Acyclic Models

Lemérange

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Lecture 1

Axiomatic Homology Theory

Homology theory has been around for more than 120 years. Its founding father was the French mathematician Henri Poincaré who gave a somewhat fuzzy definition of "homology" in 1895. In today's perspective, what he defined was close to modern "cobordism".

Thirty years later, the German mathematician Emmy Noether realised that abelian groups were the right context to study homology, rather than the then known and extensively used Betti numbers. In the decades after the advent of Poincaré's homology invariants, many different theories were developed (e.g. simplicial homology, singular homology, Čech homology etc.) by many topologists (e.g. Alexander, Čech, Eilenberg, Lefschetz, Veblen, and Vietoris) that were all called "homology theories".

In 1945, Samuel Eilenberg and Norman Steenrod gave the first (and still used) definition of what an (ordinary) (co-)homology theory should be, based on the similarities between the different, then known, theories.

1.1 The Eilenberg-Steenrod Axioms

We introduce some categorical notation that will be used to dispense ambiguities and inaccuracies often found in existing documents on homology theory. This may seem, at first sight, a bit pedantic, but in the end the readers will be convinced that it is a "must". For categorical notion, the reader is referred to "An Introduction to Category Theory" by Harold Simmons.

We denote by $\mathscr{Top}, \mathscr{Top}_{(2)}, \mathscr{Top}_{(3)}$ the categories of topological spaces, pairs of spaces (called "pairs" for short), and triples of spaces respectively, where

$\mathscr{Top}_{(2)}$ The objects of are pairs (X, A), where $X \in \mathrm{Ob}(\mathscr{Top})$ is a topological space and $A \subset X$,
a morphism $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ with $f(A) \subset B$.

$\mathscr{Top}_{(3)}$ The objects of $\mathscr{Top}_{(3)}$ are triples (X, A, B), where $X \in \mathrm{Ob}(\mathscr{Top})$ and

$B \subset A \subset X$,
a morphism $f : (X, A, B) \to (Y, A', B')$ is a continuous map $f : X \to Y$ with $f(A) \subset A'$ and $f(B) \subset B'$.

We use the term "inclusion" for maps in $\mathcal{Top}_{(2)}$ or $\mathcal{Top}_{(3)}$ to mean "inclusion in each component". If $x \in X$ is a point, we will also write (X, x) for $(X, \{x\})$ (*mutatis mutandis* for triples). Moreover, we say "space" to mean "a set endowed with a certain topology" and assume all maps to be continuous unless otherwise specified.

We get canonical inclusions

$$\mathcal{Top} \to \mathcal{Top}_{(2)} \to \mathcal{Top}_{(3)}$$

by sending each space X to $(X, \varnothing)$ and (X, A) to $(X, A, \varnothing)$: in this way we can view $\mathcal{Top}$ (resp. $\mathcal{Top}_{(2)}$) as a full subcategory of $\mathcal{Top}_{(2)}$ (resp. $\mathcal{Top}_{(3)}$). We will use this identification throughout this section and so, we will usually write X to mean $(X, \varnothing)$.

Alternatively, one could also send X to (X, X) and (X, A) to (X, A, A). It's not surprising that these two types of inclusions constitute to two adjunctions. If we denote the first inclusion by F and the second one by G the adjunctions are as follows: first inclusion by F and the second one by G the adjunctions are as follows:

$$\mathcal{Top}_{(2)} \underset{G}{\overset{U}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathcal{Top} \quad \text{and} \quad \mathcal{Top} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathcal{Top}_{(2)}$$

where $U : \mathcal{Top}_{(2)} \to \mathcal{Top}$ is the forgetful functor $(X, A) \mapsto X$ (similarly for $\mathcal{Top}_{(3)}$ and $\mathcal{Top}_{(2)}$).

We notice that $\mathcal{Top}_{(2)}$ and $\mathcal{Top}_{(3)}$ are bicomplete (i.e. have small limits and colimits) and the (co-)limits are given by taking them componentwise. Note that a category is

complete (or properly, *small complete*) if it has all limits,

cocomplete (or properly, *small cocomplete*) if it has all colimits.

For example, if we have a family $(X_j, A_j)_{j \in J}$ of objects in $\mathcal{Top}_{(2)}$ then their product is given by $(\prod_{j \in J} X_j, \prod_{j \in J} A_j)$.

Notation 1.1. We use the notation $I := [0, 1]$ to denote the unit interval.

Definition 1.2 (admissiblity)**.** A subcategory $\mathscr{C} \subset \mathcal{Top}_{(2)}$ is called *admissible for homology theory* iff

(i) $\mathscr{C}$ contains a space $\{*\}$ consisting of a single point (i.e. a final object in $\mathcal{Top}$). Furthermore, $\mathscr{C}$ contains all points (in $\mathcal{Top}$). That means that for $X \in \mathrm{Ob}(\mathscr{C})$ and $1 \simeq \{*\}$, we have

$$\mathrm{Hom}_{\mathscr{C}}(1, X) = \mathrm{Hom}_{\mathcal{Top}_{(2)}}(1, X) = \mathrm{Hom}_{\mathcal{Top}}(1, X).$$

Here 1 means a fixed one-point space in $\mathscr{C}$.

(ii) If $(X, A) \in \mathrm{Ob}(\mathscr{C})$ then the following diagram of inclusions (called the *lattice of* (X, A)) lies in $\mathscr{C}$, too:

$$\begin{array}{ccccccccc} & & & & (X,\varnothing) & & & & \\ & & & \nearrow & & \searrow & & & \\ (\varnothing,\varnothing) & \longrightarrow & (A,\varnothing) & & & & (X,A) & \longrightarrow & (X,X) \\ & & & \searrow & & \nearrow & & & \\ & & & & (A,A) & & & & \end{array}$$

Moreover, we require that for $f : (X, A) \to (Y, B)$ in $\mathscr{C}$, $\mathscr{C}$ also contains all the maps from the lattice of (X, A) to that of (Y, B), induced by f.

(iii) For any $(X, A) \in \mathrm{Ob}(\mathscr{C})$, the following diagramme lies in $\mathscr{C}$

$$(X, A) \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} (X \times I, A \times I)$$

where $\iota_t : X \to X \times I$, $x \mapsto (x, t)$ for $t \in \{0, 1\}$.

Remark 1.3. We notice that axioms (i) and (ii) imply that $\mathscr{C}$ really contains all points (i.e. also points in $\mathscr{T}\!op_2$). That means, for any $(X, A) \in \mathrm{Ob}(\mathscr{C})$ (and not only for the $(X, \varnothing)$ as in (i)) $\mathscr{C}$ contains all maps $(1, \varnothing) \to (X, A)$. The reason being that $\mathscr{C}$ contains the inclusion $(X, \varnothing) \to (X, A)$. Moreover, it follows that $\mathscr{C}$ contains I since $\mathscr{C}$ contains 1 and $1 \times I \simeq I$.

Example 1.4. The following categories are all examples of admissible categories for homology theory.

$\mathscr{T}\!op_{(2)}$), which is the largest admissible category.

The full subcategory of $\mathscr{T}\!op_{(2)}$, consisting of all pairs of compact spaces.

The subcategory of $\mathscr{T}\!op_{(2)}$, having as objects all pairs (X, A), where X is locally compact Hausdorff and $A \subset X$ is closed and as arrows all maps of pairs, satisfying that the preimage of compact subsets are compact.

Definition 1.5. A *homotopy* between two maps $f_0, f_1 : (X, A) \to (Y, B)$ in $\mathscr{C}$ is a map

$$f : (X \times I, A \times I) \to (Y, B),$$

in $\mathscr{C}$ satisfying $f_0 x = f(x, 0)$ and $f_1 x = f(x, 1)$. That means that f is an ordinary homotopy from $f_0 : X \to Y$ to $f_1 : X \to Y$, viewed as maps in $\mathscr{T}\!op$ with the additional requirement, that $f(A, t) \subset B\ \forall t \in I$. For $t \in I$, we write $f_t : (X, A) \to (Y, B)$, $x \mapsto f(x, t)$ and will loosely refer to this family of maps as a homotopy from f_0 to f_1. As always, we call f_0 and f_1 as above homotopic iff there is a homotopy f in $\mathscr{C}$ from f_0 to f_1.

With the notation from the last definition, a homotopy between f_0 and f_1 is a diagramme of the form

$$(X, A) \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} (X \times I, A \times I) \xrightarrow{f} (Y, B),$$

satisfying $f \circ \iota_0 = f_0$ and $f \circ \iota_1 = f_1$.

Definition 1.6. Let $\mathscr{C}$ be an admissible category. We define the so-called *restriction functor* $\rho : \mathscr{C} \to \mathscr{C}$ which sends (X, A) to $(A, \varnothing)$ and $f : (X, A) \to (Y, B)$ to $\rho f =: f|_B^A : (A, \varnothing) \to (B, \varnothing)$, $x \mapsto fx$. This functor is well-defined by axiom (ii) in the definition of an admissible category 1.2.

Definition 1.7. A *homology theory* on an admissible category $\mathscr{C}$ consists of a family of functors $(H_n : \mathscr{C} \to \mathscr{A})_{n \in \mathbb{Z}}$, where $\mathscr{A}$ is an abelian category and a family of natural transformations $(\partial_n : H_n \to H_{n-1} \circ \rho)_{n \in \mathbb{Z}}$. $H_n(X, A)$ is called the n^{th} *homology of* (X, A) and ∂_n the n^{th} *boundary operator* or *connecting morphism.* As mentioned before, we identify X with $(X, \varnothing)$ and in the same spirit write $H_n X$ or $H_n(X)$ for $H_n(X, \varnothing)$, which we call the n^{th} (absolute) homology of X.

Sometimes, we write f_* for $H_n(f) : H_n(X, A) \to H_n(Y, B)$ where $f : (X, A) \to (Y, B)$ and we will omit the index and write ∂ for ∂_n, "to avoid unnecessarily complicated notation". Explicitly, ∂ being a natural transformation means that the following diagramme commutes for all $f : (X, A) \to (Y, B)$ in $\mathscr{C}$.

$$\begin{array}{ccccc} (X, A) & & H_n(X, A) & \xrightarrow{\partial} & H_{n-1}A \\ \big\downarrow f & & H_n(f) \big\downarrow & & \big\downarrow H_n(f) \\ (Y, B) & & H_n(Y, b) & \xrightarrow[\partial]{} & H_{n-1}B \end{array}$$

These are required to satisfy

(i) (**Homotopy Invariance**) For each homotopy $(f_t)_{t \in I}$ in $\mathscr{C}$ we have $H_n(f_0) = H_n(f_1)$. Equivalently, with the above notation, we could also require $H_n(f)(\iota_0) = H_n(f)(\iota_1)$.

(ii) (**Long Exact Homology Sequence**) For each $(X, A) \in \mathrm{Ob}(\mathscr{C})$ we have a long exact sequence

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n A \to H_n X \to H_n(X, A) \xrightarrow{\partial} \cdots,$$

where the unnamed arrows are induced by the canonical inclusions.

(iii) (**Excision Axiom**) If $(X, A) \in \mathrm{Ob}(\mathscr{C})$, $U \subset X$ open with $\overline{U} \subset \int A$ and the standard inclusion $(X \setminus U, A \setminus U) \to (X, A)$ lies in $\mathscr{C}$. Then this inclusion induces for each $n \in \mathbb{Z}$ an isomorphism

$$H_n(X \setminus U, A \setminus U) \simeq H_n(X, A),$$

called the *excision* of U.

Some authors require a weaker form of the excision axiom instead of the one before.

(iii)* (**Weak Excision Axiom**)

If $(X, A) \in \mathrm{Ob}(\mathscr{C})$, $U \subset X$ and $f : X \to I$ is a map, satisfying $U \subset f^{-1}0 \subset f^{-1}[0,1[\subset A$ and the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ lies in $\mathscr{C}$. Then this inclusion induces for each $n \in \mathbb{Z}$ an isomorphism

$$H_n(X \setminus U, A \setminus U) \simeq H_n(X, A),$$

called the *excision* of U.

For $1 \in \mathrm{Ob}(\mathscr{C})$ a one-point space the $H_n 1$ are called the *coefficients* of the homology theory. If furthermore the following axiom is satisfied, we speak of an *ordinary* homology theory.

(iv) (**Dimension Axiom**) If $1 \in \mathrm{Ob}(\mathscr{C})$ is a one-point space then

$$H_n 1 = H_n(1, \varnothing) = 0 \quad \forall n \in \mathbb{Z} \setminus \{0\}.$$

So in an ordinary homology theory only the coefficient $H_0 1$ is of any interest. If we have chosen an isomorphism $H_0 1 \simeq G \in \mathscr{A}$ we call this an ordinary homology theory *with coefficients in* G and write $H_n(X, A; G) := H_n(X, A)$.

1.2 First Consequences

For the rest of this chapter, $(H_n : \mathscr{C} \to \mathscr{A})_{n\in\mathbb{Z}}$, $(\partial_n)_{n\in\mathbb{Z}}$ is a given (not necessarily ordinary) homology theory and all spaces and maps are assumed to be admissible (i.e. lie in $\mathscr{C}$). As a first remark we look at the homology of an empty space and at the homology of a space, relative to itself (i.e. the homology of a pair (X, X)). Using the long exact homology sequence, one easily deduces (a) in the following remark. And using the homotopy invariance axiom (and functoriality of $H_{0)}$ one deduces the first part of (b) and with the long exact homology sequence of (X, A) one proves the second part.

Recall that two spaces X and Y are called *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic id_X. That is,

$$\exists \Phi : X \times I \to X, \quad \Phi|_{X\times 0} = g \circ f, \quad \Phi|_{X\times 1} = \mathrm{id}_X,$$
$$\exists \Psi : Y \times I \to Y, \quad \Psi|_{Y\times 0} = f \circ g, \quad \Psi|_{Y\times 1} = \mathrm{id}_Y .$$

Remark 1.8. Let X be a topological space.

(a) $H_n(X, X) = 0 \ \forall n \in \mathbb{Z}$ and as a special case $H_n\varnothing = H_n(\varnothing, \varnothing) = 0 \ \forall n \in \mathbb{Z}$.

(b) If $f : A \to X$ is a homotopy equivalence then $n(f) : H_n A \to H_n X$ is an isomorphism. In particular if A is a deformation retract of X (i.e. the inclusion $i : A \rightsquigarrow X$ is a homotopy equivalence) then $H_n(i) : H_n A \to H_n X$ is an isomorphism and $H_n(X, A) = 0$.

More generally, one immediately deduces the following from part (b) of the last remark, the long exact homology sequences for (X, A) and (Y, B), and the 5-Lemma.

Remark 1.9. If $f : (X, A) \to (Y, B)$ is such that $f : X \to A$ and $f|_B^A : A \to B$ are homotopy equivalence then $H_n(f) : H_n(X, A) \to H_n(Y, B)$ is an isomorphism.

As a next step, we are going to study the homology of a finite topological sum (i.e. a coproduct of topological spaces). Of course, one will immediately ask questions about the dual situation (i.e. the homology of a product) which would lead to the definition of the so-called cross product in homology. But for now let us concentrate on the coproduct.

Theorem 1.10. *The homology functors preserve finite coproducts. Explicitly, for pairs (X_1, A_1), (X_2, A_2) let $(X, A) := (X_1 \coprod X_2, A_1 \coprod A_2)$ be their coproduct (i.e. topological sum) with the standard inclusions $i_t : (X_t, A_t) \to (X, A)$, $t \in \{1, 2\}$. Then for all $n \in \mathbb{Z}$ the diagramme*

$$H_n(X_1, A_1) \xrightarrow{H_n(i_1)} H_n(X, A) \xleftarrow{H_n(i_2)} H_n(X_2, A_2)$$

is a coproduct in $\mathcal{A}$ (and so $H_n(X, A)$ is even a biproduct since $\mathcal{A}$ is abelian). Put differently,

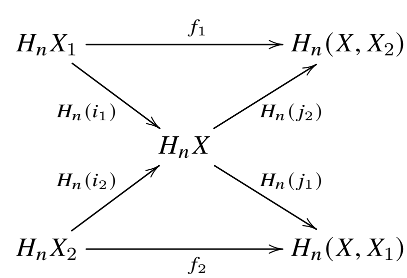
$$H_n(X_1, A_1) \oplus H_n(X_2, A_2) \xrightarrow{\binom{H_n(i_1)}{H_n(i_2)}} H_n(X, A)$$

is an isomorphism.

Proof. Consider the morphism $\binom{H_n(i_1)}{H_n(i_2)}$ from the direct sum of the long exact homology sequences for (X_1, A_1), (X_2, A_2) to the long exact homology sequence for (X, A). In view of the 5-lemma it is enough to show the proposition for the case where $A_1 = A_2 = \emptyset$. We have the standard inclusions

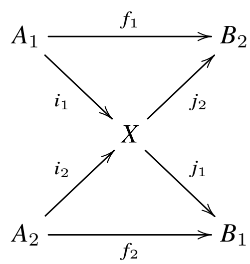
$$X_1 \xrightarrow{i_1} X \xrightarrow{j_1} (X, X_1) \quad \text{and} \quad X_2 \xrightarrow{i_2} X \xrightarrow{j_2} (X, X_2),$$

whose induced morphisms can be combined in a commutative diagramme



By the long exact homology sequences the diagonals are exact and by the excision axiom any morphism of the form $H_n(Y, B) \to H_n(Y \coprod Z, B \coprod Z)$, induced by the inclusion, is an isomorphism. So in particular, f_1 and f_2 are isomorphisms. The following lemma gives the desired result. □

Lemma 1.11. *Given a commutative diagramme*



in an abelian category with exact diagonals. Then the following conditions are equivalent:

(a) f_1 *and* f_2 *are isomorphisms;*

(b) $\binom{i_1}{i_2} : A_1 \oplus A_2 \to X$ *is an isomorphism;*

(c) $(j_1, j_2) : X \to B_1 \oplus B_2$ *is an isomorphism.*

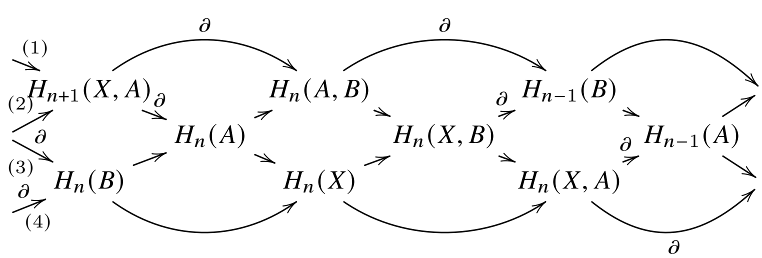
Considering the long exact homology sequence of a pair, one is forced to ask whether there is an analogue for a triple (X, A, B) and indeed there is. There are topological proofs for this but we prefer an algebraic one (even if that means that we have to draw a nasty diagramme) since it uses only the Long Exact Homology Sequence axiom.

For a triple $(X, A, B) \in \mathrm{Ob}(\mathcal{T}\!op_{(3)})$ with $(X, A), (X, B), (A, B) \in \mathrm{Ob}(\mathscr{C})$, we define another boundary operator

$$\partial : H_{n+1}(X, A) \xrightarrow{\partial} H_n A \to H_n(A, B),$$

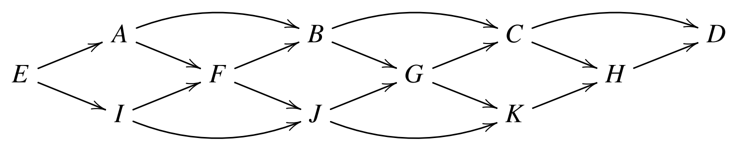
where the first morphism is the boundary map given by our homology theory and the second morphism is induced by the inclusion $(A, \varnothing) \to (A, B)$. We shall also write ∂ for this morphism as there should be no risk of confusion.

If we now consider the three pairs (X, A), (X, B), and (A, B), we can put their long exact homology sequences into a so-called *homology braid of C. T. C. Wall*



where the sequences (1), (3), and (4) are the long exact homology sequences of (X, A), (X, B), and (A, B) respectively. The sequence (2) will be called the *long* *exact homology sequence for the triple* (X, A, B). One easily checks that this is a chain complex (i.e. the composition of two morphisms is 0) and the following lemma gives us exactness.

Lemma 1.12 (Braid Lemma)**.** *Suppose we have a braid diagramme as below for abeilan groups.*



If three of the sequences are exact and the fourth is a chain complex, i.e.,

1 $E \to A \to B \to G \to K$ *is exact,*

2 $E \to I \to J \to G \to C \to D$ *is exact,*

3 $A \to F \to J \to K \to H \to D$ *is exact*

Then $I \to F \to B \to C \to H$ *is exact is exact, too.*

Definition 1.13. By the above discussion, the following definition makes sense. For each triple (X, A, B) we have an exact sequence

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \to H_n(X, B) \to H_n(X, A) \xrightarrow{\partial} \cdots$$

where ∂ is the boundary morphism of the triple (X, A, B).

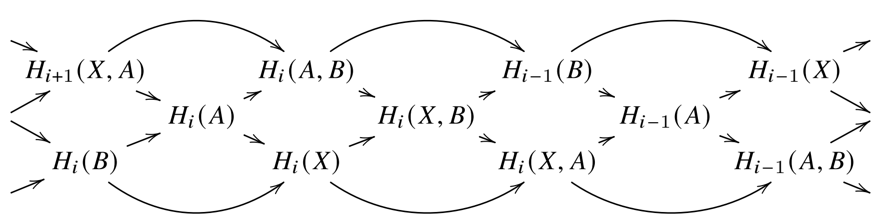
The braid diagramme of C. T. C. Wall

The above discussion of the exact sequence for a triple (X, A, B) is rather terse. We are going to be a bit more specific, following Bredon.

Theorem 1.14. *If* $B \subset A \subset X$ *and we let* $\partial_* : H_i(X, A) \to H_{i-1}(A, B)$ *be the composition of* $\partial_* : H_i(X, A) \to H_{i-1}(A)$ *with the map* $H_{i-1}(A) \to H_{i-2}(A, B)$ *induced by inclusion, then the following sequence is exact, where the maps other than* ∂_* *come from inclusions:*

$$\cdots \xrightarrow{\partial_*} H_p(A, B) \xrightarrow{i_*} H_p(X, B) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A, B) \xrightarrow{i_*} \cdots$$

Proof. There is the following commutative diagramme:

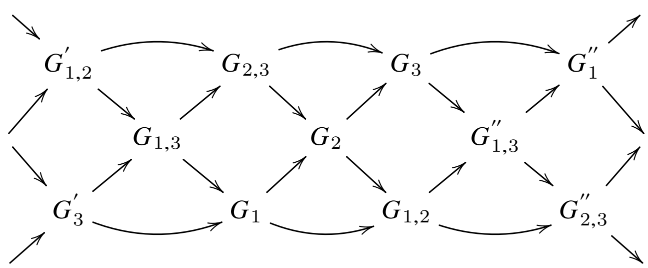


This is called a “braid” diagramme. (This kind of diagramme is due to Wall and Kervaire.) There are four braids and three of them are exact (the exact sequences for the pairs (X, A), (X, B), and (A, B)). The fourth is the sequence of the triple (X, A, B) which we wish to prove exact. This sequence is easily seen from the commutativity to be of order two except for the composition

$$H_i(A, B) \to H_i(X, B) \to H_i(X, A).$$

But this composition factors through $H_i(A, A) = 0$, so the entire sequence is of order two. The theorem now follows from the following completely algebraic fact. □

Lemma 1.15 (Wall). *Consider the following commutative braid diagramme:*



If all the three (3) braids except the one with single subscripts are exact and the fourth is of order two, then the fourth one is also exact.

Proof. The proof is by a diagramme chase. For simplicity in doing the chase we shall introduce some special notation for it. Elements of, for example, $G'_{1,2}$ will be denoted by $a'_{1,2}$, $b'_{1,2}$, etc. To indicate that an element $a_1 \in G_1$ comes from an element $b_{1,3}$, not yet defined, we just write $\exists b_{1,3} \to a_1$. To indicate that an element $a_1 \in G_1$ goes to ${}_{1,2} \in G_{1,2}$ we just write $a_1 \to b_{1,2}$, and if $b_{1,2}$ has not yet been defined, this does so. The notation $a_2 \to 0$ means that the element $a_2 \in G_2$ goes to 0 in G_3. Now we begin the chase. In following the arguments the reader will find it helpful to diagramme the elements and relations as they arise.

First we prove exactness at G_2 for the composition $G_1 \to G_2 \to G_3$. Thus suppose $a_2 \to 0_3$. Then if $a_2 \to a_{1,2}$, $a_{1,2} \to 0''_{1,3}$ by commutativity. By exactness, $\exists a_1 \to a_{1,2}$. Let $a_1 \to b_2$. Then $a_2 \to a_{1,2}$ and $b_2 \to a_{1,2}$ imply $a_2 - b_2 \to 0_{1,2}$. Thus $\exists a_{2,3} \to a_2 - b_2$. Since $a_1 \to b_2$ we have $b_2 \to 0_3$. Thus $a_2 - b_2 \to 0_3$, and so $a_{2,3} \to 0_3$. Therefore $\exists a_{1,3} \to a_{2,3}$. Let $a_{1,3} \to b_1$. Since $a_{1,3} \to a_{2,3} \to a_2 - b_2$ and $a_{1,3} \to b_1$ it follows from commutativity that $b_1 \to a_2 - b_2$. But $a_1 \to b_2$, so $a_1 + b_1 \to b_2 + (a_2 - b_2) = a_2$, as desired.

Next we prove exactness at G_3. Thus let $a_3 \to 0''_1$. Define $a_3 \to a''_{1,3}$. Then $a''_{1,3} \to 0''_1$ so that $\exists a_{1,2} \to a''_{1,3}$. Since $a_3 \to a''_{1,3} \to 0''_{2,3}$ we have $a_{1,2} \to 0''_{23}$. Consequently, $\exists a_2 \to a_{1,2}$. Let $a_2 \to b_3$. Then $a_3 - b_3 \to 0''_{1,3}$. Thus $\exists a_{2,3} \to a_3 - b_3$. Let $a_{2,3} \to c_2$. Then $a_2 + c_2 \to b_3 + (a_3 - b_3) = a_3$ as claimed.

Finally we prove exactness at G_1. Thus suppose $a_1 \to 0_2$. Then $a_1 \to 0_{1,2}$ so $\exists a_{1,3} \to a_1$. Let $a_{1,3} \to a_{2,3}$. Then $a_{2,3} \to 0_2$ so that $\exists a'_{1,2} \to a_{2,3}$. Let $a'_{1,2} \to b_{1,3}$. Then note $b_{1,3} \to 0_1$. Then we have $a_{1,3} - b_{1,3} \to 0_{2,3}$ so $\exists a'_3 \to a_{1,3} - b_{1,3}$. Now $a_{1,3} - b_{1,3} \to a_1 - 0_1 = a_1$, so that $a'_3 \to a_1$ as claimed. □

1.3 Reduced Homology

Although the coefficients $H_n 1$ are important, they do not contain any geometric information whatsoever. Because of this and for the sake of readability (so that we do not always have to carry these one-point spaces with us while doing algebraic manipulations), we want to split them off the homologies of our spaces. This leads to the idea of reduces homology.

Definition 1.16. Let X be a non-empty space and $p : X \to 1$ the unique map to a one-point space. We define the n^{th} *reduced homology* of X as

$$\widetilde{H}_n X := \ker(H_n(p) : H_n X \to H_n 1).$$

For $f : X \to Y$, we get again an induced morphism $H_n(f) : \widetilde{H}_n X \to \widetilde{H}_n Y$ in the obvious way. Like that we can extend $\widetilde{H}_n$ to a homotopy invariant functor $\mathcal{T}\!op \to \mathrm{Ab}$ (i.e. if f is a homotopy equivalence, then $H_n(f)$ is an isomorphism).

Obviously, by definition, we can calculate the reduced homology if we have the usual (i.e. non-reduced) homology given. One could ask whether it's also possible to go the other way. And indeed by some elementary algebraic facts we can. If we choose a point $x \in X$ and look at the inclusion $x : 1 \to X$ and $p : X \to 1$, we have $p \circ x = \mathrm{id}_1$ and the long exact homology sequence for (X, x) reads as

$$\cdots \to H_{n+1}(X,x) \xrightarrow{\partial} H_n 1 \xrightarrow{H_n(x)} H_n X \to H_n(X,x) \xrightarrow{\partial} H_{n-1} 1 \xrightarrow{H_n(x)} \cdots$$

Because $H_n(p) \circ H_n(x) = \mathrm{id}_{H_n 1}$ it follows that $H_n(x)$ is a monomorphism and by exactness $\operatorname{im} \partial = 0$. So we can rewrite this as a short exact sequence, which splits since $H_n(p)$ is a retraction of $H_n(x)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n 1 & \xrightarrow{H_n(x)} & H_n X & \longrightarrow & H_n(X,x) & \longrightarrow & 0 \\ & & & {\scriptstyle \mathrm{id}_{H_n 1}} \searrow & \downarrow {\scriptstyle H_n(p)} & & & & \\ & & & & H_n 1 & & & & \end{array}$$

The triangle on the left gives us an isomorphism $\binom{i}{H_n(x)} : \widetilde{H}_n X \oplus H_n 1 \xrightarrow{\sim} H_n X$, where $i : \widetilde{H}_n X \hookrightarrow H_n X$ is the standard inclusion. One plainly checks this as follows:

$$H_n X \simeq \ker H_n(p) \oplus H_n(x)(H_n 1) = \widetilde{H}_n X \oplus H_n(x)(H_n 1) \simeq \widetilde{H}_n X \oplus H_n 1.$$

By the first isomorphism theorem $H_n(j)|_{\widetilde{H}_nX} : \widetilde{H}_nX \xrightarrow{\sim} H_n(X,x)$ is an isomorphism, where $j : X \to (X,x)$ is the standard inclusion:

$$H_n(X,x) \simeq H_nX/H_n(x)(H_n1) \simeq (\widetilde{H}_nX \oplus H_n(x)(H_n1))/H_n(x)(H_n1) \simeq \widetilde{H}_nX$$

and finally these two together give

$$H_nX \simeq H_n(X,x) \oplus H_n1, \quad H_nX \xleftarrow{\binom{i}{H_n(x)}} \widetilde{H}_nX \oplus (H_n1) \xrightarrow{H_n(j)|_{\widetilde{H}_nX} \times 1_{H_n1}} H_n(X,x) \oplus H_n1,$$

which is exactly the usual splitting condition for a short exact sequence

A special case is when X is contractible. Then $H_n(x)$ is an isomorphism (by Remark 1.8) and putting this into the above short exact sequence gives us that $0 \to H_n(X,x) \to 0$ is exact from which one easily deduces the following theorem.

Theorem 1.17. *If X is contractible, then $\widetilde{H}_nX \simeq H_n(X,x) = 0 \quad \forall n \in \mathbb{Z}$.*

To finish this section, we are going to introduce the analogue of the long exact homology sequence in the reduced case. As one easily verifies, this is just a special case of the long exact homology sequence for a triple, where the triple is of the form (X,A,x), where $x \in A \subset X$ is a point.

Theorem 1.18. *(Reduced Long Exact Homology Sequence) Let (X,A) be a pair with $A \neq \varnothing$. Then the image of the boundary operator $\partial : H_{n+1}(X,A) \to H_nA$ lies in $\widetilde{H}_nA$. As a consequence, by restricting the long exact homology sequence of the pair (X,A), we get another long exact sequence for the reduced homology*

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} \widetilde{H}_nA \to \widetilde{H}_nX \to H_n(X,A) \xrightarrow{\partial} \cdots$$

Proof. Consider the unique arrow $p : X \to 1$ (resp. $p : A \to 1$ or $p : (X,A) \to (1,1)$). Then the long exact sequences of (X,A) and $(1,1)$ yield

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_{n+1}(X,A) & \xrightarrow{\partial} & \widetilde{H}_nA & \longrightarrow & \widetilde{H}_nX & \longrightarrow & H_n(X,A) & \xrightarrow{\partial} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{n+1}(X,A) & \xrightarrow{\partial} & H_nA & \longrightarrow & H_nX & \longrightarrow & H_n(X,A) & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle H_n(p)} & & \downarrow{\scriptstyle H_n(p)} & & \downarrow{\scriptstyle H_n(p)} & & \downarrow{\scriptstyle H_n(p)} & & \\
\cdots & \xrightarrow[0]{} & H_{n+1}(1,1) & \xrightarrow[0]{} & H_n1 & \xrightarrow[\sim]{} & H_n1 & \xrightarrow[0]{} & H_n(1,1) & \xrightarrow[0]{} & \cdots
\end{array}$$

In the lower long exact sequence, we have used that $H_n(1,1) = 0$ and get all the 0-morphisms. Either by exactness or by the fact that $1 \to 1$ is a homeomorphism, we conclude that $H_n1 \to H_n1$ is an isomorphism. The upper row consists simply of the kernels of the corresponding vertical morphisms $H_n(p)$ (observe that $\ker H_n(p) : H_n(X,A) \to H_n(1,1)) = H_n(X,A)$ since $H_n(1,1) = 0$). By naturality of ∂ the lower squares in the diagramme commute and so, since $H_n(p) \circ \partial = 0 \circ H_n(p)$ we have $\operatorname{im} \partial \subset \ker H_n(p)$ which proves the first part of the proposition (this actually proves more generally that the induced morphisms in the upper row are well-defined).

For the second part, we observe that all the $H_n(p)$ are epimorphisms for if we choose any point $x : 1 \to A$ (resp. $x : 1 \to X$ or $x : (1,1) \to (X,A)$), we have that $H_n(p) \circ x = \mathrm{id}_1$ and so p has a section. By a general theorem (whose proof is left as an exercise) which says that if we have an epimorphism of exact sequences then its kernel is also exact (and dually for a monomorphism of exact sequences and its cokernel) we get the exactness of the reduced long exact homology sequence. □

1.4 Homology of Spheres

In this section, we delve into the problem of calculating the homology of the spheres just from the axioms. To do so, we observe first, that we can divide the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ in an upper and lower hemisphere

$$\mathbb{D}^n_\pm := \{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n | \pm x_{n+1} \geq 0\}.$$

Obviously $\mathbb{D}^n_\pm \simeq \mathbb{D}^n$ by simply projecting $\mathbb{D}^n_\pm$ along the x_{n+1}-axis to $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Moreover, we observe that we have for any $n \in \mathbb{N}$ an inclusion

$$\mathbb{S}^{n-1} \to \mathbb{S}^n, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$$

or a little more geometric, by viewing $\mathbb{S}^{n-1}$ as the equator of $\mathbb{S}^n$. By combining these we get $\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \hookrightarrow \mathbb{S}^2 \hookrightarrow \cdots$. Let's furthermore fix the notation $e_i \in \mathbb{R}^n$ to denote the i^{th} standard basis vector having $(e_i)_j = \delta_{i,j}$ (the Kronecker delta) and with this, let's write $N := e_{n+1}$ and $S := -e_{n+1}$ for the north and south pole of $\mathbb{S}^n$ respectively. Now, for $n \in \mathbb{N}_{>0}$ we look at the commutative diagramme

$$\begin{array}{ccc} H_k(\mathbb{D}^n_-, \mathbb{S}^{n-1}) & \longrightarrow & H_k(\mathbb{S}^n, \mathbb{D}^n_+) \\ \downarrow & & \downarrow \\ H_k(\mathbb{D}^n_-, \mathbb{D}^n_- \setminus \{S\}) & \longrightarrow & H_k\mathbb{S}^n, \mathbb{S}^n \setminus \{S\}) \end{array}$$

induced by inclusions. Because $\mathbb{S}^{n-1} \hookrightarrow \mathbb{D}^n \setminus \{S\}$ and $\mathbb{D}^n_+ \hookrightarrow \mathbb{S}^n \setminus \{S\}$ are homotopy equivalences, it follows that the vertical arrows are isomorphisms (by Remark 1.9). For the bottom arrow, we can use the excision axiom with $U := \operatorname{int} \mathbb{D}^n_+$ and deduce that this is also an isomorphism. In conclusion, the top arrow has to be an isomorphism, too.

We choose $* := e_1 = (1, 0, \dots, 0) \in \mathbb{S}^{n-1} \subset \mathbb{S}^n$ and insert this isomorphism in a second diagramme

$$\begin{array}{ccccc} H_k(\mathbb{D}^n_-, \mathbb{S}^{n-1}) & \xrightarrow{\partial} & H_{k-1}(\mathbb{S}^{n-1}, *) & \simeq & \widetilde{H}_{k-1}\mathbb{S}^{n-1} \\ \simeq \downarrow & & \vdots\, \sigma_+ & & \vdots\, \sigma_+ \\ H_k(\mathbb{S}^n, \mathbb{D}^n_+) & \xleftarrow[j]{} & H_k(\mathbb{S}^n, *) & \simeq & \widetilde{H}_k\mathbb{S}^n \end{array}$$

The reduced long exact homology sequence of $(\mathbb{D}^n_-, \mathbb{S}^{n-1})$ reads as

$$\cdots \to \widetilde{H}_k(\mathbb{D}^n_-) \to H_k(\mathbb{D}^n_-, \mathbb{S}^{n-1}) \xrightarrow{\partial} \widetilde{H}_{k-1}(\mathbb{S}^{n-1}) \to \widetilde{H}_{k-1}(\mathbb{D}^n_-) \to \cdots$$

Since $\mathbb{D}^n_- \simeq \mathbb{D}^{n-1}$ is contractible, we deduce from theorem 1.17 that $\widetilde{H}_k(\mathbb{D}^n_-) = \widetilde{H}_{k-1}(\mathbb{D}^n_-) = 0$ and so ∂ is an isomorphism. By the same argument, j is an isomorphism, since $\{*\} \hookrightarrow \mathbb{D}^n_-$ is a homotopy equivalence. Now we can easily define σ_+ to be the unique isomorphism making the diagramme commute.

Lemma 1.19. $\widetilde{H}_n\mathbb{S}^0 \simeq H_n 1$ *for all* $n \in \mathbb{Z}$*.*

Proof. Choose a point $x : 1 \to \mathbb{S}^0$ and denote the other point by $y : 1 \to \mathbb{S}^0$. Let's also write $H_n(i) : \widetilde{H}_n\mathbb{S}^0 \hookrightarrow H_n\mathbb{S}^0$ for the standard inclusion. As seen in the section about the reduced homology and theorem Theorem 1.10, we get a commutative diagramme

$$\begin{array}{ccccc}
\widetilde{H}_n\mathbb{S}^0 \oplus H_n 1 & \xrightarrow[\sim]{\binom{i}{x_*}} & H_n\mathbb{S}^0 & \xleftarrow[\sim]{\binom{y_*}{x_*}} & H_n 1 \oplus H_n 1 \\
\uparrow & & & & \uparrow \\
H_n 1 & & = & & H_n 1
\end{array}$$

where the vertical arrows are the standard inclusions into the second summand. If we define the isomorphism

$$f := \begin{pmatrix} y_* \\ x_* \end{pmatrix}^{-1} \circ \begin{pmatrix} i \\ x_* \end{pmatrix} : \widetilde{H}_n\mathbb{S}^0 \oplus H_n 1 \xrightarrow{\sim} H_n 1 \oplus H_n 1,$$

we can rewrite this as a commutative diagramme

$$\begin{array}{ccccccccc}
0 & \to & H_n 1 & \rightarrowtail & \widetilde{H}_n\mathbb{S}^0 \oplus H_n 1 & \rightarrowtail & H_n\mathbb{S}^0 & \twoheadrightarrow & 0 \\
 & & \| & & \sim\downarrow f & & \downarrow \widehat{f} & & \\
0 & \to & H_n 1 & \rightarrowtail & H_n 1 \oplus H_n 1 & \rightarrowtail & H_n 1 & \twoheadrightarrow & 0
\end{array}$$

where the rows are exact and $\widehat{f}$ is the unique arrow between the cokernels, induced by f, making the diagramme commute. By the 3-lemma (which is the special case of the 5-lemma for short exact sequences) it follows that $\widehat{f}$ is an isomorphism. □

Theorem 1.20. *For all* $k \in \mathbb{Z}$ *and* $n \in \mathbb{N}$ *we have isomorphisms*

$$\widetilde{H}_k\mathbb{S}^n \simeq H_{k-n}1 \quad \text{and} \quad H_k\mathbb{S}^n \simeq H_{k-n}1 \oplus H_k 1.$$

It follows that we also have isomorphisms

$$H_k(\mathbb{D}^{n+1}, \mathbb{S}^n) \simeq \widetilde{H}_{k-1}\mathbb{S}^n \oplus H_{k-(n+1)}1.$$

Proof. As seen in the last paragraph, we have isomorphisms

$$\widetilde{H}_k\mathbb{S}^n \simeq \widetilde{H}_{k-1}\mathbb{S}^{n-1} \simeq \cdots \simeq \widetilde{H}_{k-n}\mathbb{S}^0 \simeq H_{k-n}1,$$

where we used the above lemma for the last isomorphism. By remembering ourselves that $H_k\mathbb{S}^n \simeq \widetilde{H}_k\mathbb{S}^n \oplus H_{k-(n+1)}1$ the first part of the proposition follows. For the second part, let's look at the reduced long exact homology sequence for $(\mathbb{D}^{n+1}, \mathbb{S}^n)$. Because $\mathbb{D}^{n+1}$ is contractible, this looks like

$$0 \to H_k(\mathbb{D}^{n+1}, \mathbb{S}^n) \xrightarrow{\partial} \widetilde{H}_k\mathbb{S}^n \to 0$$

and so ∂ is an isomorphism. □

Corollary 1.21. *Let $(H_n)_{n\in\mathbb{Z}}$, $(\partial_n)_{n\in\mathbb{Z}}$ be an ordinary homology theory having coefficient $H_0 1 \simeq G$, then for $n \in \mathbb{N}_{>0}$*

$$H_k\mathbb{S}^n \simeq \begin{cases} G & k \in \{0, n\} \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and similarly} \quad H_k(\mathbb{D}^{n+1}, \mathbb{S}^n) \simeq \begin{cases} G & k = n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

By noticing that an ordinary homology theory with non-trivial coefficient exists (e.g. singular homology, which will be treated in Chapter **??**), we easily deduce

Corollary 1.22 (Invariance of Dimension)**.** $\mathbb{R}^m \simeq \mathbb{R}^n \Leftrightarrow m = n$ *for* $m, n \in \mathbb{N}$.

Proof. The direction "$\Leftarrow$"is trivial and for the other direction, we assume that for $m \neq n$ we have a homeomorphism $\mathbb{R}^m \to \mathbb{R}^n$. The case where $m = 0$ or $n = 0$ is trivial and so the case $m, n \geq 1$ is left. We can extend our homeomorphism $\mathbb{R}^m \to \mathbb{R}^n$ to a homeomorphism of the one-point compactifications, which is $\mathbb{S}^m \to \mathbb{S}\mathbb{R}^n$ but since $m \neq n$ by the above theorem $H_n\mathbb{S}^m = 0$ but $H_n\mathbb{S}^n \simeq G \neq 0$. □

Corollary 1.23. $\mathbb{S}^n$ *is not contractible* $\forall n \in \mathbb{N}$.

Lecture 2

Acyclic Models

In this chapter we will be concerned with studying so-called acyclic models and will prove a form of the famous acyclic model theorem. The theory of acyclic models is in some sense a way to abstract the standard models arising in homology theory, like the standard simplices in singular homology (see Chapter 3). The acyclic model theorem will be useful in this text to prove homotopy invariance and the excision axiom for singular homology but generally finds wide applications throughout algebraic topology and homological algebra.

2.1 Models

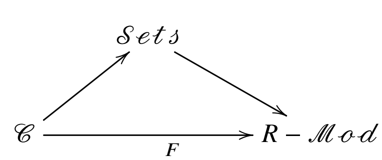
Let $\mathscr{C}$ be a category. A specified set $\mathscr{M} \subset \mathrm{Ob}(\mathscr{C})$ of objects in $\mathscr{C}$ will be called *models* of $\mathscr{C}$. From now on we fix the notation $\mathscr{M}$ to denote a set of models of a category.

Example 2.1. The intuition (and our primary use for that matter) is the following: There is a purely combinatorial theory of simplices known as simplicial sets. The easiest "models" of this theory in a topological context are the standard simplices $\Delta_q \subset \mathbb{R}^{q+1}$. And in fact, we will investigate $\{\Delta_q | q \in \mathbb{N}\}$ as models of $\mathscr{T\!op}$ in the next chapter using the tool(s) we are going to develop in this one.

Definition 2.2. A functor $F\colon \mathscr{C} \to R-\mathscr{M\!od}$ for R any ring will be called *free with models* $\mathscr{M}$ iff there is a subset $\mathscr{M}' \subset \mathscr{M}$ and for each $M \in \mathscr{M}'$ an element $e_M \in FM$ such that for every $C \in \mathrm{Ob}(\mathscr{C})$ the module FC is free and the set

$$\{(Fa)e_M \in FC | M \in \mathscr{M}', a \in \mathscr{C}(M, C)\}$$

forms a basis for FC. Put differently, the functor F factors as



where $\mathcal{Sets} \to R-\mathcal{Mod}$ is the free construction and the functor $\mathcal{C} \to \mathcal{Sets}$ maps $C \in \mathrm{Ob}(\mathcal{C})$ to the set described above and $b\colon C \to D$ in $\mathcal{C}$ to b_* with $b_*((Fa)e_M) := F(ba)e_M$. If F does map to $\mathcal{Ch}(R{-}\mathcal{Mod})$ (the category of chain maps between chain complexes of R-modules) we call F *free with models* $\mathcal{M}$ iff it is free with models $\mathcal{M}$ at each degree F_p, where for $p \in \mathbb{Z}$ $F_p\colon \mathcal{C} \to R-\mathcal{Mod}$ maps $a\colon C \to D$ to $(Fa)_p\colon (FC)_p \to (FD)_p$. One should notice that at each degree, we can have a different subset $\mathcal{M}_p \subset \mathcal{M}$ and different elements $e_M^p \in F_nM$ for $M \in \mathcal{M}_p$.

Finally, let's call a functor $F\colon \mathcal{C} \to \mathcal{Ch}(R-\mathcal{Mod})$ *a-acyclic on the models* $\mathcal{M}$ ("a-acyclic" stands for *almost acyclic)* iff for each $M \in \mathcal{M}$ the chain complex FM is exact everywhere except at the degree 0. I.e. $H_i(FM) = 0$ for all $i \in \mathbb{Z}\backslash\{0\}$. In the same spirit, we call a chain complex *a acyclic* iff it is acyclic except at degree 0.

Definition 2.3. Let $F, G\colon \mathcal{C} \to \mathcal{Ch}(R{-}\mathcal{Mod})$ be two functors and $\alpha, \beta\colon F \to G$ two natural transformations. We say that α and β are *naturally chain homotopic* or simply *naturally homotopic* iff all their components are chain homotopic in a natural way. That is for each $C \in \mathrm{Ob}(\mathcal{C})$ there is a chain map $\chi_C\colon FC \to GC$ of degree 1 (i.e. $(\chi_C)_n\colon F_nC \to G_{n+1}C, \forall n \in \mathbb{Z}$) such that

$$\alpha_C - \beta_C = \partial_{GC} \circ \chi_C + \chi_C \circ \partial_{FC},$$

where ∂_{FC} and ∂_{GC} denote the boundaries of FC and GC respectively. Furthermore, χ_C is required to be natural in C. That is, for each $a\colon C \to D$ in $\mathcal{C}$ the following diagramme commutes

$$\begin{array}{ccccc} C & \quad & FC & \xrightarrow{\chi_C} & GC \\ {\scriptstyle a}\downarrow & & {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Ga} \\ D & & FD & \xrightarrow[\chi_C]{} & GD \end{array}$$

So in some sense χ is a natural transformation $F \to G$, which is not completely honest since the components χ_C are not really arrows in $\mathcal{Ch}(R{-}\mathcal{Mod})$. To be even more formal, one could say that χ is a natural transformation $F \to S^- \circ G$ where $S^-\colon \mathcal{Ch}(R-\mathcal{Mod}) \to \mathcal{Ch}(R-\mathcal{Mod})$ is the shift functor that shifts a chain complex X by -1. So X is mapped to $X' = S^-X$, having $X'_n = X_{n+1}$ with the obvious boundaries (the arrow function of S^- is obvious, too).

2.2 The Acyclic Model Theorem

In this section we will state and prove a form of the acyclic model theorem.

Theorem 2.4 (Acyclic Model Theorem)**.** *Let $\mathcal{C}$ be a category with models $\mathcal{M}$ and $F, G\colon \mathcal{C} \to \mathcal{Ch}(R-\mathcal{Mod})$ functors which are* 0 *in negative degrees (i.e. $F_n = G_n = 0$ $\forall n \in \mathbb{Z}_{<0}$). If F is free with models $\mathcal{M}$ and G is a-acyclic on $\mathcal{M}$ and there is a natural transformation $\varphi\colon H_0F \to H_0G$ (where $H_0F, H_0G\colon \mathcal{C} \to$*

$R - \mathscr{Mod}$) then there is a natural transformation $\widetilde{\varphi}\colon F \to G$, which induces φ. Moreover, $\widetilde{\varphi}$ is unique up to natural homotopy.

Remark 2.5. For the sake of readability we will omit unnecessary indices in the following proof. We will assume that the attentive reader will still be capable of following it and fill in the details.

This proof seems a little complicated at first glance (which it really isn't). Because of this we will briefly sketch it before formalising it rigorously. For $C \in \mathrm{Ob}(\mathscr{C})$, we want to define $\widetilde{\varphi}$ as to make the diagramme

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial} & F_{n+1}C & \xrightarrow{\partial} & F_nC & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & F_0C & \twoheadrightarrow & H_0(FC) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \widetilde{\varphi}} & & & & \downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \varphi} & & \\
\cdots & \xrightarrow{\partial} & G_{n+1}C & \xrightarrow{\partial} & G_nC & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & G_0C & \twoheadrightarrow & H_0(GC) & \longrightarrow & 0
\end{array}$$

commute. We do this inductively in each degree n. To do so, we first consider the case where $C = M$ is a model and so the lower row is exact. We can use this exactness to "lift" $\widetilde{\varphi}$ from degree n to $n+1$. Afterwards we use the fact that F is free with models $\mathscr{M}$ to extend this to arbitrary $C \in \mathrm{Ob}(\mathscr{C})$.

Proof. By presumption, for each $n \in \mathbb{N}$ there is a collection $\mathscr{M}_n \subset \mathscr{M}$ and for each $M \in \mathscr{M}_n$ an element $e^n_M \in F_nM$ as in the definition of a free functor with models $\mathscr{M}$. Now, let $C \in \mathrm{Ob}(\mathscr{C})$ be arbitrary. Again by presumption, $F_0C = Z_0(FC)$ and $G_0C = Z_0(GC)$ are the cycles of FC and GC at degree 0 respectively. Thus, we get standard projections onto the 0^{th} homology modules as in the following diagramme

$$\begin{array}{ccc}
F_0C & \twoheadrightarrow & H_0(FC) \\
\downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \varphi} \\
G_0C & \twoheadrightarrow & H_0(GC)
\end{array}$$

Thus, we can augment F and G by (re)defining $F_{-1}C := H_0(FC)$ and $G_{-1}C := H_0(GC)$ and defining the standard projections as the boundary morphisms. By this, $\widetilde{\varphi}$ is defined in degree -1, where it is simply φ.

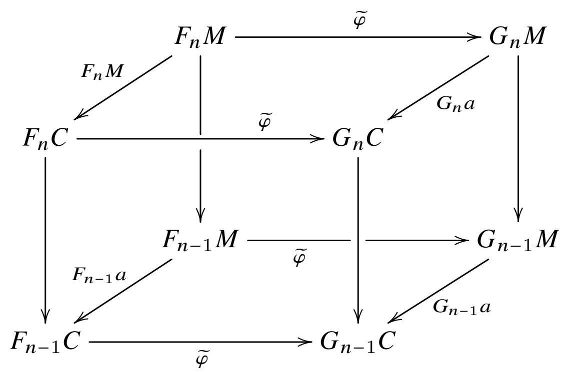
For the inductive step let's assume that $\widetilde{\varphi}$ is defined in degree $n-1$ for $n \geq 0$. For each model $M \in \mathscr{M}_n$ we consider $\widetilde{\varphi}(\partial e^n_M) \in G_{n-1}M$ (which we have already defined). Since the diagramme

$$\begin{array}{ccccc}
F_nM & \xrightarrow{\partial} & F_{n-1}M & \xrightarrow{\partial} & F_{n-2} \\
\downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \widetilde{\varphi}} \\
G_nM & \xrightarrow{\partial} & G_{n-1}M & \xrightarrow{\partial} & G_{n-2}
\end{array}$$

commutes and the lower row is exact (because G is a-acyclic on $\mathscr{M}$) we conclude that $\partial\widetilde{\varphi}(\partial e^n_M) = \widetilde{\varphi}(\partial\partial e^n_M) = 0$ and so $\widetilde{\varphi}(\partial e^n_M) \in G_{n-1}M$ must be a boundary.

I.e. we can choose $c \in G_n M$ satisfying $\partial c = \widetilde{\varphi}(\partial e^n_M)$ and define $\widetilde{\varphi} e^n_M := c$, which makes the above diagramme commute.

For $a \colon M \to C$ a morphism in $\mathscr{C}$ $(F_n a)e^n_M \in F_n C$ is a basis element of $F_n C$ and we define $\widetilde{\varphi}((F_n a)e^n_M) := (G_n a)\widetilde{\varphi} e^n_M$. We do this for every a and every M and like that define $\widetilde{\varphi}$ on the basis elements of $F_n C$ which means that we can extend it uniquely to $\widetilde{\varphi} \colon F_n C \to G_n C$. To check that $\widetilde{\varphi}$ thus defined is a chain morphism (i.e. commutes with the boundaries), we look at the following cubical diagramme



where the downward arrows are all boundary morphisms. The left and right faces of the cube obviously commute and the bottom commutes by inductive hypothesis. For the element $e^n_M \in F_n M$ the top and the back faces commute by definition of $\widetilde{\varphi}$. So the front face has to commute, too for $(F_n a)e^n_M \in F_n C$. Since this holds for all M and all a the front face commutes for all basis elements of $F_n C$ and so commutes as a whole. This defines $\widetilde{\varphi}$ in degree n. One easily checks the naturality of $\widetilde{\varphi}$, i.e. for $a \colon C \to D$ an arrow in $\mathscr{C}$ the equation $\widetilde{\varphi} \circ Fa = Ga \circ \widetilde{\varphi}$ holds.

What is left to prove is the uniqueness up to natural homotopy. So suppose that $\widetilde{\varphi}, \widetilde{\psi} \colon F \to G$ are two natural transformations inducing φ in the 0^{th} homology, or put differently, that lift $\widetilde{\varphi}$ (defined in degree -1). For each object $C \in \mathrm{Ob}(\mathscr{C})$ we must define a chain homotopy $\chi \colon FC \to GC$ from $\widetilde{\varphi}$ to $\widetilde{\psi}$ (i.e. χ is of degree 1 and satisfies $\partial\chi + \chi\partial = \widetilde{\varphi} - \widetilde{\psi}$), which is natural in C. χ is already defined in degree -1 (note that we are still working with the augmented F and G), where it is simply 0 because there holds $\widetilde{\varphi} = \widetilde{\psi} = \varphi$. Suppose now, that χ is defined in degree $n-1$ with $n \geq 0$. $F_n C$ has $\{(Fa)e^n_M\}_{M,a}$ as a basis and we notice that

$$b := \widetilde{\varphi} e^n_M - \widetilde{\psi} e^n_M - \chi(\partial e^n_M)$$

is a cycle because

$$\begin{aligned} \partial b &= \partial(\widetilde{\varphi} e^n_M - \widetilde{\psi} e^n_M - \chi(\partial e^n_M)) \\ &= \widetilde{\varphi}(\partial e^n_M) - \widetilde{\psi}(\partial e^n_M) - (\chi(\partial\partial e^n_M) + \widetilde{\varphi}(\partial e^n_M) - \widetilde{\psi}(\partial e^n_M) = 0. \end{aligned}$$

Because GM is a-acyclic, b must be a boundary, i.e. there is a $c \in G_{n+1}M$ with $\partial c = b$ and we define $\chi e^n_M := c$. By defining $\chi((F_n a)e^n_M) := (G_{n+1}a)\chi e^n_M$

we have defined χ on all basis elements and thence can extend it uniquely to $\chi : F_nC \to G_{n+1}C$. One can easily check that χ thus defined is really a chain morphism of degree 1 by using a cubical diagramme, similar to the one above. By construction this gives us a chain homotopy $\chi : \widetilde{\varphi} \simeq \widetilde{\psi}$ and it is plain to check naturality in C. □

Corollary 2.6. *If $F, G : \mathscr{C} \to \mathscr{Ch}(R - \mathscr{Mod})$ are functors which are* 0 *in negative degrees and both free and a-acyclic on $\mathscr{M}$ and there is natural isomorphism $\varphi : H_0F \widetilde{=} H_0G$, then φ can be extended to a natural isomorphism $\widetilde{\varphi} : HF \widetilde{=} HG$, where $H : \mathscr{Ch}(R - \mathscr{Mod}) \to \mathscr{Ch}(R - \mathscr{Mod})$ is the homology functor.*

Corollary 2.7. *If $F : \mathscr{C} \to \mathscr{Ch}(R - \mathscr{Mod})$ is* 0 *in negative degrees and both free with models $\mathscr{M}$ and a-acyclic on $\mathscr{M}$ and $\alpha : F \to F$ is a natural endo-transformation inducing the identity in* $0 -$ th *homology. Then α is naturally homotopic to* id_F*. In particular, for each $C \in \mathrm{Ob}(\mathscr{C})$ there is a chain homotopy $\alpha_C \simeq \mathrm{id}_{FC}$ (i.e. α_C and id_{FC} are chain homotopic).*

Proof. By the acyclic model theorem there is a natural transformation $\varphi : F \to F$ which induces $\mathrm{id}_{H_0F} : H_0F \to H_0F$ and is unique up to homotopy. But α and id_F are two such natural transformations and so α and id_F are naturally homotopic. □

Lecture 3

Singular Homology

In this chapter we are going to quickly repeat the definition of singular homology, mainly to introduce the reader to the notation used in this text. Afterwards we are going to prove the axioms for an ordinary homology theory in the case of singular homology.

3.0.1 Definitions

We repeat the definition of singular homology in this section. We do so, mainly, to fix the notation but we will also prove as a first result that the singular homology of contractible spaces vanishes in positive degrees.

3.0.2 Homotopy Invariance

As promised in chapter 2, we are going to prove, using the acyclic model theorem, that for X a topological space $\iota_0(X)$ and $\iota_1(X)$ induce the same morphisms in homology, here $\iota_t(X) : X \to X \times I$, $x \mapsto (x, t)$ for $t \in \{0, 1\}$ as defined in **??**. We are going to use the following models to apply the acyclic model theorem to singular homology.

Definition 3.1. For the rest of this ssection, we will write $\mathscr{S}$ for the collection

$$\mathscr{S} := \{\Delta_q | q \in \mathbb{N}\}$$

of objects of $\mathscr{Top}$ and call these the *standard models*. They will be key in applying the acyclic model theorem to singular homology.

Remark 3.2. The functor $S : \mathscr{Top} \to \mathscr{AbGrp}$ is free with models $\mathscr{S}$ and aacyclic (see the last paragraph of Definition 2.2) on $\mathscr{S}$. It is obviously aacyclic since all the Δ_q are contractible. The freeness has to be checked at each degree. To do so, let $p \in \mathbb{N}$ and we consider $\mathscr{S}' := \{\Delta_q\} \subset \mathscr{S}$ and the element $e_p := 1_{\Delta_p} \in S_p\Delta_p$. By definition for X any space, the set $\{\sigma_* e_p = \sigma \in S_pX | \sigma : \Delta_p \to X\}$ forms a basis for S_pX.

3.1 Barycentric Subdivision

3.2 Small Simplices and Standard Models

3.3 Excision

This section is solely devoted to proving the excision axiom for singular homology. Here our investment into the machinery of homological algebra (in the form of the acyclic model theorem) pays off again and the proof boils down to simple algebra with no geometric arguments whatsoever.