, x_0)$).

Let us now introduce the triple $(I^n, \dot{I}^n, I^{n-1})$, $n \geq 1$, where:

I^n , as always, is a single n-dimensional cube consisting of points $\boldsymbol{t} = (t_{1]}, \dots, t_n) \in \mathbb{R}^n$, for which $0 \leq t_i \leq 1$ for all $i = 1, \dots, n$;

$\dot{I}^n$ is its boundary consisting of points $\boldsymbol{t} \in I^n$, for each of which there is an index $i = 1, \dots, n$ such that either $t_i = 0$, or $t_i = 1$;

J^{n-1} - complement in the boundary $\dot{I}^n$ of the cube of its interior $(n-1)$-dimensional face I^{n-1}, defined by the equation $t_n = 0$, i.e. the set of all points $\boldsymbol{t} \in I^n$, for each of which there is an index $i = 1, \dots, n$, such that either $i < n$ and $t_i = 0$, or $t_i = 1$. See Fig 8.3.1.

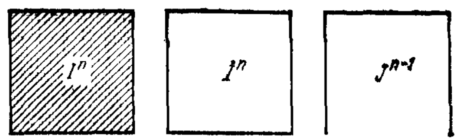


Figure 8.3.1:

It is clear that the space J^{n-1} is contractible to the point $\mathbf{0}$. Since the pairs $(I^n, \dot{I}^n)$ and $(\dot{I}^n, J^{n-1})$ are obviously cofibrations, then the corresponding deformation continues to the homotopy of the triple $(I^n, \dot{I}^n, J^{n-1})$ into itself, connecting the identical map of this triple with the map translating J^{n-1} to the point $\mathbf{0}$. This means that the subtriple $(I^n, \dot{I}^n, \mathbf{0})$ of the triple $(I^n, \dot{I}^n, J^{n-1})$ is its deformation retract and, therefore, is homotopically equivalent to it. Since the triple $(I^n, \dot{I}^n, \mathbf{0})$ (being a pointed pair) is obviously homeomorphic to the pointed pair $(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0)$, then by selecting and fixing some homeomorphism of these pairs, we can assume, that

$$\pi_n(X, A, x_0) = [(I^n, \dot{I}^n, J^{n-1}), (X, A, x_0)], \tag{8.8}$$

i.e., that

$$\pi_n(X, A, x_0) = \pi_0 \Omega^n(X, A, x_0),$$

where

$$\Omega^n(X, A, x_0) = (X, A, x_0)^{(I^n, \dot{I}^n, J^{n-1})} \subset X^{I^n}.$$

Now we can forget about the pair $(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0)$ and use the formula (8.8) to define the set $\pi_n(X, A, x_0)$.

Definition (8.8) has the advantage that for $n \geq 2$ for any two maps

$$a, b : (I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$$

the formula (5.4) of Lecture 5 well determines the map

$$a + b : (I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0),$$

moreover, as is directly verified, the formula

$$[a] + [b] = [a + b]$$

defines in the set $\pi_n(X, A, x_0)$ the addition operation with respect to which it is a group.

However, as in the case of groups $\pi_n X$, these rather tedious checks can be avoided by noting that, by applying the exponential law, the space $\Omega{-}n(X, A, x_0)$ for $n \geq 2$ is naturally homeomorphic to the loop space $\Omega\Omega^{n-1}(X, A, x_0)$ and, therefore,

$$\pi_n(X, A, x_0) = \pi_1\Omega^{n-1}(X, A, x_0).$$

Generally,

$$\pi_n(X, A, x_0) = \pi_k\Omega^{n-k}(X, A, x_0) \quad \text{for any} \quad k = 0, 1, \dots, n-1. \tag{8.9}$$

The group $\pi_n(X, A, x_0)$ is called the *n-dimensional homotopy group of the pointed pair* (X, A, x_0) (or the *n- dimensional homotopy group of the space X relative to the subspace A*; however, the latter term is gradually falling out of use now).

It is obvious that for $A = \{x_0\}$

$$\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0).$$

In this sense, the "relative" homotopy group $\pi_n(X, A, x_0)$ is a generalisation of the "absolute" group $\pi_n(X, x_0)$.

Formula (8.9) for $k = 2$ shows that

Proposition 8.10. *for $n \geq 3$ the group $\pi_n(X, A, x_0)$ is abelian.*

The group $\pi_2(X, A, x_0)$, generally speaking, is non-Abelian (however, unlike the group $\pi_1(X, x_0)$, to denote the operation in the group $\pi_2(X, A, x_0$ we will not use the multiplicative notation), and the "group" $\pi_1(X, A, x_0)$ is only a pointed set.

For $n = 0$, the group $\pi_0(X, A, x_0)$ is not defined. However, for the unity of formulations, we will sometimes express the fact that each component of the space X contains at least one component of the space A by the formula $\pi_0(X, A, x_0) = 0$.

It is clear that the correspondence $(X, A, x_0 \mapsto \pi_n(X, A, x0)$, $n \geq 1$, is a homotopy invariant functor from the category $\mathcal{T}op_2^\bullet$ to the category $\mathcal{A}b\mathcal{G}rp$ of abelian groups for $n \geq 3$, to the category $\mathcal{G}rp$ of groups for $n = 2$ and to the category $\mathcal{E}ns^\bullet$ of pointed sets for $n = 1$.

Moreover, if X_0 and A_0 are components of the spaces X and A containing the point x_0, then the embedding $(X_0, A_0, x_0) \to (X, A, x_0)$ induces for $n \geq 2$ an isomorphism of the group $\pi_n(X_0, A_0, x_0)$ on the group $\pi_n(X, A, x_0)$. Therefore, without much loss of generality, we can assume a pair (X, A) be connected, i.e. consisting of a connected space X and its connected subspace A.

According to Formula (8.9) for $k = n - 1$

$$\pi_n(X, A, x_0) = \pi_{n-1}\Omega(X, A),$$

where $\Omega(X, A) = \Omega^1(X, A, x_0)$ is the space $P(X, A, x_0)$ already familiar to us from lecture 1 of all paths in X starting in the subspace A and ending at the point x_0, i.e. the fibre $\omega^{-1}(x_0)$ of the fibration

$$\omega_1 : P(X, A) \to X, \quad u \mapsto u(1), \tag{8.11}$$

where $P(X, A)$ is the space of all paths $u : I \to X$ starting in the subspace A. This space is nothing more than a co-cylinder of the embedding $i : A \to X$, and therefore, according to Lemma 2.41 of Lecture 2 (which, however, refers to inverted cocylinders),

Proposition 8.12. *the space $P(X, A)$ is homotopically equivalent to the subspace A.*

(However, the mutually inverse homotopy equivalences $P(X, A) \to A$ and $A \to P(X, A)$ are easy to specify directly: they will be the map $\omega_0 : P(X, A) \to A$, $u \mapsto u(0)$, and the map $\sigma : A \to P(X, A)$, $a \mapsto 0_a$; it is clear that $\omega_0 \circ \sigma = \text{id}$, and $\sigma \circ \omega_0 \sim \text{id}$ by the homotopy $u \mapsto u_t$, where $u_t(\tau) = u(t\tau)$, $\tau \in I$.) So,

Proposition 8.13. *for any $n \geq 0$, the group $\pi_n P(X, A)$ is isomorphic to the group $\pi_n A$.*

Using this isomorphism (and equality (8.9) with $k = n - 1$), we obtain an exact Π-sequence from the homotopy sequence of fibration (8.11)

$$\cdots \xrightarrow{i_*} \pi_{n+1} X \xrightarrow{j_*} \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n A \xrightarrow{i_*} \pi_n X \to \cdots \tag{8.14}$$

the right end of which has the form

$$\cdots \to \underbrace{\pi_3(X, A) \xrightarrow{\partial} \pi_2 A \xrightarrow{i_*} \pi_2 X}_{\text{abelian groups}} \xrightarrow{j_*} \underbrace{\pi_2(X, A) \xrightarrow{\partial} \pi_1 A \xrightarrow{i_*} \pi_1 X}_{\text{non-abelian groups}}$$
$$\xrightarrow{j_*} \underbrace{\pi_1(X, A) \xrightarrow{\partial} \pi_0 A \xrightarrow{i_*} \pi_0 X}_{\text{pointed sets}},$$

and which is called the *homotopy sequence of the pair* (X, A).

Remark 8.15. Like the homotopy sequence of a fibration, the homotopy sequence of a pair is, by applying Remark 5.99 from Appendix to Lecture 5, a special case of the Puppe sequence.

A direct comparison of the definitions shows that in the sequence (8.14):

- the homomorphisms i_* and j_* are induced by embeddings $i : A \to X$ and $j : (A, x_0) \to (X, A)$ (i.e., more precisely, the maps $i : (A, x_0) \to (X, x_0)$ and $j : (A, x_0, x_0) \to (X, A, x_0)$);
- the homomorphism ∂ is induced by the map $\Omega^n(X, A) \to \Omega^{n-1} I$, which maps each map $(I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$ to its restriction on the face of I^{n-1} (in interpretation elements of the group $\pi_n(X, A)$ as pointed maps $(\mathbb{E}, \mathbb{S}^{n-1}) \to (X, A)$ the homomorphism ∂ is given by the restriction of these maps on the sphere $\mathbb{S}^{n-1}$);

the action of the group $\pi_1 A$ on the group $\pi_n(X, A)$, $n \geq 1$, matches the element α of the group $\pi_n(X, A)$, set by the map $(I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$, and the element ξ of the group $\pi_1 A$, set by the loop $u : (I, \dot{I}) \to (A, s_0)$, an element of the group $\pi_n(X, A)$, defined by the map $b : (I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$, for which there is a homotopy $G : I^n \times I \to X$ of the maps of pairs $(I^n, \dot{I}^n) \to (X, A)$, such that

$$G(\boldsymbol{t}, t) = u(1-t), \quad \text{if} \quad \boldsymbol{t} \in J^{n-1},\ t \in I,$$
$$G(\boldsymbol{t}, 0) = a(\boldsymbol{t}), \quad G(\boldsymbol{t}, 1) = b(\boldsymbol{t}) \quad \text{for any} \quad \boldsymbol{t} \in I^n;$$

the action of the group $\pi_1 X$ on the set $\pi_1(X, A)$ matches the element $\alpha = [a] \in \pi_1(X, A)$, where a is the path $(I, 0) \to (X, x_0)$ starting at A, and the element $\xi = [u] \in \pi_1 X$, where u is the loop $(I, \dot{I}) \to (X, x_0)$, element $\xi\alpha \in \pi_1(X, A)$, is set by au', where $u' : t \mapsto u(1-t)$.

(According to the general definition of Π-sequences, we also need the actions of the group $\pi_1 X$ on groups $\pi_n X$ and the action of the group $\pi_2(X, A)$ on groups $\pi_n(X, A)$; but it is clear that the first action is an ordinary action $R : \pi_1 X \to \operatorname{Aut} \pi_n X$, and the second can not be separately considered, since for $n = 2$ it is an action by internal automorphisms, and for $n > 2$ due to the condition d) it is induced by a homomorphism $\partial : \pi_2(X, A) \to \pi_1 A$ from the action of the group $\pi_1 A$. In this case, in the case of $n = 2$, the condition d) is reduced to the formula

$$(\partial\alpha)\beta = \alpha + \beta - \alpha \tag{8.16}$$

which should be the case for any elements $\alpha, \beta \in \pi_2(X, A)$.)

Thus, we obtain a direct construction of the sequence (8.14) that does not rely on the fibration (8.11). Of course, with this approach, the exactness of this sequence and the fact that it is a Π-sequence need independent verification. The reader is strongly recommended to do this check with all the details (special attention should be paid to formula (8.16), the direct proof of which is somewhat painstaking).

We emphasise that, thus,

Proposition 8.17. *the group $\pi_n(X, A)$ for $n \geq 3$ is a $\pi_1 A$-module.*

For the sake of unity of terminology, we will also call the group $\pi_2(X, A)$ a $\pi_1 A$-module, although this group is non-Abelian. When we need to emphasise the exceptional nature of this group, we will call it a *crossed $\pi_1 A$-module*. (In general, an additively written group G in which a multiplicative group Π acts is called a *crossed Π-module* if a homomorphism $\partial : G \to \Pi$ is given such that for any elements $\alpha, \beta \in G$ the relation (8.16) holds.)

Of course, instead of the action of the group $\pi_1 A$ on the group $\pi_n(X, A)$, we can consider the corresponding ensemble $\{\pi_n(X, A, a), a \in A\}$ on A.

An obvious generalisation of the proof of Proposition 4.25 of Lecture 4 shows that the maps $I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$ (or maps $(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}0) \to (X, A, x_0)$) are *freely homotopic*, i.e. homotopic as maps $(I^n, \dot{I}^n) \to (X, A)$ (maps $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to$

(X, A)) if and only if when they belong to the same orbit of the action of the group $\pi_1 A$. In particular, since the action of the group $\pi_1 A$ preserves the group structure on $\pi_n(X, A)$, $n \geq 2$, it follows that for $n \geq 2$ the map $(\mathbb{E}, \mathbb{S}^{n-1}) \to (X, A)$ sets the null element of the group $\pi_n * (X, A)$ if and only if when it is freely homotopic to the constant map.

On the other hand, in order for the mapping $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, A)$ to be freely homotopic to a constant map, it is sufficient (and, of course, necessary) that it be homotopic to a map $f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, A)$, such that $f(\mathbb{E}^n) \subset A$, since any such map is associated with a constant homotopy map $\boldsymbol{x} \mapsto f(t\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{E}^n$, $t \in I$, which is (by applying the condition $f(\mathbb{E}^n) \subset A$) a homotopy in the category of $\mathcal{T}op_2$.

Calling a map $f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, A)$, for which $f(\mathbb{E}^n) \subset A$, a *map contracted into* A, we get, therefore, that

Proposition 8.18. *for $n \geq 2$ the map $(\mathbb{E}^n, \mathbb{S}^{n-1}, s0) \to (X, A, x_0)$ sets the null element of the group $\pi_n(X, A)$ if and only if when it is freely homotopic to the map contracted into A.*

It is clear that this conclusion is also valid for $n = 1$.

Interestingly, the free homotopy here can be replaced by a homotopy relative to $\mathbb{S}^{n-1}$, i.e.

Proposition 8.19. *the map $(\mathbb{E}^n, \mathbb{S}^{n-1}, s0) \to (X, A, x_0)$ it is freely homotopic to a contracted map if and only if when it is homotopic to a map relative to $\mathbb{S}^{n-1}$.*

Proof. Indeed, any homotopy $f_t : (\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0) \to (X, A, x_0)$, for which $f_0 = f$ and $f_1(\mathbb{E}^n) \subset A$, we can match a homotopy $g_t : \mathbb{E}^n \to X$ fixed on $\mathbb{S}^{n-1}$, for which $g_0 = f$ and $g_1(\mathbb{E}^n) \subset A$, putting

$$g_t(x) = (F \circ \varphi)(x, t), \quad x \in \mathbb{E}?^n, \quad t \in I,$$

where $F : \mathbb{E}^n \times I \to X$ is a map $(\boldsymbol{x}, t) \mapsto f_t(\boldsymbol{x})$, and φ is a map $\mathbb{E}^n \times I \to \mathbb{E}^n \times I$ such that $\varphi(\boldsymbol{x}, t) = (x, 0)$ if $t = 0$ or $x \in \mathbb{S}^{n-1}$, and $\varphi(\boldsymbol{x}, 1) \in (\mathbb{S}^{n-1} \times I) \cup (\mathbb{E}^n \times 1)$ for any point $\boldsymbol{x} \in \mathbb{E}^n$. (For example, we can assume that

$$\varphi(\boldsymbol{x}, t) = \begin{cases} (\frac{2}{2-t}\boldsymbol{x}, t), & \text{if} \quad 0 \leq t \leq \min(2(1 - |\boldsymbol{x}|), 1), \\ (\frac{\boldsymbol{x}}{|\boldsymbol{x}|}, \frac{4(1-|\boldsymbol{x}|)t^2}{t^2+4(1-|\boldsymbol{x}|)t^2}), & \text{if} \quad 2(1 - |\boldsymbol{x}|) \leq t \leq 1, \end{cases}$$

for $(\boldsymbol{x}, t) \in \mathbb{E}^n \times I$.) □

In particular, we see that

Proposition 8.20. *the equality $\pi_n(X, A) = 0$ is equivalent to the fact that any map $(\mathbb{E}^n, \mathbb{S}^{n-1}, \boldsymbol{s}_0) \to (X, A, x_0)$ is homotopic (freely or relative to $\mathbb{S}^{n-1}$) to the contracted map.*

By analogy with the absolute case (see Definition 5.33 of Lecture 5), we will say that a pair (X, A) is *aspherical in dimension* $n \geq 0$ if $\pi_n(X, A, x_0) = 0$ for any choice of point $x_0 \in A$. In this case, we will write $\pi_n(X, A) = 0$.

Asphericity in dimension 0 (equality $\pi_0(X, A) = 0$) means that any component of the space X intersects with the subspace A. If X is connected, then this condition is always met.

Asphericity in dimension $n \geq 1$ (equality $\pi_n(X, A) = 0$) means that each map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, A)$ is homotopic (free or relative to $\mathbb{S}^{n-1}$) to the contracted map.

In particular, for asphericity in dimension 1, it is necessary that any component of space X contains at most one component of subspace A. If A is connected, then this condition is met.

A pair of (X, A) is called *n-connected* if it is aspherical in all dimensions of $\leq n$. In particular, if $n \geq 1$, then for the n-connectivity of the pair (X, A), it is necessary that each component of the space X contains one and only one component of the space A. If the pair (X, A) is connected, then this condition is automatically met.

If for a connected pair (X, A) the action of the group $\pi_1(A, x_0)$ on the group $\pi_n(X, A, x_0)$, $n \geq 2$, is trivial, then the pair (X, A) is called *homotopically simple in dimension n*. In this case, the group $\pi_n(X, A)$ is defined, whose elements are free homotopy classes of maps $(I^n, \dot{I}^n \to (X, A)$ (or $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, A))$, which for any point $x_0 \in A$ is naturally isomorphic to the group $\pi_n(X, A, x_0)$. In this case,

Proposition 8.21. *the group $\pi_1 X$ naturally acts on the group $\pi_n(X, A)$*

(moreover, for $A = \{x_0\}$, this action is the standard action of the group $\pi_1 X$ on the group $\pi_n(X, x_0) = \pi_n X$).

Proof. It is sufficient to apply Remark 8.6 to the fibration (8.11). □

Note that, as follows directly from the formula (8.16),

Proposition 8.22. *if the pair (X, A) is homotopically simple in dimension 2, then the group $\pi_2(X, A)$ is Abelian. In particular, the group $\pi_2(X, A)$ is Abelian if the space A is simply connected.*

It is also useful to keep in mind that if the space A is connected and simply connected, then the map $\pi_1 X \to \pi_1(X, A)$ is bijective, and therefore multiplication in the group $\pi_1 X$ is transferred to the set $\pi_1(X, A)$. Thus,

Proposition 8.23. *if the space A is connected and simply connected, then the set $\pi 1(X, A)$ is a group.*

It also follows directly from the definitions that for any pointed map $f : (X, A) \to (Y, B)$ and any $n \geq 1$ there is a commutative diagramme

$$\begin{array}{ccc} \pi_n(X, A) & \xrightarrow{\partial} & \pi_{n-1} A \\ \downarrow{f_*} & & \downarrow{f_*} \\ \pi_n(Y, B) & \xrightarrow[\partial]{} & \pi_{n-1} B \end{array}$$

In the language of category theory, this means that

Proposition 8.24. *the homomorphism $\partial : \pi_n(X, A) \to \pi_{n-1}A$ is a natural transformation (morphism) of the functor into a functor $\pi_n : \mathscr{T}op_2 \to \mathscr{A}b\mathscr{G}rp$*

$$\pi_{n-1} \circ \alpha : \mathscr{T}op \xrightarrow{\alpha} \mathscr{T}op \xrightarrow{\pi_{n-1}} \mathscr{A}b\mathscr{G}rp,$$

where α is the functor $(X, A) \mapsto A$.

This remark allows us to formulate axioms for relative homotopy groups similar to axioms [1]-[3] of Lecture 5 §5.15 for absolute groups (only, for example, the axiom of exactness is now formulated for pairs) and, essentially the same, inductive reasoning shows that up to isomorphism (and for non-Abelian groups up to anti-isomorphism) these axioms uniquely characterize the groups of π_n.

8.4 The five lemma

In working with exact sequences of groups, the following simple lemma, known as the *five lemma*, has unexpectedly wide applications.

Lemma 8.25. *If in the commutative diagramme*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & \downarrow{\scriptstyle\varphi_4} & & \downarrow{\scriptstyle\varphi_5} \\ B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \xrightarrow[\beta_3]{} & B_4 & \xrightarrow[\beta_4]{} & B_5 \end{array} \qquad (8.26)$$

with exact rows, the homomorphisms φ_1, φ_2, φ_4, φ_5 are isomorphisms, then the homomorphism φ_3, is also an isomorphism.

Proof. Let $\varphi_3(a_3) = 0$, where $a_3 \in A_3$. Then $\beta_3(\varphi_3)(a_3) = 0$, and therefore $\varphi_4(\alpha_3(a_3)) = 0$, i.e. $\alpha_3(a_3) = 0$. Therefore, there exists an element $a_2 \in A_2$ such that $\alpha_2(a_2) = a_3$. But then $\beta_2(\varphi_2(a_2)) = \varphi_3(\alpha_2(a_2)) = \varphi_3(a_3) = 0$, thus there is an element $b_1 \in B_1$ such that $\beta_1(b_1) = \varphi_2(a_2)$. Since φ_1 is an isomorphism, there exists an element $a_1 \in A_1$ such that $b_1 = \varphi_1(a_1)$. Then $\varphi_2(\alpha_1(a_1)) = \beta_1(\varphi_1(a_1)) = \beta_1(b_1) = \varphi_2(a_2)$ and $\alpha_1(a_1) = a_2$. Thus $a_3 = \alpha_2(a_2) = \alpha_2(\alpha_1(a_1)) = 0$, hence φ_3 *is a monomorphism.*

Similarly, since φ_4 is an isomorphism, then for any element $b_4 \in B_3$ there exists an element $a_4 \in A_4$ such that $\beta_3(b_3) = \varphi_4(a_4)$, and therefore $\varphi_5(\alpha_4)(a_4) = \beta_4(\varphi_4(a_4)) = \beta_4(\varphi_3(b_3)) = 0$. Therefore, $\alpha_4(a_4) = 0$, which means that there exists an element $a_3 \in A_3$ such that $\alpha_3(a_3) = a_4$. But then $\beta_3(\varphi_3(a_3)) = \varphi_4(\alpha_3(a_3)) = \varphi_4(a_4) = \beta_3(b_3)$, and therefore there is an element $b_2 \in B_2$ such that $\beta_2(b_2) = \varphi_3(a_3) - b_3$. Since φ_2 is an isomorphism, there exists an element $a_2 \in a_2$ such that $\varphi_2(a_2) = b_3$, and therefore $\varphi_3(\alpha_2(a_2)) = \beta_2(\varphi_2(a_2) = \beta_2(b_2)) = \varphi_3(a_3) - b_3$, i.e. $b_3 = \varphi_3(a_3 - \alpha_2(a_3))$, hence φ_3 *is an epimorphism.* □

The course of this proof is unambiguously dictated by the arrows of this diagramme, and it is quite possible to carry it out in your mind, helping yourself by moving your finger along the diagramme. This method of proof is therefore called the "diagramme chasing".

Remark 8.27. It follows from the proof of Lemma 8.25 that:

a) if φ_1 is an epimorphism, and φ_2 and φ_4 are monomorphisms, then φ3is a monomorphism;

b) if φ is a monomorphism, and φ_2 and φ_4 are epimorphisms, then φ3is an epimorphism.

This remark is sometimes useful.

To show how Lemma 8.25 works, we compare the homotopy sequence of the pointed fibration $p : E \to B$ with the homotopy sequence of the pair (E, F), where $F = p^{-1}(b_0)$ is the fibre of this fibration:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \pi_n F & \longrightarrow & \pi_n E & \longrightarrow & \pi_n(E,F) & \longrightarrow & \pi_{n-1}F & \longrightarrow & \pi_{n-1}E & \longrightarrow \cdots \\ & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle p_*}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & \\ \cdots \longrightarrow & \pi_n F & \longrightarrow & \pi_n E & \longrightarrow & \pi_n B & \longrightarrow & \pi_{n-1}F & \longrightarrow & \pi_{n-1}E & \longrightarrow \cdots \end{array} \tag{8.28}$$

An automatic verification shows that this diagramme is commutative. Therefore, according to the Five Lemma,

Proposition 8.29. *the map*

$$p_* : \pi_n(E, F) \to \pi_n B \tag{8.30}$$

for any $n \geq 2$ is an isomorphism.

Of course, this statement is easily proved directly using the axiom **CHE** (do it!), but applying the Five Lemma reduces the proof to triviality.

On the other hand, if the isomorphism (8.30) is independently established, then the statements about the exactness of homotopy sequences of fibrations and pairs turn out to be direct consequences of each other, so that the proof of the exactness of homotopy sequences of fibration from the previous lecture can, if desired, be replaced by a proof of the exactness of homotopy sequences of pairs (see above).

Remark 8.31. Although Lemma 8.25 is usually applied to diagrammes of type (8.26) consisting of abelian groups (or modules), but, as its proof shows, in which abelicity is not used, this lemma is also valid for sequences of non-Abelian groups (we actually used this when deriving the isomorphism (8.30) for $n = 2$).

For the same reasons, Lemma 8.25 remains valid if in Diagramme (8.26 the last two groups of each row are only pointed sets. Moreover, it is easy to see that Lemma 8.25 is preserved if the pointed sets are even the last three groups in each row, the groups A_2 and B_2 act on the sets A_3 and B_3, and the exactness in terms A_3 and B_3 is understood in the enhanced sense indicated above (i.e., the preimages of points in the maps α_3 and β_3 are orbits; in addition, it is assumed, of course, that the map φ_3 is a φ_2-map.

This remark is relevant for the case when the lines of diagram Diagramme (8.26) are the end segments of exact Π-sequences. In particular, it follows from it that

Proposition 8.32. *the map (8.30) is an isomorphism (a bijection for $n = 1$).*

Remark 8.33. For $n = 0$, one can only assert that $\pi_0(E, F) = 0$ if and only if $\pi_0 B = 0$.

Appendix

8.A Homotopy sequence of a triple

The isomorphism (8.30) proved in Lecture 8 can be substantially generalised.

Proposition 8.34. *For any fibration $p : E \to B$ and any subspace $A \subset B$, the homomorphism*

$$p_* : \pi_n(E, F_A) \to \pi_n(B, A), \quad n \geq 1, \quad \textit{where} \quad F_A = p^{-1}A, \tag{8.35}$$

induced by the map $p : (E, F_A) \to (B, A)$, is an isomorphism.

When $A = \{b_0\}$ the isomorphism (8.35) turns into the isomorphism (8.30) of Lecture 8.

Proposition 8.34 can be proved in many different ways. For example, it is easily proved (do it!) by a direct geometric constructions using the axiom **CHE**. But we will prefer another more instructive way.

Let (X, A, B, x_0) be an arbitrary pointed triple. By definition, X is a topological space, A is its subspace, B is a subspace of space A, and x_0 is a point of space B:

$$x_0 \in B \subset A \subset X.$$

Let, further, $i : (A, B) \to (X, B)$ and $j : (X, B) \to (X, A)$ be inclusions and

$$i_* : \pi_n(A, B) \to \pi_n(X, B) \quad \text{and} \quad j_* : \pi_n(X, B) \to \pi_n(X, A)$$

be corresponding homomorphisms of homotopy groups. Finally, let

$$\partial : \pi_n(X, A) \to \pi_{n-1}(A, B)$$

is the composition of the connecting homomorphism $\pi_n(X, A) \to \pi_{n-1}A$ from the homotopy sequence of the pair (X, A) and the homomorphism of the embedding $v\pi_{n-1}A \to \pi_{n-1}(A, B)$ from the homotopy sequence of the pair (A, B).

Definition 8.36. The sequence

$$\cdots \to \pi_n(A, B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A, B) \to \cdots \tag{8.37}$$

is called the *homotopy sequence of the triple* (X, A, B).

Proposition 8.38. *The homotopy sequence of a triple is exact.*

The sequence (8.37) to the left is infinite, and its right end has the form

$$\cdots \to \underbrace{\pi_2(A,B) \to \pi_2(X,B) \to \pi_2(X,A)}_{\text{non-abelian groups}} \to \underbrace{\pi_1(A,B) \to \pi_1(X,B) \to \pi_1(X,A).}_{\text{pointed sets}}$$

It is possible to easily determine in this sequence the actions of the groups π_2, in relation to which it will be a Π-sequence. However, we will not need these actions.

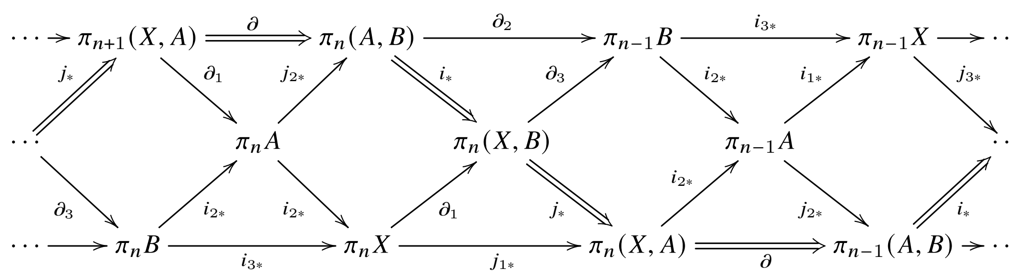
Proof. (of Proposition 8.38) Let

$$\begin{aligned}&i_1 : A \to X, \quad i_2 : B \to A, \quad i_3 : B \to X,\\ &j_1 : (X, x_0) \to (X,A), \quad j_2 : (A, x_0) \to (A,B), \quad j_3 : (X, x_0) \to (X,B)\end{aligned}$$

be inclusions, and

$$\partial_1 : \pi_{n+1}(X,A) \to \pi_n A, \quad \partial_2 : \pi_{n+1}(A,B) \to \pi_n B, \quad \partial_3 : \pi_{n+1}(X,B) \to \pi_n B$$

be connecting homomorphisms of homotopy sequences of pairs (X,A), (A,B) and (X,B). Consider the diagramme[1]



An automatic verification shows that this diagramme is commutative. It consists of four symmetrically arranged sequences, three of which, being homotopy sequences of pairs, are exact, and the fourth, highlighted in the diagramme with thickened arrows, is the sequence (8.37). At the same time, since the map $j \circ i : (A,B) \to (X,B)$ can be decomposed into the composition of inclusion $(A,B) \to (A,A) \to (X,A)$, and $\pi_n(A,A) = 0$, then $j_* \circ i_* = 0$, i.e. $\operatorname{im} i_* \subset \ker j_*$. It turns out that proposition 8.38 follows from here by a purely algebraic diagramme chasing.

Indeed, since $i_{1*} = 0$, then $i_* \circ \partial = i_* \circ j_{2*} \circ \partial_1 = j_{3*} \circ i_{1*} \circ \partial_1 = 0$, and similarly $\partial \circ j_* = j_{2*} \circ \partial_1 \circ i_* = j_{2*} \circ i_{2*} \circ \partial_3 = 0$. Thus, $\operatorname{im} \partial \subset \ker i$ and $\operatorname{im} j_* \subset \ker \partial$.

Conversely if $\gamma \in \pi_n(A,B)$ and $\gamma \in \ker i_*$, i.e. $i \circ \gamma = 0$, then $\partial_2 = 0$, and therefore $\gamma = j_{2*}\alpha$, where $\alpha \in \pi_n A$. In this case, $j_{3*} \circ i_{1*}\alpha = i_* \circ j_{2*}\alpha = i_*\gamma = 0$ and means $i_{1*}\alpha = i_{3*}\beta$, where $\beta \in \pi_n B$. Therefore, $i_{1*}(\alpha - i_{2*}\beta) = i_{1**}\alpha - i_{3*}\beta = 0$,

[1] Transcriber's note: this is called *Wall's "braid" diagramme.*

and therefore $\alpha - i_{2*}\beta = \partial_1\gamma'$, where $\gamma' \in \pi_{n+1}(X,A)$. But then $\partial_1\gamma' = j_{2*}\partial_1\gamma' = j_{2*}(\alpha - i_{2*}\beta) = j_2\alpha = \gamma$, i. e,, $\gamma \in \operatorname{im}\partial$. Thus, $\ker i_* \subset \operatorname{im}\partial$.

Similarly, if $\gamma \in \pi_n(X,A)$ and $\gamma \in \ker\partial$, i.e. $\partial\gamma = 0$, then $j_{2*}\circ\partial_1\gamma = 0$, and therefore $\partial_1\gamma = i_{2*}\beta$, where $\beta \in \pi_{n-1}B$. In this case, $i_{3*}\beta = i_{1*}\circ i_{2*}\beta = i_{1*}\circ\partial_1\gamma = 0$ and, therefore, $\beta = \partial_3\beta'$, where $\beta' \in \pi_n(X,B)$. Therefore $\partial_1(\gamma - j_*\beta = \partial_1\gamma - i_{2*}\circ\partial_2\beta' = 0$ and, therefore, $\gamma - j_*\beta' = j_{1*}\alpha$, where $\alpha \in \pi_n X$. But then $j_*(j_{3*}\alpha + \beta') = j_{1*}\alpha + j_*\beta' = \gamma$ i. e, $\gamma \in \operatorname{im} j_*$. Thus $\ker\partial \subset \operatorname{im} j_*$.

Finally, we have already seen that $\operatorname{im} i_* \subset \ker j_*$. Conversely, if $\beta \in \pi_n(X.B)$ and $\beta \in \ker j_*$ i.e. $j_*\beta = 0$, then $i_{2*}\circ\partial_2\beta = 0$, and therefore $\partial_3\beta = \partial_2\gamma$, where $\gamma \in \pi_n(A,B)$. In this case $\partial_3(\beta - i_*\gamma) = \partial_3\beta - \partial_2\gamma = 0$ and, therefore, $\beta - i_*\gamma = j_{3*}\alpha$, where $\alpha \in \pi_n X$. Therefore $j_{1*}\alpha = j_*\circ j_{3*}\alpha = j_*\beta - j_*\circ i_*\gamma = 0$ and hence $\alpha = i_{1*}\alpha'$, where $\alpha' \in \pi_n A$. But then $i_*(j_{2*}\alpha' + \gamma) = j_{3*}\circ i_{1*}\alpha' + i_*\gamma = j_{3*}\alpha + i_*\gamma = \beta$, i. e. $\beta \in \operatorname{im} i_*$ Thus, $\ker j_* \subset \operatorname{im} i_*$. □

Remark 8.39. Sequence (8.37) can be constructed for any family of $\{H_n, \partial\}$ functors $H_n : \mathscr{T}op_2^\bullet \to \mathscr{A}b\mathscr{G}rp$ and natural transformations $\partial : H_n(X,A) \to H_{n-1}(A, x_0)$. At the same time, as is directly evident from the above proof, this sequence is exact if for any pair $(X,A) \in \mathscr{T}op_2$ the sequence

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A,x_0) \to H_n(X,x_0) \to H_n(X,A) \to \cdots$$

is also exact if $H_n(A,A) = 0$ for any space $A \in \mathscr{T}op^\bullet$.

We will have occasion to use this remark next semester.

Proof. (of Proposition 8.34 Consider the diagramme

$$\begin{array}{ccccccccccc}
\cdots \to & \pi_n(F_A,F) & \to & \pi_n(E,F) & \to & \pi_n(E,F_A) & \to & \pi_{n-1}(F_A,F) & \to & \pi_{n-1}(E,F) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \pi_n(A,b_0) & \to & \pi_n(B,b_0) & \to & \pi_n(B,A) & \to & \pi_{n-1}(A,b_0) & \to & \pi_{n-1}(B,b_0) & \to \cdots
\end{array}$$

the upper line of which is the homotopy sequence of the triple (E, F_A, F), the lower line is the homotopy sequence of the pair (B,A), and the vertical homomorphisms are induced by the map p. This diagramme is obviously commutative and all its vertical homomorphisms, except for the central homomorphism $p_* : \pi_n(E,F_A) \to \pi_n(B,A)$, being homomorphisms of the form (8.30) from Lecture 8 (recall that the map $p_A : FA \to A$ is also a fibration), are isomorphisms. Therefore, according to the Five Lemma, the central homomorphism is also an isomorphism. □

Remark 8.40. Our way of proving Propositions 8.34 and 8.38 is not the easiest. It would be much easier to reverse the sequence of reasoning by first proving Proposition 8.34 purely geometrically and then by deducing Proposition 8.38

from it using the diagramme

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \pi_n(S,T) & \longrightarrow & \pi_{n-1}T & \longrightarrow & \pi_{n-1}S & \longrightarrow & \pi_{n-1}(S,T) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \pi_n(A,B) & \longrightarrow & \pi_{n-1}(X,B) & \longrightarrow & \pi_{n-1}(X,A) & \longrightarrow & \pi_{n-1}(A,B) & \longrightarrow \cdots
\end{array}$$

where $S = \Omega(X,A)$ (respectively $T = \Omega(X,B)$) is the space of paths in X starting in the subspace A (in the subspace B) and ending at the base point x_0. The top line of this diagramme is the homotopy sequence of the pair $\Omega(X,A)$, $(\Omega(X,B))$, and its bottom line is the sequence (8.37). Its mmiddle vertical homomorphisms are the isomorphisms inverse to the isomorphism (8.9) of Lecture 8 (for $k = n-1$), and the lateral homomorphisms are induced by the fibration $\omega_0 : \Omega(X,A) \to A$ and, therefore, by applying proposition prop:08-A1, are also isomorphisms (clearly, then $\omega_0^{-1}B = \Omega(X,B)$). An automatic verification shows that the diagramme in question is commutative. Therefore, the exactness of its upper line implies the exactness of its lower line. However, this proof does not allow us to make Remark 8.39.

8.B Homotopy groups of triads

It immediately follows from the exactness of the homotopy sequence of the triple that

Proposition 8.41. *for $\pi_r(A,B) = 0$ and $\pi_{r-1}(A,B) = 0$, the homomorphism induced by inclusion*

$$\pi_r(X,B) \to \pi_r(X,A) \tag{8.42}$$

is an isomorphism.

Despite its simplicity, this criterion, as we will see in its place, is very useful and allows us to obtain important and interesting geometric results. However, it is insufficient for problems in which, instead of a pair (X,B), a pair of the form (X',B) is involved, where $X' \subset X$. In order to obtain a generalisation of this criterion necessary for such problems, we must generalise the homotopy sequence of the triple accordingly.

We will call an *n-ad* a family $(X; A_1, \dots, A_{n-1})$, consisting of a topological space X and its arbitrary subspaces $A_1, \dots, A_{n-1}$. A continuous map $f : X \to Y$, for which $f(A_i) \subset B_i$ for any $i = 1, \dots, n-1$ is called an *n-ad morphism* $f : (X; A_1, \dots A_{n-1}) \to (Y; B_1, \dots, B_{n-1})$. All n-ads and their morphisms obviously constitute a category with homotopies. We will denote this category by the symbol $\mathscr{T}\!op_{[n]}$, and the corresponding homotopy category by the symbol $[\mathscr{T}\!op_{[n]}]$.

The category of n-ples $\mathscr{T}\!op_n$ introduced in Lecture 8 is a complete subcategory of the category $\mathscr{T}\!op_{[n]}$. For $n = 2$, the equality $\mathscr{T}\!op_2 = \mathscr{T}\!op_{[2]}$ takes place, but already for $n = 3$, the category $\mathscr{T}\!op_3$ is a proper subcategory of the category $\mathscr{T}\!op_{[3]}$.

If the space X is pointed and the base point x_0 lies in each space $A_1, \dots, A_{n-1}$, then the n-ad $(X; A_1, \dots, A_{n-1})$ is called a *pointed n-ad*. Pointed n-ads make up the category $\mathcal{T}op_{[n]}$, which is a complete subcategory of the category $\mathcal{T}op_{[n+1]}$.

For us, the pointed triads $(X; A, B, x_0)$ will be of particular interest, which, as a rule, we will simply denote by $(X; A, B)$. Note that for such a triad, the intersection $C = A \cap B$ is not empty.

For each pointed triad $(X; A, B)$ and any $r \geq 0$, a homomorphism is defined

$$j_* : \pi_r(A, C) \to \pi_r(X, B), \quad C = A \cap B, \tag{8.43}$$

induced by the inclusion $j : (A, C) \to (X, B)$. To obtain a criterion that this homomorphism is an isomorphism, we introduce into consideration the groups

$$\pi_r(X; A, B) = \pi_{r-1}(\Omega(X, B), \Omega(A, C)), \quad r \geq 2.$$

Definition 8.44. The group $\pi_r(X; A, B)$ is called the *r-dimensional homotopy group of the triad* $(X; A, B)$.

Of course, this "group" for $r = 2$ is only a pointed set. For $r \geq 4$, the group $\pi_r(X; A, B)$ is abelian.

By applying the exponential law, homotopy classes (in the category $\mathcal{T}op_{[4]}$ of maps can be considered as elements of the group $\pi_r(X; A, B)$

$$(I^{r-1} \times I; \dot{I}^{r-1} \times I, I^{r-1} \times 1, \widetilde{\{0\}}) \to (X; A, B, x_0), \tag{8.45}$$

where $\widetilde{\{0\}} = (I^{r-1} \times 0) \cup (\{0\} \times I)$.

It is clear that the 4-ad $(I^{r-1} \times I; \dot{I}^{r-1} \times I, I^{r-1} \times 1, \widetilde{\{0\}})$ is homotopically equivalent (in the category $\mathcal{T}op_{[4]}$) to the pointed triad $(\mathbb{E}^r; \mathbb{E}^{r-1}_+, \mathbb{E}^{r-1}_-, s_0)$, where $\mathbb{E}^{r-1}_+$, $\mathbb{E}^{r-1}_-$ are the hemispheres on which the boundary sphere $\mathbb{S}^{r-1}$ of the sphere $\mathbb{E}^r$ is split by the hyperplane $x_r = 0$. Therefore, we can consider elements of the group $\pi_r(X; A, B)$ also as homotopy classes (in the category $\mathcal{T}op^\bullet_{[3]}$) of maps

$$(\mathbb{E}^r; \mathbb{E}^{r-1}_+, \mathbb{E}^{r-1}_-) \to (X; A, B) \tag{8.46}$$

of pointed triads.

Since the permutation of the hemispheres $\mathbb{E}^{r-1}_+$ and $\mathbb{E}^{r-1}_-$ translates each map (8.46) into a similar map for the triad $(X; B, A)$, it immediately follows, that for any $r \geq 1$ there is an isomorphism

$$\pi_r(X; A, B) \approx \pi_r(X; B, A). \tag{8.47}$$

Since

$$\pi_r(X, B) = \pi_{r-1} Omega(X, B), \quad I, \pi_r(A, C) = \pi_{r-1} Omega(A, C),$$

then the homotopy sequence of the pair $(\Omega(X, B),\ \Omega(A, C))$ gives us the exact sequence

$$\cdots \to \pi_r(A, C) \xrightarrow{j_*} \pi_r(X, B) \xrightarrow{k_*} \pi_r(X; A, B) \xrightarrow{\partial} \pi_r(A, C) \to \cdots \tag{8.48}$$

which is called the *homotopy sequence of the triad* $(X; A, B)$. The homomorphism j_* of this sequence is the homomorphism (8.43) of interest to us, the homomorphism k_* is to each map $(I^r, \dot{I}^r, J^{r-1}) \to (X, B, x_0)$ this is the same map, but considered as a map of the form (8.45), and the homomorphism ∂ maps each map (8.46) to its restriction on the hemisphere $\mathbb{E}_+^{r-1}$. (To simplify the formulation, we identify maps and their homotopy classes here; we will allow ourselves this freedom of speech in the future.)

The sequence (8.48) is naturally a Π-sequence. We will provide a direct description of the relevant actions to the reader's initiative.

Due to the exactness of the sequence (8.48), the homomorphism (8.43) is an isomorphism if $\pi_{r+1}(X; A, B) = 0$ and $\pi_r(X; A, B) = 0$. Therefore, if $\pi_r(X; A, B) = 0$ for $r \leq n$ (the triad $(X; A, B)$ satisfying this condition is called *n-connected*), then the homomorphism (8.43) will be an isomorphism for $r < n$ and an epimorphism for $r = n$. (just as for a n-connected pair (A, B), the homomorphism (8.42) will be an isomorphism for $r < n$ and is an epimorphism for $r = n$).

8.C Invariance of homotopy groups under deformation retractions

For $A \subset B$, we can consider any map (8.46) as a map $(\mathbb{E}^r, \mathbb{S}^{r-1}) \to (X, A)$, and thereby obtain some homomorphism

$$\pi_r(X; A, B) \to \pi_r(X, A), \quad A \supset B. \tag{8.49}$$

An automatic verification shows that together with the identity isomorphisms $\pi_r(X, B) \to \pi_r(X, B)$ and $\pi_r(A, C) \to \pi_r(A, B)$ (note that $C = B$ for $A \supset B$) the homomorphisms (8.49) constitute a homomorphism of the sequence (8.48) in the sequence (8.37). Therefore, by applying the Five Lemma,

Proposition 8.50. *all homomorphisms (8.49) are isomorphisms.*

Thus, for $A \supset B$, the group $\pi_r(X; A, B)$ does not depend on B.

This fact can be proved in another way by noting that

Proposition 8.51. *if the subspace A of the space X is contractible ($A \searrow$ pt), then*

$$\pi_r(X, A) = \pi_r X \tag{8.52}$$

for any $r \geq 1$.

Proof. Indeed, in the exact homotopy sequence of the pair (X, A), all groups $\pi_r A$ are equal to zero, and therefore the homomorphisms $\pi_r X \to \pi_r(X, A)$ are isomorphisms. □

This proves anew the isomorphism (8.49), because if for the triad $(X; A, B)$ there is an inclusion $A \supset B$ and, therefore, $C = B$, then

$$\begin{aligned}\pi_r(X; A, B) &= \pi_r(X; B, A) = \pi_{r-1}(\Omega(X, A), \Omega(X, B)) = \pi_{r-1}(\Omega(X, A), \Omega(B, B)) \\ &= \pi_r\Omega(X, A) = \pi_r(X, A),\end{aligned}$$

since the space $\Omega(B, B) = PB$, as we know, is contractible.

The isomorphism (8.52) can be easily generalised.

Proposition 8.53. *If $(X, A) \subset (X', A')$, and the space X is a deformation retract of the space X', and the space A is a deformation retract of the space A' (i.e. $X' \searrow\!\!\searrow X$ and $A \searrow\!\!\searrow A'$), then the inclusion $(X, A) \to (X', A')$ induces isomorphisms*

$$\pi_r(X, A) = \pi_r(X', A'), \quad r \geq 1.$$

Proof. In the commutative diagramme

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \pi_r A & \longrightarrow & \pi_r X & \longrightarrow & \pi_r(X, A) & \longrightarrow & \pi_{r-1} A & \longrightarrow & \pi_{r-1} X & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \pi_r A' & \longrightarrow & \pi_r X' & \longrightarrow & \pi_r(X', A') & \longrightarrow & \pi_{r-1} A' & \longrightarrow & \pi_{r-1} X' & \longrightarrow \cdots
\end{array}$$

the vertical homomorphisms of which are induced by inclusions, and the horizontal ones are homotopy sequences of pairs (X, A) and (X', A'), all vertical homomorphisms, except the central homomorphism $\pi_r(X, A) \to \pi_r(X', A')$, are isomorphisms. Hence, according to the Five Lemma, this central homomorphism will also be an isomorphism. □

When $A' = \{x_0\}$ we get the isomorphism (8.52).

It is clear that a similar statement (with almost the same proof) holds for triads: if $(X; A, B) \subset (X'; A', B')$ and each space$X$, A, B and $C = A \cap B$ is a deformation retract of the space X', A', B' and $C' = A' \cap B'$, respectively, then

$$\pi_r(X; A, B) = \pi_r(X'; A', B')$$

for any $r \geq 1$.

8.D Homotopy sequence of 4-ads

Is it possible, by analogy with homotopy groups of triads, to define homotopy groups of n-ad for $n \geq 4$? For example, it seems natural to define the homotopy groups $\pi_r(X; A, B, Y)$ of an arbitrary pointed 4-ad $(X; A, B, Y)$ by the formula

$$\pi_r(X, A, B, Y) = \pi_{r-1}(\Omega(X, Y); \Omega(X, E), \Omega(A, D))$$

(having meaning for any $r \geq 2$), where $D = Y \cap A$ and $E = Y \cap B$. For these groups there is an exact sequence

$$\cdots \to \pi_r(X; A, E) \to \pi_r(X; A, Y) \to \pi_r(X; A, B, Y) \to \pi_{r-1}(X; A, E) \to \cdots,$$

being nothing more than the sequence (8.48) for the triad $(\Omega(X, Y); \Omega(X, E), \Omega(A, D))$ (just as the sequence (8.48) itself was nothing more than a homotopy sequence of the pair $(\Omega(X, B), \Omega(A, C))$). However, by rearranging $\Omega(X, E)$ and $\Omega(A, D)$

(why does the group $\pi_r(X; A, B, Y)$ not change), we similarly get the exact sequence

$$\cdots \to \pi_r(D; D \cap E) \to \pi_r(Y, E) \to \pi_r(X; A, B, Y) \to \pi_{r-1}(D; D \cap E) \to \cdots,$$

from the comparison of which with the homotopy sequence of the triad $(Y; D, E)$, it immediately follows by applying the Five Lemma that $\pi_r(X; A, B, Y) \approx \pi_r(Y; D, E)$. Thus, the introduced groups are reduced to groups of triads.

Nevertheless, this attempt turned out to be useful to us, because we now see that

Proposition 8.54. *for any pointed* 4*-ad* $(X; A, B, Y)$ *there is an exact sequence*

$$\cdots \to \pi_r(X; A, E) \xrightarrow{i_*} \pi_r(X; A, Y) \xrightarrow{\partial} \pi_{r-1}(Y; D, E) \xrightarrow{j_*} \pi_{r-1}(X; A, E) \to \cdots, \tag{8.55}$$

where $D = Y \cap A$, $E = Y \cap B$.

An automatic verification shows that, as expected, the homomorphisms i_* and j_* of this sequence are induced by the inclusions

$$i : (X; A, E) \to (X; A, Y) \quad \text{and} \quad j : (Y; D, E) \to (X; A, E).$$

As for the binding homomorphism ∂, it is a composition of the permutation isomorphism $\pi_r(X; A, Y) \approx \pi_r(X; Y, A)$, the connecting homomorphism $\pi_r(X; Y, A) \to \pi_{r-1}(Y, D)$ from the homotopy sequence of the triad $(X; Y, A)$, the homomorphism $\pi_{r-1}(Y, D) \to$
$pir-1(Y; E, D)$ from the homotopy sequence of the triad $(Y; E, D)$ and the isomorphism of the permutation $\pi_{r-1}(K; E, D) \approx \pi_{r-1}(X; D, E)$.

Note the special case of the sequence (8.55) that occurs when $Y \supset B$, i.e. when $E = B$:

$$\cdots \to \pi_r(X; A, B) \xrightarrow{i_*} \pi_r(X; A, Y) \xrightarrow{\partial} \pi_{r-1}(Y; D, B) \xrightarrow{j_*} \pi_{r-1}(X; A, B) \to \cdots. \tag{8.56}$$

Of course, the failure of our attempt to construct meaningful (non-reducible to homotopy groups of triad) homotopy groups n-ad for $n \geq 4$ does not mean that such groups cannot be constructed. We will leave the clarification of this question to the reader.

Lecture 9

The value of fibrations in the theory of homotopy groups is determined mainly by the presence of a homotopy sequence for them. Therefore, it is advisable to study in general the maps for which this sequence can be written.

9.1 Weak fibrations

Definition 9.1. A map $p : E \to B$ is called a *weak fibration* if for any point $e_0 \in E$ and any $n > 0$ it induces an isomorphism

$$p_* : \pi_n(E, F_{b_0}, e_0) \xrightarrow{\approx} \pi_n(B, b_0), \quad b_0 = p(e_0), \quad F_{b_0} = p^{-1}(b_0),$$

and if each component of the space E that passes into the component of space B containing the point b_0 intersects with F_{b_0}.

Replacing in the homotopy sequence of pairs (E, F_{b_0}) the group $\pi_n(E, Fb_0)$ with the isomorphic group $\pi_n B$, we get an exact sequence

$$\cdots \to \pi_n F_{b_0} \xrightarrow{i_*} \pi_n E \xrightarrow{p_*} \pi_n B \xrightarrow{\partial} \pi_{n-1} F_{b_0} \to \cdots, \tag{9.2}$$

called the *homotopy sequence of the weak fibration* $p : E \to B$ *at the point* e_0. Therefore, for calculations with homotopy groups, weak fibrations are no worse than ordinary fibrations (which we will now allow ourselves to call *strong fibrations*).

Remark 9.3. It should be borne in mind that in the literature on topology, the term "weak fibration" is used in many different senses. For example, Spanier (see [12], p. 482) calls weak fibrations *fibrations in the sense of Serre* (maps satisfying the axiom **CH** only with respect to cube). On the contrary, weak fibrations in the sense of Definition 9.1 Dold and Thom (to whom the merit of introducing this concept belongs) call *quasi-fibrations*.

Of course, any fibration is a weak fibration. The converse, generally speaking, is not true.

Example 9.4. Let E be a subset of the plane consisting of two horizontal segments connected by a vertical segment, B be a horizontal segment of double length and $p : E \to B$ is the projection (see Fig. 9.1.1).

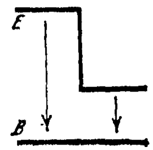


Figure 9.1.1:

It is directly verified that this projection is a weak fibration. However, it will not be a fibration (even in the sense of Serre), since the path $id : B \to B$ in B is not covered by any path in E.

As we know, if the map $p : E \to B$ is a fibration, then for any subspace $A \subset B$ the induced map $p_A = p|_A : F_A \to A$, where $F_A = p^{-1}A$, is also a fiibration (this fact, in particular, was significantly used in the proof of proposition 8.34 from the Appendix to the previous lecture). The analogous statement for weak fibrations is, in general, incorrect (and, therefore, the more general statement that a map induced by a weak fibration is a weak fibration is incorrect).

Example 9.5. Let B be a half-plane $x > 0$ of the plane $\mathbb{R}^2$, and E be the same half–plane, but with a section $0 < x < l$, $y = 0$, from which only its upper side is left. Obviously, the natural projection $p : E \to B$ is a weak fibration (because all homotopy groups of spaces E, B and F_{b_0}, $B_0 \in B$, in dimensions > 0 are equal to zero). At the same time, the preimage F_A of the circle $A : (x-1)^2 + y^2 = 1/4$ is a segment, and therefore the projection $p_A : F_A \to A$ cannot be a fibration - even a weak one (because $\pi_1(F_A) = 0$, whereas $\pi_1 A \neq 0$).

Following Dold and Thom, we will call a subspace $A \subset B$ *distinguished* (with respect to a given surjective map $p : E \to B$) if the map $p_A : F_A \to B$ is a weak fibration.

Now it is easy to see by actually repeating verbatim the proof of Proposition 8.34 from the Appendix to the previous lecture that, when restricted to distinguished sets, this proposition also holds for weak fibrations i.e.

Proposition 9.6. *for any weak fibration $p : E \to B$ and any distinguished subspace of $A \subset B$ the homomorphism*

$$p_* : \pi_n(E, F_A) \to \pi_n(B, A), \quad n > 0,$$

induced by the map $p : (E, F_A) \to (B, A)$, is an isomorphism (for any choice of base points $e_0 \in F_A$ and $b_0 = p(e_0)$).

It is natural to expect that similar reservations are needed when generalising, to weak fibrations, Theorem 1.71 of Appendix to Lecture 1and, therefore, from the existence for the space B of a cover (even enumerable) consisting of open distinguished sets, it does not follow yet, in general, hat the map $p : E \to B$ is a weak fibration (i.e. that the space B is distinguished.)

Example 9.7. Let B be the plane $\mathbb{R}^2$, and E be the same plane, with a section $0 < x < 1$, $y = 0$, from which only its upper side is left. According to what was said in Example 9.5, the half-lane $x > 0$ is distinguished with respect to the natural projection $p : E \to B$. By symmetry, the half-plane $x < 1$ is also marked, which together with the half-plane $x > 0$ is an open cover of the plane B. At the same time, if the projection p were a weak fibration, then due to the fact that all the sets $p^{-1}(b_0)$, $B_0 \in B$, are points and therefore aspherical, it would would be an isomorphism of $pi_1 E \approx \pi_1 B$. But $\pi_1 B = 0$, and the group $\pi_1 E$ is nontrivial, since the loop bypassing the cut is not homotopic to zero (prove it!). Therefore, the projection of $p : E \to B$ cannot be a fibration - even a weak one.

At the same time, it turns out that

Proposition 9.8. *if for a surjective map $p : E \to B$ there is a covering of the space B consisting of distinguished open sets U_α such that any of their finite intersections are also distinguished, then the map $p : E \to B$ will be a weak fibration.*

Moreover, the following slightly more general theorem, due to Dold and Thom, is valid.

Theorem 9.9. *A surjective map $p : E \to B$ will be a weak fibration if for the space B there exists a covering consisting of distinguished closed sets U_α such that every intersection $U_\alpha \cap U_\beta$ is a combination of covering elements.*

Note that the covering U_α is not supposed to be enumerable.

For the proof of Theorem 9.9 we will assume four lemmas, the first of which has an independent interest.

9.2 The additional lemma

Along with the standard constructions of the sum of elements of homotopy groups described in Lectures 5 and 6, there are many other constructions for this, which, although less elegant, are more convenient in certain situations.

Descriptions of these constructions are known by the common name of additional lemmas. We will not dive deep into the swamp of these lemmas and for now we will limit ourselves to only one of them, which is necessary to prove Theorem 9.9.

This lemma uses the fact that for any pointed pair (X, A, x_0), we can consider homotopy classes of maps $(I^n, \dot{I}^n, \mathbf{0}) \to (X, A, x_0)$ as elements of the group $\pi_n(X, A, x_0)$.

Let K^n e a subcube of the cube I^n consisting of points $\boldsymbol{t} = (t_1, \dots, t_n)$ for which

$$1/4 \le t_1 \le 1/2, \dots, 1/4 \le t_{n-1} \le 1/2, 0 \le t_n \le 1/4$$

(see Fig. 9.2.1).

It is clear that pair $(I^n, \dot{I}^n \cup K^n)$ is a cofibration, with $\dot{I}^n \cup K^n \searrow \dot{I}^n$). Therefore, any map $(I^n, \dot{I}^n, \mathbf{0}) \to (X, A, x_0)$, is homotopic (in the category $\mathscr{T\!op}_2^\bullet$) to the map that translates the cube K^n to A, and, therefore, to the map that translates the cube K^n to the point x_0.

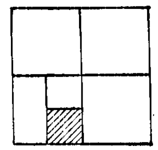


Figure 9.2.1:

Let, further, $\dot{K}^n$ be the boundary of the cube K^n, and L^{n-1} be the union of all its faces other than the face $K_1^{n-1} = K^n \cap \dot{I}^n$, given by the equation $t_n = 0$. Let, finally, φ_n be the linear map

$$t_1 \mapsto 4t_1 - 1, \ldots, t_{n-1} \mapsto t_{n-1} - 1, t_n \mapsto 4t_n$$

from the triple $(K^n, \dot{K}^n, L^{n-1})$ to the triple $(I^n, \dot{I}^n, J^{n-1})$.

The additional lemma we need may be now formulated as follows.

Lemma 9.10. *Let the element $\alpha \in \pi_n(X, A)$ is given by the map $f : (I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$, and the element $\beta \in \pi_n(X, A)$ is given by the map $g : (I^n, \dot{I}^n, \mathbf{0}) \to (X, A, x_0)$, such that $g(K) = x_0$. Then the map $h : (I^n, \dot{I}^n, \mathbf{0}) \to (X, A, x_0)$, defined (obviously, well) by the formula*

$$h(\boldsymbol{t}) = \begin{cases} (f \circ \varphi_n)(\boldsymbol{t}), & \text{if } \ \boldsymbol{t} \in K^n, \\ g(\boldsymbol{t}), & \text{if } \ \boldsymbol{t} \notin K^n, \end{cases} \quad \boldsymbol{t} \in I^n. \tag{9.11}$$

will set the element $\alpha + \beta \in \pi_n(X, A)$.

Proof. It is clear that the pairs $(\dot{I}^n, K^{n-1} \cup j^{n-1})$ and $(I^n, K^n \cup \dot{I}^n)$ are cofibrations. A standard reasoning therefore shows that there is a homotopy

$$g_t : (I^n, \dot{I}^n, \mathbf{0}) \to (X, A, x_0)$$

such that

$$g_0 = g, \quad g_1(J^{n-1}) = x_0, \quad g_1(K^n) = x_0$$

for any $t \in I$. Replacing the map g in formula (9.11) with the map g_t, we obviously get a homotopy $h_t : (I^n, \dot{I}^n, \mathbf{0}) \to (X, A, x_0)$, connecting the map $h = h_0$ with the map h_1, constructed by the map g_1. Therefore, without loss of generality, we can assume in Lemma 9.10 that the map g has the additional property that $g(J^{n-1}) = x_0$, i.e., it is., like the map f, a map of the standard form $(I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$ (and, of course, still satisfies the condition $g(K^n) = x_0$, which ensures the correctness of formula (9.11)). Moreover, for similar reasons, we can even require that $g(\boldsymbol{t}) = x_0$ for $0 \leq t1 \leq 1/2$, i.e. that the map g be the sum const $+g'$, a constant map const and a map $g' : (I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$ defined by the formula

$$g'(\boldsymbol{t}) = g\left(\frac{1 + t_1}{2}, t_2, \ldots, t_n\right), \quad \boldsymbol{t} = (t_1, t_2, \ldots, t_n) \in I^n.$$

But then the map h will obviously be the sum $f' + g'$ where $f' : (I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$ is defined by the formula

$$f'(\boldsymbol{t}) = \begin{cases} (f \circ \varphi_n)(\frac{t_1}{2}, t_2, \ldots, t_n), & \text{if} \quad (\frac{t_1}{2}, t_2, \ldots, t_n) \in K^n, \\ x_0, & \text{if} \quad (\frac{t_1}{2}, t_2, \ldots, t_n) \notin K^n, \end{cases}$$

and g' as above. Since $\beta = [g]^\bullet = [\text{const} + g'] = [g']^\bullet$, it follows that to prove Lemma 9.10 it is sufficient to prove that $\alpha = [f']^\bullet$ i.e. that there is a homotopy $f_t : (I^n, \dot{I}^n, J^{n-1}) \to (X, A, x_0)$, connecting the map f to the map f'. Since such a homotopy can be defined, for example, by formulae

$$f'(\boldsymbol{t}) = \begin{cases} (f \circ \varphi_n)(\frac{t_1}{2}, t_2, \ldots, t_n), & \text{if} \quad (\frac{t_1}{2}, t_2, \ldots, t_n) \in K^n, \\ x_0, & \text{if} \quad (\frac{t_1}{2}, t_2, \ldots, t_n) \notin K^n, \end{cases}$$

$$f_t(\boldsymbol{t}) = \begin{cases} f(\frac{t+2t_1+1}{1+t}, \frac{t+4t_2-1}{1+3t}, \ldots, \frac{t+4tn-1-1}{1+3t}, \frac{4t_n}{1+3t}), \\ \quad \text{if} \quad \frac{1-t}{2} \le t_1 \le 1, \frac{1-t}{4} \le t_2 \le \frac{1+t}{2}, \ldots, \frac{1-t}{4} \le t_{n-1} \le \frac{1+t}{2}, 0 \le t_n \le \frac{1+3t}{4}, \\ x_0, \quad \text{otherwise}, \end{cases}$$

Lemma 9.10 is thus fully proved. □

We will say that the map h is obtained from the map g *by pasting the map* f.

9.3 The main lemma

Now we can prove the fundamental lemma Dold and Thom, revealing the internal "homotopy" springs of the validity of the axiom **CHE** for cube maps. This lemma specifies the conditions on the mapping $p : E \to B$ and the subspace $A \subset B$, under which for any map $g : (I^{n+1}, I^n) \to (B, A)$, any homotopy $h_t : (J^n, \dot{I}^n) \to (B, A)$, having the property that $h_0 = g|_{J^n}$, and any map $\overline{h}_1 : (J^n, \dot{I}^n) \to (E, F_A)$, $F_A = p^{-1}A$, covering the map h_1 (i.e. such that $\overline{h}_1 = p \circ h_1$), there is a homotopy $g_t : (I^{n+1}, I^n) \to (B, A)$ and a map $\overline{g} : (I^{n+1}, I^n) \to (E, F_A)$ such that

$$\overline{g}|_{J^n} = \overline{h}, \quad {}_0 = g, \quad g_1 = p \circ \overline{g}$$

and$g_t|_{J^n} = h_t$ for any $t \in I$.

Lemma 9.12. *If for each point $e_0 \in F_A$ and an induced map p, the homomorphism*

$$p_n : \pi_n(E, F_A, e_0) \to \pi_n(B, b_0), \quad b_0 = p(e_0),$$

is a monomorphism, and the homomorphism

$$p_{n+1} : \pi_{n+1}(E, F_A, e_0) \to \pi_{n+1}(B, b_0)$$

is an epimorphism, then for any g, h_t and $\overline{h}_1$ there are g_t and $\overline{g}$.

Before proving this lemma, we note that the pair $(J^n, \dot{I}^n)$ is obviously homeomorphic to the pair $(I^n, \dot{I}^n)$. (Indeed, for any homeomorphism $\varphi : (I^n, \dot{I}^n) \to (\mathbb{E}^n, \mathbb{S}^{n-1})$ the formula

$$\psi(\boldsymbol{t}, t) = \begin{cases} \frac{1}{2}\varphi(\boldsymbol{t}), & \text{if} \quad t = 1, \\ \frac{2-t}{2}\varphi(\boldsymbol{t}), & \text{if} \quad \boldsymbol{t} \in \dot{I}^n, \end{cases}$$

will determine the homeomorphism of the pair $(J^n, \dot{I}^n) = ((\dot{I}^n \times I) \cup (I^n \times 1), \dot{I}^n \times 0)$ with the pair $(J^n, \dot{I}^n)$. So the map $\psi^{-1} \circ \varphi$ will be a homeomorphism $(I^n, \dot{I}^n) \to (\mathbb{E}^n, \mathbb{S}^{n-1})$. Therefore, if we fix the homeomorphism $(I^n, \dot{I}^n) \to (J^n, \dot{I}^n)$, then for any pair (X, A) each pointed map $f : (J^n, \dot{I}^n) \to (X, A)$ will define some element of the group $\pi_n(X, A)$. Moreover, homeomorphism $(I^n, \dot{I}^n) \to (J^n, \dot{I}^n)$ we can consider (why?) a restriction of some homeomorphism $(I^{n+1}, J^n) \to (I^{n+1}, I^n)$, from which it follows that each map $g : (I^{n+1}, I^n) \to (X, A)$ we can consider as a homotopy from I^n to X, the initial map of which is the map $g|_{J^n}$(considered as a map $(I^n, \dot{I}^n) \to (X, A)$). Since, when identifying $I^{n+1} = I^n \times I$, the subset J^n of the cube I^{n+1} is identified with the subset $(\dot{I}^n \times I) \cup (I^n \times 1)$ of the product $I^n \times I$, and by the condition $g(I^n) \subset A$, this homotopy will be the homotopy of the maps of pairs $(I^n, \dot{I}^n) \to (X, A)$, and its end image will be a map drawn in A. Therefore, its initial map will set the zero element of the group $\pi_n(X, A)$. This proves that

Proposition 9.13. *the map $f : (J^n, \dot{I}^n) \to (X, A)$ sets the null element of the group $\pi_n(X, A)$ if and only if when there is a map $g : (I^{n+1}, I^n) \to (X, A)$ such that $f = g|_{J^n}$*

Proof. (of Lemma 9.12) Let α be an element of the $\pi_n(E, F_A)$, defined by the map $\overline{h_1} : (J^n, \dot{I}^n) \to (E, F_A)$ (or, more precisely, the map $\overline{h}_1 : (J^n, \dot{I}^n, \mathbf{0}) \to (E, F_A, e_0)$, where $e_0 = \overline{h}_1(\mathbf{0})$. Then the element $p_n(\alpha) \in \pi_n(B, A)$ will be set by the map $h_1 : (J^n, \dot{I}^n) \to (B, A)$, and therefore by the homotopy map $h_0 : (J^n, \dot{I}^n) \to (B, A)$. But by the condition $h_0 = g|_{J^n}$, where $g : (I^{n+1}, I^n) \to (B, A)$, from which, according to the remark just made, it follows that the map h_0 specifies the zero element of the group $\pi_n(X, A)$. Thus, $p_n(\alpha) = 0$, and therefore, due to the assumed injectivity of the homomorphism p_n, equality $\alpha = 0$ takes place in the group $\pi_n(E, F_A)$.

Therefore, there is a map $g' : (I^{n+1}, I^n) \to (E, F_A)$ such that $g'|_{J^n} = \overline{h}_1$ It is clear that the formula

$$G(\boldsymbol{t}, t_{n+1}, t_{n+2}) = \begin{cases} g(\boldsymbol{t}, t_{n+2}), & \text{if} \quad t_{n+1} = 0, \\ h_{t_{n+1}}(\boldsymbol{t}, t_{n+2}), & \text{if} \quad \boldsymbol{t} \in \dot{I} \text{ or } t_{n+1} = 1, \\ (p \circ g')(\boldsymbol{t}, t_{n+2}), & \text{if} \quad t_{n+1} = 1, \end{cases}$$

where $\boldsymbol{t} \in I^n$, and $t_{n+1}, t_{n+2} \in I$ well defines some map $G : (J^{n+1}, \dot{I}^{n+1}) \to (B, A)$. Let β be the corresponding element of the group $\pi_{n+1}(B, A)$. Since the homomorphism p_{n+1} is by condition an epimorphism, there is a map $f : (I^{n+1}, \dot{I}^n, J^{n+1}) \to (E, F_A, e_0)$, which has the property that the map $p \circ f$ specifies the element β of the group $\pi_{n+1}(B, A)$.

Let $\overline{K}^{n+1}$ be the image of a subcube K^{n+1} of the cube I^{n+1} under the homeomorphism $\chi : (I^{n+1}, \dot{I}^{n+1}) \to (J^{n+1}, \dot{I}^{n+1})$, by which we interpret the map G as the map $(I^{n+1}, \dot{I}^{n+1}) \to (B, A)$. It is clear that by choosing the homeomorphism χ accordingly, we can assume without loss of generality that $\overline{K}^{n+1}$ is contained in the face $t^{n+1} = 1$ of the cube I^{n+2}, i.e. after identifying this face with the cube I^{n+1}, in the cube I^{n+1} (and does not intersect with J^n). In addition, it can be assumed that the map g' translates $\overline{K}^{n+1}$ to the point e_0. Then the map G, considered as a map $(I^{n+1}, \dot{I}^{n+1}) \to (B, A)$, will translate the cube K^{n+1} to the point e_0, and therefore will allow pasting the map f. The corresponding map H will, according to Lemma 9.10, set the element $-\beta + \beta = 0$ of the group $\pi_n(B, A)$, and therefore there will be a map $H' : (I^{n+2}, I^{n+1}) \to (B, A)$, coinciding on J^{n+1} with the transformation H. By applying the identification $I^{n+2} = I^{n+1} \times I$ we can consider this map as a homotopy $g_t : I^{n+1} \to B$. In this case, the relation $H'|_{J^{n+1}} = H$ will be equivalent to the relations $g_0 = g$, $g_t|_{J^n} = h_t$, $t \in I$, and $g_1 = p \circ \overline{g}$, where $\overline{g}$ is the map $(I^{n+1}, I^n) \to (E, F_A)$, coinciding outside $\overline{K}^{n+1}$ with the map g', and on $\overline{K}^{n+1}$ with the map f, considered by applying the homeomorphism $\overline{K}^{n+1} \to K^{n+1} \xrightarrow{\varphi_{n+1}} I^{n+1}$ as a map $\overline{K}^{n+1} \to E4$.

To complete the proof, it remains to note that since $\overline{K}^{n+1}$ does not intersect with J^n, the map g coincides on J^n with the map $\overline{g}$, and therefore with the map $\overline{h}_1$. □

9.4 Covering homotopies for weak fibrations

To deduce Theorem 9.9 from Lemma 9.12, we will need a general methodological (or, better to say, elementary geometric) lemma concerning cubable sets (see Appendix to Lecture 7).

To shorten the formulations, we will call the open covering $\{U_\alpha, \alpha \in A\}$ of a topological space X an *additively saturated covering*, if for any indices $\alpha, \beta \in A$ the intersection $U_\alpha \cap U_\beta$ is the union of the covering elements $\{U_\alpha\}$.

Lemma 9.14. *Let Q be a compact cubable set, X be a topological space, $\{U\alpha, \alpha \in A\}$ be an additively saturated open covering of the space X, and $F : Q \to X$ be a continuous map. There is a cubilage to the set Q and a map $\varphi : K \to \Lambda$ such that:*

(i) for any cube $\sigma \in K$ there is an inclusion

$$F(\sigma) \subset U_{\varphi(\sigma)};$$

(ii) if the cube $\sigma \in K$ is a face of the cube $\tau \in K$, then $U_{\varphi(\sigma)} \subset U_{\varphi(\tau)}$.

(iii) In addition, given finite family $\{C_\beta, \beta \in B \subset A\}$ of closed sets $C_\beta \in Q$, such that $F(C_\beta) \in U_\beta$ for any $\beta \in B$, then the map f can be selected so that when $\sigma \cap C_\beta \neq \varnothing$, the inclusion

$$U_{\varphi(\sigma)} \subset U_\beta, \quad \sigma \in K, \ \beta \in B.$$

takes place.

Proof. Let's call the cubilage K of a set Q *satisfactory with dimension* $n \geq 0$ if on the set K^n of all its cubes of dimension greater than or equal to n a map $\varphi : K^n \to \Lambda$ is given such that the conditions (i), (ii) and (iii) are fulfilled for all cubes $\sigma \in K^n$. The statement of the lemma means that there is a satisfactory cubic with dimension 0. Since for n, greater than the dimension of the set Q, any cubilage of this set is obviously satisfactory from dimension n, to prove the lemma, it is therefore necessary to prove that the existence of a cubilage satisfying from dimension $n+1$ implies the existence of a cubilage satisfying from dimension n.

With this in mind, consider an arbitrary cubilage K of the set Q, satisfactory with dimension $n+1$. Let σ be an arbitrary n-dimensional cube of this cubicle, and let $x \in \sigma$. Due to the conditions imposed on the covering$\{U_\alpha\}$, there is an element $U_{\alpha(x)}$ of this covering containing the point $f(x) \in X$ such that $U_{\alpha(x)} \subset U_{\varphi(\tau)}$ for any $(n+1)$-dimensional cube $\tau \in K$ having the cube σ as its edge and $U_{\alpha(x)} \subset U_\beta$ for any index $\beta \in B$ that has the property that $x \in C_\beta$. By applying the continuity of the map $F : Q \to X$ and the compactness of the sets C_β the point $x \in Q$ has in Q a neighbourhood of $V(x)$ such that $F(V(x)) \subset U_{\alpha(x)}$ and $V(x) \cap C_\beta = \varnothing$ if $x \notin C_\beta$ The neighbourhoods $V(x)$, $x \in \sigma$, make up the open covering of the cube σ, and therefore there is a small cubicle K_σ of this cube such that each cubicle K_σ is contained in at least one neighbourhood of the form $V(x)$. Since any smaller cubicle of a cube also has this property, by applying the finiteness of cubilage K, we can assume that for all n-dimensional cubes $\sigma \in K$ cubes K_σ have the same fineness N.

Consider the cubilage K_1 of the fineness N of the set Q Any of its cube σ_1 of dimension greater than or equal to n is contained in a single cube $\sigma \in K$ of minimum dimension $\geq n$. In the case when $\dim \sigma \geq n+1$ (and therefore the index $\varphi(\sigma)$ is defined), we will put $\varphi_1(\sigma_1) = \varphi(\sigma)$. If $\dim \sigma = n$, then by construction $\sigma_1 \in K_\sigma$, and therefore there is a point $x \in \sigma$, which has the property that $\sigma_1 \subset V(x)$. Arbitrarily choosing such a point x, we will put $\varphi_1(\sigma 1) = \alpha(x)$. A direct check shows that the cubilage K_1 with respect to the so-constructed map $\varphi_1 : K_1^{(n)} \to \Lambda$ is a cubilage satisfactory from dimension n. □

Now we can consider the question of what remains of the axiom **CH** in the case when the map $p : E \to B$ satisfies the conditions of Theorem 9.9.

Lemma 9.15. *Let's say for the map* $p : \to B$ *there is an additively saturated open covering* $\{U_\alpha; \alpha \in A\}$ *of the space* B*, consisting of the distinguished sets, and and for a compact cubable set* P *and a continuous map* $F : P \times I \to B$ *for a finite family* $\{C_\beta, \beta \in B \subset A\}$ *of closed sets* $C_\beta \subset P \times I$ *such that* $F(C_\beta) \subset U_\beta$ *for each* $\beta \in B$ *is given. Then for any commutative diagramme of the form*

$$\begin{array}{ccc} P & \xrightarrow{\overline{f}} & E \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle p} \\ P \times I & \xrightarrow[F]{} & B \end{array}$$

there is a map $\overline{F} : P\times I \to E$ *such that* $\overline{F}\circ\sigma_0 = \overline{f}$ *and a homotopy* $G_t : P\times I \to B$ *such that* $G_0 = F$, $G_1 = p \circ \overline{F}$, $G_t \circ \sigma_0 = F \circ \sigma_0$, $G_t(C_\beta) \subset U_\beta$ *for any* $t \in I$, $\beta \in B$.

Proof. According to Lemma 9.14 (applied to the map F and the curated set $Q = P \times I$), there is a cubilage K of the set $Q = P \times I$ and a map $f : K \to A$ having properties (i), (ii) and (iii). We will strengthen Lemma 9.15 by requiring that for any cube $\sigma \in K$ and any $t \in I$ there is an inclusion $G_t(\sigma) \subset U_{\varphi(\sigma)}$. By applying the property (iii), this will obviously ensure that we have the inclusion $G_t(C_\beta) \subset U_\beta$.

Each cube $\sigma \in K$ has the form $\tau \times \rho$, where τ belongs to some cubilage of the set P, and ρ belongs to some cubilage of the segment I, i.e. it is either one of a finite number of points of the form $k2^{-N}$,$k = 0, 1, \ldots, 2^N$,or one of the segments $I_k = [k2^{-N}, (k + 1)2^{-N}]$, $0 \le k \le 2^N - 1$. Let's focus our attention on cubes of the form $\tau \times I_k$. We will arrange these cubes, assuming that $\tau_1 \times I_{k_1} < \tau_2 \times I_{k_2}$ if $k_1 < k_2$, and for $k_1 = k_2$ if $\dim\tau_1 < \dim\tau_2$ (cubes with $k_1 = k_2$ and $\dim\tau_1 = \dim\tau_2$ are ordered in an arbitrary way). We will build the map F and the homotopy G_t separately on each cube of the form $\tau \times I_k$, assuming that they have already been built on the previous cubes. At each step (including the initial one) of this construction, after identifying the cube $\tau \times I_k$ with the cube $I^n \times I = In + 1$, $n = \dim\tau$, we will deal with the situation of Lemma 9.12, where the role of the space B will be played by the set $U = U_{\varphi(\tau\times I_k)}$, the role of the space E is its preimage $F_U = p^{-1}U$, the role of the map p is the restriction p_U of the map p given to us on this preimage and the role of the subspace A is the set $V = U_{\varphi(\sigma_k)}$, $\sigma_k = \tau \times \{k2^{-N}\}$. The conditions imposed in Lemma 9.12 and a of the homomorphisms p_n and P_{n+1} will be fulfilled due to the distinguished sets U and V (note that if the set $V \subset U \subset B$ is distinguished with respect to the map $p : E \to B$, then it will be distinguished in relation to the map $p_U : F_U \to U$). The condition that the map $\overline{F}$ and the homotopy G_t are constructed for the previous cubes of the cubilage K will give us the map g, h_t and $\overline{k}_1$ and the map $\underline{g_t}$ and $\overline{g}$ will allow you to extend $\overline{F}$ and G_t to the cube $\tau \times I$. Thus, the map $\overline{F}$ and the homotopy G_t will be constructed by induction on all cubes $\tau \times I$, i.e. on the entire set $Q = P \times I$. □

9.5 Proof of the Dold-Thom theorem

Let us now turn directly to the proof of Theorem 9.9.

Proof. (of Theorem 9.9) It is enough to show that the map $p : E \to B$, satisfying the conditions of Theorem 9.9, has the property that for any point $e_0 \in E$ and any point containing $b_0 = p(e_0)$ of an element U_{α_0} of the covering $\{U_\alpha\}$ induced by the map $p : (E, F_{U_{\alpha_0}}, e_0) \to (B, U_{\alpha_0}, b_0)$, the homomorphism

$$p_* : \pi_n(E, F_{U_{\alpha_0}}, e_0) \to \pi_n(B, U_{\alpha_0}, b_0), \quad n > 0,$$

is an isomorphism. Indeed, then in the commutative diagramme

$$\begin{array}{ccccccccccc}
\cdots \to \pi_{n+1}(E, F_{U_{\alpha_0}}) & \to & \pi_n(F_{U_{\alpha_0}}, F_{b_0}) & \to & \pi_n(E, F_{b_0}) & \to & \pi_n(E, F_{U_{\alpha_0}}) & \to & \pi_{n-1}(E, F_{U_{\alpha_0}}) \to \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to \pi_{n+1}(B, F_{b_0}) & \to & \pi_n(U_{\alpha_0}) & \to & \pi_n(B) & \to & \pi_n(B, U_{\alpha_0}) & \to & \pi_{n-1}(U_{\alpha_0}) \to \cdots
\end{array}$$

the horizontal lines of which are homotopy sequences of the triple $(E, F_{U_{\alpha_0}}, F_0)$ and of the pair (B, U_{α_0}) respectively, and vertical homomorphisms are induced by the map p, all vertical homomorphisms, except the central one $\pi_n(E, F_{b_0}) \to \pi_n B$, are isomorphisms. Therefore, by applying the Five Lemma, the last homomorphisms will also be isomorphisms, i.e. the map $p : E \to$B will be a weak fibration.

p_* *is a monomorphism.* Let ξ be an element of the group $\pi_n(E, F_{U_{\alpha_0}}, e_0)$ such that $p_*\xi = 0$. The condition $p_*\xi = 0$ means that to the map $f : (I^n, \dot{I}^n, J^{n-1}) \to (E, F_{U_{\alpha_0}}, e_0)$, of the defining element ξ, there exists a homotopy $f_t : (I^n, \dot{I}^n) \to (B, U_{\alpha_0})$ such that $f_0 = p \circ f$, and $f_1 =$ const. Let's apply Lemma 9.15, taking for P the cube I^n (and, therefore, for Q the cube I^{n+1}), for $\overline{f}$ the map f, and for F the homotopy f_t (interpreted as the map $I^n \times I \to B$). For the family $\{C_\alpha\}$, we will take the family consisting of a single set $C_{\alpha_0} = (\dot{I}^n \times I) \times (I^n \times 1) = J^n$ (so $B = \{\alpha_0\}$). It is clear that all the conditions of Lemma 9.15 are fulfilled, and therefore there is a map $\overline{F} : I^{n+1} \to E$, such that $\overline{F} \circ \sigma_0 = f$, and a homotopy $G_t : I^{n+1} \to B$, such that $G_0 = F$, $G_1 = p \circ F$, $G_t \circ \sigma_0 = f_0$ and $G_t(J^n) \subset U_{\alpha_0}$ for any $t \in I$, where, as always, $\sigma_0 : I^n \to I^{n+1}$ is the inclusion $\boldsymbol{t} \to (\boldsymbol{t}, 0)$. Since $G_1 = p \circ \overline{F}$ and $G_1(J^n) \subset U_{\alpha_0}$, then the map $\overline{F}$, interpreted as a homotopy from I^n to B, is actually a homotopy (free) from $(I^n, \dot{I}^n)$ to (E, U_{α_0}), connecting the map f with the map pulled down to U_{α_0}). Since the existence of such a homotopy implies, as we know (see Lecture 8), equality $\xi = 0$, this proves that the homomorphism p_* is a monomorphism.

p_* *is an epimorphism.* The elements of the group $\pi_n(B, U_{\alpha_0})$ can be considered as homotopy classes of maps $F : (I^n, \dot{I}^n, I^{n-1}) \to (B, U_{\alpha_0}, b_0)$, i.e. maps $F : I^n \to B$ such that $F(J^{n-1}) \subset U_{\alpha_0})$ and $F(I^{n-1}) = b_0$. Each such map F, together with the map $f = \text{const}_{e_0} : I^{n-1} \to E$, satisfies (for $P = I^{n-1}$) the conditions of Lemma 9.15 (with $B = \{\alpha_0\}$ and $C_{\alpha_0} = J^{n-1}$). Therefore, there is a map $\overline{F} : I^n \to E$ such that $\overline{F} \circ \sigma_0 = \text{const}_{e_0}$, (i.e. $\overline{F}(I^{n-1}) = e_0$), and a homotopy $G_t : I^n \to B$ such that $G_0 = F$, $G_1 = p \circ \overline{F}$, $G_t \circ \sigma_0 = \text{const}_{e_0}$ and $G_t(J^{n-1}) \subset U_\alpha$ for any $t \in I$. The conditions that the homotopy G_t satisfies mean that this homotopy is a homotopy of maps $(I^n, \dot{I}^n, I^{n-1}) \to (B, U_{\alpha_0}, b_0)$, connecting the map $F$to the map $p \circ \overline{F}$. Therefore, firstly, $p \circ \overline{F}$ is a map $(I^n, \dot{I}^n, I^{n-1}) \to (B, U_{\alpha_0}, b_0)$ and therefore sets some element of the group $\pi_n(B, U_{\alpha 0})$, and secondly, that element coincide with the element $\xi \in \pi_n(B, U_{\alpha_0})$, set by the map F. Therefore, the map $\overline{F}$ will be a map $(I^n, \dot{I}^n, I^{n-1}) \to (E, F_{U_{\alpha_0}}, e_0)$, and the element $\overline{\xi}$ of the group $\pi_n(E, F_{U_{\alpha_0}})$ will have the property that $p_*(\overline{\xi}) = \xi$.

Thus, Theorem 9.9 is fully proved. □

9.6 James' lemma

We will apply Theorem 9.9 to the proof of a useful lemma first established by James. This lemma involves two pointed spaces X and Y, a reduced cone $C^\bullet X$ over the space X, a pointed map $f : X\times Y \to Y$ and the space $E = (C^\bullet X\times Y)\cup_f Y$, obtained by gluing the space $C^\bullet X\times Y$ to the space Y by the map f (since $X \subset C^\bullet$, then $X \times Y \subset C^\bullet X \times Y$). We will say that the space E is obtained by applying *James' constructions to the map f.*

Let $p : E \to S^\bullet X$ be a map defined (obviously well) by formulae

$$p([x,,t]]^C, y) = [xx,t]^S, \quad p(y) = x_0, \quad x \in X,\ y \in Y,\ t \in I.$$

Proposition 9.16 (James' lemma)**.** *If the base point $x_0 \in X$ has a contractible* $\operatorname{rel} x_0$ *neighbourhood U_0 and for any point $x \in X$, the map $y \mapsto f(x,y)$, $y \in Y$, from Y to Y is a homotopy equivalence, then the map p is a weak fibration.*

Let's first prove the following lemma.

Lemma 9.17. *Under the conditions of Proposition 9.16, for the base X of the cone $C^\bullet X$, there exists a neighbourhood U, of which it is a strong deformation retraction.*

Proof. Let U be a subset (obviously open) of the cone $C^\bullet X$, consisting of points $[x,t]$ that either $x \in U_0$ or $t \notin [1/3, 2/3]$. Clearly the formula

$$g_\tau[x,t] = [x, \rho(t,\tau)], \quad [x,t] \in U,$$

where

$$\rho(t,\tau) = \begin{cases} 0, & \text{if} \quad 0 \le t \le \frac{\tau}{3}, \\ \frac{3t-\tau}{3-2\tau} & \text{if} \quad \frac{\tau}{3} \le t \le \frac{3-\tau}{3}, \\ 1, & \text{if} \quad \frac{3-\tau}{3} \le t \le 1, \end{cases}$$

defines a homotopy fixed on X that connects the identity map $g_0 = \mathrm{id} : U \to U$ with the map $g_1 : U \to U$, which is a map to the union of $X \cup C^\bullet U_0$, where $C^\bullet U_0$ is the cone contained in $C^\bullet X$ over the neighbourhood U_0. On the other hand, since the neighbourhood U_0 is contractible $\operatorname{rel} x_0$, it is a strong deformation retract of the cone $C^\bullet U_0$, i.e. there is a homotopy $h_\tau : C^\bullet U_0 \to C^\bullet U_0$ fixed on U_0 such that $h_0 = \mathrm{id}$ and $h_1(C^\bullet U_0) = U_0$. Defining this homotopy on X, assuming that $h_\tau = \mathrm{id}$ on X, and putting

$$f_\tau = \begin{cases} g_{2\tau}, & \text{if} \quad 0 \le \tau \le 1/2, \\ h_{2\tau-1} \circ g_1, & \text{if} 1/2 \le \tau \le 1, \end{cases}$$

obviously, we will well define a homotopy fixed on X connecting the identity map $U \to U$ with the map $i \circ r$, where $i : X \to U$ is an inclusion, $r : U \to X$ is some retraction (representing the map $h_1 \circ g_1$, considered as a map in X). Therefore, $U \searrow\!\!\searrow X$. □

Let V be the image of the neighbourhood U of the factorisation map $C^{\bullet}X \to S^{\bullet}X$. Since considered as a map $(C^{\bullet}X, X) \to (S^{\bullet}X, x_0)$, this map is a relative homeomorphism, the retraction $r : U \to X$ defines a strong deformation retraction $V \to x_0$. Thus,

Proposition 9.18. *the neighbourhood Vis contractible* $\operatorname{rel} x_0$.

Consider the set $F_V = p^{-1}V$.

Lemma 9.19. *For any point $v \in V$ the set $F_v = p^{-1}(v)$ is a deformation retract of the set F_V.*

Proof. By applying the standard identifications $F_V = ((U \setminus X) \times Y) \cup Y$, and

$$p(u, y) = u^S, \quad p(y) = x_0,$$

for any point $u \in U \setminus X$ and $y \in Y$, where u^S is the image of the point u in V (i.e. $u^S = [x, t]^s$, if $u = [x, t]^C$), and x_0 is the base point of the suspension $S^{\bullet}X$. At the same time, the homotopy $f_\tau : U \to U$ constructed above obviously defines a homotopy $F_V \to F_V$ fixed on Y, connecting the identity map with the map $\bar{i} \circ \bar{g}$, where $\bar{i} : Y \to F_V$ is an inclusion, and $\bar{f}$ is a retraction $F_V \to Y$, defined (obviously well) by the formulae

$$\bar{r}(u, y) = f(ru, y), \quad \bar{r}(y) = y, \quad u \in U \setminus X, \ y \in Y.$$

Since $F_{x_0} = Y$, this proves Lemma 9.19 for $v = x_0$.

Let $v \neq x_0$. Then $F_v = u \times Y$ where is $u \in U$ is (obviously, the only) point such that $u^S = v$. The restriction $\bar{r}_v$ of the retraction $\bar{r}$ to F_v differs from the homotopy equivalence $y \mapsto f(x, y)$ only by the homeomorphism $(u, y) \mapsto y$, where $x = r(u)$, and therefore is itself a homotopy equivalence. Let $g : Y \to F_v$ be the inverse of the homotopy equivalence, and $j : F_v \to F_V$ be the inclusion. Then $(g \circ \bar{r}) \circ j = g \circ \bar{r}_v \sim \mathrm{id}$, and $j \circ (g \circ \bar{r}) \sim \bar{i} \circ \bar{r} \circ j \circ g \circ \bar{r} = \bar{i} \circ \bar{r}_V \circ g \circ \bar{r} \sim \bar{i} \circ \bar{r} \sim \mathrm{id}$, and hence the map j is a homotopy equivalence. Since the pair (F_V, F_v) is obviously a closed cofibration, this is possible only if $F_V \searrow\!\!\searrow F_v$ (Corollary 2.30 of Proposition 2.29 of Lecture 2). □

Proof. (of Proposition 9.16) Let W be the complement in $S^{\bullet}X$ of the point x_0 (or, equivalently, the complement in $C^{\bullet}X$ of the space X). The set W is open, its preimage $p^{-1}W \subset E$ is the product $W \times Y$, and the map p on $F_W = p^{-1}W$ is the projection $W \times Y \to Y$. The intersection $V \cap W$ has similar properties, of course. This means that the sets $V \cap W$ and W are open and distinguished. Therefore, by applying Theorem 9.9, to prove Proposition 9.16, it is sufficient to prove that the set V is distinguished, i.e. that for any point $v \in V$ and any $n \geq 0$ the map p induces an isomorphism p_* of the group $\pi_n(F_V, F_v)$ and the group $p_n(V, v)$. But due to the contractibility of the neighbourhood V and the deformation retractibility of the set F_V onto the fibre F_v just proved (see Lemma 9.19), both these groups are null. Consequently, the map p_* is automatically an isomorphism. □

James found a remarkable application of Proposition 9.16 to the theory of homotopy groups. We will deal with it in the next lecture.

Appendix

For fibrations, Theorem 1.71 from the Appendix to Lecture 1 is valid, and for weak fibrations, theorem 9.9 of Lecture 9 is valid. The question is natural: does a similar theorem hold for homotopy fibrations?

We will show that the answer to this question is yes, moreover, that in this respect homotopy fibrations behave exactly the same as strong fibrations.

Preliminarily, we will need to prove another characteristic property of homotopy fibrations.

9.A The axiom delayed covering homotopy

We will call a homotopy $F : X \times B$ *delayed* if there is a number $t_0 > 0$ such that for $0 \le t \le t_0$ for any point $x \in X$ the equality $F(x,t) = F(x,0)$ is valid.

Definition 9.20. It is said that the map $p : E \to B$ satisfies the axiom *delayed covering homotopy* (in short, the axiom **CDH**) if for any diagramme of the form

$$\begin{array}{ccc} X & \xrightarrow{\overline{f}} & E \\ \sigma_0 \big\downarrow & \overset{\overline{F}}{\nearrow} & \big\downarrow p \\ X \times I & \xrightarrow[F]{} & B \end{array} \tag{9.21}$$

in which the homotopy F is a delayed homotopy, there is a covering homotopy $\overline{F}$.

The following somewhat unexpected proposition is valid, which illuminates the concept of homotopy fibration in a new way.

Proposition 9.22. *A map $p : E \to B$ is a homotopy fibration if and only if when it satisfies the axiom delayed covering homotopy.*

Proof. In the place of Diagramme (9.21), the diagramme, in which the homotopy F is replaced by the homotopy $F' : X \times I \to B$, defined by the formula

$$F'(x,t) = F(x,(1-t_0)t + t_0), \quad x \in X,\ 0 \le t \le 1.$$

is also commutative. Therefore, in the case when the map $p : E \to B$ is a homotopy fibration, for Diagramme (9.21) modified accordingly, there is a covering homotopy $\overline{F}' : X \times I \to E$, which has the property that $\overline{F}' \circ \sigma_0 \sim_B \overline{f}$. Denoting by $G : X \times I \to E$ the homotopy over B connecting the map $\overline{f}$ with the map $\overline{F}' \circ \sigma_0$, we put

$$\overline{F}(x,t) = \begin{cases} G\left(x, \frac{t}{t_0}\right), & \text{if} \quad 0 \le t \le t_0, \\ \overline{F}'\left(x, \frac{t-t_0}{1-t_0}\right), & \text{if} \quad t_0 \le t \le 1, \end{cases} \quad x \in X, \ 0 \le t \le 1.$$

An automatic check shows that in this way we obtain a homotopy $\overline{F} : X \times I \to E$, which closes Diagramme (9.21). Thus, the homotopy fibration $p : E \to B$ satisfies the axiom **CDH**.

Conversely, assuming that the map $p : E \to B$ satisfies the axiom **CDH**, consider an arbitrary Diagramme (9.21) (with, generally speaking, non-delayed homotopy F). It is clear that the formula

$$F^*(x,t) = \begin{cases} F(x,0), & \text{if} \quad 0 \le t \le 1/2, \\ F(x, 2t-1), & \text{if} \quad 1/2 \le t \le 1, \end{cases} \quad x \in X, \ 0 \le t \le 1.$$

defines a delayed homotopy $F^* : X \times I \to B$ (with $t_0 = 1/2$), for which the commutative diagram (9.21) also takes place (with F replaced by F^*). Therefore, for the homotopy F^*, there is a covering homotopy $\overline{F}^* : X \times I \to E$. By putting

$$\begin{aligned} G(x,t) &= \overline{F}^*\left(x, \frac{1}{2}\right), \\ \overline{F}(x,t) &= \overline{F}^*\left(x, \frac{1+t}{2}\right), \end{aligned} \quad x \in X, \ 0 \le t \le 1, \tag{9.23}$$

obviously, we will get homotopies $G : X \times I \to E$ and $\overline{F} : Xx \times I \to E$ such that G is a homotopy over B connecting the map $\overline{f}$ with the map $\overline{F} \circ \sigma_0$, and the homotopy $\overline{F}$ covers the homotopy F. Therefore, the map $p : E \to B$ is a homotopy fibration. □

Let $\operatorname{Cocyl}(F, \overline{f})$ be a subspace of the cocylinder $\operatorname{Cocyl} \overline{f}$ consisting of points (x, ud), $x \in X$, $u : I \to E$, such that

$$(p \circ u)(t) = F(x,t)$$

for any point $(x,t) \in X \times I$ (and, of course, $u(0) = \overline{f}(x)$).

Corollary 9.24. *The map $p : E \to B$ is a homotopy fibration if and only if when, for any diagram (diag:09-A1) with delayed homotopy, the projection F*

$$\operatorname{Cocyl}(F, \overline{f}) \to X$$

has a cross section.

9.B The axiom hyper-weak covering homotopy extension

Similarly, the axiom **WCHE** can be transferred to the case of homotopy fibrations.

Definition 9.25. Let for the diagramme

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\overline{f}} & E \\ {\scriptstyle \widetilde{\sigma}_0}\downarrow & \overset{\overline{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ X\times I & \xrightarrow[F]{} & B \end{array} \tag{9.26}$$

where $A \subset X$, $\overline{A} = X \times 0 \cup A \times I$ and σ_0 is an inclusion, there exists a map $\overline{F}' : (\widetilde{U\times I})_{t_0} \to E$, closing the diagramme

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\overline{f}} & E \\ {\scriptstyle \widetilde{\sigma}_0}\downarrow & \overset{\overline{F}'}{\nearrow} & \downarrow{\scriptstyle p} \\ (\widetilde{U\times I})_{t_0} & \xrightarrow[F']{} & B \end{array} \tag{9.27}$$

in which $t_0 > O$, U is some closed functional neighbourhood (cf. 2.25) of A in X, $(\widetilde{U\times I})_{t_0} = (X \times [0, t_0]) \cup (U \times I)$ and F' is the restriction of the map F on $(\widetilde{U\times I})_{t_0}$. Then, if there is a homotopy $\overline{F} : X \times I \to E$, closing the diagram (9.26), then it is said that the map $p : E \to B$ satisfies the axiom *hyper-weak covering homotopy extension* (in short, the axiom **HWCHE**).

Proposition 9.28. *A map $p : E \to B$ is a homotopy fibration if and only if when it satisfies the axiom hyper-weak covering homotopy extension.*

Proof. Let Diagramme (9.26) be given for a homotopy fibration $p : E \to B$, for which there exists Diagramme (9.27) with a closing map F'.

It is clear that without loss of generality, one can put $t_0 = 1/2$. Let φ denote the function $X \to I$ which has the property that $\varphi = 1$ on A and $\varphi = 0$ outside U, and we consider - obviously, a commutative diagramme

$$\begin{array}{ccc} X\times[0,1/2] & \xrightarrow{\overline{g}} & E \\ {\scriptstyle \sigma_0}\downarrow & \overset{\overline{G}}{\nearrow} & \downarrow{\scriptstyle p} \\ X\times[0,1/2]\times I & \xrightarrow[G]{} & B \end{array} \tag{9.29}$$

where the maps $\overline{g}$ and G are defined by the formulae

$$\begin{aligned} \overline{g}(x,\tau) &= \overline{F}'(x, \min(\tau + \varphi(x))), \\ G(x,\tau,t) &= F(x, \min(t + \tau - t\tau + \varphi(x)), 1), \quad 0 \le t \le 1, \end{aligned} \quad x \in X,\ 0 \le \tau \le 1/2$$

(the formula for g makes sense, since $\varphi(x) = 0$ for $x \notin U$). Since the map p is a homotopy fibration, there exists a map $\overline{G} : (X \times [0, 1/2]) \times X \to E$ such that $p \circ \overline{G} = G$ and $\overline{G} \circ \sigma_0 \sim_B \overline{g}$.

Let $H : X \times [0, 1/2] \times E$ be a homotopy over B (i.e. such that the point $(p \circ H)(x, t, \tau)$ does not depend on t) connecting the map $\overline{g}$ with the map $\overline{G} \circ \sigma_0$. Then the formula

$$\overline{F}(x,t)) = \begin{cases} \overline{F}'(x,t), & \text{if} \quad 0 \le t \le \varphi(x), \\ H(x, t-\varphi(x), 2(t-\varphi(x)), & \text{if} \quad 0 \le t-\varphi(x) \le 1/2, \\ \overline{G}\left(x, \frac{1}{2}, 2(t-\varphi(x)-1\right), & \text{if} \quad 1/2 \le t-\varphi(x) \le 1, \end{cases}$$

will (see Fig. 9.B.1) determine the homotopy $F : X \times I \to E$, which closes Diagramme (9.26).

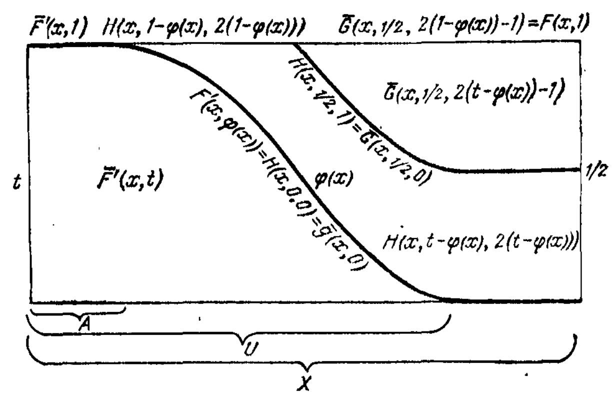


Figure 9.B.1:

This proves that any homotopy bundle satisfies the axiom c.

Conversely, since any diagramme of the form (9.21) is a diagramme (9.26) with $A = \varnothing$ and since - assuming that the homotopy F is delayed - for the corresponding diagramme (9.27) (with $U = \varnothing$), the map F' can be given by the formula

$$\overline{F}'(x,t) = \overline{f}(x), \quad x \in X, \ 0 \le t \le t_0,$$

then any map $p : E \to B$ satisfying the axiom **HWCHE** will satisfy the axiom **CDH** and, therefore, will be a homotopy fibration. □

Remark 9.30. Note that in the last argument we used the axiom **HWCHE** only when $A = \varnothing$.

9.C Dold's theorem for homotopy fibrations

Now we are ready to prove for homotopy fibrations an analogue of Theorem 1.71 from the Appendix to Lecture 1.

Theorem 9.31. *If for the map $p : E \to B$ there exists a enumerable covering $\{U_\alpha, \alpha \in A\}$ of the space B such that each map*

$$p_\alpha = p|_{p^{-1}(U_\alpha)} : p^{-1}(U_\alpha) \to U_\alpha$$

is a homotopy fibration, then the map $p : E \to B$ will also be a homotopy fibration.

Caveat: The proof is quite involved.

Proof. According to Corollary 9.24 of Proposition 9.28, we must prove that for any diagram (9.21) with delayed homotopy F, the projection

$$q : \mathrm{Cocyl}(F, \overline{f}) \to X, \quad (x, u) \mapsto x,$$

has a cross section. To do this, it is enough to prove that for some numbered cover $\{W_\beta$ of the space X, each map

$$q_\beta = q|_{q^{-1}(W_\beta)} : q^{-1}(W_\beta) \to W_\beta$$

is a weak map, because then, by applying Lemma 1.63 from the Appendix to Lecture 1, the map q will also be weak, and therefore there will be a section for it. We prove this by taking $\{W_\beta\}$ the preimage for homotopy F, considered as a map $X \to B^I$ from covering $\{V_\beta; \beta \in B\}$ of the space B^I, corresponding by applying Corollary 1.67 of Lemma 1.64 from the Appendix to Lecture 1 to the covering $\{U_\alpha\}$ of the space B.

According to the identification described in the Appendix to Lecture 1 for any point $x \in W_\beta$ let $F(x) \in V_\beta \subset B^I$ can be considered as a sequence $(u_1^x, \ldots, u_{n_\beta}^x)$ of paths $u_1^x : I \to U_{\alpha_1}, \ldots, u_{n_\beta}^x : I \to U_{n_\beta}^x$ where $\alpha_1, \ldots, \alpha_{n\beta}$ are some indexes of A (depending only on the index $\beta \in B$), and $u_i^x(0) = u_{i-1}^x(1)$ for any $i > 1$. Accordingly, any path $v : I \to E$ for which $p \circ v = F(x)$ can be identified with the sequence $(v_1, \ldots, v_{n_\beta})$ of paths $v_1 : I \to p^{-1}(U_{\alpha_1}), \ldots, v_{n_\beta} : I \to p^{-1}(U_{\alpha_{n_\beta}})$, such that $p \circ v_1 = u_1^x, \ldots, p \circ v_{n_\beta} = u_{n_\beta}^x$ and $v_i(0) = v_{i-1}(1)$ for $i > 1$, and, therefore, points from sequences of the form $(x, v_1, \ldots, v_{n_\beta})$, where $x \in W_\beta$ ad $v_1(0) = \overline{f}(x)$ (note that if $x \in W_\beta$, then $p\overline{f}(x) = F(x, 0) = U_1^x(0) = U_{\alpha_1}$, which means $\overline{f}(x) \in p^1(U_{\alpha_1})$). Therefore, each section $s : W_\beta \to \mathrm{Cocyl}(F, \overline{f})$ of the map q over the subspace W_β will be given by n_β continuous maps

$$s_i : W_\beta \to p^{-1}(U_{\alpha_i})^I, \quad i = 1, \ldots, n_\beta,$$

satisfying the relations for any point $x \in W_\beta$

$$\begin{aligned} &s_1(x)(0) = \overline{f}(x), \\ &s_i(x)(0) = s_{i-1}(x))(1) \quad \text{for} \quad i > 1, \\ &p \circ s_i(x) = u_i^x \quad \text{for any} \quad i = 1, \ldots, n_\beta. \end{aligned} \tag{9.32}$$

However, it is convenient here to consider the maps s_i as homotopies $u_\beta \times I \to p^{-1}(U_{\alpha_i})$ and, accordingly, the maps $F_i : x \mapsto u_i^x$ as homotopies $U_\beta \times I \to U_{\alpha_i}$. Then the relations (9.32) will be equivalent to commutativity for each $i = 1, \dots, n_\beta$ of the diagramme

$$\begin{array}{ccc} W_\beta & \xrightarrow{s_{i-1}\circ\sigma_i} & p^{-1}(U_{\alpha_i}) \\ {\scriptstyle\sigma_0}\big\downarrow & \overset{s_i}{\nearrow} & \big\downarrow{\scriptstyle p_i} \\ W_\beta \times I & \xrightarrow[F_i]{} & U_{\alpha_i} \end{array} \qquad p_i = p_{\alpha_i}, \tag{9.33}$$

in which, for $t = 1$, the map $s_{i-1} \circ \sigma_1$ should be understood as the map $\overline{f}$.

If the section s is set only on the subspace $A \subset W_\beta$ then, of course, there will be commutative diagrammes obtained from Diagrammes (9.33) by replacing W_β with A.

We see, therefore, that in order to prove the weakness of the map q_β, we need to prove for an arbitrary subspace $A \subset W_\beta$ that from the existence for some of its functional neighbourhood U (in W_β) of homotopies $\overline{s}_i : U \times I \to p^{-1}(U_{\alpha_i},$ $i = 1, \dots, n_\beta$ such that commutative diagrammes

$$\begin{array}{ccc} U & \xrightarrow{\overline{s}_{i-1}\circ\sigma_i} & p^{-1}(U_{\alpha_i}) \\ {\scriptstyle\sigma_0}\big\downarrow & \overset{\overline{s}_i}{\nearrow} & \big\downarrow{\scriptstyle p_i} \\ U \times I & \xrightarrow[F_i]{} & U_{\alpha_i} \end{array} \tag{9.34}$$

take place take place where, for $i = 1, \overline{s}_{i-1} \circ \sigma_1$ means the map $\overline{f}|_U$ implies the existence of homotopies $s_i : W_\beta \times I \to p^{-1}(U_{\alpha_i})$, $i =, \dots, n_\beta$, such that there are commutative diagrammes (9.33) and

$$s_i|_A = \overline{s}_i|_A \quad \text{for any} \qquad i = 1, \dots, n_\beta.$$

To this end, we denote by φ a function $W_\beta \to I$ having the property that $\varphi = 0$ on A and $\varphi = 1$ outside U, and we will introduce into consideration the set

$$U_i = \varphi^{-1}\left(\left[0, 1 - \frac{i}{n_\beta}\right]\right), \quad i = 1, \dots, n_\beta.$$

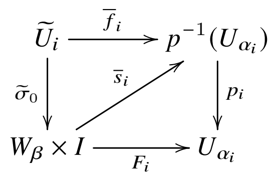
It is clear that

$$A \subset U_{n_\beta} \subset \cdots \subset U_{i+1} \subset U_i \subset \cdots \subset U_1 \subset U$$

and the set U_i for any $i < n_\beta$ is a functional neighbourhood of the set U_{i+1}, and the set U is a functional neighbourhood of the set and U_1. We will construct homotopies s_i by induction on i, additionally requiring that for each $i = 1, \dots, n_\beta$ the following equality holds

$$s_i|_{U_i \times I} = \overline{s}_i|_{U_i \times i}$$

that is, so that the commutative Diagramme 9.35 takes place



(9.35)

where $\widetilde{U}_i = (W_\beta \times 0) \cup (U_i \times I)$, $\widetilde{\sigma}_0$ is an inclusion, and $\overline{f}_i$, are maps defined by the formulae

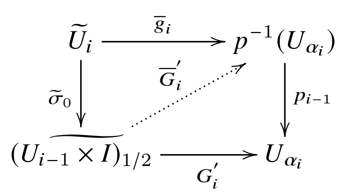
$$\overline{f}_i(x,t) = \begin{cases} s_{i-1}(x,1), & \text{if} \quad t = 0, \\ \overline{s}_i(x,t) & \text{if} \quad x \in U_i, \end{cases}$$

(for $i = 1$ instead of $s_{i-1}(x,1)$ it is necessary, of course, to write $\overline{f}(x)$).

If the map p_i were fibrations, then the homotopies s_i would immediately be built on the basis of the axiom **WCHE** (see similar arguments in Appendix to Lecture 1), which, by the way, would give us a new proof of Theorem 1.71 from the Appendix to Lecture 1. However, in the current situation, we can only use the axiom of the **HWCHE**, and therefore our design should be somewhat thinner.

First of all, let us note that by doubling, if necessary, the number n_β, we can assume that for each index $i > 1$ and any point $x \in W_\beta$ there is not only the inclusion $u_i^x \subset U_{\alpha_i}$, but also the inclusion $u_i^x \subset U_{\alpha_{i-1}}$. In other words, without loss of generality, we can also consider each homotopy F_i to be a homotopy from W_β to $U_{\alpha_{i-1}}$ (and hence the maps $\overline{s}_i$, s_i, and $\overline{f}_i$ from Diagrammes (9.34) and (9.35) as maps in $p^{-1}(U_{\alpha_{i-1}})$).

With this in mind and assuming that for some $i > 1$ the homotopy of $si - 1$ has already been built, let's consider a commutative diagramme



in which

$$\widetilde{U}_i = (W_\beta \times 0) \cup (U_i \times I), \quad (\widetilde{U_{i-1} \times I})_{1/2} = (W_\beta \times [0,1/2]) \cup (U_i \times I),$$

and $\overline{g}_i$ is a map defined by the formula

$$\overline{g}_i(x,t) = \begin{cases} \overline{f}_i(x,2t), & \text{if} \quad 0 \le t \le 1/2, \\ \overline{f}_{i+1}(x,2t-1) & \text{if} \quad 1/2 \le t \le 1, \end{cases} \quad (x,t) \in \widetilde{U}i$$

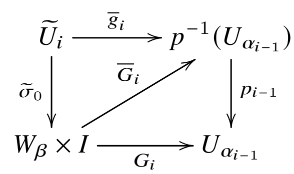
and G_i' is the restriction on $(\widetilde{U_{i-1} \times I})_{1/2}$ of the homotopy $G_i : W_\beta \times I \longrightarrow U_{\alpha_{i-1}}$ specified by the formula

$$G_i(x,t) = \begin{cases} F_i(x,2t), & \text{if} \quad 0 \le t \le 1/2, \\ F_{i+1}(x, 2t-1) & \text{if} \quad 1/2 \le t \le 1, \end{cases} \qquad (x,t) \in W_\beta \times I.$$

It is directly verified that the map $\overline{G}_i'$ that closes this diagramme can be given by the formula

$$\overline{G}_i'(x,t) = \begin{cases} s_i(x,2t), & \text{if} \quad 0 \le t \le 1/2, \\ \overline{s}_{i+1}(x, 2t-1) & \text{if} \quad x \in U_i. \end{cases}$$

Therefore, according to the axiom of the **HWCHE** (for $X = W_\beta$, $A = U_i$, $U = U_{i-1}$ and $p = p_{i-1}$) for the diagramme



there is a closing homotopy $\overline{G}_i : W_\beta \times I \longrightarrow p^{-1}(U_{\alpha_{i-1}})$ and it is clear that the homotopy

$$s_i(x,t) = \overline{G}_i\left(x_i, \frac{t+1}{2}\right), \qquad (x,t) \in W_\beta \times I,$$

where the point $s_i(x,t)$ is considered as a point of $p^{-1}(U_{\alpha_i}$, closes Diagramme (9.35).

Thus, to complete the proof of Theorem 9.31, it remains to justify only the initial step of induction, i.e., to construct the map s_1.

If for any $t \in I$ there is an equality

$$\overline{s}_1(x,t) = \overline{f}(x) \quad \text{for} \quad \varphi(x) = 1, \tag{9.36}$$

such that the map s_1 can be given by the formula

$$s_1(x,t) = \begin{cases} \overline{s}_1(x,t), & \text{if} \quad x \in U, \\ \overline{f}(x), & \text{if} \quad x \notin U. \end{cases}$$

Therefore, Theorem 9.31 will be proved if we show that the fulfilment of condition (9.36) can always be achieved by appropriately transforming the homotopies $\overline{s}_i$ (without, of course, changing them on $A \times I$). Since we will not use the sets U_{α_i} in this transformation, it is advisable to move from the homotopy $\overline{s}_i$ to the complete homotopy $\overline{F} : U \times I \longrightarrow$ composed of them. The conditions imposed above on homotopy $\overline{s}_i$ (the commutative diagramme (9.34) for the homotopy $\overline{F}$ mean that its initial map $\overline{F} \circ \sigma_0$ is a restriction on U of the map $\overline{f}$ and that this homotopy covers over U the homotopy $F: X \times I \to B$. As for the condition (9.36, for the homotopy $\overline{F}$ it has the form

$$\overline{F}(x,t) = \overline{f}(x) \quad \text{for} \quad \varphi(x) = 1 \quad \text{and} \quad 0 \le t \le 1/n_\beta. \tag{9.37}$$

Thus, from the homotopy $\overline{F}: U \times I \to E$ satisfying the relations $\overline{F} \circ \sigma_0 = \overline{f}|_U$ and $p \circ \overline{F} = F|_{U\times I}$, we must, while maintaining these relations, proceed to a homotopy satisfying, in addition, the condition (9.37).

We will define a new homotopy by matching

$$(x,t) \mapsto \overline{F}(x, \alpha(x,t)), \quad (x,t) \in U \times I,$$

where α is a continuous function $U \times I \to I$ such that:

a) $\alpha(x,0) = 0$ for any point $x \in U$ (this ensures the relation $\overline{F} \circ \sigma_0 = \overline{f}|_U$;

b) if $t_0 > 0$ is the number provided for the delay condition of the homotopy F (i.e. such that $F(x,t) = F(x,0)0$ for $0 \le t \le t_0$; note that this is the first time we use this condition), then the function $\alpha_x : t \mapsto \alpha(x,t)$ maps the segment $[0,t_0]$ to itself, and on the segment $[t_0,1]$ is the identity map (this requirement ensures that the relation $p \circ \overline{F} = F|_{U\times I}$ is preserved);

c) $\alpha(x,t) = t$ for $\varphi(x) = 0$ and any t (ensures the invariance on A);

d) $\alpha(x,t) = 0$ if $\varphi(x) = 1$ and $0 \le t \le 1n_\beta$ (ensures the fulfilment of the condition (9.37)).

Assuming for simplicity that $t_0 = 1/2$, and assuming that $n_\beta > 2$ (both assumptions obviously do not lose generality), for $\varphi(x) \le \frac{n_\beta - 1}{n_\beta}$ we will take the identity map $I \to I$ as the function α_x, and for $\varphi(x) \ge \frac{n_\beta - 1}{n_\beta}$ we will take a function linear on each interval with ends at points 0, $1-\varphi(x)$, $1/n_\beta$, 1/2, 1 and translating these points into points 0, $1-\varphi(x)$, $1-\varphi(x)$, 1/2, 1 (see Fig. 9.C.1).

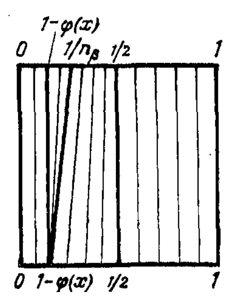


Figure 9.C.1:

It is easy to see that all the conditions a) - d) will be fulfilled at the same time.

Thus, Theorem 9.31 is fully proved. □

Lecture 10

10.1 Suspension homomorphism and suspension sequence

The pointed map $S^\bullet f : S^\bullet A \to S^\bullet X$, obtained by applying the functor $S^\bullet$ to the pointed map $f : A \to X$, is also denoted by the symbol Ef, and its homotopy class $S^\bullet[f]^\bullet = [S^\bullet f]$ is denoted by the symbol $E[f]^\bullet$ (cf. Remark 3.41 of Lecture 3). This notation is especially convenient for $A = \mathbb{S}^n$, when the homotopy classes $[f]^\bullet$ are elements of the group $\pi_n X$. In this case, by applying the identification $S^\bullet \mathbb{S}^n = \mathbb{S}^{n+1}$, we can consider the map Ef to be a map $\mathbb{S}^{n+1} \to S^\bullet X$ and, therefore, the homotopy class $E\alpha$ is an element of the group $\pi_{n+1} S^\bullet X$. The resulting map

$$E : \pi_n X \to \pi_{n+1} S^\bullet X, \quad \alpha \mapsto E\alpha,$$

is called a *suspension map* of homotopy groups.

Remark 10.1. In the literature, the symbol S or its variations (say, S^* or Σ) is also used to indicate the map E.

In the interpretation of the elements of the group $\pi_n X$ as map classes $f : (I^n, \dot{I}^n) \to (X, x_0)$ the map E is given by the correspondence $f \mapsto Ef$, where Ef is the map $(I^{n+1}, \dot{I}^{n+1}) \to (S^\bullet X, x_0)$ defined by the formula

$$(Ef)(t, \boldsymbol{t}) = [f(\boldsymbol{t}), t], \quad t \in I, \ \boldsymbol{t} \in I^n,$$

(the cube I^{n+1} is identified here with the product $I \times I^n$).

An automatic check shows that the formula $i(x) = u_x$, where $u_x(t) = [x, t]$, $t \in I$, $x \in X$, defines a homeomorphic map (embedding) of the space X into the space $\Omega S^\bullet X$ (which is nothing more than an adjoint to the identity map $S^\bullet X \to S^\bullet X$). The homomorphism $i_* : \pi_r X \to i_r \Omega S^\bullet X$ induced by this selection is defined by the formula $i_*[f]^\bullet = [g]^\bullet$, where the map $g : \mathbb{S}^r \to \Omega S^{bullet} X$ maps the point $x \in \mathbb{S}^r$ to the path $t \mapsto [fx, t]$ of the space $\S^\bullet X$. On the other hand, as we know, there is an isomorphism $h : \pi_r \Omega S^\bullet X \to \pi_{r+1} S^\bullet X$, which corresponds to the class $[g]^\bullet$ the class $[\overline{g}]^\bullet$ of the map $\overline{g} : \mathbb{S}^{r+1} = S^\bullet \mathbb{S}^r \to S^\bullet X$, defined by the formula $\overline{g}[x, t] = g(x)(t)$. Therefore, the composite homomorphism $h \circ i_* : \pi_r \to$

$\pi_{r+1}S^\bullet X$ maps to the class $[f]^\bullet$ the class $[\overline{f}]^\bullet$ of the map $\overline{f} : [x,t] \mapsto [fx,t]$, i.e. the map $S^\bullet f$. This proves that

$$E = h \circ i_*.$$

In particular, it follows that the *suspension map E is a homomorphism* (which, of course, is easily and directly verified). In addition, by introducing an exact homotopy sequence of the pair $(\Omega S^\bullet X, X)$ into consideration

$$\cdots \to pi_r X \xrightarrow{i_*} \pi_r \Omega S^\bullet X \xrightarrow{j_*} \pi_r(\Omega S^\bullet X, X) \xrightarrow{\partial} \pi_{r-1} X \to \cdots$$

and replacing the group $\pi_n \Omega S^\bullet X$ in it with the isomorphic group $\pi_{n+1} S^\bullet X$, we get the exact sequence

$$\cdots \to pi_n X \xrightarrow{E} \pi_{n+1} \Omega S^\bullet X \xrightarrow{H} \pi_n(\Omega S^\bullet X, X) \xrightarrow{P} \pi_{n-1} X \to \cdots \tag{10.2}$$

where $H = j_* \circ h^{-1}$ and $P = \partial$. This sequence is called the *suspension sequence* (or *EHP-sequence*[1]) of the space X and serves as a powerful means of studying the homomorphism E.

Of course, for specific calculations, it is necessary to obtain sufficiently complete information about the rather mysterious homomorphisms H and P and, in particular, about the group $\pi_r(\Omega S^\bullet X, X)$.

The first step in this direction is based on the transfer of the notion of a free monoid known from algebra to the category $\mathcal{T}op^\bullet$.

10.2 A universal monoid of a pointed space

Let X be an arbitrary pointed set. We will call each expression of the form

$$x_1 x_2 \cdots x_n, \tag{10.3}$$

a *word* over X. We will call the word *reduced* if none of the points $x_1, \dots, x_n$ is the base point x_0 of the space X. An *empty word* is a (given) word by definition. We will denote the set of all the above words with the symbol JX. It is obviously a monoid with respect to the operation of juxtaposing words each other. The unit of this monoid is the empty word $\varnothing$.

However, it is more convenient to define the monoid JX somewhat in another way, by noting that when all the base points included in an arbitrary word are thrown out, an unambiguously defined reduced word is obtained.

$$X^\infty = \coprod_{n=0}^{\infty} X^n, \quad X^n = \overbrace{X \times \cdots \times X}^{n}$$

in which all words (10.3) with respect to the equivalence relation $u \sim v$, if after throwing out the marked points from the words u and v, the same reduced word is obtained.

[1] This terminology is by G. Whitehead.

(In the language of algebra, this means that JX is obtained from a free monoid over X by superimposing the relation $x_0 = 1$.)

The identification of the set X with the set X^1 obviously includes X in JX. With respect to this embedding, the monoid JX has the universal property, i.e. for any monoid M and any pointed map $\varphi : X \to M$ (meaning that its unit is based in M), there is a unique monoid homomorphism $\Phi : JX \to M$, coinciding on X with the map φ:

$$\begin{array}{ccc} JX & & \\ {\scriptstyle\subset}\uparrow & \overset{\Phi}{\searrow} & \\ X & \xrightarrow[\varphi]{} & M \end{array} \tag{10.4}$$

In other terminology, this means that

Proposition 10.5. *the monoid JX is a free monoid generated by the set X*

(in the category of monoids, whose morphisms are homomorphisms of monoids that translate the unit into the unit).

Now suppose that the set X is a topological (pointed) space. Then the topological space will also be the set X^∞ of all words (as a disjoint union of topological spaces X^n, $n \geq 0$), and hence its coset space JX (with respect to the coset topology). In this case, the space X will obviously be a subspace of the space JX and in the case when in diagramme (10.4) the monoid M is a topological monoid, and the map φ is a continuous map, the map Φ will also be continuous. Nevertheless, it is impossible to assert that JX is a free topological monoid generated by the space X for the simple reason that, generally speaking, - for example, in the case when X is a field of rational numbers, - the multiplication

$$JX \times JX \to JX \tag{10.6}$$

in the monoid JX is not continuous, and therefore this monoid simply will not be a topological monoid. (Here we again encounter the general defect of the category $\mathcal{T}op$, which is already known to us by the example of the exponential law. As in the case of the exponential law, to eliminate it, you need to go either to canonical spaces or to canonical maps.)

We will call the monoid JX the *universal monoid* of the pointed space X.

It is easy to see that for any continuous map $f : X \to Y$, the map $f^\infty : X^\infty \to Y^\infty$, defined by the formula $x_1 \cdots x_n \mapsto y_1 \cdots y_n$ where $y_1 = f(x_1), \dots y_n = f(x_n)$, is continuous and compatible with the factorisation map $X^\infty \to JX$ and $Y^\infty \to JY$. Therefore, this map induces some continuous map

$$Jf : JX \to JY,$$

and it is obvious that the correspondences $X \mapsto JX$, $f \mapsto J_f$ constitute a J functor from the category $\mathcal{T}op^\bullet$ to the category of pointed topological spaces that are simultaneously monoids.

At the same time, it is clear that

Proposition 10.7. *the functor J, considered as a functor $\mathscr{T}op^\bullet \to \mathscr{T}op^\bullet$, is a homotopy functor*

(for any homotopy $f_t : X \to Y$, the maps Jf_t make up a homotopy from JX to JY), and therefore its homotopy is defined by $[\mathscr{T}op^\bullet] \to [\mathscr{T}op^\bullet]$, which we will denote by the same symbol is J.

10.3 Filtration of universal monoids

Let $n \geq 0$, and let J_nX be the set of all given words $a = x_1 \cdots x_k$ where $k \leq n$. This set is the coset of the set X^n, and we will provide it with the appropriate coset topology.

Let $U_1, \dots, U_k$, $k \leq n$, be open sets of the space X that do not contain its base point x_0, and let U_0 be an arbitrary (open) neighbourhood of the point x_0. For any monotone map

$$\lambda : [1, \dots, k] \to [1, \dots, n] \tag{10.8}$$

we will introduce into consideration the open set $V_n^\lambda = V_n^\lambda(U_0; U_1, \dots, U_k)$ of the space X^n defined by the formula

$$V_n^\lambda = U_1^\lambda \times \cdots \times U_n^\lambda,$$

where

$$U_i^\lambda = \begin{cases} U_i, & \text{if} \quad j = \lambda(i), \\ U_0, & \text{if} \quad j \notin \operatorname{im} \lambda, \end{cases} \qquad i = 1, \dots, n.$$

Let $V_n = V_n(U_0; U_1, \dots, U_k)$ be the union of all possible sets V_n^λ and $W_n = W_n(U_0; U_1, \dots, U_k)$ is its image by the factorisation map $X^n \to J_nX$. It is clear that the preimage of the set W_n of the map $X^n \to J_nX$ is just the set V_n. Therefore

Proposition 10.9. *all sets $W_n = W_n(U_0; U_1, \dots, U_k)$ are open in J_nX.*

Now let Y be an arbitrary topological space, and W be a subset of the product $J_nX \times Y$ that its complete preimage V in $X^n \times Y$ of the natural map

$$X^n \times Y \to J_nX \times Y \tag{10.10}$$

is open in $X^n \times Y$. Let's show that

Proposition 10.11. *if a point x_0 is closed in X, then for any point $(a, y) \in J_nX \times Y$ there exists a set $W_n = W_n(U_0; U_1, \dots, U_k)$ and an open set $U \subset Y$ such that*

$$(a, y) \in W_n \times U \subset W. \tag{10.12}$$

Proof. Indeed, let $a = x_1 \cdots x_k$, where $x_1, \ldots, x_k \neq x_0$. For any monotone map (10.8), we define a point $a^\lambda = (x_1^\lambda, \ldots, x_k^\lambda)$ of the space X by the formula

$$x_j^\lambda = \begin{cases} x_i, & \text{if} \quad j = \lambda(i), \\ x_0, & \text{if} \quad j \notin \operatorname{im}\lambda, \end{cases} \qquad j = 1, \ldots, n.$$

It is clear that $(\alpha^\lambda, y) \in V$, and therefore the points $x_1^\lambda, \ldots, x_k^\lambda$ and y have neighbourhoods $V_1^\lambda, \ldots, V_k^\lambda$ and U^λ such that the neighbourhood $V_1^\lambda \times \cdots \times V_k^\lambda \times U^\lambda$ of the point (α^λ, y) is contained in V. We will put $U = \cap_\lambda U_\lambda$ where $\cap_\lambda$ means the intersection across all maps (10.8) and

$$U_0 = \cap_\lambda \cap_{j \in \operatorname{im}\lambda} V_j^\lambda \quad U_i = (X \setminus x_0) \cap \cap_\lambda V_{\lambda(i)}^\lambda, \quad i = 1, \ldots, k.$$

Then it is clear that the set $W_n = W_n(U_0; U_1, \ldots, U_k)$ is defined and $(a, y) \in W_n \times U \subset W$. □

The proven statement entails two simple but important consequences.

First, since all sets of the form $W_n(U_0; U_1, \ldots, U_k)$ are open in $J_n X$, the inclusion (10.12) means that the point (a, λ) is the inner point of the set W. Therefore, due to the arbitrariness of this point, the set W is open. By definition, this means that the map (10.10) is an epimorphism.

In particular, for any $n, m \geq 0$, the epimorphisms will be the maps $X^n \times X^m \to J_n X \times X^m$ and $J_n X \times X^m \to J_n X \times J_m X$, and therefore their component will be an epiomorphism

$$X^n \times X^m \to J_n X \times J_m X.$$

Thus, in the commutative diagrammme

$$\begin{array}{ccc} X^n \times X^m & \longrightarrow & X^{n+m} \\ \downarrow & & \downarrow \\ J_n X \times J_m X & \longrightarrow & J_{n+m} X \end{array}$$

the horizontal arrows of which represent the map of juxtaposition of words, both vertical arrows are epiomorphisms. Since the upper arrow is a continuous map, it follows that the lower arrow is also a continuous map. Thus, it is proved that

Proposition 10.13. *for any $n, m \geq 0$ the map*

$$J_n X \times J_m X \to J_{n+m} X,$$

induced by multiplication in JX, is continuous.

Secondly, for $Y = \mathrm{pt}$, i.e. for the natural epiomorphism $X^n \to J_n X$, we get that for any set W open in $J_n X$ and any of its points $a \in W$ there is a set of the form W_n such that $a \in W_n \subset W$. By definition, this means that

Proposition 10.14. *the sets $W_n = W_n(U_0; U_1, \ldots, U_k)$ form the basis of the topology of the space $J_n X$.*

The analoguue of the set $W_n(U_0; U_1, \ldots, U_k)$ in the space JX are the sets $W(U_0; U_1 \ldots, U_k)$ - images of the factorisation map $X^\infty \to JX$ of the set

$$V = \coprod_{n=k}^{\infty} V_n^\lambda(U_0; U_1 \ldots, U_k), \quad x_0 \in U_0; \quad U_1, \ldots, U_k \in X \setminus x_0,$$

where λ runs through all sorts of monotone maps (10.8). It is clear that the preimage of the set W in X^∞ is just the set of V. Since the set V is obviously open in X^∞, it follows that all sets $W(U_0; U_1, \ldots, U_k)$ are open in JX.

But it's easy to see that

$$J_n \cap W(U_0; U_1, \ldots, U_k) = \begin{cases} W_n(U_0; U_1, \ldots, U_k), & \text{if} \quad n \geq k, \\ \varnothing & \text{if} \quad n < k. \end{cases}$$

Therefore (under the previous assumption of the closure of the point x_0 in the space X), the topology of the space J_nX coincides with the topology induced in J_nX by the topology of the space JX, i.e.

Proposition 10.15. *for any $n \geq 0$ the space J_nX is a subspace of the space JX.*

(In fact, this is true without any conditions for the point x_0, but due to our general attitude to avoid general-topological pathologies, we will not prove this.)

In addition, it is easy to see (under the same - now essential - assumption of the closure of the point x_0) that

Proposition 10.16. *for any $n \geq 0$ the subspace J_nX is closed in the space JX.*

Proof. Indeed, if the point $a \in JX$ does not lie in J_nX, i.e. it has the form $a = x_1 \ldots x_k$, where $k > n$ and $x_1, \ldots, x_k \neq 0$, then its neighbourhood $W(U_0; U_1, \ldots, U_k)$, where $U_0 = X$, and $U_1 = \cdots = U_k = X \setminus x_0$, does not intersect with J_nX. □

Moreover,

Proposition 10.17. *the space JX is a free union (direct limit) of its subspaces J_nX, $n \geq 0$: $JX = \varinjlim_n J_nX$.*

i.e. the set $W \subset JX$ if and only if it is open in JX when for any $n \geq 0$ the intersection $W \cap J_nX$ is open in J_nX.

Proof. Indeed, if W is open in JX, then $W \cap J_nX$ is open in J_nX, because J_nX is a subspace of the space JX. Conversely, let $W \cap J_nX$ be open in JX for any $n \geq 0$, and let V be the complete preimage of the set W of the factorisation map $X^\infty JX$, and W_n is the complete preimage of the set $W \cap J_nX$ of the factorisation map $X^n \to J_nX$. By convention, all sets V_n are open in X^n. But it is clear that $V = \cup_{n=0}^\infty V_n$, and since $X^\infty = \coprod_{n=0}^\infty X^n$, then, consequently, the set V is open in X^∞ Therefore, W is open in JX. □

An increasing sequence

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n \subset \cdots$$

of subspaces of the space Y are called *filtration of the space* Y if all subspaces Y_n are closed, and the space Y is their free union (= inductive limit). Thus, summing up everything proved, we can say that

Proposition 10.18. *the spaces J_nX constitute a filtration of the space JX.*

If $\{Y_n\}$ is a filtration of the space Y, then the map $f : Y \to B$ is continuous if all its restrictions $f|_{Y_n}$ are continuous. With this in mind, consider for any point $a \in JX$ the maps

$$\begin{aligned} L_a :& JX \to JX, \quad u \mapsto au, \\ R_a :& JX \to JX, \quad u \mapsto ua. \end{aligned}$$

If $a \in JmX$, then for any $n \geq 0$ the map $L_a|_{J_nX}$ decomposes into a composition of continuous maps

$$\begin{aligned} J_nX \to J_mX \times J_nX \to J_{m+n}X \subset JX, \\ u \mapsto (a,u) \mapsto au \end{aligned}$$

and therefore continuous. For similar reasons, the map $R_a|_{J_nX}$ is continuous. Therefore,

Proposition 10.19. *the maps $L_a, R_a : JX \to JX$ are continuous.*

For any $n \geq 1$, the factorisation map $X^n \to J_{nX}$ is obviously a relative homeomorphism $(X^n, X^n_{n-1}) \to (J_nX, J_{n-1}X)$, where, as in the Appendix to Lecture 5, X^n_{n-1} denotes a subspace a space X^n consisting of points, at least one coordinate of which is the base point x_0 (and for which therefore the corresponding cofibre - cf. §4.14 - X^n/X^n_{n-1} is an n-tuple of the mixed power

$$x^{\wedge n} = \overbrace{X \wedge \cdots \wedge X}^{n}$$

of the pointed space X). Being a relative homeomorphism, the map $X^n \to J_nX$ induces a homeomorphism of cofibre. Thus, for any $n \geq 1$ the space $J_nX/J_{n-1}X$ is homeomorphic to the space $X^{\wedge n}$.

10.4 The case of well-pointed spaces

Definition 10.20. A filtration $\{Y_n\}$ of the space Y is called a *cofibration filtration* (or *Borsuk filtration*) if for each $n \geq 1$, the pair (Y_n, Y_{n-1}) is a cofibration.

It is easy to see that

Proposition 10.21. *for any well-pointed space X, the filtration $\{J_nX\}$ of the space JX is a cofibration filtration.*

Proof. Indeed, since the factorisation map $X^n \to J_nX$ represents a relative homeomorphism $(X^n, X^n_{n-1}) \to (J_nX, J_{n-1}X)$, then in the case when the space X is well-pointed (and therefore the pair $(X^n X^n_{n-1}) = (X, x_0)^n$ is a cofibration), each pair $(J_nX, J_{n-1}X)$ by applying Lemma 4.41 of Lecture 4 will be a cofibration. □

Lemma 10.22. *For any cofibration filtration$\{Y_n\}$ of a space Y all pairs (Y_m, Y_n), $m \geq n \geq 0$, as well as all pairs (Y, Y_n), $n \geq 0$, are cofibrations.*

Proof. For each pair (Y_m, Y_n), $m \geq n \geq 0$, the inclusion $Y_n \to Y_m$ is a composition of inclusions $Y_n \to_{n+1} \to Y_{n+2} \to \cdots \to Y_m$, each of which is, by the condition, a cofibration. Therefore, the inclusion $Y_n \to Y_m$ will also be a cofibration. This proves Lemma 10.22 for a pair (Y_m, Y_n).

Less formally, this reasoning can be stated as follows. The statement that the pair (Y_m, Y_n) is a cofibration means that the homotopy $f_t : Y_n \to Z$ extends to Y_m if its initial map f_0 extends to Y_m. But if the map f_0 is extended to Y_m, then it is thus extended to Y_{n+1} (if $n + 1 \leq m$). Therefore, since the pair (Y_{n+1}, Y_n) is by convention a cofibration, the homotopy $f_t$can be extended to Y_{n+1}. Applying the same reasoning to this extension, we get an extension of the homotopy f_t to Y_{n+2}, etc., until we reach Y_m.

If now the map f_0 is extended to all Y, then this construction will give us the extension of the homotopy f_t to each subspace Y_m, $m \geq n$, and thus give us the required homotopy $\overline{f}_t : Y \to Z$ (its continuity is ensured by the fact that the space Y is a free union of spaces Y_m, on each of which the homotopy $\overline{f}_t$ is continuous).

This proves Lemma 10.22 for pairs (Y, Y_n). □

With regard to filtration $\{J_nX\}$, we get from here that

Proposition 10.23. *if the space X is well-pointed, then all pairs (J_mX, J_nX), $m \geq n \geq 0$, as well as all pairs (JX, J_nX), $n \geq 0$, are cofibrations.*

Since $J_0X = \{\varnothing\}$, we see in particular that for any well-pointed space X, the space JX is also well-pointed (with respect to the base point φ).

10.5 Meridian maps

Let's return now to the space $\Omega S^\bullet X$. For now, we will only assume that the base point x_0 of the space X is functionally separable, i.e. that there is a continuous function $\varphi : X \to I$ such that $f(x_0) = 0$ and $\varphi(x) \neq 0$ if $x \neq 0$. (In particular, this condition is satisfied if the space X is well-pointed.)

Using the function φ, for ach point $x \in X$ we associate the Moore loop u_x of the space $S^\bullet X$ of length $\varphi(x)$ defined by the formula

$$u_x(t) = \left[x, \frac{t}{\varphi(x)}\right], \quad \text{if} \quad 0 \leq t \leq \varphi(x).$$

Thus, the point $u_x(t)$, when changing t from 0 to $\varphi(x)$, runs through the meridian of the suspension $S^\bullet X$ passing through the point x. On this basis, we will call the map $x \mapsto y_x$ from X to $\Omega^M S^\bullet X$ a *meridian map.*

The composition of the meridian map $X \to \Omega^M S^\bullet X$ with the inclusion

$$\Omega^M S^\bullet X \to \Omega^M S^\bullet X \times \mathbb{R}^+, \quad u \mapsto (u^\#, a) \quad \text{(see Lecture 3)}$$

is obviously given by the formula $x \mapsto (u_x^{(0)}, \varphi(x))$, where $u_x^{(0)}$ is a loop $t \mapsto [x,t]$. Since the map $x \mapsto u_x^{(0)}$ from X to $\Omega S^\bullet X \subset (S^\bullet X)^I$ is adjoint to the factorisation map $X \times I \to S^\bullet X$, $(x,t) \mapsto [x,t]$, and therefore continuous, it follows that the map $x \mapsto (u_x^{(0)}, \varphi(x))$ is continuous. Hence,

Proposition 10.24. *the meridian map $X \to \Omega^M S^\bullet X$ is continuous.*

Therefore, due to the universality of the monoid JX, this map uniquely extends to

$$\boldsymbol{i} : JX \to \Omega^M S^\bullet X,$$

which we will also call the *meridian map.* By definition, it is an algebraic homomorphism of monoids.

Now let (X, x_0) and (Y, y_0) be two pointed spaces with functionally distinguished base points, and let $\varphi : X \to I$, $\psi : Y \to I$ be functions such that $\varphi(x_0) = \psi(y_0) = 0$ and $\varphi(x) \neq 0$, $\psi(y) \neq 0$ if $x \neq x_0$, $y \neq y_0$. The map $JX \to \Omega^M S^\bullet X$ constructed using the function φ is denoted by $\boldsymbol{i}_\varphi$, and the map $JY \to \Omega^M S^\bullet Y$ constructed using the function ψ is denoted by $\boldsymbol{i}_\psi$. Then it is easy to see that

Proposition 10.25. *for any pointed map $f : X \to Y$ the following diagramme*

$$\begin{array}{ccc} JX & \xrightarrow{Jf} & JY \\ {\scriptstyle \boldsymbol{i}_\varphi}\downarrow & & \downarrow{\scriptstyle \boldsymbol{i}_\psi} \\ \Omega^M S^\bullet X & \xrightarrow[\Omega^M S^\bullet f]{} & \Omega^M S^\bullet Y \end{array}$$

is homotopy commutative.

Proof. Indeed, by matching each point $(x,t) \in X \times I$ the Moore loop $F(x,t) \in \Omega^M S^\bullet Y$ of length $a(x,t) = (1-t)\varphi(x) + t\psi(f(x))$, defined by the formula

$$F(x,t)(\tau) = \left[f(x), \frac{\tau}{a(x,t)} \right], \quad 0 \leq \leq a(x,t),$$

we will get (as it is easy to see, a continuous) map $F : X \times I \to \Omega^M S^\bullet Y$, which has the property that $(x,0) = S^\bullet f \circ u_x$ and $F(x,1) = u_{f(x)}$. The extension of this map to JX will therefore be a homotopy from JX to $\Omega^M S^\bullet Y$, connecting the map $\Omega^M S^\bullet f \circ \boldsymbol{i}_\varphi$ to the map $\boldsymbol{i}_\psi \circ Jf$. □

For $X = Y$ and $f = \text{id}$, we obtain, in particular, that,

Proposition 10.26. *up to homotopy, the map $\boldsymbol{i} = \boldsymbol{i}_\varphi$ does not depend on the choice of the function φ, and passing to homotopy classes (i.e., to the category $[\mathscr{Top}^\bullet]$), that the homotopy class $[\boldsymbol{i}]^\bullet$ is a morphism (natural transformation) of the functor J into the functor $\Omega^M S^\bullet$ considered as functors from $[\mathscr{Top}^\bullet]$ to $[\mathscr{Top}^\bullet]$).*

Remark 10.27. At first glance, it seems that the dependence of the map $\boldsymbol{i}$ on φ can be eliminated by moving from the space $\Omega^M S^\bullet X$ to the space $\Omega S^\bullet X$, i.e. by combining the map $\boldsymbol{i}$ with the retraction $g : \Omega^M S^\bullet X \to \Omega S^\bullet X$, $u \mapsto u^\#$, since the composition $X \to \Omega S^\bullet X$ of the map $X \to \Omega^M S^\bullet X$ with retraction r is given by the correspondence $x \mapsto u_x^{(0)}$ and therefore does not depend on the choice of the function φ (and is the map $\boldsymbol{i}$ discussed at the beginning of this lecture). However, due to the non-associativity of the multiplication in $\Omega S^\bullet X$, the resulting map $JX \to \Omega S^\bullet X$ still depends on the choice of the function φ (although, of course, only up to homotopy), and therefore the transition from $\Omega^M S^\bullet X$ to $\Omega S^\bullet X$ does not give any advantages.

10.6 James' theorem and the transformation of the suspension sequence.

The importance of the meridian map for the theory of homotopy groups is determined by the following remarkable theorem of James.

Theorem 10.28. *For each connected well-pointed space X, the homomorphism*

$$\boldsymbol{i}_* : \pi_n JX \to \pi_n \Omega^M S^\bullet X$$

induced by the map $\boldsymbol{i} : JX \to \Omega^M S^\bullet X$, is an isomorphism for any $n \geq 0$.

Remark 10.29. If the space X is numerically locally contractible, then, as Puppe showed, the map $\boldsymbol{i}$ is even a homotopy equivalence. However, proving this statement is somewhat troublesome. We will present the general technique on which Puppe's reasoning is based in addition to this lecture, but we will leave the details to the reader (see note 10.C in the Appendix).

The meridian map is, of course, (identical on X) a map of a pair (JX, X) into a pair $(\Omega^M S^\bullet X, X)$ and therefore induces homomorphism of the corresponding homotopy sequences. By applying Theorem 10.28 and the five lemma, this homomorphism will be an isomorphism, i.e. all homomorphisms

$$\boldsymbol{i}_* : \pi_n(JX, X) \to \pi_n(\Omega^M S^\bullet X, X), \quad n \geq 0$$

will be isomorphisms. Since the groups $\pi_n(\Omega^M S^\bullet X, X)$ are obviously isomorphic to the groups $\pi_n(\Omega S^\bullet X, X$, we see that Theorem 10.281 allows the suspension sequence of the space X to be rewritten in the following significantly simpler form:

$$\cdots \to \pi_n(X) \xrightarrow{E} \pi_{n+1} S^\bullet X \xrightarrow{H} \pi_n(JX, X) \xrightarrow{P} \pi_{n-1} X \to \cdots \tag{10.30}$$

(here the homomorphisms H and P differ, of course, from the homomorphisms H and P of the corresponding isomorphisms in (10.2)). We will achieve a further (and final) simplification of this sequence in Lecture 18[2]

10.7 The Moore variant of the Serre fibration

To prove Theorem 10.28, we will need a Moore variant of the space PX.

For any pointed space X, we denote by $P^M X$ the space of all Moore paths of the space X starting at the base point x_0. The operation of multiplying Moore paths obviously determines the action of the monoid $\Omega^M X$ on the space $P^M X$, i.e. the continuous map

$$\Omega^M X \times P^M X \to P^M X, \quad (u, v) \mapsto uv,$$

having the property that, for any Moore path $v \in P^M X$ holds,

(i) the equality $u_1(u_2 v) = (u_1 u_2)v$, where u_1, u_2 are arbitrary Moore loops of $\Omega^M X$, and

(ii) the equality $ev = v$, where e is the unit of the monoid $\Omega^M X$.

We will consider the space $P^M X$ s a pointed space with a base point e.

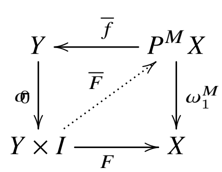
It is clear that the formula $\omega_1^M(v) = v(a)$, where $v(a)$ is the end of the Moore path v, defines a continuous pointed map

$$\omega_1^M : P^M X \to X,$$

and it is easy to see that, like the map $\omega_1 : PX \to X$,

Proposition 10.31. *the map ω_1^M is a fibration.*

Proof. Indeed, for any commutative diagramme



The covering homotopy $\overline{F}$ can be defined by the formula

$$\overline{F}(y, t) = \overline{f}(y) \cdot F_{y,t}, \quad y \in Y, \ t \in I,$$

where $F_{y,t}$ is the restriction of the path $\tau \mapsto F(y, \tau)$, $\tau \in Y$, on the segment $[0, t]$ (so that the length of the path $\overline{F}(y, t)$ is $a_y + t$, where a_y is the length of the path $\overline{f}(y)$). □

[2] The transcriber guesses that Postnikov means Chapter 8 of "Cellular Homootopy".

The fibre of the fibration ω_1^M is obviously the space of the Moore loops $\Omega^M X$. Like the space PX,

Proposition 10.32. *the space $P^M X$ is contractible.*

Proof. Indeed, putting

$$f_t(v) = v|_{[0,ta]}, \quad v \in P^M X, \ 0 \le t \le 1,$$

where a is the length of the path v, we will obviously get a homotopy $f_t : P^M X \to P^M X$, for which $f_0 = \text{const}$ and $f_1 = \text{id}$. □

10.8 Proof of James' theorem

It is clear that if Theorem 10.28 holds for a space X, then it holds for any space homotopically equivalent to X. On the other hand, it is easy to see that every space is homotopically equivalent to a space whose base point x_0 has a contractible neighbourhood rel x_0 (it is enough to glue a whisker[3] and apply Proposition 2.18 of Lecture 2). Therefore, when proving Theorem 10.28, it can be assumed without loss of generality that in the space X the base point has a contractible neighbourhood U_0, i.e. that this space satisfies the condition imposed in James' lemma (Lecture 9, Proposition 9.16).

With this in mind, let's consider the space E obtained by applying James' construction to the (as we know, continuous) left shift map $L : X \times JX \to JX$, $(x, u) \mapsto xu$, and the corresponding map $p : E \to S^\bullet X$. Since the space X is by condition connected, each map $L_x : u \mapsto xu$ is homotopic to the identity map, and therefore is a homotopy equivalence. Thus, all the conditions of James' lemma are fulfilled, and therefore, according to this lemma,

Proposition 10.33. *the map $p : E \to S^\bullet X$ is a weak fibration.*

Since the map L is surjective, the space $E = (C^\bullet X \times JX) \cup_L JX$ is the coset space of the product $C^\bullet X \times JX$ with respect to the minimum equivalence relation in which $(x, u) \sim (x_0, xu)$. Let E_n be the image in E of the subspace $C^\bullet X \times J_n X \subset C^\bullet X \times JX$ by the factorisation map $C^\bullet X \times JX \to E$.

It is clear that the subspace $E_0$is naturally homeomorphic to the cone $C^\bullet X$. In addition, for any $n > 1$, the preimage of the subspace $E_{n-1} \subset E_n$, by the the factorisation map $C^\bullet X \times J_n X \to E_n$ is obviously the subspace $(C^\bullet X \times J_{n+1} X) \cup (x_0 \times J_n X)$, from which it directly follows that this map represents a relative homeomorphism of the pair

$$(C^\bullet X \times J_n X, (C^\bullet X \times J_{n-1} X) \cup (x_0 \times J_n X)) = (C^\bullet X, x_0) \times (J_n X, J_{n-1} X)$$

for the pair (E_n, E_{n-1}). Therefore, for any $n \ge 1$ the pair (E_n, E_{n-1}) is a cofibration, and its cofibre E_n/E_{n-1} is homeomorphic to the cofibre $(C^\bullet X \times J_n X)/((C^\bullet X \times J_{n-1} X) \cup (x_0 \times J_n X))$.

[3]This technique is called the "whiskering"

On the other hand, it is clear that for any two pairs (X, A) and (Y, B) there is a natural homeomorphism

$$(X \times Y)/((X \times B) \cup (A \times Y)) = (X/A) \wedge (Y/B)$$

that is,

Proposition 10.34. *the cofibre of the direct product of two pairs is the product of their cofibres.*

Therefore, in particular, the cofibre E_n/E_{n-1} is homeomorphic to the product

$$C^{\bullet}X \wedge (J_nX/J_{n-1}X) = C^{\bullet}X \wedge X^{\wedge^n}.$$

But it is easy to see that

Proposition 10.35. *if a pointed space X is contractible, then for any pointed space Y the smash product $X \wedge Y$ is also contractible.*

Proof. Indeed, if $\mathrm{id}_X \sim \mathrm{const}$, then $\mathrm{id}_{X\wedge Y} = \mathrm{id}_X \wedge \mathrm{id}_Y \sim \mathrm{const} \wedge \mathrm{id}_Y$. It follows directly from the definitions that $\mathrm{const} \wedge = \mathrm{const}$ for any map g. So $\mathrm{id}_{X\wedge Y} \sim \mathrm{const}$, i.e. the space $X \wedge Y$ is contractible. □

Since the cone $C^{\bullet}X$ is contractible, this statement applies, in particular, to the product $C^{\bullet}X \wedge X^{\wedge^n}$. This proves that

Proposition 10.36. *for any $n \geq 1$ the cofibre E_n/E_{n-1} is contractible.*

But it is obvious that

Proposition 10.37. *if for a cofibration (X, A) the subspace A and the cofibre X/A are contractible, then the space X is also contractible.*

Proof. (since - see lemma 4.46 of Lecture 4 - it is homotopically equivalent to the space X/A). □

Since the space $E_0 = C^{\bullet}X$ is contractible, it follows by obvious induction that

Proposition 10.38. *for any $n \geq 0$ the space E_n is contractible.*

Further, from the fact that the spaces J_nX constitute the filtration of the space JX, it directly follows that the spaces E_n constitute the filtration of the space E. Therefore, for any compact set $C \subset E$, there exists an $n \geq 0$ such that $C \subset E_n$ (otherwise, all sets $(E_n/E_{n-1}) \cap C$ would not be empty and, choosing a point in each of them, we would get an infinite discrete subset in C, which is impossible due to the compactness of C). Since each sphere is $\mathbb{S}^r$ is compact, hence it follows that for any map $f : \mathbb{S}^r \to E$ there exists an $n \geq 0$ such that $f(\mathbb{S}^r) \subset E_n$. Therefore, since the space E_n is contractible, the map f is null-homotopic (as a map in E_n, and therefore as a map in E). This proves that

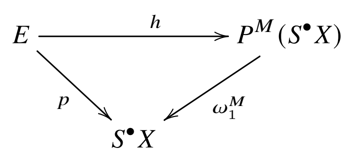
Proposition 10.39. *$\pi_r E = 0$ for any $r \geq 0$, i.e. that the space E is aspherical in all dimensions (∞-connected).*

Now we can proceed directly to the proof of Theorem 10.28.

Proof. (of Theorem 10.28) Let's compare the weak fibration $p : E \to S^\bullet X$ with the strong fibration $\omega_1 : P^M(S^\bullet X) to S^\bullet X$. Naturally generalising the meridian map, we have each point $a = [x, t]^C \in C^\bullet X$ is comparable to the Moore path $u_a \in P^M(S^\bullet X)$ of length $t\varphi(x)$ (where, as above, φ is an arbitrary function $X \to I$, equal to zero only at the base point $x_0 \in X$), putting

$$u_a(\tau) = \left[x, \frac{\tau}{\varphi(x)}\right]^S, \quad 0 \leq \tau \leq t\varphi(x).$$

It is clear that the correspondence $(a, u) \mapsto \boldsymbol{i}(u)u_a$ well defines the map $h : E \to P^M(S^\bullet X)$, for which the following diagramme is communicative



and which therefore induces a homomorphism of the homotopy sequence of the weak fibration p into the homotopy sequence of the fibration ω_1^M, which is an identity map on the groups $\pi_n S^\bullet X$. Since on the groups $\pi_n E$ this homomorphism is an isomorphism (since the groups $\pi_n P^M(S^\bullet X)$, like the groups $\pi_n E$, are zero), it follows from here, by applying the five lemma, that the map h induces isomorphism of homotopy groups of fibres.

To complete the proof of the theorem, it remains to note that above the point x_0, the fibre of the weak fibration p is the space JX, the fibre of the fibration ω_1 is the space $\Omega^M S^\bullet X$ and that on JX the map h coincides with the map $\boldsymbol{i}$. $\square$

Appendix

In connection with the above proof of James' theorem, the question naturally arises whether the space E involved in it will be not only aspherical in all dimensions, but also contractible. In this Appendix, we will outline Milnor's general theory designed to answer these kinds of questions.

10.A Telescope normalisation of filtrations

Let X be an arbitrary topological space and let

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

be some of its filtration. The *telescope* T *over* $\{X_n\}$ is the subspace of the product $X \times \mathbb{R}^+$ (where $\mathbb{R}^+$ is the set of all non-negative real numbers) consisting of points (x, r), $x \in X$, $r \in \mathbb{R}^+$, such that $x \in X_n$ if $n \leq r < n + 1$. The telescope T can be visually depicted in Fig. 10.A.1. It can also be imagined as the result

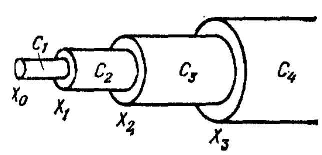


Figure 10.A.1:

of gluing the reversed cylinders $C_{n+1} = \mathrm{Cyl}(i_n)$ of the inclusions $i_n : X_n \to X_{n+1}$ over the spaces X_n contained in them.

It is easy to see that the subspaces T_n of the telescope T consisting of points (x, r) for which $r \leq n$, make up its filtration

$$T_0 \subset T_1 \subset \cdots \subset T_n \subset \cdots$$

This filtration is said to be a *telescopic normalisation* of the filtration $\{X_n\}$. Obviously, this filtration is a cofibration filtration.

Note that

$$T_n = C_0 \cup \cdots \cup C_{n-1}, \quad \text{for} \quad n \geq 1$$

(where $T_0 = X_0$).

The projection $p : (x, r) \mapsto x$ obviously translates each subspace T_n into the corresponding subspace X_n (i.e., as they say, preserves the filtration of the spaces T and X). This means that there is a commutative diagramme

$$\begin{array}{ccccccccccc}
T_0 & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_n & \longrightarrow & T_{n+1} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle p_0} & & {\scriptstyle p_1}\downarrow & & & & {\scriptstyle p_n}\downarrow & & {\scriptstyle p_{n+1}}\downarrow & & \\
X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & X_{n+1} & \longrightarrow & \cdots
\end{array}$$

the horizontal arrows of which are inclusions, and the vertical ones are induced by the map p.

We will embed X_n into T_n by identifying each point $x \in X_n$ with the point $(x, n) \in T_n$ (this embedding is obviously consistent with the inclusions $X_n \subset C_{n-1}$ and $C_{n-1} \subset T_n$). Then the map $p_n : T_n \to X_n$ will be a retraction. Moreover, it is easy to see that

Proposition 10.40. *the map $p_n : T_n \to X_n$ is a strict deformation retraction.*

Proof. Indeed, the map $f_t : (x, r) \mapsto (x, (n - r)t + y)$ constitutes a homotopy fixed on X_n from T_n to T_n, connecting the identity map $\mathrm{id} : T_n \to T_n$ with the map $(x, r) \mapsto (x, n)$, which is a composition of the map p_n and the embedding $X_n \to T_n$. □

Thus, we see that

Proposition 10.41. *all maps $p_n : X_n \to T_n$ are homotopy equivalences.*

Nevertheless, the complete map $p : X \to T$ will not be a homotopy equivalence, in general.

Definition 10.42. The filtration $\{X_n\}$ is called *homotopically correct* if the map $p : X \to T$ is a homotopy equivalence. In this case, it is also said that the space X is the *homotopy limit* of the subspaces X_n.

10.B Homotopy equivalence of homotopy limits

Let X and Y be two spaces with filtrations $\{X_n\}$ and $\{Y_n\}$, respectively, and TX and TY be the corresponding telescopes. Then for any filtration-preserving map $f : X \to Y$ the formula

$$(Tf)(x, r) = (fx, r), \quad (x, r) \in TX,$$

obviously defines some kind of telescope-filtration map $Tf :: TX \to TY$.

On the other hand, since the map $f : X \to Y$ preserves filtrations, it defines some map for any $n \geq 0$

$$f_n : X_n \to Y_n.$$

Let's first consider the special case when $X = Y$ and $X_n = Y_n$.

10.C Milnor's theorem

Proposition 10.43 (Milnor). *If the map $f : X \to X$, preserving the filtration $\{X_n\}$ has the property that for any $n \geq 0$ the map $f_n : X_n \to X_n$ is homotopic to the identity map, then the map*

$$Tf : TX \to TY$$

is a homotopy equivalence.

Proof. Let $f_{n,t} : X_n \to X_n$ be a homotopy connecting the map id_{X_n} to the map f_n. Then the formula

$$h_t(x,r) = \begin{cases} (f(x), n + s(2t+1)), & \text{if} \quad 0 \leq t \leq 1/2,\ 0 \leq s \leq 1/2, \\ (f(x), n + 2(1-s)t + 2), & \text{if} \quad 0 \leq t \leq 1/2,\ 1/2 \leq s \leq 1, \\ (f_{n,2-2t}(x), n + 2s), & \text{if} \quad 1/2 \leq t \leq 1,\ 0 \leq s \leq 1/2, \\ (f_{n,1-(3-4s)(2t-1)}(x), n+1), & \text{if} \quad 1/2 \leq t \leq 1,\ 1/2 \leq s \leq 3/4, \\ (f_{n+1,1-(4s-3)(2t-1)}(x), n+1), & \text{if} \quad 1/2 \leq t \leq 1,\ 3/4 \leq s \leq 1, \end{cases}$$

where $n = [r]$ (and, therefore, $x \in X_n$), and $s = r - n$, will be well (see Fig. 10.C.1) define the homotopy $h_t : TX \to TX$ connecting he map Tf with the

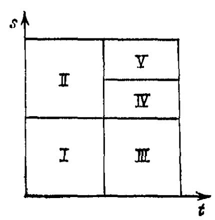


Figure 10.C.1:

map $h = h_1$ defined by the formula

$$h(x,r) = \begin{cases} (x, n + 2s), & \text{if} \quad 0 \leq s \leq 1/2, \\ (f_{n,4s-2)}(x), n+1), & \text{if} \quad 1/2 \leq s \leq 3/4, \\ (f_{n+1,4-4s(x)}, n+1), & \text{if} \quad 3/4 \leq s \leq 1, \end{cases}$$

where still $n = [r]$ and $s = r - n$. Therefore, it is sufficient to prove that the homotopy equivalence is the map h.

With this in mind, we note that by applying the equalities

$$h\left(x, n + \frac{1}{2}\right) = h(x, n+1) = (x, n+1)$$

the following formula

$$h(x,r) = \begin{cases} (x, n+2s), & \text{if} \quad 0 \leq s \leq 1/2, \\ h\left(x, n + \frac{3-2s}{2}\right), & \text{if} \quad 1/2 \leq s \leq 1, \end{cases}$$

well defines some continuous map

$$g : TX \longrightarrow TX.$$

Let

$$\varphi_t(x,r) = \begin{cases} (x, n + (1+6t)s), & \text{if } 0 \le t \le 1/2,\ 0 \le s \le 1/4, \\ (x, n + 2(1-s)(t+s)), & \text{if } 0 \le t \le 1/2,\ 1/4 \le s \le 1, \\ h(x, n+t), & \text{if } 1/2 \le t \le 1,\ t/2 \le s \le (3-2t)/2, \\ (h \circ g)(x,r), & \text{if } 1/2 \le t \le 1, \text{ and either } 0 \le s \le t/2, \text{ or } (3-2t)/2 \le s \le 1. \end{cases}$$

It is automatically verified (see Fig. 10.C.2) that this formula well defines a

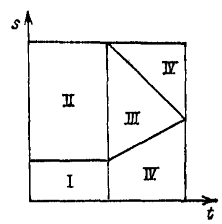


Figure 10.C.2:

homotopy $\varphi_t : TX \to TX$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = h \circ g$. Similarly, the following formula

$$\psi_t(x,r) = \begin{cases} (x, n + (1+6t)s), & \text{if } 0 \le t \le 1/2,\ 0 \le s \le 1/4, \\ (x, n + 2(1-s)(t+s)), & \text{if } 0 \le t \le 1/2,\ 1/4 \le s \le 1, \\ g(x, n+t), & \text{if } 1/2 \le t \le 1,\ t/2 \le s \le (3-2t)/2, \\ (g \circ h)(x,r), & \text{if } 1/2 \le t \le 1, \text{ and either } 0 \le s \le t/2, \text{ or } (3-2t)/2 \le s \le 1. \end{cases}$$

well defines a homotopy $\psi_t : TX \to TX$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 = g \circ h$.

So the map h is indeed a homotopy equivalence (with the inverse homotopy equivalence g). □

Corollary 10.44. *If*

a) the filtration $\{X_n\}$ of the space X is homotopically correct;

b) the filtration-preserving map $f : X \to X$ has the property that for any $n \ge 0$ the map $f_n : X_n \to X_n$ is homotopic to the identity map,

then the map $f : X \to X$ is a homotopy equivalence.

Proof. In the (obviously commutative) diagramme below

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & X \end{array}$$

all arrows except the lower one are homotopy equivalences. Therefore, the lower arrow will also be a homotopy equivalence. □

Remark 10.45. Under the conditions of Corollary 8.39, the map f will not, in general, be homotopic to the identity map.

Corollary 10.46. *Let X and Y be topological spaces with filtrations $\{X_n\}$ and $\{Y_n\}$, respectively, and let $f : X \to Y$ be a continuous filtration-preserving map. If*

a) the filtrations $\{X_n\}$ and $\{Y_n\}$ are homotopically correct;

b) for any $n \geq 0$, the map $f_n : X_n \to Y_n$ is a homotopy equivalence;

c) there is a filtration-preserving map $g : Y \to X$ such that for every $n \geq 0$ the map $g_n : Y_n \to X_n$ is a homotopy equivalence inverse to the homotopy equivalence $f_n : X_n \to Y_n$,

then the map $f : X \to Y$ is a homotopy valence.

Proof. The map $g \circ f : X \to Y$ satisfies the conditions of Corollary 10.44. Therefore, it is a homotopy equivalence. Similarly, the map $f \circ g : Y \to Y$ is homotopy equivalence. Therefore, the map f will also be a homotopy equivalence (with the inverse homotopy equivalence $h \circ g : Y \to X$, where $h : X \to X$ is the homotopy equivalence inverse to the homotopy equivalence $g \circ f : X \to X$). □

Remark 10.47. The map g (though being a homotopy equivalence) will not, in general, be homotopically equivalent to the inverse of the homotopy equivalence f.

Corollary 10.48. *If the space X has a homotopically correct filtration $\{X_n\}$ such that for every $n \geq 0$ the space X_n is contractible, then the space X is also contractible.*

Proof. This corollary is a special case of Corollary 10.46, corresponding to the case when $Y = \text{pt}$ (and $Y_n = \text{pt}$ for any $n \geq 0$). □

Interestingly, the conclusion of Corollary 10.46 follows only from conditions a) and b), so that condition c) is actually superfluous. To prove this, we will need the following lemma, which explains, by the way, why we call the filtration $\{T_n\}$ the normalisation of the filtration $\{X_n\}$.

Lemma 10.49. *The telescopic normalization $\{T_n\}$ of an arbitrary filtration $\{X_n\}$ is homotopically correct.*

Proof. The telescope $T(T)$ of the telescope $T = T(X)$ consists of points of the form (x, r, s), where is $xinX$, and $r, s \in \mathbb{R}^+$, having the property that $x \in X_{[r]}$ and $r \leq [s]$. Therefore, the formula

$$q(x, r) = (x, r, r + 1), \quad (x, r) \in T,$$

defines some map $q : T \to T(T)$, which is a section of the projection $p : T(T) \to T$, $(k, r, s) \mapsto (x, r)$, i.e. such that $p \circ q = \mathrm{id}$. In addition, the correspondence $(x, r, s) \mapsto (x, r, (1-t)s + t(r+1))$, $0 \le t \le 1$, will determine the homotopy $T(T)$ in $T(T)$, connecting the identity map to the map $q \circ p$. This proves that the projection p is a homotopy equivalence, and, therefore, the filtration $\{T_n\}$ is homotopically correct, □

Theorem 10.50 (Milnor)**.** *Let X and Y be topological spaces with filtrations $\{X_n\})$ and $\{Y_n\}$, respectively, and let $f : X \to Y$ be a continuous filtration map.*

a) the filtrations $\{X_n\}$ and $\{Y_n\}$ are homotopically correct;

b) for any $n \ge 0$, the map $f_n : X_n \to Y_n$ is a homotopy equivalence,

then the map f is a homotopy equivalence, too.

Proof. Let $g_n : Y_n \to X_n$ be the homotopy equivalence inverse to the homotopy equivalence $f_n : X_n \to Y_n$, and let $i_n : X_n \to X_{n+1}$ and $j_n : Y_n \to Y_{n+1}$ be inclusions. Then

$$i_n \circ g_n \sim g_{n+1} \circ f_{n+1} \circ i_n \circ g = g_{n+1} \circ j_n \circ g_n \sim g_{n+1} \circ j_n,$$

i.e., the following diagramme

$$\begin{array}{ccc} Y_n & \longrightarrow & Y_{n+1} \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle g_{n+1}} \\ X_n & \longrightarrow & X_{n+1} \end{array}$$

the horizontal arrows of which are inclusions, is homotopically commutative. Let $F_n : Y_n \times I \to X_{n+1}$ be a homotopy connecting the map $i_n \circ g_n$ with the map $g_{n+1} \circ j_n$. By putting

$$h(y, r) = \begin{cases} (g_n(y), n + 2s), & \text{if} \quad 0 \le s \le 1/2, \\ (F_n(y, 2s-1), n1), & \text{if} \quad 1/2 \le s \le 1, \end{cases}$$

where $n = [r]$ and $s = r - n$, we will well define a filtration-preserving continuous map $h : TY \to TX$, which has the property that for the map $h \circ Tf : TX \to X$ for any $n > 0$, the following commutative diagramme takes place

$$\begin{array}{ccc} X_n & \longrightarrow & T_n X \\ {\scriptstyle g_n \circ f_n}\downarrow & & \downarrow{\scriptstyle h \circ T f_n} \\ X_n & \longrightarrow & T_n X \end{array}$$

the horizontal arrows of which indicate the inclusion $x \mapsto (x, n)$, $x \in X_n$. Since this embedding, as we know, is a homotopy equivalence, and the map $g_n \circ f_n$ is homotopic to the identity map, it follows that every map $(h \circ T f)_n : T_n X \to T_n Y$ is homotopic to the identity map, and since by applying Lemma 10.49 the filtration $\{T_n(X)\}$ is homotopically correct, then, according to Corollary refcor:10-A1, the map $h \circ Tf$ will be a homotopy equivalence.

For similar reasons, the map $Tf \circ h$ will be a homotopy equivalence. Therefore, the map Tf also represents a homotopy equivalence (with the inverse homotopy equivalence $h \circ k$, where k is the homotopy equivalence, the inverse of the homotopy equivalence $Tf \circ h$). By applying condition a), this proves the theorem (cf. with the proof of Corollary 10.44). □

10.D Homotopy exactness of cofibration filtrations

There are several different criteria for homotopy exactness of filtration. We will prove the following criterion, which was first mentioned, apparently, by Puppe.

Proposition 10.51. *Any cofibration filtration $\{X_n\}$ is homotopically correct.*

Proof. First of all, we will build some filtration-preserving map $q : X \to TX$, and then we prove that it is a homotopy equivalence with the inverse homotopy equivalence $p : TX \to X$.

To construct the map q, it is sufficient for all $n \geq 0$ to construct continuous maps $q_n : X_n \to T_n$, having the property that for each $n \geq 0$ the following diagramme

$$\begin{array}{ccc} X_n & \longrightarrow & X_{n+1} \\ {\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle q_{n+1}} \\ T_n & \longrightarrow & T_{n+1} \end{array} \tag{10.52}$$

the horizontal arrows of which are inclusions, are commutative. Indeed, by putting $q(x) = q_n(x)$, if $x \in X_n$, we will uniquely define then the map $q : X \to T$, which will be continuous, because the maps q_n are continuous, and the space X is a free union of spaces X_n.

We will construct the maps $q_n : X_n \to T_n$ by induction on n, taking for q_0 the identity map of the space $X_0 = T_0$.

To carry out the induction, we additionally require that there be a homotopy $q_{n,t} : X_n \to T_n$ for any $n \geq 0$, connecting the map $X_n \to T_n$, $x \mapsto (x, n)$, with the map $q_n : X_n \to T_n$. To satisfy this condition for $n = 0$, is it sufficient for any $t \in I$ put $q_{0,t} = \mathrm{id}$.

Suppose that for some $n \geq 0$ the map $q_n : X_n \to T_n$ and the homotopy $q_{n,t} : X_n \to T_n$ have already been constructed. Let $q_{n,t}(x) = (\overline{q}_{n,t}(x), r_{n,t}(x))$, where $\overline{q}_{n,t}(x) \in X_n$ and $0 \leq r_{n,t}(x) \leq n$ for each point $x \in X_n$ (and, of course, $q_{n,t}(x) \in X_k$ if $k \leq r_{n,t}(x) \leq k+1$). We define the homotopy of $Q_{n,\tau} : X_n \times I \to T_{n+1}$ of the map $Q_n : (x,t) \mapsto q_{n,t}(x)$, considered as a map in T_{n+1}, by putting

$$Q_{n,\tau}(x,t) = (\overline{q}_{n,t}(x), r_{n,t}(x) + \tau)$$

for any $x \in X_n$ and $t, \tau \in I$. Let $Q^{(0)}_{n,\tau}$ be the homotopy from X_n to T_{n+1}, defined by the formula $Q^{(0)}_{n,\tau}(x) = Q_{n,\tau}(x, 0)$. Since by the condition $\overline{q}_{n,0}(x) = n$ and $Q^0_{n,\tau}(x) = n$, then $Q^{((0)}_{n,\tau}(x) = (x, n + \tau)$ and, in particular, $Q^{(0)}_{n,1}(x) = (x, n + 1)$. This means that the map $Q^{(0)}_{n,1}$ represents a limit on X_n of the map $j_{n+1} : X_{n+1} \to T_{n+1}$. Since the pair (X_{n+1}, X_n) is by convention a cofibration, it follows that the homotopy $Q^{(0)}_{n,\tau}$ is a restriction on X_n of a homotopy $\overline{Q}_{n,\tau} : X_{n+1} \to T_{n+1}$ such that $\overline{Q}_{n,1} j_{n+1}$. For the homotopy $H_t : (x, \tau) \mapsto Q_{n,\tau}(x, t)$ from $X_n \times I$ to T_{n+1}, this means that its initial map $H_0 : (x, \tau) \mapsto Q_{n,\tau}(x, 0) = Q^{(0)}_{n,\tau}(x)$ is a restriction onto the $X_n \times I$ of the map $X_{n+1} \times I \to T_{n+1}$ defined by the formula $(x, \tau) \mapsto \overline{Q}_{n,\tau}(x)$. Therefore, since the pair $(X_{n+1} \times I, X_n \times I)$ is also a cofibration, this homotopy is a restriction on $X_n \times I$ of some homotopy $\overline{H}_t : (x, \tau) \mapsto \overline{H}_t(x, \tau)$, $x \in X_{n+1}$, $t, \tau \in I$, which has the property that $\overline{H}_0(x, \tau) = \overline{Q}_{n,\tau}(x)$ for any point $(x, \tau) \in X_{n+1} \times I$ (see Fig. 10.D.1). Since for $x \in X_n$ the equality $\overline{H}_1(x, 0) = H_1(x, 0) = Q_{n,0}(x, 1) =$

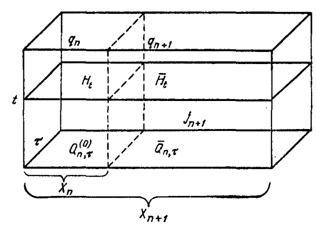


Figure 10.D.1:

$Q_n(x, 1) = q_n X$ holds, the map $q_{n+1} : x \mapsto \overline{H}_1(x, 0)$ in Diagramme (10.52 is commutative. In addition, putting for any point $(x, t) \in X_{n+1} \times I$

$$Q_{n+1}(x, t) = \begin{cases} \overline{Q}_{n,1-2t}(x), & \text{if } \quad 0 \le t \le 1/2, \\ \overline{H}_{2t-1}(x, 0), & \text{if } \quad 1/2 \le t \le 1, \end{cases} \tag{10.53}$$

we will get a homotopy $Q_{n+1} : X_{n+1} \times I \to T_{n+1}$ connecting the map $x \mapsto Q_{n+1}(x, 0) = \overline{Q}_{n,1}(x) = j_{n+1}(x)$ with the map $q_{n+1} : x \mapsto \overline{H}_1(x, 0)$.

Thus, all maps q_n are completely constructed by induction. Therefore, the map q is also constructed.

Now we need to show that $p \circ q \sim \mathrm{id}$ and $q \circ p \sim \mathrm{id}$. To do this, it is again sufficient to construct homotopies $f_{n,t} : X_n \to X_n$ and $g_{n,t} : T_n \to T_n$, connecting the identity maps id_{X_n}, and id_{T_n} with the maps $p_n \circ q_n : X_n \to X_n$ and $q_n \circ p_n : T_n \to T_n$ respectively, for any $t \in I$ and every $n \ge 0$ there exist commutative diagrammes

$$\begin{array}{ccc} X_n & \longrightarrow & X_{n+1} \\ {\scriptstyle f_{n,t}}\downarrow & & \downarrow{\scriptstyle f_{n+1,t}} \\ X_n & \longrightarrow & X_{n+1} \end{array} \qquad \begin{array}{ccc} T_n & \longrightarrow & T_{n+1} \\ {\scriptstyle g_{n,t}}\downarrow & & \downarrow{\scriptstyle g_{n+1,t}} \\ T_n & \longrightarrow & T_{n+1} \end{array}$$

the horizontal arrows of which are inclusions.

Let's first consider the homotopy $f_{n,t} : \mathrm{id} \sim p_n \circ q_n$. By construction, the maps $p_n \circ q_{n,t} : X_n \longrightarrow X_n$ also constitute a homotopy from X_n to X_n, connecting the map $p_n \circ q_{n,0} = p_n \circ j_n = \mathrm{id}$ with the map $p_n \circ q_{n,1} = p_n \circ q_n$. This means that the map $F_n : X_n \times I \longrightarrow X_n$, $(x, t) \mapsto f_{n,t}(x)$ coincides on $(X_n \times 0) \cup (X_n \times 1)$ with the map $p_n \circ Q_n : X_n \times I \longrightarrow Xn$, $(x, t) \mapsto (p_n \circ q_{n,t})(x)$. As an additional condition facilitating the construction by induction of the homotopy $f_{n,t}$, we will require that for any $n \geq 0$ there exists a homotopy $\Phi_n : X \times I \times I \longrightarrow X_n$, fixed on $(X \times 0) \cup (X \times 1)$ connecting the map $p \circ Q_n$ with the map F_n.

Assuming now that the homotopies F_n and Φ_n have already been constructed, consider the homotopy $p_{n+1} \circ Q_{n+1}$, where Q_{n+1} is the homotopy given by formulae (10.53).

So for $x \in X_n$ the homotopy $p_{n+1} \circ Q_{n+1}$ is given by the formula

$$p_{n+1} \circ Q_{n+1}(x,t) = \begin{cases} x, & \text{if} \quad 0 \leq t \leq 1/2, \\ (p_n \circ Q_n)(x, 2t-1), & \text{if} \quad 1/2 \leq t \leq 1, \end{cases}$$

then the formula

$$\overline{\Phi}_n = \begin{cases} x, & \text{if} \quad 0 \leq t \leq \frac{1-\tau}{2}, \\ \Phi\left(x, \frac{2t+\tau-1}{1+\tau}, \tau\right), & \text{if} \quad \frac{1-\tau}{2} \leq t \leq 1, \end{cases}$$

where $x \in X_n$, $t, \tau \in I$, defines (well due to the identity $\Phi(x, 0, \tau) = x$) fixed on $(X_n \times 0) \cup (X \times 1)$ the homotopy $\overline{\Phi}_n : X_n \times I \times I \longrightarrow X_n$, connecting the homotopy $p_{n+1} \circ Q_{n+1}$ on $X_n \times I$ with the homotopy F_n. Therefore, - since the pair $(X_{n+1} \times I, \widetilde{X}_n)$, where $\widetilde{X}_n = (X_{n+1} \times 0) \cup (X_n \times I) \cup (X_{n+1} \times 1)$, is, according to Proposition 2.19 of Lecture 2, a cofibration, - the homotopy $\overline{\Phi}$ is a restriction fixed on $(X_{n+1} \times 0) \cup (X_{n+1} \times 1)$ of the homotopy $\Phi_{n+1} : X_{n+1} \times I \times I \longrightarrow X_n$, connecting the homotopy $p_{n+1} \circ Qn+1$ with some homotopy F_{n+1}, and coinciding with the homotopy F_n on $X_n \times I$.

Thus, homotopies F_n are constructed for all $n \geq 0$.

The construction of the homotopy $G_n : (z,t) \mapsto g_{n,t}(z)$, $z \in T_n$, $t \in I$ is carried out similarly. As an additional condition, we require that for any $n \geq 0$ there exists a homotopy $\Psi_n : T_n \times I \times I \longrightarrow T_n$ fixed on $(T_n \times 0) \cup (T_n \times 1)$, connecting the homotopy $G_n : T_n \times I \longrightarrow T_n$ with the homotopy $\widehat{Q}_n : T_n \times I \longrightarrow T_n$ defined for any $z = (x, r) \in T_n$ and each $t \in I$ by the formula

$$\widehat{Q}_n(x,t) = \begin{cases} (x, 2(n-r)t + r), & \text{if} \quad 0 \leq t \leq 1/2, \\ Q_n(x, 2t-1), & \text{if} \quad 1/2 \leq t‘1, \end{cases}$$

and also connecting maps id_{T_n} and $q_n \circ p_n$.

Putting for any point $(z, t, \tau) \in T_n \times I \times I$, $z = (x, r)$,

$$\overline{\Psi}_n(z,t,\tau) = \begin{cases} (x, 2(n-r+1-2\tau)t + r), & \text{if} \quad 0 \leq t \leq 1/2, \quad 0 \leq \tau \leq 1/2, \\ (x, n+3-4t-2\tau), & \text{if} \quad 1/2 \leq t \leq \frac{3-2\tau}{4}, \quad 0 \leq \tau \leq 1/2, \\ Q_n(x, \frac{4t+2\tau-3}{2\tau+1}), & \text{if} \quad \frac{3-2\tau}{4} \leq t \leq 1, \quad 0 \leq \tau \leq 1/2, \\ \Psi_n(z, t, 2\tau - 1), & \text{if} \quad 1/2 \leq \tau \leq 1, \end{cases}$$

we get a homotopy $\overline{\Psi}_n : T_n \times I \times I \to T_n$ fixed on $(T_n \times 0) \cup (T_n \times 1)$, connecting the homotopy $\widehat{Q}_{n+1}$ with the homotopy G_n on $T_n \times I$. Since the pair (T_{n+1}, T_n), and therefore the pair $(T_{n+1} \times I, \widetilde{T}_n)$, where $\widetilde{T}_n = (T_{n+1} \times 0) \cup (T_n \times I) \cup (T_{n+1} \times 1)$, is a cofibration, the homotopy $\overline{\Psi}_n$ is a restriction of the homotopy $\Psi_{n+1} : T_{n+1} \times I \times I \to T_{n+1}$ fixed on $T_{n+1} \times 0) \cup T_{n+1} \times 1$, connecting the homotopy $\widehat{Q}_{n+1}$ with a homotopy G_{n+1} such that $G_{n+1}|_{T_n \times I} = G_n$.

Proposition 10.51 is thus fully proved. □

Remark 10.54. According to Proposition 10.51 and the Corollary 10.48 of Proposition 10.43, the space E involved in the proof of James' theorem is contractible. Therefore, the map $h : E \to P^M(S^\bullet X)$ is a homotopy equivalence and, therefore, according to Proposition 2.57 of the Appendix to Lecture 2 - assuming that p is a (homotopy at least) fibration, it will be a fibre homotopy equivalence. In particular, the homotopy equivalence will be the restriction of this map on the fibre JX of the fibration p, i.e. the meridian map $\boldsymbol{i}$. Thus, to prove Puppe's theorem (see Remark 9.3 of Lecture 9), it is sufficient to prove that the map $p : E \to S^\bullet X$ is not a weak, but a strong fibration. Recalling the proof of Lemma 9.17 of Lecture 9 and Theorem 9.31 from the Appendix to Lecture 9, we see that for this, in turn, it is sufficient to prove that under the conditions of this lemma, the map $p_V : p^{-1}V \to V$ induced by p is a homotopy fibration (two-element covering $\{Y, W\}$ is obviously enumerable). Pupe shows that for a enumerable locally contractible space X, the map p_V is homotopically equivalent (over V) to some homotopy fibration $q : \widetilde{V} to V$ and, therefore (lemma 2.51 from the Appendix to Lecture 2), is itself a homotopy fibration. At the same time, for $\widetilde{V}$ you can take the space $(p^{-1}V \times I) \cup_f (V \times Y)$, obtained by gluing the product $p^{-1}V \times I$ to the space $V \times Y$ by the map $f : (u, 1) \mapsto (p(u), \overline{r}(u))$, $u \in p^{-1}V$, where $\overline{r} : p^{-1}V \to Y$ is the retraction constructed in the proof of Lemma 9.17 of Lecture 9, and the fibration $q : \widetilde{V} \to V$ is set by the correspondence $(v, y) \mapsto v$, $u \in p^{-1}V$, $t \in I$, $v \in V$.

Detailed conduct of the relevant reasoning we'll leave it to the reader.

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