Differentiable Group Actions

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Foreword

The reference for Section 5 and Section 6 is A. Borel's 'Seminar on Transformation Groups', Chapter VII (by G.E. Bredon) and Chapter VIII (by R.S. Palais). The other four sections use some results from: -

- N. Bourbaki : General Topology, Part I
- C. Chevalley : Theory of Lie Groups, I.
- Hewitt and Ross : Abstract Harmonic Analysis, I.
- S. Lang : Introduction to Differentiable Manifolds
- D. Husemoller : Fiber Bundles

In these notes, "smooth" will mean C^∞, and a manifold will always be assumed to be finite-dimensional paracompact and Hausdorff. An acquaintance with the exponential map is assumed.

If G is a group and H a subgroup of G, then G/H (resp. $H\backslash G$) denote the set of left (resp. right) cosets of G modulo H.

1 Introduction: Transformation Groups

A *(right) transformation group* is a set X together with a group G and a function (or *action*) $\varphi\colon X \times G \to X$ such that :-

T1. $\varphi(x, 1) = x \; \forall x \in X$

T2. $\varphi(x, gh) = \varphi(\varphi(x, g), h) \; \forall x \in X, g, h \in H$

Such a transformation group will be denoted by $\varphi\colon X \times G \to X$, i.e., by the action. We speak of G *acting on* X.

Writing $\varphi(y, g)$ as $y.g$, T1 $\times$ T2 becomes :-

$$\begin{array}{lll} \text{T1.} & x.1 = x & \forall x \in X \\ \text{T2.} & x.(gh) = (x.g).h & \forall x \in X, g, h \in H \end{array}$$

Denote by $\varphi_y\colon G \to X$ the map $g \mapsto y.g$, for $y \in X$, and denote by $g\colon X \to X$ th map $x \mapsto x.g$ for $g \in G$. Note that $g\colon X \to X$ is a bijection, with inverse $g^{-1}\colon X \to X$.

The action is

effective if for each $g \in G$, $g \neq 1$, there exists $x \in X$ such that $x.g \neq x$

free if $x.g = x \Rightarrow g = 1$

transitive if $\forall$ pairs $x, y \in X$, there exists $g \in G$ such that $y = x.g$.

Note that every free action is effective.

The set $G_x = \{g \in G\colon x.g = x\}$ is a subgroup of G, called the *isotropy group* of x.

Define an equivalence relation $\sim$ on X by:

$$x \sim y \Leftrightarrow \exists g \in G \quad \text{such that} \quad x.g = y.$$

The equivalence classes are called *orbits*; the equivalence class of $x \in X$ - the *orbit through* x - is $\{xG = x.g\colon g \in G\}$. The set of equivalence classes, denoted X/G, is called the *orbit set*. Further there is a canonical projection

$$p\colon X \to X/G, \quad x \mapsto xG.$$

Proposition 1.1. *There is a natural bijection*

$$\theta\colon G_x \backslash G \to xG, \quad G_x.g \mapsto x.g.$$

Proof.

$$G_x.g = G_x.h \Leftrightarrow hg^{-1} \in G_x \Leftrightarrow x.hg^{-1} = x \Leftrightarrow x.h = x.g.$$

□

A *topological group* is a group G endowed with a topological space structure such that :-

TG1. The map $G \to G, g \mapsto g^{-1}$, is continuous.

TG2. The map $G \times G \to G, (g,h) \mapsto gh$, is continuous.

If G is a topological group, then a subset $H \subset G$ is a *(topological) subgroup* of G if H is an abstract subgroup of the abstract group G, and H, given the subspace topology, satisfies TG1 and TG2.

A *topological transformation group (TTG)* is a topological space X together with a topological group G and a continuous action $\varphi\colon X \times G \to X$ satisfying T1 and T2. X is called a *G-space*.

Notice that $g\colon X \to X$ is a homeomorphism.

Proposition 1.2. *Let $\varphi\colon X \times X \to X$ be a TTG, If X is Hausdorff, then isotropy groups are closed.*

Proof. Take $x \in X$. $\varphi_x\colon G \to X$ is the composite

$$G \xrightarrow{\ i\ } X \times G \longrightarrow X$$

where $i(g) = (x,g)$, and is therefore continuous. So if X is Hausdorff, then $\{x\}$ is closed, thus $\varphi_x^{-1}(x)$ is closed; and $\varphi_x^{-1}(x) = G_x$. □

As remarked above, we have a canonical projection $p\colon X \to X/G$. We give X/G the identification topology given by p (i.e., U is defined to be open in $X/G \Leftrightarrow p^{-1}(U)$ is open in X). X/G with this topology, is called the *orbit space*. Further, p is continuous.

Proposition 1.3. *$p\colon X \to X/G$ is an open map.*

Proof. Let V be open in X. We want to prove that $p(V)$ is open in X/G, i.e., that $p^{-1}p(V)$ is open in X. We have :-

$$\begin{aligned} p^{-1}p(V) &= \{x \in X : p(x) = p(v), \quad \text{for some} \quad v \in V\} \\ &= \{x \in X : x = v.g, \quad \text{for some} \quad v \in V \quad \text{and some} \quad g \in G\} \\ &= \{\cup V.g \colon g \in G\} \end{aligned}$$

But each $V.g$ is open since $g\colon X \to X$ is a homeomorphism, so $p^{-1}p(V)$ is open. □

Proposition 1.4.

$$\theta\colon G_x \backslash G \to xG, \quad G_x.y \mapsto x.g$$

is a continuous bijection.

Proof. We have only to show θ is continuous. We have the commutative diagram,

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi_x\ } & xG \\ {\scriptstyle \Pi}\downarrow & \nearrow{\scriptstyle \theta} & \\ G_x\backslash G & & \end{array}$$

where $\Pi\colon G \to G_x\backslash G$ is the projection $g \mapsto G_x.g$.

$G_x\backslash G$ has (by definition) the identification topology given by Π. Thus if φ_x is continuous, then θ is continuous. □

If G is a topological group and H a (topological) subgroup, then H acts continuously on G by right translations, i.e.,

$$\varphi\colon G \times H \to G, \quad (g, h) \mapsto gh.$$

The orbit through $g \in G$ is gH and the orbit space G/H - the space of left cosets of G by H - is called an *homogeneous space*.

A *Lie group* is a group G with a smooth manifold structure such that

L1. The map $G \to G, g \mapsto g^{-1}$, is smooth

L2. The map $G \times G \to G, (g, h) \mapsto gh$, is smooth

A *differential* (or *Lie*) *transformation group (DTG)* is a smooth manifold M together with a Lie group G and a smooth action $\varphi\colon M \times G \to M$ satisfying T1 and T2. M is called a *G-manifold* or a *smooth G-space*.

Notice that $g\colon M \to M, x \mapsto x.g$ is a diffeomorphism.

Example 1.5. We list examples.

(1) *Lie groups* :

- $\mathbb{R}^n, \mathbb{C}^n$ under addition
- $\{0\}, S^1, S^3$ under multiplication
- the classical groups $\mathbf{GL}_n(\mathbb{R}), \mathbf{GL}_n(\mathbb{C}), \mathbf{SL}_n(\mathbb{R}), \mathbf{O}_n, \mathbf{SO}_n, \mathbf{SU}_n$
- discrete groups, e.g., $\mathbb{Z}$

(2) *Topological groups* :

- All Lie groups
- $\mathbb{Q}^n$ under addition
- groups of all homeomorphisms $X \to X$, where X is a compact topological space
- the group having the compact open topology

(3) *Homogeneous spaces* :

- Spheres $S^{n-1} = \mathbf{O}_n/\mathbf{O}_{n-1}$
- Stiefel manifolds $V_k^n = \mathbf{O}_n/\mathbf{O}_{n-k}$
- Grassmann manifolds $G_k^n = \mathbf{O}_n/(\mathbf{O}_{n-k} \times \mathbf{O}_k)$

(4) *Differential transformation groups* :

- X a Riemannian manifold, G the group of all isometries of X

- If X is a compact smooth manifold and ν is a vector field on X, then a unique section $\varphi\colon X \times \mathbb{R} \to X$ such that for each $x \in X$, the tangent at x of the curve t $\varphi(x,t)$ is $s(x)$ - i.e., $\partial\varphi/\partial t = s \circ \varphi$.

(5) *Topological transformation groups* : Let G be a topological space, H be a subgroup of G, then G acts on the homogeneous space $H\backslash G$ by $\varphi\colon H\backslash G \to H\backslash G, (Hg_1, g_2) \mapsto Hg_1g_2$

See C. Chevalley "Theory of Lie Groups I" for (1 - 3), and S. Lang "introduction to Differentiable Manifolds, Chapter IV" for (4).

2 Analytic Topology of Topological Groups and Topological Transformation Groups

Proposition 2.1. *(1) If H is an abstract subgroup of a topological group G, then $\bar{H}$ is a (topological) subgroup of G. Further if H is normal, then $\bar{H}$ is normal.*

(2) If H is a normal subgroup of G, then G/H is a topological group.

Proof. (1) Denote by $\theta\colon G \times G \to G$ the map $(x, y) \mapsto xy^{-1}$. Take $g, g' \in \bar{H}$ and consider $g'g^{-1}$. Let U be any neighbourhood of $g'g^{-1}$, then by the continuity of θ (which follows from the definition of a topological group) there exist neighbourhoods V, V' of g, g' respectively, such that $V'V^{-1} \subset U$. Since $g, g' \in \bar{H}$, there exist $h \in V \cap H$ and $h' \in V^{-1} \cap H$ such that $h'h^{-1} \in (V'V^{-1}) \cap H \subset U \cap H$, i.e., $U \cap H \neq \varnothing$. So if $g'g^{-1} \in \bar{H}$, then $\bar{H}$ is a subgroup.

Now suppose H is normal. Take $x \in \bar{H}$ and consider $a \times a^{-1}$, where $a \in G$. Let U be a neighbourhood of $a \times a^{-1}$, then since the map $y \mapsto a^{-1}ya$ is a homeomorphism, $a^{-1}Ua$ is a neighbourhood of x. Hence $a^{-1}Ua \cap H \neq \varnothing$ which implies $a^{-1}(U \cap H)a \neq \varnothing$ (since H is normal) thus $U \cap H \neq \varnothing$ so $a^{-1}xa \in \bar{H}$, i.e., $a\bar{H}a^{-1} \subset \bar{H}$.
(2) This is straightforward. □

Proposition 2.2. *If H is an open subgroup of G, then H is closed and G/H is discrete.*

Proof. If H is open then Hg is open for all $g \in G$ so $H = G \setminus \cup_{g \notin H} Hg$ is closed. Any point in G/H is both open and closed. □

Separation

From now onwards, a topological group will always be assumed to be T_0 (i.e., if x, y are two distinct points then either there exists a neighbourhood of x not containing y or there exists a neighbourhood of y not containing x).

Proposition 2.3. *Topological groups are T_1 (i.e., points are closed).*

Proof. Let G be a topological group. Let $a \in \{\bar{1}\}, a \neq 1$ then for all neighbourhoods of A of a, $A \cap \{1\} \neq \varnothing$, i.e., $1 \in A$. Since G is T_0, there exists a neighbourhood B of 1, such that $a \in B$ and hence $a \notin B \cap B^{-1}$ - contradiction. So $\{\bar{1}\} = \{1\}$, i.e., $\{1\}$ is closed, thus all points of G are closed (since for any $x \in G$ the map $p_x\colon G \to G, y \mapsto xy$, is a homeomorphism), i.e., G is T_1. □

Proposition 2.4. *(1) If H is a subgroup of the topological subgroup G, then G/H is T_1 if and only if H is closed in G.*

(2) T_1 homogeneous spaces are regular.

Proof. (1) G/H has the identification topology given by the canonical projection $p\colon G \to G/H$. Hence H is closed in G if and only if $\{1_{G/H}\}$ is closed in G/H if and only if all points of G/H are closed if and only if G/H is T_1.
(2) Let G/H be a T_1 homogeneous space. Let C be closed in G/H and let $x \in G/H \setminus C$. Since

$$\varphi\colon G \times G/H \to G/H, \quad (g_1, g_2H) \mapsto g_1g_2H$$

is continuous, $(1, x)$ has an open neighbourhood $U \times V$ mapped into $G/H \setminus C$ by φ (since $G/H \setminus C$ is a neighbourhood of x). Then $U^{-1}C$ and V are disjoint sets; V is an open neighbourhood of x and we show that $U^{-1}C$ is an open neighbourhood of C.

The map $G \to G, g \mapsto g^{-1}$ is a homeomorphism so U^{-1} is open. If $y = xH \in C$, then $U^{-1}x$ is open in G so $U^{-1}y$ is open in G/H, since $p\colon G \to G/H$ is open by Proposition 1.3. □

Proposition 2.5. *If H is a subgroup of the topological group G, then G/H is Hausdorff if and only if H is closed in G.*

Proof. Let $g_1H, g_2H \in G/H$ such that $g_1H \neq g_2H$, i.e., $g_1g_2^{-1} \notin H$ where H is closed. Then map $G \times G \to G, (x, y) \mapsto xy^{-1}$ is continuous and so taking a neighbourhood W of $g_1g_2^{-1}$ such that $W \cap H = \varnothing$ (which is possible since H is closed), there exist neighbourhoods U of g_1, V of g_2 such that $UV^{-1} \subset W$.

Now $P(U), P(V)$ are open neighbourhoods of g_1H, g_2H respectively, where $p\colon G \to G/H$ is the projection, and further $P(U)$ and $P(V)$ are disjoint. For if $pU \cap pV \neq \varnothing$ then there exist $u \in U, v \in V$ such that $uH = vH$ if and only if $uv^{-1} \in H$ - contradicting $UV^{-1} \subset W$ and $W \cap H = \varnothing$. So G/H is Hausdorff.

If G/H is Hausdorff then G/H is T_1 so H is closed, by Proposition 2.4. □

Corollary 2.6. *H is a closed subgroup of G if and only if G/H is T_3 (i.e., regular and Hausdorff).*

Using Proposition 2.3 and putting $H = \{1\}$ in Proposition 2.6, we have :-

Corollary 2.7. *Topological groups are T_3.*

Remark 2.8. 2.4, 2.5, 2.6 did not use the fact that G was T_0.

Compactness

Proposition 2.9. *Let a topological group G act on a locally compact space X, such that X/G is Hausdorff. Then X/G is locally compact and for any compact $K' \subset X/G$, there exists a compact $K \subset X$ such that $p(K) = K'$, where $p\colon X \to X/G$ is the projection.*

Proof. Take $xG \in X/G$. X is locally compact so x has a compact neighbourhood A. p is open and continuous thus $p(A)$ is a compact neighbourhood of $p(x) = xG$. So X/G is locally compact.

Let K' be compact in X/G. For each $y \in K'$, let Vy be a compact neighbourhood of some point of $p^{-1}(y)$ in X (thus $f(Vy)$ is a compact neighbourhood of y). There are a finite number of $y_i \in K'$ such that the $f(Vy_i)$ cover K'.

Let K_1 be the compact set $\cup_i Vy_i$ in X. We have $K' \subset f(K_1)$ and hence $K = K_1 \cap f^{-1}(K')$ is compact (since X/G Hausdorff, K' is closed, so $K_1 \cap f^{-1}(K')$ is closed in K_1) and $f(K) = K'$. □

We quote the next two results, which can be found in Hewitt and Ross "Abstract Harmonic Analysis I":

Proposition 2.10. *(1) Let H be a subgroup of the topological group G. If H and G/H are (locally) compact, then G is (locally) compact. (See p. 39)*

(2) A locally compact topological group is paracompact, and hence normal. (See p. 76).

Connectedness

If G is a topological group, denote by G_o the component of G which contains 1.

Proposition 2.11. *If G is a topological group then G_o is a closed normal subgroup.*

Proof. G_o is closed by definition. The maps $G \to G, x \mapsto x^{-1}$ and $x \mapsto a^{-1}xa$, for some $a \in G$, are homeomorphisms leaving 1 fixed, thus they map G_o into G_o. The map $\psi \colon G \times G \to G, (x, y) \mapsto xy$ is continuous. As G_o is connected, $G_o \times G_o$ is connected thus $f(G_o \times G_o)$ is connected; and $f(G_o \times G_o)$ contains 1. So G_o is a normal subgroup. □

Proposition 2.12. *If H is a connected subgroup of G and A is a connected subset of G/H, then $p^{-1}(A)$ is a connected subset of G, $p \colon G \to G/H$ being the projection.*

Proof. Suppose $p^{-1}(A) = P \cup Q$, where P, Q are disjoint and open in $p^{-1}(A)$. Since each orbit is connected (because H is), each of P, Q is a union of orbits, $P = p^{-1}(B), Q = p^{-1}(C)$ say. Then $A = B \cup C, B \cap C = \varnothing$ and B, C are open in $B \cup C = A$. So one of P, Q is empty. □

Corollary 2.13. *If H is a subgroup of the topological group G, then :-*

(1) If H and G/H are connected, then G is connected

(2) The only connected subsets of G/G_o are points (i.e., G/H_o is totally disconnected*).*

Proposition 2.14. *Let G be a connected topological group and U any open subset of G. Then U generates the abstract group G.*

Proof. Let H be the subgroup generated by U. Then H contains a neighbourhood (in G) of each $u \in U$ and hence contains a neighbourhood of each of its points. H is thus an open subgroup. By Proposition 2.2, G/H is discrete, and is connected, so G/H has only one point. □

Proposition 2.15. *let G be a connected topological group and D a discrete normal subgroup. Then D is contained in the centre of G.*

Proof. The map $x \mapsto x^{-1}dx$, for some $d \in G$, is a homeomorphism $G \to G$. If $d \in D$ then $\{d\}$ is a neighbourhood of d and so there exists a neighbourhood U of 1 such that $U^{-1}dU \subset \{d\}$, i.e., $x^{-1}dz = d$ for all $x \in U$. Using (2.14), we see that $y^{-1}dy = d$ for all $y \in G$. □

Proper Actions

For this section, the reader is referred to Bourbaki : "General Topology", part 1, Chapter I §10 and Chapter III §4.

A continuous map $f\colon X \to Y$ is *proper* if f is closed and $f^{-1}(y)$ is compact for each $y \in Y$.

An action $\varphi\colon X \times G \to X$ of a TTG is *proper* if

$$(1, \varphi)\colon X \times G \to X \times X, \quad (x, g) \mapsto (x, x.g)$$

is a proper map.

We will assume the following result (proved in Bourbaki Chapter I §10.1 and §10.2) :-

Axiom :

The composite of two proper maps is proper.

Proposition 2.16. *If a map $f\colon X \to Y$ is proper, then for any compact set $K \subset U$, $f^{-1}(K)$ is compact.*

Proof. The map $f_K\colon f^{-1}(K) \to K, f_K = f|f^{-1}(K)$, is proper (true for any set $K \subset Y$), and so is the map $K \to P$, where P is a one-point space (since K is compact). Hence the composite

$$f^{-1}(K) \xrightarrow{f_K} K \longrightarrow P$$

is proper, thus $f^{-1}(K)$ is compact. □

Theorem 2.17. *An action $\varphi\colon X \times G \to X$ of a TTG, where X is Hausdorff, is proper so φ is closed and all isotropy groups of X are compact.*

Proof. The closed condition on φ is obvious.

If φ is proper then the map

$$(1, \varphi)\colon X \times G \to X \times X, \quad (x, g) \mapsto (x, x.g),$$

is proper so

$$(1,\varphi)^{-1}(x,y) = \{(x,g) \in \{x\} \times G \colon x.g = y\} = G_{x,y}$$

is compact for all $(x,y) \in X \times X$. In particular $G_{x,x} = \{x\} \times G_x$ is compact for all $x \in X$, so G_x is compact for all $x \in X$. □

Example 2.18. If $\varphi\colon X \times G \to X$ is a proper action of a TTG, and $G = \mathbb{R}$ or $\mathbb{Z}$, then $\forall x \in X, G_x = \{1\}$.

Proposition 2.19. *Let G be a topological group acting on a Hausdorff space X and let $K \subset G$ be a compact set. Then $p\colon X \times K \to X, (x,s) \mapsto x.s$ is proper.*

Proof. p is the composite

$$X \times K \xrightarrow{\alpha} X \times K \xrightarrow{prd} X$$

where $\alpha(x,g) = (x.g, x)$ and α is a homeomorphism. As K is compact, the projection : $X \times K \to X$ is proper, and hence p is proper. □

Corollary 2.20. *With the notation of 2.19,*

(1) If A is a closed (compact) subset of X, then $A.K$ is a closed (compact) subset of X

(2) $p\colon X \to X/K$ is proper.

We also deduce :-

Theorem 2.21. *If G is a compact topological group acting on a Hausdorff space X, then the action is proper. Further, $p\colon X \to X/G$ is proper.*

Corollary 2.22. *If G is a compact topological group acting on a Hausdorff space X, then X/G is (locally) compact if and only if X is (locally) compact.*

Theorem 2.23. *If a topological group G acts properly on a space X, then X/G is Hausdorff. Further, X is Hausdorff.*

Proof. Let $\varphi\colon X \times G \to X$ be the action. If φ is proper, then $(1,\varphi)\colon X \times G \to X \times X$ is proper, and in particular closed. So the set

$$C = \{(x, x.g) \in X \times X \colon \forall x \in X, g \in G\}$$

is closed in $X \times X$. But $C = (p \times p)^{-1}(\Delta)$, where $p \times p\colon X \times X \to X/G \times X/G$ and Δ is the diagonal in X/G. Hence $\Delta(p \times p)(C)$ is closed (since X/G has the identification topology determined by p) thus X/G is Hausdorff.

Since G is T_1, the map $\theta\colon X \to X \times G, x \mapsto (x,1)$, is a homeomorphism onto a closed subset of $X \times G$ and therefore is proper. Composing θ with the map $X \times G \to X \times X, (x,g) \mapsto (x, x.g)$, which is proper by hypothesis, we get a proper map $X \to X \times X, x \mapsto (x,x)$ so $\Delta(X)$ is closed, thus X is Hausdorff. □

Proposition 2.24. *Let G be a compact topological group acting on a space X, and let I be a G-invariant subset of X. Then any neighbourhood of I contains a G-invariant neighbourhood of I.*

Proof. Let V be an open set containing I. Then $W = X \setminus p^{-1}(p(X \setminus V))$ is G-invariant and $W \subset V$, where $p\colon X \to X/G$ is the projection. By Theorem 2.21, p is proper so W is open; further, $I \subset W$. □

The following result will be very useful :-

Theorem 2.25. *Let G act properly on X, $\varphi\colon X \times G \to X$, then the following hold :-*

(1) Isotropy groups are compact.

(2) $\varphi_x\colon G \to X, g \mapsto x.g$, is a proper map, for each $x \in X$.

(3) Orbits are closed.

(4) The natural map $\theta\colon G_x\backslash G \to x.G$, $G_x g \mapsto x.g$ is a homeomorphism, for each $x \in X$.

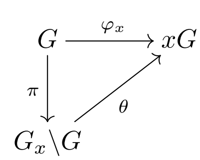
Proof. (1) Apply Theorem 2.23 to Theorem 2.17.

(2) If $y \in X, \varphi_x^{-1}(y) = \{g \in G\colon x.g = g\}$ - which was proved to be compact in the proof of Theorem 2.17.

By Theorem 2.23 X is Hausdorff. Hence, F is closed in G, thus $\{x\} \times F$ is closed in $X \times G$ hence $(x, \varphi_x(F)) = (i, \varphi)(x, F)$ is closed in $X \times X$ and so in $\{x\} \times X$. So $\varphi_x(F)$ is closed in X. Thus φ_x is proper.

(3) $\{x\} \times G$ is closed in $X \times G$ so $x.G$ is closed in X.

(4) We have the commutative diagram :-



θ has already shown to be a continuous bijection, so it is sufficient to show that it is closed. F is closed in $G_x\backslash G$ if and only if $\pi^{-1}(F)$ closed in G so $\varphi_x(\pi^{-1}(F))$ is closed in $x.G$ implies $\theta(F)$ is closed in $x.G$. □

Corollary 2.26. *If G acts properly on a compact space X, then X/G and G are compact.*

Proof. $p\colon X \to X/G$ is continuous so $p(X) = X/G$ is compact (this is true for any action). By part (2) to Theorem 2.25, $\varphi_x\colon G \to X$ is proper; by part (3), $x.G$ is closed in X and is therefore compact. $G = \varphi_x^{-1}(x.G)$ and is therefore compact. □

We now quote two useful results; proofs will be found in Bourbaki Chapter III.

Theorem 2.27. *Let G be a locally compact group acting on a Hausdorff space X. Then*
G acts properly on X.
$\Leftrightarrow$
for each pair of points $x, y \in X$, and neighbourhoods V_x of x, V_y of y such that $\{g \in G\colon V_y \cap V_x.g \neq \varnothing\}$ has compact closure (see §4.4 Prop. 7)
$\Leftrightarrow$
for all compact sets $K, L \subset X, \{g \in G\colon K.g \cap L \neq \varnothing\}$ has compact closure. (See §4.5 Theorem 1)

If G is a locally compact group acting on a Hausdorff space X, then $x \in X$ is a *wandering point* if it has a neighbourhood V_x such that $\{g \in G\colon V_x.g \cap V_x \neq \varnothing\}$ has a compact closure, or equivalently, if there exists a compact subset $K \subset G$ such that $g \notin K$ implies $V_x.g \cap V_x = \varnothing$.

It follows that the action is proper if and only if all points of X are wandering points. The set of all wandering points is clearly open.

An action $\varphi\colon X \times G \to X$ is called a *principal bundle* if φ is free and proper. If $\varphi\colon X \times G \to X$ is a proper action and G is a discrete group, φ is said to be *properly discontinuous* (thus isotropy groups are finite).

3 G-Vector Bundles

Let V be a real (or complex), finite dimensional vector space, and let $\psi\colon V \times G \to V$be a TTG such that $\forall g \in G$, the map $g\colon V \to V, v \mapsto v.g$ is linear (and hence a linear isomorphism). The action ψ is called a *linear action* and V is called a *representation space of* G *(or* G*-module).*

In the case $\psi\colon V \times G \to V$ is a DTG, ψ is called a *smooth linear action* and V is called a *smooth representation space of* G.

Let G be a topological group. A G*-vector bundle* is a real (or complex) vector bundle $p\colon E \to X$ with finite dimensional fibre, together with TTG's $\varphi\colon X \times G \to X$, $\psi\colon E \times G \to E$, such that

(1) Then following diagram is commutative

$$\begin{array}{ccc} E \times G & \xrightarrow{\psi} & E \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle p} \\ X \times G & \xrightarrow[\varphi]{} & X \end{array}$$

i.e., $\varphi(p(e), g) = p\psi(e, g)$ for all $e \in E, g \in G$ which is written, $p(e).g = p(e.g)$. p is thus a morphism of G-spaces (or an *equivariant map*).

(2) The induced action of ψ on each fibre is linear, i.e., given $x \in X, v, w \in p^{-1}(x)$, then $(\lambda v + \mu w).g = \lambda(v.g) + \mu(w.g)$, for all $g \in G, \lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

Example 3.1. Let $\varphi\colon M \times G \to M$ be a DTG, then we have a canonical action $d\varphi\colon TM \times G \to TM$ given by $((x, v), g) \mapsto (x.g, dgx(v))$. It is easy to see that the projection $\pi\colon TM \to M$ together with the DTG's $\varphi\colon M \times G \to M$ and $d\varphi\colon TM \times G \to TM$ is a smooth G-vector bundle.

Let $p\colon E \to X$ be a vector bundle. Denote by $E \times_X E$ the set

$$\{(v, w) \in E \times E\colon p(v) = p(w)\}.$$

In the case X is a topological space, $E \times_X E$ is a subspace of $E \times E$; in the case X is a manifold, $E \times_X E$ is a submanifold of $E \times E$ (see for instance Lang : Introduction to Differentiable Manifolds).

A vector bundle $p\colon E \to X$, with finite dimensional fibre, is said to have *Riemann structure* if there exists a continuous map

$$<,>\colon E \times_X E \to \mathbb{R}, \quad (v, w) \mapsto < v, w >,$$

such that for each $x \in X$,

$$<,> |p^{-1}(x) \times p^{-1}(x)\colon p^{-1}(x) \times p^{-1}(x) \to \mathbb{R}$$

is an inner product; the Riemann structure is said to be *smooth* if $<,>\colon E \times_X E \to \mathbb{R}$ is smooth. Note that if X is paracompact (which is the case if X is a manifold) then any vector bundle $\varphi\colon E \to X$ has a Riemann structure (see Husemoller "Fibre Bundles").

One object of this section is to prove that if $p\colon E \to X$ is a G-vector bundle with Riemann structure, where G is compact, then there exists a G-invariant Riemann structure on the bundle; that is, there exists a Riemann structure $<,>\colon E \times_X E \to \mathbb{R}$ such that

$$\forall (v, w) \in E \times_X E \quad \forall f \in G, < v, w >=< v.g, w.g > .$$

We need the following result (for the proof see Hewitt and Ross "Abstract harmonic Analysis I" and Chevalley "Theory of Lie Groups I", Chapter V.)

Axiom

(a) If G is a compact topological group, then there exists a linear map $\int_G\colon C(G, \mathbb{R}) \to \mathbb{R}$ (where $C(G, \mathbb{R})$ is the space of continuous maps $G \to \mathbb{R}$) such that :-

  (I1) If $f\colon G \to \mathbb{R}$ is non-negative, then $\int_G f > 0$
  (I2) If $f\colon G \to \mathbb{R}$ is non-negative and not identically zero, then $\int_G f > 0$
  (I3) If $f\colon G \to \mathbb{R}$ is identically 1, then $\int_G f = 1$
  (I4) For any $g \in G, f \in C(G, \mathbb{R})$, $\int_G f \circ p_g = \int_G f = \int f \circ \lambda_g$, where $p_g\colon G \to G, \lambda_g\colon G \to G$ are the right, left translation of G by g.
  (I5) For any $f \in C(G, \mathbb{R}), \int_G f = \int_G f \circ i$, where $i\colon G \to G, g \mapsto g^{-1}$

(b) Further, if G is a Lie group, then $\int_G\colon C(G, \mathbb{R}) \to \mathbb{R}$ is the usual integral defined on compact, oriented manifolds (recall that all Lie groups are oriented).

Proposition 3.2. *Let G be a compact group. If $f\colon M \times G \to IR$ is continuous (resp. smooth), then*

$$F\colon M \to IR, \quad x \mapsto \int_G f(x, g)dg$$

is continuous (resp. smooth).

Proof. (a) The continuous case. For each $(x, g) \in M \times G$ and $\varepsilon > 0$, by the continuity of f there exist neighbourhoods $V_{x,g}$ of $x \in X$ and $U_{x,g}$ of $g \in G$ such that

$$(y, h) \in V_{x,g} \times U_{x,g} \Rightarrow |f(y, h) - f(x, g)| < \varepsilon.$$

For a fixed $x \in X$, $\{u_{x,g}\colon g \in G\}$ is a cover for G, so there exists a finite subcover $U_{x,g_1}, \cdots, U_{x,g_k}$; put $V_x = \cap_{i=1}^k V_{x,g_i}$. Then

$$y \in V_x \Rightarrow |f(y, g) - f(x, g)| < \varepsilon \quad \forall g \in G.$$

Denote by $f_y\colon G \to \mathbb{R}$, the map $g \mapsto f(y, g)$ and let $\|\|$ be the sup norm on $C(G, \mathbb{R})$. We have :-

$$y \in V_x \Rightarrow |f_y(g) - f_x(g)| < \varepsilon, \quad \forall g \in G \Rightarrow \|f_y - f_x\| < \varepsilon$$
$$\Rightarrow \left|\int_G (f_y(g) - f_x(g))dg\right| \leqq \int_G \|f_y - f_x\|dg < \int_G \varepsilon dg = \varepsilon \int_G 1dg = \varepsilon$$

i.e.,

$$y \in V_x \Rightarrow |\int_G f(y,g)dg - \int_G f(x,g)dg| < \varepsilon,$$

i.e., F is continuous.
(b) The smooth case. We only have to prove the result in the case $M = U$, an open set in some Euclidean space. If (V, ψ) is a chart of G and $\beta\colon U \times V \to \mathbb{R}$ is a smooth map, then the map

$$B\colon U \to \mathbb{R}, \quad u \mapsto \int_V \beta(u,g)dg = \int_{\psi(V)} \beta(u, \psi^{-1}(x))dx,$$

the usual Lebesgue integral, is smooth: for the proof see Dieudonné P. 172 (Leibnitz's rule).

Let

$$\{(v_i, \varphi_i)\colon i = 1, \ldots, p\}$$

be a finite collection of charts of G such that the V_i covers G, and let

$$\{(\psi_i)\colon i = 1, \ldots, p\}$$

be an associated smooth partitions of unity. Then maps

$$U \times V_i \to \mathbb{R}, \quad (x,g) \mapsto \psi_i(g) f(x,g)$$

for $i = 1, \ldots, p$, are smooth, and by the result above so are the maps

$$U \to \mathbb{R}, \quad x \mapsto \int_{V_i} \psi_i(g) f(x,g)dg$$

for $i = 1, \ldots, p$ so the map

$$U \to \mathbb{R}, \quad x \mapsto \sum_{i=1}^{p} \int_{V_i} \psi(g) f(x,g)dg$$

is smooth. But

$$\sum_{i=1}^{p} \int_{V_i} \psi_i(g) f(x,g)dg = \sum_{i=1}^{p} \int_G \psi_i(g) f(x,g)dg =$$
$$\int_G \sum_{i=1}^{p} \psi_i(g) f(x,g)dg = \int_G f(x,g)dg = F(x).$$

Hence $f\colon M \to \mathbb{R}, \quad x \mapsto \int_G f(x,g)dg$ is smooth. $\square$

Note that the group properties of G were not used in the proof of this theorem.

The existence of partitions of unity follows from the paracompactness of manifolds (See Lang.)

Proposition 3.3. *Let $V \times G \to V$ be a continuous (resp. smooth) linear action, where G is compact and V is a finite dimensional real vector space. Then there exists a continuous (resp. smooth) G-invariant inner product on V.*

Proof. Let $<,>^*$ be an inner product on V. We have a continuous (resp. smooth) map

$$V \times V \times G \to \mathbb{R}, \quad (v, w, g) \mapsto < v.g, W.g >^*;$$

it follows from Proposition 3.2 therefore that the map

$$<,>: V \times V \to \mathbb{R}, \quad \text{given by} \quad (v, w) \mapsto \int_G < v.g, w.g >^* \, dg$$

is continuous (resp. smooth).

We show that $<,>$ is an inner product. The bilinearity of $<,>$ follows from that of $<,>^*$ and the linearity of the integral; $<,>$ is symmetric since $<,>^*$ is. For any $v \in V$,

$$\begin{aligned} < v, v >> 0 &\Leftrightarrow \int_G < v.g, v.g >^* \, dg > 0 \\ &\Leftrightarrow < v.g, v.g >^* > 0 \quad \text{for some} \quad g \in G \quad \Leftrightarrow v.g \neq 0 \Leftrightarrow v \neq 0. \end{aligned}$$

So $<,>$ is an inner product. It is G-invariant since for any $v, w \in V, k \in G$,

$$\begin{aligned} < v.k, w.k > &= \int_G < v.kg, w.kg > dg = \int_G < v.h, w.h >^* \, k^{-1} dh, \quad \text{where} \quad h = kg \\ &= \int_G < v.h, w.h >^* \, dh = < v, w > . \end{aligned}$$

□

Theorem 3.4. *Let $p \colon E \to X$ be a (smooth) G-vector bundle, where G is compact, with a (smooth) Riemann structure; then there exists a (smooth) G-invariant Riemann structure on $p \colon E \to X$.*

Proof. Let the given Riemann structure be $<,>^* \colon E \times_X E \to \mathbb{R}$. We have a continuous (resp. smooth) map

$$E \times_X E \times G \to \mathbb{R}, \quad (v, w, g) \mapsto < v.g, w.g >^* .$$

By Proposition 3.2, the map

$$<,> \colon E \times_X E \to \mathbb{R}, \quad (v, w) \mapsto \int_G < v.g, w.g >^* \, dg,$$

is continuous (resp. smooth); by Proposition 3.3, for any $x \in X$,

$$<,> |p^{-1}(x) \times p^{-1}(x)$$

is a G-invariant inner product. □

Remark 3.5. Let $\varphi\colon M \times G \to M$ be a DTG, where G is compact. Then the canonical projection $\pi\colon TM \to M$ together with the DTG's

$$\varphi\colon M \times G \to M, \quad d\varphi\colon TM \times G \to TM,$$

is a G-vector bundle; by Theorem 3.4, we can give the bundle $\pi TM \to M$ a G-invariant Riemann structure. It follows that for any $g \in G$, the map $g\colon M \to M$ is such that for any $x \in M$, $dg_x\colon N_x \to M_{x.g}$ preserves the Riemann structure. Since the exponential map depends only on the metric, we have for all $x \in M, g \in G$, the commutative diagram,

$$\begin{array}{ccc} M_x & \xrightarrow{dg_x} & M_{x.g} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ M & \xrightarrow[g]{} & M \end{array}$$

Using this result we can deduce,

Theorem 3.6. *Let $\varphi\colon M \times G \to M$ be a DTG, where G is compact, and let $p \in M$ be a stationary point (i.e., $p.g = p$ for all $g \in G$). Then there exists a (G-invariant) neighbourhood U of p in M such that U is isomorphic as a G-space to an open set in a linear representation space of G.*

Proof. Give M a G-invariant metric. By Proposition 2.24 there exists a G-invariant neighbourhood U of p, so $\varphi|U \times G\colon U \times G \to U$ is a DTG.

Assume U is small enough such that U is the diffeomorphic image of a dice V, centre 0, in M_P under exp. The action of $d\varphi\colon TM \times G \to TM$ restricted to $M_P \times G$ is such that $((p, v), g) \mapsto (p, dgp(v))$, since p is a stationary point. M has a G-invariant metric, so $d\varphi|M_P \times G$ sends $V \times G$ onto V; we then have the commutative diagram,

$$\begin{array}{ccc} V \times G & \xrightarrow{d\varphi} & V \\ {\scriptstyle\exp\times 1}\downarrow{\scriptstyle\approx} & & {\scriptstyle\approx}\downarrow{\scriptstyle\exp} \\ U \times G & \xrightarrow[\varphi]{} & U \end{array}$$

$d\varphi|M_P \times G\colon M_P \times G \to M_P$ is a linear section and thus M_P is a linear representation space of G; the above commutative diagram assures the result. $\square$

4 Local Triviality

First we state the Rank Theorem; proofs can be found in Dieudonné 'Foundations of Modern Analysis', and in Flett 'Modern Analysis'.

Theorem 4.1 (The Rank Theorem (Vector Spaces))**.** *Let E be an m-dimensional, and F an n-dimensional, real vector space, A an open neighbourhood of $a \in E$, and $f\colon E \to F$ a C^q-map such that for all $x \in A$, df_x has rank p, for some fixed integer $p > 0$. Then there exist,*

(1) An open neighbourhood $U \subset A$ of a and a C^q-diffeomorphism $u\colon U \to I^m$ (the m-cube).

(2) An open neighbourhood $V \supset f(A)$ of $f(a)$ and a C^q-diffeomorphism $v\colon I^n \to V$, such that $f|U = v \circ i \circ u$, where $i\colon I^m \to I^n$ is the map $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_p, 0, \ldots, 0)$.

From 4.1 we deduce :-

Theorem 4.2 (The Rank Theorem (Smooth Manifolds))**.** *Let M be a smooth m-manifold and N a smooth n-manifold, A an open neighbourhood of $a \in M$, and $f\colon M \to N$ a smooth map such that for all $x \in A$, df_x has rank p, for some fixed integer $p > 0$. Then there exists a neighbourhood W of 0 in M_a and diffeomorphisms*

$$u^*\colon W \approx u^*(W) \subset M$$
$$v^*\colon fu^*(W) \approx df_a(W)$$

where $u^*(W)$ is a neighbourhood of a and $df_a|W = v^* f u^*|W$.

Proof. We can assume A is such that (A, φ) is a chart of M around a, for some φ, so that $f(A) \subset B$, where (B, ψ) is a chart of N around $f(a)$ for some ψ. We have the commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \varphi\downarrow & & \downarrow\psi \\ \varphi(A) & \xrightarrow[\tilde{f}=\psi f \varphi^{-1}]{} & \psi(B) \end{array}$$

and we can apply 4.1 to $\tilde{f}$ on $\varphi(A)$.

Hence there exists an open neighbourhood $U' \subset \varphi(A)$ of $\varphi(a)$ and open neighbourhood $V' \supset \tilde{f}\varphi(A)$ of $f\varphi(a)$, and diffeomorphisms $u'\colon U' \to I^m$, $v'\colon I^n \to V'$, such that $\tilde{f}|U' = v' \circ i \circ u'$, where

$$i\colon I^m \to I^n, \quad (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_p, 0, \ldots, 0)$$

It follows the there are open neighbourhoods $U \subset A$ of a and $V \subset B$ of $f(a)$ such that the following commutes,

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle u}\downarrow\approx & & \approx\uparrow{\scriptstyle v} \\ I^m & \xrightarrow[i]{} & I^n \end{array}$$

where u, v are diffeomorphisms. Hence, $df_a = dv(iu(a)) \circ i \circ du_a$, where

$$\begin{aligned} du_a &\colon M_a \approx \mathbb{R}^m \\ i &\colon \mathbb{R}^m \to \mathbb{R}^n \\ dv(iu(a)) &\colon \mathbb{R}^n \to N_{f(a)} \end{aligned}$$

so

$$df_a = dv(iu(a)) \circ v^{-1} \circ (f|U) \circ u^{-1},$$

on the open set $(du_a)^{-1})(I^n)$, i.e.,

$$df_a|W = v^* fu{*}|W,$$

where W is an open neighbourhood of 0 in M_a and

$$\begin{aligned} &u{*}\colon W \approx u{*}(W) \quad \text{a neighbourhood of } a \text{ in } M \\ &u{*}\colon fu^*(W) \approx df_a(W) \end{aligned}$$

□

Let $f\colon X \to X$ be a continuous map. We say f is *locally trivial* if for each $x \in X$ there exists a set $U \subset X$ containing x such that $f|U\colon U \to f(U)$ is a homeomorphism onto a neighbourhood of $f(x)$ in Y. U is called a *local cross section* at x.

Theorem 4.3. *Let $\varphi\colon M \times G$ be a DTG and let $x \in M$ have trivial isotropy group (i.e., $G_x = \{1\}$). Then the map*

$$\varphi_x\colon g \to M, \quad g \mapsto x.g$$

is an immersion.

Proof. Let $p_g\colon G \to G$ denote the right translation of G by g, i.e., $p_g(h) = hg$ for all $h \in G$, and recall that we denote by $g\colon N \to M$ the diffeomorphism given by $x \mapsto x.g$ for all $x \in M$. We have a commutative diagram,

$$\begin{array}{ccc} 1 & \overset{p_g}{\rightsquigarrow} & g \\ {\scriptstyle \varphi_x}\wr & & \wr{\scriptstyle \varphi_x} \\ x & \underset{g}{\rightsquigarrow} & x.g \end{array}$$

which gives rise to the commutative diagram,

$$\begin{array}{ccc} G_1 & \xrightarrow[\approx]{dp_g^1} & G_g \\ {\scriptstyle d\varphi_x 1}\downarrow & & \downarrow{\scriptstyle d\varphi_x g} \\ M_x & \xrightarrow[dg_x]{\approx} & M_{x.g} \end{array}$$

so, rank of $d\varphi_x g$ is the same for all $g \in G$. □

Corollary 4.4. *Let G be a Lie group acting smoothly and properly on the smooth manifold M, and let $x \in M$ have trivial isotropy group. Then the map*

$$\varphi_x \colon G \to M, \quad g \mapsto x.g$$

is an imbedding.

Proof. Since the action of G on M is proper, $\varphi_x \colon G \to M$ is proper by Theorem 2.25; in particular φ_x is closed. By Theorem 4.3, φ_x is an injective immersion. Hence by the rank theorem, φ_x is locally equivalent to $d\varphi_x^1$ (i.e., there exist suitable diffeomorphisms u^*, v^* such that $\varphi_x = d\varphi_x^1 \circ v^*$ on a suitable neighbourhood). But φ_x is injective and hence so is $d\varphi_x^1 g$ for all $g \in G$, thus φ_x is an immersion. □

We now come to the main result of this section :-

Theorem 4.5. *Let $\varphi \colon M \times G \to M$ be a DTG, where the action of φ is free and proper. Then $p \colon M \to M/G$, the canonical projection, is locally trivial.*

Proof. For each $x \in M$, we have an immersion $\varphi_x \colon G \to M$ (by Theorem 4.3), and so $d\varphi_x^1 : G_1 \to M_x$ is injective. (In fact, φ_x is an imbedding by Corollary 4.4). Thus $M_x = T_x \oplus 1(G_1)$, where T_x is the orthogonal compliment of $d\varphi_x^1(G_1)$ in M_x; note that $d\varphi_x^1(G_1)$ is the tangent space of the orbit through x, at x. Choose a disc (i.e., a closed ball) $D \subset T_x$ such that D is a neighbourhood of 0 in T_x and such that D is so small to be mapped diffeomorphically by exp into a chart neighbourhood of x; thus we can assume $\exp D$ is in a Euclidean space. Note that $x \in \exp D$.

We claim that there exists a disc $D^* \subset D$ (D^* is a neighbourhood of 0 in T_x) such that $p|\exp D^*$ maps $\exp D^*$ homeomorphically onto a neighbourhood of $p(x)$ in M/G, which will prove the theorem. As the action is proper, M/G is Hausdorff by Theorem 2.23. Hence since D^* is compact, $p|\exp D^*$ is closed, and it is therefore sufficient to prove :-

(1) $p|\exp D^*$ is injective

(2) $p(\exp D^*)$ is a neighbourhood of $p(x)$ in M/G.

(1) We shall show that there exists a disc $D^* \subset D$ such that $\varphi|(\exp D^*) \times G$ is injective, from which it follows that

$$x_1.G = x_2.G \Rightarrow x_1 = x_2 \quad \text{for all} \quad x_1, x_2 \in \exp D^*,$$

and so that

$$p(x_1) = p(x_2) \Rightarrow x_1 = x_2 \quad \text{for all} \quad x_1, x_2 \in \exp D^*.$$

That is, we shall show that there exists a disc $D^* \subset D$ such that orbits which intersect $\exp D^*$ only intersect at single points, i.e., that

$$y.g = z, \quad y, z \in \exp D^*, g \in G \Rightarrow y = z, g = 1.$$

By Theorem 2.27, the set

$$K = \overline{\{g \in G\colon\ \exp D \cap (\exp D)g \neq \varnothing\}}$$

is compact; so we know that

$$y.g = z, \quad y, z \in \exp D^* \Rightarrow g \in K.$$

The set

$$J = \{(y, z, g) \in \exp D \times \exp D \times K\colon y.g = z\}$$

is a closed (and therefore compact) subset of $\exp D \times \exp D \times K$ - since it is the inverse image of the diagonal in $\exp D \times \exp D$ under the map

$$\exp D \times \exp D \times K \to \exp D \times \exp D, \quad (y, z, g) \mapsto (y.g, z).$$

Further J does not meet $\{x\} \times \{x\} \times (K \setminus \{1\})$; by the Hausdorff property of M, it follows that J is disjoint from some neighbourhood W of $\{x\} \times \{x\} \times (K \setminus \{1\})$ in $\exp D \times \exp D \times (K \setminus \{1\})$ (and therefore in $M \times M \times K$). We can assume $W = \exp D^* \times \exp D^* \times L$, where $D^* \subset D$ is small enough disc. Then $\varphi| \exp D^* \times G$ is injective.
(2) Since p is an open map (Proposition 1.3), it is sufficient to prove $(\exp D^*)G$ is a neighbourhood of x in M. By taking partial differentials, we have that

$$d\varphi(x, 1)\colon M_x \times G_1 \to M_x$$

is given by

$$d\varphi(x, 1)(a, v) = a + d\varphi_x^1(v).$$

So the restriction

$$d\varphi(x, 1)|D^* \times G_1\colon D^* \times G_1 \to D^* \oplus d\varphi_x^1(G_1)$$

is an isomorphism, since $d\varphi_x^1$ is injective and D^* lies in the orthogonal compliment of $d\varphi_x^1(G_1)$. As D^* lies in a neighbourhood of 0 in T_x, $D^* \oplus d\varphi_1^1(G_1)$ is a neighbourhood of 0 in

$$T_x \oplus d\varphi_1^1(G_1) = M_x.$$

So the rank of $d\varphi(x,1)|D^* \times G_1$ is $\dim M$.

If $d\varphi(x,1)|D^* \times G_1$ then $d\varphi(y,g)|D^* \times G_1$ has maximal rank, for (y,g) in some neighbourhood N of $(x,1)$ in $M \times G$, and since D^* is compact, we can assume $(\exp D^* \times U) \subset N$, where U is some neighbourhood of 1 in G. By the Rank Theorem, it follows that if $\varphi(\exp D^* \times U)$ is a neighbourhood of x, then $(\exp D^*)G$ is a neighbourhood of x. □

Corollary 4.6. *If G is a Lie group and H a closed Lie subgroup, then the canonical projection $p\colon g \to G/H$ is locally trivial.*

Proof. The action

$$G \times H \to G, \quad (g,h) \mapsto gh$$

is smooth and free. We show it is proper: the result then follows from Theorem 4.5. The map

$$(i,\varphi)\colon G \times G \to G \times G, \quad (g_1,g_2) \mapsto (g_1, g_1g_2)$$

is a homeomorphism and is therefore proper. As H is closed in G, $G \times H$ is closed in $G \times G$. So

$$(i,\varphi)|G \times H\colon G \times H \to G \times G$$

is also proper, which holds if and only if the action

$$G \times H\colon \to G \times G, \quad (g,h) \mapsto gh$$

is proper (by definition). □

N.B. If U is a local cross section of $p\colon G \to G/H$ at 1, then $H \cap U = \{1\}$.

Remark 4.7 (G/H has a G-manifold structure). First, we put a manifold structure on G/H. For each $x \in G$, there exists a set U_x, containing x such that $p|U_x\colon U_x \to p(U_x)$ is a homeomorphism onto a neighbourhood of $p(x)$. Let (V,θ) be a chart of G/H around x; then $(p(V) \cap p(U_x), t\theta q_x)$ is defined to be a chart of G/H around $p(x)$ where $q_x\colon p(u_x) \to U_x$ is $(p|v_x)^{-1}$ and t is a "straightening" map from the relevant Euclidean space to itself (recall that p is an open map). Clearly the set of all such charts forms a smooth atlas for G/H.

Finally, we have the commutative diagram,

$$\begin{array}{ccc} G \times G & \xrightarrow{\alpha} & G \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle p} \\ G/H \times G & \xrightarrow[\tilde{\alpha}]{} & G/H \end{array}$$

where $\alpha(g_1,g_2) = g_2g_1$ and $\tilde{\alpha}(g_1H, g_2) = g_2g_1H$.

The manifold structure defined on G/H dictates that p be smooth, and in fact, is a local diffeomorphism, so $\tilde{\alpha}$ is smooth; and we have already shown in Section 1 that $\tilde{\alpha}\colon G/H \times G \to G/H$ is a transformation group.
So G/H is a G-manifold.

Theorem 4.8. *Let $\varphi\colon M \times G \to M$ be a DTG and let $x \in M$ have (closed) isotropy group H. Then the map*

$$\theta\colon H\backslash G \to M, \quad Hg \mapsto x.g$$

is an immersion.

Proof. H being a closed subgroup, is thus a Lie subgroup (see C. Chevalley 'Theory of Lie Groups I', pp 130-135). Hence $p\colon G \to H\backslash G$ is a local diffeomorphism. We have the commutative diagram,

$$\begin{array}{ccc} G & \xrightarrow{p} & H\backslash G \\ & \searrow^{\varphi_x} & \downarrow{\scriptstyle\theta} \\ & & M \end{array}$$

where p is a local diffeomorphism and φ_x is smooth. So θ is smooth.

Let $\rho_{[g]}\colon H\backslash G \to H\backslash G$ be the right translation of $H\backslash G$ by $[g] = \rho(g)$. Then $\rho_{[g]}[h] = [hg].\rho_{[g]}$ is a diffeomorphism by the remark above. We have he commutative diagram,

$$\begin{array}{ccc} H\backslash G_{[1]} & \xrightarrow[\approx]{d\rho^{[1]}_{[g]}} & H\backslash G_{[g]} \\ {\scriptstyle d\theta[1]}\downarrow & & \downarrow{\scriptstyle d\theta[g]} \\ M_x & \xrightarrow[dg_x]{\approx} & M_{x.g} \end{array}$$

Hence the rank of $d\theta[g]$ is the same for all $[g] \in H\backslash G$. By the rank theorem, $d\theta[g]$ is locally equivalent to θ, for all $[g] \in H\backslash G$. As θ is injective, so is $d\theta[g]$ for all $[g] \in H\backslash G$, so θ is an immersion. □

Corollary 4.9. *Let $\varphi\colon M \times G \to M$ be a DTG, where the section of φ is proper, and let $x \in M$ have isotropy group H, Then*

$$\theta\colon H\backslash G \to M, \quad Hg \mapsto x.g,$$

is an imbedding.

5 Slices

Let $\varphi\colon M \times G \to M$ be a DTG, and H a closed subgroup of G. A *(smooth) H-slice in M* is a subset S of M such that

(1) S is invariant under H.

(2) $Sg \cap S \neq \varnothing \Rightarrow g \in H$.

(3) If U is a local cross-section at 1 of the projection $p\colon G \to H\backslash G$, then $\varphi|S \times U\colon S \times U \to M$ is a diffeomorphism onto some neighbourhood in M.

In the case of a TTG, the concept of an H-slice can be defined analogously - "diffeomorphism" in (3) being replaced by "homeomorphism".

Note: By (2), $s \in S \Rightarrow G_B \subset H$. Further, if $x \in M$, then a *slice at x* is a G_x-slice S which contains x, and such that, with notation of (3), $\varphi|S \times U \to M$ is a diffeomorphism onto a neighbourhood of x in M ($\varphi(S \times U)$ automatically contains x).

Lemma 5.1. *Let $f\colon V \times Z \to V$ be a continuous map, where V is a metric space and Z a topological space. Let K be a compact subset of Z and let $v \in V$; then given $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\operatorname{dist}(v, w) < \delta \Rightarrow \operatorname{dist}(f(v, z), f(w, z)) < \varepsilon \quad \textit{for all} \quad z \in K.$$

Proof. By the continuity of f, there exist a $\delta(x) > 0$ and an open neighbourhood N_z of z such that

$$f\colon B(u, \delta(z)) \times N_z \to B(f(v, z), \varepsilon/2),$$

where $B(a, \lambda)$ denotes the open λ-disc centred at a. Note that if $z' \in N_{z'}$, then

$$\operatorname{dist}(v, w) < \delta(z) \Rightarrow \operatorname{dist}(f(v, z'), f(w, z^1)) < \varepsilon.$$

The open sets N_z form a cover for K and K is compact, so there exists a finite subcover $\{N_{z_1}, \cdots, N_{z_k}\}$ for K. Put

$$\delta = \min\{\delta(z_i)\colon i = 1, \ldots, k\}.$$

Then for all $z' \in K$,

$$\operatorname{dist}(v, w) < \delta \Rightarrow \operatorname{dist}(f(v, z'), f(w, z^1)) < \varepsilon$$

□

Theorem 5.2 (Existence of Slices)**.** *Let $\varphi\colon M \times F \to M$ be a DTG, where the section of φ is proper. Then there exists a smooth slice at each point of M.*

Proof. Take $x \in M$ and let x have isotropy group H. The action of φ is proper, so H is compact. Thus we can give M an H-invariant metric. By Theorem 4.8, the map

$$\theta\colon H\backslash G \to M, \quad Hg \mapsto x.g,$$

is an immersion (it is in fact an imbedding by Corollary 4.9) so

$$d\theta 1\colon (H\backslash G)_1 \to M_x$$

is injective. Note that $d\theta 1(H\backslash G)_1$ is the tangent space to the orbit of x at x. Let T_x be the orthogonal subspace of $d\theta 1(H\backslash G)_1$ in M_x and take a disc D in T_x such that D is a neighbourhood of 0 in T_x and such that D is so small that it is mapped diffeomorphically by exp. Put $S = \exp D$. We shall show that by restricting D if necessary, S is a slice at x, that is the conditions (1), (2) and (3) are satisfied. As in the proof of Theorem 4.5, we can assume S is contained in some chart neighbourhood of x, i.e., that S is in a Euclidean space.

(1) Since M has an H-invariant metric, for $h \in H$, we have the commutative diagram,

$$\begin{array}{ccc} M_x & \xrightarrow{dh_x} & M_x \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ M & \xrightarrow[h]{} & M \end{array}$$

where $M_x = T_x \oplus d\theta 1(H\backslash G)_1$.

Recall that the differential of exp is the identity map, from which it follows that exp maps $d\theta 1(H\backslash G)_1$ - the tangent space of $x.G$ at x - onto $x.G$. Hence, $x.G$ is invariant under h, so $d\theta 1(H\backslash G)_1$ is invariant under dh_x. Since M has an H-invariant metric, T_x is invariant under dh_x, thus $dh_x(D) = D$, which means $S = \exp \circ dh_x(D) = h(S) = S.h$. So S is H-invariant.

(2) We want to show that if $S.g \cap S \neq \varnothing$ then $g \in H$.

Let U be a local cross-section of $p\colon G \to H\backslash G$ at 1, such that $p(U)$ is open (which exists by Corollary 4.6). Then $p^{-1}p(U) = HU$, and since $p(U)$ is open in $H\backslash G$, HU is open in G (since $H\backslash G$ has the identification topology). HU is thus an open neighbourhood of H; and $G \setminus HU$ is closed. As φ is proper, $\varphi_x\colon G \to M$ is proper and hence closed. So $x.(G \setminus HU)$ is closed in M and further $x \notin x.(G \setminus HU)$.

The action of φ is proper, so there exists a neighbourhood V_x if x such that

$$K = \overline{\{g \in G\colon V_x g \cap V_x \neq \varnothing\}}$$

is compact (Theorem 2.27), and we can assume D is small enough such that $S \subset V_x$, (Note that $H \subset K$).

It follows from Lemma 5.1 that given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\operatorname{dist}(x, y) < \delta) \Rightarrow \operatorname{dist}(x.g, y.g) < \varepsilon, \quad \text{for all} \quad g \in K.$$

Let $\mathrm{dist}(x, x.(G\backslash HU)) = \varepsilon$, then by restricting D if necessary we can assume that the radius of S is $< \delta < \varepsilon/2$, where δ is such that

$$\mathrm{dist}(x, y) < \delta) \Rightarrow \mathrm{dist}(x.g, y.g) < \varepsilon/2, \quad \text{for all} \quad g \in K.$$

Suppose $S.g \cap S \neq \varnothing$, then there exist $s, t \in S$ such that $s.g = t$. Now

$$\begin{aligned} \mathrm{dist}(x, x.g) &< \mathrm{dist}(x, t) \quad \text{and} \\ \mathrm{dist}(t, x.g) &= \mathrm{dist}(x, t) + \mathrm{dist}(s.g, x.g) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $g \notin G \setminus HU$. i.e., $g \in HU$. So $g = hu$, for some $h \in H, u \in U$. We have that $t = s.g = sh.u = s^1.u$, where $sh = s^1 \in S$ - since S is H-invariant.

If $u \neq 1$ (then $u \notin H$ since $U \cap H = \{1\}$ - see N.B. to Corollary 4.6), then $x.u \neq x$ and so there exists a neighbourhood W of x such that $W.u \cap W = \varnothing$. Restricting D if necessary, we can assume $S \subset W$ implies $S.u \cap S = \varnothing$. Hence

$$t = s^1.u, \quad t, s^1 \in S, u \in U \Rightarrow u = 1.$$

So $g = h$.

(3) We want to show that if U is a local cross-section of $p\colon G \to H\backslash G$ at 1, then $\varphi|S \times U\colon S \times U \to S.U$ is a diffeomorphism onto a neighbourhood of x in M.

(a) $\varphi|S \times U\colon S \times U \to S.U$ is a homeomorphism. Since φ is continuous and closed (since it is proper) we only have to show that it is injective. Suppose $s.u = t.v$, where $s, t \in S$, $u, v \in V$, then $uv^{-1} \in H$ (by (2)), $uv^{-1} = h$ say. So $u = hv$ and $p(u) = p(hv) = p(v)$ if and only if $u = v$ since $p|U\colon U \to p(U)$ is a homeomorphism, so $s = t$.

(b) $\varphi|S \times U\colon S \times U \to S.U$ is a homeomorphism onto a neighbourhood of x. Let $\tilde{\varphi}\colon S \times p(U) \to S.U$ be the map defined by $\tilde{\varphi}(s, H.u) = s.u$ (which is well-define since $U \cap H = \{1\}$), so we have the commutative diagram,

$$\begin{array}{ccc} S \times p(U) & \xrightarrow{\tilde{\varphi}} & S.U \\ {\scriptstyle 1\times p|U}\uparrow & \nearrow{\scriptstyle \varphi} & \\ S \times U & & \end{array}$$

As $\varphi|S \times U$ is injective, so is $\tilde{\varphi}$. Since φ is smooth and $p|U\colon U \to p(U)$ is a diffeomorphism, $\tilde{\varphi}$ is smooth.

Recall that the tangent space to S at x is T_x (since the differential of exp is the identity)

$$d\tilde{\varphi}(x, 1)\colon T_x \times p(U)_1 \to M_x$$

is given by $d\tilde{\varphi}(x, 1)(a, b) = a + d\tilde{\varphi}_x 1(b)$, where

$$\tilde{\varphi}_x\colon p(U) \to S.U \quad H.u \mapsto u.x$$

i.e., $\tilde{\varphi}_x = \theta|p(U)$.

As $P(U) \subset H\backslash G$, T_x lies in the orthogonal complement of $d\tilde{\varphi}_x(P(U)_1)$ so $d\tilde{\varphi}(x,1)$ is injective. Now $P(U)$ is a neighbourhood of 1 in $H\backslash G$, so $p(U)_1 = (H\backslash G)_1$ implies

$$d\theta(x,1)\colon T_x \times p(U)_1 \approx T_x \oplus d\tilde{\varphi}_x 1(p(U)_1) = T_x \oplus d\theta 1(H\backslash G)_1 = M_x.$$

So $d\tilde{\varphi}(x,1)$ is surjective, and is therefore an isomorphism. $d\tilde{\varphi}(y,Hg)$ is therefore an isomorphism for (y,Hg) in some neighbourhood N of $(x,1)$ in $S \times p(U)$. Thus by the Rank Theorem $\tilde{\varphi}(N)$ is a neighbourhood of x in M, hence $\tilde{\varphi}(S \times U) = S.U$ is a neighbourhood of x in M.

(c) $\varphi|S \times U \to S.U$ is an immersion. By (b), there exists a neighbourhood N of $(x,1)$ in $S \times p(U)$ such that $d\tilde{\varphi}(y,Hg)$ is an isomorphism for $(y,Hg) \in N$. As S is compact, we can take S so small such that $S \times \{1\} \subset N$.

We have the commutative diagram,

$$\begin{array}{ccc} S_s \times p(U)_{Hu} & \xrightarrow{\tilde{\varphi}(s,Hu)} & M_{su} \\ {\scriptstyle 1\times d\tilde{\rho}_u 1}\uparrow \approx & & \approx \uparrow {\scriptstyle dus} \\ S_z \times p(U)_1 & \xrightarrow[d\tilde{\varphi}(s,1)]{\approx} & M_s \end{array}$$

where

$$\tilde{\rho}_u\colon p(U) \to p(U), \quad Hv \mapsto Hvu, \quad \text{and} \quad u \in U.$$

Hence $d\tilde{\varphi}(s,Hu)$ is an isomorphism - and in particular injective - for all (s,Hu) in $S \times p(U)$. $\tilde{\varphi}$ is thus an immersion, so is φ (by the commutativity of diagram of (b)).

□

N.B.

(1) In constructing a slice at point $x \in M$, we have thrown away the original metric on M and replaced it by a G_x-invariant one. Thus the exponential map used in constructing the slice, will not necessarily be the same as the exponential map used to construct a slice at another point y when M has a G_y-invariant metric.

(2) The slice theorem shows in effect that points with isotropy groups of the same dimension (whose orbits have the same dimension and vice versa) are locally equivalent, in the sense that they have neighbourhoods diffeomorphic to $D \times U$, where D is a disc of dimension $= \dim M - \dim(\textit{orbit})$ and U is a local cross-section of $p\colon G \to H\backslash G$ at 1, of dimension $= \dim(\textit{orbit})$ (since $P|U\colon U \to p(U)$ is a homeomorphism onto a neighbourhood and $\theta\colon H\backslash G \approx x.G$).

A continuous map $f\colon X \to X'$ between G-spaces X, X' is *equivariant* if for all $x \in X, g \in G$, we have $f(x.g) = f(x).g$.

Theorem 5.3. *Let $f\colon X \to X'$ be a continuous (resp. smooth) equivariant map between G-spaces (resp. G-manifolds) X, X', and let S' be an H-slice (resp. smooth H-slice) in X'. Then $S = f^{-1}(S')$ is an H-slice (resp. smooth H-slice) in X.*

Proof. We verify that conditions (1) - (3) for slices, hold for S.
(1) If $s \in S, h \in H$ then $f(s.h) = f(s).h \in S'$. Indeed if S' is H-invariant, then $s.h \in S$, so s is H-invariant.
(2) If $s = t.g$ where $s, t \in S, g \in G$, then $f(s) = f(t).g$ implies $g \in H$ since $f(s), f(t) \in S'$. So $S.g \cap S \neq \varnothing$, thus $g \in H$.
(3)

(a) In the continuous case, all we have to show is that $\tilde{\varphi} = \varphi | S \times U\colon S \times U \to S.U$ is a homeomorphism onto a neighbourhood of X, where $\varphi\colon X \times G \to X$ is the action of G on X, and U is a local cross-section of $p\colon G \to H\backslash G$ at 1. Let $\varphi'\colon X' \times G \to X'$ be the action of G on X'.

Note that $S.U \subset f^{-1}(S'.U)$. We have the commutative diagram,

$$\begin{array}{ccc} S \times U & \xrightarrow{f \times 1} & S' \times U \\ {\scriptstyle \tilde{\varphi}} \downarrow & & \approx \downarrow {\scriptstyle \varphi' | S' \times U} \\ f^{-1}(S'.U) & \xrightarrow[f]{} & S'.U \end{array}$$

- the commutativity follows from the equivariance of f.

Clearly $\tilde{\varphi}$ is onto $f^{-1}(S'.U)$ i.e., $S'.U = f^{-1}(S'.U)$. $\tilde{\varphi}$ is also injective, since if $s_1.u_1 = s_2.u_2$ where $S_1, S_2 \in S, u_1, u_2 \in U$, then $s_1 = s_2.u_2u_1^{-1}$ if and only if $u_2u_1^{-1} = 1$ (since $U \cap H = \{1\}$) if and only if $u_1 = u_2$ and $s_1 = s_2$. Since $\tilde{\varphi}$ is continuous and closed, it is a homeomorphism. Now, $S'U$ is a neighbourhood in X', so is $f^{-1}(S'U)$ in X. Hence $f^{-1}(S'U) = S.U$ $\tilde{\varphi}\colon S \times U \to S.U$ is thus a homeomorphism onto a neighbourhood in X.

(b) In the differentiable case we have to show further that $\tilde{\varphi}$ is a diffeomorphism. As $\tilde{\varphi}$ is smooth, we show its inverse $\tilde{\varphi}^{-1}$ is smooth. Consider the two maps

$$\psi_1\colon S.U \to U, \quad s.u \mapsto u$$
$$\psi_2\colon S.U \to S, \quad s.u \mapsto s$$

where $s \in S$ and $u \in U$. So ψ_1 is the composite:

$$S.U \xrightarrow{f} S'U \quad \approx \quad S'U \xrightarrow{proj} U$$

where $s.u \mapsto f(s).u \mapsto (f(s), u) \mapsto u$, and is therefore smooth; and ψ_2 is the composite:

$$S.U \xrightarrow{1 \times \psi_1} S.U \times U \xrightarrow{1 \times inv.} S.U \times u^{-1} \xrightarrow{\varphi} S$$

where $s.u \mapsto (s.u, u) \mapsto (s.u, u^{-1}) \mapsto s$, and so ψ_2 is smooth. And $\tilde{\varphi}^{-1}\colon S.U \to S \times U$ is given by $s.u \mapsto (s.u)$, thus $s.u \mapsto (\psi_2(s.u), \psi_1(s.u))$, so $\tilde{\varphi}^{-1}$ is smooth.

□

Theorem 5.3 is the first step in extending the slice existence theorem to the topological case, but before we can continue we need the following Lemmas:-

Lemma 5.4. *Let G be a compact Lie group and H a closed subgroup. Then there exist a linear representation space, V of G and an element $v \in V$ such that the isotropy group of v, G_v, is equal to H.*

Proof. See Borel 'Seminar on Transformation Groups', Chap. VIII (by Palais), Prop. 2.2 □

Lemma 5.5. *Let $\varphi\colon X \times G \to X$ be a TTG, where G is compact, and let $x_0 \in X$. Then there exists an equivariant map of X into a linear representation space of G, which is injective on $x_0.G$.*

Proof. By Lemma 5.4 there exist a linear representation space V of G and an element $v \in V$ such that $G_v = G_{x_0}$; hence the map

$$f\colon x_0.G \to vG, \quad x_0.g \mapsto vg$$

is well-defined, continuous and equivariant. Since $x_0.G$ is a compact subset of X, and X is Hausdorff, we can apply Tietze's Extension Theorem to obtain a continuous extension $\tilde{f}\colon X \to V$ of f; note that $\tilde{f}$ is injective on $x.G$.

The map

$$G \to V, \quad g \mapsto \tilde{f}(x.g)g^{-1}$$

is continuous, since it the composite

$$g \mapsto ((x.g), g^{-1}) \mapsto (\tilde{f}(x.g)g^{-1}) \mapsto \tilde{f}(x.g)g^{-1}.$$

As G is compact (see Proposition 3.2) the map $F\colon X \to V$ is continuous, where

$$F(x) = \int_G \tilde{f}(x.g)g^{-1}dg.$$

We show that F is equivariant and is injective on $x_0.G$. Now

$$\begin{aligned} F(x.h) &= \int_G \tilde{f}(x.hg)g^{-1}dg = \int_G \tilde{f}(x.k)k^{-1}hdk, \quad \text{where} \quad k = hg \\ &= (\int_G \tilde{f}(x.k)k^{-1}dk)h = F(x)h, \quad \text{for all} \quad x \in X, h \in G. \\ F(x_0.h) &= \int_G \tilde{f}(x_0.hg)g^{-1}dg = \int_G f(x_0.hg)g^{-1}dg = \int_G f(x_0.h)dg \\ &= (\int_G f(x_0)dg).h = (\int_G vdg).h = v.h, \quad \text{for all} \quad h \in G. \end{aligned}$$

so F is injective on $x_0.G$, since $G_v = G_x$. □

We are now in a position to extend 5.2 to the case of topological transformation groups, where the group of the action is a compact Lie group.

Theorem 5.6 (Existence of Slices)**.** *Let $\varphi\colon X \times G \to X$ be a TTG, where G is a compact Lie group. Then there exists a slice at each point of X.*

Proof. Take $x \in X$. By Lemma 5.5 there exists an equivariant map $F\colon X \to V$, where V is a linear representation of G, such that F is injection on $x.G$ and that $G_x = G_{F(x)}$. V is a smooth G-space and so there exists a (smooth) $G_{F(x)}$-slice S' at $F(x)$. By Theorem 5.3, $F^{-1}(S')$ is a $G_{F(x)}$-slice at x; but $G_{F(x)} = G_x$. □

6 Orbit Types and Principal Orbits

Let G be a group. We define an equivalence relation on the subgroups of G by :-

$$H_1 \sim H_2 \Leftrightarrow \exists g \in G \quad \text{such that} \quad H_1 = g^{-1} H_2 g.$$

The equivalence classes of this relation (i.e., the conjugacy classes of the subgroups) are called *orbit types*.

Let $\varphi\colon X \times G \to X$ be a TTG. The *orbit type of a point* $x \in X$ is defined to be the orbit type of the isotropy group of x. Note that if $x \in X$ has isotropy group H, then $x.g \in x.G$ has isotropy group $g^{-1}Hg$; so the points on a particular orbit have the same orbit type. The *(orbit) type of an orbit in* X is the orbit type of any point that lies on the orbit.

Two orbits, of perhaps different G-spaces, are called *equivalent* if there exists an equivariant homeomorphism (i.e., a continuous equivariant map with continuous inverse) mapping one orbit to the other.

Proposition 6.1. *Two orbits have the same type if and only if they are equivalent.*

Proof. Let Γ be an orbit in the G-space X, and let Γ' be an orbit in the G-space X'. If Γ, Γ' have the same orbit type, then there exist $x \in \Gamma, x' \in \Gamma'$ such that $G_x = G_{x'}$. The mapping

$$f\colon \Gamma \to \Gamma' \quad x.g \mapsto x'.g$$

is well defined, bijective and equivariant. It is continuous with a continuous inverse because of the commutative diagram,

$$\begin{array}{ccc} x \times G & \longrightarrow & x.G \\ {\scriptstyle\approx}\downarrow & & \downarrow{\scriptstyle f} \\ x' \times G & \longrightarrow & x'.G \end{array}$$

This proves the first part.

Now suppose there exists an equivariant homeomorphism $f\colon \Gamma \to \Gamma'$. Then for each $x \in \Gamma$, the fact that $f\colon x.g \mapsto f(x).g$ assures that $G_x \subset G_{f(x)}$. As F is an equivariant homeomorphism, so is f^{-1}. Thus $G_{f(x)} \subset G_x$. Hence $G_x = G_{f(x)}$, so Γ, Γ' have the same orbit type. □

We now come to our main result for orbit type :-

Theorem 6.2. *let $\varphi\colon M \times G \to M$ be a DTG, where the action of φ is proper. Then M has locally a finite number of orbit types.*

Proof. In the case M is 0-dimensional, for each $x \in M$, $\{x\}$ is a neighbourhood of x and $\{x\}$, since it contains just one point, contains just one orbit type. So the assertion holds in this case.

Now suppose that the assertion holds in the case M is a k-manifold, for all k such that $0 \leqq k \leqq n$. This implies that if M is compact, M has a finite number of orbit types. Thus, if $M = S^k, 0 \leqq k \leqq n$, M has a finite number of orbit types. We shall show that this last result implies that the theorem holds in the case M is an $n+1$-manifold. This will prove the theorem.

Let M be an $(n+1)$-manifold and let $x \in M$. Let H be the isotropy group of x and give M an H-invariant metric (recall that if the action is proper, then isotropy groups are compact). We can construct a slice S at x (as in Theorem 5.2), so x has a neighbourhood (e.g., $S.U$ where U is a local cross-section at 1 of $p\colon G \to H\backslash G$) such that every orbit in the neighbourhood meets S. So it is sufficient to show that there are only a finite number of orbit type in S. Recall that if $s \in S$, then $G_s \subset H$, thus we only have to consider the action of H on S, i.e., we can restrict our attention to the DTG - $\varphi | M \times H\colon M \times H \to M$.

Note that x is a stationary pint of H , and further S is H-invariant. By Theorem 3.6 we see that$S$ is isomorphic, as an H-space, to the disc $D = \exp^{-1} S$, where $\exp\colon M_x \to M$. Recall that D is a neighbourhood in N_x, where N_x is the normal space to $x.G$ at x. H leaves $0 \in D$ fixed, acts linearly and isometrically on D, and leaves D in N_x; so H acts orthogonally on D. Clearly, all points on the same open radius (i.e., a radius excluding the centre point) of D have the same isotropy group, so the different orbit types occur on the boundary of D and at 0. The boundary of D is S^k, for some $k, 0 \leqq k \leqq n$ and the inductive hypothesis assures that there are only a finite number of orbit types on S^k; further, there is just one orbit type at 0 - namely the one determined by H. D has thus a finite number of orbit types, so does S. □

In order to introduce the concept of a "principal orbit", we need the following proposition :-

Proposition 6.3. *Let G be a compact Lie group and let H be a proper Lie subgroup. Then either*
(1) $\dim H < \dim G$
or
(2) $\dim H = \dim G$*, and H has fewer components than G.*

Proof. First note that the compactness of G implies that G has a finite number of components. Also $H \subset G$ dictates $\dim H \leqq \dim G$.

Suppose $\dim H = \dim G$. Then H contains a neighbourhood of 1 in G, so H contains G_0. As H is a proper subgroup of G, H/G_0 is a proper subgroup of G/G_0 (recall that G_0 is a closed normal subgroup , by Proposition 2.11). G/G_0 is compact discrete (since G_0 is open and closed), and is therefore finite. But the components of G are the cosets of G_0. □

Let $\varphi\colon M \times G \to M$ be a DTG, where the action of φ is proper. Since the action is proper, the isotropy groups are compact. In the class of all isotropy groups of the action, we can therefore speak of a particular isotropy group H being "minimal" in the senses that :-

(1) $\dim H$ is as small as possible.

(2) subject to (1) the number of components of H is as small as possible.

An orbit of the action is called a *principal orbit* if there is a point on the orbit whose isotropy group is minimal in the sense described above. Note that condition (1) implies the dimension of the principal orbit is as large as possible.

Proposition 6.3 assures that the principal orbit always exist.

Notation:

If $\varphi\colon M \times G \to M$ is a DTG, where the action of φ is proper, the set of points in M lying on the principal orbits of the action is denoted by $P(M, G)$.

Lemma 6.4. *Let $\varphi\colon X \times G \to X$ is a TTG and let S be an H-slice in X. Then the map*

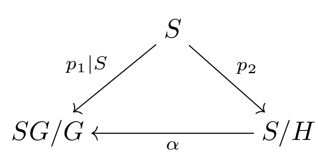
$$\alpha\colon S/H \to SG/G, \quad sH \mapsto sG$$

is a homeomorphism.

Proof. We have the flowing two restrictions of φ, and their associated projections from the corresponding restrictions of X to their corresponding orbit space :-

$$\varphi_1\colon SG \times G \to SG; \quad p_1\colon SG \to SG/G, \quad s.g \mapsto s.G$$
$$\varphi_2\colon S \times H \to S; \quad p_2\colon S \to S/H, \quad s \mapsto s.H.$$

Note that $p_1(S) = SG/G$. For $s_1, s_2 \in S$, $s_1.G = s_2.G$ if and only if $s_1 = s_2.g$ for some $g \in G$ if and only if $s_1 = s_2.g$ for some $g \in H$ (condition (2) of the Section 5) if and only if $s_1.H = s_2.H$ So α is injective. We have the commutative diagram,



Since $p_1|S$ and p_2 are onto, so is α; and since $p_1|S$ and p_2 are open and continuous, again, so is α. Thus $\alpha\colon S/H \to SG/G$ is a homeomorphism. □

We now come to the main (and rather surprising) result of this section :-

Theorem 6.5 (Principal Orbits)**.** *Let $\varphi\colon M \times G \to M$ be a DTG, where M* is connected *and the action of φ is proper. Then $P(M, G)$ is an open dense set in M, whose image in M/G is connected. Further, all principal orbits are of the same type.*

Proof. In the case M is 0-dimensional, M has just one point, the theorem follows trivially. We assume that the result holds in the case M is a connected k-manifold, for all k such that $0 \leqq k \leqq n$, and show that this implies that the result holds in the case M is a connected $n+1$-manifold. This will prove the theorem.

(1) $P(M,G)$ *is open :* Take $x \in P(M,G)$ and let S be a slice at x. Then SG is a neighbourhood of x and every orbit in SG meets S. For $s \in S$, $G_s \subseteq G_x$, but $G_s \not\subset G_x$ because G_x is minimal. Hence $G_s = G_x$, so $s \in P(M,G)$ and thus $SG \subset P$.

(2) $P(M,G)$ *is dense:* Since M is connected, it is enough to show that $\overline{P(M,G)}$ is open. Take $x \in \overline{P(M,G)}$ and let S be a slice at x. Recall that SG is a neighbourhood of x and every orbit of SG meets S, hence S meets $P(M,G)$ say in y. Put $G_x = H$ and consider the action of H on S, i.e., $\varphi|S \times H \colon S \times H \to S$. For $s \in S$, $G_s \subset G_y = H$ implies $G_s = H_s$. Hence $s \in S$ lies on a principal orbit of the action $\varphi|SG \times G \colon SG \times G \to SG$ if and only if s lies on a principal orbit of the action $\varphi|S \times H \to S$, i.e., $s \in P(SG,G)$ if and only if $s \in P(S,H) \cap S$ if and only if $sg \in P(SG,G)$ for all $g \in G$.

(a) In the case $\dim s \neq 0$, we have that

$$S \setminus x \approx \exp^{-1}(S \setminus x) = (\exp^{-1} S) \setminus 0 \approx S^k \times]0,1]$$

for some $k, 0 \leqq k \leqq n$, where $\exp \colon M_x \to M$. We have the commutative diagram,

$$\begin{array}{ccc}
(S^k \times]0,1]) \times H & \xrightarrow{\varphi^*} & S^k \times]0,1] \\
{\scriptstyle q\times 1}\downarrow \approx & & \approx \downarrow {\scriptstyle q} \\
((\exp^{-1} S) \setminus 0) \times H & \xrightarrow{\tilde{\varphi}} & (\exp^{-1} S) \setminus 0 \\
{\scriptstyle \exp\times 1}\downarrow \approx & & \approx \downarrow {\scriptstyle \exp} \\
(S \setminus x) \times H & \xrightarrow[\varphi]{} & S \setminus x
\end{array}$$

where $\tilde{\varphi}, \varphi^*$ are the actions induced by φ, and the map q, is the homeomorphism

$$(\exp^{-1} S) \setminus 0 \to S^k \times]0,1].$$

As remarked in the proof of Theorem 6.2 H acts orthogonally on D, which implies the induced action of H on the cylinder $S^k \times]0,1]$ is one of rotation about its axis, i.e., H acts orthogonally on S^k but does not act on $]0,1]$.

We are saying in fact that

$$\varphi^* = (\varphi_0^*, 1) \colon (S^k \times]0,1]) \times H \to S^k \times]0,1], \quad ((s,t),h) \mapsto (\varphi_0^*(s,h),t))$$

for all $s \in S, t \in]0,1], h \in H$ for some orthogonal action $\varphi_0^* \colon S^k \times H \to S^k$.

The principal orbits of $\varphi_0^* \colon S^k \times H \to S^k$ are, by the inductive hypothesis dense in S^k, for $k \neq 0$; when $k = 0$, there is either just one orbit or the isotropy group of both points of S^0 is H, thus $P(S^0, H) = S^0$, in either case. Hence the principal orbits of

$$\varphi^*(S^k \times]0,1]) \times H \to S^k \times]0,1]$$

are dense, so the principal orbits of

$$\varphi|(S \setminus x) \times H \colon (S \setminus x) \times H \to (S \setminus x)$$

are dense in $S \setminus x$, thus $P(SG, G) \cap S$ is dense in S (by above), thus $P(M, G) \cap S$ is dense in S. Hence $S \subset \overline{P(M,G)}$ and so $SG \subset \overline{P(M,G)}$. Thus $P(M, G)$ is open.

(b) In the case $\dim S = 0$, $x.G$ has the same dimension as M (since $\dim S = 0$ implies $S = \{x\}$, thus $x.G$ is a neighbourhood in M). The map

$$\theta \colon H\backslash G \to x.G \subset M, \quad Hg \mapsto x.g$$

is a diffeomorphism (Corollary 4.9) so $x.G$ is open by the inverse function theorem. Since the action of φ is proper, $x.G$ is closed (Theorem 2.25). Thus the connectedness of M implies $x/G = M$, $x.G$ being the only orbit is thus principal.

(3) $P(M, G)/G$ *is connected* : Suppose $P(M, G)/G = U \cup V$, where U, V are disjoint open sets in $P(M, G)/G$. Then $P(M, G) = p^{-1}(U) \cup p^{-1}(V)$, where $p \colon M \to M/G$ is the canonical projection. By (2),

$$M = \overline{P(M,G)} = \overline{p^{-1}(U)} \cup \overline{p^{-1}(V)}$$

so there exists $x \in \overline{p^{-1}(U)} \cap \overline{p^{-1}(V)}$ since M is connected.

Let S be a slice at x. If $\dim S = 0$, it follows from (2b) that $P(M, G)/G$ is a one point set and is thus connected. So we can suppose $\dim S \neq 0$.

From (2), we have that $P(SG, G) \cap S = P(S, H)$; and if $p_1 \colon SF \to SG/G$ is the canonical projection, the relation

$$p_1(s.g) = p_1(s) = s.G, \quad \text{for all} \quad s \in S, g \in G$$

assures $(P(SG, G) \cap S)/G = P(SG, G)/G$; Applying Lemma 6.4, we have that

$$P(S, H)/H \approx (P(SG, G) \cap S)/G = P(SG, G)/G;$$

in particular

$$P(SG \setminus x.G, G)/G \approx P(S \setminus x, H)/H$$

and we know that

$$P(S \setminus x, H)/H = P(S^k, H) \times]0, 1])/H = P(S^k, H)|H \times]0, 1].$$

The inductive hypothesis implies $P(S^k, H)/H$ is connected which implies $P(SG \setminus x.G, G)/G$ is connected, and so $P(SG, G)/G$ is connected. Since

$P(SG,G) = P(M,G) \cap SG$ and SG and $P(M,G)$ are neighbourhoods in M, $P(SG,G)/G$ is thus a connected neighbourhood of $P(x)$ in M/G. Now $p(x) \in \bar{U} \cap \bar{V}$, but $P(SG,G)/G \subset P(M,G)/G = U \cup V$, so $P(SG,G)/G$ lies entirely in U or entirely in V, thus either U or V is empty. So $P(M,G)/G$ is connected.

(4) *The principal orbits are of the same type.* The proof of (1) implies that each point of p has a neighbourhood, in M, in which all the principal orbits are of the same type. Hence the orbits of a given type form an open set in M and therefore its image in M/G, and thus in $P(M,G)/G$ is open. Thus $P(M,G)/G$ is the disjoint union of open sets, each open set being the image in M/G of the open set in M consisting of points in $P(M,G)$ of a particular orbit type. As $P(M,G)/G$ is connected, all but one of these disjoint open sets is empty. So all the principal orbits have the same type.

□