Introduction to Singular Homology Theory

Lemérange

Preface

These notes are the authour's attempt to make an *accessible* account on singular homology theory:

- Singular homology theory is introduced and is shown to satisfy certain axioms (of Eilenberg-Steenrod), and cellular spaces and their homology are explained.
- Singular cohomology theory is introduced together with certain products, and Poincaré duality and its application for manifolds are explained.

The authour has no claim for originality, possibly except for organisation. In fact, these notes are a hodgepodge of materials from literally tens of textbooks and lecture notes.

An excuse

There is made no special provision for category theory and homological algebra: in fact no section is devoted to category theory, and homological algebra required to understand universal coefficient theorems are given in an ad hoc fashion.

It is true that those two disciplines have originated from algebraic topology, but, without much care, the former tends be "abstract nonsense" (an expression coined by N. Steenrod) and the latter reduces to boredom as in the quotes bellow.

A recall by M. M. Postnikov

The assistant dean ... called me in to say, "*Postnikov, you need to have an advisor for your senior thesis. Who do you want, Lusternik or Alexandrov?*" At this point a very strange thing happened. Without a moment's hesitation I blurted out, "*I want Lev Semenovich Pontryagin!*" The reason why this was so strange is that I did not know Pontryagin, and had only taken his very boring, formalistic course of homological algebra.

An excerpt from "An Introduction to Homological Algebra" by J. J. Rotman

When I was a graduate student, Homological Algebra was an unpopular subject. The general attitude was that it was a grotesque formalism, boring to learn, and not very useful once one had learned it.

An excerpt from "Algebra: Chapter 0" by P. Aluffi

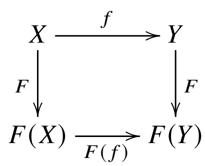
Proving the snake lemma is something that should not be done in public, and it is notoriously useless to write down the details of the verification for others to read: the details are all essentially obvious, but they lead quickly to a notational quagmire. Such proofs are collectively known as the sport of "diagram chase", best executed by pointing several fingers at different parts of a diagram on a blackboard, while enunciating the elements one is manipulating and stating their fate.

Contents

Lecture 1

Introduction

Topology is the study of topological spaces (e.g. subsets of $\mathbb{R}^n$) and continuous maps between them. The basic idea of *algebraic topology* is to study *functors* F from topological spaces to groups (or some other type of algebraic category). This means is that for every topological space X, we assign a group $F(X)$, and to each continuous map $f : X \to Y$, we assign a group homomorphism $F(f) : F(X) \to F(Y)$

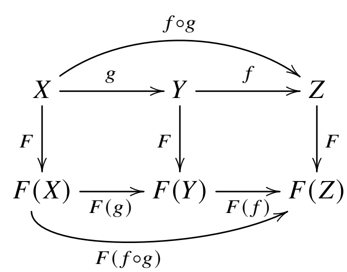


such that for any pair of composable maps

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

we have

$$F(f \circ g) = F(f) \circ F(g)$$



and also that identity maps are sent to identity maps:

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}\,.$$

To see how this sort of thing may be useful, observe that if two spaces X and Y are isomorphic (i.e. homeomorphic), then $F(X)$ and $F(Y)$ must be isomorphic for every functor F. It turns out that the most powerful way to prove that two spaces X, Y are *not* homeomorphic is to find a functor such that $F(X)$ and $F(Y)$ are not isomorphic.

For another application, we begin with a definition. A subset $A \subset X$ is called a *retract*, if there exists a continuous map $r : X \to A$ such that $f(a) = a$ for all $a \in A$: for example, the inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^2$ as the x-axis is a retract using the map $r(x, y) = (x, 0)$. Inclusion of sets defines an injective map $i : A \hookrightarrow X$. If A is a retract in X, then there exists r such that

$$r \circ i = \mathrm{id}_A .$$

For any functor, this means that

$$F(r) \circ F(i) = F(r \circ i) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)} .$$

In particular, this means that $F(i)$ must be injective when $A \subset X$ is a retract (if not, $F(r) \circ F(i) = \mathrm{id}_{F(A)}$ would not be injective, a contradiction). Using this idea, we will prove that the unit circle $\mathbb{S}^1$ is not a retract inside the unit disk $\mathbb{D}^2$.

The kinds of functors we will learn about in this course are the (*singular*) *homology and cohomology functors*. These functors come in families labelled by non-negative integers called the *degree* (also called *dimension*): $H_0, H_1, H_2, \ldots$ for homology and $H^0, H^1, H^2, \ldots$ for cohomology. Both homology and cohomology take values in abelian groups, though we will also study variations that take values in vector spaces.

The historical motivation for homology theory came from vector calculus. (The development of homology theory is usually attributed to the work of Poincaré in the late 19th to the early 20th, though the subject didn't really come into it's own until the 1930's through the work of numerous other mathematicians including S. Eilenberg and H. Whitney.) Recall that there are various versions of the Fundamental Theorem of Calculus (Stokes' Theorem, Green's Theorem, the Divergence Theorem) that equate an integral over a manifold (curve, surface, solid, etc.) with an integral over its boundary (set of points, a curve, surface, respectively). Homology emerged, from (more or less intuition-driven) efforts of mathematicians including J. W. Alexander, S. Lefschetz and O. Veblen, to understand how many "independent" submanifolds there are with respect to a given domain. Roughly speaking, the 0-homology $H_0(X)$ is generated by points in X, the 1-homology $H_1(X)$ is generated by (oriented) closed curves in X, the 2-homology is generated by (oriented) closed surfaces, and so on. The homology class is trivial if the curve, surface, etc. is the boundary of a surface, solid, etc ....

To see how this might work, consider the disconnected subset $X \subset \mathbb{R}^2$ pictured in Figure 1.1.

A point p in one component cannot be joined by a continuous path to a point q in another component. It follows that p and q determine different elements $[p]$ and $[q]$ in $H_0(X)$. We will show that there is an isomorphism $H_0(X) \cong \mathbb{Z}^n$ where n is the number of path-components of X.

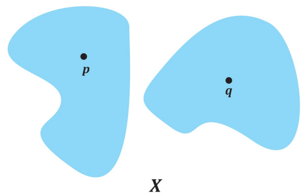


Figure 1.1: A space X with two path components

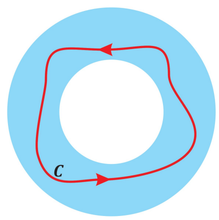


Figure 1.2: Loop in an annulus

Consider now a annulus A in $\mathbb{R}^2$ (Figure 1.2).

The closed loop C represents an element in $H_1(A)$. It is intuitively clear that $\mathbb{S}^1$ is not the boundary of a surface in A, so C represents a non-trivial element $[C]$ in $H_1(A)$. Indeed, we will show that $H_1(A) \cong \mathbb{Z}$ and that C represents one of the generators (the other generator is obtained by reversing the orientation on C). On the other hand, if we take a union of C with a curve D that winds around the annulus in the opposite direction, we see that together they form the boundary of a surface (Figure 1.3).

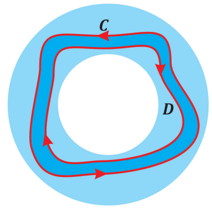


Figure 1.3: Two loop in an annulus bound a surface

There are many different kinds of homology that are defined in different ways. The approach we will take in this course is called *singular homology*. Singular homology has some great theoretical advantages over others (such as simplicial homology and cellular homology), but has the drawback of being difficult to calculate directly. Indeed, it will take some time before we establish the fact that oriented submanifolds determine homology classes which has been the basis of today's lecture. (It is possible to define a kind of homology theory using oriented manifolds directly, called *bordism*. One of the reasons that this approach is not standard in introductory courses is that the theory of manifolds gets complicated in dimensions greater than two.) Instead, singular homology is based on "singular simplices" which we will be introduced in Chapter 3.

Lecture 2

Review of Point-Set Topology

We collect some basic facts from general topology that will be required in this course. Proofs of these results can be found in any introductory textbook on general topology (e.g, Jänich *Topology*, and Viro et al. *Elementary Topology Problem Textbook*.)

Definition 2.1. A *topological space* (or simply *space*) (X, τ) is a set X and a collection τ of subsets of X, called the *open sets*, satisfying the following conditions:

i) $\varnothing$ and X are open,

ii) Any union of open sets is open,

iii) Any finite intersection of open sets is open.

A set is called *closed* if its complement is open. Usually, we will denote the topological space (X, τ) simply by X.

Example 2.2 (Euclidean Topology). An *open ball* in $\mathbb{R}^n$ is a set of the form

$$B = B_\epsilon(p) := \{x \in \mathbb{R}^n | \|x - p\| < \epsilon\}$$

for some $p \in \mathbb{R}^n$ and $\epsilon > 0$. A subset $U \subset \mathbb{R}^n$ is called *open* if it is a union of open balls. Equivalently, U is open if for every $p \in U$, there exists an open ball B such that $p \in B \subset U$.

In the example above, we say that open balls form a basis for the Euclidean topology. More generally, a collection of open sets $\mathfrak{B}$ in a topological space X is called a *basis* if every other open set in X is a union of sets in $\mathfrak{B}$.

Definition 2.3. A *continuous map* $f : X \to Y$ between topological spaces is a map of sets for which pre-images of open sets are open. I.e.

$$U \subset Y \quad \text{is open} \quad \Rightarrow \quad f^{-1}(U) := \{x \in X | f(x) \in U\} \subset X \quad \text{is open}$$

Definition 2.4. A *homeomorphism* is a continuous bijection $f : X \to Y$ such that the inverse f^{-1} is also continuous. This is the notion of isomorphism for topological spaces.

We will only rarely need use the abstract Definition 2.3 explicitly. More often, we will make use of certain properties of continuous functions, including the following.

Proposition 2.5. *Let X, Y and Z be topological spaces.*

- *The identity map $\mathrm{id}_X : X \to X$ is continuous.*
- *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the composition $g \circ f : X \to Z$ is continuous.*
- *Any constant map $f : X \to Y$ is continuous.*

The first two conditions above make (topological spaces + continuous maps) into a category. We will speak more about categories later.

2.1 New spaces from old

Most of the topological spaces we encounter in this course are constructed from $\mathbb{R}^n$ using the operations below.

Definition 2.6. Let X be a topological space and $A \subset X$ a subset. The *subspace topology* on A is the topology for which $V \subset A$ is open if and only if $V = A \cap U$ for some open set U in X.

Example 2.7. Any subset of $\mathbb{R}^n$ acquires a subspace Euclidean topology. Unless otherwise stated, we will always assume subsets of $\mathbb{R}^n$ to have this topology.

The *inclusion map* $i : A \hookrightarrow X$ is continuous (with respect to the subspace topology). In fact, we have the following special property: A map $f : Y \to A$ from a topological space Y is continuous if and only if the composition $i \circ f : Y \to X$ is continuous.

Definition 2.8. The *product space* $X \times Y$ of two spaces X and Y is the Cartesian product of sets $X \times Y$, with a *basis* of open sets of the form $U \times V$ where $U \subset X$ and $V \subset Y$ are both open.

The above definition iterates to define products of any *finite* number of spaces (infinite products require a different definition).

Example 2.9. The n-fold product $\overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n}$ is homeomorphic to $\mathbb{R}^n$ with the Euclidean topology.

The key property of product spaces is that a map

$$F : Z \to X \times Y$$

is continuous if and only if the coordinate functions $F = (F_1, F_2)$ are continuous as maps from Z to X and to Y respectively.

Definition 2.10. Let $\{X_\alpha\}$ be a (possibly infinite) collection of spaces indexed by α. The *coproduct space* or mphdisconnected union $\coprod_\alpha X_\alpha$ is the *disjoint* union of the sets X_α with $U \subset \coprod_\alpha X_\alpha$ is open if and only if $U \cap X_\alpha$ is open for all $X\alpha$.

The inclusions $i_{\alpha_0} : X_{\alpha_0} \hookrightarrow \coprod_\alpha X_\alpha$ are all continuous. A map $F : \coprod_\alpha X_\alpha \to Y$ is continuous if and only if the composition $F \circ i_\alpha : X_\alpha \to Y$ are continuous for all $X\alpha$.

Definition 2.11. An *equivalence relation* on a set X is a relation $\sim$ satisfying, for all $x, y \in X$

(i) $x \sim x$

(ii) $x \sim y$ implies $y \sim x$

(iii) $x \sim y$ and $y \sim z$ implies $x \sim z$.

Given any relation R on X, we can generate the "smallest" equivalence relation $\sim_R$ such that xRy implies $x \sim_R y$. Explicitly, we define $x \sim_R y$ if and only if there exists a finite sequence $\{x_i \in X\}_{i=0}^n$ for $n \geq 0$ satisfying

$$\begin{aligned} &x_0 = x, \\ &x_n = y \quad \text{and}, \\ &x_i R x_{i-1} \quad \text{or} \quad x_{i-1} R x_i \quad \text{for all} \quad i = 1, \ldots, n. \end{aligned}$$

Given $x \in X$, the equivalence class of x is

$$[x] := \{y \in X | x \sim y\}$$

Notice that $[x] = [y]$ if and only if $x \sim y$. The equivalence classes determine a partition of X into disjoint sets. Let $E := \{[x] | x \in X\}$ be the set of equivalence classes (we will sometimes denote $E = X/\sim$). There is a canonical map

$$Q : X \to E, \quad x \mapsto [x]$$

called the *coset map*.

Definition 2.12. Let X be a topological space and let $\sim$ be an equivalence relation on the set underlying X. The *coset topology* or *induced topology* on E is the topology for which $U \subset E$ is open if and only if $Q^{-1}(U)$ is open in X.

Observe that $Q : X \to E$ is continuous and that a map $f : E \to Y$ is continuous if and only if $f \circ Q : X \to Y$ is continuous.

Example 2.13. Suppose X and Y are topological spaces, $A \subset X$ is a subspace, and $f : A \to Y$ is a continuous map. Define an equivalence relation on the coproduct $X \coprod Y$ generated by $f(a) \sim a$ for all $a \in A$. We say that the coset space $(X \coprod Y)/\sim$ is *obtained by attaching X to Y along A using f*.

Definition 2.14. Suppose G is a group, and X is a set. A *left action* of G on X is a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$, with the following properties:

(i) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $x \in X$ and all $g_1, g_2 \in G$.

(ii) $1 \cdot x = x$ for all $x \in X$.

Similarly, a *right action* is a map $X \times G \to X$, written $(x, g) \mapsto x \cdot g$, with the same properties except that composition works in reverse: $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$.

Any right action determines a left action in a canonical way, and vice versa, by the correspondence

$$g \cdot x = x \cdot g^{-1}.$$

Definition 2.15. For any $x \in X$, the set $G \cdot x = \{g \cdot x : g \in G\} \subset X$ is called the *orbit* of x. The action is said to be *transitive* if for every pair of points $x, y \in X$, there is a group element g such that $g \cdot x = y$, or equivalently if the orbit of each point is the entire space X. The action is said to be *free* if the only element of G that fixes any point in $X$is the identity; that is, if $g \cdot x = x$ for some x implies $g = 1$.

Example 2.16. Suppose X is a topological space, and G is a group that acts on X via homeomorphisms, i.e. $G \subset \mathrm{Aut}(X)$. Define two points in X equivalent if they lie in the same orbit of G. The coset space in this case is called the *orbit space* and is denoted X/G.

2.2 Connectedness and Path-Connectedness

Let I denote the unit interval $[0, 1] \subset \mathbb{R}$ with the Euclidean topology.

Definition 2.17. A space X is called *path-connected* if for any two points $p, q \in X$ there exists a continuous map $\gamma : I \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Definition 2.18. A space X is called *connected* if there is no proper subset $A \subset X$ which is both open and closed. ("Proper" means other than X or $\varnothing$, which are always both open and closed).

Observe that if $A \subset X$ is both open and closed, then the complement A^c is also both open and closed, and there is a natural isomorphism $A \coprod A^c \cong X$. Thus spaces that are not connected can be decomposed into a disconnected union of nonempty spaces.

Proposition 2.19. *Path-connected spaces are connected.*

The converse of Proposition 2.19 is not true in general. However all the connected spaces we encounter in this course will also be path-connected, as connected and not path-connected spaces are pathological.

The famous "topologist's sine curve" makes up a typical pathology: consider subsets T_0 and T_+ in $\mathbb{R}^2$.

$$T_0 = \{(x, y)|x = 0 \text{ and } y \in [-1, 1]\},$$
$$T_+ = \{(x, y)|x \in (0, 2/\pi] \text{ and } y = \sin(1/x)\}$$

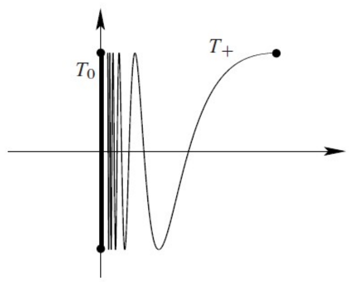


Figure 2.1: A topologist's sine curve

Connectedness and path-connectedness are preserved under the following operations

- A product of (path-)connected spaces is (path-)connected.
- The continuous image of a (path-)connected space is (path-)connected.
- Let $\{U_\alpha\}$ be a covering of X such that each U_α is (path-)connected and the intersection $\cap_\alpha U_\alpha$ is non-empty. Then X is (path-)connected.

2.3 Covers and Compactness

Definition 2.20. An open (closed) cover of a topological space X is a collection of open (resp. closed) sets $\{U_\alpha\}$ such that the union $\cup_\alpha c = X$.

Proposition 2.21. *Let $\{U_\alpha\}$ be either an open cover or a finite closed cover of X. A map of sets*

$$f : X \to Y$$

between topological spaces is continuous if and only if the restrictions $f|_{U_\alpha} : U_\alpha \to Y$ are continuous for all α (where U_α has the subspace topology).

The preceding proposition will be used in two ways: to test if a map f is continuous by considering the restrictions, and also to construct a map f by gluing together continuous maps defined on the U_α that agree on overlaps.

Definition 2.22. A space X is called *compact* if every open cover $\{U_\alpha\}$ of X contains a finite subcover. I.e., there exists a finite collection $\{U_1, \ldots, U_n\} \subset \{U_\alpha\}$ such that $\cup_{i=1}^n U_i = X$.

Proposition 2.23. *A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Compactness is preserved under the following:

- A closed subspace of a compact space is compact.
- A finite union of compact spaces is compact.
- A product of compact spaces is compact.
- If $f : X \to Y$ is continuous and X is compact, then the image $f(X) \subset Y$ is compact.

2.4 Metric spaces and the Lebesgue number lemma

Definition 2.24. Let X be a set. A *metric* on X is a function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

called the distance or metric function, satisfying

1. $d(x, x') = 0 \Leftrightarrow x = x'$ (d separates points)
2. $d(x, x') = d(x', x)$ (d is symmetric)
3. $d(x, x'') \leq d(x, x') + d(x', x'')$ (the triangle inequality)

A metric space (X, d) determines a *metric topology* on X, which is generated by the basis of open balls $B_\epsilon(p) = \{x \in X | d(x, p) < \epsilon\}$. If A is a subset of a metric space X then A becomes a metric space by restriction. The metric topology on A is the same as the subspace topology on A.

The following result will come up repeatedly.

Lemma 2.25 (Lebesgue number Lemma). *Let $\mathscr{A}$ be an open covering of a compact metric space X. There exists $\delta > 0$, called the* Lebesgue number*, such that for all $p \in X$, the open ball $B_\delta(p)$ is contained in some $U \in \mathscr{A}$.*

2.5 Hausdorff spaces

Definition 2.26. A space X is called *Hausdorff* if for any pair of distinct points $p, q \in X$, there exist open sets U, V such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$.

Proposition 2.27. *Any metric space is Hausdorff. In particular, any subset of $\mathbb{R}^n$ is Hausdorff.*

The Hausdorff property is preserved under the following:

- Products of Hausdorff spaces are Hausdorff.
- Subspaces of Hausdorff spaces are Hausdorff.
- Coproducts of Hausdorff spaces are Hausdorff.

Lecture 3

Singular Homology

In this chapter we are going to quickly review the definition of singular homology, mainly to introduce the reader to the notation used in this text. Afterwards we are going to prove the so called "axioms for an ordinary homology theory" in the case of singular homology.

3.1 Simplices

The *standard q-simplex* Δ_q is the simplex spanned by the zero vector $e_0 = \vec{0}$ and the standard basis vectors $e_1, \ldots, e_q$ in $\mathbb{R}^q$ (Figure 3.1). Thus,

$$\Delta_q := \{(t_1, \ldots t_n) | t_i \geq 0 \quad \forall i = 1, \ldots, q, \quad \text{and} \quad \sum_i t_i \leq 1\}$$

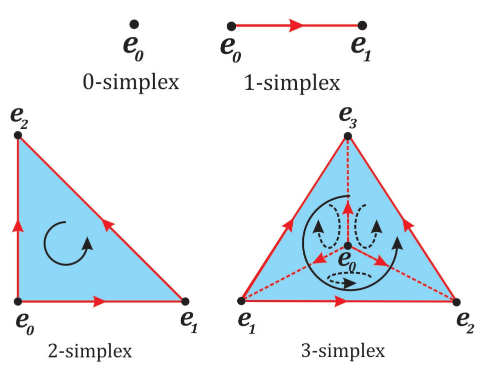


Figure 3.1: The standard simplices

If X is a topological space, a *singular q-simplex* (or simply *simplex*) in X is a (continuous) map

$$\sigma : \Delta_q \to X.$$

Thus a singular 0-simplex in X is simply a point in X, a singular 1-simplex in X is a continuous path in X, etc. We can think of singular simplices as probes used to study the space X.

Example 3.1. Let $v_0, \dots, v_q$ be a set of $q+1$-vectors in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Define

$$[v_0, \dots, v_q] : \Delta_q \to \mathbb{R}^n, \quad (t_1, \dots, t_n) \mapsto (1 - t_1 - \dots - t_q)v_0 + t_1 v_1 + \dots + t_q v_q.$$

We call $[v_0, \dots, v_q]$ the *affine simplex* defined by $v_0, \dots, v_q$. Slightly abusing notation, define the face maps for $0 \le i \le q$, define by

$$F_q^i : \Delta_{q-1} \to \Delta_q$$

by $F_q^i = [e_0, \dots, \widehat{e_i}, \dots, e_q]$ where the $\widehat{e_i}$ means "omit e_i". More specifically, F_q^i is the affine map that sends and therefore maps Δ_{q-1} homeomorphically onto

$$\begin{array}{ccc} e_0 & \mapsto & e_0 \\ \cdots & & \cdots \\ e_{i-1} & \mapsto & e_{i-1} \\ e_i & \mapsto & e_{i+1} \\ \cdots & & \cdots \\ e_{q-1} & \mapsto & e_q \end{array}$$

the boundary face of Δ_q opposite the vertex e_i.

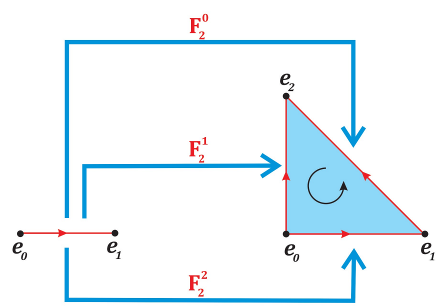


Figure 3.2: Faces of the standard 2-simplex

The *i-th face of a singular q-simplex* $\sigma : \Delta_q \to X$ is the $q-1$-simplex

$$\sigma^{(i)} : \Delta_{q-1} \to X$$

defined by composition with the face map:

$$\sigma^{(i)} := \sigma \circ F_q^i.$$

3.2 Chains, cycles, and boundaries

Define $S_q(X)$ to be the free Abelian group generated by singular q-simplices. The elements of $S_q(X)$ are called *singular chains* and are formal linear combinations of the form

$$\sum_\sigma a_\sigma \sigma$$

where the coefficients $a_\sigma \in \mathbb{Z}$ and the sum is over a finite number of singular q-simplices σ. By convention, $S_q(X) = 0$ for $q < 0$.

Caveat: Unless X is "small" (finite, say), $S_q(X)$, $q \geq 0$ is huge as a set.

The boundary map $\partial_q : S_q(X) \longrightarrow S_{q-1}(X)$ is a homomorphism, defied on singular simplices by

$$\partial_q(\sigma) = \sum_{i=0}^{q} (-1)^i \sigma^{(i)} = \sum_{i=0}^{q} (-1)^i \sigma \circ F_q^i$$

and extended linearly to all $S_q(X)$ by the rule

$$\partial_q(\sum \sigma a_\sigma \sigma) = \sum_\sigma \partial_q(\sigma).$$

We will often drop the subscript and write $\partial = \partial_q$ when it is unlikely to cause confusion.

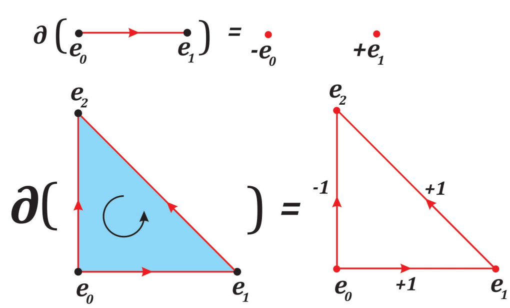


Figure 3.3: Faces of the standard 2-simplex

Example 3.2. Let σ_1 and σ_2 be singular 2-simplices in X. Then $-2\sigma_1 + 3\sigma_2 \in S_2(X)$ is a 2-chain and

$$\begin{aligned}\partial_2(-2\sigma_1 + 3\sigma_2) &= -2\partial_2(\sigma_1) + 3\partial_2(\sigma_2) \\ &= -2(\sigma_1^{(0)} - \sigma_1^{(1)} + \sigma_1^{(2)}) + 3(\sigma_2^{(0)} - \sigma_2^{(1)} + \sigma_2^{(2)}) \\ &= -2\sigma_1^{(0)} + 2\sigma_1^{(1)} - 2\sigma_1^{(2)} + 3\sigma_2^{(0)} - 3\sigma_2^{(1)} + 3\sigma_2^{(2)}\end{aligned}$$

is a 1-chain in $S_1(X)$.

The boundary map can be understood schematically from Figure 3.3, but be careful not to confuse singular simplices (which are maps) with their images (which are sets).

Proposition 3.3. *The composition $\partial_{q-1} \circ \partial_q : S_q(X) \to S_{q-2}(X)$ is the zero map. Dropping subscripts, we write this*

$$\partial^2 = 0.$$

Caveat: A bare-hand brute-force approach will turn out to be a notational nightmare. Note that 3.4 below circumvents difficulties in the proof.

Proof. Since $S_q(X)$ is generated by simplices, it suffices to check that $\partial_{q-1} \circ \partial_q(\sigma) = 0$ for all q-simplices σ.

First we note that the face maps satisfy the commutation relation

$$F^i_q \circ F^j_{q-1} = F^j_q \circ F^{i-1}_{q-1} \quad \text{when} \quad i > j, \tag{3.4}$$

as can be seen immediately by observing that the vertices of Δ_{q-2} are mapped according to the following chart: In other words, both $F^i_q \circ F^j_{q-1}$ and $F^j_q \circ F^{i-1}_{q-1}$

$$\begin{array}{ccccccccccc}
 & F^j_{q-1} & & F^i_q & & \qquad & & F^{i-1}_{q-1} & & F^j_q & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 & & e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} & & e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} \\
e_j & \mapsto & e_{j+1} & \mapsto & e_{j+1} & & e_j & \mapsto & e_j & \mapsto & e_{j+1} \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
e_{i-2} & \mapsto & e_{i-1} & \mapsto & e_{i-1} & & e_{i-2} & \mapsto & e_{i-2} & \mapsto & e_{i-1} \\
e_{i-1} & \mapsto & e_i & \mapsto & e_{i+1} & & e_{i-1} & \mapsto & e_i & \mapsto & e_{i+1} \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
e_{q-2} & \mapsto & e_{q-1} & \mapsto & e_q & & e_{q-2} & \mapsto & e_{q-1} & \mapsto & e_q
\end{array}$$

are equal to the affine simplex $[e_0, \ldots, \widehat{e_j}, \ldots, \widehat{e_i}, \ldots, e_q]$.

Using this, we compute

$$
\begin{aligned}
\partial_{q-1} \circ \partial q(\sigma) &= \partial_{q-1}(\sum_{i=0}^{q}(-1)^i \sigma^{(i)}) \\
&= \sum_{i=0}^{q}(-1)^i \partial_{q-1}(\sigma \circ F_q^i) \\
&= \sum_{i=0}^{q}(-1)^i \sum_{j=0}^{q-1}(-1)^j (\sigma \circ F_q^i \circ F_{q-1}^j) \\
&= \sum_{i=0}^{q}\sum_{j=0}^{q}(-1)^{i+j}(\sigma \circ F_q^i \circ F_{q-1}^j) \\
&= \sum_{0 \le i \le j \le q-1}(-1)^{i+j}(\sigma \circ F_q^i \circ F_{q-1}^j) + \sum_{0 \le j \le i \le q}(-1)^{i+j}(\sigma \circ F_q^i \circ F_{q-1}^j) \\
&= \sum_{0 \le i \le j \le q-1}(-1)^{i+j}(\sigma \circ F_q^i \circ F_{q-1}^j) - \sum_{0 \le j \le i-1 \le q-1}(-1)^{i-1+j}(\sigma \circ F_q^j \circ F_{q-1}^{i-1}) \\
&= 0
\end{aligned}
$$

as these two sums cancel term by term by changing the index of the first sum by $i \mapsto j$, $j \mapsto i-1$. □

A geometric illustration of $\partial^2 = 0$ is provided for $q = 2$ in Figure fig:3-4.

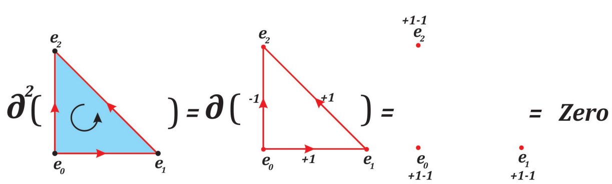


Figure 3.4: $\partial^2 = 0$ for $q = 2$

Definition 3.5. The group of *q-cycles* $Z_q(X)$ is the kernel of ∂_q:

$$Z_q(X) := \{\alpha \in S_q(X) | \partial(\alpha) = 0\}.$$

The group of *q-boundaries* $B_q(X)$ is the image of ∂_{q+1}:

$$B_q(X) := \{\partial(\beta) | \beta \in S_{q+1}(X)\}.$$

By Proposition 3.3, $B_q(X)$ is a subgroup of $Z_q(X)$: $B_q(X) \triangleleft Z_q(X)$. The qth degree *singular homology* of X is the coset group:

$$H_q(X) := Z_q(X)/B_q(X).$$

Example 3.6. The homology of a point. If $X = \{\text{pt}\}$ is a single point, then there is only one singular simplex in each degree, which is the constant map $\sigma_q : \Delta_q \longrightarrow \{\text{pt}\}$. The chain groups are[1]

$$C_q(\{\text{pt}\}) = \mathbb{Z}\sigma_q \cong \mathbb{Z}.$$

The boundary map satisfies (for $q \geq 1$)

$$\begin{aligned} \partial_q(\sigma_q) &= \sum_{i=0}^{q} (-1)^i \sigma q^{(i)} \\ &= \sum_{i=0}^{q} (-1)^i \sigma q - 1 \\ &= \begin{cases} \sigma_{q-1} & \text{if} \quad q \quad \text{is even} \\ 0 & \text{if} \quad q \quad \text{is odd} \end{cases} \end{aligned}$$

while $\partial_0(\sigma_0) = 0$.

It follows that for $q \geq 1$

$$Z_q(\{\text{pt}\}) = B_q(\{\text{pt}\}) = \begin{cases} 0 & \text{if} \quad q \quad \text{is even} \\ C_q(\{\text{pt}\}) & \text{if} \quad q \quad \text{is odd} \end{cases}$$

while $Z_0(\{\text{pt}\}) \cong \mathbb{Z}$ and $B_0(\{\text{pt}\}) = 0$. Thus

$$H_q(\{\text{pt}\}) = Z_q(\{\text{pt}\})/B_q(\{\text{pt}\}) = \begin{cases} 0 & \text{if} \quad q \geq 1 \\ \mathbb{Z} & \text{if} \quad q = 0 \end{cases}$$

Remark 3.7. A space X for which $H_q(X) \cong H_q(\{\text{pt}\})$ for all q is called *acyclic*, meaning no cycles that are not also boundaries.

Proposition 3.8. *Let $\{X_k\}$ be the set of path components of a space X (indexed by k). Then*

$$H_q(X) = \oplus_k H_q(X_k)$$

for all $q \geq 0$.

[1] The case of a point is highly unusual in this respect. For most spaces Y, $C_q(Y)$ is huge, with an uncountable rank.

Proof. Because the standard q-simplex is path connected (indeed convex), the image of a singular q-simplex $\sigma : \Delta_q \to X$ must be path connected and in particular must lie within one of the path components of X. It follows that for all q we have a canonical decomposition,

$$C_q(X) = \oplus_k C_q(X_k).$$

Moreover, it is clear that the boundary map ∂ respects this decomposition, so that

$$Z_q(X) = \oplus_k Z_q(X_k) \quad \text{and} \quad B_q(X) = \oplus_k B_q(X_k)$$

and finally that

$$\begin{aligned} H_q(X) &= Z_q(X)/B_q(X) \\ &= (\oplus_q Z_q(X_k))/(\oplus_q B_q(X_k)) \\ &= \oplus_k (Z_q(X_k)/B_q(X_k)) \\ &= \oplus_k H_q(X_k). \end{aligned}$$

□

We denote by $\pi_0(X)$ the set of path components of a space X.

Proposition 3.9. *There is a canonical isomorphism*

$$H_0(X) = \mathbb{Z}\pi_0(X).$$

Thus $H_0(X) \cong \mathbb{Z}^n$ where n is the number of path components of X.

Proof. By Proposition 3.8, it suffices to show that if X is path connected, then there is a canonical isomorphism

$$H_0(X) = \mathbb{Z}.$$

Recall that a singular 0−simplex is the same thing as a point in X. Thus

$$S_0(X) = Z_0(X) = \oplus_{p \in X} \mathbb{Z}p.$$

The standard one simplex Δ_1 is equal to the unit interval $[0,1] \subset \mathbb{R}$, so a singular 1-simplex is a continuous path in $\sigma : [0,1] \to X$. Since X is path-connected, for any two points $p, q \in X$, there exists a path σ such that $\sigma(0) = p$ and $\sigma(1) = q$. Consequently, the boundary satisfies

$$\partial(\sigma) = \sigma(1) - \sigma(0) = p - q \in S_0(X).$$

and thus

$$B_0(X) = \mathrm{Span}_{\mathbb{Z}}\{p - q | p, q \in X\} \subset \oplus_{p \in X} \mathbb{Z}p.$$

Observe that $B_0(X)$ is equal to the kernel of the homomorphism

$$\epsilon : \oplus_{p \in X} \mathbb{Z}p \to \mathbb{Z}$$

defined on generators by $\epsilon(p) = 1$. It follows that ϵ descends to a homomorphism

$$H_0(X) = Z_0(X)/B_0(X) \cong \mathbb{Z}.$$

□

3.2.1 Homology as a functor

Suppose that $f : X \to Y$ is a continuous map. If σ is a q-simplex for X, then the composition $f \circ \sigma$ is a q-simplex for Y. This defines a homomorphism

$$S_q(f) : S_q(X) \to S_q(Y), \quad S_q(f)(\sum_\sigma a_\sigma \sigma) = \sum_\sigma a_\sigma f \circ \sigma.$$

Clearly $S_q(\mathrm{id}\, X) = \mathrm{id}_{S_q(X)}$ and $S_q(f \circ g) = S_q(f) \circ S_q(g)$ for composable continuous maps f and g. Thus S_q is functor from topological spaces to abelian groups. It allows commuting with the boundary map.

Lemma 3.10. $\partial_q \circ S_q(f) = S_{q-1}(f) \circ \partial_q$.

Proof. It is enough to check for simplices.

$$\begin{aligned} \partial_q S_q(f)(\sigma) &= \partial(f \circ \sigma) \\ &= \sum_{i=0}^{q} (-1)^i (f \circ \sigma)(i) \\ &= \sum_{i=0}^{q} (-1)^i f \circ \sigma \circ F_q^i \\ &= \sum_{i=0}^{q} (-1)^i f \circ \sigma^{(i)} \\ &= S_q(f)(\sum_{i=0}^{q} (-1)^i \sigma^{(i)}) \\ &= S_q(f)(\partial \sigma) \end{aligned}$$

using associativity of composition. □

It follows then $S_q(f)(Z_q(X)) \subset Z_q(Y)$ and $S_q(f)(B_q(X) \subset B_q(Y)$ and thus induces a homomorphism between the coset groups

$$H_q(f) : H_q(X) \to H_q(Y).$$

Now, it easily follows from the fact that S_q is a functor that H_q is a *functor from topological spaces to abelian groups.* It is common to use short hand

$$f_* = H_q(f)$$

though we will try to avoid doing so, as "$*$" is overused.

3.3 Homotopy Invariance

Recall that two continuous maps $f, g : X \to Y$ are said to be *homotopic* if there exists a continuous map

$$h : X \times I \to Y$$

where $I = [0,1]$ is the unit interval and both $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Intuitively, two maps are homotopic if one can be continuously deformed into the other.

The goal of this section is to prove the following theorem.

Theorem 3.11. *Let f and g be homotopic maps from X to Y. For all $q \geq 0$, the induced maps on homology are equal: $H_q(f) = H_q(g)$.*

Recall that two spaces X and Y are called *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic id_X. That is,

$$\exists \Phi : X \times I \to X, \quad \Phi|_{X\times 0} = g \circ f, \quad \Phi|_{X\times 1} = \mathrm{id}_X,$$
$$\exists \Psi : Y \times I \to Y, \quad \Psi|_{Y\times 0} = f \circ g, \quad \Psi|_{Y\times 1} = \mathrm{id}_Y .$$

Corollary 3.12. *If X and Y are homotopy equivalent, then $H^*_q(X)(X) \cong H^*_q(Y)$ for all $q \geq 0$.*

Proof. By Theorem 3.11 and functoriality, we have

$$H_q(f) \circ H_q(g) = H_q(f \circ g) = H_q(\mathrm{id}_Y) = \mathrm{id}_{H_q(Y)}$$

and similarly $H_q(g) \circ H_q(f) = \mathrm{id}_{H_q(X)}$. The $H_q(f)$ and $H_q(g)$ are inverse isomorphisms between $H_q(X)$ and $H_q(Y)$. □

Recall that a space is called *contractible* if it is homotopy equivalent to a point. Examples of contractible spaces include all convex subspaces of $\mathbb{R}^n$. By Corollary 3.12, a contractible space X satisfies $H_q(X) = 0$ for $q \geq 1$ and $H_0(X) = \mathbb{Z}$ (i.e. contractible spaces are acyclic).

Before proving Theorem 3.11, it will be helpful to introduce some abstract (= purely algebraic) ideas about chain complexes.

3.3.1 Chain complexes and chain homotopy

Definition 3.13. A *chain complex* (*of abelian groups*)

$$C := (C_q, \partial_q)_{q\in\mathbb{Z}}$$

is a sequence of abelian groups $(C_q)_{q\in\mathbb{Z}}$ and homomorphisms $\partial_q : C_q \to C_{q-1}$ such that $\partial_{q-1} \circ \partial_q = 0$ for all $q \in \mathbb{Z}$.

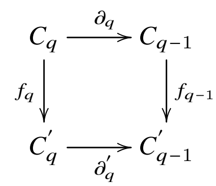
$$\cdots \xrightarrow{\partial_{q+3}} C_{q+2} \xrightarrow{\partial_{q+2}} C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} C_{q-2} \xrightarrow{\partial_{q-2}} \cdots$$

Typically $C_q = 0$ for $q < 0$.

Example 3.14. The singular chain complex $S(X) = (S_q(X), \partial_q)_{q\in\mathbb{Z}}$ is *indeed* a chain complex.

We define $Z_q(C) = \ker(\partial_q)$, $B_q(C) = \mathrm{im}(\partial_{q+1})$ and $H_q(C) = Z_q(C)/B_q(C)$, called respectively the *q-chains*, *q-boundaries*, and *q-homology groups* of the chain complex. If $z \in Z_q(C)$, denote by $[z] \in H_q(C)$ the coset represented by z.

Definition 3.15. A *morphism of chain complexes* (or a *chain map* for short) $f : C \to C^{'}$ is a sequence of homomorphisms $(f_q : C_q \to C^{'}_q)_{q\in\mathbb{Z}}$ that commutes with boundary maps, i.e., that satisfies $f_{q-1}\partial_q = \partial^{'}_q f_q$ for all q.



Example 3.16. Given a continuous map $\phi : X \to Y$, the morphisms $S_q(\phi) : S_q(X) \to S_q(Y)$ determine a chain morphism $S(\phi) : S(X) \to S(Y)$.

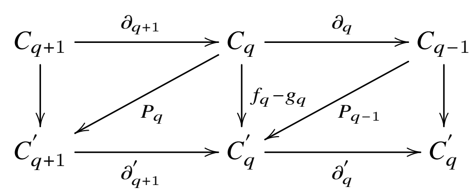
A chain map $f : C \to C^{'}$ induces a homomorphism in homology $H_q(f) : H_q(C) \to H_q(C^{'})$ for all $q \in \mathbb{Z}$ by the same reasoning as in §3.2.1 by the rule

$$H_q(f)([z]) = [f_q(z)].$$

Thus each H_q is a functor from chain complexes to abelian groups.

Let $f, g : C \to C^{'}$ be two chain maps. A *chain homotopy* between f and g is a sequence of homomorphism $(P_q : C_q \to C_{q+1})^{'})_{q\in\mathbb{Z}}$ such that

$$\partial^{'}_{q+1}P_q + P_{q-1}\partial_q = f_q - g_q.$$



The chain maps f and g are called *chain homotopic* if there exists a chain homotopy between them.

Proposition 3.17. *If chain maps $f, g : C \to C^{'}$ are chain homotopic, then the induced maps on homology are equal: $H_q(f) = Hq(g)$ as maps from $H_q(C)$ to $H_q(C^{'})$.*

Proof. Let $[z] \in H_q(C)$ be represented by $z \in Z_q(C)$. Then, since $\partial_q(z) = 0$,

$$f_q(z) - g_q(z) = P_{q-1}\partial_q(z) + \partial^{'}_{q+1}P_q(z) = \partial^{'}_{q+1}P_q(z)$$

is a boundary. Thus

$$H_q(f)([z]) - H_q(g)([z]) = [f_q(z)] - [g_q(z)] = [f_q(z) - g_q(z)] = [\partial^{'}_{q+1}P_q(z)] = 0$$

so $H_q(f)([z]) = H_q(g)([z])$. □

The prism operator

For $t \in I = [0,1]$, define

$$\iota_t : X \hookrightarrow X \times I, \quad \iota_t(x) = (x, t). \tag{3.18}$$

Lemma 3.19. *The two maps $\iota_0, \iota_1 : X \to X \times I$, determine chain homotopic chain morphisms $S(\iota_0)$ and $S(\iota_1)$.*

Proof. The first step is to define a decomposition of $\Delta_q \times I \subset \mathbb{R}^{n+1}$ into $(q+1)$-simplices. Denote the vertices lying in $\Delta_q \times \{0\}$ by $v_0, \ldots, v_q$ and those lying in $\Delta_q \times \{1\}$ by $w_0, \ldots, w_q$. For each i, the image of the affine simplex

$$[v_0, \ldots, v_i, w_{i+1}, \ldots, w_q] : \Delta_q \to \Delta_q \times I$$

can be thought of as the graph of a map from Δ_q to I, because composing the projection $\Delta_q \times I \to \Delta_q$ is the identity map. These graphs slice $\Delta_q \times I$ into the images of affine $(q+1)$-simplices

$$[v_0, \ldots, v_i, w_i, \ldots, w_q] : \Delta_{q+1} \to \Delta_q \times I, \quad i \in \{0, \ldots, q\}.$$

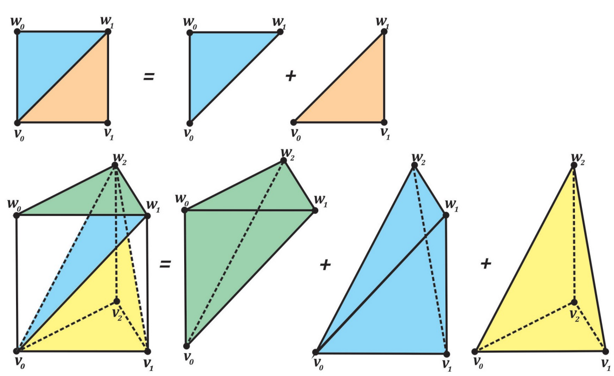


Figure 3.5: Prism decomposition

For arbitrary X, define the *prism operator* $P_q : S_q(X) \to S_{q+1}(X \times I)$ on simplices by

$$P_q(\sigma) := \sum_{i=0}^{q} (-1)^i (\sigma \times \mathrm{id}_I) \circ [v_0, \ldots, v_i, w_i, \ldots, w_q].$$

This will be our chain homotopy.

$$\begin{aligned}\partial_{q+1}P_q(\sigma) &= \partial_{q+1}\sum_{i=0}^{q}(-1)^i(\sigma\circ \mathrm{id}_I)\circ[v_0,\dots,v_i,w_i,\dots,w_q]\\ &= \sum_{i=0}^{q}\sum_{j\le i}(-1)^{i+j}(\sigma\circ \mathrm{id}_I)\circ[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_q]\\ &+\sum_{i=0}^{q}\sum_{j\ge i}(-1)^{i+j+1}(\sigma\circ \mathrm{id}_I)\circ[v_0,\dots,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_q]\end{aligned}$$

On the other hand,

$$\begin{aligned}P_{q-1}\partial_q(\sigma) &= P_{q-1}\sum_{i=0}^{q}(-1)^j(\sigma\circ\circ[e_0,\dots,\widehat{e}_j,\dots,e_q])\\ &= \sum_{i=0}^{q}\sum_{j\le i}(-1)^{i+j+1}(\sigma\circ \mathrm{id}_I)\circ[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_q]\\ &+\sum_{i=0}^{q}\sum_{j\ge i}(-1)^{i+j}(\sigma\circ \mathrm{id}_I)\circ[v_0,\dots,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_q]\end{aligned}$$

Adding together, we get

$$\begin{aligned}\partial_{q+1}P_q(\sigma)+P_{q-1}\partial_q(\sigma) &= \sum_{i=0}^{q}(\sigma\circ \mathrm{id}_I)\circ[v_0,\dots,\widehat{v}_i,w_i,\dots,w_q]\\ &-\sum_{i=0}^{q}(\sigma\circ \mathrm{id}_I)\circ[v_0,\dots,v_i,\widehat{w}_i,\dots,w_q]\\ &= \sigma\circ \mathrm{id}_I\circ[w_0,\dots,w_q]-\sigma\circ \mathrm{id}_I\circ[v_0,\dots,v_q]\\ &= \iota_1\circ\sigma-\iota_0\circ\sigma.\end{aligned}$$

□

It follows that

$$\partial_{q+1}P_q+P_{q-1}\partial_q = S_q(\iota_1)-S_q(\iota_0).$$

Proof. (of Theorem 3.11) A homotopy between two maps $f, g : X \to Y$ is a map $h : X \times I \to Y$ such that $f = h\circ\iota_0$ and $g = h\circ\iota_1$. By Lemma 3.19, $S_q(\iota_1)$ and $S_q(\iota_0)$ are chain homotopic, so Proposition 3.17 implies $H_q(\iota_0) = H_q(\iota_1)$. Finally we see that

$$H_q(f) = H_q(h)\circ H_q(\iota_0) = H_q(h)\circ H_q(\iota_1) = H_q(g).$$

□

In fact, it is not hard to show that $S(f)$ and $S(g)$ are chain homotopic via the chain homotopy $S_{q+1}(H)\circ P_q$.

3.4 Relative Homology and the long exact homology sequence

A topological pair (X, A) consists of a space X and a subspace $A \subset X$. A pair (X, A) gives rise to an inclusion of chain groups $S_q(A) \leq S_q(X)$ (technically the inclusion map $i : A \hookrightarrow X$ determines an injective homomorphism $S_q(i)$). Define the relative chain group of the pair to be the coset group

$$S_q(X, A) := S_q(X)/S_q(A).$$

The relative chain groups combine to form the relative chain complex

$$\cdots S_{q+1}(X, A) \xrightarrow{\overline{\partial}_{q+1}} S_q(X, A) \xrightarrow{\overline{\partial}_q} S_{q-1}(X, A) \xrightarrow{\overline{\partial}_{q-1}} \cdots$$

where the boundary map is defined by the following commutative diagramme

$$\begin{array}{ccc} S_q(X) & \xrightarrow{\partial_q} & S_{q-1}(X) \\ \downarrow & & \downarrow \\ S_q(X, A) & \xrightarrow[\overline{\partial}_q]{} & S_{q-1}(X, A) \end{array}$$

where the vertical arrows are coset maps.

Note that $\overline{\partial}_q$ is well defined because ∂_q sends $S_q(A)$ to $S_{q-1}(A)$ and that $\overline{\partial}^2 = 0$ because $\partial^2 = 0$. It follows that we can define relative cycles, relative boundaries, and relative homology as described in 3.3.1, which are denoted

$$Z_q(X, A), \quad B_q(X, A), \quad H_q(X, A)$$

respectively. Geometrically, a relative cycle in $Z_q(X, A)$ is represented by a chain in $Z_q(X)$ whose boundary lands in $S_{q-1}(A)$.

Remark 3.20. Observe that if $A = \emptyset$, then $S_q(A) = 0$ for all q. It follows that $S_q(X, \emptyset) = S_q(X)$ and that $H_q(X, \emptyset) = H_q(X)$. Thus it is possible to think of homology as just a special case of relative homology.

A *map of topological pairs*

$$f : (X, A) \to (X', A')$$

is a continuous map $f : X \to X'$ such that $f(A) \subset A'$. Such a map determines a morphism of chain complexes $S(f) : S(X, A) \to S(X', A')$ and thus also a homomorphism on homology

$$H_q(f) : H_q(X, A) \to H_q(X', A').$$

The following properties are proven similarly to their counterparts for $H_q(X)$.

- H_q is functor from topological pairs to Abelian groups. I.e. $H_q(f) \circ H_q(g) = H_q(f \circ g)$ and $H_q(\mathrm{id}_{(X,A)}) = \mathrm{id}_{H_q(X,A)}$.
- If $\{X_k\}$ is the set of path components of X and $A_k = A \cap X_k$, then there is a canonical isomorphism $H_q(X, A) = \oplus_k H_q(X_k, A_k)$.
- Let $h : X \circ I \to X'$ be a homotopy between maps such that $h_t(a) := h(a, t) \in A'$ for all $a \in A$ and $t \in I$. Then $H_q(h_0) = H_q(h_1)$ as homomorphisms from $H_q(X, A)$ to $H_q(X', A')$.

The coset morphisms $S_q(X) \to S_q(X, A)$ fit together into a morphism of chain complexes $j : S(X) \to S(X, A)$. Combined with inclusion chain morphism $i : S(A) \to S(X)$ we get a commutative diagramme (Recall that $S_q(X, A) = S_q(X)/S_q(A)$.)

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & S_{q+1}(A) & \longrightarrow & S_q(A) & \longrightarrow & S_{q-1}(A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\ \cdots & \longrightarrow & S_{q+1}(X) & \longrightarrow & S_q(X) & \longrightarrow & S_{q-1}(X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\ \cdots & \longrightarrow & S_{q+1}(X,A) & \longrightarrow & S_q(X,A) & \longrightarrow & S_{q-1}(X,A) & \longrightarrow & \cdots \end{array} \tag{3.21}$$

By functoriality, these chain morphisms give rise to homology homomorphisms $H_q(A) \to H_q(X) \to H_q(X, A)$ for all $q \geq 0$. The most important property of relative homology is the existence of a *connecting homomorphism*

$$H_q(X, A) \xrightarrow{\partial} H_{q-1}(A),$$

(denoted by ∂ using abuse of notation). To show is, we need some homological algebra as below.

3.4.1 Some homological algebra

For a space X we defined (see 3.14) the "singular chain complex" $S(X) = (S_q(X), \partial_q)_{q \in \mathbb{Z}}$ of X. From that point, the definition of the homology groups $H_q(X)$ and some simple properties were derived completely algebraically. Such "chain complexes" can, and will, occur in other contexts. Accordingly, it is very useful to abstract the algebraic part of the process, in order to apply it to future situations.

Recall that (see 3.13) a *chain complex* (*of abelian groups*)

$$C := (C_q, \partial_q)_{q \in \mathbb{Z}}$$

is a sequence of abelian groups $(C_q)_{q \in \mathbb{Z}}$ and homomorphisms $\partial_q : C_q \to C_{q-1}$ such that $\partial_{q-1} \circ \partial_q = 0$ for all $q \in \mathbb{Z}$.

As defined in 3.5, the *homology* of a chain complex $C = (C_q, \partial_q)_{q\in\mathbb{Z}}$ is

$$H(C) := (\ker \partial_q : (C_q \to C_{q-1})/\operatorname{im} \partial_{q+1} : (C_{q+1} \to C_q))_{q\in\mathbb{Z}}$$

Thus $H_q(X) = H_q(\Delta(X))$.

As defined in 3.15, a chain map $f : A_* \to B_*$ between two chain complexes $A_* = \{A_q\}_{q\in\mathbb{Z}}$ and $B_* = \{B_q\}_{q\in\mathbb{Z}}$ is a collection of homomorphisms $f_q : A_q \to B_q$such that $f \circ \partial = \partial \circ f$. In other words, a chain map is a "ladder" of homomorphisms which commutes:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & A_{q+1} & \xrightarrow{\partial} & A_q & \xrightarrow{\partial} & A_{q-1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow f & & \downarrow f & & \downarrow f & & \\
\cdots & \xrightarrow[\partial]{} & B_{q+1} & \xrightarrow[\partial]{} & B_q & \xrightarrow[\partial]{} & B_{q-1} & \xrightarrow[\partial]{} & \cdots
\end{array}$$

A chain map $f : A_* \to B$ induces a homomorphism of graded groups $f_* : H_*(A_*) \to H_{**}B_*$ by $f[a] = [f(a)]$, such that $(f \circ g)_* = f_* \circ g_*$ and $1_* = 1$.

Definition 3.22. A sequence of groups $A \xrightarrow{i} B \xrightarrow{j} C$ is called *exact* if $\operatorname{im}(j) = \ker(j)$.

Exact sequences are common and fundamental in algebraic topology. Note that an exact sequence of the form $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ means that i is an isomorphism of A onto a subgroup of B and j induces an isomorphism of $B/i(A)$ onto C. Also note that to say that $k : A \to B$ is an isomorphism (onto), is the same as to say that $0 \to A \xrightarrow{k} B \to 0$ is exact.

Theorem 3.23. *A "short" exact sequence* $0 \to A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \to 0$ *of chain complexes and chain maps induces a "long" exact sequence*

$$\cdots \xrightarrow{\partial_*} H_p(A_*) \xrightarrow{i_*} H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\partial_*} H_{p-1}(A_*) \xrightarrow{i_*} \cdots \qquad (3.24)$$

where $\partial_*[[c]] = [[i^{-1} \circ \partial \circ j^{-1}(c)]]$ *and is called the "connecting homomorphism".*

Proof. The arguments in this proof are of a type called "diagramme chasing" consisting, in this case, of carrying elements around in the diagramme

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{p+1} & \xrightarrow{i} & B_{p+1} & \xrightarrow{j} & C_{p+1} & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & A_p & \xrightarrow{i} & B_p & \xrightarrow{j} & C_p & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & A_{p-1} & \xrightarrow{i} & B_{p-1} & \xrightarrow{j} & C_{p-1} & \longrightarrow & 0
\end{array}$$

We will see this one through in detail, but later such arguments will be abbreviated, since they are almost always straightforward to one with previous experience.

∂_* is well defined : We begin by checking that the definition given for ∂_* really does define a unique homomorphism $H_p(C_*) \longrightarrow H_{p-1}(A_*)$. Suppose given $c \in C_p$ such that $\partial c = 0$. Since j is onto there is a $b \in B_p$ with $c = j(b)$. Then $j(\partial b) = \partial(j(b)) = \partial(c) = 0$. By exactness there is a unique element $a \in A_{p-1}$ such that $i(a) = \partial b$. Then $i(\partial a) = \partial(i(a)) = \partial\partial b = 0$. Thus $\partial a = 0$ since i is a monomorphism. Therefore $[[a]] \in H_{p-1}(A*)$ is defined. As indicated $\partial_*[[c]]$ is defined to be $[[a]]$.

We must show that this does not depend on the choices of b and of c within its homology class. First suppose $c = j(b^{'})$, so that $j(b - b^{'}) = 0$. Then $b - b^{'} = i(a_0)$ for some $a_0 \in A_p$. Thus $\partial b - \partial b^{'} = \partial(i(a_0)) = i(\partial a0)$. But the left-hand side of this equation is $\partial b - \partial b^{'} = i(a) - i(a^{'}) = i(a - a^{'})$. It follows that $a - a^{'} = \partial a_0$ and so $a \sim a^{'}$ as desired.

We now consider the effect of changing c within its homology class. Let $c^{'} = c + \partial c^{''}$. Then we can set $c = j(b)$ and $c^{''} = j(b^{''})$. Let $b^{'} = b + \partial b^{''}$. We calculate $j(b^{'}) = j(b) + j(\partial b^{''}) = c + \partial c^{''} = c^{''}$ But $\partial b^{'} = \partial b + \partial\partial b^{''} = \partial b$ and so ∂b and $\partial b^{'}$, being equal, pull back to the same thing under i^{-1}.

One also must show that ∂_* is a homomorphism. But for two classes c and $c^{'}$, we can trace the definition back for both and, at any stage, the addition of the elements going into the definition work for the sum $c + c^{'}$ which proves this contention.

The "long" sequence 3.24 is exact : First, we show it is of "order two" (i.e., the composition of adjoining homomorphisms is zero). There are three cases. First, $j_* i_* = (j \circ i)_* = 0_* = 0$.

Second, consider $\partial_* j_*[[b]]$, where $\partial b = 0$. By definition of ∂_*, this is obtained by taking b, then applying ∂ to it, giving $\partial b = 0$, and pulling this back (to 0) to A_*.

Third, consider $i\partial_*$. This is the result of taking an element of $C*$, pulling it back to B_*, taking ∂ of it, pulling that back to A_* (this being the ∂_* part) and then pushing this out to B_* again. But this element of B_* is, by construction, ∂ of something, which has homology class 0, as claimed.

Now we must show that an element in the kernel of one of the maps i_*, j_* or ∂_* is in the image of the preceding one. Again the proof of this has three cases.

First, we show the exactness at $H_*(B_*)$. Suppose that $j_*[[b]] = 0$. This means that $j(b) = \partial c$ for some $c \in C_*$. Let $b^{'} \in B_*$ be such that $j(b^{'}) = c$. Then

$$j(b - \partial b^{'}) = j(b) - j(\partial b^{'}) = \partial c - \partial(j(b^{'})) = \partial c - \partial c = 0.$$

This shows that we could have taken the representative b of its homology class to be such that $j(b) = 0$. For this choice, then, $b = i(a)$ for some $a \in A_*$ (and $\partial a = 0$ since it maps, by the monomorphism i, into $\partial b = 0$). Thus $[[b]] = i_*[[a]]$ as claimed.

Second, for the exactness at $H_*(A_*)$, suppose that $i_*[[a]] = 0$. Then $i(a) = \partial b$ for some $b \in B_*$. Then put $c = j(b)$. We have $\partial c = \partial j(b) = j(\partial b) = j(i(a)) = 0$. Thus c represents a homology class, and by construction of ∂_*, $\partial_*[[c]] = [[a]]$.

Third, for the exactness at $H_*(C_*)$, suppose that $\partial_*[[c]] = 0$. Then for an element $b \in B_*$ for which $j(b) = c$, there is an $a \in A*$ such that $i(a) = \partial b$, by the construction of ∂_*, and a must be a boundary since it represents $\partial_*[[c]] = 0$. Thus let $a = \partial a^{'}$. Then $\partial i(a^{'}) = i(\partial a^{'}) = i(a) = \partial b$. Accordingly, $\partial(b - i(a^{'})) = 0$ and $j(b - i(a^{'})) = c - 0 = c$. Therefore $j_*[[b - i(a^{'})]] = [[c]]$ as required. □

The connecting homomorphism in the long exact homology sequence satisfies an important naturality property.

Proposition 3.25 (Naturality of the Connecting Homomorphism)**.** *Suppose*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_* & \xrightarrow{F} & D_* & \xrightarrow{G} & E_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} & & \\ 0 & \longrightarrow & C^{'}_* & \xrightarrow[F^{'}]{} & D^{'}_* & \xrightarrow[G^{'}]{} & E^{'}_* & \longrightarrow & 0 \end{array} \tag{3.26}$$

is a commutative diagramme of chain maps in which the horizontal rows are exact. Then the following diagramme commutes for each p:

$$\begin{array}{ccc} H_p(E_*) & \xrightarrow{\partial *} & H_{p-1}(C_*) \\ {\scriptstyle\epsilon_*}\downarrow & & \downarrow{\scriptstyle\kappa_*} \\ H_p(E^{'}_*) & \xrightarrow[\partial *]{} & H_{p-1}(C^{'}_*). \end{array}$$

Proof. Let $[e_p] \in H_p(E_*)$ be arbitrary. Then $\partial * [e_p] = [c_{p-1}]$, where $Fc_{p-1} = \partial d_p$ for some d_p such that $Gd_p = e_p$. Then by commutativity of (3.26),

$$\begin{aligned} F^{'}(\kappa c_{p-1}) &= \delta F c_{p-1} = \delta\partial d_p = \partial(\delta d_p); \\ G^{'}(\delta d_p) &= \epsilon G d_p = \epsilon e_p. \end{aligned}$$

By definition, this means that

$$\partial_* \epsilon_* [e_p] = \partial_*[\epsilon_* e_p] = [\kappa c_{p-1}] = \kappa * [c_{p-1}] = \kappa * \partial * [e_p].$$

which was to be proved. □

The following lemma is helpful for calculations:

Lemma 3.27 (The 5-lemma)**.** *Consider a commutative diagramme of abelian groups*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ {\scriptstyle\cong}\downarrow{\scriptstyle\alpha} & & {\scriptstyle\cong}\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & {\scriptstyle\cong}\downarrow{\scriptstyle\delta} & & {\scriptstyle\cong}\downarrow{\scriptstyle\epsilon} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

where the rows are exact. If α, β, γ and ϵ are all isomorphisms, then γ is also an isomorphism.

Proof. This is a fairly straightforward diagramme chase. First suppose $a_3 \in \ker(\gamma)$. Then a_3 maps into $\ker(\delta) = 0$, so that a_3 comes from some $a_2 \in 2$ by exactness. If we push a_2 to $b_2 \in B2$ then that goes to 0 in B_3 and thus comes from some $b_1 \in B_1$, and in turn that can be lifted to $a_1 \in A_1$. But a_1 maps to a_2, since the images of these in B_2 are equal. But then a_1 maps to a_3 and so the latter is 0 by exactness. This shows that γ is a monomorphism.

Now, forgetting the above notation, let $b_3 \in B_3$. Map this to $b_4 \in B_4$ and pull it up to $a_4 \in A_4$. This must map to 0 in A_5 since it goes to 0 in B_5. By exactness, there is an $a_3 \in A_3$ mapping to a_4. If we map this to B_3 and subtract it from the original b_3 ($= b_3 - \gamma(a_3)$) we conclude that this goes to 0 in B_4. Accordingly, we may as well assume that the original b_3 maps to 0 in B_4, and hence comes from some $b_2 \in B_2$. Pulling this up to A_2 and pushing it into A_3 gives us an element that maps to b_3, showing that γ is onto. □

Let's get back to topology. Recall that if $A \subset X$ be a pair of spaces, then $S_q(A)$ is a subgroup of $S_q(X)$ and the inclusion is a chain map. As we defined $S_q(X, A) = S_q(X)/S_q(A)$, we have am exact sequence of chain complex

$$0 \to S(A) \to S(X) \to S(X, A) \to 0 \tag{3.28}$$

Since we defined $H_q(X, A) = H_q(S(X, A))$, by applying Theorem 3.23 to the sequence 3.28 we obtain

Theorem 3.29. *The following sequence is exact.*

$$\cdots \to H_{q+1}(X, A) \xrightarrow{\partial} H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots$$

It is called the long exact homology sequence *associated to the pair (X, A).*

Denote by $\pi_0(X, A)$ the set of path components of X that do not intersect A.

Proposition 3.30. *There is a canonical isomorphism*

$$H_0(X, A) \cong \mathbb{Z}\pi_0(X, A).$$

In particular, $H_0(X, A) \cong \mathbb{Z}^m$ where m is the number of path components of X that do not intersect A.

Proof. Let $i : A \hookrightarrow X$ denote the inclusion of A into X. Then we have an exact sequence

$$H_0(A) \xrightarrow{H_0(i)} H_0(X) \xrightarrow{H_0(j)} H_0(X, A) \to 0.$$

Exactness implies that $H_0(X, A) \cong H_0(X)/\mathrm{im}(H_0(i))$. We know (Proposition 3.9) that $H_0(X) = \mathbb{Z}\pi_0(X)$ and the image of $H_0(i)$ is generated by those path components of X that contain a path component of A. The result follows. □

One of the great merits of the long exact homology sequence is that it is functorial with respect to maps of pairs.

Proposition 3.31. *Let $f : (X, A) \to (Y, B)$ be a map of pairs. This induces homomorphisms on homology such that the following diagramme commutes.*

$$\begin{array}{ccccccccccccc}
\longrightarrow & H_{q+1}(X,A) & \xrightarrow{\partial} & H_q(A) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X,A) & \xrightarrow{\partial} & H_{q-1}(A) & \longrightarrow \\
& \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \\
\longrightarrow & H_{q+1}(Y,B) & \xrightarrow[\partial]{} & H_q(B) & \longrightarrow & H_q(Y) & \longrightarrow & H_q(Y,B) & \xrightarrow[\partial]{} & H_{q-1}(B) & \longrightarrow
\end{array}$$

Proof. The commutativity of squares *not* involving ∂ are evident, as all involved homomorphisms are induced from commuting continuous maps. The commutativity of squares involving ∂ can be proven using Proposition 3.25. □

Proposition 3.32. *If $f : (X, A) \to (Y, B)$ is a map of pairs such that two out of three families of induced maps on homology*

$$\{H_q(A) \to H_q(B)\}_{q\in\mathbb{Z}},$$
$$\{H_q(X) \to H_q(Y)\}_{q\in\mathbb{Z}},$$
$$\{H_q(X,A) \to H_q(Y,B)\}_{q\in\mathbb{Z}},$$

are isomorphisms in all degree, then the remaining family is isomorphisms in all degrees.

Proof. Apply the 5-lemma 3.27 to the diagramme:

$$\begin{array}{ccccccccccccc}
\longrightarrow & H_{q+1}(X,A) & \xrightarrow{\partial} & H_q(A) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X,A) & \xrightarrow{\partial} & H_{q-1}(A) & \longrightarrow \\
& \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \\
\longrightarrow & H_{q+1}(Y,B) & \xrightarrow[\partial]{} & H_q(B) & \longrightarrow & H_q(Y) & \longrightarrow & H_q(Y,B) & \xrightarrow[\partial]{} & H_{q-1}(B) & \longrightarrow
\end{array}$$

□

Example 3.33. If a map of pairs $f : (X, A) \to (Y, B)$ restricts to homotopy equivalences between X and Y and between A and B, then $f_* : H_q(X, A) \to H_q(Y, B)$ is an isomorphism in all degrees.

3.4.2 Reduced Homology

It is sometimes convenient to use a modified version of singular homology called *reduced homology*. For any space X, there exists a unique map to a point $\epsilon : X \to \{\text{pt}\}$ called "augmentation". Define the reduced homology

$$\widetilde{H}_q(X) := \ker H_q(\epsilon).$$

It is an easy consequence of (3.6) that if X has n path components,

$$\widetilde{H}_q(X) \cong \begin{cases} H_q(X) & \text{if} \quad q \geq 1 \\ \mathbb{Z}^{n-1} & \text{if} \quad q = 0 \end{cases}$$

More canonically, $\widetilde{H}_0(X)$ is the kernel of the map $\mathbb{Z}\pi_0(X) \to \mathbb{Z}$ that sends each generator to 1. For relative homology we define

$$\widetilde{H}_q(X, A) = H_q(X, A)$$

if $A \neq \varnothing$. Basically, reduced homology is designed so that $H_q(\{\text{pt}\}) = 0$ for all degrees without exception and this sometimes makes calculations less clumsy.

Functoriality, homotopy invariance, and the long exact sequence all work for relative homology. In particular, if A is non-empty, then we have a long exact sequence

$$\cdots \to \widetilde{H}_{q+1}(X, A) \xrightarrow{\partial} \widetilde{H}_q(A) \to \widetilde{H}_q(X) \to \widetilde{H}_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots \tag{3.34}$$

Remark 3.35. If X is a path-connected space and $p \in X$, then the long exact sequence defines a natural isomorphism

$$H_q(X, p) = \widetilde{H}_q(X, p) \cong \widetilde{H}_q(X)$$

in all degrees.

3.5 Excision

The last property we need before we can do practical calculations is called *excision*. Given an ordered pair (X, A) we say a subspace of $B \subset A$ can be *excised* if the inclusion $(X \setminus B, A \setminus B) \hookrightarrow (X, A)$ induces isomorphisms

$$H_q(X \setminus B, A \setminus B) \cong H_q(X, A)$$

in all degrees q.

Theorem 3.36. *If the closure of B is contained in the interior of A: ($\overline{B} \subset \operatorname{int} A$), then B can be excised.*

Corollary 3.37. *If $V \subset B \subset A$ and*

1. V can be excised, and

2. the inclusion $(X \setminus B, A \setminus B) \hookrightarrow (X \setminus V, A \setminus V)$ determine homotopy equivalences $X \setminus B \hookrightarrow X \setminus V$ and $A \setminus B \hookrightarrow A \setminus V$,

then B can be excised.

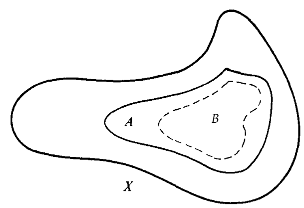


Figure 3.6: Excisive pair $(A,\ B)$

Proof. We want to prove that $H_q(X \setminus B, A \setminus B) \to H_q(X, A)$ is an isomorphism. By functoriality, it is enough to show that the homomorphisms $H_q(X \setminus B, A \setminus B) \to H_q(X \setminus V, A \setminus V)$ and $H_q(X \setminus V, A \setminus V) \to H_q(X, A)$are isomorphisms. The fist is an isomorphism by by homotopy invariance (Example 3.33) and the second is an isomorphism because V can be excised. □

Before we give the proof for Theorem 3.36, we do some actual calculations.

Proposition 3.38. *The homology groups of the unit sphere $\mathbb{S}^n$ for $n \geq 1$ satisfy*

$$H_q(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & if q = 0 \text{ or } q = n \\ 0 & otherwise \end{cases}$$

Proof. It will be more convenient to work with reduced homology, so our goal is to prove that

$$\widetilde{H}_q(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbb{E}^n_+$ and $\mathbb{E}^n_-$ denote the upper and lower closed hemispheres of $\mathbb{S}^n$. Note that for $n \geq 1$, $\mathbb{E}^n_+ \cap \mathbb{E}^n_- \cong \mathbb{S}^{n-1}$. We claim by an excision that

$$\widetilde{H}_q(\mathbb{S}^n, \mathbb{E}^n_-) \cong \widetilde{H}_q(\mathbb{E}^n_+, \mathbb{S}^{n-1}), \quad \forall q \in \mathbb{Z}, n \geq 1.$$

Here we are excising the interior of the lower hemisphere $\mathbb{E}^n_-$. This does not satisfy the hypotheses of Theorem 3.36, but a slightly smaller open disk does and then we can apply Corollary 3.37. Now consider the long exact sequences (LES) associated to these pairs. Because $\mathbb{E}^n_+$ is contractible, the LES of the pair $(\mathbb{E}^n_+, \mathbb{S}^{n-1})$ breaks into isomorphisms

$$0 \to \widetilde{H}_q(\mathbb{E}^n_+, \mathbb{S}^{n-1}) \xrightarrow{\sim} \widetilde{H}_{q-1}(\mathbb{S}^{n-1}) \to 0$$

for all $n \geq 1$ and all $q \in \mathbb{Z}$. Likewise, the LES of $(\mathbb{S}^n, \mathbb{E}^n_-$ gives rise to isomorphisms

$$0 \to \widetilde{H}_q(\mathbb{S}^n) \xrightarrow{\sim} \widetilde{H}_q(\mathbb{S}^n, \mathbb{E}^n_-) \to 0$$

for all $n \geq 1$ and all $q \in \mathbb{Z}$. Combined, we obtain isomorphisms

$$\widetilde{H}_q(\mathbb{S}^n) \cong \widetilde{H}_{q-1}(\mathbb{S}^{n-1})$$

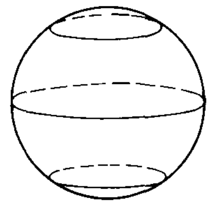


Figure 3.7: $(\mathbb{S}^n, \mathbb{E}^n_-)$ vs $(\mathbb{E}^n_+, \mathbb{S}^{n-1})$

for all $n \geq 1$ and all $q \in \mathbb{Z}$.

Since $\mathbb{S}^0$ is a disconnected union of two points, it follows that

$$\widetilde{H}_q(\mathbb{S}^0) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

The result now follows by induction. □

Proposition 3.38 hints at the special role that spheres play in homology. Later, in §3.7, we will consider a special class of spaces built out of spheres called *cellular spaces* that are particularly well suited to algebraic topology.

Theorem 3.39 (Brouwer Fixed Point Theorem)**.** *Let $f : \mathbb{E}^n \to \mathbb{E}^n$ be a continuous map from a closed n-disk $\mathbb{E}^n$ to itself. There exists $p \in \mathbb{E}^n$ such that $f(p) = p$.*

Proof. We resort to contradiction, so suppose that no such p exists. Then $f(x) \neq x$ for all $x \in \mathbb{E}^n$ and we can define a continuous map $r : \mathbb{E}^n \to \mathbb{S}^{n-1}$ as illustrated in Figure 3.8. Notice that for points $x \in \mathbb{S}^{n-1}$, $r(x) = x$. This implies

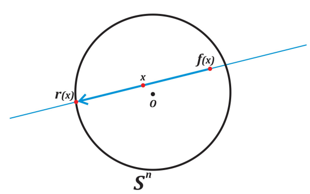


Figure 3.8: Brouwer retraction

that r is retract. In particular, this means $r : H_n(\mathbb{E}^n) \to H_n(\mathbb{S}^{n-1})$ is surjective which contradicts the fact that $H_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ and $H_n(\mathbb{E}^n) = 0$. □

3.5.1 Subdivision

Caveat : This §3.5.1 is quite involved.

We wish to prove Theorem 3.36, i.e, that singular homology satisfies the Excision

Axiom. First we will indicate the difficulty with doing this, and outline the remedy, and finally we go into the detailed proof.

Suppose, for example, $U \subset A \subset X$ with $\overline{U} \subset \mathrm{int}(A)$ and we wish to "excise" U. If all singular simplices which are not completely within A miss U completely, then we could just discard any simplex in A. Thus the problem is with "large" singular simplices, those touching both $X \setminus A$ and U. These sets are "separated," i.e., their closures do not meet. Thus if we could somehow "subdivide" a singular simplex into smaller simplices (a chain) which satisfy the above condition then we might be able to make excision work.

We are going to define an operator Υ called "subdivision" on $\Delta_i(X)$ and a chain homotopy T from Υ to the identity.

Recall that the standard q-simplex $\Delta_q \subset \mathbb{R}^{n+1}$. Let $L*(\Delta_q)$ be the subcomplex of $\Delta_*(\Delta_q)$ generated by the affine singular simplices, i.e., singular simplices of the form $\sigma : \Delta_p \to \Delta_q$ such that $\sigma(\sum_i \lambda_i e_i) = \sum_i \lambda_i v_i$ where $\sum_i \lambda_i = 1$ and $v_i = \sigma(e_i)$. We denote such affine singular simplices by $\sigma = [v_0, \ldots, v_p]$.

Now we define the "cone operator" which takes an affine simplex and forms the "cone" on it from some point, producing a simplex of one higher dimension. Let $v \in \Delta_q$ and let $\sigma = [v_0, \ldots, v_p] : \Delta_p \to \Delta_q$ be affine. The cone on σ from v is then defined to be $v\sigma = [v, v_0, \ldots, v_p] : \Delta_{p+1} \to \Delta_q$. For a chain $c = \sum_\sigma n_\sigma L_p(\Delta_q)$, let $vc = \sum_\sigma n_\sigma v\sigma \in L_{p+1}(\Delta_q)$. Taking $c \mapsto vc$ gives a homomorphism

$$L_p(\Delta_q) \to L_{p+1}(\Delta_q).$$

(By definition, $v_0 = 0$.) If $p > 0$ then we compute

$$\begin{aligned}\partial[v, v_0, \ldots, v_p] &= [v_0, \ldots, v_p] - \sum_i (-1)^i [v, v_0, \ldots, \widehat{v}_i, \ldots, v_p] \\ &= [v_0, \ldots, v_p] - v(\partial[v_0, \ldots, v_p]).\end{aligned}$$

If $p = 0$ then $\partial v\sigma = \sigma - [v]$. Thus, for a 0-chain c, $\partial vc = c - \epsilon(c)[v]$ where ϵ is the augmentation, of Subsection 3.4.2, assigning to 0-chain the sum of its coefficients. Thus we have that

$$\partial(vc) = \begin{cases} c - v(\partial c) & \text{if } \deg(c) > 0, \\ c - \epsilon(c)[v] & \text{if } \deg(c) = 0. \end{cases}$$

We now define the "barycentric subdivision" operator $\Upsilon : L_p(\Delta_q) \to L_p(\Delta_q)$ inductively by

$$\Upsilon(\sigma) = \begin{cases} \underline{\sigma}(\Upsilon(\partial c)) & \text{for } p > 0, \\ \sigma & \text{for } p = 0, \end{cases}$$

where $\underline{\sigma}$ denotes the "barycentre" of the affine simplex σ, i.e., $\sigma = (\sum_{i=0}^p v_i)/(p+1)$ for $\sigma = [v_0, \ldots, v_p]$. This defines Υ on a basis of $L_p(\Delta_q)$ and thus we extend it linearly to be a homomorphism. See Figure 3.9

Lemma 3.40. $\Upsilon : L_p(\Delta_q) \to L_p(\Delta_q)$ *is a chain map.*

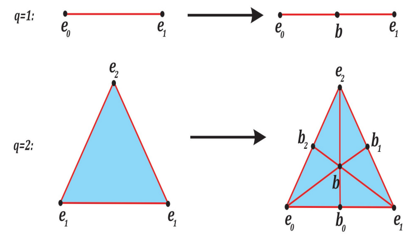


Figure 3.9: Barycentric subdivision

Proof. We shall show that $\Upsilon(\partial\sigma) = \partial(\Upsilon(\sigma))$ inductively on p where σ is an affine p-simplex.
Case: $p = 0$
$\Upsilon(\partial\sigma) = \Upsilon(0) = 0$, while $\partial(\Upsilon(\sigma)) = \partial\sigma = 0$, since there are no $(-l)$-chains.
Case: $p = 1$:
$\Upsilon(\partial\sigma) = \partial\sigma$ while $\partial(\Upsilon(\sigma)) = \partial(\underline{\sigma}(\Upsilon(\partial\sigma))) = \partial(\underline{\sigma}(\partial\sigma)) = \partial\sigma - \epsilon(\partial\sigma)[\sigma] = \partial\sigma$.
Case: $p > 1$:
Assuming that the formula is true for chains of degree $< p$, we have $\partial(\Upsilon(\sigma)) = \partial(\underline{\sigma}\Upsilon(\partial\sigma)) = \Upsilon(\partial\sigma) - \underline{\sigma}(\partial\Upsilon\partial\sigma) = \Upsilon(\partial\sigma)$ since $\partial\Upsilon\partial\sigma = \Upsilon\partial\partial\sigma = 0$ by the inductive assumption. □

Now we define $T : L_p(\Delta_q) \to L_{p+1}(\Delta_q)$ by induction on the formula

$$T\sigma = \sigma(\Upsilon\sigma - \sigma - T(\partial\sigma)),$$

and $T = 0$ for $p = 0$.

We wish to show that T is a chain homotopy from id to Υ, i.e, $\partial T + T\partial = \Upsilon - \mathrm{id}$. For $p = 0$ we compute

$$\partial T\sigma + T\partial\sigma = \partial(\underline{\sigma}(\Upsilon\sigma - \sigma)) = 0$$

since $\Upsilon\sigma = \sigma$ for $p = 0$. For the same reason $(\Upsilon - 1)\sigma = 0$.

For $p > 0$ we compute

$$\partial T\sigma = (\Upsilon\sigma - \sigma - T\partial\sigma) - \underline{\sigma}(\partial\Upsilon\sigma - \partial\sigma - \partial T\partial\sigma). \tag{3.41}$$

The term $\partial T\partial\sigma = (\Upsilon - \mathrm{id} - T\partial)(\partial\sigma) = (\Upsilon\partial\sigma - \partial\sigma)$ so that the entire right-hand term of (3.41) vanishes, which yields the claimed formula. Thus T is a chain homotopy from id to Υ.

We are now done for affine chains in Δ_q. We now transfer these results to general singular chains of X.

We wish to define $\Upsilon : \Delta_p(X) \to \Delta_p(X)$ and $T : \Delta_p(X) \to \Delta_{p+1}(X)$ such that:

(1) (naturality) $\Upsilon S(f)(c) = S(f)(\Upsilon c)$ and $T(S(f)(c)) = S(f)(T(c))$ for $f : X \to Y$;

(2) Υ is a chain map and $\partial T + T\partial = T - \mathrm{id}$;

(3) Υ and T extend the previous definition on affine chains; and

(4) $\Upsilon\sigma$ and $T\sigma$ are chains in $\mathrm{im}(\sigma)$.

Note that (4) follows from (1): we list it for stress.

Thus let $\sigma : \Delta_p \to X$. Then we have $\sigma = S(\sigma)(\mathrm{id}_{\Delta_p})$ and, of course, $\mathrm{id}_{\Delta_p} \in L_p(\Delta_p)$. We define

$$\begin{aligned} \Upsilon\sigma &= S(\sigma)(\Upsilon\,\mathrm{id}_{\Delta_p}), \\ T\sigma &= S(\sigma)(T\,\mathrm{id}_{\Delta_p}). \end{aligned}$$

Of course, one must check that these coincide with the previous definitions when σ is affine, but this is obvious because Υ and T were defined on affine simplices using only affine operations. Property (4) is also clear, so this settles (3) and (4).

To show naturality (1) we compute $\Upsilon S(f)\sigma = \Upsilon S(f \circ \sigma)(\mathrm{id}_{\Delta_p}) = S(f \circ \sigma)(\Upsilon\,\mathrm{id}_{\Delta_p}) = S(f)(S(\sigma)(\Upsilon\,\mathrm{id}_{\Delta_p})) = S(f)(\Upsilon\sigma)$, and similarly for T.

It remains to prove property (2). To show that Υ is a chain map, we compute

$$\begin{aligned} \Upsilon\partial\sigma &= \Upsilon(\partial(S(\sigma)(\mathrm{id}_{\Delta_p}) \\ &= \Upsilon(S(\sigma)(\partial\,\mathrm{id}_{\Delta_p}) && \text{(since } S(\sigma) \text{ is a chain map)} \\ &= S(\sigma)(\Upsilon(\partial\,\mathrm{id}_{\Delta_p})) && \text{(naturality)} \\ &= S(\sigma)(\partial(\Upsilon\,\mathrm{id}_{\Delta_p})) && \text{(since } \mathrm{id}_{\Delta_p} \text{ is affine)} \\ &= \partial(S(\sigma)(\Upsilon\,\mathrm{id}_{\Delta_p})) && \text{(since } S(\sigma) \text{ is a chain map)} \\ &= \partial(\Upsilon\sigma) && \text{(by definition).} \end{aligned}$$

Similarly, for the formula involving T we compute

$$T\partial\sigma = T(S(\sigma)(\partial\,\mathrm{id}_{\Delta_p}) = S(\sigma)(T\partial\,\mathrm{id}_{\Delta_p})$$

and

$$\partial T\sigma = \partial S(\sigma)(T\partial\,\mathrm{id}_{\Delta_p}) = S(\sigma)(\partial T\,\mathrm{id}_{\Delta_p})$$

so that

$$(T\partial{+}\partial T)(\sigma) = S(\sigma)((T\partial{+}\partial T)\,\mathrm{id}_{\Delta_p}) = S(\sigma)((\Upsilon{-}\mathrm{id})\,\mathrm{id}_{\Delta_p}) = (\Upsilon{-}\mathrm{id})S(\sigma)(\mathrm{id}_{\Delta_p}) = \Upsilon\sigma{-}\sigma$$

Corollary 3.42. *For $k \geq 1$, $\Upsilon^k : \Delta_p(X) \to \Delta_p(X)$ is chain homotopic to the identity.*

Proof. This follows from $\Upsilon^2 \sim \Upsilon \circ \mathrm{id} \sim \mathrm{id} \circ \mathrm{id} \sim \mathrm{id}$, etc. Another way to show it, which displays the chain homotopy explicitly is to note that

$$\begin{aligned} \Upsilon^k - \mathrm{id} &= \Upsilon^k - \Upsilon^{k-1} + \Upsilon^{k-1} - \cdots - \mathrm{id} = (\Upsilon^{k-1} + \Upsilon^{k-2} + \cdots + \Upsilon + \mathrm{id})(\Upsilon - \mathrm{id}) \\ &= G(\Upsilon - \mathrm{id}) = G(T\partial + \partial T) = (GT)\partial + \partial(GT). \end{aligned}$$

□

Let us denote this chain homotopy GT (in the second proof) by T_k and note that it is natural.

Lemma 3.43. *If $\sigma = [v0, \ldots, v_p]$ is an affine simplex of Δ_q then any simplex in the chain $\Upsilon\sigma$ has diameter at most $(p/(p+1))\operatorname{diam}(\sigma)$.*

Proof. A simplex in $\Upsilon\sigma$ has the form $\underline{\sigma}\tau$ where τ is a simplex of $\Upsilon(\partial\sigma)$, i.e., τ is a simplex of $\Upsilon(\sigma^{(i)})$ for some i. Thus a simplex of $\Upsilon\sigma$ has the form $[\underline{\sigma}_0, \underline{\sigma}_1, \underline{\sigma}_2, \ldots]$ where $\sigma = \sigma_0 > \sigma_1 > \sigma 2 \ldots$, using $\alpha > \beta$ to mean that β is a proper face of σ. Each barycentre $\underline{\sigma}_i$ is the average of some of the v_k. If $j > i$ then $\underline{\sigma}_j$ is the average of some of *these* v_k. Thus by reordering the vertices, the lemma comes down to the following:

If $w_1, \ldots, w_k \in \mathbb{R}^q$ with $m < k \le p+1$ then

$$\left\| \frac{1}{k}\sum_{i=1}^{k} w_i - \frac{1}{m}\sum_{i=1}^{m} w_i \right\| \le \frac{p}{p+1} \max\|w_i - w_j\|.$$

Since $x/(x+1)$ is an increasing function and $m < k \le p+1$, it suffices to show that the left-hand side of this inequality is at most $((k-1)/k)\max\|w_i - w_j\|$. We calculate

$$\begin{aligned} \left\| \frac{1}{k}\sum_{i=1}^{k} w_i - \frac{1}{m}\sum_{i=1}^{m} w_i \right\| &= \left\| \frac{1}{m}\sum_{i=1}^{m} w_i - \frac{1}{k}\sum_{i=1}^{m} w_i - \frac{1}{k}\sum_{i=m+1}^{k} w_i \right\| \\ &= \left\| \frac{k-m}{km}\sum_{i=1}^{m} w_i - \frac{1}{k}\sum_{i=m+1}^{k} w_i \right\| \\ &= \frac{k-m}{k}\left\| \frac{1}{m}\sum_{i=1}^{m} w_i - \frac{1}{k-m}\sum_{i=m+1}^{k} w_i \right\|. \end{aligned}$$

Both terms in the norm of the last expression are in the convex span of the w_i and so this entire expression is at most $((k-m)/k)\max\|w_i - w_j\| \le ((k-1)/k)\max\|w_i - w_j\|$. □

Corollary 3.44. *Each affine simplex in $\Upsilon^k(\mathrm{id}_{\Delta_p}) \in\in L_q(\Delta_q)$ has a diameter of at most $(p/(p+1))^k \operatorname{diam}(\Delta_q)$, which approaches 0 as $k \to \infty$.*

Corollary 3.45. *Let X be a space and $\boldsymbol{U} = \{U_\alpha\}$ be an open covering of X. Let σ be a singular p-simplex of X. Then $\exists k \ni \Upsilon^k(\sigma)$ is $\boldsymbol{U}$-small. That is, each simplex in $\Upsilon^k(\sigma)$ has image in some U_α.*

Proof. This is an easy consequence of Corollary 3.44 and the Lebesque Lemma (Lemma 2.25 of Chapter 2). □

Definition 3.46. Let $\boldsymbol{U}$ be a collection of subsets of X whose interiors cover X. Let $\Delta_*^{\boldsymbol{U}}(X) \subset \Delta_* X)$ be the subcomplex generated by the $\boldsymbol{U}$-small singular simplices and let $H_*^{\boldsymbol{U}}(X) = H_*(\Delta_*^{\boldsymbol{U}}(X))$.

Theorem 3.47. *The map $H_*^{\boldsymbol{U}}(X) \to H_*(X)$ generated by inclusion is an isomorphism.*

Proof. First we show the map to be a monomorphism. Let $c \in \Delta_*^{\boldsymbol{U}}(X)$ with $\partial c = 0$. Suppose that $c = \partial e$ for some $e \in \Delta_{p+1}(X)$. We must show that $c = \partial e'$ for some $e' \in \Delta_{p+1}^{\boldsymbol{U}}(X)$. There is a k such that $\Upsilon^k(e) \in \Delta_{p+1}^{\boldsymbol{U}}(X)$ and

$$\Upsilon^k(e) - e = T_k(\partial e) + \partial T_k(e) = T_k(c) + \partial T_k(e).$$

Thus

$$\partial \Upsilon^k(e) - \partial e = \partial T_k(c),$$

so that

$$c = \partial e = \partial(\Upsilon^k(e) - T_k(c)) \in \partial(\Delta_*^{\boldsymbol{U}}(X))$$

by the naturality of T_k.

Now we shall show the map to be onto. Let $c \in \Delta_p(X)$ with $\partial c = 0$. We must show that there is a $c' \in \Delta_p^{\boldsymbol{U}}(X)$ such that $c \sim c'$. There is a k such that $\Upsilon^k \in \Delta_p^{\boldsymbol{U}}(X)$. Then

$$\Upsilon^k(c) - c = T_k(\partial c) + \partial T_k(c) = \partial T_k(c).$$

Thus $c' = \Upsilon^k(c)$ works. □

We remark that it can be shown that the isomorphism of Theorem 3.47 is induced by a chain equivalence.

To discuss the relative case of this result, put

$$\Delta_*^{\boldsymbol{U}}(X, A) = \Delta_*^{\boldsymbol{U}}(X)/\Delta_*^{\boldsymbol{U}\cap A}(A)$$

where $\boldsymbol{U} \cap A$ is the set of intersections of members of $\boldsymbol{U}$ with A. We have the commutative diagramme

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta_*^{\boldsymbol{U}\cap A}(A) & \longrightarrow & \Delta_*^{\boldsymbol{U}}(X) & \longrightarrow & \Delta_*^{\boldsymbol{U}}(X, A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta_*(A) & \longrightarrow & \Delta_*(X) & \longrightarrow & \Delta_*(X, A) & \longrightarrow & 0 \end{array}$$

This induces a commutative "ladder" in homology

$$\begin{array}{ccccccccccc} \longrightarrow & H_i^{\boldsymbol{U}\cap A}(A) & \longrightarrow & H_i^{\boldsymbol{U}}(X) & \longrightarrow & H_i^{\boldsymbol{U}}(X, A) & \longrightarrow & H_{i-1}^{\boldsymbol{U}\cap A}(A) & \longrightarrow & H_{i-1}^{\boldsymbol{U}}(X) & \longrightarrow \\ & \downarrow\approx & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow\approx & \\ \longrightarrow & H_i(A) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X, A) & \longrightarrow & H_{i-1}(A) & \longrightarrow & H_{i-1}(X) & \longrightarrow \end{array}$$

Thus $H_i^{\boldsymbol{U}}(X, A) \xrightarrow{\approx} H_i(X, A)$ follows from the 5-lemma.

Now we are prepared to prove the Excision Axiom. Note that the following statement of it is slightly stronger than the axiom itself.

Theorem 3.48 (Excision)**.** *If $B \subset A \subset X$ with $\overline{B} \subset \operatorname{int}(A)$ then the inclusion $(X \setminus B, A \setminus B) \hookrightarrow (X, A)$ induces an isomorphism $H_*(X \setminus B, A \setminus B) \xrightarrow{\approx} H_*(X, A)$.*

Proof. Let $\boldsymbol{U} = \{A, X \setminus B\}$. Then $X = \operatorname{int}(A) \cup (X \setminus \overline{B}) = \operatorname{int}(A) \cup \operatorname{int}(X \setminus B)$. Thus we have $H_*^{\boldsymbol{U}}(X, A) \xrightarrow{\approx} H_*(X, A)$. Note that

$$\Delta_*^{\boldsymbol{U}}(X) = \Delta_*(A) + \Delta_*(X \setminus B)$$

as a subgroup of $\Delta_*(X)$. (The sum is not direct.) Also

$$\Delta_*(A \setminus B) = \Delta_*(A) \cap \Delta_*(X \setminus B).$$

By one of the Noetherian isomorphisms it follows that inclusion induces the isomorphism

$$\Delta_*(X \setminus B)/\Delta_*(A \setminus B) \xrightarrow{\approx} \Delta_*^{\boldsymbol{U}}(X)/\Delta_*(A).$$

Thus the inclusion maps induce

$$\begin{array}{ccc} \Delta_*(X \setminus B)/\Delta_*(A \setminus B) & \xrightarrow{\approx} & \Delta_*^{\boldsymbol{U}}(X)/\Delta_*(A) \\ & \searrow \quad \swarrow & \\ & \Delta_*(X)/\Delta_*(A) & \end{array}$$

This diagramme of chain complexes and chain maps induces the following diagramme in homology:

$$\begin{array}{ccc} H_*(X \setminus B, A \setminus B) & \xrightarrow{\approx} & H_*^{\boldsymbol{U}}(X, A) \\ & {\scriptstyle \text{incl}_*}\searrow \quad \swarrow{\scriptstyle \approx} & \\ & H_*(X, A) & \end{array}$$

It follows that the map marked incl_* is an isomorphism. □

3.5.2 Mapping cylinders and cones

A subspace $A \subset X$ is called a *deformation retract* if there is a homotopy $h : X \times I \to A$ such that $h(x, 0) = x$ and $h(x, 1) \in A$ for all $x \in X$ and $h(a, t) = a$ for all $a \in A$ and $t \in I$. Note that the map $r : X \to A$ defined by $r(x) = h(x, 1)$ is a retraction of X onto A, and h is a homotopy between id_X and $i \circ r$, whence the inclusion $i : A \hookrightarrow X$ is a homotopy equivalence.

A *closed* subspace $A \subset X$ is called a *neighbourhood deformation retract* if there exists an open neighbourhood $A \subset U \subset X$ such that A is a deformation retract of U. In other words, there exist continuous functions $u : X \to I$ and $h : X \times I \to X$ such that:

(i) $A = u^{-1}(0)$;

(ii) $h(x, 0) = x$ for all $x \in X$;

(iii) $h(a,t) = a$ for all $a \in A$, $t \in I$;

(iv) $h(x,1)$ lies in A for all $x \in u^{-1}([0,1))$.

Example 3.49. The inclusion of $\mathbb{S}^1 \subset \mathbb{R}^2$ is a neighbourhood deformation retract because it includes as a deformation retract into an open annulus.

Proposition 3.50. *Let $A \subset X$ be a neighbourhood deformation retract that intersects every path component of X. Then there is a canonical isomorphism*

$$H_q(X,A) \cong \widetilde{H}_q(X/A)$$

where X/A is the coset space of X obtained by shrinking all A to a point by identification.

Proof. Since $A \hookrightarrow U$ is a homotopy equivalence, we know by Example 3.33 that

$$H_q(X,A) \cong H_q(X,U)$$

for all q. By excision (Theorem 3.48)

$$H_q(X \setminus A, U \setminus A) \cong H_q(X,U).$$

On the other hand, if we denote by A/A the point in X/A that A is collapsed to, it is not hard to see that U/A deformation retracts onto A/A. Thus

$$\widetilde{H}_q(X/A) \cong H_q(X/A, A/A) \cong H_q((X/A)\setminus(A/A).(U/A)\setminus(A/A)) \cong H_q(X\setminus A, U\setminus A)$$

where the first isomorphism follows from Remark 3.35 since X/A is path connected. □

Example 3.51. The coset space $\mathbb{D}^n/\partial\mathbb{D}^n$ is homeomorphic to the sphere $\mathbb{S}^n$. It follows that

$$H_q(\mathbb{D}^n, \partial\mathbb{D}^n) \cong \widetilde{H}_q(\mathbb{S}^n)$$

which was basically what we used in the proof of Proposition 3.38.

Example 3.52. If X contains a contractible neighbourhood deformation retraction A, then $H_q(X) \cong H_q(X/A)$ for all q. (Indeed, one may show that $X \to X/A$ is a homotopy equivalence.)

The hypotheses of Proposition 3.50 hold in many situations, but not always, so it is convenient to have a construction that works in general. Let $f : Y \to X$ be a continuous map. The *mapping cylinder* associated to f is the coset space (or adjunction)

$$\mathrm{Cyl}(f) := (Y \times I) \sqcup_f X = ((Y \times I) \sqcup X)/\sim$$

where the relation is generated by $(y,1) \sim f(y)$ for all $y \in Y$. The inclusion $X \hookrightarrow \mathrm{Cyl}(f)$ is a homotopy equivalence with homotopy inverse $\mathrm{Cyl}(f) \to X$, $x \mapsto x$ and $(y,t) \mapsto f(x)$.

The *mapping cone* of $f : Y \to X$ is the cosetspace

$$\mathrm{Cone}(f) := \mathrm{Cyl}(f)/(Y \times \{0\})$$

Proposition 3.53. *Given any map of spaces $f : YtoX$ such that the image of f intersects every path component of X, we can define a long exact sequence in homology*

$$\cdots \to \widetilde{H}_{q+1}(\mathrm{Cone}(f)) \to \widetilde{H}_q(Y) \to \widetilde{H}_q(X) \to \widetilde{H}_q(\mathrm{Cone}(f)) \to \widetilde{H}_{q-1}(Y) \to \cdots$$

in case $i : Y \hookrightarrow X$ is a subspace inclusion, this is canonically isomorphic to the long exact sequence of the pair

$$\cdots \to \widetilde{H}_{q+1}(X,Y) \to \widetilde{H}_q(Y) \to \widetilde{H}_q(X) \to \widetilde{H}_q(X,Y) \to \widetilde{H}_{q-1}(Y) \to \cdots$$

Proof. The subspace $Y \times \{0\}$ is a closed subset of $\mathrm{Cyl}(f)$ and is a deformation retract of the open subset $Y \times [0,1)$, so $Y \times \{0\}$ is a neighbourhood deformation retract in $\mathrm{Cyl}(f)$. Thus we have a canonical isomorphisms

$$\widetilde{H}_q(\mathrm{Cone}(f)) \cong \widetilde{H}_q(\mathrm{Cyl}(f), Y \times \{0\}).$$

Since $Y \times \{0\}$ is homotopy equivalent to Y and $\mathrm{Cyl}(f)$ is homotopy equivalent to X, a long exact sequence can be obtained from the LES of the pair $(\mathrm{Cyl}(f); Y \times \{0\})$ by replacing groups with isomorphic groups.

In case $f : Y \to X$ is a subspace inclusion, then the homotopy equivalence $\mathrm{Cyl}(f) \to X$ sending $(y,t) \in Y \times I$ to $f(y)$ restricts to a homeomorphism from $Y \times \{0\}$. The resulting morphism of long exact sequences

$$\begin{array}{ccccccccc} H_q(Y\times\{0\}) & \to & H_q(\mathrm{Cyl}(f)) & \to & H_q(\mathrm{Cyl}(f), Y\times\{0\}) & \to & H_{q-1}(Y\times\{0\}) & \to & H_{q-1}(\mathrm{Cyl}(f)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_q(Y) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X,Y) & \longrightarrow & H_{q-1}(Y,B) & \longrightarrow & H_{q-1}(X) \end{array}$$

which must be an isomorphism by the 5-Lemma. □

Example 3.54. (*The wedge sum*) Let X_k be a collection of spaces containing base points $p_k \in X_k$. The *wedge product* (or *bouquet*) is the space

$$\vee_k X_k = (\sqcup_k X_k)/\sim$$

where we identify basepoints $p_i \sim p_j$ for all i, j. If the base points neighbourhood deformation retracts (= well-pointed) then

$$\widetilde{H}(\vee_k X_k) \cong \oplus_k \widetilde{H}(X_k)$$

by Proposition 3.50.

3.6 Applications to spheres: the degree of a map

Recall that our calculation of $H_n(\mathbb{S}^n)$ relied on the following sequence of isomorphisms

$$\widetilde{H}_n(\mathbb{S}^n) \xrightarrow{\cong} \widetilde{H}_n(\mathbb{S}^n, \mathbb{E}^n_-) \xleftarrow{\cong} \widetilde{H}_n(\mathbb{E}^n_+, \mathbb{S}^{n-1}) \xrightarrow{\cong} \widetilde{H}_n(\mathbb{S}^{n-1})$$

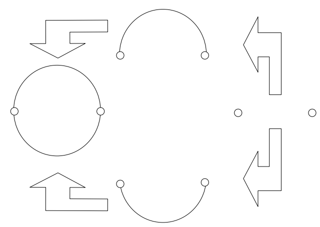


Figure 3.10: Constructing a cycle generating $H_1(\mathbb{S}^1)$

We can use this to construct a cycle representing the generator of $H_1(\mathbb{S}^1)$ by the Figure 3.10. Indeed, the cycle we have constructed is the barycentric subdivision of a simplex $\sigma : \Delta_1 \to \mathbb{S}^1$ that winds once around the circle. It is not hard to show that (exercise) that any "chain" of 1-simplices that wraps once around the circle also represents the generator of $H_1(\mathbb{S}^1)$.

Recall that $\widetilde{H}_n(\mathbb{S}^n) \cong \mathbb{Z}$. Given a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$, the induced map $f_* : \widetilde{H}_n(\mathbb{S}^n) \to \widetilde{H}_n(\mathbb{S}^n)$ must be of the form $f_*(\alpha) = d\alpha$ for some integer $d \in \mathbb{Z}$. We call $d = \deg(f)$ the *degree* of the map f.

Since H_n is a functor, we see immediately that $\deg(\mathrm{id}_{\mathbb{S}^n}) = 1$, that $\deg(f \circ g) = \deg(f)\deg(g)$ for two maps $f, g : \mathbb{S}^n \to \mathbb{S}^n$, and that homotopic maps have the same degree. (It is also true that two maps from $\mathbb{S}^n$ to $\mathbb{S}^n$ are homotopic if and only if they have the same degree. The proof of this requires Hurewicz's theerem in homotopy theory.)

Proposition 3.55. *A map $f : \mathbb{S}^n \to \mathbb{S}^n$ that is not surjective has degree zero.*

Proof. Suppose $p \in \mathbb{S}^n$ is not in the image of f. Then f factors through the inclusion map $\mathbb{S}^n \to \mathbb{S}^n \setminus \{p\} \to \mathbb{S}^n$ so by functoriality $H_q(f)$ factors through $H_q(\mathbb{S}^n \setminus \{p\}) \cong 0$ and thus must be zero. □

Given a space X, define the *suspension* $SX := X \times I/\!\sim$ to be the coset of $X \times I$ where $\sim$ collapses $X \times \{0\}$ and $X \times \{1\}$ to distinct points. If $f : X \to Y$ is a map, define the suspension of f

$$Sf : SX \to SY, \quad Sf(x,t) = (f(x),t).$$

This defines the *suspension functor* from spaces to spaces.

Lemma 3.56. *The suspension of a sphere satisfies $S\mathbb{S}^n = \mathbb{S}^{n+1}$. Given a map $f : \mathbb{S}^n \to \mathbb{S}^n$, the suspension $Sf : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ satisfies $\deg(f) = \deg(Sf)$.*

Proof. The homeomorphism $S\mathbb{S}^n = \mathbb{S}^{n+1}$ is pretty clear; this is the picture where $\mathbb{S}^n$ includes into $\mathbb{S}^{n+1}$ as the equator. Because the long exact homology sequence

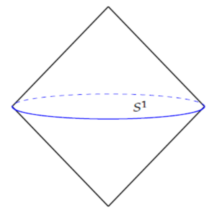


Figure 3.11: $S\mathbb{S}^n = \mathbb{S}^{n+1}$ for $n = 1$

is functorial with respect to pairs and the excision isomorphism is canonical, we obtain a commutative diagramme with horizontal arrows isomorphisms

$$\begin{array}{ccccccc}
\widetilde{H}_n(\mathbb{S}^n) & \longrightarrow & \widetilde{H}_n(\mathbb{S}^n, \mathbb{E}-^n) & \longrightarrow & \widetilde{H}_n(\mathbb{E}+^n, \mathbb{S}^{n-1}) & \longrightarrow & \widetilde{H}_n(\mathbb{S}^{n-1}) \\
\downarrow (Sf)_* & & \downarrow (Sf)_* & & \downarrow (Sf)_* & & \downarrow f_* \\
\widetilde{H}_q(\mathbb{S}^n) & \longrightarrow & \widetilde{H}_n(\mathbb{S}^n, \mathbb{E}-^n) & \longrightarrow & \widetilde{H}_n(\mathbb{E}+^n, \mathbb{S}^{n-1}) & \longrightarrow & \widetilde{H}_n(\mathbb{S}^{n-1})
\end{array}$$

so $S(f)$ and f have the same degree. □

Proposition 3.57. *Let $r_n : \mathbb{S}^n \to \mathbb{S}^n$ be a restriction of a reflexion on $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Then* $\deg(f) = -1$.

Proof. For $n \geq 1$, we can identify $r_n = Sr_{n-1}$, so by induction it suffices to prove the case $n = 0$. In this case, $\mathbb{S}^0 = \{N, S\}$ is a pair of points and r_0 transposes them. The points represent 0-simplices and $\widetilde{H}_0(\mathbb{S}^0)$ is generated by $[N] - [S]$. We have

$$(r_0)_*([N] - [S]) = [r_0(N)] - [r_0(S)] = [S] - [N] = -([N] - [S])$$

so $\deg(r_0) = -1$. □

We define the *antipodal map* on $\mathbb{S}^n$ by $x \mapsto -x$.

Proposition 3.58. *If $f : \mathbb{S}^n \to \mathbb{S}^n$ is a map with no fixed points (i.e. there is no point $p \in \mathbb{S}^n$ such that $f(p) = p$), then f is homotopic to the antipodal map. In particular,* $\deg(f) = (-1)^{n+1}$.

Proof. If f has no fixed points, then the path $tf(x) - (1-t)x$ does not pass through the origin. It follows that

$$h : \mathbb{S}^n \times I \to \mathbb{S}^n, \quad h_t(x) = \frac{tf(x) - (1-t)x}{|tf(x) - (1-t)x|}$$

is a homotopy joining the antipodal map h_0 to $f = h_1$. Finally, note that the antipodal map is equal to a composition of $(n+1)$ reflexions on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ so it has degree $(-1)^{n+1}$ by Proposition 3.57. □

Theorem 3.59 (Hairy Ball Theorem)**.** *Every continuous vector field on an even dimensional sphere has a zero.*

Proof. A continuous vector field on $\mathbb{S}^n$ is equivalent to a map $V : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $V(x)$ is orthogonal to x for all $x \in \mathbb{S}^n$. If a non-vanishing vector field V exists, then we can define an associated map $f : \mathbb{S}^n \to \mathbb{S}^n$ by $f(x) = V(x)/|V(x)|$ which has no fixed points. By Proposition 3.58, this implies that $\deg(f) = (-1)^{n+1}$. One the other hand, since $f(x)$ and x are always orthogonal, we can build a homotopy

$$h : \mathbb{S}^n \times I \to \mathbb{S}^n, \quad h_t(x) = \cos(t\pi/2)x + \sin(t\pi/2)f(x)$$

between the identity map and f, from with we conclude that $deg(f) = 1$. If n is even, this leads to a contradiction. □

Remark 3.60. In contrast with Theorem 3.59, if n is odd $\mathbb{S}^n$ always admits a non-vanishing vector field. This is because $\mathbb{S}^{2m-1} \subset \mathbb{R}^{2m} = \mathbb{C}^m$ and we can use complex scalar multication to rotate each vector by 90 degrees. Explicitly, $V(x1, y_1, \ldots, x_n, y_n) = (-y1, x_1, \ldots, -y_n, x_n)$.

Alternative approach For $n \geq 0$, the homology group $\widetilde{H}_n(\mathbb{S}^n)$ is isomorphic to $\mathbb{Z}$. There are two possible isomorphisms $\widetilde{H}_n(\mathbb{S}^n) \cong \mathbb{Z}$ depending on a choice of generator. A choice of this generator is called an (*global*) *orientation* of $\mathbb{S}^n$.

Given a point $p \in \mathbb{S}^n$ and an open neighbourhood $p \in U \subset \mathbb{S}^n$, we have canonical isomorphisms

$$H_n(\mathbb{S}^n) \xrightarrow{\cong} H_n(\mathbb{S}^n, \mathbb{S}^n \setminus \{p\}) \xleftarrow{\cong} H_n(U, U \setminus \{p\}). \tag{3.61}$$

composing the long exact sequence of the pair $(\mathbb{S}^n, \mathbb{S}^n \setminus \{p\})$ with excision. A choice of orientation for $H_n(U, U \setminus \{p\}) \cong \mathbb{Z}$ is called a *local orientation* of $\mathbb{S}^n$ at p. Because the isomorphism (3.61) is natural, an orientation of $\mathbb{S}^n$ determines a local orientations at all points $p \in \mathbb{S}^n$, and vice versa.

Now suppose that $f : \mathbb{S}^n \to \mathbb{S}^n$ is a map and for some point $p \in \mathbb{S}^n$ the preimage $f^{-1}(p)$ is a finite set of points $\{q_1, \ldots, q_k\} \subset \mathbb{S}^n$.[2] Suppose further that for some open neighbourhood $p \in U$ the preimage $f^{-1}(U)$ is a disjoint union of open sets $V_1 \cup \cdots \cup V_k$ for which $q_i \in V_i$. For each i, the restriction of f induces homomorphism

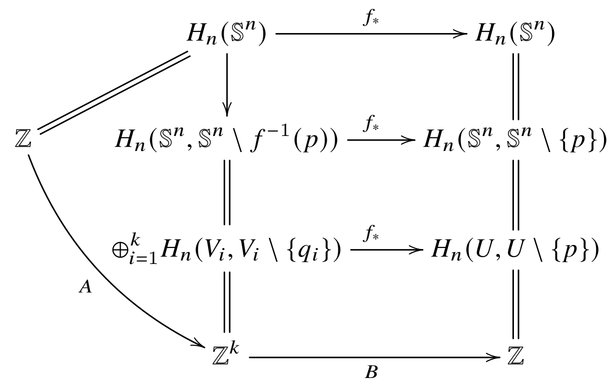
$$H_n(V_i, V_i \setminus \{p\}) \to H_n(U, U \setminus \{p\}).$$

Since both groups are isomorphic to $\mathbb{Z}$, the homomorphism must be multiplication by an integer d_i which we call the *local degree.*

Proposition 3.62. *Under the conditions above, the degree of f is the sum of the local degrees:* $\deg(f) = \sum_{i=1}^{k} d_i$.

[2]Such a point always exists if f is differentiable (Sard's Theorem)

Proof. Fix an orientation $\mathbb{Z} = H_n(\mathbb{S}^n$ and use this to impose local orientations at all points. We have a commutative diagramme where natural and orientation isomorphisms are indicated by double lines.



In matrix notation, we have

$$A = [1\ 1\ \cdots\ 1]^t \quad B = [d_1\ d2\ \cdots\ d_k]$$

So the composition $d_1 + \cdots + d_k$ is the degree of f. □

In the simplest case, p and U can be chosen so that f restricts to local homemorphisms $V_i \to U$. In this case the local degrees are all ± 1, so the degree is obtained by counting points $q_1, \ldots, q_k$ with signs according to whether f is locally orientation preserving or reversing.

Example 3.63. We can construct a map $\mathbb{S}^n \to \mathbb{S}^n$ of degree $d \geq 2$ as follows. Let $A \subset \mathbb{S}^n$ be the complement of d disjoint open disks B_i in $\mathbb{S}^n$. Let

$$q : \mathbb{S}^n \to \mathbb{S}^n = X/A \cong \vee_d \mathbb{S}^n$$

be the quotient map. The orientation on $\mathbb{S}^n$ induces local orientations and hence global orientations on each sphere in the wedge sum. Let

$$p : \vee_d \mathbb{S}^n \to \mathbb{S}^n$$

map each sphere by a degree 1 homeomorphism to $\mathbb{S}^n$.

The preimage $(pq)^{-1}(y)$ of a generic point $y \in \mathbb{S}^n$ consists of a single point in each disk B_i each with local degree is 1 because pq is a local homeomorphism. Therefore $\deg(pq) = d$. By precomposing pq with a reflexion, we can construct a map of degree $-d$.

Consider the map given $d \in \mathbb{Z}$

$$w_d : \mathbb{S}^1 \to \mathbb{S}^1, \quad w_d(e^{i\theta}) = e^{id\theta}$$

for $d \geq 1$ we can see by Figure 3.12 that $\deg(w_d) = d$.

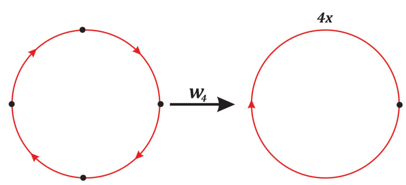


Figure 3.12: The winding map w_4

Note that w_{-d} is equal to the composition of w_d with a reflexion, so $\deg(w_{-d}) = -d$. By suspension, we construct maps

$$\mathbb{S}^n(w_d) : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$$

of degree d for any integer.

Theorem 3.64 (Fundamental Theorem of Algebra)**.** *A complex polynomial function $f(z)$ of degree $d \geq 1$ has a complex root. Hence $\mathbb{C}$ is algebraically closed.*

Proof. The case $d = 1$ is obvious, so suppose that $d \geq 2$. We assume f is monic for simplicity so $f(z) = z^d + O(z^{d-1})$. Assume that $f(z)$ has no complex roots. Then there is a well-defined, continuous map

$$g : \mathbb{C} \to \mathbb{S}^1, \quad g(z) = \frac{f(z)}{|f(z)|} \left(= \frac{z^d + O(z^{d-1})}{|z^d + O(z^{d-1}|} \right).$$

Define a homotopy $h : \mathbb{S}^1 \times I \to \mathbb{S}^1$ by

$$h_t(e^{i\theta}) = g\left(\frac{t}{1-t} e^{i\theta}\right).$$

for $t < 1$ and extend by continuity for $t = 1$. We have $h_0(e^{i\theta}) = g(0)$ is a constant and thus $\deg(h_0) = 0$. On the other hand, for large values of z, $g(z)$ becomes dominated by the highest degree terms in the numerator and denominator, so in the limit $t \to 1$, we have

$$h_1(e^{i\theta}) = e^{id\theta}$$

so $\deg(h_1) = d$, which contradicts degree being a homotopy invariant. □

3.7 Cellular homology

3.7.1 Cellular spaces

Let

$$\mathbb{D}^n := \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$$

denote the *unit disk* or *closed n-cell* with boundary

$$\mathbb{S}^{n-1} = \partial\mathbb{D}^n := \{x \in \mathbb{R}^n | |x| = 1\}.$$

Given a topological space X and a continuous map $f : \mathbb{S}^{n-1} \to X$, we may construct a new space

$$Y := (X \coprod \mathbb{D}^n)/\sim$$

where we quotient by the equivalence relation generated by $p \sim f(p)$ for all $p \in \mathbb{S}^{n-1}$. We say that Y is obtained from X by *attaching an n-cell*; the map f is called the *attaching map*. More generally, if we have a collection of maps $f_\alpha : \mathbb{S}^{n-1} \to X$, then we construct

$$Y = (X \coprod (\coprod_\alpha \mathbb{D}^n_\alpha))/\sim$$

where $p \sim f_\alpha(p)$ for all $p \in \mathbb{S}^{n-1}$ and α.

A *cellular space* (also called *CW-complex*) is a space that is constructed inductively by attaching cells. For instance,

- A 0-dimensional cellular space X_0 is a discrete set of points (i.e. a disconnected union of 0-cells).
- A 1-dimensional cellular space X_1 is a space constructed by attaching a collection of 1-cells to X_0.
- A 2-dimensional cellular space X_2 is constructed by attaching 2-cells to X_1.
- and so on ...

In general, a cellular space X may have cells in arbitrarily high dimensions, in which case it is called ∞-dimensional. Each n-cell determines a *characteristic map* $\phi_\alpha : \mathbb{D}^n \to X$. A subset $S \subset X$ is open/closed if and only if $\phi_\alpha^{-1}(S) \subset \mathbb{D}^n$ is open/closed for all cells.

Example 3.65. A wedge of n-spheres $\vee_I \mathbb{S}^n$ is constructed by attaching I many n-cells onto point $X_0 = \{p\}$ by the only possible attaching map $f : \mathbb{S}^{n-1} \to \{p\}$.

Example 3.66. The torus $\mathbb{S}^1 \times \mathbb{S}^1$ can be constructed by attaching a 2-cell onto a wedge of two circles $X = \mathbb{S}^1 \vee \mathbb{S}^1$. If we denote by a and b the loops defined by the two circles in X, then the attaching map $f : \mathbb{S}^1 \to X$ is the loop $a \cdot b \cdot a^{-1} \cdot b^{-1}$. See the left side of Figure 3.13.

Example 3.67. More generally, the genus g surface Σ_g is constructed by gluing a 2-cell to a wedge of $2g$ circles. If the loops defined by the circles are called $a_1, b_1, \dots a_g, b_g$, then the attaching map sends $\mathbb{S}^1$ to the concatenation $\prod_{i=1}^g [a_i, b_i]$, where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator. See the right side of Figure 3.13 for $g = 2$.

A subset $A \subset X$ is called a *cellular subspace* if it is a closed union of cells (that is, of images of characteristic maps). Given a cellular subspace $A \subset X$, the coset X/A defined by identifying all points in A with each other, is naturally a cellular space called a *coset cellular space* of X.

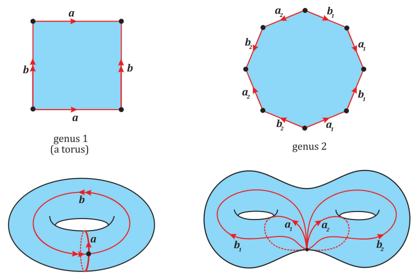


Figure 3.13:

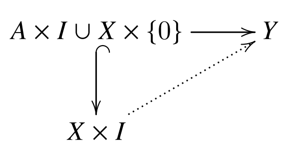
3.7.2 Cofibration

One of the fundamental questions in topology is the “extension problem.” This asks for criteria for being able to extend a map $g : A \longrightarrow Y$ defined on a subspace A of X to all of X. Of course, this cannot always be done as is shown by the case $A = Y = \mathbb{S}^n$, $X = \mathbb{D}^{n+1}$.

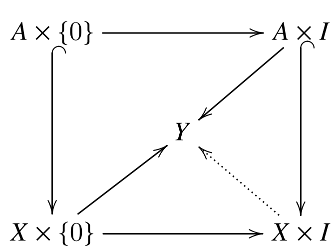
It is natural to ask whether or not this problem is in the scope of algebraic topology. That is, does the answer depend only on the homotopy class of g? The answer to this is “not generally” as is shown by the space $X = [0, 1]$, $A = \{0\} \cup \{1/n | n = 1, 2, \ldots\}$, and $Y = CA$, the cone on A. The map g which is the canonical inclusion of A in Y cannot be extended to X, since the extension would have to be discontinuous at $\{0\}$. However, $g \sim g'$, where g' is the constant map of A to the vertex of the cone, and g' obviously extends to X.

However, it turns out that some very mild conditions on the spaces will ensure that this problem is homotopy theoretic, as we now discuss.

Definition 3.68. Let (X, A) and Y be given spaces. Then (X, A) is said to have the *homotopy extension property* with respect to Y if the following diagramme can always be completed to be commutative:

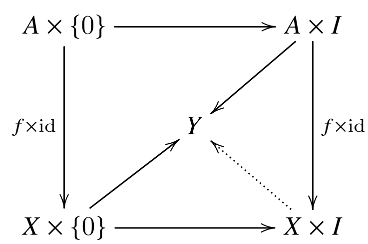


Note that one can also depict this with the following type of diagramme:



If (X, A) has the homotopy extension property with respect to Y then extensibility of maps $g : A \to Y$ clearly depends only on the homotopy class of g.

Definition 3.69. Let $f : A \to X$ be a map. Then f is called a *cofibration* if one can always fill in the following commutative diagramme:



for *any* space Y.

Note that if f is an inclusion then this is the same as the homotopy extension property for all Y. That attribute is sometimes referred to as the "absolute homotopy extension property."

Theorem 3.70. *For an inclusion $A \hookrightarrow X$ the following are equivalent:*

(1) The inclusion map $A \hookrightarrow X$ is a cofibration.

(2) $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

Proof. (1) $\Rightarrow$ (2): consider the diagramme of Definition 3.69 with $Y = A \times I \cup X \times \{0\}$. The filled-in map is the desired retraction.

(2) $\Rightarrow$ (1): composing the retraction of (2) with a map $A \times I \cup X \times \{0\} \to Y$ gives the homotopy extension property for all Y, which, as mentioned, is equivalent to (1). ◻

Corollary 3.71. *If A is a cellular subspace of a cellular space X, then the inclusion $A \hookrightarrow X$ is a cofibration.*

Proof. One constructs a retraction $((A \cup X^{(}r)) \times I) \cup (X \times \{0\}) \to (A \times I) \cup (X \times \{0\})$ by induction on r. If it has been defined for the $(r - l)$-skeleton then extending it over an r-cell is simply a matter of extending a map on $\mathbb{S}^{r-1} \times I \cup \mathbb{D}^r \times \{0\}$ over $\mathbb{D}^r \times I$, which can always be done because the pair $(\mathbb{D}^r \times I, \mathbb{S}^{r-1} \times I \cup \mathbb{D}^r \times \{0\})$ is homeomorphic to $(\mathbb{D}^r \times I, \mathbb{D}^r \times \{0\})$, see Figure 3.14.

These maps for each cell fit together to give a map on the r-skeleton because of the topology (See 3.7.1) on $X \times I$. The union of these maps for all r gives a map on $X \times I$, again because of the topology of $X \times I$. ◻

The main technical result for proving that particular inclusions are cofibrations is the following rephrasing of 3.5.2. Note that conditions (1) and (2) always hold if X is metric.

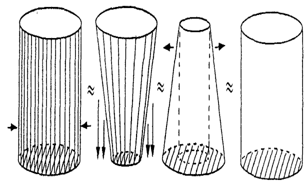


Figure 3.14:

Theorem 3.72. *Assume that $A \subset X$ is closed and that there exists a neighbourhood U of A and a map $\phi : X \to I$, such that:*

(1) $A = \phi^{-1}(0)$*;*

(2) $\phi(X \setminus U) = \{0\}$*; and*

(3) U deforms to A through X with A fixed. That is, there is a map $H : U \times I \to X$ such that $H(a,t) = a$ for all $a \in A$, $H(u,0) = u$, and $H(u,1) \in A$ for all $u \in U$.

Then the inclusion $A \subset X$ is a cofibration. The converse also holds.

Proof. We can assume that $\phi = 1$ on a neighbourhood of $(X \setminus U$, by replacing ϕ with $\min(2\phi, 1)$. It suffices to show that there exists a map

$$\Phi : U \times I \to X \times \{0\} \cup A \times I$$

such that $\Phi(x,0) = (x,0)$ for $x \in U$ and $\Phi(a,t) = (a,t)$ for $a \in A$ and all t, since then the map $r(x,t) = \Phi(x, t(l - \phi(x)))$ for $x \in U$ and $r(x,t) = (x,0)$ for $x \notin U$ gives the desired retraction $X \times I \to A \times I \cup X \times \{0\}$.

We define Φ by

$$\Phi(u,t) = \begin{cases} H(u, t/\phi(u)) \times \{0\} & \text{for} \quad \phi(u) > t, \\ H(u,1) \times \{t - \phi(u)\} & \text{for} \quad \phi(u) \leq t. \end{cases}$$

We need only show that Φ is continuous at those points $(u,0)$ such that $\phi(u) = 0$, i.e., at points $(a,0)$ for $a \in A$.

Note that $H(a,t) = a$ for all $t \in I$ Thus, for W a neighbourhood of a, there is a neighbourhood $V \subset W$ of a such that $H(V \times I) \subset W$. Therefore, $t < \epsilon$ and $u \in V$ imply that $\Phi(u,t) \in W \times [0,\epsilon]$, and hence that Φ is continuous.

We will now prove the converse.

Let $r : X \times I \to (A \times I) \cup (X \times \{0\})$ be a retraction, let $s(x) = r(x,1)$ and put $U = s^{-1}(A \times (0,1])$. Let p_X, p_I be the projections of $X \times I$ to its factors. Then put $H = p_X \circ f : U \times I \to X$. This satisfies (3). For (1) and (2), put $\phi(x) = \max_{t \in I} |t - p_I r(x,t)|$ which makes sense since I is compact. That this satisfies (1) and (2) is clear and it remains to show that ϕ is continuous. Let $f(x,t) = |t - p_I r(x,t)|$ and $f_t(x) = f(x,t)$, all of which are continuous. Then

$$\phi^{-1}(-\infty, b]) = \{x | f(x,t) < b \text{ for all } t\} = \cap_{t \in I} f_t^{-1}((-\infty, b])$$

is an intersection of closed sets and so is closed. Similarly

$$\phi^{-1}([a, \infty)) = \{x | f(x,t) > a \text{ for some } t\} = p_X(f^{-1}([a, \infty)))$$

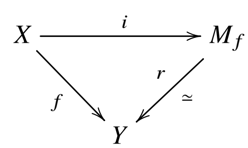
is closed since p_X is closed. Since the complements of the intervals of the form $[a, \infty)$ and $(-\infty, b]$ give a subbase for the topology of $\mathbb{R}$, the contention follows. □

It can be shown that, in the situation of Theorem 3.72, $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$. See May "A Concise Course in Algebraic Topology", pp. 43-44.

Suppose that $f : X \to Y$ is any map. Recall that the "mapping cylinder" M_f of f is defined to be the coset space

$$M_f = ((X \times I) + Y)/((x, 0) \sim f(x)).$$

The inclusion $i : X \hookrightarrow M_f$ clearly satisfies Theorem 3.72 and hence is a cofibration. Also, the retraction $r : M_f \to Y$ is a homotopy equivalence with homotopy inverse being the inclusion $Y \hookrightarrow M_f$. The diagramme



commutes. This shows that *any* map f is a cofibration, up to a homotopy equivalence of spaces.

Also recall the definition of the "mapping cone" of $f : X \to Y$ as the coset space

$$C_f = M_f/(X \times \{1\}) \approx M_f \cup CX.$$

In the case of an inclusion $i : A \hookrightarrow X$, we have $C_i = X \cup CA$. There is the map

$$C_i \xrightarrow{h} X/A,$$

defined as the coset map $X \cup CA \to (X \cup CA)/CA$ composed with the inverse of the homeomorphism $X/A \to (X \cup CA)/CA$. It is natural to ask whether h is a homotopy equivalence. This is not always the case, but the following gives a sufficient condition for it to be so.

Theorem 3.73. *If $A \subset X$ is closed and the inclusion $i : A \hookrightarrow X$ is a cofibration then $h : Ci \to X/A$ is a homotopy equivalence. In fact, it is a homotopy equivalence of pairs*

$$(X/A, *) \simeq (C_i, CA) \simeq (C_i, v),$$

where v is the vertex of the cone.

Proof. The mapping cone $C_i = X \cup CA$ consists of three different types of points, the vertex $v = \{A \times \{1\}\}$, the rest of the cone $\{(a,t)|0 < t < 1\}$ where $(a,0) = a \in A \subset X$, and points in X itself, which we identify with $X \times \{0\}$ to simplify definitions of maps.

Define $f : A \times I \cup X \times \{0\} \to C_i$, as the collapsing map and extend f to $\overline{f} : X \times I \to C$, by the definition of cofibration. Then $\overline{f}(a,1) = v$, $\overline{f}(a,t) = (a,t)$ and $\overline{f}(x,0) = x$.

Put $\overline{f}_t = \overline{f}|_{X\times\{t\}}$. Since $\overline{f}_1(A) = \{v\}$, there is the factorisation $\overline{f}_1 = g \circ j$, where $j : X \to X/\mathrm{A}$ is the coset map and $g : X/A \to C_i$. (g is continuous by definition of the coset topology.)

We claim that g is a homotopy equivalence and a homotopy inverse to h.

First we will prove that $hg \simeq \mathrm{id}$. There is the homotopy $h\overline{f}_1 : X \to X/A$. For all t, this takes A into the point $\{A\}$. Thus it factors to give the homotopy

$$hg \simeq \{\overline{f}_1\} \simeq \{h\overline{f}_0 c = \{j\} = \mathrm{id}\,.$$

Next we will show that $gh \simeq \mathrm{id}$. For this, consider $W = (X \times I)/(A \times \{1\})$ and the maps illustrated in Figure 3.15. The map $\overline{f}'$ is induced by $\overline{f}$. The map

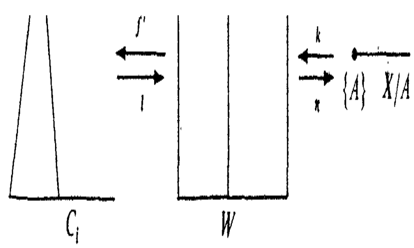


Figure 3.15: A homotopy equivalence and homotopy inverse

k is the "top face" map. We see that

$$\begin{aligned}
\overline{f}' \circ l &= \mathrm{id}, \\
\pi \circ k &= \mathrm{id} \quad \text{(which we don't need)}, \\
k \circ \pi &\simeq \mathrm{id}, \\
\overline{f}' \circ k &= g \quad \text{(definition of g)}, \\
\pi \circ l &= h.
\end{aligned}$$

Hence $g \circ h = \overline{f}' \circ (k \circ \pi) \circ l \simeq \overline{f}' \circ l = \mathrm{id}$, as claimed. □

A non-example of Theorem refthm:Br7-1-6 is $A = \{0\} \cup \{1/n | n = 1, 2, \dots\}$, and $X = [0,1]$. Here C_i, is not homotopy equivalent to X/A, which is a one-point union of an infinite sequence of circles with radii going to zero. (C_i has homeomorphs of circles joined along edges, but the circles do not tend to a point and so any prospective homotopy equivalence $X/A \to C_i$ would be discontinuous at the image of $\{0\}$ in X/A.)

As a corollary we have

Proposition 3.74. *If $A \subset X$ is closed and the inclusion $A \hookrightarrow X$ is a cofibration then the map $j : (X, A) \to (X/A, *)$ induces isomorphisms*

$$H_*(X,A) \xrightarrow{\approx} H_*(X/A,*) \approx \widetilde{H}(X/A).$$

Hence we have a long exact sequence in homology

$$\cdots \widetilde{H}_{q+1}(X/A) \to H_q(A) \to H_q(X) \to \widetilde{H}_q(X/A) \to H_{q-1}(A) \to \cdots$$

Proof. $H_*(X/A,*) \approx H_*(C_i, CA) \approx H_*(X \cup (A \times [0,1/2]), A \times A \times [0,1/2]) \approx H_*(X,A)$. □

A nonexample is $X = \mathbb{S}^2$ with $A \subset X$ the "$\sin(1/x)$" subspace pictured in Figure 3.16. Here $X/A \approx \mathbb{S}^2 \vee \mathbb{S}^2$, so that $\widetilde{H}_2(X/A) \approx \mathbb{Z} \oplus \mathbb{Z}$. But $H_1(A) = 0 =$

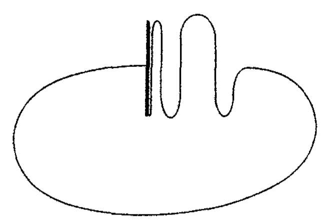


Figure 3.16: A pseudo-circle

$H_2(A)$, so that $H_2(X,A) \approx H_2(X) \approx \mathbb{Z}$. It follows that the inclusion $A \to \mathbb{S}^2$ is not a cofibration.

Let us recall the notion of the pointed category and some notational items. The pointed category has, as objects, spaces with a base point $*$, and, as maps, those maps of spaces preserving the base point. There is also the category of pairs of pointed spaces. There is also the notion of homotopies in this category, those homotopies which preserve the base point.

If $f : X \to Y$ is a pointed map then the reduced mapping cylinder of f is the coset space M_f of $(X \times I) \cup Y$ modulo the relations identifying $(x, 0)$ with $f(x)$ and identifying the set $\{*\} \times I$ to the base point of M_f.

The reduced mapping cone is the coset of the reduced mapping cylinder M_f gotten by identifying the image of $X \times \{1\}$ to a point, the base point.

The one-point union of pointed spaces X and Y is the coset space $X \vee Y$ of the disjoint union $X + Y$ obtained by identifying the two base points.

The wedge, or smash, product is the pointed space $X \wedge Y = (X \times Y)/(X \vee Y)$.

The circle $\mathbb{S}^1$ is defined as $I/\partial I$ with base point $\{\partial I\}$.

The reduced suspension of a pointed space X is $SX = X \wedge \mathbb{S}^1$. It can also be considered as the coset space $(X \times I)/[(X \times \partial I) \cup (\{*\} \times I)]$.

$\mathbb{S}^n \wedge \mathbb{S}^n$ is the one-point compactification of $\mathbb{R}^n \times \mathbb{R}^m$ and hence is homeomorphic to $\mathbb{S}^{n+m}$. Thus we can, and will in this article, redefine $\mathbb{S}^n$ inductively by letting $\mathbb{S}^{n+1} = S\mathbb{S}^n$. Also note that

$$S(SX) = (SX) \wedge \mathbb{S}^1 = (X \wedge \mathbb{S}^1) \wedge \mathbb{S}^1 = X \wedge \mathbb{S}^2, \quad \text{etc.}$$

The preceding results of this section can all be rephrased in terms of the pointed category. Extending the proofs is elementary, mostly a matter of seeing that the unreduced versions become the reduced versions by taking the coset of spaces by sets involving the base point. For example, Proposition 3.74 would say that if A is a closed, pointed, subspace of the pointed space X and if the inclusion $i : A \to X$ is a cofibration (same definition since the base point is automatically taken care of) then $X/A \simeq C_i$, where the latter is now the reduced mapping one, and the homotopies involved must preserve the base points.

Definition 3.75. A base point $x_0 \in X$ is said to be *non-degenerate* if the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration. A pointed Hausdorff space X with non-degenerate base point is said to be *well-pointed.*

Any pointed manifold or cellular space is clearly well-pointed. An example of pointed space that is not well-pointed is $\{0\} \cup \{1/n | n \geq 1\}$ with 0 as base point. The reduced suspensions of this also fail to be well-pointed.

If $A \hookrightarrow X$ is a cofibration then X/A, with base point $\{A\}$, is well-pointed as follows easily from Theorem 3.72.

If a whisker is appended at the base point of any pointed space X, then changing the base point to the other end of the whisker provides a well-pointed space. (This is, of course, just the mapping cylinder of the inclusion of the base point into X.)

Theorem 3.76. *If X is well-pointed then so are the reduced cone CX and the reduced suspension SX. Moreover, the collapsing map $\Sigma X \to SX$, of the unreduced suspension to the reduced suspension, is a homotopy equivalence.*

Proof. Denote the base point of X by $*$. Consider a homeomorphism

$$h : [I \times I, (I \times \{0\}) \cup (\partial I \times I)] \xrightarrow{\approx} (I \times I, I \times \{0\})$$

which clearly exists. Then the induced homeomorphism

$$\mathrm{id} \times h : X \times I \times I \xrightarrow{\approx} X \times I \times I$$

carries $(X \times I \times \{0\}) \cup (X \times \partial I \times I)$ to $X \times I \times \{0\}$. Hence it takes $A = (X \times I \times \{0\}) \cup (X \times \partial I \times I) \cup (\{*\} \times I \times I)$ to $(X \times I \times \{0\}) \cup (\{*\} \times I \times I)$. Therefore, the pair $(X \times I \times I, A)$ is homeomorphic to the pair $I \times [X \times I, (X \times \{0\}) \cup (\{*\} \times I)]$. Since $(X \times \{0\}) \cup (\{*\} \times I)$ is a retract of $X \times I$ by the definition of "well-pointed," it follows that A is a retract of $X \times I \times I$. This implies that the inclusion $(X \times \partial I) \cup (\{*\} \times I) \hookrightarrow X \times I$

is a cofibration. Therefore, $SX = (X \times I)/[(X \times \partial I) \cup (\{*\} \times I)]$ is well-pointed. A similar argument using a homeomorphism $[I \times I, (I \times \{0\}) \cup (\{1\} \times I)] \xrightarrow{\approx} (I \times I, I \times \{0\})$ shows that the inclusion $[(X \times \{1\}) \cup (\{*\} \times I)] \hookrightarrow X \times I$ is a cofibration and so $CX = (X \times I)/[X \times \{1\}) \cup (\{*\} \times I)]$ is well-pointed.

The fact that $(X \times \partial I) \cup (\{*\} \times I) \hookrightarrow X \times I$ is a cofibration implies that the induced inclusion $I \approx \{*\} \times I \hookrightarrow (X \times I)/\{X \times \{0\}, X \times \{1\}\} = \Sigma X$ is a cofibration by an easy application of Theorem 3.72. By Theorem 3.73, $\Sigma X \simeq \Sigma X \cup CI \simeq \Sigma X/I = SX$ via the collapsing map. □

The cellular subspace $X_n \subset X$ consisting of all cells of dimension $\leq n$ is called the *n-skeleton* of X (by convention $X_{-1} = \emptyset$). If X is infinite dimensional, the topology on X satisfies $S \subset X$ is open (resp. closed) if and only if $S \cap X_n$ is open (resp. closed) in X_n for all n. In particular, a map $f : X \to Y$ is continuous if and only if the restrictions $f_n : Xn \to Y$ are continuous for all n. We say X has the *direct limit topology* with respect to X_n.

Lemma 3.77. *Let X be a cellular space and $C \subset X$ a compact subspace. Then C is contained within finitely many cells of X.*

Proof. Choose a sequence of points $x_i \in C$ lying in distinct cells. We will show that the set $S := \{x_i\}$ is finite. We begin by showing S is closed.

First observe that

$$S \subset X \text{ is closed } \Leftrightarrow S \cap X_n \text{ is closed in } X, \forall n$$

We use induction on n.

Clearly $S \cap X_0$ is closed in X_0 hence in X, because every subset of X_0 is closed. Assume by induction that $S \cap X_{n-1}$ is closed in X. Thus for any characteristic map

$$\phi_\alpha : \mathbb{D}^k \to X$$

the pre-image $\phi_\alpha^{-1}(S \cap X_{n-1})$ is closed in D^k. For $k < n$, the pre-image $\phi_\alpha^{-1}(S \cap X_n) = \phi_\alpha^{-1}(S \cap X_{n-1})$ is closed in $\mathbb{D}^k$. For $k = n$, the pre-image $\phi_\alpha^{-1}(S \cap X_n) \subset \mathbb{D}^n$ equals $\phi_\alpha^{-1}(S \cap X_{n-1})$ plus at most one point, thus it is a union of two closed sets, hence is closed in $\mathbb{D}^n$. We deduce that $S \cap X_n$ is closed in X_n hence also in X. By induction, this holds for all n so S is closed in X.

The same argument shows that every subset of S is also closed, so S has the discrete topology. But S is a closed subset of the compact set C, so it is compact. We conclude that S is finite. □

3.7.3 The Compact-Open Topology

Let X be a locally compact Hausdorff space, and Y any Hausdorff space. By Y^X we mean the set of *continuous* functions $X \to Y$.

Definition 3.78. The *compact-open topology* on Y^X is the topology generated by the sets $M(K, U) = \{f \in Y^X | f(K) \subset U\},]$ where $K \subset X$ is compact and $U \subset Y$ is open.

Recall that "generated" here means that these sets form a subbasis for the open sets. Henceforth, unless otherwise noted, Y^X will always be given the compact-open topology.

Lemma 3.79. *Let $\boldsymbol{K}$ be a collection of compact subsets of X containing a neighbourhood base at each point of X. Let $\boldsymbol{B}$ be a subbasis for the open sets of Y. Then the sets $M(K,U)$, for $K \in \boldsymbol{K}$ and $U \in \boldsymbol{B}$, form a subbasis for the compact-open topology.*

Proof. Note that $M(K,U) \cap M(K,V) = M(K, U \cap V)$, which implies that it suffices to consider the case in which $\boldsymbol{B}$ is a basis. We need to show that the indicated sets form a neighbourhood basis at each point $f \in Y^X$. Thus it suffices to show that if $K \subset X$ is compact and $U \subset Y$ is open, and $f \in M(K,U)$, then there exist $K_1, \ldots, K_n \in \boldsymbol{K}$ and $U_1, \ldots, U_n \in \boldsymbol{B}$ such that $f \in \cap M(K_i, U_i) \subset M(K,U)$.

For each $x \in K$, there is an open set $U_x \in \boldsymbol{B}$ with $f(x) \in U_x \subset U$, and there exists a $K_x \in \boldsymbol{K}$ which is a neighbourhood of x such that $f(K_x) \subset U_x$. Thus $f \in M(K_x, U_x)$.

By the compactness of K there exist points $x_1, \ldots, x_n$ such that $K \subset K_{x_1} \cup \cdots \cup K_{x_n}$. Then $f \in \cap M(K_{x_i}) \subset M(K,U)$. □

Proposition 3.80. *For X locally compact Hausdorff, the "evaluation map" $e : Y^X \times X \to Y$, defined by $e(f,x) = f(x)$, is continuous.*

Proof. If f and x are given, let U be an open neighbourhood of $f(x)$. Since f is continuous, there is a compact neighbourhood K of x such that $f(K) \subset U$. Thus $f \in M(K,U)$ and $M(K,U) \times K$ is taken into U by the evaluation e. Since $M(K,U) \times K$ is a neighbourhood of (f,x) in Y^X, we are done. □

Theorem 3.81. *Let X be locally compact Hausdorff and Y and T arbitrary Hausdorff spaces. Given a function $f : X \times T \to Y$, define, for each $t \in T$, the function $f_t : X \to Y$ by $f_t(x) = f(x,t)$. Then f is continuous $\Leftrightarrow$ both of the following conditions hold:*

(a) each f_t is continuous; and

(b) the function $T \to Y^X$ taking t to f_t is continuous.

Proof. The implication $\Leftarrow$ follows from the fact that f is the composition of the map $X \times T \to Y^X \times X$ taking $(x,t)$to (f_t, x), with the evaluation $Y^X \times X \to Y$.

For the implication $\Rightarrow$, (a) follows from the fact that f_t is the composition $X \to X \times T \to Y$ of the inclusion $x \mapsto (x,t)$ with f. To prove (b), let $t \in T$ be given and let $f_t \in M(K,U)$. It suffices to show that there exists a neighbourhood W of t in T such that $t' \in W \to f_{t'} \in M(K,U)$. (That is, it suffices to prove the conditions for continuity for a subbasis only.)

For $x \in K$, there are open neighbourhoods $V_x \subset X$ of x and $W_x \subset T$ of t such that $f(V_x \times W_x) \subset U$. By compactness, $K \subset V_{x_1} \cup \cdots \cup V_{x_n} = V$ say. Put $W = W_{x_1} \cap \cdots \cap W_{x_n}$. Then $f(K \times W) \subset f(V \times w) \subset U$, so that $t' \in W \Rightarrow f_{t'} \in M(K,U)$ as claimed. □

This theorem implies that a homotopy $X \times I \to Y$, with X locally compact, is the same thing as a path $I \to Y^X$ in Y^X.

An often used consequence of Theorem 3.81 is that in order to show a function $T \to Y^X$ to be continuous, it suffices to show that the associated function $X \times T \to Y$ is continuous.

Theorem 3.82 (The Exponential Law). *Let X and T be locally compact Hausdorff spaces and let Y be an arbitrary Hausdorff space. Then there is the homeomorphism*

$$Y^{X\times T} \xrightarrow{\approx} (Y^X)^T$$

taking f to f^*, where $f^*(t)(x) = f(x,t) = f_t(x)$.

Proof. Theorem 3.81 says that the assignment $f \mapsto f^*$ is a bijection. We must show it and its inverse to be continuous. Let $U \subset Y$ be open, and $K \subset X$, $K' \subset T$ compact. Then

$$\begin{aligned} f \in M(K \times K', U) &\Leftrightarrow (t \in K', x \in K \Rightarrow f_t(x) = f(x,t) \in U) \\ &\Leftrightarrow (t \in K' \Rightarrow f_t \in M(K,U)) \\ &\Leftrightarrow f^* \in M(K', M(K,U)). \end{aligned}$$

Now the $K\times K'$ give a neighbourhood basis for $X\times T$. Therefore the $M(K\times K', U)$ form a subbasis for the topology of $Y^{X\times T}$.

Also, the $M(K,U)$ give a subbasis for Y^X and therefore the $M(K', M(K,U))$ give a subbasis for the topology of $(Y^X)^T$.

Since these subbases correspond to one another under the exponential correspondence, the theorem is proved. □

Proposition 3.83. *If X is locally compact Hausdorff and Y and W are Hausdorff then there is the homeomorphism*

$$Y^X \times W^X \xrightarrow{\approx} (Y \times W)^X$$

given by $(f,g) \mapsto f\underline{\times}g = (f \times g) \circ \text{diag}$, where $\text{diag} : U \to U \times U$ is the diagonal map.

Proof. This is clearly a bijection. If $K, K' \subset X$ are compact, and $U \subset Y$ and $V \subset W$ are open then we have

$$\begin{aligned} (f,g) \in M(K,U) \times M(K',V) &\Leftrightarrow (x \in K \Rightarrow f(x) \in U) \text{ and } (x \in K' \Rightarrow g(x) \in V) \\ &\Leftrightarrow (x \in K \Rightarrow (f\underline{\times}g)(x) \in U \times W) \\ &\quad\ \text{and } (x \in K' \Rightarrow (f\underline{\times}g)(x) \in Y \times V) \\ &\Leftrightarrow (f\underline{\times}g) \in M(K, U \times W) \cap M(K', Y \times V). \end{aligned}$$

Thus $(f,g) \mapsto f\underline{\times}g$ is open.

Also, $(f,g) \in M(K,U) \times M(K,V) \Leftrightarrow (f\underline{\times}g) \in M(K, U \times V)$, which implies that the function in question is continuous. □

Proposition 3.84. *If X and T are locally compact Hausdorff spaces and Y is an arbitrary Hausdorff space then there is the homeomorphism*

$$Y^{X+T} \xrightarrow{\approx} Y^X \times Y^T$$

taking f to $(f \circ i_X, f \circ i_Y)$.

Proof. Let

$$i_X : X \to X + T, \quad i_T : T \to X + T$$

be the inclusions. Now define a function $\theta : Y^X \times Y^T \to (Y+Y)^{X+T}$ by $\theta(\lambda, \mu) = \lambda + \mu$, where $\lambda : X \to Y$ and $\mu : T \to Y$, and consider the composite functions

$$\phi : Y^{X+T} \xrightarrow{\Delta} Y^{X+T} \times Y^{X+T} \xrightarrow{\mathrm{id}_{i_X} \times \mathrm{id}_{i_T}} Y^X \times Y^T$$

and

$$\psi : Y^X \times^T \xrightarrow{\theta} (Y+Y)^{X+T} \xrightarrow{\nabla^{\mathrm{id}}} Y^{X+T},$$

where Δ is the diagonal map, and $\nabla : Y + Y \to Y$ is the folding map. Given $\nu : X + T \to Y$, $\phi(\nu) = (\nu i_X, \nu i_T)$, and given $\lambda : X \to Y$ and $\mu : T \to Y$, $\psi(\lambda, \mu) = \nabla(\lambda + \mu)$. Thus $\phi\psi$ and $\psi\phi$ are identity functions, and the only point that remains in showing that ϕ is a homeornorphism is to show that θ is continuous.

To do so, consider the set $M_{K,U}$, where $K \subset X + T$ is compact and $U \subset Y + Y$ is open. Now

$$\begin{aligned} \theta^{-1} &= \{(\lambda,\mu) | (\lambda+\mu)(K) \subset U\} \\ &= \{(\lambda,\mu) | (\lambda)(K \cap X) \subset U \cap (Y \times y_0) \text{ and } (\mu)(K \cap T) \subset U \cap (y_0 \times Y)\} \end{aligned}$$

where y_0 is a non-degenerate point of Y (i.e., a point such that $\{y_0\} \hookrightarrow Y$ is a cofibration) and X and T are identified with their images in $X + T$. Certainly $U_1 = U \cap (Y \times y_0)$ and $U_2 = U \cap (y_0 \times Y)$ are open, since U is the intersection with $Y + Y$ of an open set in $Y \times Y$. But since X and T are Hausdorff, so is $X \times T$ and $X + T$: thus K, X and T are closed in $X + T$, so that $K \cap X$ and $K \cap T$ are closed and hence compact. That is, $\theta^{-1} M_{K,U}) = M_{K \cap X, U_1} \times M_{K \cap T, U_2}$ so θ is continuous. Hence ϕ is a homeomorphism. □

Theorem 3.85. *For X locally compact and both X and Y Hausdorff, Y^X is a covariant functor of Y and a contravariant functor of X.*

Proof. A map $\phi : Y \to Z$ induces $\phi^X : Y^X \to Z^X$, by $\phi^X = \phi \circ f$. We must show that ϕ^X is continuous. By Theorem 3.81 it suffices to show that $Y^X \times X \to Z$, taking (f, x) to $\phi(f(x))$, is continuous. But this is the composition $\phi \circ e$ of ϕ with the evaluation, which is continuous.

Next, for $\psi : X \to T$, both spaces locally compact, we must show that $Y^\psi : Y^T \to Y^X$, taking f to $f \circ \psi$, is continuous. It suffices, by Theorem 3.81, to show that $Y^T \times X \to Y$, taking (f, x) to $f(\psi(x)$, is continuous. But this is just the composition $e \circ (\mathrm{id} \times \psi)$, which is continuous. □

Corollary 3.86. *For $A \subset X$ both locally compact and X, Y Hausdorff, the restriction $Y^X \to Y^A$ is continuous.*

Theorem 3.87. *For X, Y locally compact, and X, Y, Z Hausdorff, the function*

$$Z^Y \times Y^X \to Z^X$$

taking (f, g) to $f \circ g$, is continuous.

Proof. It suffices, by Theorem 3.81, to show that the function $Z^Y \times Y^X \times X \to Z$, taking (f, g, x) to $(f \circ g)(x)$, is continuous. But this is the composition $e \circ (\mathrm{id} \times e)$. □

All of these things, and the ones following, have versions in the pointed category, the verification of which is trivial.

We finish this section by showing that, for Y metric, the compact-open topology is identical to a more familiar concept.

Lemma 3.88. *Let Y be a metric space, let C be a compact subset of Y, and let $U \supset C$ be open. Then there is an $\epsilon > 0$ such that $B_\epsilon(C) \subset U$.*

Proof. Cover C by a finite number of balls of the form $B_{\epsilon(x_i)}(x_i)$ such that $B_{2\epsilon(x_i)}(x_i) \subset U$. Put $\epsilon = \min(\epsilon(x_i))$. Suppose $x \in B_\epsilon(C)$. Then there is a $c \in C$ with $\mathrm{dist}(x, c) < \epsilon$ and an i such that $dist(c, x_i) < \epsilon(x_i)$. Thus $x \in B_{2\epsilon(x_i)}(x_i) \subset U$. □

Theorem 3.89. *If X is compact Hausdorff and Y is metric then the compact-open topology is induced by the uniform metric on Y^X i.e., the metric given by* $\mathrm{dist}(f, g) = \sup\{\mathrm{dist}(f(x), g(x)) | x \in X\}$.

Proof. For $f \in Y^X$, it suffices to show that a basic neighbourhood of f in each of these topologies contains a neighbourhood of f in the other topology.

Let $\epsilon > 0$ be given. Let $N = B_\epsilon(f) = \{g \in Y^X | \mathrm{dist}(f(x), g(x)) < \epsilon$ for all $x \in X\}$. Given x, there is a compact neighbourhood N_x of x such that $p \in N_x \Rightarrow f(p) \in B_{\epsilon/2}(f(x))$. Cover X by $N_{x_1} \cup \cdots N_{x_k}$. We claim that

$$V = M(N_{x_1}, B_{\epsilon/2}(f(x_1))) \cap \cdots \cap M(N_{x_k}, B_{\epsilon/2}(f(x_k))) \subset N.$$

To see this, let $g \in V$, i.e., $x \in N_{x_i} \Rightarrow g(x) \in B_{\epsilon/2}(f(x_i))$. But $f(x) \in B_{\epsilon/2}(f(x_i))$ and so it follows that $g \in V \Rightarrow \mathrm{dist}(f(x), g(x)) < \epsilon$ for all x. That is, $V \subset N$.

Conversely, suppose that $f \in M(K_1, U_1) \cap \cdots \cap M(K_r, U_r)$, i.e., $f(K_i) \subset U_i$ for $i = 1, \ldots, r$. By Lemma reflem:Br7-2-11, there is an $\epsilon > 0$ such that $B_\epsilon(f(Ki)) \subset U_i$ for all $i = 1, \ldots, r$. If $x \in K_i$ then $B_\epsilon(f(x)) \subset B_\epsilon(f(K_i)) \subset U_i$. Therefore, if $g \in B_\epsilon(f)$ and $x \in K_i$ then $g(x) \in B_\epsilon(f(x)) \subset U_i$. Thus $g \in M(K_i, U_i)$ for all i and so $B_\epsilon(f) \subset \cap M(K_i, U_i)$. □

Corollary 3.90. *If X is locally compact Hausdorff and Y is metric then the compact-open topology on Y^X is the topology of uniform convergence on compact sets. That is, a net $f_\alpha \in Y^X$ converges to $f \in Y^X$ in the compact-open topology $\Leftrightarrow$ $f_\alpha|_K$ converges uniformly to $f|_K$ for each compact set $K \subset X$.*

Proof. For $\Rightarrow$ recall from Corollary 3.86 that $Y^X \to Y^K$ is continuous. Thus $f_\alpha|_K \to f|_K$ in the compact-open topology. But Y^K has the topology of the uniform metric and so $f_\alpha|_K$ converges to $f|_K$ uniformly.

For $\Leftarrow$, suppose that $f_\alpha|_K$ converges uniformly to $f|_K$ for each compact $K \subset X$. Let $f \in M(K, U)$. Then there exists an $\epsilon > 0$ such that $B_\epsilon(f(K)) \subset U$. There is an α such that $\beta > a \Rightarrow \text{dist}(f_\beta(x), f(x)) < \epsilon$ for all $x \in K$. That is, $f_{beta}(x) \in B_\epsilon(f(K)) \subset U$. Thus $\beta > \alpha \Rightarrow f_{beta} \in M(X, U)$. This implies that f_α converges to f in the compact-open topology. □

3.7.4 Cellular space propaganda

We present some results (without proof) showing that many interesting spaces are either homeomorphic or homotopy equivalent to cellular spaces.

Definition 3.91. A *real analytic function* $f : \mathbb{R}^n \to \mathbb{R}$ is a infinitely differentiable function such that at every point $p \in \mathbb{R}^n$, f equals its Taylor series at p on some positive radius. A *real analytic set* $X \subset \mathbb{R}^n$ is the solution set of a finite collection of equations $f_1(x) = \cdots = f_n(x) = 0$, for f_i real analytic.

Example 3.92. Polynomial functions, exponential functions, trigonometric functions, etc. are real analytic.

Theorem 3.93 (Lojasiewicz 1964)**.** *Every real analytic set $X \subset \mathbb{R}^n$ is homeomorphic to a cellular space.*

Let X and Y be two topological spaces. Let $C(X, Y)$ be the set of all continuous maps from X to Y with the compact-open topology.

Example 3.94. The space $LY = C(\mathbb{S}^1, Y)$ is called the *free loop space* of Y.

Theorem 3.95 (Milnor 1959)**.** *If X and Y are cellular spaces and X is compact, then $C(X, Y)$ is homotopy equivalent to a cellular space.*

Definition 3.96. A topological space X is called a (*topological*) *n-manifold* if it is Hausdorff and if every point $p \in X$ is contained in an open neighbourhood $p \in U \subset X$ that is homeomorphic to $\mathbb{R}^n$.

- Every open set in $\mathbb{R}^n$ is an n-manifold.
- The sphere $\mathbb{S}^n$ is an n-manifold.
- Surfaces of any genus are 2-manifolds.
- The product of an m-manifold and an n-manifold is an $m + n$-manifold.

An example of a space that is locally Euclidean but is not a manifold is constructed by taking two copies of the real line $\mathbb{R} \coprod \mathbb{R} = \mathbb{R} \times \{a, b\}$ and forming the coset by $(t, a) \sim (t, b)$ if $t \neq 0$. This space looks locally like $\mathbb{R}$, but the points $(0, a)$ and $(0, b)$ cannot be separated by open sets. See Figure 3.17.

Theorem 3.97. *Every compact n-manifold is homotopy equivalent to a cellular space. It remains an open question whether or not every compact n-manifold is homeomorphic to a cellular space.*

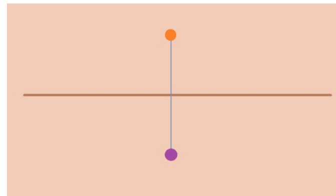


Figure 3.17: The line with two origins

3.7.5 Cellular Homology

In this section, we introduce *cellular homology*, which is a homology theory "bespoke" for cellular spaces, and show that it is isomorphic to singular homology.

Cellular homology is often the most useful for computing. In fact computation is much easier than the standard singular homology (for cellular spaces).

Before giving a precise definition, we will give a rough sketch of the reasoning.

If X is a cellular space with n-skeleton X_n, the cellular-homology modules are defined as the homology groups H_q of the "cellular chain complex"

$$\cdots \to C_{n+1}(X_{n+1}, X_n) \to C_n(X_n, X_{n-1}) \to C_{n-1}(X_{n-1}, X_{n-2}) \to \cdots,$$

where X_{-1} is taken to be the empty set.

The group

$$C_n(X_n, X_{n-1})$$

is free abelian, with generators that can be identified with the n-cells of X. Let e_n^α be an n-cell of X, and let $\chi_n^\alpha : \partial e_n^\alpha \cong \mathbb{S}^{n-1} \to X_{n-1}$ be the attaching map. Then consider the composition

$$\chi_n^{\alpha\beta} : \mathbb{S}^{n-1} \xrightarrow{\cong} \partial e_n^\alpha \xrightarrow{\chi_n^\alpha} X_{n-1} \xrightarrow{q} X_{n-1}/\left(X_{n-1} \setminus e_{n-1}^\beta\right) \xrightarrow{\cong} \mathbb{S}^{n-1},$$

where the first map identifies $\mathbb{S}^{n-1}$ with ∂e_n^α via the characteristic map Φ_n^α of e_n^α, the object e_{n-1}^β is an $(n-1)$-cell of X, the third map q is the coset map that collapses $X_{n-1} \setminus e_{n-1}^\beta$ to a point (thus wrapping e_{n-1}^β into a sphere $\mathbb{S}^{n-1}$), and the last map identifies $X_{n-1}/\left(X_{n-1} \setminus e_{n-1}^\beta\right)$ with $\mathbb{S}^{n-1}$ via the characteristic map Φ_{n-1}^β of e_{n-1}^β.

The boundary map

$$\partial_n : C_n(X_n, X_{n-1}) \to C_{n-1}(X_{n-1}, X_{n-2})$$

is then given by the formula

$$\partial_n(e_n^\alpha) = \sum_\beta \deg\left(\chi_n^{\alpha\beta}\right) e_{n-1}^\beta,$$

where $\deg\left(\chi_n^{\alpha\beta}\right)$ is the degree of $\chi_n^{\alpha\beta}$ and the sum is taken over all $(n-1)$-cells of X, considered as generators of $C_{n-1}(X_{n-1}, X_{n-2})$.

Now we move on to the details:

Lemma 3.98. *If X is a cellular space, then:*

(a) $H_q(X_n, X_{n-1})$ is zero if $q \neq n$ and is an free abelian group with generators corresponding to the n-cells when $q = n$.

(b) $H_q(X_n) = 0$ for $q > n$. Thus $H_q(X) = 0$ for $q > \dim(X)$.

(c) The inclusion $i : X_n \hookrightarrow X$ induces an isomorphism $H_q(i) : H_q(X_n) \to H_q(X)$ for $q < n$.

Proof. By Proposition 3.74, we have an isomorphism $H_q(X_n, X_{n-1}) \cong \widetilde{H}_q(X_n/X_{n-1})$ and X_n/X_{n-1} is a wedge of spheres indexed by the n-cells of X. Property (a) follows.

Property (b) is proven by induction. Clearly true for $n = 0$. Now suppose it has been proven for $n-1$. The long exact sequence of the pair contains

$$\to H_q(X_{n-1}) \to H_q(X_n) \to H_q(X_n, X_{n-1}) \to$$

where both $H_q(X_{n-1}) = H_q(X_n, X_{n-1}) = 0$ for $q > n$ by induction and property (a). Thus $H_q(X_n) = 0$ as well.

To prove property (c), consider the exact sequence

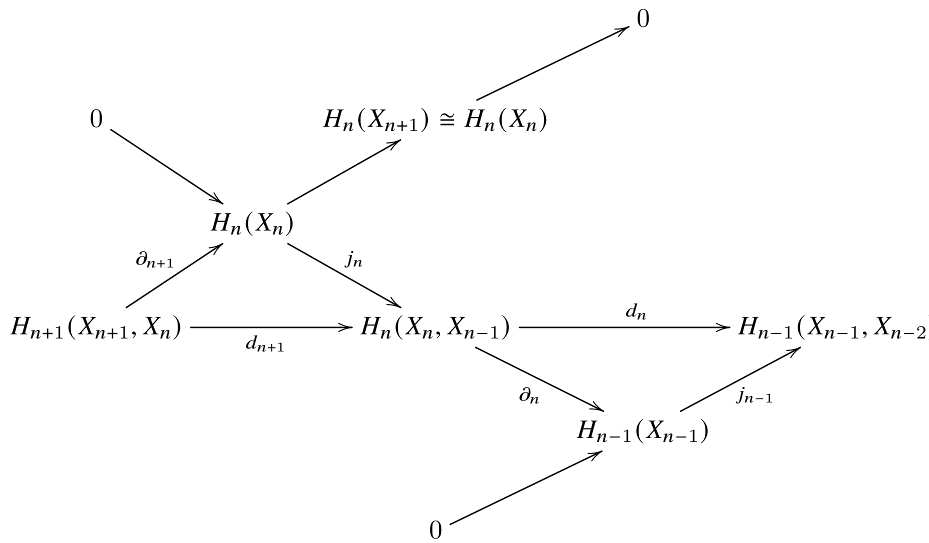
$$H_{q+1}(X_{n+1}, X_n) \to H_q(X_n) \to H_q(X_{n+1}) \to H_q(X_{n+1}, X_n).$$

By (a), the two groups on the end vanish if $q < n$ so $H_q(X_n) \cong H_q(X_{n+1})$. Repeating this argument, we get

$$H_q(X_n) \cong H_q(X_{n+1}) \cong H_q(X_{n+2}) \cong \cdots$$

which suffices if X is finite dimensional. To take care of the infinite dimensional case, observe that Lemma 3.77 implies that every chain in $S_q(X)$ must be in the image of $S_q(X_n)$ for some n (since the union of images of simplices occurring in the chain is a compact subset of X). Thus every cycle $Z_q(X)$ arises as the image of a cycle in $Z_q(X_n)$ for some n, and every boundary in $B_q(X_n)$ arises as the image of a boundary in $B_q(X_n)$ for some n. The result follows. □

Define a homomorphism $d_n : H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$ by the commutative diagramme



where the diagonal maps occur in the long exact sequences of pairs. Notice that $d_n \circ d_{n+1} = 0$ because it factors through $\partial_n \circ j_n = 0$. Thus $(H_n(X_n, X_{n-1}), d_n)_{n\in\mathbb{Z}}$ forms a chain complex, called the *cellular chain complex*. The homology of the cellular chain complex is called the *cellular homology*.

Theorem 3.99. *The cellular homology groups are naturally isomorphic to the singular homology groups.*

Proof. From the diagramme, we may identify $H_n(X) \cong H_n(X_n) = \mathrm{im}(\partial_{n+1})$. Since j_n is injective, this is isomorphic to $\mathrm{im}(j_n) = \mathrm{im}(d_{n+1})$. By exactness, this is the same as $\ker(\partial_n) = \mathrm{im}(d_{n+1})$. Finally, because j_{n-1} is injective, this is equal to $\ker(d_n) = \mathrm{im}(d_{n+1})$. □

Theorem 3.99 is very useful for calculations, because it allows us to replace the usually uncountably infinite rank $S_q(X)$ by the - at most countable and often finite rank - $H_n(Xn, X_{n-1})$. Before getting started with examples, we want a more direct understanding of the boundary maps d_n. Denote by $\{e^n_\alpha\}_\alpha$ the set of n-cells of a cellular space X, so that $H_n(X_n, X_{n-1})$ is the free abelian group generated by $\{e^n_\alpha\}_\alpha$.

Proposition 3.100. *For $n > 1$, the cellular boundary map satisfies*

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha,\beta} e^{n-1}_\beta$$

where $d_{\alpha,\beta}$ is the degree of the map

$$\mathbb{S}^{n-1}\alpha \longrightarrow X_{n-1} \longrightarrow \mathbb{S}^{n-1}_\beta$$

defined by composing the attaching map of e^n_α with the coset map $X_{n-1} \longrightarrow \mathbb{S}^{n-1}_\beta = X_{n-1}/(X_{n-1} \setminus e^n_\beta)$.

Proof. The proof is based on the following commutative diagramme

$$\begin{array}{ccccc}
H_n(\mathbb{D}^n_\alpha, \partial\mathbb{D}^n_\alpha) & \xrightarrow[\cong]{\partial} & \widetilde{H}_{n-1}(\partial\mathbb{D}^n_\alpha) & \xrightarrow{(\Delta_{\alpha,\beta})_*} & \widetilde{H}_{n-1}(\mathbb{S}^{n-1}_\beta) \\
\downarrow{\scriptstyle \Phi_{\alpha^*}} & & \downarrow{\scriptstyle f_{\alpha^*}} & & \uparrow{\scriptstyle q_{\beta *}} \\
H_n(X_n, X_{n-1}) & \xrightarrow{\partial_n} & \widetilde{H}_{n-1}(X_{n-1}) & \xrightarrow{q_*} & \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \\
& \searrow{\scriptstyle d_n} & \downarrow{\scriptstyle j_{n-1}} & & \downarrow{\scriptstyle \cong} \\
& & H_{n-1}(X_{n-1}/X_{n-2}) & \xrightarrow[\cong]{} & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{array}$$

where

- Φ_{α^*} is the characteristic map for e^n_α and f_α the attaching map.
- $q : X_{n-1} \to X_{n-1}/X_{n-2}$ is the coset map.
- $q_\beta : X_{n-1}/X_{n-2} \to \mathbb{S}^{n-1}_\beta$ is the coset map obtained by collapsing everything belonging to the complement of the cell e^{n-1}_β to a point, the resulting coset (sphere) being identified with $\mathbb{S}^{n-1}_\beta = \mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta$ via the characteristic map Φ_β.
- $\Delta_{\alpha,\beta} = q_\beta \circ q \circ f_\alpha$ is, the attaching map of e^n_α followed by the coset map $X_{n-1} \to \mathbb{S}^{n-1}_\beta$ collapsing the complement of $e^{n^1}_\beta$ inn X_{n-1} to a point.

The map Φ_{α^*} sends the generator $[\mathbb{D}^n_\alpha] \in H_n(\mathbb{D}^n_\alpha, \partial\mathbb{D}^n_\alpha)$ to a generator of the $\mathbb{Z}$ summand of $H_n(X_n, X_{n-1})$ corresponding to $e^n\alpha$. Letting $e^n\alpha$ denote this generator, commutativity of the left half of the diagramme then gives $d_n(e^n\alpha = j_{n-1}f_{\alpha^*}\partial[\mathbb{D}^n_\alpha]$. In terms of the basis for $H_{n-1}(X_{n-1}, X_{n-2})$ corresponding to the cells e^{n-1}_β, the map q_{β_*} is the projection of $\widetilde{H}_{n-1}(X_{n-1}/X_{n-2})$ onto its $\mathbb{Z}$ summand corresponding to e^{n-1}_β. Commutativity of the diagramme then yields the formula for d_n given above. □

3.7.6 Examples

Example 3.101. A *genus g surface* σ_g is constructed by attaching a 2-cell to a wedge of $2g$ circles using an attaching map $\prod_{i=1}^g [a_i, b_i]$. The cellular complex is:

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z}.$$

The boundary map d_1 is zero, because each 1-cell meets the 0-cell twice and the attaching map $\mathbb{S}^0 \to \{\text{pt}\}$ sends $\widetilde{H}_0(\mathbb{S}^0)$ to zero by definition. The boundary map d_2 is also zero, because the attaching map winds around each loop twice, but in opposite directions, giving total degree zero. It follows that

$$H_q(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if} \quad q = 0, 2 \\ \mathbb{Z}^{2g} & \text{if} \quad q = 1 \\ 0 & \text{otherwise} \end{cases}$$

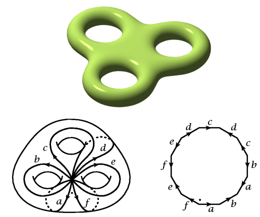


Figure 3.18: Σ_g with $g = 3$

Example 3.102. The *non-orientable surface* N_g of genus $g \geq 0$ is constructed by attaching a single 2-cell to a wedge sum of $g+1$ circles $a_0, \ldots, a_g$ by the attaching map $a_0^2 \cdots a_g^2$. The surface N_0 in the real projective plane and N_1 is the Klein bottle. The cellular chain complex is

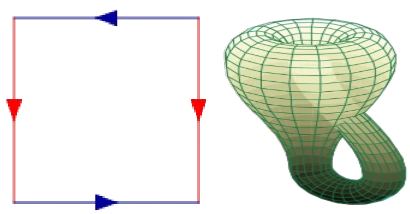


Figure 3.19: Klein bottle N_g with $g = 1$

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{g+1} \xrightarrow{d_1} \mathbb{Z}.$$

As before, $d_1 = 0$. The attaching map for the 2-cell winds twice around each circle in the same direction and thus has degree 2 for each 1-cell. Consequently, $d_2(n) = (2n, \ldots, 2_n)$. If we do a change of basis for $\mathbb{Z}^{g+1}$ using generators $(1,1,\ldots,1), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)$ then with respect to the new basis, $d_2(n) = (2, 0, \ldots, 0)$.

$$H_q(N_g) = \begin{cases} \mathbb{Z} & \text{if } \quad q = 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^g & \text{if } \quad q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 3.103. A emphproduct of spheres $\mathbb{S}^m \times \mathbb{S}^n$ with $m, n \geq 1$ has the structure of a cellulr space with four cells, in dimensions 0, m, n, and $m+n$. To see this observe that $\mathbb{D}^{m+n} \cong \mathbb{D}^m \times \mathbb{D}^n$ and

$$\mathbb{S}^m \times \mathbb{S}^n \cong (\mathbb{D}^m \times \mathbb{D}^n)/\text{si} \cong (\mathbb{D}^m/\sim) \times (\mathbb{D}^n/\sim)$$

where $\sim$ is generated by the relations $(x, y) \sim (x, y')$ if $y, y' \in \partial\mathbb{D}^n$ and $(x, y) \sim (x', y)$ if $x, x' \in \partial\mathbb{D}^m$. Since the relations occur only in the boundary of $\mathbb{D}^m \times \mathbb{D}^n$, this can be understood as attaching a $(m+n)$-cell to the coset of

$$\partial(\mathbb{D}^m \times \mathbb{D}^n) = (\partial\mathbb{D}^m \times \mathbb{D}^n) \cup (\mathbb{D}^m \times \partial\mathbb{D}^n)$$

which is identified as the wedge sum $\mathbb{S}^m \vee \mathbb{S}^n$, which is a cellular space with cells in dimension 0, m, and n (compare the case of a torus $\mathbb{S}^1 \times \mathbb{S}^1$).

Suppose that $n \geq m$ and $n > 1$. Then the $(n+1)$-skeleton is equal to $\mathbb{S}^m \vee \mathbb{S}^n$ so by Lemma 3.98, $H_q(\mathbb{S}^m \times \mathbb{S}^n) = H_q(\mathbb{S}^m \vee \mathbb{S}^n)$ for $q \leq n$. It follows that the boundary map in the cellular chain complex is trivial and that

$$H_q(\mathbb{S}^m \times \mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if} \quad q = 0, m, n, \text{ or } m+n \\ 0 & \text{otherwise} \end{cases}$$

Example 3.104. The complex projective space $\mathbb{C}P^n$ is the set of one dimensional vector subspaces of $\mathbb{C}^{n+1}$. It may also be construct as the coset of the sphere $\mathbb{S}^{2n+1}$ by the relation $v \sim \lambda v$ where $\lambda \in \mathbb{S}^1$ is a unit scalar (= a complex number of norm 1.)

We can construct $\mathbb{C}P^n$ inductively by attaching a $2n$-dimensional cell to $\mathbb{C}P^{n-1}$. To see this, consider the embedding from $\mathbb{D}^{2n} \subset \mathbb{C}^n$ to $\mathbb{C}^n \times \mathbb{C}$ by

$$w \mapsto (w, \sqrt{1 - |w|^2}).$$

The boundary of $\mathbb{D}^{2n}$ is sent to the unit sphere in $\mathbb{S}^{2n-1} \subset \mathbb{C}^n \times \{0\}$ and there is a one-to-one correspondence between the interior of $\mathbb{D}^n$ and the one-dimensional subspaces of $\mathbb{C}^n \times \mathbb{C}$ not contained in $\mathbb{C}^n \times \{0\}$. Thus $\mathbb{C}P^n$ is obtained by attaching $\mathbb{D}^{2n}$ to $\mathbb{C}P^{n-1}$ by the coset map $\mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$.

Since $\mathbb{C}P^n = e_0 \cup e_2 \cup \cdots \cup e_{2n}$ only has cells in even dimension, this means the boundary maps in the cellular chain complex are necessarily zero, and we obtain

$$H_q(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if} \quad 0 \leq q \leq 2n \quad \text{and is even} \\ 0 & \text{otherwise} \end{cases}$$

Example 3.105. The real projective space needs more care compared with the complex case. The space $\mathbb{R}P^n$ is defined as the set of all lines in $\mathbb{R}^{n+1}$ through he origin.

We claim that as a cellular space, $\mathbb{R}P^n$ can be seen as the union of one k cell for each $k \leq n$. We prove this by induction. First we notice that the statement is trivially true for $n = 0$. Then consider that $\mathbb{R}P^n \cong \mathbb{S}^{n-1}/{\sim}$ where $v \sim -v$ for all $v \in \mathbb{S}^n$. Note that this is the same as saying that $\mathbb{R}P^n \cong \mathbb{D}^n/{\sim}$ where $v \sim -v$ for all $v \in \partial\mathbb{D}^n = \mathbb{S}^{n-1}$. In other words, $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e_n$. Thus $\mathbb{R}P^n = e_0 \cup e_1 \cup \cdots \cup e_n$. Notice that for each k, the k-skeleton of $\mathbb{R}P^k$ is $\mathbb{R}P^{k-1}$. Thus we analyse the following chain complex:

$$0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathbb{Z} \to 0$$

To understand the boundary maps, we must determine the degrees of the composition of the attaching map $\phi_k : \partial e_k \cong \mathbb{S}^{k-1} \to \mathbb{R}P^{k-1}$ and the coset map $q : \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong \mathbb{S}^{k-1}$.

Notice that the preimage of $\mathbb{R}P^{k-2}$ under the map ϕ_k is clearly $\mathbb{S}^{k-2} \subset \mathbb{S}^{k-1}$. Moreover, we have that $\mathbb{S}^{k-1} \setminus \mathbb{S}^{k-2} = \mathbb{B}_1^{k-1} \sqcup \mathbb{B}_2^{k-1}$, two $k-1$ open balls. These, under ϕ_k, are mapped homeomorphically to $\mathbb{R}P^{k-1} \setminus \mathbb{R}P^{k-2}$. One easily notices that the image $\mathbb{B}_1^{k-1}$ under ϕ_k is the "top" part of $\mathbb{R}P^k$, i.e. ϕ_k restricted to $\mathbb{B}_1^{k-1}$ is the identity map (which has degree 1).

On the other hand, ϕ_k restricted to $\mathbb{B}_2^{k-1}$ is the antipodal map, as the second open ball is mapped to the "lower part" of $\mathbb{R}P^k$. As we saw in Proposition 3.58, the degree of the antipodal map of $\mathbb{S}^{k-1}$ is $(-1)^k$. Therefore the map $\phi_k \circ q$ can be seen as the sum of the identity and the antipodal map, hence we can conclude that its degree is $1 + (-1)^k$.

Thus we get that

$$d_k(e_k) = \begin{cases} 2e_{k-1} & \text{if} \quad k \text{ is even} \\ 0 & \text{if} \quad k \text{ is odd.} \end{cases}$$

Therefore, if k is even $\ker(d_k) = \{0\}$ and $\operatorname{im}(d_k) \cong \mathbb{Z}$; while when k is odd, $\ker(d_k) \cong \mathbb{Z}$ and $\operatorname{im}(d_k) = \{0\}$.

The chain complex is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

if n is odd and

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

if n is even. We obtain homology groups,

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if} \quad q = 0 \text{ or } q = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if} \quad 0 < k < (n-1) \text{ and is odd} \\ 0 & \text{otherwise} \end{cases}$$

3.8 Mayer-Vietoris Sequence

The Mayer-Vietoris sequence is an alternative to the long exact sequence of a pair (X, A) that sometimes more convenient to use.

Let X be a topological space and let $A, B \subset X$ be a pair of subspaces such that $A \cup B = X$. We consider the open covering $\mathcal{U} = \{A, B\}$. We need the following maps:

$$\begin{array}{ccccc} & & A & & \\ & \overset{i_1}{\nearrow} & & \overset{\kappa_1}{\searrow} & \\ A \cap B & & & & X \\ & \underset{i_2}{\searrow} & & \underset{\kappa_2}{\nearrow} & \\ & & B & & \end{array}$$

Note that by definition, the sequence

$$0 \to S_*(A \cap B) \xrightarrow{(i_1,i_2)} S_*(A) \oplus S_*(B) \to S_*^{\mathfrak{U}}(X) \to 0$$

is exact. Here, the second map is

$$(\alpha 1, \alpha_2) \mapsto \kappa_1(\alpha_1) - \kappa_2(\alpha_2).$$

Theorem 3.106 (Mayer-Vietoris). *There is a long exact sequence*

$$\cdots \xrightarrow{\delta} H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \to \cdots$$

A similar sequence exists for reduced homology

$$\cdots \to \widetilde{H}_n(A \cap B) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(A \cap B) \to \cdots$$

Proof. The proof follows from Theorem 3.23 with Theorem 3.48 applied to $S_*^{\mathfrak{U}}(X) \sim S_*(X)$. □

Proposition 3.107. *For a path connected space X, there is a canonical isomorphism*

$$\widetilde{H}_{q+1}(SX) \cong \widetilde{H}_q(X)$$

for all $q \in \mathbb{Z}$, where SX denotes the suspension of X.

Proof. Note that $SX = (X \times I)/(X \times \{0,1\})$ and regard $A = (X \times I)/(X \times \{0\})$ and $B = (X \times I)/(X \times \{1\})$. Then A and B form a covering of SX, $A \cup B = SX$ and $A \cap B = X \times \mathring{I} \sim X$.

Since A and B are contractible, $\widetilde{H}_*(A) = \widetilde{H}_*(B) = 0$, and $\widetilde{H}_*(A \cap B) = \widetilde{H}_*(X)$. Apply Theorem 3.106 to the triad $(A \cup B = SX, A, B)$ and the result follows. □

3.9 Homology with coefficients

So far we have developed singular homology theory for integer coefficients, meaning that our chains are finite formal sums $\sum_\sigma a_\sigma \sigma$ with coefficients $a_\sigma \in \mathbb{Z}$. More generally, it is possible (and useful) to work with coefficients in any commutative ring R with identity $1 \in R$; in particular, this means we have a canonical ring homomorphism $\mathbb{Z} \to R$. The most interesting cases are when $R = \mathbb{Z}/n\mathbb{Z}$ is the ring of integers modulo n, or when R is a field, such as $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}/p\mathbb{Z}$ for prime p.

To that end, we need some preparations.

3.9.1 Tensor Products

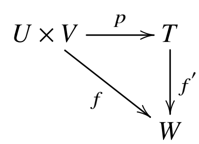
The tensor product may be defined for any pair of bimodules, but we shall examine the simpler case of modules over a commutative ring with unity. This is enough for many applications like algebraic topology.

Let R a commutative ring and U, V, W be any R-modules. We shall write them as left modules, although it is only a matter of notation whether a module over a commutative ring is regarded as a left or right module.

We want to consider bilinear mappings from U, V to W, i.e. maps

$$f : U \times V \to W, \tag{3.108}$$

such that f is R-linear in each argument. Our object will be to construct a K-module T and a bilinear map $p : U \times V \to T$ which is universal for all bilinear maps (3.108), in the sense that to any bilinear mapping f as in (3.108) there corresponds a unique linear mapping $f^{'} : T \to W$ such that the accompanying diagramme commutes.



A module T with these properties is called a tensor product of U and V and is denoted by $U \otimes_R V$ or simply $U \otimes V$. If it exists it is unique up to isomorphism, as universal object and we shall speak of *the* tensor product.

To prove the existence of T we form the free R-module A on the set $U \times V$ (without the module structure); in A we consider the submodule B generated by all the elements

$$\begin{aligned} &(u + u^{'}, v) - (u, v) - (u^{'}, v), \quad (u, v + v^{'}) - (u, v) - (u, v^{'}) \quad (u, u^{'} \in U, v, v^{'} \in V) \\ &(\alpha u, v) - \alpha(u, v), \quad (u, \alpha v) - \alpha(u, v), \quad \alpha \in R. \end{aligned} \tag{3.109}$$

There is a map $p : U \times V A/B$, obtained by taking the inclusion map $U \times V \to A$, followed by the natural homomorphism $A \to A/B$. This map p is bilinear, for the elements (3.109) generating B were just chosen to ensure this. We set $T = A/B$ and claim that T, with the map p, is the required tensor product. Let $f : U \times V \to W$ be any bilinear mapp; regarded as a set map, i.e. ignoring bilinearity, it may be extended to a unique homomorphism $f_1 : A \to W$, because A is free on the elements (u, v). We claim that $\ker f_1 \supset B$; for we have

$$\begin{aligned} &f_1[(u + u^{'}, v) - (u, v) - (u^{'}, v)] = f[(u, v + v^{'})] - f[(u, v)] - f[(u, v^{'})] = 0, \\ &f_1[(\alpha u, v) - \alpha(u, v)] = f[(\alpha u, v) - f[\alpha(u, v)] = 0, \end{aligned}$$

by the bilinearity of f, and similarly for the other relations. Hence f_1 may be taken via T, by the factor theorem, and this provides the required map $f : T \to W$. This map $f^{'}$ is unique since its values are determined on the images of (u, v) in T and these form a generating set. Our conclusions may be summed up as follows:

Theorem 3.110. *Let U, V be modules over a commutative ring R. Then there exists an R-module $U \otimes V$ together with a bilinear map $p : U \times V \to U \otimes V$ which is universal for bilinear maps from $U \times V$ to R-modules.*

The image of (u, v) in $U \otimes V$ is denoted by $u \otimes v$. Thus $U \otimes V$ is an R-module with generating set $\{u \otimes v u \otimes v | u \in U, v \in V\}$ and defining relations

$$(u + u^{'}) \otimes v = u \otimes v + u^{'} \otimes v, \quad u, u^{'} \in U,$$
$$u \otimes (v + v^{'}) = u \otimes v + u \otimes v^{'}, \quad v, v^{'} \in V,$$
$$(\alpha u) \otimes v = u \otimes (\alpha v) = \alpha(u \otimes v), \quad \alpha \in R.$$

There is another way of expressing Theorem 3.110 which is often useful. Theorem 3.110 states in effect that for any R-modules U, V, W there is a natural bijection between the set of bilinear maps $U \times V \to W$ and the set of homomorphisms $U \otimes V \to W$. Now a map $f : U \times V \to W$ is linear in the second variable iff for each $u_0 \in U$, the map $V \to U \times V \to W$ given by $v \mapsto (u_0, v) \mapsto f(u_0, v)$ is linear. Further, f is bilinear iff in addition the map $U \to \mathrm{Hom}_R(V, W)$ given by $u \mapsto f(u, -)$ is linear, i.e. $f \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$. Hence there is a natural bijection

$$\mathrm{Hom}_R(U, \mathrm{Hom}_R(V, W)) \cong \mathrm{Hom}_R(U \otimes_R V, W). \tag{3.111}$$

This is easily verified to be an isomorphism of R-modules. The property expressed in (3.111) is known as *adjoint associativity.*

From the definition it is easy to check that tensor products satisfy the associative and commutative laws:

Proposition 3.112. *Let U, V, W be any R-modules, where R is a commutative ring. Then*

$$U \otimes V \cong V \otimes U, \tag{3.113}$$
$$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W. \tag{3.114}$$

Proof. The rule $(u, v) \mapsto v \otimes u$ is a bilinear map $U \times V \to V \otimes U$, and hence gives rise to a homomorphism $\alpha : U \otimes V \to V \otimes U$, in which $u \otimes v \mapsto v \otimes u$. The general element of $U \otimes V$ has the form $\sum u_i \otimes v_i$, and it follows that $\alpha : \sum u_i \otimes v_i \mapsto \sum v_i \otimes u_i$. The same argument shows that $\beta : \sum v_i \otimes u_i \mapsto \sum u_i \otimes v_i$ is a homomorphism; clearly it is inverse to α, hence α is an isomorphism and (3.113) follows.

The proof of (3.114) is quite similar. We consider the map $\alpha : U \times V \times W \to U \otimes (V \otimes W)$ given by $(w, v, w) \mapsto u \otimes (v \otimes w)$. For fixed w this is bilinear in u, v and hence gives rise to a map $\alpha^{''} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, in which $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$. The inverse map is constructed in the same way and this shows $\alpha^{''}$ to be an isomorphism, which proves (3.113). □

We observe that it is possible to define $U \otimes V \otimes W$ directly by the universal property for trilinear maps, and a similar proof will show that it is isomorphic to either of the modules in (3.114). The same holds for more than three factors; this is just the generalised associative law. We shall therefore omit brackets in repeated tensor products.

Next we prove a 'distributive law':

Proposition 3.115. *For any R-modules U, $V^{'}$, $V^{''}$ we have*

$$U \otimes (V^{'} \oplus V^{''}) \cong (U \otimes V^{'}) \oplus (U \otimes V^{''}). \tag{3.116}$$

Proof. We show that the module on the right of (3.116) satisfies the universal property of the tensor product. A bilinear map from $U \times (V' \otimes V'')$ is given by $(u, v', v'') \mapsto (u \otimes v', u \otimes v'')$. If $f : U \times (V' \otimes V'') \to W$ is any bilinear map, then

$$f(u, v', v'') = f(u, v') + f(u, v''),$$

and the expression on the right can be regarded as a map from $(U \otimes V') \oplus (U \otimes V'')$. Thus f is uniquely factored by the standard bilinear map, and the result follows. □

The definition of the tensor product by a universal property is useful for proving the existence of mappings from $U \otimes V$ to an R-module, for we need only find the appropriate bilinear map from $U \times V$. It also has the merit of generality; but the definition is not such that it allows the structure of $U \otimes V$ to be read off. For example, if r, s are coprime integers, then $(\mathbb{Z}/r\mathbb{Z}) \otimes (\mathbb{Z}/s\mathbb{Z}) = 0$. This is seen as follows. Since r, s are coprime, there exist $m, n \in \mathbb{Z}$ such that $mr + ns = 1$. Now for any $a \in \mathbb{Z}/r\mathbb{Z}$, $b \in \mathbb{Z}/s\mathbb{Z}$ we have

$$a \otimes b = mr(a \otimes b) + ns(a \otimes b) = m(ra \otimes b) + n(a \otimes sb) = 0_{(\mathbb{Z}/r\mathbb{Z}) \otimes (\mathbb{Z}/s\mathbb{Z})}.$$

It follows that $(\mathbb{Z}/r\mathbb{Z}) \otimes (\mathbb{Z}/s\mathbb{Z}) = 0$, because the tensor product is generated by elements of the form $a \otimes b$.

It is important to bear in mind that the general element of $U \otimes V$ is not of the form $u \otimes v$, but is a swm of such terms: $\sum u_i \otimes v_i$. For example, if V is a free R-module, with basis $e_1 \ldots, e_n$ then every element of $U \otimes V$ can be written uniquely in the form $\sum u_i \otimes e_i$ $(u_i \in U)$, i.e. $U \otimes R^n \cong U^n$. To prove this fact, let us first take the case $n = 1$:

$$U \otimes R \cong U. \tag{3.117}$$

We have a bilinear map $\theta : (u, \lambda) \mapsto u\lambda$ from $U \times R$ to U, and if $F : U \times \to W$is any bilinear mapping, then $f(u, \lambda) = f(u\lambda, 1)$, hence $f = \theta f'$, where $f' : u \mapsto f(u, 1)$, and clearly f' is the only map with this property. Thus U satisfies the universal property of Theorem 3.110 and (3.117) follows. Now $U \otimes R^n \cong U^n$ follows by induction on n, using the distributive law (Proposition 3.115). Thus we obtain

Proposition 3.118. *For any R-module U over a commutative ring R, the tensor product with a free R-module of rank n is a direct sum of n copies of U:*

$$U \otimes R^n \cong U^n. \tag{3.119}$$

By symmetry a corresponding result holds for the first factor, and combining the two, we obtain

Corollary 3.120. *If U and V are free R-modules of finite rank over a commutative ring R, say $U \cong R^m$, $V \cong R^n$, then $U \otimes V \cong R^{mn}$. In particular, this applies to finite-dimensional vector spaces over a field, and we then have* $\dim(U \otimes V) = \dim U \cdot \dim V$.

Explicitly, if $e_1, \dots, e_m$ is a basis for U and $f_1, \dots, f_n$ is a basis for V, then the elements $e_i \otimes f_j$ $(i = 1, \dots, m, j = 1, \dots, n)$ form a basis for $U \otimes V$.

We record a property noted before (3.117), namely the independence property of the tensor product:

Proposition 3.121. *Let U be any R-module and V be a free R-module with basis $e_1, \dots, e_n$ over a commutative ring R. Then every element of $U \otimes V$ is unique of the form*

$$\sum u_i \otimes e_i, \quad \textit{where} \quad u_i \in U. \tag{3.122}$$

Caution is needed in applying this result. Thus if $\sum u_i \otimes v_i = 0$ in $U \otimes V$ and the v_i, are linearly independent over R, then it does not follow that the u_i must vanish. If the submodule generated by the v_i, is denoted by $V^{'}$ (so that the v_i form a basis for $V^{'}$) then all we can conclude is that the u_i all vanish if $\sum u_i \otimes v_i = 0$ in $U \otimes V^{'}$. Now the inclusion $V^{'} \to V$ induces the homomorphism

$$U \otimes V^{'} \to U \otimes V. \tag{3.123}$$

which however may not be injective. For example, the inclusion $2\mathbb{Z} \to \mathbb{Z}$ is injective, but it does not remain so on tensoring with $\mathbb{Z}/2\mathbb{Z}$. If $Z/2\mathbb{Z}$, $\mathbb{Z}$, $2\mathbb{Z}$ are generated by e, f, $f^{'}$ respectively, then $(\mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z}) \otimes 2\mathbb{Z}$ are both isomorphic to $\mathbb{Z}/2\mathbb{Z}$, by (3.117), with generators $e \otimes f$, $e \otimes f^{'}$ respectively. But $f^{'}$ maps to $2f$ and $e \otimes f^{'} \mapsto e \times 2f = 2e \otimes f = 0$. Thus (3.123) is the zero map in this case. A more precise analysis of this phenomenon will be undertaken in homological algebra. For the moment we note that (3.123) is certainly injective if $V^{'}$ is a direct summand in V, by Proposition 3.115; so in that case we can identify $U \otimes V^{'}$ with its image in $U \otimes V$. We note that this always holds when R is a field.

Let us next consider the effect of the tensor product on homomorphisms. Given any R-linear maps $\alpha : U \to U^{'}$, $\beta : V \to V^{'}$, there is a unique R-linear map $\alpha \otimes \beta : U \otimes V \to U^{'} \otimes V^{'}$ such that the left-hand square of the diagramme below commutes:

$$\begin{array}{ccccc} U \times V & \xrightarrow{\alpha\times\beta} & U^{'} \times V^{'} & \xrightarrow{\alpha^{'}\times\beta^{'}} & U^{''} \times V^{''} \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda^{'}} & & \downarrow{\scriptstyle\lambda^{''}} \\ U \otimes V & \xrightarrow[\alpha\otimes\beta]{} & U^{'} \otimes V^{'} & \xrightarrow[\alpha^{'}\otimes\beta^{'}]{} & U^{''} \otimes V^{''} \end{array} \tag{3.124}$$

For the map $(u, v) \mapsto \alpha u \otimes \beta v$ from $U \times V$ to $U^{'} \otimes V^{'}$ is bilinear, and hence can be taken via $U \otimes V$, by the universal property of $U \otimes V$.

If $\alpha^{'} : U^{'} \to U^{''}$, $\beta^{'} : V^{'} \to V^{''}$ is another pair of homomorphisms, we obtain a commutative diagramme (3.124). Since $(\alpha^{'} \times \beta^{'})(\alpha \times \beta)(u, v) = (\alpha^{'}\alpha u, \beta^{'}\beta v)$ for any $u \in U$, $v \in V$, we have $(\alpha^{'} \times \beta^{'})(\alpha \times \beta) = \alpha^{'}\alpha \times \beta^{'}\beta$, and it follows from the diagramme (3.124) that

$$\alpha^{'}\alpha \otimes \beta^{'}\beta = (\alpha^{'} \otimes \beta^{'})(\alpha \otimes \beta) \tag{3.125}$$

In the special case $V'' = V' = V$, $\beta'' = \beta' = 1$, (3.125) reduces to

$$\alpha'\alpha \otimes 1 = (\alpha' \otimes 1)(\alpha \otimes 1), \tag{3.126}$$

and together with the obvious equation $1 \otimes 1 = 1$ this shows that the assignment $U \mapsto U \otimes V$ is a functor from R-modules to R-modules, for any given V. By symmetry the assignment $V \mapsto U \otimes V$ is also a functor for fixed U. Thus the tensor product is a bifunctor.

The above diagramme shows that there is a correspondence between pairs of maps $(\alpha, \beta) \in \mathrm{Hom}_R(U, V) \times \mathrm{Hom}\, R(V, V)$ and maps $\alpha \otimes \beta \in \mathrm{Hom}_R(U \otimes V, U' \otimes V')$. So we have a map $(\alpha, \beta) \mapsto \alpha \otimes \beta$ which is clearly bilinear; by the universal property of the tensor product it induces a linear map

$$\mathrm{Hom}_R(U, U') \otimes \mathrm{Hom}_R(V, V') \rightarrow \mathrm{Hom}_R(U \otimes V, U' \otimes V'). \tag{3.127}$$

We remark that for a pair of maps $\alpha : U \rightarrow U'$, $\beta : V \rightarrow V'$ the expression $\alpha \otimes \beta$ is ambiguous: it may mean the element of the left of (3.127) or the induced homomorphism from $U \otimes V$ to $U' \otimes V'$, and one of these is mapped to the other in (3.127). It will usually be clear from the context which interpretation is intended; in some important cases the mapping (3.127) is an isomorphism and the ambiguity disappears. For example, when U and V are free of finite rank, say $U = R^m$, $V = R^n$, then (3.127) reduces to $U'^m \otimes V'^n \cong (U \otimes V)^{mn}$, by a double application of Proposition 3.115, together with the relation

$$\mathrm{Hom}_R(R^n, U) \cong U^n,$$

which follows by associating with $(u_1, \ldots, u_n) \in U^n$ the map $e_i \mapsto u_i$, where $e_1, \ldots, e_n$ is the standard basis of R^n. In particular, when $U' = U$, $V' = V$, we obtain

Proposition 3.128. *If U, V are free modules of finite rank (over a commutative ring R), then the map (
refeq:C4-8-14) induces the isomorphism*

$$\mathrm{End}_R(U) \otimes \mathrm{End}_R(V) \cong End_R(U \otimes V).$$

When we come to consider tensor products over a non-commutative ring, the corresponding construction leads in the first instance to abelian groups rather than modules. Thus let R be any ring, U be a right R-module and V be a left R-module, and for any abelian group W consider maps $f : U \times V \rightarrow W$ which are *biadditive*, i.e. additive in each argument, and *R-balanced*, i.e.

$$f(ur, v) = f(u, rv) \quad \text{for all} \quad u \in U, v \in V, r \in R.$$

A map which is biadditive and R-balanced will again be called *R-bilineary* or simply *bilinear*, if the ring R is clear from the context. We can again construct $U \otimes V$, now merely an abelian group, universal for R-balanced biadditive maps from $U \times V$ to abelian groups. The existence is proved as before, $U \times V = A/B$, where A is the free abelian group on $U \times V$ and B is the subgroup generated by

$$(w + u', v) - (u, v) - (u', v), \quad u, u' \in U,$$
$$(u, v + v') - (u, v) - (u, v'), \quad v, v' \in V,$$
$$(ur, v) - (u, rv), \quad r \in R.$$

Suppose now that U is an (S, R)-bimodule and V is an (R, T)-bimodule, for some rings S, T. Then the tensor product $U \otimes V$ just defined may be regarded as an (S, T)-bimodule in the following way. Take $s \in S$ and consider the map $\lambda_s : U \times V \to U \otimes V$ defined by

$$\lambda_s : (u, v) \mapsto su \otimes v.$$

Clearly this is biadditive and balanced; e.g. to prove the latter, we have $s(ur) \otimes v = (su)r \otimes v = su \otimes rv$, by the bimodule property of U. It follows that λ induces a homomorphism $U \otimes V \to U \otimes V$ which is simply denoted by s; thus we have

$$s(\sum u_i \otimes v_i) = \sum su_i \otimes v_i. \tag{3.129}$$

If we do this for each $s \in S$ we obtain a left S-module structure on $U \otimes V$, for we have, for any $s, s' \in S$,

$$(ss')(u \otimes v) = (ss')u \otimes v = s(s'u) \otimes v = s[' u \otimes v] = s[s'(u \otimes v)],$$

and of course $1(u \otimes v) = u \otimes v$. Similarly we can define a right T−module structure on $U \otimes V$ such that $(u \otimes v)t = u \otimes vt$ for $t \in T$, and $U \otimes V$ is an (S, T)-bimodule, because

$$s[(u \otimes v)t] = s[u \otimes vt] = su \otimes vt = (su \otimes v)t = [s(u \otimes v)]t.$$

Given any (S, T)-bimodule W, we can as before regard any homomorphism $f : U \times V \to W$ which is S-linear in the first, T-linear in the second argument and R-balanced, as defining for each $u \in U$ a T-linear map $f_u : v \mapsto f(u, v)$. The set of all these T-linear maps has a natural (S, R)-bimodule structure induced from $\mathrm{Hom}_T(V, W)$ and the map $u \mapsto f_u$ is a homomorphism of (S, R)-bimodules; $ur \mapsto f_{ur}$ and $f(wr, v) = f(u, rv)$ because f is R-balanced. Thus the natural homomorphism (3.111) leads to an isomorphism of S-bimodules, again called *adjoint associativity*:

$$\mathrm{Hom}_T(U \otimes_R, W) \cong \mathrm{Hom}_R(U, \mathrm{Hom}_T(V, W)) \quad ({}_SU_R, V_T, {}_SW_T) \tag{3.130}$$

By symmetry we likewise have an isomorphism of T-bimodules:

$$\mathrm{Hom}_S(U \otimes_R, W) \cong \mathrm{Hom}_R(U, \mathrm{Hom}_S(V, W)) \tag{3.131}$$

Like the hom-functor, the tensor product is not an exact functor; however it is right exact:

Proposition 3.132. *For any ring R, the tensor product $U \otimes_R V$ is right exact in each variable.*

Proof. By symmetry it will be enough to show that $- \otimes V$ is right exact. Given an exact sequence of right R-modules:

$$U^{'} \to U \to U^{''} \to 0,$$

we have to show that for any right R-module V, the sequence

$$U^{'} \otimes V \xrightarrow{\alpha\otimes 1} U \otimes V \xrightarrow{\beta\otimes 1} U^{''} \otimes V \to 0$$

is exact. Clearly $\beta\otimes 1$ is surjective and $(\beta\otimes 1)(\alpha\otimes 1) = 0$, i.e. $\operatorname{im}\alpha\otimes 1 \subset \ker\beta\otimes 1$ and it remains to show that equality holds. Since $\operatorname{im}\alpha = \ker\beta = X$, say, it is clear that $\operatorname{im}\alpha \otimes 1$ is the subgroup of $U \otimes V$ generated by all products $x \otimes v$ $(x \in X, v \in V)$. Further, each $u^{''} \in U^{''}$ can be written as $u^{''} = \beta u$ for some $u \in U$, which is unique $\operatorname{mod} X$, so we have a bilinear map $U^{''} \times V \to (U \otimes V)/(\operatorname{im}\alpha \otimes 1)$ given by $(u^{''}, v) \mapsto u \otimes v$, where $u \in U$ is such that $\beta u = u^{''}$. We thus obtain a homomorphism $f : U^{''} \otimes V \to (U \otimes V)/(\operatorname{im}\alpha \otimes 1)$ which maps $\beta u \otimes v$ to the residue class $u \otimes v$ mod $\operatorname{im}\alpha \otimes 1$, and so has the form $f(\beta \otimes 1)$ on $U \otimes V$. Hence it vanishes on $\ker\beta \otimes 1$ and so $\operatorname{im}\alpha \otimes 1 = \ker\beta \otimes 1$, as claimed. □

The following description of the relations in a general tensor product is often useful:

Proposition 3.133. *Let R be a ring and U be a right R-module generated by a family (u_λ, $\lambda \in I$ with defining relations $\sum u_\lambda a_{\lambda\mu} = 0$, $\mu \in J$. If V is a left R-module with a family (x_λ) of elements indexed by I, almost all zero, such that*

$$\sum u_\lambda \otimes k_\lambda = 0 \quad in \quad U \otimes V, \tag{3.134}$$

then there exist elements $y_\mu \in V$, almost all zero, such that

$$x_\lambda = \sum a_{\lambda\mu} y_\mu. \tag{3.135}$$

Proof. By hypothesis U has a presentation

$$0 \to L \xrightarrow{\alpha} F \xrightarrow{\beta} U \to 0,$$

where F is free on a family (f_λ), $\lambda \in I$, and I is the submodule generated by the elements $\sum f_\lambda a_{\lambda\mu}$. Tensoring with V and observing that this operation is right exact, we obtain an exact sequence

$$L \otimes V \xrightarrow{\alpha^{'}} F \otimes V \xrightarrow{\beta^{'}} U \otimes V \to 0.$$

By hypothesis, $\beta^{'}(\sum f_\lambda \otimes x_\lambda) = \sum_\lambda \otimes x_\lambda = 0$, hence by exactness, as L is generated by the elements $f_\lambda a_{\lambda\mu}$

$$\sum f_\lambda \otimes x_\lambda = \alpha^{'}(\sum f_\lambda a_{\lambda\mu} \otimes y_\mu)$$

for some elements $y_\mu \in V$, almost all zero. Now α' is the homomorphism induced by the inclusion $L \to F$ and F is free on the f_λ. Equating coefficients in $F \otimes V$, we obtain (3.135). □

Here it is important to bear in mind the hypothesis that (u_λ) is a generating set of U and $\sum u_\lambda a_{\lambda\mu} = 0$ is a family of defining relations. If (3.134) holds for some elements in $U \otimes V$ we cannot conclude that (3.135) follows; in fact this comes close to a criterion for U to be *flat* (i.e. $U \otimes -$ to be exact). We note that the result may be stated in matrix form as follows: Let U be a right R-module with presentation matrix A, relative to a generating family u (written as a row), so that $uA = 0$. If x is a column vector over V with almost all components 0, such that $u \otimes x = 0$, then there exists a column vector y over V with almost all components 0 such that $x = Ay$.

3.9.2 Homology with Arbitrary Coefficients

First, recall that abelian groups are $\mathbb{Z}$-modules. Let G be an abelian group and X a topological space. We define the homology of X with G-coefficients, denoted $H_*(X;G)$, as the homology of the chain complex

$$C_i(X;G) = C_i(X) \otimes G \tag{3.136}$$

consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ $(\sigma_i : \Delta_i \to X, \eta_i \in G)$, and with boundary maps given by

$$\partial_i^G := \partial_i \otimes \mathrm{id}_G\,.$$

Since ∂_i satisfies $\partial_i \circ \partial_{i+1} = 0$ it follows that $\partial_i^G \circ \partial_{i+1}^G = 0$, so $(C_*(X;G), \partial_i^G)$ forms indeed a chain complex. We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the natural way. Define relative chains with G-coefficients by $C_i(X,A;G) := C_i(X;G)/Ci(A;G)$, and reduced homology with G-coefficients via the augmented chain complex

$$\cdots \xrightarrow{\partial_{i+1}^G} C_i(X;G) \xrightarrow{\partial_i^G} \cdots \xrightarrow{\partial_2^G} C_1(X;G) \xrightarrow{\partial_1^G} C_0(X;G) \xrightarrow{\epsilon} G \to 0$$

where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i \in G$. Notice that $H_q(X) = H_q(X;\mathbb{Z})$ by definition.

By studying the chain complex with G-coefficients, it follows that

$$H_q(\mathrm{pt};G) = \begin{cases} G & q = 0 \\ 0 & q \neq 0. \end{cases}$$

Nothing (other than coefficients) needs to change in describing the relationships between relative homology and reduced homology of coset spaces, so we can compute the homology of a sphere as before by induction and using the long exact sequence of the pair $(\mathbb{D}^n, \mathbb{S}^n)$ to be

$$H_q(\mathbb{S}^n;G) = \begin{cases} G & q = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

We can build cellular homology with G-coefficients in the same way, defining

$$C_i^G(X) := H_i(X_i, X_{i-1}; G) \cong G^{\text{card}\, i-\text{cells}}$$

The cellular boundary maps are given by:

$$d_i^G(\sum_\alpha \eta_\alpha e_\alpha^i) = \sum_{\alpha,\beta} \eta_\alpha d_{\alpha\beta} e^{i-1}\beta,$$

where $d_{\alpha\beta}$ is as before the degree of a map $\delta_{\alpha\beta} : \mathbb{S}^{i-1} \to \mathbb{S}^{i-1}$. This follows from the easy fact that if $f : \mathbb{S}^k \to \mathbb{S}^k$ has degree m, then $f_* : H_k(\mathbb{S}^k; G) \cong G \to H_k(\mathbb{S}^k; G) \cong G$ is the multiplication by m. As it is the case for integers, we get an isomorphism

$$H_i^{CW}(X; G) \cong H_i(X; G)$$

for all i.

One of the great advantages in working with coefficients in a field F is that homology are now vector spaces. This means for instance that short exact sequences always split and this can simplify a lot of calculations.

Example 3.137. We compute $H_*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ using the cellular homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Notice that over $\mathbb{Z}$ the cellular boundary maps are $d_i = 0$ or $d_i = 2$ depending on the parity of i,

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$
$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

and therefore with $\mathbb{Z}/2\mathbb{Z}$-coefficients all of boundary maps vanish.

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0=2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0=2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0=2} \cdots \xrightarrow{0=2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0=2} \mathbb{Z}/2 \to 0.$$

Therefore,

$$H_q(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & q = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.138. Fix $n > 0$ and let $g : \mathbb{S}^n \to \mathbb{S}^n$ be a map of degree m. Define the cellular space

$$X = \mathbb{S}^n \cup_g e^{n+1},$$

where the $(n+1)$-cell e^{n+1} is attached to $\mathbb{S}^n$ via the map g. Let f be the coset map $f : X \to X/\mathbb{S}^n$. Define $Y = X/\mathbb{S}^n = \mathbb{S}^{n+1}$. The homology of X can be easily computed by using the cellular chain complex:

$$0 \xrightarrow{d_{n+2}} \mathbb{Z} \xrightarrow[m]{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Therefore,

$$H_i(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/m\mathbb{Z} & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, as $Y = \mathbb{S}^{n+1}$, we have

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that f induces the trivial homomorphisms in homology with $\mathbb{Z}$-coefficients (except in degree zero, where f_* is the identity). So it is natural to ask if f is homotopic to the constant map. As we will see below, by considering $\mathbb{Z}/m\mathbb{Z}$-coefficients we can show that this is not the case.

Let us now consider $H_*(X;\mathbb{Z}/m\mathbb{Z})$ where m is, as above, the degree of the map g. We return to the cellular chain complex level and observe that we have

$$0 \xrightarrow{d_{n+2}} \mathbb{Z}/m\mathbb{Z} \xrightarrow[m]{d_{n+1}} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d_0} 0$$

Multiplication by m is now the zero map, so we get

$$H_i(X;\mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & i = 0, n, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, as already discussesd,

$$H_i(Y;\mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

We next consider the induced homomorphism $f_* : H_{n+1}(X;\mathbb{Z}/m\mathbb{Z}) \to H_{n+1}(Y;\mathbb{Z}/m\mathbb{Z})$. The claim is that this map is injective, thus non-trivial, so f cannot be homotopic to the constant map. As noted before, we have an isomorphism $\widetilde{H}_{n+1}(Y;\mathbb{Z}/m\mathbb{Z}) \simeq H_{n+1}(X,\mathbb{S}^n;\mathbb{Z}/m\mathbb{Z})$. This leads us to consider the long exact sequence of the pair $(X,\mathbb{S}^n)$ in dimension $n+1$. We have

$$\cdots \to H_{n+1}(\mathbb{S}^n;\mathbb{Z}/m\mathbb{Z}) \to H_{n+1}(X;\mathbb{Z}/m\mathbb{Z}) \xrightarrow{f} H_{n+1}(X,\mathbb{S}^n;\mathbb{Z}/m\mathbb{Z}) \to \cdots$$

But, $H_{n+1}(\mathbb{S}^n;\mathbb{Z}/m\mathbb{Z}) = 0$ and so f_* is injective on $H_{n+1}(X;\mathbb{Z}/m\mathbb{Z})$. Since $H_{n+1}(X;\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \neq 0$ and $H_{n+1}(X,\mathbb{S}^n;Zm) \simeq \widetilde{H}_{n+1}(Y;\mathbb{Z}/m\mathbb{Z})$ it follows that f_* is not trivial on $H_{n+1}(X;\mathbb{Z}/m\mathbb{Z})$, which proves our claim.

3.9.3 The functor Tor and the Universal Coefficient Theorem

Suppose that we are given $H_*(X;\mathbb{Z})$. Can we compute $H_*(X;\mathbb{Z}/2\mathbb{Z})$? This is non-obvious. Consider the map $\mathbb{R}P^2 \to \mathbb{S}^2$ that pinches $\mathbb{R}P^1$ to a point. Now $H_2(\mathbb{R}P^2;\mathbb{Z}) = 0$, so in H_2 this map is zero. But in $\mathbb{Z}/2\mathbb{Z}$-coefficients, in dimension 2, this map gives an isomorphism. This shows that there is no *functorial* determination of $H_*(X;\mathbb{Z}/2\mathbb{Z})$ in terms of $H_*(X;\mathbb{Z})$; the effect of a map in integral homology does not determine its effect in mod2 homology. So how do we go between different coefficients?

Let R be a commutative ring and M an R-module, and suppose we have a chain complex C_* of R-modules. It could be the singular complex of a space, but it doesn't have to be. Let us compare $H_n(C_*) \otimes M$ with $H_n(C_* \otimes M)$. (Here and below we will just write $\otimes$ for $\otimes_R$.) The latter thing gives homology with coefficients in M. How can we compare these two? Let us investigate, and build up conditions on R and C_* as we go along.

First, there is a natural map

$$\alpha : H_n(C_*) \otimes M \longrightarrow H_n(C_* \otimes M),$$

sending $[z] \otimes m$ to $[z \otimes m]$. We propose to find conditions under which it is injective. The map α fits into a commutative diagramme with exact columns like this:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H_n(C_*) \otimes M & \xrightarrow{\alpha} & H_n(C_* \otimes M) \\
\uparrow & & \uparrow \\
Z_n(C_*) \otimes M & \longrightarrow & Z_n(C_* \otimes M) \\
\uparrow & & \uparrow \\
C_{n+1} \otimes M & \xrightarrow[=]{} & C_{n+1} \otimes M
\end{array}$$

Now, $Z_n(C_* \otimes M)$ is a submodule of $C_n \otimes M$, but the map $Z_n(C) \otimes M \to C_n \otimes M$ need not be injective (!) unless we impose more restrictions. If we can guarantee that it is, then a diagramme chase shows that α is a monomorphism.

So let us assume that R is a PID and that C_n is a free R-module for all n. Then the submodule $B_{n-1}(C_*) \subset C_{n-1}$ is again free, so the short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Zn(C_*) & \longrightarrow & C_n & \longrightarrow & B_{n-1}(C_*) & \longrightarrow & 0 \\
 & & & & & \searrow^{d} & \downarrow & & \\
 & & & & & & C_{n-1} & &
\end{array}$$

splits. So $Z_n(C_*) \to C_n$ is a splitting monomorphism, and hence $Zn(C_*) \otimes M \to C_n \otimes M$ is too.

In fact, a little thought shows that this argument produces a splitting of the map α.

Now, α is not always an isomorphism. But it certainly is if $M = R$, and it is compatible with direct sums, so it certainly is if M is free. The idea is now to "resolve" M by frees, and see where that idea takes us.

Following is a discussion on Tor functor required for understanding Universal Coefficient Theorem. Those readers who are familiar with the theorem can directly jump to 3.151.

Definition 3.139. A *free resolution* of an abelian group H is an exact sequence:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0,$$

with each F_n a free abelian group.

Example 3.140. The following are free resolutions of $\mathbb{Z}/5\mathbb{Z}$.

$$\cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \to \mathbb{Z}/\mathbb{Z} \to 0$$
$$\cdots \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/\mathbb{Z} \to 0$$

Theorem 3.141 (Existence of free resolutions)**.** *Every abelian group A has a free resolution*

$$\cdots \to 0 \to R \to F \to A \to 0$$

where F and R are free abelian.

Caveat: A is not supposed to be finitely generated!

Before proving Theorem 3.141 we recall

Theorem 3.142. *Every subgroup of a free abelian group is a free abelian group.*

Proof. (A sketch)

Given a real vector space V, we can put the direct limit topology on it (subsets are closed if and only if their intersection with any finite dimensional subspace is closed). This is a contractible topological group.

If A is a free abelian group, then A is a discrete subgroup of the associated real vector space $(\mathbb{R} \otimes A)$ and the coset space has fundamental group A. Any covering space is a coset of $(\mathbb{R} \otimes A)$ by a discrete subgroup B of A.

So the question boils down to showing: Any discrete subgroup of a vector space (with the direct limit topology) is free abelian.

Let us say that a *partial basis* is a set S of elements of B such that

- S is linearly independent, and
- S generates $B \cap \mathrm{Span}(S)$.

Then partial bases are a partial order under containment, and Zorn's lemma implies that there is a maximal element S. We show that S is a basis of B as a free abelian group.

S is linearly independent by construction, so it generates a free abelian group, and hence it suffices to show that it generates all B. If b in B is not in S, then it is not in $\mathrm{Span}(S)$. Let S' be $(S \cup \{b\})$. Then $\mathrm{Span}(S')/\mathrm{Span}(S)$ is a 1-dimensional vector space and the image of $B \cap \mathrm{Span}(S')$ must be discrete, because otherwise $\mathrm{Span}(S')$ would contain an element $(rb + v)$ for v in $\mathrm{Span}(S)$ that we could use to generate a non-discrete subset of B. (If v is a combination of $w_1 \cdots w_n$ in S, then it suffices to check that any subgroup of the finite-dimensional space $\mathrm{Span}(w_1 \cdots w_n, b)$ requiring more than n generators is indiscrete.)

Thus any lift of a generator of $B \cap \mathrm{Span}(S')$ would extend to a larger generating set, contradicting maximality. □

Proof. (of Theorem 3.141)
We have a surjection

$$\mathbb{Z}[A] \xrightarrow{p} A$$

given by $p(a) = a$. Let $F = \mathbb{Z}[A]$ and $R = \ker p$. Then

$$\cdots \to 0 \to R \to F \xrightarrow{p} A \to 0$$

is a free resolution of A. □

Before proceeding further, we need some definition and relevant properties of Tor.

Definition 3.143. Let A and B be abelian groups and

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to A \to 0$$

be a free resolution of A. Let $\partial_0 : F_0 \to 0$ be the 0 map so we have a chain complex (F, ∂)

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

Tensoring with B we get the chain complex $F \otimes B$

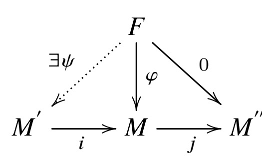
$$\cdots \xrightarrow{\partial_3 \otimes \mathrm{id}_B} F_2 \otimes B \xrightarrow{\partial_2 \otimes \mathrm{id}_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes \mathrm{id}_B} 0$$

We define $\mathrm{Tor}_n(A, B) = H_n(F \otimes B)$.

Theorem 3.144 (Tor is well-defined)**.** $\mathrm{Tor}_n(A, B)$ *is independent of the free resolution F of A.*

Proof of Theorem 3.144 requires a substantial amount of preparation.

Lemma 3.145 (Free abelian groups are projectiv)**.** *Suppose we have commutative*



with

1. *F is a free abelian group*

2. *$M' \xrightarrow{i} M \xrightarrow{j} M''$ is exact*

Then there is a homomorphism ψ making the diagramme commute.

Proof. Let $\{e_\alpha\}$ be a free basis for F. Then $j\varphi(e_\alpha) = 0(e_\alpha) = 0$ so $\varphi(e_\alpha) \in \ker j = \operatorname{im} i$. Thus there is $m'_\alpha \in M'$ such that $i(m'_\alpha) = \varphi(e_\alpha)$. Let $\psi(e_\alpha) = m'_\alpha$. □

Lemma 3.146. *If F and $F^{'}$ are two free resolutions of A. Then there is a chain map $f : F \to F^{'}$ which is a chain homotopy equivalence.*

Proof. We have

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial 1} & F_0 & \xrightarrow{\varepsilon} & A & \xrightarrow{0} & 0 \\ &&&&&& \Big\downarrow{\scriptstyle \mathrm{id}_A} && \\ \cdots & \xrightarrow[\partial_2^{'}]{} & F_1 & \xrightarrow[\partial_1^{'}]{} & F_0 & \xrightarrow[\varepsilon^{'}]{} & A & \xrightarrow[0]{} & 0 \end{array}$$

By applying Lemma 3.145 with $\varphi = \mathrm{id}_A \circ \varepsilon$ we obtain $\psi : F_0 \to F_0^{'}$ with $\varepsilon^{'}\psi = \mathrm{id}_A\, \varepsilon$. Let $f_0 = \psi$ and apply Lemma 3.145 with $\varphi = f_0\partial_1$ to get $f_1 : F_1 \to F_1^{'}$. Continuing inductively, we obtain a chain map $f : F \to F^{'}$.

Now we will show that the chain homotopy type of f is unique, i.e, if $g : F \to F_0$ is another chain map extending $\mathrm{id}_A : A \to A$ then there is a chain homotopy T between f and g. Let $\tau = g - f$ and assume inductively that $\partial_{n+1}^{'} T_n + T_{n-1}\partial_n = \tau_n$. Then we have the following (possibly non-commutative) diagramme

$$\begin{array}{ccccccc} & & F_{n+1} & \xrightarrow{\partial_{n+1}} & F_n & \xrightarrow{\partial_n} & F_{n-1} \\ & & {\scriptstyle \tau_{n+1}}\Big\downarrow & {\scriptstyle T_n}\swarrow & \Big\downarrow{\scriptstyle \tau_n} & \swarrow{\scriptstyle T_{n-1}} & \\ F_{n+2}^{'} & \xrightarrow[\partial_{n+2}^{'}]{} & F_{n+1}^{'} & \xrightarrow[\partial_{n+1}^{'}]{} & F_n^{'} & & \end{array}$$

Consider the map $(\tau_{n+1} - T_n\partial_{n+1}) : F_{n+1} \to F_{n+1}^{'}$. We have

$$\begin{aligned} \partial_{n+1}^{'}(\tau_{n+1} - T_n\partial_{n+1}) &= \partial_{n+1}^{'}\tau_{n+1} - \partial_{n+1}^{'}T_n\partial_{n+1} \\ &= \partial_{n+1}^{'}\tau_{n+1} - (\tau_n - T_{n-1}\partial_n)\partial_{n+1} \\ &= \partial_{n+1}^{'}\tau_{n+1} - \tau_n\partial_{n+1} + T_{n-1}\partial_n\partial_{n+1} \\ &= 0. \end{aligned}$$

Applying Lemma 3.145 with $\varphi = \tau_{n+1} - T_n\partial_{n+1}$, let $T_{n+1} = \psi : F_{n+1} \to F_{n+2}^{'}$. Then $\partial_{n+2}^{'}T_{n+1} = \varphi = \tau_{n+1} - T_n\partial_{n+1}$. Hence

$$\partial_{n+2}^{'}T_{n+1} + T_n\partial_{n+1} = \tau_{n+1} = g_{n+1} - f_{n+1}.$$

We may start the induction in degree -2 where all groups are 0. Thus the chain homotopy class of $f : F \to F$ is unique. Similarly we get $f^{'} : F^{'} \to F$. $\mathrm{id}_F : F \to F$ extends $\mathrm{id}_A : A \to A$, so $f^{'} \circ f$ is chain homotopic to id_F. That is, f is a chain homotopy equivalence. □

Proof. (of Theorem 3.144) Let F and $F^{'}$ be two free resolutions of A. By Lemma 3.146 we have a chain homotopy equivalence

$$f : F \to F^{'}$$

Hence there is a chain map $f^{'} : F^{'} \longrightarrow F$ such that $f^{'} \circ f$ is chain homotopic to id_F. Tensoring with B we get $(f^{'} \otimes \mathrm{id}_B) \circ (f \circ \mathrm{id}_B)$ is chain homotopic to $\mathrm{id}_F \otimes \mathrm{id}_B$. Thus

$$(f \otimes \mathrm{id}_B)_* : H_n(F \otimes B) \longrightarrow H_n(F^{'} \otimes B)$$

is an isomorphism, which means $\mathrm{Tor}_n(A, B) = H_n(F \otimes B) \cong H_n(F^{'} \otimes B)$ is well-defined. □

As we saw in Theorem 3.141 every abelian group A has a free resolution

$$\cdots \longrightarrow 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

So we have the chain complexs F

$$\cdots \longrightarrow 0 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

and $F \otimes B$

$$\cdots \longrightarrow 0 \xrightarrow{\partial_2 \otimes \mathrm{id}_B} F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} F_0 \otimes B \xrightarrow{\partial_0 \otimes \mathrm{id}_B} 0$$

$$\mathrm{Tor}_n(A, B) = H_n(F \otimes B) = \begin{cases} \frac{F_0 \otimes B}{\mathrm{im}(\partial_1 \otimes \mathrm{id}_B)}, & n = 0 \\ \ker(\partial_1 \otimes \mathrm{id}_B), & n = 1 \\ 0, & n \neq 0, 1 \end{cases}$$

We can say more about $\mathrm{Tor}_0(A, , B)$.

We have the exact sequence

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

Which remains exact after tensoring with B so we get exact

$$F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathrm{id}_B} A \otimes B \longrightarrow 0$$

Hence

$$\mathrm{Tor}_0(A, B) \cong \frac{F_0 \otimes B}{\mathrm{im}(\partial_1 \otimes \mathrm{id}_B)} \cong A \otimes B.$$

Since $\mathrm{Tor}_1(A, B)$ is the only (possibly) new object we define

Definition 3.147. $\mathrm{Tor}(A, B) = \mathrm{Tor}_1(A, B)$.

Remark 3.148. Note that if we have an exact sequence

$$0 \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

Then we have an(other) exact sequence

$$F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathrm{id}_B} A \otimes B \longrightarrow 0$$

and hence still another exact one

$$0 \longrightarrow \ker(\partial_1 \otimes \mathrm{id}_B) \longrightarrow F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathrm{id}_B} A \otimes B \longrightarrow 0$$

and hence an exact sequece

$$0 \to \mathrm{Tor}(A,B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} F_0 \otimes B \xrightarrow{\varepsilon \otimes \mathrm{id}_B} A \otimes B \to 0$$

In particular if $\mathrm{Tor}(A,B) = 0$ then tensoring with B preserves exactness.

Example 3.149. Let us compute $\mathrm{Tor}(\mathbb{Z}/60\mathbb{Z}, \mathbb{Z}/42\mathbb{Z})$. First, consider a free resolution F of $\mathbb{Z}/60\mathbb{Z}$

$$\cdots \to 0 to \mathbb{Z} \xrightarrow{\times 60} \mathbb{Z} \to \mathbb{Z}/60\mathbb{Z} \to 0.$$

Then $F \otimes \mathbb{Z}/42\mathbb{Z}$ is

$$0 \to \mathbb{Z} \otimes (\mathbb{Z}/42\mathbb{Z}) \xrightarrow{(\times 60)\otimes 1} \mathbb{Z} \otimes (\mathbb{Z}/42\mathbb{Z}) \to 0$$

Simplifying, we have

$$0 \to \mathbb{Z}/42\mathbb{Z} \xrightarrow{(\times 60)} \mathbb{Z}/42\mathbb{Z} \to 0$$

Hence

$$\mathrm{Tor}(\mathbb{Z}/60\mathbb{Z}, \mathbb{Z}/42\mathbb{Z}) \cong \ker(\times 60) \cong 7\mathbb{Z}/42\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z} \cong \frac{\mathbb{Z}}{\gcd(42,60)\mathbb{Z}}$$

Proposition 3.150 (Properties of Tor). *(1)* $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.

(2) $\mathrm{Tor}(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B)$.

(3) $\mathrm{Tor}(A,B) = 0$ *if* A *or* B *is free or torsion free.*

(4) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(A_{\mathrm{tor}}, B)$ *where* A_{tor} *is the torsion subgroup of* A.

(5) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{\times n} A)$

(6) The short exact sequence

$$0 \to B \to C \to D \to 0$$

yields a natural exact sequence

$$0 \to \mathrm{Tor}(A,B) \to \mathrm{Tor}(A,C) \to \mathrm{Tor}(A,D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

Proof. (2): $\mathrm{Tor}(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B)$.
Let F_α be a free resolution of A_α. Then $\oplus_\alpha F_\alpha$ is a free resolution of A_α. Thus

$$\begin{aligned} \mathrm{Tor}(\oplus_\alpha A_\alpha, B) &\cong H_1((\oplus_\alpha F_\alpha) \otimes B) \\ &\cong H_1(\oplus_\alpha (F_\alpha \otimes B)) \\ &\cong \oplus_\alpha H_1(F_\alpha \otimes B) \\ &\cong \oplus_\alpha \mathrm{Tor}(A_\alpha, B). \end{aligned}$$

(5): $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{\times n} A)$.
Using the free resolution of $\mathbb{Z}/n\mathbb{Z}$,

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

Tensoring with A simplifies to

$$\cdots \to 0 \to A \xrightarrow{\times n} A \to 0$$

The result follows.

(3): $\mathrm{Tor}(A, B) = 0$ if A or B is free (we will address torsion free later.)
Suppose A is free. Use the free resolution of A

$$\cdots \to 0 \to 0 \to A \to A \to 0$$

we have $\mathrm{Tor}(A, B) = \ker(0 \to A \otimes B) = 0$.
Suppose $B = \mathbb{Z}$. Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with B remains exact.
Suppose $B \cong \oplus_\alpha \mathbb{Z}$. Then tensoring an exact free resolution $0 \to F_1 \to F_0 \to A \to 0$ with B is a direct sum of exact sequences which is exact.

(6): The short exact sequence

$$0 \to B \to C \to D \to 0$$

yields a natural exact sequence

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0.$$

Choose a free resolution F of the form $0 \to F_1 \to F_0 \to A \to 0$. All the terms of F are free so tensoring $0 \to B \to C \to D \to 0$ with F_n remains exact. So we get a short exact sequence of chain complexes

$$0 \to (F \otimes B) \to (F \otimes C) \to (F \otimes D) \to 0$$

Apply Snake Lemma (see the proof for Theorem 3.23) to get the natural exact sequence above.

(1): $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$. Consider the six term exact sequence from part (6) coming from the short exact sequence:

$$0 \to F_1 \to F_0 \to B \to 0$$

$$0 \to \mathrm{Tor}(A, F_1) \to \mathrm{Tor}(A, F_0) \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0.$$

F_0 and F_1 are free so by part (3) $\mathrm{Tor}(A, F_1) \cong \mathrm{Tor}(A, F_0) \cong 0$. So we have

$$0 \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0.$$

Together,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}(A,B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\ & & & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathrm{Tor}(B,A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0 \end{array}$$

We will define a homorphism $\gamma : \mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$ preserving commutativity:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}(A,B) & \xrightarrow{\alpha} & A \otimes F_1 & \xrightarrow{\beta} & A \otimes F_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\ & & \vdots\gamma & & \tau\downarrow\cong & & \mu\downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathrm{Tor}(B,A) & \xrightarrow[\alpha']{} & F_1 \otimes A & \xrightarrow[\beta']{} & F_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0 \end{array}$$

Let $x \in \mathrm{Tor}(A, B)$ We claim that $\tau\alpha(x) \in \mathrm{im}\,\alpha'$.

By commutativity $\beta'\tau\alpha(x) = \mu\beta\alpha(x) = \mu(0) = 0$, so we have $\tau\alpha(x) \in \ker\beta' = \mathrm{im}\,\alpha'$. By injectivity of α' there is a unique $x' \in \mathrm{Tor}(B, A)$ with $\alpha'(x_0) = \tau\alpha(x)$. Set $\gamma(x) = x'$. As γ takes 0 to 0 and sums to sums so it is a homomorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(A,B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 \\ \cong\downarrow & & \downarrow\cong & & \downarrow\gamma & & \tau\downarrow\cong & & \mu\downarrow\cong \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(B,A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A \end{array}$$

Add some trivial groups and homomorphisms to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(A,B) & \longrightarrow & A \otimes F_1 & \longrightarrow & A \otimes F_0 \\ \cong\downarrow & & \downarrow\cong & & \downarrow\gamma & & \tau\downarrow\cong & & \mu\downarrow\cong \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(B,A) & \longrightarrow & F_1 \otimes A & \longrightarrow & F_0 \otimes A \end{array}$$

Now apply Five Lemma 3.27 to show $\tau : \mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$ is an isomorphism.

(3): $\mathrm{Tor}(A, B) = 0$ if A or B is torsion free.
Assume B is torsion free applying part 1. Let

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \to A$$

be a free resolution of A. Then we get an exact sequence

$$0 \to \mathrm{Tor}(A, B) \to F_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} F_0 \otimes B$$

We claim $\partial_1 \otimes \mathrm{id}_B$ is injective: Suppose $\sum_i f_i \otimes b_i \in \ker\partial_1 \otimes \mathrm{id}_B$. Then $\sum_i(\partial_1 f_i) \otimes b_i = 0$ in $F_0 \otimes B$. Hence in $\mathbb{Z}[F_0 \times B]$ we have

$$\sum_i(\partial_1 f_i, b_i) = \sum_j(f_j^0, b_j^1+b_j^2)-(f_j^0, b_j^1)-(f_j^0, b_j^2)+\sum_k(f_k^1+f_k^2, b_k^0)-(f_k^1, b_k^0)-(f_k^2, b_k^2)$$

Let $B_0 \subset B$ be the subgroup generated by the finite set $\{b_i, b_k^0, b_j^1, b_j^2\}$. Then in $\mathbb{Z}[F_0 \times B]$ we have

$$\sum_i (\partial_1 f_i, b_i) = \sum_j (f_j^0, b_j^1 + b_j^2) - (f_j^0, b_j^1) - (f_j^0, b_j^2) + \sum_k (f_k^1 + f_k^2, b_k^0) - (f_k^1, b_k^0) - (f_k^2, b_k^2)$$

Therefore in $F_0 \otimes B_0$

$$\sum_i (\partial_1 f_i) \otimes b_i = 0$$

Let $B_0 \subset B$ is torsion free and finitely generated so free abelian. Then

$$F_1 \otimes B_0 \xrightarrow{\partial_1 \otimes \mathrm{id}_{B_0}} F_0 \otimes B_0$$

is injective. Thus in $F_1 \otimes B_0$

$$\sum_i f_i \times b_i = 0.$$

Hence in $\mathbb{Z}[F_1 \times B0]$ we have

$$\sum_i (f_i, b_i) = \sum_n (f_n^3, b_n^4 + b_n^5) - (f_n^3, b_n^4) - (f_n^3, b_n^5) + \sum_k (f_m^4 + f_k^5, b_m^3) - (f_m^4, b_m^3) - (f_m^5, b_m^3)$$

This equality holds in $\mathbb{Z}[F_1 \times B]$, so in $F_1 \otimes B$

$$\sum_i f_i \times b_i = 0.$$

It follows that $\mathrm{Tor}(A, B) = \ker(\partial_1 \otimes \mathrm{id}_B) = 0$. □

Having prepared to handle Tor, let us return to the discussion of Universal Coefficient Theorem. Let

$$0 \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M. Again, we are using the assumption that R is a PID, to guarantee that $\ker(F_0 \to M)$ is free. Again using the assumption that each C_n is free, we get a short exact sequence of chain complexes

$$0 \to C_* \otimes F_1 \to C_* \otimes F_0 \to C_* \otimes M \to 0.$$

In homology, this gives a long exact sequence. Unsplicing it gives the left-hand column in the following diagramme.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\operatorname{coker}(H_n(C_*\otimes F_1)\to H_n(C_*\otimes F_0)) & \xrightarrow{\cong} & \operatorname{coker}(H_n(C_*)\otimes F_1\to H_n(C_*)\otimes F_0) \\
\downarrow & & \downarrow \\
H_n(C_*\otimes M) & \xrightarrow{=} & H_n(C_*\otimes M) \\
\downarrow{\scriptstyle\partial} & & \downarrow \\
\ker(H_{n-1}(C_*\otimes F_1)\to H_{n-1}(C_*\otimes F_0)) & \xrightarrow{\cong} & \ker(H_{n-1}(C_*)\otimes F_1\to H_{n-1}(C_*)\otimes F_0) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The right hand column occurs because α is an isomorphism when the module involved is free. But

$$\operatorname{coker}(H_n(C_*)\otimes F_1\to H_n(C_*)\otimes F_0)=H_n(C_*)\otimes M$$

and

$$\ker(H_{n-1}(C_*)\otimes F_1\to H_{n-1}(C_*\otimes F_0)=\operatorname{Tor}_1^R(H_{n-1}(C_*,M).$$

We have proved the following theorem.

Theorem 3.151 (Universal Coefficient Theorem)**.** *Let R be a PID and C_* a chain complex of R-modules such that C_n is free for all n. Then there is a natural short exact sequence of R-modules*

$$0\to H_n(C_*)\otimes M\xrightarrow{\alpha}H_n(C_*\otimes M)\xrightarrow{\partial}\operatorname{Tor}_1^R(H_{n-1}(C_*),M)\to 0$$

that splits (but not naturally).

Example 3.152. The pinch map $\mathbb{R}P^2\to\mathbb{S}^2$ induces the following map of universal coefficient short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\mathbb{R}P^2)\otimes(\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H_2(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\cong} & \operatorname{Tor}_1(H_1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & H_2(\mathbb{S}^2)\otimes(\mathbb{Z}/2\mathbb{Z}) & \xrightarrow[\cong]{} & H_2(\mathbb{S}^2;\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & Tor_1(H_1(\mathbb{S}^2);\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

This shows that the splitting of the universal coefficient short exact sequence cannot be made natural, and it explains the mystery that we began with.

Lecture 4

Singular Cohomology

4.1 Cohomology

4.1.1 Dual moodules

Let R be a commutative ring and let M be an R-module. The *dual module*

$$M^\vee := \mathrm{Hom}_R(M, R)$$

is the set of R-module homomorphisms from M to R. $M^\vee$ is an R module under addition and scalar multiplication of functions. There is a natural isomorphism

$$(\oplus_i M_i)^\vee \cong \prod_i M_i^\vee,$$

defined by the rule $(\phi_1, \phi_2 \ldots,)(m_1 + m_2 + \cdots) = \sum_i \phi_i(m_i)$. In particular, for free modules we have

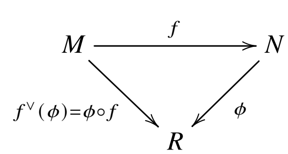
$$(\oplus_i R)^\vee = \prod_i R^\vee = \prod_i R. \tag{4.1}$$

More directly, (4.1) holds because a homomorphism out of a free module is specified by listing where the free generators are sent.

Given an R-module homomorphism $f : M \to N$, define the transpose

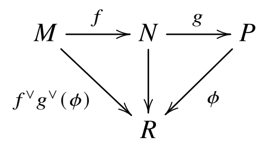
$$f^\vee : N^\vee \to M^\vee$$

which sends $\phi \in N^\vee = \mathrm{Hom}_R(N, R)$ to $f^\vee(\phi) = \phi \circ f$.



Dualisation is a *contravariant functor* from R-modules to R-modules. This means that $\mathrm{id}_M^\vee = \mathrm{id}_{M^\vee}$ and $(g \circ f)^\vee = f^\vee \circ g^\vee$. The first is obvious and the second follows from associativity of composition:

$$(g \circ f)^\vee(\phi) = \phi \circ (g \circ f) = (\phi \circ g) \circ f = f^\vee(g^\vee(\phi)) = (f^\vee \circ g^\vee)(\phi)$$



4.1.2 Cohomology

Given a chain complex of R-modules

$$C_\bullet : \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

we form the dual chain complex (or *cochain* complex)

$$C^\bullet : \cdots \leftarrow C^{n+1} \xleftarrow{\delta_{n+1}} C^n \xleftarrow{\delta_n} C^{n-1} \xleftarrow{\delta_{n-1}} \cdots$$

where $C^n = C_n^\vee$ and $\delta_n = \partial_n^\vee$. Thus cochain complexes are really huge sets.

Note that

$$\delta_n \circ \delta_{n+1} = \partial_n^\vee \circ \partial_{n+1}^\vee = (\partial_{n+1} \circ \partial_n)^\vee = 0^\vee = 0.$$

We can now define *cocycles* $Z^n := \ker(\delta_{n+1})$, *coboundaries* $B^n := \mathrm{im}(\delta_n)$ and *cohomology*

$$H^n := Z^n / B^n.$$

Let us calculate several examples. Take the coefficient ring $R = \mathbb{Z}$.

Example 4.2. Let $C_\bullet$ be the chain complex $\cdots \to 0 \to \mathbb{Z} \to 0 \to \cdots$ where $\mathbb{Z}$ is at the k-th position. Then the homology groups are

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k. \end{cases}$$

The corresponding cochain complex $C^\bullet$ is $\cdots \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots$ where $\mathbb{Z}$ is still at the k-th position. Then

$$H_n(C^\bullet) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k. \end{cases}$$

At first glance cohomology seems completely dual to homology, and therefore seemingly redundant. But in fact the situation is more subtle.

Example 4.3. Let $C_\bullet$ be the chain complex $\cdots \to 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0 \to \cdots$ where $\mathbb{Z}$ are at the k-th and $(k-1)$-th positions. Then the homology groups are

$$H_n(C^\bullet) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & n = k - 1 \\ 0 & n \neq k. \end{cases}$$

The corresponding cochain complex $C^\bullet$ is $\cdots \leftarrow 0 \leftarrow \mathbb{Z} \xleftarrow{m} \mathbb{Z} \leftarrow 0 \leftarrow \cdots$ where $\mathbb{Z}$ are still at the k-th and $(k-1)$-th positions. But

$$H^n(C^\bullet) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & n = k \\ 0 & n \neq k-1. \end{cases}$$

These examples show the difference of the free part $\mathbb{Z}$ and torsion part $\mathbb{Z}/m\mathbb{Z}$. Actually suppose C_n is free abelian for all n and $H_n(C)$ is finitely generated for all n. And suppose $H_n(C) = F_n \oplus T_n$ where F_n is free abelian and T_n is torsion. Then $H^n(C) = F_n \oplus T_{n-1}$. This is the simplest form of Universal coefficient theorem which determines cohomology groups with arbitrary coefficients from homology with $\mathbb{Z}$ coefficients.

Remark 4.4. For non-abelian group G, we could still define (co)homology, but the point is that usually $H^n(C;G)$ do not have a group structure when $n > 0$, since $\operatorname{im}\delta$ need not be a normal subgroup of $\ker\delta$.

4.1.3 Universal Coefficient Theorem

Chain map and chain homotopy

Let us recall

Definition 4.5. A map $f : C \to D$ of chain complexes C and D is a sequence of homomorphisms $f = \{f_n : C_n \to D_n\}$ such that $\partial_n \circ f_n = f_{n-1} \circ \partial_n$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ & & {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & D_n & \xrightarrow[\partial_n]{} & D_{n-1} & \longrightarrow & \cdots \end{array}$$

Proposition 4.6. *A chain map $f : C \to D$ induces a homomorphism $H(f) : H_n(C) \to H_n(D)$ of homology groups.*

Proof. By definition, f_n takes cycles in $Z_n(C)$ to cycles in $Z_n(D)$ and takes boundaries to boundaries. Hence it induces a homomorphism $H(f) : H_n(C) \to H_n(D)$. □

Definition 4.7. Two maps of chain complexes $f, g : C \to D$ are *chain homotopic* (denoted by $f \simeq g$) if there exists a sequence of maps $T_\bullet = \{T_n : C_n \to D_{n+1}\}$ such that

$$\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n = f_n - g_n.$$

Note that $T_\bullet$ *is not a chain map in any sense.* The definition just tells that $\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n$ is a chain map, which is equal to $f_n - g_n$.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \Downarrow & {\scriptstyle T_n}\swarrow & {\scriptstyle f_n}\Downarrow{\scriptstyle g_n} & \swarrow{\scriptstyle T_{n-1}} & \Downarrow & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow[\partial_{n+1}]{} & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots \end{array}$$

We will drop the subscripts for boundary maps later on if no confusion is likely.

Proposition 4.8. *If f and g are chain homotopic, then their induced homomorphisms of homology are equal.*

Proof. Let T be a chain homotopy. For any $z_n \in Z_n(C)$, we have

$$H(f)[z_n] - H(g)[z_n] = [f_n(z_n) - g_n(z_n)] = [\partial \circ T(z_n) + T \circ \partial(z_n)] = [\partial T(z_n)] = 0.$$

□

So chain homotopy is an equivalence relation on chain complexes.

Definition 4.9. Two chain complexes C and D are called *chain homotopy equivalent* ($C \simeq D$), if there are chain maps $f : C \to D$ and $g : D \to C$ such that

$$g \circ f \simeq \mathrm{id}_C : C \to C, \quad f \circ g \simeq \mathrm{id}_D : D \to D.$$

Each of them is called a *chain homotopy equivalence*.

The next result follows from Proposition 4.8.

Proposition 4.10. *Every chain homotopy equivalence induces an isomorphism of homology groups. So if $C \simeq D$, then $H_\bullet(C) = H_\bullet(D)$.*

The converse also holds if C and D are complexes of abelian groups.
All the above discussion works for cochain complexes *mutatis mutandis*.

Hom **functor**

Notice that the functor $\mathrm{Hom}(-,G)$ is the key for cohomology. By definition, $\mathrm{Hom}(H,G)$ is the set of all homomorphisms from H to G. It is an abelian group as well, and called *homomorphism group*. Let us have a look in a more abstract viewpoint. It is a contravariant functor, which means $f : A \to B$ induces $f^* : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$ and if furthermore we have $g : B \to C$ then $(g \circ f)^* = f^+ \circ g^*$. The above discussion tells us that $\mathrm{Hom}(-,G)$ is also a contravariant functor from Chain complexes to cochain complexes.

The functor $\mathrm{Hom}(-,G)$ has the following properties:

(a) $\mathrm{Hom}(\oplus_i A_i, G) = \prod_i \mathrm{Hom}(A_i, G)$;

(b) Left exactness: If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence, then the induced sequence

$$\mathrm{Hom}(A,G) \xleftarrow{f^*} \mathrm{Hom}(B,G) \xleftarrow{g^*} \mathrm{Hom}(C,G) \leftarrow 0$$

is exact. In other words,

$$0 \to \mathrm{Hom}(C,G) \xrightarrow{g^*} \mathrm{Hom}(B,G) \xrightarrow{f^*} \mathrm{Hom}(A,G) \tag{4.11}$$

is exact.

We show the property (b).

Proof. We must show that $\text{Hom}(C,G) \to \text{Hom}(B,G)$ is injective and that the image of $\text{Hom}(C,G) \to \text{Hom}(B,G)$ is the kernel of the map $\text{Hom}(B,G) \to \text{Hom}(A,G)$.

Let $\phi \in \text{Hom}(C,G)$ such that $g^*(\phi) = 0$. Then $\phi \circ g(b) = 0$ for all $b \in B$. However, g is surjective so $\phi(c) = 0$ for all $c \in C$, and $\phi = 0$. Thus g^* is injective.

Since $g \circ f$ is zero $f^* \circ g^*$ is also zero, and $\text{im}(g^*) \subset \ker(f^*)$. To show the other inclusion, let $\psi \in \text{Hom}(B,G)$ such that $f^*\psi = 0$. Then $\psi(f(a)) = 0$ for all $a \in A$, and we can define a homomorphism $\overline{\psi} : C \to G$ by $\overline{\psi}(c) = \psi(b)$ for some $b \in B$ with $g(b) = c$. This is well defined since for two elements b and b' such that $g(b) = g(b')$ there is an $a \in A$ with $b = b' + a$. Then $\psi(b) = \psi(b' + f(a)) = (b')$, thus $\ker(f^*) \subset \text{im}(g^*)$. Now we have shown $\text{im}(g^*) = \ker(f^*)$, i.e., the sequence 4.11 is exact at $\text{Hom}(B,G)$. □

Example 4.12. $\text{Hom}(\mathbb{Z}/m\mathbb{Z}, G) = \{g \in G | mg = 0\}$. In fact, take Hom for the exact sequence $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$, we obtain

$$0 \leftarrow G \xleftarrow{m} G \leftarrow \text{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \leftarrow 0.$$

By (b), the sequence is exact *except possibly at the leftmost term*. So $\text{Hom}(\mathbb{Z}/m\mathbb{Z}, G) = \ker(G \xrightarrow{m} G) = \{g \in G | mg = 0\}$. And when $mG \neq G$, the sequence is not exact.

Thus the functor $\text{Hom}(-, G)$ is *left exact*, but not exact. (Functors that preserve exact sequences are called *exact functors*.)

To state universal coefficient theorem, we need to introduce the functor Ext, which measures the failure of Hom to be an exact functor. It is defined from a *free resolution* of the abelian group.

First, recall Definition 3.139.

Definition 4.13. A free resolution of an abelian group B is a chain complex

$$\cdots \to F_2 \to F_1 \to F_0 \to 0$$

of free groups with a map $F_0 \to B$ such that

$$\cdots \to F_2 \to F_1 \to F_0 \to B \to 0$$

is exact.

Remark 4.14. If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of free abelian groups then

$$0 \to \text{Hom}(A, F'') \to \text{Hom}(A, F) \to \text{Hom}(A, F') \to 0$$

is also a short exact sequence for any abelian group A. So on free abelian groups $\text{Hom}(-, A)$ is an exact functor.

A free resolution is a way of replacing a (possibly) very complicated abelian group B with much simpler groups where $\mathrm{Hom}(-, A)$ is an exact functor.

For a chain complex C, let $H^n(C; A)$ be the cohomology of the chain complex

$$\cdots \to \mathrm{Hom}(C_{i-1}, A) \to \mathrm{Hom}(C_i; A) \to \mathrm{Hom}(C_{i+1}; A) \to \cdots$$

Let us recall Lemma 3.146..

Lemma 4.15. *Given free resolutions F of B and $F^{'}$ of $B^{'}$, a homomorphism $\alpha : B \to B^{'}$ can be extended to a chain map from F to $F^{'}$. This chain map is unique up to chain homotopy.*

$\mathrm{Ext}(H; G)$, which measures the failure of Hom to be an exact functor is defined from a free resolution of the abelian group $H : 0 \to F_1 \to F_0 \to H \to 0$. Recall that we can assume $F_i = 0$ for $i > 1$. This could be obtained in the following way. Choose a set of generators for H and let F_0 be a free abelian group with basis in one-to-one correspondence with these generators. Then we have a surjective homomorphism $f_0 : F_0 \to H$. The kernel of f_0 is free as a subgroup of a free abelian group. We let F_1 be the kernel and the inclusion to F_0 as f_1. It is an exact chain complex sequence. In summary, constructing a free resolution is equivalent to choosing a presentation for A.

Take its dual cochain complex by $\mathrm{Hom}(F; G)$, which may no longer exact, so could have its cohomology group, temporarily denoted by $H^n(F; G)$. For the above constructed resolution, $H^n(F; G) = 0$ for $n > 1$. So the only interesting group is $H^1(F; G)$. As we will show, it is independent of the resolution. There is a standard notation for that: $\mathrm{Ext}(H; G)$. The element in this group could also be interpreted as the isomorphism class of extensions of G by H, i.e. $0 \to G \to J \to H \to 0$.

Now, we are ready to state the universal coefficient theorem.

Theorem 4.16. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then for each n, there is a natural short exact sequence:*

$$0 \to \mathrm{Ext}(H_{n-1}(C); G) \to H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C); G) \to 0.$$

The sequence splits, so we have

$$H^n(C; G) = \mathrm{Ext}(H_{n-1}(C); G) \oplus \mathrm{Hom}(H_n(C); G).$$

But the splitting is not natural.

Proof of universal coefficient theorem, step 1

There is a natural choice of free resolution of the homology group

$$0 \to B_n(C) \xrightarrow{i_n} Z_n(C) \xrightarrow{q} H_n(C) \to 0.$$

So the $H^1(F; G)$ for this free resolution is exactly $\mathrm{coker}(i_n^* : \mathrm{Hom}(Z_n; G) \to \mathrm{Hom}(B_n; G))$ by definition. Now, let us prove universal coefficient theorem in two steps.

Step 1 Derive the split short exact sequence

$$0 \to \operatorname{coker} i_{n-1}^* \to H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C);G) \to 0.$$

Step 2 Prove $H_1(F;G)$ depends only on H and G, but not the resolution. So $\operatorname{coker} i_{n-1} = \operatorname{Ext}(H_{n-1};G)$.

We start with Step 1.

Lemma 4.17. *There is a natural homomorphism*

$$h : H^n(C;G) \to \operatorname{Hom}(H_n(C);G).$$

Proof. We first have the map in the cycle level.

For any cocycle $\alpha \in Z^n$ and any cycle $z \in 2Z_n$, we let $h(\alpha)(z) = \alpha(z)$. $\alpha \in Z^n$ means $\delta\alpha = 0$, i.e $\alpha\partial = 0$. In other words, α vanishes on B_n. So h descends to a map from Z^n to $\operatorname{Hom}(H_n;G)$.

Next if $\alpha \in B^n$, then $\alpha = \delta\beta = \beta\partial$. Hence α is zero on Z_n. Thus there is a well defined coset map $h([\alpha])([z]) = \alpha(z)$ from $H^n(C;G)$ to $\operatorname{Hom}(H_n(C);G)$. This is a homomorphism since:

$$h([\alpha+\beta])([z]) = (\alpha+\beta)(z) = \alpha(z) + \beta(z) = h[\alpha]([z]) + h[\beta]([z]).$$

□

Now there is a split short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0.$$

It splits since B_{n-1} is free (for any generator of B_{n-1}, one could map it to a preimage of ∂. So it is not canonically chosen.). Thus we have $p : C_n \to Z_n$ whose restriction to Z_n is the identity.

We have the commutative diagramme:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & B_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow[\partial]{} & B_{n-1} & \longrightarrow & 0
\end{array}$$

Since the dual of a split short exact sequence is a split short exact sequence (the splitting exactness of Hom), the following commutative diagramme has exact rows:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & Z^n & \longleftarrow & C^n & \xleftarrow{\delta} & B^{n-1} & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle 0} & & \\
0 & \longleftarrow & Z^{n+1} & \longleftarrow & C^{n+1} & \xleftarrow[\delta]{} & B^n & \longleftarrow & 0
\end{array}$$

This is a part of a short exact sequence of chain complexes. From this, we have a long exact sequence (because the differential of the complexes B and Z are trivial)

$$B^n \overset{i_n^*}{\longleftarrow} Z^n \leftarrow H^n(C;G) \leftarrow B^{n-1} \overset{i_{n-1}^*}{\longleftarrow} Z^{n-1}.$$

The connecting homomorphism is i_n^* by definition: one takes an element of Z^n, pulls back to C^n, applies δ to get an element in C^{n+1}, then pulls back to B^n. That is, we first extend a homomorphism $f : Z_n \to G$ to $f^{'} : C_n \to G$, then composes it with ∂, finally view it as a map from B_n. So it is nothing but the restriction of f from Z_n to B_n.

Or we could also see from its dual operation: given $b \in B_n$, so $b = \partial c$, then the first step maps it to c, second takes ∂, thus gets b back which is in Z_n. The composition is the inclusion i_n.

Hence we have

$$0 \leftarrow \ker i_n^* \leftarrow H_n(C;G) \leftarrow \operatorname{coker} i_{n-1}^* \leftarrow 0.$$

The final step for Step 1 is

Lemma 4.18. $\ker(i_n^*) = \operatorname{Hom}(H_n(C), G)$.

Proof. Since the elements of $\ker(in^*)$ are homomorphisms $Z_n \to G$ that vanish on B_n, that is they are homomorphisms $H_n = Z_n/B_n \to G$. □

Under this identification, the natural map h is the map $0 \leftarrow \ker i_n^* \leftarrow H^n(C;G)$. And the short exact sequence splits because of the induced map p^*.

4.1.4 Ext **functor**

Recall that to complete Step 2, we only need to prove that for any two (2-step) free resolutions of abelian group H, the homology groups are isomorphic. Then the notation $\operatorname{Ext}(H;G)$ is well defined.

We have the following.

Lemma 4.19. *Suppose given free resolutions F and $F^{'}$ of abelian groups H and $H^{'}$. Then every homomorphism $\alpha : H \to H^{'}$ could be extended to a chain map from F to $F^{'}$:*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ \cdots & \longrightarrow & F_2^{'} & \xrightarrow[f_2^{'}]{} & F_1^{'} & \xrightarrow[f_1^{'}]{} & F_0^{'} & \xrightarrow[f_0^{'}]{} & H^{'} & \longrightarrow & 0 \end{array}$$

Furthermore, any two such chain maps extending α are chain homotopic.

Proof. Since F_i's are free, it suffices to define α_i on a basis of F_i. Given $x \in F_0$, $\alpha(f_0(x)) \in H_0$. Since $f_0^{'}$ is surjective, we have $x_0^{'} \in F_0^{'}$ such that $f_0^{'}(x^{'}) = \alpha(f_0(x))$. We define $\alpha_0(x) = x^{'}$.

Let us define α_1. For $x \in F_1$, $\alpha_0(f_1(x))$ lies in $\operatorname{im} f_1^{'} = \ker f_0^{'}$ since $f_0^{'}\alpha_0 f_1 = \alpha f_0 f_1 = 0$. So define $\alpha_1(x) = x^{'}$ such that $\alpha_0(f_1(x)) = f_1^{'}(x^{'})$. Other α_i could be constructed inductively in a similar way.

To check any such chain maps are chain homotopic, we will only give a proof for the case of 2-step free resolutions, i.e when $F_i = F^{'} = 0$ for $n > 1$. This is the case we need since we are dealing with abelian groups.

If β_i is another extension of α, then we want to find a chain homotopy between $\{\alpha_i\}$ and $\{\beta_i\}$, that is, maps $T_0 : F_0 \to F1^{'}$ and $T_{-1} : H \to F_0^{'}$ such that $\alpha_i - \beta_i = f_{i+1}^{'}T_i + T_{i-1}f_i$ for $i = 0, 1$. We let $T_{-1} = 0$. We let $T_0(x) = x^{'}$ such that $f_1^{'}(x^{'}) = \alpha_0(x) - \beta_0(x)$. This can be done because $f_0^{'}\beta_0(x) = f_0^{'}\alpha_0(x) = \alpha f_0(x)$ and $\operatorname{im} f_1^{'} = \ker f_0^{'}$. Hence $\alpha_0 - \beta_0 = f_1^{'}T_0$.

To check $\alpha_1 - \beta_1 = T_0 f_1$, we only need to check the relation after composing $f_1^{'}$ which is injective. It is nothing but $f_1^{'}(\alpha_1 - \beta_1) = (\alpha_0 - \beta_0)f_1$. □

Corollary 4.20. *For any two free resolutions F and $F^{'}$ of H, $H^n(F;G) = H^n(F^{'};G)$.*

Proof. It follows from above lemma and (cohomology version of) Proposition 4.10 by taking $\alpha = \mathrm{id} : H \to H$ and by looking at the composition of two chain maps, one from F to $F^{'}$ and the other from $F^{'}$ to F. □

Hence we finished Step 2 and thus the proof of Theorem 4.16 is complete.

Now the calculation of cohomology groups is reduced to that of Ext. So we will list the properties of Ext.

Proposition 4.21. Ext *has the following computational properties:*

1. $\mathrm{Ext}(H \oplus H^{'}, G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H^{'}, G)$,

2. $\mathrm{Ext}(H, G) = 0$ *if H is free abelian,*

3. $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.

Proof. 1. Take the direct sum of the free resolutions.

1. Use $0 \to H \to H \to 0$ as the resolution to calculate.

3. Use $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ and the calculation in Example 4.12 □

Corollary 4.22. *If the homology groups of chain complex C of free abelian groups are finitely generated and $H_n(C) = F_n \oplus T_n$ where F_n is free and T_n is torsion, then $H^n(C;\mathbb{Z}) = F_n \oplus T_{n-1}$.*

Proof. It follows from Theorem 4.16 and Proposition 4.21, and the fact $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = 0$, $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. □

Next property shows how Ext functor remedies the left exactness of Hom functor.

Proposition 4.23. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Then there is a six-term exact sequence*

$$\begin{aligned} 0 &\to \operatorname{Hom}(C,G) \to \operatorname{Hom}(B,G) \to \operatorname{Hom}(A,G) \\ &\to \operatorname{Ext}(C,G) \to \operatorname{Ext}(B,G) \to \operatorname{Ext}(A,G) \to 0 \end{aligned}$$

Proof. Any abelian group B has a free resolution with only two terms.

$$0 \to F_1 \to F_0 \to B \to 0.$$

The group F_0 has a generator for each generator of B and F_1 has a generator for each relation of B.

The short exact sequence

$$0 \to A \to B \to C \to 0.$$

can be extended to a short exact sequence of chain complexes

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_1^A & \longrightarrow & F_1^B & \longrightarrow & F_1^C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_0^A & \longrightarrow & F_0^B & \longrightarrow & F_0^C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Applying $\operatorname{Hom}(-,G)$ to the F_i we get a short exact sequence of chain complexes, and this short exact sequence gives a long exact sequence in cohomology

$$\begin{aligned} 0 &\to H^0(f^C,G) \cong \operatorname{Hom}(C,G) \to H^0(F^B,G) \cong \operatorname{Hom}(B,G) \to H^0(F^A,G) \cong \operatorname{Hom}(A,G) \\ &\to H^1(F^C,G) = \operatorname{Ext}(C,G) \to H^1(F^B,G) = \operatorname{Ext}(B,G) \to H^1(F^A,G) = \operatorname{Ext}(A,G) \to 0 \end{aligned}$$

□

Universal coefficient theorem and Künneth formula for homology

Instead of Hom, we apply $\otimes$ to a free resolution $0 \to F_1 \to F_0 \to H \to 0$. This operation is right exact. Recall that, by similar idea as Ext, we have used Tor to measure its non-exactness, i.e. Tor is the first (and the only non-trivial) homology of the new complex. For the reader's convenience we record Theorem 3.151 again:

Theorem 4.24 (Universal Coefficient Theorem)**.** *If C is a chain complex of free abelian groups, then there are natural short exact sequences*

$$0 \to H_n(C) \otimes G \to H_n(C,G) \to \operatorname{Tor}(H_{n-1}(C),G) \to 0$$

for all n. These sequences split, though not naturally.

Finally, we want to remark that the universal coefficient theorem in homology is a special case of the Künneth theorem. We first introduce the tensor products of chain complexes.

Let (C, ∂) and (D, ∂) be chain complexes, where C_i and D_i are zero for $i < 0$. The tensor product of chain complexes is

$$(C \otimes D)_n = \oplus_{p+q=n} C_p \otimes Dq,$$

with differential

$$\partial(c_p \otimes d_q) = (\partial c_p) \otimes d_q + (-1)^p c_p \otimes (\partial d_q).$$

This indeed defines a chain complex since

$$\partial\partial(c_p \otimes d_q) = \partial((\partial c_p) \otimes d_q + (-1)^p c_p \otimes (\partial d_q)) = (-1)^{p-1} \partial c_p \otimes \partial dq + (-1)^p \partial c_p \otimes \partial d_q = 0.$$

Tensor product of chain maps is defined as $(f \otimes g)(c_p \otimes d_q) = (f_p c_p \otimes (g_q d_q)$. An easy checking shwos it commutes with ∂. We also know that chain homotopy is compatible with tensor products.

Theorem 4.25 (Künneth formula)**.** *For a free chain complex C and an arbitrary chain complex D, there is a natural short exact sequence*

$$0 \to \oplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \oplus_{p+q=n-1} \operatorname{Tor}(H_p(C), H_q(D)) \to 0.$$

It splits but not canonically.

The proof of this theorem requires acyclic models theorem which is beyond scope of this note: see the note "Acyclic Models" (in preparation.)

Note that we obtain Theorem 4.24 by taking D as the complex $D_0 = G$ and $D_i = 0$ for $i \neq 0$ in Theorem 4.25.

4.1.5 Singular cohomolgy

This is an explicit example in geometry of choosing the chain complex and cochain complex.

Recall that a standard n-simplex is the convex set $\Delta_n \subset \mathbb{R}^{n+1}$ consisting of all $(n+1)$-tuples of real numbers $(v_0, \ldots, v_n)$ with $v_i \geq 0$, $v_0 + \cdots + v_n = 1$. A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$. The singular chain group $S_n(X)$ is the free abelian group generated by the singular n-simplices. The boundary homomorphism $\partial : S_n(X) \to S_{n-1}(X)$ is defined as $\partial(\sigma) = \sum_i (-1)^i \sigma[v_0, \cdots, \widehat{v}_i, \ldots, v_n]$ where $\widehat{v}_i$ implies to omit v_i. Then we were able to define the singular homology group of X, and denoted by $H_n(X)$.

Definition 4.26. The *singular cohomology* of a space X, denoted $H^q(X; G)$ $q \geq 0$, is the cohomology of the singular cochain complex $S^q(X) = \operatorname{Hom}(S_q(X), G)$ with the coboundary $\delta : S^q(X; G) \to S^{q+1}(X; G)$ being the dual of ∂, thus any $\phi \in S^q(X; G)$, $\delta\phi$ is the composition $S_{q+1}(X) \xrightarrow{\partial} S_q(X) \xrightarrow{\phi} G$.

Explicitly, for $\sigma : \Delta_{n+1} \to X$,

$$\delta\phi(\sigma) = \sum_i (-1)^i \phi(\sigma|_{[v_0,\dots \widehat{v_i},\dots,v_{n+1}]}).$$

Geometrically, we can understand a singular cochain $\xi \in S^q(X,G)$ as a function that assigns a scalar to every singular simplex $\sigma : \Delta_q \to X$. The pairing

$$S^q(X;G) \times S_q(X;G) \to G$$

is sometimes called *integration*, because it is an algebraic analogue of integrating a differential form over parametrised manifold.

For example,

- A 0-cochain is simply a function (of sets) $f : X \to R$, since 0-simplices correspond to points in X.
- A 1-cochain assigns a scalar to every continuous path $\gamma : [0,1] \to X$.
- A 2-cochain assigns a scalar to every map $\sigma : \Delta_2 \to X$.

The analogue of Stoke's Theorem follows just by definition. If $\alpha \in S_q(X;G)$ and $\xi \in S_{q-1}(X;G)$ then

$$\xi(\partial\alpha) = (\delta\xi)(\alpha).$$

or in integral notation

$$\int_{\partial\alpha} \xi = \int_\alpha \delta\xi.$$

For example, given a 1 simplex $\gamma : [0,1] \to X$ and 0-cochain $f : X \to R$, we have

$$\int_\gamma \delta f = \int_{\partial\gamma} f = f(\gamma(1) - \gamma(0)) = f(\gamma(1)) - f(\gamma(0)).$$

This example really illustrate why the boundary of a 1-simplex requires signs: to recover the Fundamental Theorem of Calculus.

An easy consequence is that this integration pairing descends to homology and cohomology: we will discuss this in 4.2.1.

Recall that in multivariate calculus, you multiply (wedge) differential forms together, and cohomology becomes a ring. This is still true in more general approaches such as singular cohomology. On the homology side, one has an intersection pairing, but this is harder to describe and only available for really "nice" spaces.

Perhaps another feature of cohomology worth mentioning is that is contravariant: cohomology classes pullback from the target to the source under a map of spaces. This important in the theory of characteristic classes, where such classes are pulled back from to maps to certain universal spaces. Such classes measure the amount of "twisting" of bundles.

For a pair (X, A), we could also talk about relative groups. Let $S_q(X, A) = S_q(X)/S_q(A)$ and $S_q(X, A; G) = \mathrm{Hom}(S_q(X, A); G)$. The elements of $S^q(X, A; G)$ are n-cochains taking the value 0 on singular n-simplices in A. Thus

$$\ldots \to S^{q-1}(X, A; G) \to S^q(X, A; G) \to S^{q+1}(X, A; G) \to \cdots$$

is obtained by dualising the singular chain complex of X.

Hence we have relative groups $H^\bullet(X, A; G) := H(S^\bullet(X, A; G))$. A map of pairs $f : (X, A) \to (Y, B)$ induces homomorphisms $f^\bullet : S^\bullet(Y, B; G) \to S^\bullet(X, A; G)$ and $H^\bullet(f) : H^\bullet(Y, B; G) \to H^\bullet(X, A; G)$.

We could describe relative cohomology groups in terms of exact sequences. Recall that the short exact sequence

$$0 \to S_q(A) \xrightarrow{i} S_q(X) \xrightarrow{j} S_q(X, A) \to 0$$

splits. By the splitting exactness of Hom functor, the dual

$$0 \leftarrow S^q(A, G) \xleftarrow{i^*} S^q(X, G) \xleftarrow{j^*} S^q(X, A; G) \leftarrow 0$$

is a split exact sequence as well. Since i^* and j^* commute with δ, it induces a long exact sequence of cohomolgy groups

$$\cdots \to H^q(X, A; G) \xrightarrow{j^*} H^q(X; G) \xrightarrow{i^*} H^q(A; G) \xrightarrow{\delta} H^{q+1}(X, A; G) \to \cdots .$$

Let us describe the connecting homomorphism $H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G)$. For $\phi \in Z^n(A; G)$, we first extend it to a cochain $\overline{\phi} \in S^n(X; G)$, by assigning the value 0 on singular simplices not in A. Then $\delta^X(\overline{\phi}) = \overline{\phi}\partial \in S^{n+1}(X; G)$. It is $S^{n+1}(X, A; G)$ because the original ϕ is a cocycle in A, i.e taking the value 0 in $B^n(A)$, which means $\delta^X(\overline{\phi}) = \overline{\phi}\partial$ takes the value 0 on $S^{n+1}(A)$. Finally it is $Z^{n+1}(X, A; G)$ because $\delta^{X,A}(\delta^X\overline{\phi}) = \delta^X(\delta^X\overline{\phi}) = 0$. Its class $[\delta^X\overline{\phi}] \in H^{n+1}(X, A; G)$ is $\delta[\phi]$.

A more general long exact sequence is for a triple (X, A, B), induced by

$$0 \leftarrow S^q(A, B; G) \xleftarrow{i^*} S^q(X, B; G) \xleftarrow{j^*} S^q(X, A; G) \leftarrow 0$$

When B is a point, it induces the long exact sequence for reduced cohomology.

4.1.6 The Eilenberg-Steenrod Axioms for cohomology

For simplicity, we omit the coefficient G in our notation. A cohomology theory consists of 3 functions:

1. For any integer $n \geq 0$ and any pair of spaces (X, A), we have an abelian group $H^n(X, A)$.

2. For any integer $n \geq 0$ and any map of pairs $f : (X, A) \to (Y, B)$ (f maps A to B), we obtain homomorphism $H(f)^n : H^n(Y, B) \to H^n(X, A)$.

3. For any integer $n \geq 0$ and any pair of spaces (X, A), we have a connecting homomorphism $\delta : H^n(A) \to H^{n+1}(X, A)$.

These functions satisfy the following 7 axioms:

1. *Unit*: If $\mathrm{id} : (X, A) \to (X, A)$ is the identity, $H(\mathrm{id})$ is the identity.
2. *Composition*: $H(g \circ f) = H(f) \circ H(g)$.
3. *Naturality* : Given $f : (X, A) \to (Y, B)$, the following diagramme commutes:

$$\begin{array}{ccc} H^n(A) & \xleftarrow{H(f)^n|_A} & H^n(B) \\ \delta \downarrow & & \downarrow \delta \\ H^{n+1}(X, A) & \xleftarrow[H(f)^{n+1}]{} & H^{n+1}(Y, B) \end{array}$$

4. *Exactness*: The following sequence is exact:

$$\cdots \to H^n(X, A) \xrightarrow{H(j)} H^n(X) \xrightarrow{H(i)} H^n(A) \xrightarrow{\delta} H^{n+1}(X, A) \to \cdots,$$

   where $i : A \to X$ and $j : (X, \varnothing) \to (X, A)$ are inclusions.
5. *Homotopy*: If $f \simeq g$ are homotopic maps of pairs, then $H(f) = H(g)$.
6. *Excision*: Given (X, A) and $U \subset X$ such that $\overline{U} \subset \mathrm{int}(A)$. Then the inclusion $i : (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms in cohomology.
7. *Dimension*: Let P be a one-point space, then

$$H^n(P) = \begin{cases} G & n = 0 \\ 0 & n \neq 0 \end{cases}$$

There are many generalised cohomology theories which satisfy all the axioms except probably the dimension axiom. The Eilenberg-Steenrod uniqueness theorem says that there is a unique cohomology theory satisfying all the axioms in the category of finite cellular spaces. Unfortunately the proof is beyond the scope of the current note. But we could prove a weaker version.

Suppose $H^n(X, A)$ and $K^n(X, A)$ are cohomology theories, and $\phi : H^\bullet(X, A) \to K^\bullet(X, A)$ is a natural transformation of cohomology theories, i.e. it commutes with induced homomorphisms and with coboundary homomorphisms in long exact sequence of pairs.

Theorem 4.27 (Weak form of Eilenberg-Steenrod Uniqueness)**.** *Suppose $\phi : H^\bullet(X) \to K^\bullet(X)$ is an isomorphism when $X = \{\mathrm{pt}\}$. Then ϕ is an isomorphism for any finite cellular spaces.*

We will give a proof in §§4.1.7.

A few more words about Eilenberg-Steenrod axioms. If we want to prove uniqueness for a larger category, more conditions are needed for the uniqueness. For the category of cellular spaces, an eighth axiom is added to guarantee its uniqueness.

Milnor's Additivity axiom in "On axiommatic homology theory" Let $X = \sqcup X\alpha$ be disjoint union. Then the induced homomorphism $\prod_\alpha i_\alpha^* : H^n(X) \to \prod_\alpha H^n(X_\alpha)$ is an isomorphism.

This axiom has force only if there are infinitely many X_α. The finite sum case is a corollary of the following Mayer-Vietoris Sequence. There are also examples of cohomology theories which are not additive (James and Whitehead: "Homology with zero coeffients").

4.1.7 Mayer-Vietoris Sequences

By using Eilenberg-Steenrod Axioms, we can form the Mayer-Vietoris Sequence for cohoomoogy just as we did for homology. The model in our mind is $X = A \cup B$ with A and B open in X. But for the ease of application, we use the following setting.

Definition 4.28. For $U \subset A \subset X$, the map $(X \setminus U, A \setminus U) \to (X, A)$ is an excision if the induced homomorhism $H^n(X, A) \to H^n(X \setminus U, A \setminus U)$ is an isomorphism for all n.

Example 4.29. The inclusion $(\mathbb{D}^n_+, \mathbb{S}^{n-1}) \to (\mathbb{S}^n, \mathbb{D}^n_-)$ is an excision. We cannot apply the Excision Axiom directly. But look at the exact sequences from exactness axiom corresponding to the two pairs, we have $H^{i+1}(\mathbb{S}^n, \mathbb{D}^n_-) = H^{i+1}(\mathbb{S}^n) = H^i(\mathbb{S}^{n-}) = H^{i+1}(\mathbb{D}^n_+, \mathbb{S}^{n-1})$ when $i > 0$ by noticing $H^n(\mathbb{D}^n) = H^n(\mathrm{pt})$. The rest of two identities also follow from the exact sequence by knowing $H^0(\mathbb{D}^n) = H^0(\mathbb{S}^n) = G$.

Definition 4.30. Suppose $A, B \subset X$, such that

$A \cup B = X$

Both $(A, A \cap B) \to (X, B)$ and $(B, A \cap B) \to (X, A)$ are excisions.

Then $(X; A, B)$ is an *excisive triad*.

Example 4.31. Let $X = A \cup B$ with A and B open in X. Then $(A, A \cap B) = (X \setminus U, B \setminus U)$ where $U = B \setminus (A \cap B)$. Since $X \setminus U = A$, U is closed and $U \subset B = \int(B)$. Then the excision axiom shows this is an excision.

Example 4.32. $(\mathbb{S}^n; \mathbb{D}^n_+, \mathbb{D}^n_-)$ is an excisive triad.

Theorem 4.33. *Suppose $(X; A, B)$ is an excisive triad. Then there is a long exact sequence*

$$\cdots \to H^n(X) \xrightarrow{(j_A^*, j_B^*)} H^n(A) \oplus H^n(B) \xrightarrow{i_A^* - i_B^*} H^n(A \cap B) \xrightarrow{\delta} H^{n+1}(X) \to \cdots$$

Recall that a similar Mayer-Vietoris sequence was derived for homology as the long exact sequence associated to the short exact sequence

$$0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*(A + B) \to 0.$$

We could prove it similarly for singular cohomology say, but not for a general cohomology theory.

We need the following purely algebraic lemma:

Lemma 4.34 (Barratt-Whitehead)**.** *Suppose we have the following commutative diagramme with exact rows*

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \gamma n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \\
\cdots & \longrightarrow & A_n' & \xrightarrow[f_n']{} & B_n' & \xrightarrow[g_n']{} & C_n' & \xrightarrow[h_n']{} & A_{n-1}' & \longrightarrow & \cdots
\end{array}
$$

in which every third map $\gamma_i : C_i \to C_i'$ is an isomorphism. Then there exists a long exact sequence

$$\cdots \to A_n \xrightarrow{(f_n, \alpha_n)} B_n \oplus A_n \xrightarrow{\beta_n - f_n'} B_n' \xrightarrow{h_n \gamma_n^{-1} g_n'} A_{n-1} \to \cdots$$

This is a standard diagramme chasing argument, and we only sketch the proof. (See *The First Non-Vanishing Group of an (n+l)-AD* M. Barratt, J. Whitehead Published 1 July 1956 Mathematics Proceedings of The London Mathematical Society.)

Proof. That this is a chain complex is easy to check in terms of commutativity of the diagramme. To check the exactness, we have 3 parts:

1. $\ker(\beta_n - f_n') \subset \operatorname{im}(f_n, \alpha_n)$: Assume $\beta_n(b) = f_n'(d)$.
2. $\ker(h_n \gamma_n^{-1} g_n') \subset \operatorname{im}(\beta_n - f_n')$: Let $e \in \ker(h_n \gamma_n^{-1} g_n')$. By exactness at C_n, we have $b \in B_n$ such that $g_n(b) = \gamma_n^{-1} g_n'(e)$. So $\beta_n(b) - e \in \ker g_n' = \operatorname{im} f_n'$. Choose any element d in it, we have $e = \beta_n(b) - f_n'(d)$.
3. $\ker(f_{n-1}, \alpha_{n-1}) \subset \operatorname{im}(h_n \gamma_n^{-1} g_n')$: First find an element in C_n. Then an element in B_n'.

□

Proof. (of Theorem 4.33) We have

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{n-1}(X) & \xrightarrow{j_B^*} & H^{n-1}(B) & \xrightarrow{\delta} & H^n(X,B) & \xrightarrow{h_n} & H^n(X) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle j_A^*} & & \downarrow{\scriptstyle i_B^*} & & \downarrow{\scriptstyle \gamma n} & & \downarrow{\scriptstyle j_A^*} & & \\
\cdots & \longrightarrow & H^{n-1}(A) & \xrightarrow[i_A^*]{} & H^{n-1}(A \cap B) & \xrightarrow[\delta]{} & H^n(A, A \cap B) & \xrightarrow[h_n']{} & H^n(A) & \longrightarrow & \cdots
\end{array}
$$

where γ_n is an isomorphism since $(X; A, B)$ is an excisive triad. Hence we have the Mayer-Vietoris sequence by Barratt-Whitehead lemma. □

Example 4.35. (Unreduced cones and unreduced suspensions) Given a space X, the *unreduced cone* of X is

$$C(X) := \frac{X \times [0,1]}{(x,1) \sim (y,1), \forall x, y \in X}$$

and the unreduced suspension is

$$\Sigma(X) := \frac{X \times [0,1]}{(x,0) \sim (y,0), (x,1) \sim (y,1), \forall x, y \in X}$$

Then $(\Sigma(X); C_+(X), C_-(X))$ is an excisive triad. Use the Mayer-Vietoriis sequence and the fact $C_+(X) \simeq C_-(X) \simeq \text{pt}$, we have $H^k(X) = H^{k+1}(\Sigma X)$. If X is connected, $H^1(\Sigma X) = 0$ and $H^0(\Sigma X) = \mathbb{Z}$.

As a special case, $\Sigma \mathbb{S}^n = \mathbb{S}^{n+1}$, so it can be used to calculate the cohomology of $\mathbb{S}^n$ inductively. The reader may wan to to complete the calculation.

Given a continuous map $f : \mathbb{S}^{n-1} \to A$ for $n \geq 1$. We have

$$X = C(f) := A \cup_f \mathbb{D}^n = \frac{A \sqcup \mathbb{D}^n}{f(x) \sim x, \forall x \in \mathbb{S}^{n-1}}.$$

Notice that if $f \simeq g : \mathbb{S}^{n-1} \to A$, then $C(f) \simeq C(g)$. Also f extends to a map $f : \mathbb{D}^n \to C(f)$.

Proposition 4.36. *1. The inclusion $A \to X$ induces isomorphisms $H^q(A) = H^q(X)$ for $q \neq n, n-1$.*

2. There is an exact sequence

$$0 \to H^{n-1}(X) \to H^{n-1}(A) \xrightarrow{f^*} H^{n-1}(\mathbb{S}^{n-1}) \to H^n(X) \to H^n(A) \to 0$$

Proof. Use the cohomology sequence for pairs (or Mayer-Vietoris sequence). For both items, use $H^i(X, A) = H^i(\mathbb{D}^n, \mathbb{S}^{n-1}) = H^{i-1}(\mathbb{S}^{n-1})$ and previous calculation for $H^*(\mathbb{S}^n)$. □

Example 4.37. Recall $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus 0)/\sim$ where $x \sim y$ if there exists $\lambda \in \mathbb{C}\setminus$ such that $\lambda x = y$. We write $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_f \mathbb{D}^{2n}$, where $f : \mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$ is the natural factorisation map. The above sequence tells us hat for $0 < m < 2n$,

$$0 \to H^{2n-1}(\mathbb{S}^{2n-1}) \to H^{2n}(\mathbb{C}P^n) \to 0,$$
$$0 \to H^m(\mathbb{C}P^n) \to H^m(\mathbb{C}P^{n-1}) \to 0.$$

$H^0(\mathbb{C}P^n) = \mathbb{Z}$ since it is path connected. By induction we have

$$H^m(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & m = 0, 2, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Theorem 4.27 Now we can finish the proof as announced.

Proof. We use induction. The dimension axiom gives us the dimension 0 case. Assume it is done for all complexes with dimension less than or equal to $n-1$, and X be a cellular space of diimension n. Note that X is obtained by attaching n-cells to an $(n-1)$ cellular space. So we could do these attachment one by one (but A will possibly have dimension n then).

Hence the statement is reduced to the following situaton: Suppose the result is true for A, and prove the statement is true for $X = C(f) := A \cup_f \mathbb{D}^n$. Item 1 of Proposition 4.36 tells us that $\phi : H^q(X) to K^q(X)$ is an isomorphism for $q \neq n-1, n$. For the rest of q's, look at

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & H^{n-1}(X) & \longrightarrow & H^{n-1}(A) & \longrightarrow & H^{n-1}(\mathbb{S}^{n-1}) & \longrightarrow & H^n(X) & \longrightarrow & H^n(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \alpha & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & K^{n-1}(X) & \longrightarrow & K^{n-1}(A) & \longrightarrow & K^{n-1}(\mathbb{S}^{n-1}) & \longrightarrow & K^n(X) & \longrightarrow & K^n(A) & \longrightarrow & 0 \end{array}$$

As we have shown in the suspension calculation, α is also an isomorphism. So the five Lemma completes the proof. □

The five lemma also shows that $H^\bullet(X, A) = K^\bullet(X, A)$. Notice that for infinite cellular spaces, we need Milnor additivity in the above argument. A telescope argument (i.e, taking colimits) could reduce an infinite dimensional case to a finite dimensional case.

4.2 Products

In this section, we take the coefficients in a commutative ring R with a unit. The most common choices are $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Q}$.

4.2.1 Cup Product for Singular Cohomology

There is a product structure in cohomology. We start with singular cohomology. Let $\sigma : \Delta^{p+q} \to X$ be a singular simplex. By the *front p-face* or ${}_p\sigma$, we mean $\sigma|_{[v_0,\ldots,v_p]}$. Similarly, by the *back q-face* or σ_q, we mean $\sigma|_{[v_p,\ldots,v_{p+q}]}$.

Then we define the cup product at chain level:

Definition 4.38. Given $\phi \in S^p(X)$ and $\psi \in S^q(X)$, the *cup product* $\phi \cup \psi \in S^{p+q}(X)$ is defined by

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[v_0,\ldots,v_p]}) \cdot (\psi|_{[v_p,\ldots,v_{p+q}]}).$$

The reader may want to consider $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ and $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and apply the cup product on them to grasp the idea. (Note that their homology groups are isomorphic, that is, homology can't distinguish these non-homeomorphic spaces.)

Lemma 4.39. *For $\sigma : S^p(X)$ and $\psi \in S^q(X)$, we have*

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^p \phi \cup \delta\psi.$$

Proof. For $\sigma : \Delta^{m+n+1} \to X$, we have

$$\delta\phi \cup \psi(\sigma) = \sum_{i=0}^{p+1}(-1)^i \phi(\sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_{p+1}]})\psi(\sigma|_{[v_{p+1},\ldots,v_{p+q+1}]})$$
$$(-1)^p\phi \cup \delta\psi(\sigma) = \sum_{i=p}^{p+q+1}(-1)^i \phi(\sigma|_{[v_0,\ldots,v_p]})\psi(\sigma|_{[v_p,\ldots,\widehat{v}_i,\ldots,v_{p+q+1}]})$$

Add and cancel the last of the first sum and the first of the second sum, we have $\delta(\phi \cup \psi)$. □

Corollary 4.40. *There is a well defined cup product*

$$H^p(X) \times H^q(X) \xrightarrow{\cup} H^{p+q}(X).$$

Proof. $Z^p(X) \times Z^q(X) \xrightarrow{\cup} Z^{p+q}(X)$. If $\phi \in Z^p(X)$, $\phi \cup \delta\psi = (-1)^k\delta(\phi \cup \psi)$, thus $Z^p(X) \times B^q(X) \xrightarrow{\cup} B^{p+q}(X)$. Similarly $B^p(X) \times Z^q(X) \xrightarrow{\cup} B^{p+q}(X)$ □

Relative cup products Let (X, A) be a pair of spaces. The formula which specifies the cup product by its effect on a simplex

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[e_0,\ldots,e_p]}) \cdot (\psi|_{\sigma[e_p,\ldots,e_{p+q}]})$$

extends to relative cohomology.

For, if $\sigma : \Delta^{p+q} \to X$ has image in A, then so does any restriction of σ. Thus, if either φ or ψ vanishes on chains with image in A, then so does $\varphi \cup \psi$. Hence we get relative cup product maps

$$H^p(X;R) \times H^q(X,A;R) \to H^{p+q}(X,A;R)$$
$$H^p(X,A;R) \times H^q(X;R) \to H^{p+q}(X,A;R)$$
$$H^p(X,A;R) \times H^q(X,A;R) \to H^{p+q}(X,A;R).$$

More generally, assume we have two open subsets A and B of X. Then the formula for $\varphi \cup \psi$ on cochains implies that the cup product yields a map

$$S^p(X,A;R) \times S^q(X,B;R) \to S^{p+q}(X,A+B;R)$$

where $S^p(X,A+B;R)$ denotes the subgroup of $S^p(X;R)$ of cochains which vanish on sums of chains in A and chains in B.

The natural inclusion

$$S^p(X,A \cup B;R), \hookrightarrow S^p(X,A+B;R)$$

induces an isomorphism in cohomology. For we have a map of long exact cohomology sequences

$$\begin{array}{ccccccccc} H^p(A \cup B) & \longrightarrow & H^p(X) & \longrightarrow & H^p(X,A \cup B) & \longrightarrow & H^{p+1}(A \cup B) & \longrightarrow & H^{p+1}(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^p(A + B) & \longrightarrow & H^p(X) & \longrightarrow & H^p(X,A + B) & \longrightarrow & H^{p+1}(A + B) & \longrightarrow & H^{p+1}(X) \end{array}$$

where we omit the coefficients. The subdivision argument 3.47 and our results on cohomology of free chain complexes imply that $H^p(A \cup B; R) \xrightarrow{\cong} H^p(A+B; R)$ is an isomorphism for every p. Thus, the Five-Lemma implies that

$$H^p(X, A \cup B; R) \xrightarrow{\cong} H^p(X, A + B; R)$$

is an isomorphism as well. Thus composition with this isomorphism gives a cup product map

$$H^p(X, A; R) \times H^q(X, B; R) \to H^{p+q}(X, A \cup B; R).$$

Now one can check that all the formulae we proved for the cup product also hold for the relative cup products.

Lemma 4.41. *For a map $f : X \to Y$, the induced maps $H^n(f) : H^n(Y; R) \to H^n(X; R)$ satisfy $H^n(f)(\alpha \cup \beta) = H^n(f)(\alpha) \cup H^n(f)(\beta)$, and similarly in the relative case.*

Proof. Let α and β represented by $\phi \in S^p$ and $\psi \in S^q$ respectively. Let $\sigma : \Delta^{p+q} \to X$. Then $f(\sigma)$ is a $(p+q)$-simplex in Y. So

$$\begin{aligned} H^*(f)(\phi \cup \psi)(\sigma) &= (\phi \cup \psi)(f(\sigma)) \\ &= \phi(f(\sigma|_{[v_0,\dots,v_p]})(\psi(\sigma)|_{[v_p,\dots,v_{p+q}]}) \\ &= H^*(f)(\phi)(\sigma|_{[v_0,\dots,v_p]})H^*(f)\psi(\sigma|_{[v_p,\dots,v_{p+q}]}) \\ &= H^*(f)(\phi)(\sigma) \cup H^*(f)(\psi)(\sigma). \end{aligned}$$

□

It is easy to check that the cup product is associative (even at chain level): $(\phi \cup \psi) \cup \tau = \phi \cup (\psi \cup \tau)$. We also know there is a unit: Let $1 \in S^0(X)$ be the function taking value 1 on any point of X, then $1 \cup a = a$.

Proposition 4.42. *If $\alpha \in H^p(X; R)$ and $\beta \in H^q(X; R)$ then*

$$\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha.$$

Proving this proposition requires a substantial effort.

Proof. For a singular p-simplex $\sigma : [v_0, \dots, v_p] \to X$, let $\overline{\sigma} = \sigma \circ r$ where r is the linear map determined by $r(v_i) = v_{n-i}$. Then we define $\rho : S^p(X) \to S^p(X)$ by $\rho(\sigma) = \epsilon_p \overline{\sigma}$ where $\epsilon_p = (-1)^{\frac{p(p+1)}{2}}$.

We want to show that ρ is a chain map which is chain homotopic to the identity. Once we have that, the theorem follows:

$$\begin{aligned} (S^*(\rho)\phi \cup S^*(\rho)\psi)(\sigma) &= \phi(\epsilon_p \sigma|_{[v_p,\dots,v_0])\epsilon_q} \psi\sigma|_{[v_{p+q},\dots,v_p]}) \\ S^*(\rho^*)(\psi \cup \phi)(\sigma) &= \epsilon_{p+q} \psi(\sigma|_{[v_{p+q},\dots,v_p]})\phi(\sigma|_{[v_p,\dots,v_0]}) \end{aligned}$$

with $\epsilon_{p+q} = (-1)^{pq}\epsilon_p\epsilon_q$, showing that

$$S^*(\rho)\phi \cup S^*(\rho)\psi = (-1)^{pq} S^*(\rho^*)(\psi \cup \phi)$$

at chain level. Since ρ induces identity in cohomology, we have $\alpha \cup \beta = (-1)^{pq}\beta \cup \alpha$ when passing to the cohomology level. Now the proof reduces to the following two lemmas. □

Lemma 4.43. *ρ is a chain map.*

Proof. For an n-simplex σ,

$$\begin{aligned}\partial\rho(\sigma) &= \epsilon_n \sum_i (-1)^i \sigma|_{[v_n,\ldots,\widehat{v}_{n-i},\ldots,v_0]}\\ \rho\partial(\sigma) &= \rho(\sum_i (-1)^i \sigma_{[v_0,\ldots,\widehat{v}_i\ldots,v_n]})\\ &= \epsilon_{n-1}\sum_i (-1)^{n-i}\sigma_{[v_n,\ldots,\widehat{v}_{n-i},\ldots v_0]}\end{aligned}$$

They are equal since $\epsilon_n = (-1)^n \epsilon_{n-1}$. □

Lemma 4.44. *The chain map ρ defined above is chain homotopic to the identity, and so it induces the identity homomorphism in cohomology.*

Proof. We need the fact that there is a natural division of $[v_0 \ldots v_n] \times I$ into $n+1$ simplices. (Note that the product of two simplices is not a simplex in general.) If we denote $(v_i, 0)$ by v_i and $(v_i, 1)$ by w_i, then these simplices are

$$\sigma_i = [v_0, \ldots, v_i, w_i, \ldots w_n].$$

Thus tracing out along the bottom face until the position i, then jumping to the top face, tracing out the rest starting with the position i.

We define the prism operator $P : S_n(X) \to S_{n+1}(X)$ by

$$P(\sigma) = \sum_i (-1)^i \epsilon_{n-i} (\sigma \circ \pi)|_{[v_0,\ldots,v_i,w_n,\ldots,w_i]},$$

where $\pi : \Delta \times I \to \Delta$ is a projection. We will leave out $\sigma \circ \pi$ for the sake of notational simplicity in the remainder of this proof.

$$\begin{aligned}\partial P(\sigma) &= \sum_{j\le i} (-1)^i (-1)^j \epsilon_{n-i} [v_0, \ldots, \widehat{v}_j, \ldots, v_i, w_n, \ldots, w_i]\\ &+ \sum_{j \ge i} (-1)^i (-1)^{i+1+n-j} \epsilon_{n-i} [v_0, \ldots, v_i, w_n, \ldots, \widehat{w}_j, \ldots, w_i]\\ P\partial(\sigma) &= \sum_{i \le j} (-1)^i (-1)^j \epsilon_{n-i-1} [v_0, \ldots, v_i, w_n, \ldots, \widehat{w}_j, \ldots, w_i]\\ &+ \sum_{i \ge i} (-1)^{i-1} (-1)^j \epsilon_{n-i} [v_0, \ldots, \widehat{v}_j, \ldots, v_i, w_n, \ldots, w_i]\end{aligned}$$

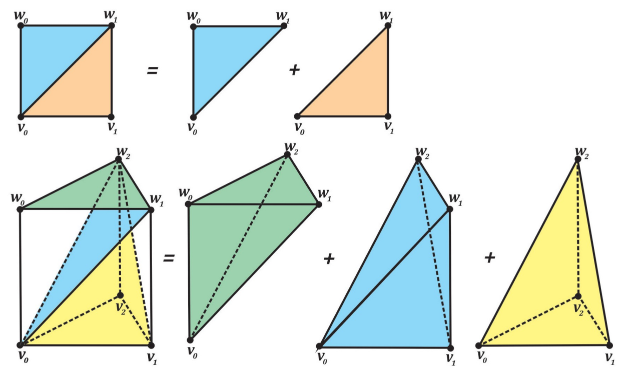


Figure 4.1: Prism decomposition for $n = 1$ and $n = 2$

Since $\epsilon_{n-i} = (-1)^{n-i}\epsilon_{n-i-1}$, the terms with $i \neq j$ cancel in the two sums. The terms with $j = i$ give

$$\epsilon_n[w_n, \dots, w_0] + \sum_{i>0} \epsilon_{n-i}[v_0, \dots, v_{i-1}, w_n, \dots, w_i]$$
$$+ \sum_{i<n} (-1)^{n+i-1}\epsilon_{n-i}[v_0, \dots, v_u, w_n, \dots, w_{i+1}] - [v_0, \dots, v_n]$$

The two summations cancel, as replacing i by $i-1$ in the second sum produces a new sign $(-1)^{n+i}\epsilon_{n-i+1} = -\epsilon_{n-i}$. Thus the remaining two terms are just

$$\partial P(\sigma) + P\partial(\sigma) = \epsilon_n[w_n, \dots, w_0] - [v_0, \dots, v_n] = \rho(\sigma) - \sigma.$$

Hence P is the chain homotopy between ρ and the identity. □

Remark 4.45. This proof should be read using picture like Figure 4.1. P is the map from a simplex to get an oriented cylinder with a simplicial division compatible with the original simplex. Then ∂P is the oriented boundary of the cylinder. $P\partial$ is the part of the boundary without the top face and the bottom face and with the opposite sign. Cancellations could also be interpreted: the cancellation on $i \neq j$ is the cancellation on the boundary without the top face and the bottom face. Then second cancellation above is the intersection face of different simplices in the division with different orientation.

To summarise the above discussion, we have obtained

Theorem 4.46 (The cohomology ring). *Let X be a topological space and R be a commutative ring with unity. Then*

$$H^\bullet(X; R) = \oplus_i H^i(X; R)$$

is a graded (skew) commutative ring with identity under the cup product, i.e. if $\alpha \in H^m(X;R)$ *and* $\beta \in H^n(X;R)$ *then* $\alpha \cup \beta = (-1)^{mn}\beta \cup \alpha \in H^{m+n}(X;R)$. *Moreover* $H^\bullet(X;R)$ *is a graded* R*-algebra.*

Example 4.47. $H^\bullet(\mathbb{S}^n;\mathbb{Z}) = \frac{\mathbb{Z}[a_n]}{a_n^2}$, where the generator of H^0 corresponds to 1 and the generator of H^n is a_n.

Actually, we are ready to calculate the cohomology ring of projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$. But we would postpone these after we have studied Poincaré duality. We remark that all the above results extend to relative case, i.e the case where $H^\bullet(X;R)$ being replaced by $H^\bullet(X,A;R)$.

Cap product

For any space X, there is a bilinear pairing operation between cochains and chains.

Definition 4.48. Let $a \in S^q(X)$ and $\sigma \in S_{p+q}(X)$. Then the cap product $a \cap \sigma \in S_p(X)$ is defined by

$$a \cap \sigma = \langle a, \sigma_{[v_p,\dots,v_{p+q}]}\rangle \cdot \sigma_{[v_0,\dots,v_p]}, \quad \text{or} \quad a \cap \sigma = \langle a, \sigma_q\rangle \cdot {}_p\sigma.$$

The cap product at chain level has the following properties.

Proposition 4.49. *1.* Duality*: For* $a, b \in S^\bullet(X)$, $c \in S_\bullet(X)$, *we have*

$$\langle a \cup b, c\rangle = \langle a, b \cap c\rangle.$$

2. Associativity*: For* $a, b \in S^\bullet(X)$, $c \in S_\bullet(X)$, *we have*

$$(a \cup b) \cap c = a \cap (b \cap c).$$

3. Existence of unit*:*

$$1 \cap c = c.$$

4. Naturality*: Let* $f : X \to Y$ *be a map. For* $b \in S^\bullet(Y)$ *and* $c \in S_\bullet(X)$, *we have*

$$b \cap (H_*(f)c) = H_*(f)(H^*(f)b \cap c).$$

All these are easy to derive from Definition 4.48 and the properties of the cup product.

To define cap product on (co)homology, we need

Proposition 4.50. *For* $a \in S^q(X)$ *and* $\sigma \in S_{p+q}(X)$, *we have*

$$\partial(a \cap \sigma) = (-1)^p(\delta a) \cap \sigma + a \cap (\partial\sigma).$$

Proof. For simplicity, we omit the notation σ in the calculation.

$$\begin{aligned} a \cap \partial[v0, \dots, v_{p+q}] &= \sum_{i=0}^{p+q} (-1)^i a \cap [v_0, \dots, \widehat{v}_i, \dots, v_{p+q}] \\ &= \sum_{i=0}^{p} (-1)^i \langle a, [v_p, \dots, v_{p+q}]\rangle [v_0, \dots, \widehat{v}_i, \dots v_p] \\ &+ (-1)^{p-1} \sum_{i=p-1}^{p+q} (-1)^{i-p+1} \langle a \cap [v_{p-1}, \dots, \widehat{v}_i, \dots, v_{p+q}]\rangle [v_0, \dots, v_{p-1}] \\ &= \partial(a \cap \sigma) + (-1)^{p-1} \langle a \cap \partial[v_{p-1}, \dots, v_{p+q}]\rangle [v_0, \dots, v_{p-1}] \\ &= \partial(a \cap \sigma) + (-1)^{p-1} (\delta a) \cap \sigma. \end{aligned}$$

Hence it induces the cap product between homology and cohomology:

$$H^q(X) \times H_{p+q}(X) \xrightarrow{\cap} H_p(X).$$

□

De Rham cohomology

Let M be a smooth manifold of dimension n. Let $\Omega(M)$ be the (real linear) space of q-forms, $d : \Omega^q(M) \to \Omega^{q+1}(M)$ be the exterior differential. Then we have the de Rham cochain complex

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

We denote he space of closed (exact) k-forms, i.e k-forms ω with $d\omega = 0$ ($\omega = d\eta$ respectively), by $Z^k_{dR}(M)$ (and $B^k_{dR}(M)$ respectively). We denote its cohomology by $H^\bullet_{dR}(M)$, which is called the *de Rham cohomology* of M.

The cup product for de Rham cohomology is just the wedge product $\omega \wedge \eta$. Since for k-forms ω and η we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

we know it descends to a product on cohomology by the same argument as Corollary 4.40. A notable fact for de Rham cohomology is that the cup product is graded (skew) commutative even at the chain level: $x \wedge y = (-1)^{|x||y|} y \wedge x$.

There are several reasons prevent us from exploiting the Eilenberg-Steenrod uniqueness theorem to claim it is isomorphic to the singular cohomology $H^\bullet(X; R)$. The main reason is that the de Rham cohomology is only defined for smooth manifolds. The lack of relative version of de Rham theory could be compensated by Thom isomorphism (4.4.6).

However, de Rham theorem does ensure this isomorphism.

Theorem 4.51 (de Rham). *$H^\bullet_{dR}(M) = H^\bullet(M; \mathbb{R})$ as cohomology rings.*

More precisely, we need two facts for this theorem:

1. Let $S_q^{sm}(M;\mathbb{R})$ be the real space spanned by smooth singular q-simplices $\sigma : \Delta^q \to M$, then the inclusion
$$S_\bullet^{sm}(M;\mathbb{R}) \to S_\bullet(M;\mathbb{R})$$
is a chain homotopy equivalence. Then its dual
$$S^\bullet(M;\mathbb{R}) \to S_{sm}^\bullet(M;\mathbb{R})$$
is a cochain homotopy equivalence.

2. We could take integration of a q-form on a singular chain of dimension q, this is a bilinear function
$$\Omega^q(M) \times S_q^{sm}(M;\mathbb{R}) \to \mathbb{R}, \quad (\omega, \sigma) \mapsto \int_\sigma \omega.$$
Stokes' theorem tells us
$$\int_{\partial\sigma} = \int_\sigma d\omega..$$

In other words, exterior differentials are dual to boundary maps. This provides us a cochain map
$$\Omega^\bullet(M) \to S_{sm}^\bullet(M;\mathbb{R}).$$
If we show that this is a cochain homotopy equivalence, we will finish the proof of de Rham theorem. The proof proceeds as the same pattern as the proof of Poincaré duality which we will provide in the next section.

4.2.2 Cross product and Künneth formula

We want to understand the cohomology ring of a product space. Let us first define the cross product in cohomology.

Definition 4.52. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be projections. We define the cross product by
$$x \times y = p_1^*(x) \cup p_2^*(y) \in H^{m+n}(X \times Y; R)$$
for $x \in H^m(X;R)$ and $y \in H^n(Y;R)$.

To be more precise, for $\sigma : \Delta^{m+n} \to \times Y$, let $\sigma_1 = p_1 \circ \sigma : \Delta^{m+n} \to X$ and $\sigma_2 = p_2 \circ \sigma : \Delta^{m+n} \to Y$. Then for $\phi \in S^m(X)$ and $\psi \in S^n(Y)$, we have
$$(\phi \times \psi)(\sigma) = \phi(\sigma_1|_{[v_0,\dots,v_m]})\psi(\sigma_2|_{[v_m,\dots,v_{m+n}]})$$

We could also extend this to a relative version
$$H^m(X;A) \times H^n(Y;B) \to H^{m+n}(X \times Y; (X \times B) \cup (A \times Y)).$$

Since $\times$ is bilinear, it factors through the tensor product (by the universal property) to give a linear map (also denoted by $\times$)
$$H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \to H^\bullet(X \times Y; R).$$

Proposition 4.53. *For any $a, c \in H^\bullet(X;R)$ and $b, d \in H^\bullet(y;R)$ with $c \in H^m(X;R)$ and $c \in H^n(Y;R)$ we have*

$$(a \times b) \cup (c \times d) = (-1)^{mn}(a \cup c) \times (b \cup d).$$

Proof. $(a \times b) \cup (c \times d) = (p_1^* a \cup p_2^* b) \cup (p_1^* c \cup p_2^* d) = (-1)^{mn} p_1^*(a \cup c) p_2^*(b \cup d) = (-1)^{mn}(a \cup c) \times (b \cup d)$. □

Example 4.54. Using Proposition 4.53 with induction on n, we can determine the cohoomlogy ring $H^\bullet(T^n)$ where $T = \mathbb{S}^1$. T^n is the product of n copies of $\mathbb{S}^1$ whose cohomology as a ring is $\mathbb{Z}[x]/x^2$; there are no interesting cup products. By the Künneth formula, the cohomology of T^n is the graded tensor product, as algebras, of n copies of $\mathbb{Z}[x]/x^2$ (note that all the cohomology groups involved are free). This is precisely the exterior algebra $\Lambda_{\mathbb{Z}}[x_1, \ldots, x_n] = \mathbb{Z}[x_1, \ldots, x_n]/(x_i^2, x_i x_j + x_j x_i)$, with each generator in degree 1. In particular, $H^k(T^n) \cong \Lambda_{\mathbb{Z}}^k(H^1(T^n)$ naturally, and under this isomorphism the cup product corresponds to the wedge product.

Remark 4.55. The above example is a special case of

1. Hopf's theorem which asserts that the cohomology algebra of an H-space is a Hopf algebra.
2. The structure theorem which asserts that a Hopf algebra over a field of characteristic 0 is a free skew-commutative graded algebra.

The following Künneth formula (in a sense) generalises the previous example.

Theorem 4.56. *The cross product $H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \to H^\bullet(X \times Y;R)$ is an isomorphism of rings if X and Y are cellular spaces and $H^k(Y;R)$ is a finitely generated free R-module for all k.*

We would resort to Theorem 4.27. We show Theorem 4.56 for finite cellular spaces. (For general cellular spaces,, we need Milnor additivity axiom as we mentioned before.) Consider the following functors:

$$h^n(X, A) = \oplus_{i+j=n}(H^i(X, A; R) \otimes_R H^j(Y;R)),$$
$$k^n(X, A) = H^n(X \times T, A \times Y; R).$$

We have $\phi : h^n(X, A) \to k^n(X, A)$ given by the cross product. So we need to show

1. $h^\bullet$ and $k^\bullet$ cohomology theories.
2. π is a natural transformation.

Proof. (of Theorem 4.56) First we check that $h^\bullet$ and $k^\bullet$ are cohomology theories. All axioms are easy to verify. A few words for exactness axiom. The exactness for $k^\bullet$ is trivial. For $h^\bullet$, it is where we use the freeness of $H^k(Y;R)$.

The naturality of ϕ with respect to maps between spaces is from the naturality of cup products. To show the naturality with respect to the coboundary maps, we have to check the following diagramme commutes. Note that we omit R in the diagramme.

$$\begin{array}{ccc} H^k(A)\times H^\ell(Y) & \xrightarrow{\delta\times\mathrm{id}} & H^{k+1}(X,A)\times H^\ell(Y) \\ \times\downarrow & & \downarrow\times \\ H^{k+\ell}(A\times Y) & \xrightarrow[\delta]{} & H^{k+\ell+1}(X\times Y, A\times Y) \end{array}$$

To check this, start with an element (a,b) represented by cocycles $\varphi\in S^k(A)$ and $\psi\in S^\ell(Y)$. Extend φ to a cochain $\overline{\varphi}\in S^k(X;R)$. Then (φ,ψ) maps rightward to $(\delta\overline{\varphi},\psi)$ and then downward to $p_X^*(\delta\overline{\varphi})\cup p_Y^*(\psi)$. On the other direction, (φ,ψ) maps downward to $p_X^*(\varphi)\cup p_Y^*(\psi)$ and then rightward to $\delta(p_X^*(\overline{\varphi})\cup p_Y^*(\psi)) = p_X^*(\delta\overline{\varphi})\cup p_Y^*(\psi)$.

Note that the symbol δ stands for either the chain level and the connecting homomorhism, which might be confusing. □

Example 4.57. Now it is more straightforward to show

$$H^\bullet(T^n) = \Lambda_{\mathbb{Z}}[x_1,\dots,x_n].$$

Similarly, one could also show

$$H^\bullet(\mathbb{S}^n\times\mathbb{S}^m) = \frac{[a_n,a_m]}{(a_n^2,a_m^2)}.$$

An alternative way to define cross and cup products

We outline a construction of cup products for cellular spaces. Note that by Eilenberg-Steenrod uniqueness, this cup product is the same as that of singular cohomology.

The cross product in this setting is quite natural. Start with that of chain level. Take cells $e^i\in X$ and $e^j\in Y$, then we could send it to the product cell $e^i\times e^j$ in $X\times Y$. (Note that, unlike simplices, products of cellular spaces are again cellular spaces.) One could extend this map by tensor product from $C_\bullet(X)\otimes C_\bullet Y$ to $C_\bullet(X\times Y)$. Then for a pair of cocycles z_1,z_2 of X and Y, it thus yields a cocycle $z_1\times z_2$. This defines

$$H^{ii}(X)\times H^j(Y)\to H^{i+j}(X\times Y).$$

One could check it is the same as our previous defined cross product.

Then by using the diagonal map $\Delta: X\to X\times X$, $x\mapsto(x,x)$, we can define the cup product as the composition

$$H^k(X)\times H^\ell(X)\xrightarrow{\times} H^{k+\ell}(X)\xrightarrow{\Delta^\bullet} H^{k+\ell}(X).$$

This is our previous cup product since

$$\Delta^\bullet(a\times b) = \Delta^\bullet(p_1^*(a)\cup p_2^*(b)) = a\cup b.$$

But unfortunately, this cup product is not defined at the level of cellular cochains, which prevents us to prove the properties. To resolve this issue, one need to find a cellular map which is homotopic to Δ. This is in general true for maps between cellular spaces. And for our case $X \to X \times X$, the map is actually a slight modification of P (called *Alexander-Whitney chain approximation*) used in the proof of graded commutativity of the cup product. We leave here and will not go through the detail.

4.2.3 Ljusternik-Schnirelmann category

Definition 4.58. The *Ljusternik-Schnirelmann category* $\mathrm{cat}(X)$ of a topological space X is defined to be the smallest integer k such that there is an open covering $\{U_i\}_{1\leq i\leq k}$ of X such that each inclusion $U_i \hookrightarrow X$ is null-homotopic (we say U_i is *contractible in* X), i.e homotopic to a constant map.

Example 4.59. Notice the subtly of the definition. $\mathbb{S}^{n-1}$ is contractible in $\mathbb{D}^n$, although $\mathbb{S}^{n-1}$ itself is not a contractible space.

Example 4.60. $\mathrm{cat}(\mathbb{S}^1) = 2$. Actually, for any suspension $\Sigma(X)$, $\mathrm{cat}(\Sigma(X)) \leq 2$ since it can be covered by two contractible sets $C_+(X)$ and $C_-(X)$.

Notice that $\mathrm{cat}(M) < \infty$ if M is a compact manifold since it can be covered with finitely many sets homeomorphic to open discs. And in fact $\mathrm{cat}(M) \leq \dim M + 1$. In general, there is no reason for $\mathrm{cat}(X)$ to be finite.

Definition 4.61. The *cup length* $\mathrm{Cl}(X)$ of a topological space X is defined to be

$$\mathrm{Cl}(X) := \max\{n|\exists \alpha_i \in H^m(X) \text{ with } m_i > 0 \text{ such that } \alpha_1 \cup \ldots \alpha_n \neq 0\}.$$

Proposition 4.62. *For any space X, we have* $\mathrm{Cl}(X) < \mathrm{cat}(X)$.

Proof. Suppose $\mathrm{cat}(X) = n$, so $X = \cup_{j=1}^n U_j$ with each U_k contractible in X. We denote the inclusion by i_k. Since i_k is nullhomotopic, its induced homomorphism $ik^\bullet$ can be decomposed as $H^\bullet(X) \to H^\bullet(\mathrm{pt}) \to H^\bullet(U_k)$. So when $q > 0$, $i_k^\bullet = 0 : H^q(X) \to H^q(U_k)$. By the cohomology exact sequence for the pair (X, U_k), we know $j_k^\bullet : H^q(X, U_k) \to H^q(X)$ is surjective. So for any $\xi_k \in H^\bullet](X)$, we have $\eta_k \in H^\bullet(X, U_k)$ such that $\xi_k = j_k^\bullet(\eta_k)$.

Now look at the commutative diagramme

$$\begin{array}{ccccc}
H^\bullet(X,U_1) & \times\cdots\times & H^\bullet(X,U_n) & \xrightarrow{\cup} & H^\bullet(X,\cup_{k=1}^n U_k) \\
\downarrow{\scriptstyle j_1^\bullet} & & \downarrow{\scriptstyle j_n^\bullet} & & \downarrow{\scriptstyle j^\bullet} \\
H^\bullet(X) & \times\cdots\times & H^\bullet(X) & \xrightarrow[\cup]{} & H^\bullet(X)
\end{array}$$

Hence $\xi_1 \cup \ldots \xi_n = j_1^\bullet(\eta_1) \cup \ldots \cup j_n^\bullet(\eta_n) = j^\bullet(\eta_1 \cup \ldots \cup \eta_n) = 0$. The last equality is because of the fact that $H^\bullet(X, \cup_{k=1}^n U_k) = H^\bullet(X, X) = 0$. □

Example 4.63. An n-torus T^n has $\mathrm{cat}(T^n) = n + 1$.

Example 4.64. So for any suspension, $\mathrm{Cl}(\Sigma(X)) = 1$ if X is not weakly contractible. This tells us that *the cup product does not commute with the suspension and hence is not a stable property.*

Ljusternik-Schnirelmann used the notion of cat to study critical points. Their main theorem is the following

Theorem 4.65. *Let M be a smooth connected compact manifold and $f : Mto\mathbb{R}$ be a smooth function. Then f has at least* $\mathrm{cat}(x)$ *critical points.*

Example 4.66. Any smooth function on T^2 has at least 3 critical points. We can construct a smooth function on T^2 with exactly 3 critical points. In fact, using the viewpoint in the proof of the following theorem, we can construct a vector field on torus with 3 singular points.

Actually, Ljusternik-Schnirelmann category is an example of general category (It has nothing to do with "category theory"). We assume X to be a locally contractible path connected space.

Definition 4.67. A *category* is an assignment $\nu : \mathfrak{P}(X) \to \mathbb{N}_0$ (where $\mathfrak{P}(X)$ denotes the set of all subsets in X, i.e. the power set, and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$) satisfying the following axioms:

- *Continuity:* for every $A \in \mathfrak{P}(X)$ there exists an open set $U \supset A$ such that $\nu(A) = \nu(U)$.
- *Monotonicity:* if $EA, \in \mathfrak{P}(X)$ with $A \subset B$ then $\nu(A) \leq \nu(B)$.
- *Subadditivity:* for any $A, B \in \mathfrak{P}(X)$ we have $\nu(A \cup B) \leq \nu(A) + \nu(B)$.
- *Naturality:* if $\phi : X \to Y$ is a homeomorphism then for any $a \in \mathfrak{P}(X)$, $\nu_Y(\phi(A)) = \nu_X(A)$.
- *Normalization:* $\nu(\varnothing) =$, and if $A = \{x0, \dots, x_n\}$ is a finite set then $\nu(A) =$.

To prove Theorem 4.65, we prove the following more general proposition first.

Proposition 4.68. *Let X be a locally contractible path connected compact metric space, and ϕ_t be a global flow on X. Suppose there exists a Lyapunov function $\Phi : X \to \mathbb{R}$ such that Φ strictly decreases along non-constant orbits of ϕ_t. Then Φ has at least $\nu(X)$ critical points where ν is any category.*

Proof. Let $X^c := \Phi^{-1}(-\infty, c]$. A critical value for Φ is a value such that $\Phi^{-1}(c)$ contains a constant orbit. If c is not critical, then for sufficiently small $\delta > 0$, we can find $t > 0$ such that

$$\phi_t(X^{c+\delta}) \subset X^{c-\delta},$$

since Φ strictly decreases away from the constant orbits. If c is a critical level and U is a neighbourhood of $\Phi^{-1}(c)$ then for small $\delta > 0$, we have, for $t > 0$

$$\phi_t(X^{c+\delta} \setminus U) \subset X^{c-\delta}.$$

By naturality and monotonicity, $\nu(X^{c+\delta} \setminus U) \leq \nu(X^{c-\delta})$.

For $j = 1, \ldots, N = \nu(X)$, let

$$c_j := \sup\{c | \nu(X^c) < j\}.$$

Then $c_1 = \min\{\Phi\}$ and $c_N = \max\{\Phi\}$. Note that c_j is a critical value of Φ for each j.

Now we want to prove either $c_j < c_{j+1}$ or $\Phi^{-1}(c_j)$ contains infinitely many critical points. If the latter happens, the theorem follows immediately, so we assume the latter does not happen. Suppose $\Phi^{-1}(c_j) = \{x_0, \ldots, x_n\}$. Then by the continuity axiom, there exists a neighbourhood U of $\{x_0, \ldots, x_n\}$ such that $\nu(U) = 1$.

Then by the subadditivity axiom, we have

$$\begin{aligned} \nu(X^{c_j+\delta}) &\leq \nu(X^{c_j+\delta} \setminus U) + 1 \\ &\leq \nu(X^{c_j-\delta}) + 1 \\ &\leq j. \end{aligned}$$

Hence $c_{j+1} \geq c_j + \delta > c_j$. Thus $\{c_1 < \cdots < c_N\}$ are $N = \nu(X)$ different critical points. This completes the proof. □

Now we can complete the proof of Theorem 4.65.

Proof. (of Theorem 4.65) Give M a Riemannian metric and let ∇f denote the gradient of f with respect to this metric, i.e. the unique vector field determined by

$$\langle (\nabla f)_p, V_p \rangle = df_p(V_p)$$

for every vector field V. The critical points of f are precisely the zeros of ∇f. Let ϕ_t be the associated flow of ∇f, i.e.

$$\frac{d\phi_t(p)}{dt} = -(\nabla f)_{\phi_t(p)}.$$

We claim that f is a Lyapunov function for ϕ_t:

$$\begin{aligned} \frac{d}{dt}(f \circ \phi_t(p)) &= df\left(\frac{d\phi_t(p)}{dt}\right) \\ &= \langle (\nabla f)_{\phi_t(p)}, \frac{d\phi_t(p)}{dt} \rangle \\ &= -\langle \frac{d\phi_t(p)}{dt}, \frac{d\phi_t(p)}{dt} \rangle \\ &\leq 0 \end{aligned}$$

with equality holds if and only if $\frac{d\phi_t(p)}{dt} = 0$, i.e. p is a critical point of f, and $\phi_t(p)$ is a constant orbit p. This completes the proof. □

4.2.4 Higher products

Later we will see that the linking number of two spheres $\mathbb{S}^p$ and $\mathbb{S}^q$ in $\mathbb{R}^{p+q+1}$ will be understood as the cup product of the cohomolgy ring of the complement $H^\bullet(\mathbb{R}^{p+q+1}\setminus(\mathbb{S}^p\cup\mathbb{S}^q))$. There are links with 3 components with each two of them are unlinked, but nonetheless all three are linked. The most famous example is the Borromean rings. This rather complicated linking phenomenon for three

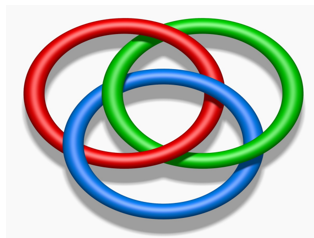


Figure 4.2: Borromean rings

or more spheres suggests the existence of a higher cup product: the Massey product. We start with Massey triple product.

Assume $[u]$, $[v]$, $[w]$ are cohomology classes of dimension p, q and r respectively, represented by $u \in Z^p(X)$, $v \in Z^q(X)$ and $w \in Z^r(X)$. If $[u] \cup [v] = 0 = [v] \cup [w]$, then we introduce a (set of) new cohomology classes. For the notation, we introduce $\overline{u} = (-1)^{1+\deg u}u$.

Since $[u][v] = 0$, we have $s \in C^{p+q+1}(X)$ such that $\delta s = \overline{u} \cup v$. Similarly, we have $t \in C^{q+r-1}(X)$ such that $\delta t = \overline{y} \cup w$. The element $\overline{s} \cup w + \overline{u} \cup t$ determines a cocycle in $X^{p+q+r-1}(X)$:

$$\begin{aligned}\delta(\overline{s} \cup w + \overline{u} \cup t) &= (-1)^{p+q}\delta s \cup w + (-1)^p \overline{u} \cup \delta t \\ &= (-1)^{p+q}\overline{u} \cup v \cup w + (-1)^{p+q+1}\overline{u} \cup v \cup w = 0\end{aligned}$$

We define the Massey triple product as the set of all such cohomology classes

$$\langle [u], [v], [w] \rangle = \{[\overline{s} \cup w + \overline{u} \cup t] | \delta s = \overline{u} \cup v, \delta t = \overline{v} \cup w\}.$$

There are indeterminacy from the different choices of representatives. But we can identify them as the following.

Proposition 4.69. *The Massey triple product* $\langle [u], [v], [w] \rangle$ *is an element of the factor group* $H^{p+q+r-1}(X)/([u] \cup H^{q+r-1}(X) + H^{p+q-1}(X) \cup [w])$.

Proof. We need to show different choices of u, v, w, s, t do not affect the coset in $H^{p+q+r-1}(X)$ given above. We only check that of s, as other cases can be done in a similar fashion.

If s and s' are chosen such that $\delta s = \overline{u} \cup v = \delta s'$, then

$$(\overline{s} \cup w + \overline{u} \cup t) - (\overline{s}' \cup w + \overline{u} \cup t) = (\overline{s} - \overline{s}' \cup w,$$

which resides in $H^{p+q-1}(X) \cup [w]$ as a cohomology class. □

The Massey product was used to prove the Jacobi identity for Whitehead product in homotopy groups.
Remark by the transcriber: A purely homotopy-theoretic argument not resorting to cohomology is available: see "Cellular Homotopy", M. M. Postnikov.

We can define higher order Massey products. When two triple products $\langle [u], [v], [w] \rangle$ and $\langle [v], [w], [x] \rangle$ are defined, and if $0 \in \langle [u], [v], [w] \rangle$ and $0 \in \langle [v], [w], [x] \rangle$, then we can find

$$\delta Y_1 = \bar{t}_0 \cup w + \bar{u} \cup t_1, \quad \delta Y_2 = \bar{t}_1 \cup x + \bar{v} \cup t_2$$

where $\delta t_0 = \overline{u} \cup v$, $\delta t_1 = \overline{v} \cup w$, $\delta t_2 = \overline{w} \cup x$. Then we form a subset $\langle [u], [v], [w], [x] \rangle$ in $H^{|u|+|v|+|w|+|x|-2}(X)$ whose elements are

$$\overline{u} \cup Y_2 + \bar{t}_0 \cup t_2 + \overline{Y}_1 \cup x.$$

This is called a *fourfold product.*

It may be better to understand the whole picture by matrices

$$\begin{bmatrix} u & s & \\ & v & t \\ & & w \end{bmatrix} \quad \begin{bmatrix} u & t_0 & Y1 & \\ & v & t_1 & Y_2 \\ & & w & t_2 \\ & & & x \end{bmatrix}$$

We can inductively define n-fold Massey product: $\langle [a_{1,1}, [a_{2,2}, \ldots, [a_{nn}] \rangle$ to be the set of elements of the forms

$$\overline{a}_{1,1} a_{2,n} + \overline{a}_{1,2} a_{3,n} + \ldots + \overline{a}_{1,n-1} a_{n,n}$$

for all solutions of the equations

$$\delta a_{i,j} = \overline{a}_{i,i} a_{i+1,j} + \overline{a}_{i,i+1} a_{i+2,j} + \ldots + \overline{a}_{i,j-1} a_{j,j}, \quad 1 \le i \le j \le n, \ (i,j) \ne (1,n)$$

Hence, to ensure the set is non-empty, we need the vanishing of many lower order Massey product.

We remark that the Massey products are defined for (homology) of a diferential graded algebra (DGA) A. It is a graded algebra $A = \oplus_{k \ge 0} A^k$ with a differential $d : A \to A \to A$ of degree $+1$, such that

1. A is graded commutative, i.e,

$$x \cdot y = (-1)^{k\ell} y \cdot x, \quad x \in A^k. \ y \in A^\ell$$

2. d is a derivation, i.e.

$$d(x \cdot y) = dx \cdot y + (-1)^k x \cdot y, \quad x \in A^k,$$

3. $d^2 = 0$.

Examples include cohomology ring with 0 as its differential and de Rham complex $(\Omega^\bullet, d)$ on a manifold.

Rational homotopy theory of Quillen and Sullivan is built to understand (real) homotopy groups by the structure of DGA. For example, a manifold on which all Massey products vanish is a *formal manifold*: its real homotopy type follows ("formally") from its real cohomology ring. Deligne-Griffiths-Morgan-Sullivan proved that all Kähler manifolds are formal: see "Real Homotopy Theory of Kähler Manifolds", Inventiones Mathematicae, Volume: 29, No. 3, Year: 1975, Pages: 245-274.

4.3 Poincaré Duality

We will prove the Poincaré duality in this section. For a compact n-manifold M without boundary, this asserts that $H^p(M^n)$ is isomorphic to $H_{n-p}(M^n)$. It is the most important result in this course, and has lots of important applications. Poincaré's original proof used the idea of dual cell structures and rather intuitive and geometrically intricate. Unfortunately, we will lose some generality if we use this method.

Hence, we use the proof by Milnor. The basic idea of Milnor's proof is very natural as explained bellow. We know that any n-manifold is a union of open subsets, each of which is homeomorphic to $\mathbb{R}^n$. It is natural to first prove (certain version of) the theorem for $\mathbb{R}^n$, and then use Mayer-Vietoris sequences to prove the case of a finite union of open subsets. Finally, it passes to the case of an infinite union by a direct limit argument. We will then state and prove a more general version which is applicable to noncompact manifolds since we have to first deal with $\mathbb{R}^n$. For this reason, we need to introduce cohomology with compact supports.

4.3.1 Cohomology with compact supports

Let M be a topological space. The *singular cochains with compact support* on M is defined as $\alpha \in S^p(M)$ such that there is a compact set $K \subset M$ such that $\alpha \in S^p(M, M \setminus K) \subset S^p(M)$, i.e. $\alpha|_{M\setminus K} = 0$. Write $S_c^p(M)$ for the set of all these cochains. Note that δ preserves $S_c^\bullet(M)$. Hence

Definition 4.70. $H_c^p(M) := H^p(S_c^\bullet(M))$ is the cohomology with compact support of M.

Observe that if M is compact $H_c^\bullet(M) = H^\bullet(M)$. We will need to calculate $H_c^\bullet(M)$ in general for the proof of Poincaré duality. We would like to understand it by relative cohomology groups. So we introduce the definition of the direct limit.

Definition 4.71. A *directed system of abelian groups* $\{G_a | a \in A\}$ is a collection of abelian groups indexed by a partially ordered set A satisfying

1. For all $a, b \in A$ there exists $c \in A$ such that $a \leq c$ and $b \leq c$.

2. For all $a \le b$ there exists a homomorphism $f_{ab} : G_a \to G_b$ such that $f_{aa} = \mathrm{id}$ and if $a \le b \le c$, $f_{ac} = f_{bc} \circ f_{ab}$.

Recall that a partially ordered set (or poset) is a set A along with a binary relation $\le$ which is reflxive, antisymmetric and transitive:

1. Reflexive: $a \le a$;
2. Antisymmetric: if $a \le b$ and $b \le a$, then $a = b$;
3. Transitive: if $a \le b$ and $b \le c$, then $a \le c$.

A partially ordered set with the property 1 in Definition 4.71 is called a *directed set*. The main example of directed sets in our mind is the set of compact subsets (of a set) with partial ordering $\subset$ and f is the inclusion.

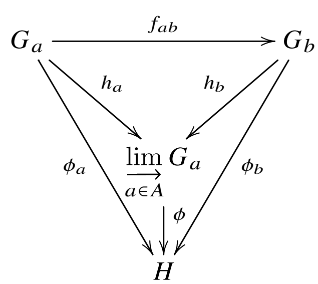
Definition 4.72. Given a direct system $\{G_a | a \in A\}$, the *direct Limit* (or *colimit*) is defined to be

$$\varinjlim_{a\in A} G_a := (\oplus_{a\in A} G_a)/N$$

where $N \subset \oplus_{a\in A} G_a$ is the subgroup generated by $x - f_{ab}(x)$ where $x \in G_a$ and $a \le b$.

The direct limit has the following universal property which actually characterises the direct limit.

Proposition 4.73 (The universal property of direct limits)**.** *Any homomorphisms $\phi_a : G_a \to H$ such that $\phi_a = \phi_b \circ f_{ab}$ for any pair $a \le b$ factor through $\varinjlim_{a\in A} G_a$. That is, there exists a unique homomorphism $\phi : \varinjlim_{a\in A} G_a \to H$ such that $\phi \circ h_a = \phi_a$ for all $a \in A$. Here $h_a : G_a \to \varinjlim_{a\in A} G_a$ are the inclusion maps.*



Proof. For any $g \in G_a$, we denote its equivalence class in $\varinjlim_{a\in A} G_a$ as $[g]$. So $h_a(g) = [g]$.

Let us first construct a $\phi : [g] \mapsto \phi_a(g)$. It is easy to check that it makes the diagramme commutes. Let us check it is well defined. Suppose there are $g_1 \in G_a$, $g_2 \in G_b$ such that $[g_1] = [g_2]$. Then by definition there is a c such that $a \le c$ and $b \le c$, and $f_{ac}(g_1) = f_{bc}(g_2)$. Then we see that

$$\phi_a(g_1) = \phi_c \circ f_{ac}(g_1) = \phi_c \circ f_{bc}(g_2) = \phi_b(g_2).$$

Then we prove the uniqueness. Suppose there is another $\phi' : \varinjlim_{a\in A} G_a \to H$ such that $\phi' \circ h_a = \phi_a$ for all $a \in A$. Then

$$\phi'([g]) = \phi' \circ h_a(g) = \phi_a(g) = \phi([g]).$$

In other words, $\phi' = \phi$. □

Now we can give an alternative definition of $H_c^\bullet$ in terms of direct limit. The compact subsets $K \subset M$ form a directed set under inclusion. For $K \subset L$, we have the inclusion $(M, M \setminus L) \to (M, M \setminus K)$, and thus the homomorphism $H^p(M, M \setminus K) \to H^p(M, M \setminus L)$. Hence we have the direct limit

$$\varinjlim_{K\subset M} H^p(M, M \setminus K).$$

We sow that this is equal to $H_c^p(M)$ we defined at beginning. It is easy to see that $H_c^p(M) \subset \varinjlim_{K\subset M} H^p(M, M\setminus K)$ by definition. To see the reverse inclusion, note hat each element $\varinjlim_{K\subset M} H^p(M, M \setminus K)$ is represented by a cocycle in $S^p(M, M \setminus K)$ for some compact K, hence the inclusion at cochain level. And such a cocycle is zero in $\varinjlim_{K\subset M} H^p(M, M\setminus K)$ if and only if it is in $B^p(M, M\setminus L)$ for some compact $L \supset K$, hence the inclusion passes to the cohomology level.

Example 4.74. We compute $H_c^\bullet(\mathbb{R}^n)$. Since every compact subset of $\mathbb{R}^n$ is contained in a closed ball $\mathbb{D}_R(0)$ of some radius $R \in \mathbb{N}$, we have

$$\varinjlim_{R\in\mathbb{N}} H^p(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{D}_R(0)) = \varinjlim_{K\in\mathbb{R}^n} H^p(\mathbb{R}^n, \mathbb{R}^n \setminus K)$$

Now for any $R > 0$, $\mathbb{R}^n \setminus \mathbb{D}_R(0)$ is homotopy equivalent to $\mathbb{S}^{n-1}$. Hence the long exact sequence for pairs $(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{D}_R(0))$ gives us

$$H^m(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{D}_R(0)) = \begin{cases} \mathbb{Z} & m = n \\ 0 & m \neq n \end{cases}$$

Since the map $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{D}_R(0)) \to H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{D}_{R+1}(0))$ corresponds to the inclusions are isomorphism, we conclude

$$H_c^m(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & m = n \\ 0 & m \neq n \end{cases}$$

This example tells us that $H_c^\bullet(M)$ is *not* a cohomology theory in the sense of Eilenberg-Steenrod. In fact it is not a homotopy invariant: a one-point space is compact, so $H_c^\bullet(pt) = H^\bullet(\mathrm{pt})$, but $H_c^\bullet(\mathbb{R}^n) \neq H^\bullet(\mathbb{R}^n)$. Actually, it is not even a functor: the constant map $\mathbb{R}^n \to \mathrm{pt}$ does not induce a map on cohomology with compact support.

4.3.2 Orientations for Manifolds

Let M be an n-manifold. For each $x \in M$, choose an open ball U with $x \in U$. Then by excision, we have

$$H_n(M, M \setminus x) \simeq H_n(U, U \setminus x) = \mathbb{Z}.$$

Definition 4.75. A *local orientation* μ_x *for* M *at* x is a choice of one of the two possible generators for $H_n(M, M \setminus x)$.

If $x, y \in U$ then we have homomorphisms ρ_x and ρ_y induced by the inclusion of pairs

$$H_n(M, M \setminus x) \xleftarrow{\rho_x} H_n(M, M \setminus U) \xrightarrow{\rho_y} H_n(M, M \setminus y).$$

So a generator for $H_n(M, M \setminus U)$ gives a local orientation at any point in U.

Definition 4.76. An *orientation* of M is a function $x \mapsto \mu_x$ subject to the following continuity condition: given any point $x \in M$, there exists a neighbourhood U of x and an element $\mu_u \in H_n(M, M \setminus U)$ such that $\rho_y(\mu_U) = \mu_y$ for each $y \in U$.

We say M is *orientable* if there exists an orientation. And if the orientation is fixed, M is *oriented*.

One could translate this into the connectedness of the double cover

$$\widetilde{M} = \{\mu_x | x \in M, \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

We will not go through this construction, see §VI-7 of "Topology and Geometry" by G. Bredon for details. For a "compensation" we summarise two useful criteria.

Proposition 4.77. *1. If M is simply connected, then M is orientable.*

2. Suppose $H^1(M; \mathbb{Z}/2\mathbb{Z}) = 0$, then M is orientable.

Example 4.78. 1. $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{C}P^n$ are orientable.

2. If M and N are orientable, then $M \times N$ is orientable.

To determine which manifolds are not orientable, we have the following lemma. For the proof, we need relative Mayer-Vietoris sequence:

$$\cdots \to H_n(X, A \cap B) \to H_n(X, A) \oplus H_n(X, B) \to H_n(X, A \cup B) \to H_{n-1}(X, A \cap B) \to \cdots$$

Lemma 4.79. *Let M be a n-manifold, and $K \subset M$ be a compact subset. Then*

1. *$H_i(M, M \setminus K = 0$ for $i > n$;*

2. *Suppose $x \mapsto a_x$ is an orientation of M. Then there is a unique class $a_K \in H_n(M, M \setminus K)$ whose image in $H_n(M, M \setminus x)$ is a_x for all $x \in K$.*

Proof. We use the relative Mayer-Vietoris sequence for a triple (X, A, B):

$$\cdots \to H_p(X, A \cap B) \to H_p(X, A) \oplus H_p(X, B) \to H_p(X, A \cup B) \to \cdots$$

We break down our proof in 4 steps:

Step 1 Suppose the lemma holds for K_1, K_2 and $K_1 \cap K_2$. We want to prove it holds for $K_1 \cup K_2$ as well. Then taking $X = M$, $A = M \setminus K_1$, $B = M \setminus K_2$ we have

$$\cdots \to H_{p+1}(M, M\setminus(K_1 \cap K_2)) \to H_p(M, M\setminus K) \to H_p(M, M\setminus K_1) \oplus H_p(M, M\setminus K_2) \to \cdots$$

By assumption, if $p > n$ the left side and right side terms are both zero, so $H_p(M, M \setminus K) = 0$. For the second statement, we know the map $H_n(M, M \setminus K_1) \oplus H_n(M, M \setminus K_2) \to H_n(M, M \setminus (K_1 \cap K_2))$ is the difference map $a_{K_1} - a_{K_2}$. By uniqueness, it has to be zero. So we have a_K from the exact sequence. This is unique because, $H_{n+1}(M, M \setminus (K_1 \cap K_2)) = 0$.

Step 2 We reduce the problem to the case $M = \mathbb{R}^n$. Any compact set $K \subset M$ can be written as $K_1 \cup \cdots \cup K_m$ where each K_i is contained in a neighbourhood which is homeomorphic to a ball in $\mathbb{R}^n$. Then applying step 1 and induction on $K_1 \cup \cdots \cup K_{m-1}$, K_m and their intersection.

Step 3 Suppose $M = \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ is a compact convex subset. For any point $x \in K$, let S be a large $(n-1)$-sphere with centre x. Then S is a deformation retract of both $\mathbb{R}^n \setminus x$ and $\mathbb{R}n \setminus K$. Hence the map

$$H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K) \to H_i(\mathbb{R}^n, \mathbb{R}^n \setminus x)$$

is an isomorphism for each i.

Induction also shows that the lemma holds when K is a finite union of compact convex sets.

Step 4 Now suppose $K \subset \mathbb{R}^n$ is an arbitrary compact subset and $\beta \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$. We choose a relative cycle z with $[z] = \beta$. Let $C \in \mathbb{R}^n \setminus K$ be the union of the images of the boundary of singular simplices in z. C is compact (since z is closed in the absolute sense), the distance from K to C is some real number $\delta > 0$.

Cover K by finitely many balls with centres in K and radii $< \delta$. Let N be the union of these balls and so $K \subset N$ and z defines a class $\beta_N \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ such that the restriction $\rho_K(\beta_N) = \beta$.

If $i > n$ then by step 3, $\beta_N = 0$ so $\beta = 0$. This, together with step 2, finishes the first part of the lemma.

If $i = n$, 1 and step 3 also construct a_N and then $a_K = \rho_K(a_N)$ such that $\rho_x(a_N) = a_x$ and $\rho_x(a_K) = a_x$. We prove the uniqueness: if $a_K^{'}$ is another choice, let $\beta = a_K - a_K^{'}$. Then $\rho_x(\beta) = 0$ for any $x \in K$, especially when x is one of the centres of the balls to define N. By step 3 again, β is zero on these balls and thus on N. Hence $a_K - a_K^{'} = \beta = 0 \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus K)$.

□

When M is closed (i.e. compact without boundary), taking $K = M$ and we have

Corollary 4.80. *Suppose M is a connected closed n-manifold. Then*

1. $H_i(M) = 0$ *if* $i > n$;

2. *M is orientable if and only if $H_n(M) = \mathbb{Z}$. If M is not orientable, then $H_n(M) = 0$.*

Proof. We need some remarks for non-orientable case. If $H_n(M) \neq 0$, take a cycle $z \neq 0$. We take a cell decomposition of M. Then at two sides of any $(n-1)$-dimensional cell, the coefficient of z is the same. Since M is connected, the coefficient on all n-cells are the same. This gives us an orientation on M. □

Example 4.81. $\mathbb{R}P^{2n}$ is not orientable, since $H_{2n}(\mathbb{R}P^{2n}) = 0$. $\mathbb{R}P^{2n+1}$ is orientable since $H_{2n+1}(\mathbb{R}P^{2n+1}) = \mathbb{Z}$.

In particular, if M itself is compact, then there is one and only one $\mu_M \in H_n(M)$ with the required property. This class $\mu = \mu_M$ is called the *fundamental homology class* of M.

4.3.3 Poincaré duality theorem

The Poincaré duality for compact manifolds can be stated now.

Theorem 4.82. *Let M be a compact and oriented n-manifold, then the homomorphism*

$$D : H^p(M) \to H_{n-p}(M), \quad \alpha \mapsto \alpha \cap \mu_M$$

is an isomorphism.

It actually follows from a more general theorem (which we will prove), for any oriented manifolds. Before stating the result, we need to explain the notations.

First observe that for any pair (X, A), the cap product gives rise to a paring

$$S^i(X, A) \otimes S_n(X, A) \to S_{n-i}(X)$$

and hence to a pairing

$$H^i(X, A) \otimes H_n(X, A) \to H_{n-i}(X).$$

For oriented M, we define the duality map

$$D : H_c^p(M) \to H_{n-p}(M)$$

as follows. For any $a \in H_c^p(M) = \varinjlim H^p(M, M \setminus K)$, choose a representative $a' \in H^p(M, M \setminus K)$ and set

$$D(a) = a' \cap \mu_K.$$

This is well defined since for $K \subset L$, we have the restriction

$$\rho_K : H_n(M, M \setminus L) \to H_n(M, M \setminus K)$$

with $\rho_K(\mu_L) = \mu_K$. Then the naturality of the cap product tells us that the following diagramme commutes:

$$\begin{array}{ccc} H^i(M, M \setminus K) & \longrightarrow & H^i(M, M \setminus L) \\ & {\scriptstyle \cap\mu_K} \searrow \quad \swarrow {\scriptstyle \cap\mu_L} & \\ & H_{n-i}(M) & \end{array}$$

Theorem 4.83. *Let M be an oriented n-manifold, then the homomorphism*

$$D : H_c^p(M) \to H_{n-p}(M)$$

is an isomorphism.

Proof. We break down our proof in 5 steps:

Step 1 We first prove the theorem for the case $M = \mathbb{R}^n$. Given a closed ball $B \subset \mathbb{R}^n$, we know that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) = \mathbb{Z}$ with generator μ_B. Hence $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B) = \mathbb{Z}$ and by the universal coefficient theorem, the homomorphism $h : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B; \mathbb{Z}) \to \hom(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B; \mathbb{Z})$ is an isomorphism. Then there exists a generator a such that $\langle a, \mu_B \rangle = 1$. Now the identity

$$\langle 1 \cup a, \mu_B \rangle = \langle 1, a \cap \mu_B \rangle$$

shows that $a \cap \mu_B$ is a generator of $H_0(\mathbb{R}^n) = \mathbb{Z}$. Thus $\cap\mu_B$ gives an isomorphism $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \to H_0(\mathbb{R}^n)$ for all B. Hence by the universal property of direct limit, the map D is an isomorphism in the case $i = n$. The cases $i \neq n$ is obvious since it maps 0 to 0.

Step 2 Suppose $M = U \cup V$ and that the theorem holds for U, V and $U \cap V$. We first construct Mayer-Vietoris sequence for H_c^*:

$$\cdots \to H_c^{p-1}(M) \to H_c^p(U \cap V) \to H_c^p(U) \oplus H_c^p(V) \to H_c^p(M) \to \cdots$$

This is obtained from relative Mayer-Vietoris sequence

$$H^p(M, M \setminus (K \cap L)) \to H^p(M, M \setminus K) \oplus H^p(M, M \setminus L) \to H^p(M, M \setminus (K \cup L))$$

and excisions

$$\begin{aligned} H^p(M, M \setminus (K \cap L)) &= H^p(U \cap V, (U \cap V) \setminus (K \cap L)) \\ H^p(M, M \setminus K) &= H^p(U, U \setminus K) \\ H^p(M, M \setminus L) &= H^p(V, V \setminus L) \end{aligned}$$

Now if we know the following diagramme of exact sequence is commutative (up to sign), then Five lemma will finish this step.

$$\begin{array}{ccccccc} H_c^p(U \cap V) & \longrightarrow & H_c^p(U) \oplus H_c^p(V) & \longrightarrow & H_c^p(M) & \xrightarrow{\delta} & H_c^{p+1}(U \cap V) \\ \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D \\ H_{n-p}(U \cap V) & \longrightarrow & H_{n-p}(U) \oplus H_{n-p}(V) & \longrightarrow & H_{n-p}(M) & \xrightarrow[\partial]{} & H_{n-p-1}(U \cap V) \end{array}$$

The left and the middle squares are easily seen to commute at the chain level. Much less simple is the rightmost square, which we will show commutes up to sign.

Notice we only need to show the following is commutative

$$\begin{array}{ccc} H^p(M, M\setminus(K\cup L)) & \xrightarrow{\delta} & H^{p+1}(U\cap V, (U\cap V)\setminus(K\cap L)) \\ \downarrow{\cap\mu_{K\cup L}} & & \downarrow{\cap\mu_{K\cup L}} \\ H_{n-p}(M) & \xrightarrow[\partial]{} & H_{n-p-1}(U\cap V) \end{array}$$

Let $A = M \setminus K$ and $B = M \setminus L$. Then the map δ is obtained from the short exact sequence

$$0 \to S^*(M, A) \cap S^*(M, B) \to S^*(M, A) \oplus S^*(M, B) \to S^*(M, A \cap B) \to 0.$$

We use the fact $S^*(M, A \cup B) \to S^*(M, A) \cap S^*(M, B)$ induces an isomorphism on cohomology. For a cocycle $\phi \in S^*(M, A \cap B)$, we write $\phi = \phi_A - \phi_B$ for $\phi_A \in S^*(M, A)$ and $\phi_B \in S^*(M, B)$. Then $\delta[\phi]$ is represented by the cocycle $\delta\phi_A = \delta\phi_B \in S^*(M, A) \cap S^*(M, B)$. Similarly if $z \in S_*(M)$ represents a homology class then $\partial[z] = [\partial z_U]$, where $z = z_U - z_V$ with $z_U \in S_*(U)$ and $z_V \in S_*(V)$.

Via barycentric subdivision, the class $\mu_K \cup L$ can be represented by a chain α which is the sum $(\alpha = \alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K})$ of chains in three open sets $U \setminus L$, $U \cap V$, and $V \setminus K$ respectively. By uniqueness of $\mu_{K\cap L}$ the chain $\alpha_{U\cap V}$ represents $\mu_{K\cap L}$, since the other two chains lie in the complement of $K \cap L$. Similarly the chain $\alpha_{U\setminus L} + \alpha_{U\cap V}$ represents μ_K.

Now let ϕ be a cocycle representing an element in $H^p(M, M \setminus (K \cup L))$. By δ, it maps to $\delta\phi_A$. Continuing downward to the bottom, we obtain $\delta\phi_A \cap \alpha_{U\cap V}$, which represents the same homology class as $(-1)^{n-p-1}\phi_A \cap \partial\alpha_{U\cap V}$, since

$$\partial(\phi_A \cap \alpha_{U\cap V}) = (-1)^{n-p}\delta phi_A \cap \alpha_{U\cap V} + \phi_A \cap \partial\alpha_{U\cap V}.$$

For the other way, ϕ is first mapped to $\phi \cap \alpha \in H_{n-p}(M)$. Write is as a sum of a chain in U and a chain in V:

$$\phi \cap \alpha = \phi \cap \alpha_{U\setminus L} + \phi \cap (\alpha_{U\cap V} + \alpha_{V\setminus K})$$

and by definition $\partial[\phi \cap \alpha] = [\partial(\phi \cap \alpha_{U\setminus L})] \in H_{n-k-1}(U \cap V)$. Now we have

$$\partial(\phi \cap \alpha_{U\setminus L}) = \phi \cap \partial\alpha_{U\setminus L} = \phi_A \cap \partial\alpha_{U\setminus L} = -\phi_A \cap \partial\alpha_{U\cap V}.$$

The second equality is because ϕ_B is zero on $M \setminus L$. The last equality follows from $\alpha_{U\setminus L} + \alpha_{U\cap V} = \mu_K$ which is a chain in $U \setminus K$. This completes step 2.

Step 3 Suppose M is the union of a direct system of open subsets $\{U_i\}_{i\in I}$ with the property that if K is a compact subset of M then K is contained in some U_i. Then if we have

$$H_c^p(M) = \varinjlim_{i\in I} H_c^p(U_i), \quad H_{n-p}(M) = \varinjlim_{i\in I} H_{n-p}(U_i),$$

and know the theorem hodls for each U_i, then the theorem follows for M since the direct limit preserves isomorphisms.

We prove $H_p(M) = \varinjlim_{i\in I} H_p(U_i)$, as the other follows by the same argument. By the universal property of directed limit, we have a homomorphism

$$\varinjlim_{i\in I} H_p(U_i) \to H_p(M).$$

We show it is surjective: if $z \in S_p(M)$ is a cycle, then there exists a compact set such that $[z] \in \operatorname{im}(H_p(K) \to H_p(X))$. Assume $K \subset U_i$. Then $[z] \in \operatorname{im}(H_p(U_i) \to H_p(M)$ and so $[z] \in \operatorname{im}(\varinjlim_{i\in I} H_p(U_i) \to H_p(M))$.

T show the injectivity, take a cycle z in U_i and assume it is a boundary of K in X. Take j such that $K \subset U_j$, then its inclusion into $\varinjlim_{i\in I} H_p(U_i)$ is 0.

Step 4 Suppose M is an open subset of $\mathbb{R}^n$. If M is convex, it follows from step 1 since then M is homeomorphic to $\mathbb{R}^n$. We can find convex open sets $V_1, V_2, \dots$ such that $M = \cup_{i=1}^{\infty} V_i$ (for example, take open discs whose centres have rational coordinates). Then by step 2, the theorem holds for $V_1 \cup \cdots \cup V_r$ for each r. And by step 3 the theorem holds for $\cup_r(\cup_{i=1}^r V_i) = \cup_{i=1}^{\infty} V_i = M$.

Step 5 M is arbitrary. Consider the family of all open subsets U of M such that Poincaré duality holds for U. This family is nonempty. In view of step 3, we can apply Zorn's lemma to this family to choose a maximal open set V belonging to it. If $V \neq M$, then there is an open subset $B \subset M$ such that B is homeomorphic to $\mathbb{R}^n$, and B is not contained in V. We apply step 2 and step 4 (for the intersection) to conclude Poincaré duality also holds for $V \cup B$, contradicting the maximality of V. Thus $V = M$.

□

We also have Poncaré duality for non-orientable manifolds, but only for $\mathbb{Z}/2\mathbb{Z}$ coefficient. Let M be an arbitrary n-manifold. For each point $x \in M$, μ_x denotes the unique non-zero element of the local homology group $H_n(M, M \setminus x; \mathbb{Z}/2\mathbb{Z})$. And for each compact subset K, the same argument as Lemma 4.79 gives us the unique element μ_K of $H_n(M, M \setminus K; \mathbb{Z}/2\mathbb{Z})$ such that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$. Now we define a homomorphism

$$H^p(M, M \setminus K; \mathbb{Z}/2\mathbb{Z}) \to H_{n-p}(M; \mathbb{Z}/2\mathbb{Z}), \quad x \mapsto x \cap \mu_K.$$

This induces the homomorphism

$$D_2 : H_c^p(M; \mathbb{Z}/2\mathbb{Z}) \to H_{n-p}(M; \mathbb{Z}/2\mathbb{Z}).$$

Theorem 4.84. *For any n-manifold M, the homomorphism*

$$D_2 : H_c^p(M; \mathbb{Z}/2\mathbb{Z}) \to H_{n-p}(M; \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism.

4.4 Applications of Poincaré Duality

4.4.1 Intersection form, Euler characteristic

Intersection pairing

Let M be a closed connected orientable n-manifold. We let $\mu \in H_n(M)$ be the orientation, i.e. the unique element such that the image of μ in $H_n(M, M \setminus x)$ is a generator.

Now we have a pairing on cohomology ring induced by cup product:

$$\langle , \rangle : H^k(M) \times H^{n-k}(M) \xrightarrow{\cup} H^n(M) \xrightarrow{\cap \mu} \mathbb{Z}.$$

In other words, $\langle a, b \rangle = (a \cup b)(\mu)$. This map is called the *intersection form.* One could also define the pairing on homology by taking Poincaré duality.

Poincaré duality simply tells us that the intersection form is non-singular when we take the free part.

Corollary 4.85. *Suppose we take coefficients in a field F. Then the intersection form*

$$\langle , \rangle : H^k(M; F) \times H^{n-k}(M; F) \to F$$

is non-singular. We have the same conclusion if we look at the the pairing

$$\frac{H^k(M; \mathbb{Z})}{\text{Tor}} \times \frac{H^{n-k}(M; \mathbb{Z})}{\text{Tor}} \to \mathbb{Z}.$$

Here "the intersection form is nonsingular" means both $\langle \alpha, \cdot \rangle$ and $\langle \cdot, \alpha \rangle$ are isomorphisms if α is non-zero.

Proof. Consider the composition

$$H^{n-k}(M; R) \xrightarrow{h} \text{Hom}_R(H_{n-k}(M; R), R) \xrightarrow{D^*} \text{Hom}_R(H^k(M; R), R)$$

Recall that h is an isomorphism for the above two cases. And here D^* is the Hom-dual of the Poincaré duality map $D : H^k \to H_{n-k}$. Note that

$$D^*(h(\alpha))(\beta) = \alpha \cap (\beta \cap \mu) = (\alpha \cup \beta)(\mu) = \langle \alpha, \beta \rangle.$$

Since both D^* and h are isomorphisms, their composition is an isomorphism and hence the intersection form is non-singular. □

Later we will just write $H^m(M)$ for the free part if there will arise no confusion.

Corollary 4.86. *Let M be an even dimensional ($n = 2m$) orientable manifold. Then the pairing*

$$H^m(M) \times H^m(M) \to \mathbb{Z}$$

is unimodular: if we choose a basis $u_1, \ldots, u_k \in H^m(M)$, then the matrix $A = (a_{ij})$ with $a_{ij} = \langle u_i, u_j \rangle$ has $\det A = \pm 1$.

And when m is even, it is symmetric; m is odd, it is anti-symmetric.

Proof. The second conclusion is easy, so we will prove only the first one. Take a basis $u_1, \ldots, u_k \in H^m(M)$, then we know there is a dual basis $v_1, \ldots, v_k$ such that $\langle u_i, v_j \rangle = \delta_{ij}$. Let $A = (a_{ij})$ where $a_{ij} = \langle u_i, u_j \rangle$. Let B be the matrix of base change: $v_j = \sum_k b_{kj} u_k$. Then

$$\delta_{ij} = \langle u_i, v_j \rangle = \sum_k b_{kj} \langle u_i, u_k \rangle = \sum_k a_{ik} b_{kj},$$

i.e. $AB = I$. Since both A, B are of $\mathbb{Z}$ coefficients, $\det A = \pm 1$. □

Now, if an orientable M has dimension $4m$, the intersection pairing is a symmetric unimodular bilinear form. So the eigenvalues are all real numbers. We will denote the numbers of its positive and negative eigenvalues by b_{2m}^+ and b_{2m}^- respectively. Their sum is the Betti number b_{2m}. Their difference is an important invariant called *signature*, denote by

$$\sigma(M) = b_{2m}^+ - b_{-2m}^-.$$

For orientable manifolds of dimensions other than $4m$, we define $\sigma(M) = 0$.

There is another viewpoint of the intersection pairing, from homology. If $x, y \in H_*(M)$, $\xi, \eta \in H^*(M)$ and $x = D\xi$, $y = D\eta$. Then we define

$$x \cdot y = \langle \xi, \eta \rangle = (\xi \cup \eta)(\mu_M)$$

which is also a non singular pairing by the above corollary.

Assume X, Y are closed oriented submanifold of dimensions i and j respectively with $i + j = n$. We also assume they intersect transversally, i.e. at each point $x \in X \cap Y$,

$$T_x X + T_x Y = T_x M.$$

Then the intersection is also a submanifold of dimension 0, thus a finite number of points. Then each $x \in X \cap Y$ has a sign $\epsilon(x)$ determined by comparing the orientations of $T_x X + T_x Y$ and $T_x M$. Let $a = i_*(\mu_X)$ and $b = i_*(\mu_Y)$ where i is the inclusion. Then the intersection number can be calculated as

$$a \cdot b = \sum_{X \in X \cap Y} \epsilon(x). \tag{4.87}$$

Intuitively, this is very clear. We choose a singular decomposition of manifold M such that it also induces singular decomposition of X and Y. So each intersection point will be a vertex of singular simplices. Since oriented means we have $\mu_M = \sum s_i$ where s_i are all simplices. And we also have similar formula for μ_X and μ_Y comprising sub-simplices of s_i, but may have a sign. Then $PD^{-1}(a) \cup PD^{-1}(b)(\sum s_i)$ is nonzero only when s_i contains intersection points. And at each intersection point, and any s_i containing it, the evaluation is just a check of whether the orientations are matched. To make this argument rigorous, we need to make the definition of the cohomology class $PD^{-1}(a)$ clearer so that it satisfies the property we want: restricting to each normal direction is just 1. This needs the Thom isomorphism theorem.

Betti numbers and Euler characteristic

Let us introduce Betti numbers and Euler characteristic. For a cellular space M with finitely many cells in each dimension, let

$$b_i = \mathrm{rank}(H_i(M;\mathbb{Z})) = \dim H_i(M;\mathbb{Q}) = \mathrm{rank}(H^i(M;\mathbb{Z})) = \dim H^i(M;\mathbb{Q}).$$

$b_i(M)$ is called the *i-th Betti number*. The *Euler characteristic* of M is

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i(M).$$

More generally, we have the Poincaré series $P_M(t) = \sum_{i=0}^{\infty} b_i(M)t^i$, and $\chi(M) = PM(-1)$.

Given a finite cellular structure on M, let $\alpha_i(M)$ be the number of i-cells, then we have $\chi = \sum_i (-1)^i \alpha_i(M)$. If we let $Q_M(t) = \sum_i \alpha_i t^i(M)$, this actually follows from the following more general result

Theorem 4.88. *Let M be a finite cellular space. Then*

$$Q_M(t) - P_M(t) = (1+t)R(t).$$

Proof. First if we have a short exact sequence for Abelian groups

$$0 \to A \to B \to C \to 0$$

and A, B, and C are finitely generated, then $rankB = \mathrm{rank}\, A + \mathrm{rank}\, C$.

Next we look at short exact sequences

$$0 \to Z_q \to C_q \to B_{q-1} \to 0$$
$$0 \to B_q \to Z_q \to H_q \to 0$$

We have

$$\sum_q (\mathrm{rank}\, C_q)t^q = \sum_q (\mathrm{rank}\, Z_q)t^q + t\sum_q (\mathrm{rank}\, B_q)t^q$$
$$\sum_q (\mathrm{rank}\, Z_q)t^q = \sum_q (\mathrm{rank}\, B_q)t^q + \sum_q (\mathrm{rank}\, H_q)t^q$$

Adding the both sides will yield

$$Q_M(t) = P_M(t) + (1+t)\sum_q (\operatorname{rank} B_q) t^q.$$

□

From this alternative definition of Euler characteristic, it is easy to see that $\chi(M)$ has some counting property:

Proposition 4.89. *1. If A and B are subspaces of a finite cellular space M, $\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B)$.*

2. If $\widetilde{M}$ is a k-sheeted covering of M, then $\chi(\widetilde{M}) = k\chi(M)$.

Then we have the following restrictions on Euler characteristic of manifolds.

Proposition 4.90. *1. If M is an odd dimensional closed manifold then $\chi(M) = 0$.*

2. If M is an orientable $4k+2$-dimensional manifold, then $\chi(M)$ is even.

Proof. If M is orientable and of odd dimension, then by Poincaré duality, $\chi(M) = 0$. If M is not orientable, we know its double covering $\widetilde{M}$ we constructed before is orientable and $\chi(M) = \frac{1}{2}\chi(\widetilde{M}) = 0$.

For the second statement, we know $b_i = b_{n-i}$ by Poincaré duality. So

$$\chi(M) \equiv b_{2k+1} \bmod 2.$$

We want to prove b_{2k+1} is even. Choose a basis of $H^{2k+1}(M)$, and let A be the matrix of intersection form under this basis. $\alpha \cup \beta = -\beta \cup \alpha$ when both are $2k+1$ dimensional. So A is antisymmetric: $A = -A^t$. So

$$\det A = \det A^t = \det(-A) = (-1)^{b_{2k+1}} \det A.$$

Since the matrix is non-degenerate, b_{2k+1} must be even. □

A non-orientable manifold need not satisfy the second statement. For example, $\chi(\mathbb{R}P^2) = 1$.

4.4.2 Calculation of cohomology rings

Cohomology ring of $\mathbb{C}P^n$

We have calculated the *homology group*

$$H_m(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & m = 0, 2, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

We claim the *cohomology ring*

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{n+1}}$$

where α has degree 2, i.e $\alpha \in H^2(\mathbb{C}P^n;\mathbb{Z})$. Since the inclusion $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ induces an isomorphism on H^i for $i \leq 2n-2$, we can see by induction on n that $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ is generated by α^i for $i < n$. By the corollary, there is an integer m such that the product $\alpha \cup m\alpha^{n-1} = m\alpha^n = 1$. Hence $m = \pm 1$, and our conclusion follows.

Wee see that the ring structure of cohomology can distinguish spaces with the same (co)homology groups. $\mathbb{S}^2 \times \mathbb{S}^4$ has the same cohomology groups as $\mathbb{C}P^3$. But by Künneth formula,

$$H^*(\mathbb{S}^2 \times \mathbb{S}^4;\mathbb{Z}) = \frac{\mathbb{Z}[\alpha,\beta]}{\alpha^2,\beta^2}.$$

This is not isomorphic to $H^*(\mathbb{C}P^3;\mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^4}$.

A similar calculation of cohomology rings for complex projective spaces works for Quaternionic projective space $\mathbb{H}P^n = (\mathbb{H}^{n+1} \setminus 0)/\sim$ where $x \sim y$ if there exists $\lambda \in \mathbb{H} \setminus 0$ such that $\lambda x = y$. We conclude that

$$H^*(\mathbb{H}P^n;\mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{n+1}}$$

where α has degree 4.

The situation for Cayley projective spaces is more restrictive. The division algebra of Cayley numbers $\mathbb{O}$ is not associative. We can form $\mathbb{O}P^1 = \mathbb{S}^8$ and $\mathbb{O}P^2$ and we have

$$H^*(\mathbb{O}P^2;\mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^3}.$$

Cohomology ring of $\mathbb{R}P^n$ and Borsuk-Ulam theorem

Recall that

$$H_m(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0 \text{ or } m = n = 2k+1 \\ \mathbb{Z}/2\mathbb{Z} & m \text{ odd, } 0 < m < n \\ 0 & \text{otherwise} \end{cases}$$

Hence by universal coefficient theorem, we know

$$H^m(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & 0 \leq m \leq n \\ 0 & \text{otherwise} \end{cases}$$

Using Theorem 4.84, and a similar argument for $\mathbb{C}P^n$ as above, we conclude that

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = \frac{\mathbb{Z}/2\mathbb{Z}[x]}{x^{n+1}}.$$

Now as an application of this calculation, we have

Lemma 4.91. *Suppose we have a continuous map $f : \mathbb{R}P^m \to \mathbb{R}P^n$ such that $f_* \neq 0 : H_1(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z}) \to H_1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$. Then $m \leq n$.*

Proof. Since $H^1(X;G) = \mathrm{Hom}(H_1(X);G)$, we know $f^* \neq 0 : H^1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z})$.

Take $\xi \neq 0 \in H^1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$, then $\eta = f^*(\xi) \neq 0 \in H^1(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z})$. By the calculation of cohomology ring, we know $\eta^m = f^*(\xi^m) \neq 0$. So $\xi^m \neq 0 \in H^m(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$, which means $m \leq n$. □

Lemma 4.92. *Let σ be a path connecting a pair of antipodal points $\mathbb{S}^n$. Then under the factor map $\pi : \mathbb{S}^n \to \mathbb{R}P^n$, the σ becomes a singular cycle $\pi_*(\sigma)$ representing a nonzero element in $H_1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$.*

Proof. We use the cellular decomposition induced from the natural one of $\mathbb{S}^0 \subset \mathbb{S}^1 \subset \cdots \subset \mathbb{S}^n$. Let σ be the path connecting the two points of $\mathbb{S}^0$ (in $\mathbb{S}^i$, $i \leq n$). When $n = 1$, $\pi_*(\sigma)$ rotates along $\mathbb{R}P^1 = \mathbb{S}^1$ odd number of times. It is nontrivial in $H_1(\mathbb{R}P^1;\mathbb{Z}/2\mathbb{Z})$.

When $n > 1$, take a path τ with the same end points of σ in $\mathbb{S}^1 \subset \mathbb{S}^n$. We proceed by induction: $\pi_*(\tau)$ represents a nonzero element $H_1(\mathbb{R}P^1;\mathbb{Z}/2\mathbb{Z})$. Since the inclusion $H_1(\mathbb{R}P^1;\mathbb{Z}/2\mathbb{Z}) \to H_1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ is an isomorphism, $\pi_*(\tau)$ is nonzero in $H_1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ as well. On the other hand, $\sigma - \tau$ is a singular cycle in $\mathbb{S}^n$ with $n > 1$, so it is a boundary. Hence $\pi_*(\sigma)$ is nonzero in $H_1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$. □

Theorem 4.93. *There is no continuous map $f : \mathbb{S}^{n+1} \to \mathbb{S}^n$ such that $f(-x) = -f(x)$.*

Proof. If there is such a map, then it induces a map $g : \mathbb{R}P^{n+1} \to \mathbb{R}P^n$. Take a path σ connecting a pair of antipodal points in $\mathbb{S}^{n+1}$. It is mapped to a path $f*(\sigma)$ connecting some antipodal pair of $\mathbb{S}^n$. Hence by Lemma 4.92, $g_* \neq 0 : H_1(\mathbb{R}P^{n+1};\mathbb{Z}/2\mathbb{Z}) \to H_1(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$. Then Lemma 4.91 finishes the proof. □

Corollary 4.94 (Borsuk-Ulam)**.** *Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be a continuous map. Then there exists $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$.*

Proof. If not, g defined as the following is well-defined since $\|f(x) - f(-x)\|$ is non-zero.

$$g : \mathbb{S}^n \to S^{n-1}, \quad g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

Then we have $g(-x) = -g(x)$, contradicting with Theorem 4.93. □

Corollary 4.95 (Ham sandwich)**.** *Let $A_1, \ldots, A_m$ be m measurable sets in $\mathbb{R}^m$. Then we have a hyperplane P which bisects each A_i.*

Proof. Consider in $\mathbb{R}^{m+1}$ fix $x_0 \in \mathbb{R}^{m+1} \setminus \mathbb{R}^m$. For any vector $v \in \mathbb{S}^m \subset \mathbb{R}^{m+1}$, construct a hyperplane orthogonal to v and passing through x_0. It divides $\mathbb{R}^{m+1}$ and hence $\mathbb{R}^m$ into two parts. We record the volume of the $A_i \subset \mathbb{R}^m$ in the half space determined by the direction of v, by $f_i(v)$. Thus we have a continuous map

$$f : \mathbb{S}^m \to \mathbb{R}^m, \quad v \mapsto (f_1(v), \ldots, f_m(v)).$$

By corollary 4.94, there exists a v such that $f(v) = f(-v)$. This hyperplane bisects each A_i. □

Another application is the Ljusternik-Schnirelmann category $\mathrm{cat}(\mathbb{R}P^n) = n+1$. This is because $\mathrm{Cl}(\mathbb{R}P^n) = n$ (4.61), so $\mathrm{cat}(\mathbb{R}P^n) \geq n+1$. On the other hand, it is not hard to construct a smooth function on $\mathbb{R}P^n$ with $n+1$ critical points.

Cohomology ring of lens spaces

Given an integer $m > 1$ and integers $\ell_1, \ldots, \ell_n$ relatively prime to m, we define the *lens space* $L = L_m(\ell_1, \ldots, \ell_n)$ as the orbit space $\mathbb{S}^{2n-1}/(\mathbb{Z}/m\mathbb{Z})$ of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ by the action ρ of $\mathbb{Z}/m\mathbb{Z}$, given by the following

$$\rho(z_1, \ldots, z_n) = (\exp(2\pi\ell_1/m)z_1, \ldots, \exp(2\pi\ell_n/m)z_n).$$

Since ℓ_i is coprime to m, the action of $\mathbb{Z}/m\mathbb{Z}$ over $\mathbb{S}^{2n-1}$ is free. Thus the projection $\mathbb{S}^{2n-1} \to L$ is a covering map. Note that when $m = 2$, ρ is the antipodal map and $L_2 = \mathbb{R}P^{2n-1}$.

L has a CW structure with one cell e_k for each $k \leq 2n-1$ and the resulting cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Therefore

$$H_k(L_m(\ell_1, \ldots, \ell_n)) = \begin{cases} \mathbb{Z} & k = 0, 2n-1 \\ \mathbb{Z}/m\mathbb{Z} & k \text{ odd}, 0 < k < 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

By the universal coefficient theorem,

$$H^k(L_m(\ell_1, \ldots, \ell_n); \mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z} & 0 \leq k \leq 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

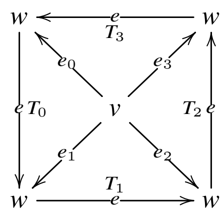
Since the cohomology group only depends on m and n, later on we will denote the lens space by L_m^{2n-1} or simply L^{2n-1}. To calculate the cup product, we let $\alpha \in H^1(L^{2n-1}; \mathbb{Z}/m\mathbb{Z})$ and $\beta \in H^2(L^{2n-1}; \mathbb{Z}/m\mathbb{Z})$ be generators.

We claim that $H^{2i}(L^{2n-1}; \mathbb{Z}/m\mathbb{Z})$ is generated by β^i and $H^{2i+1}(L^{2n-1}; \mathbb{Z}/m\mathbb{Z})$ is generated by $\alpha\beta^i$.

We proceed by induction, so we assume the claim holds for L^{2n-1} and want to show it holds for L^{2n+1}. Using the inclusion $L^{2n-1} \to L^{2n+1}$ which induces isomorphisms in cohomology for $0 \leq k \leq 2n-1$ by comparing the cellular chain complexes, we may assume the claim holds for $H^k(L^{2n+1}; \mathbb{Z}/m\mathbb{Z})$ with $k \leq 2n-1$. By corollary 4.85, there exists $\lambda \in \mathbb{Z}/m\mathbb{Z}$ such that $\beta \cup \lambda\alpha\beta^{n-1} = \lambda\alpha\beta^n$ generates $H^{2n+1}(L^{2n+1}; \mathbb{Z}/m\mathbb{Z})$. So λ has to be a generator of $\mathbb{Z}/m\mathbb{Z}$ and therefore $\alpha\beta^n$ is a generator of $H^{2n+1}(L^{2n+1}; \mathbb{Z}/m\mathbb{Z})$. It also implies that β^n is a generator of $H^{2n}(L^{2n}; \mathbb{Z}/m\mathbb{Z})$, otherwise $\alpha\beta^n$ would have order less than m.

To complete the calculation of the ring $H^*(L^{2n-1}; \mathbb{Z}/m\mathbb{Z})$, we need to compute α^2. By graded commutativity, we have $\alpha \cup \alpha = -\alpha \cup \alpha$. So if m is odd, $\alpha^2 = 0$.

When $m = 2k$, we claim $\alpha^2 = k\beta$. We use the fact that the 2-skeleton $\mathbb{S}^1 \cup_{f_m} e^2$ of L^{2n-1} is the circle $\mathbb{S}^1$ attached by a 2-cell with a map of degree m. We first get the 2-skeleton a cellular structure (in fact, a Δ-complex structure) by subdivide an m-gon into m triangles T_i around a central vertex v, and identify all the outer edges by rotations of the m-gon. We call the faces in a counter-clockwise order $T_0, \ldots, T_{m-1}$ and the the rays from v which bound T_i by e_i and e_{i+1}.



Then we choose a representative ϕ for α which assigns value 1 to the boundary edge. The condition that ϕ is a cocycle means $\phi(e_i) + \phi e) = \phi(e_{i+1})$, which means we can take $\phi(e_i) = i$ in $\mathbb{Z}/m\mathbb{Z}$. Then by definition of the cup product, $(phi \cup \phi)(T_i) = \phi(e_i)\phi(e) = i$. Since $0 + 1 + \cdots + (m-1) \equiv k$ in $\mathbb{Z}/m\mathbb{Z}$, we know $\phi \cup \phi$ evaluates as k on ΣT_i. This means $\alpha^2 = k\beta$.

Hence we have

$$H^*(L^{2n-1}; \mathbb{Z}/(2k+1)\mathbb{Z}) = \frac{\mathbb{Z}/(2k+1)\mathbb{Z}[\alpha, \beta]}{\alpha^2 = 0, \beta^n = 0}$$

$$H^*(L^{2n-1}; \mathbb{Z}/2k\mathbb{Z}) = \frac{\mathbb{Z}/2k\mathbb{Z}[\alpha, \beta]}{\alpha^2 = k\beta, \beta^n = 0}$$

4.4.3 Degree and Hopf invariant

Degree

We can define degree of a map $f : M^n \to N^n$ between closed oriented connected manifolds. For orientable manifolds $H^n(M) = H^n(N) = \mathbb{Z}$, so $f_* : H_n(M) \to H_n(N)$ maps the generator μ_M to an integer multiple k of μ_N. We call this $k := \deg(f)$ the *degree* of the map f. The degree has natural composition property: $\deg(f)\deg(g) = \deg(f \circ g)$. Since $H^n(M) = H^n(N) = \mathbb{Z}$ as well, we can define degree as the corresponding integer for the cohomology $f^* : H^n(N) \to H^n(M)$. Apparently, these two definitions yield the same number.

Example 4.96. A reflexion of $\mathbb{S}^n$ along a great circle has degree -1, since it changes the orientation. Hence the antipodal map a sending $x \mapsto -x$ has degree $(-1)^{n+1}$ since it is a composition of $n+1$ reflexions.

This example has lots of corollaries. We only show a few.

Corollary 4.97. *1. If $f, g : \mathbb{S}^n \to \mathbb{S}^n$ are maps such that $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$ then f is homotopic to $a \circ g$.*

2. If $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points then it is homotopic to the antipodal map, and thus has degree $(-1)^{n+1}$.

Proof. *1*: By assumption

$$x \mapsto \frac{(1-t)f(x) - tg(x)}{\|(1t)f(x) - tg(x)\|}$$

is a well defined homotopy from f to $a \circ g$.
2: Set $g = \mathrm{id}$ in *1*. □

We can also get some information for group actions on $\mathbb{S}^n$. First note that $\mathbb{S}^{2n-1}$ can be viewed as the unit sphere in $\mathbb{C}^n$. Thus it admits a free action of $\mathbb{S}^1$, e.g, $z \mapsto e^{i\theta}z$. Especially, it tells us that any finite subgroup $\mathbb{Z}/m\mathbb{Z} \subset \mathbb{S}^1$ can act on $\mathbb{S}^{2n-1}$ freely. However, the situation for $\mathbb{S}^{2n}$ is quite different.

Corollary 4.98. *Suppose a group G acts freely on $\mathbb{S}^{2n}$. Then $G \leq \mathbb{Z}/2\mathbb{Z}$.*

Proof. By assumption, each non-trivial element $g \in G$ has no fixed point, thus has degree -1 by the above corollary. Hence there is at most one such element, otherwise the composition would give a map of degree 1 which has to be trivial. □

Proposition 4.99. *Given $f : \mathbb{C}P^n \to \mathbb{C}P^n$, there exists an integer k such that* $\deg f = k^n$.

Proof. Let u be a generator of $H^2(\mathbb{C}P^n)$, then $f*(u) = ku$ for some constant k. Hence $f^*(u^n) = f^*(u)^n = k^n u^n$. By definition, $\deg f = k^n$. □

Proposition 4.100. *If $f : \mathbb{S}^{2n} \to \mathbb{C}P^n$ with $n > 1$ then* $\deg(f) = 0$.

Proof. $f^*(u) = 0$ since $H^2(\mathbb{S}^{2n}) = 0$. So $f^*(u^n) = f^*(u)^n = 0$. □

The last proposition suggests: if there is a map $f : M \to N$ with $\deg(f) \neq 0$, then M may well be topologically more complicated than N.

Example 4.101. If $f : \mathbb{S}^n \to \mathbb{S}^n$ is a continuous map, and $\Sigma f : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ is the suspension of f then $\deg \Sigma f = \deg f$.

In fact, if $X \to X$ is a continuous map and

$$\Sigma X = X \times [-1,1]/(X \times \{-1\}, X \times \{1\})$$

denotes the suspension of X, then $\Sigma f := f \times \mathrm{id}_{[-1,1]}/\sim$, with the same equivalence as in ΣX. Note that $\Sigma \mathbb{S}^n = \mathbb{S}^{n+1}$.

The Suspension Theorem states that

$$\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X).$$

We show this fact by using the Mayer-Vietoris sequence for the decomposition

$$\Sigma X = C_+X \cup_X C_-X,$$

where C_+X and C_-X are the upper and lower cones of the suspension joined along their bases:

$$\cdots \to \widetilde{H}_{n+1}(C_+X) \oplus \widetilde{H}_{n+1}(C_-X) \to \widetilde{H}_{n+1}(\Sigma X) \to \widetilde{H}_n(X) \to \widetilde{H}_n(C_+X) \oplus \widetilde{H}_n(C_-X) \to \cdots$$

Since C_+X and C_-X are both contractible, the end groups in the above sequence are both zero. Thus, by exactness, we get $\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X)$, as desired.

Let $C_+\mathbb{S}^n$ denote the upper cone of $\Sigma\mathbb{S}^n$. Note that the base of $C_+\mathbb{S}^n$ is $\mathbb{S}^n \times \{0\} \subset \Sigma\mathbb{S}^n$. Our map f induces a map $C_+f : (C_+\mathbb{S}^n, \mathbb{S}^n) \to (C_+\mathbb{S}^n, \mathbb{S}^n)$, giving the factor map Σf. The long exact sequence of the pair $(C_+\mathbb{S}^n, \mathbb{S}^n)$ in homology gives the following commutative diagramme:

$$\begin{array}{ccccccc} 0 \longrightarrow \widetilde{H}_{i+1}(C_+\mathbb{S}^n, \mathbb{S}^n) & \cong & \widetilde{H}_{i+1}(C_+\mathbb{S}^n/\mathbb{S}^n) & \xrightarrow[\sim]{\partial} & \widetilde{H}_i(\mathbb{S}^n) \longrightarrow 0 \\ & & \downarrow{\scriptstyle (\Sigma f)_*} & & \downarrow{\scriptstyle f_*} \\ & & \widetilde{H}_{i+1}(\mathbb{S}^{n+1}) & \xrightarrow[\sim]{\partial} & \widetilde{H}_i(\mathbb{S}^n) \end{array}$$

Note that $C_+\mathbb{S}^n/\mathbb{S}^n \cong \mathbb{S}^{n+1}$ so the boundary maps ∂ at the top and the bottom of the diagramme are the same map. So by the commutativity of the diagramme, since f_* is defined by multiplication by some integer m, $(\Sigma f)_*$ as well, is multiplication by the same integer m.

Hopf invariant

Hopf invariant is a kind of degree when studying the maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$.

Given a map $f : \mathbb{S}^m \to \mathbb{S}^n$ with $m \geq n$, we can form a cellular space

$$C(f) := \mathbb{S}^n \cup_f \mathbb{D}^{m+1} = \frac{\mathbb{S}^n \sqcup \mathbb{D}^{m+1}}{f(x) \sim x, \forall x \in \mathbb{S}^m}$$

The homotopy type of $C(f)$ depends only on the homotopy class of f. We can use Proposition 4.36 to calculate its (co)homology group. For example, if $m = n$ and f has degree d, then $H^n(C(f)) = \mathbb{Z}/|d|\mathbb{Z}$, which detects degree up to sign.

When $m > n$, we calculate that the cohomology of $C(f)$ has $\mathbb{Z}$ in dimensions 0, n and $m+1$. Especially when $m = 2n-1$, we have chance to use cup product to detect something nontrivial. n this case, choose generators $\alpha \in H^n(C(f))$ and $\beta \in H^{2n}(C(f))$, then the ring structure of $H^*(C(f))$ is determined by $\alpha^2 = H(f)\beta$ for an integer $H(f)$ which is called the *Hopf invariant* of f.

If f is a constant map then $C(f) = \mathbb{S}^n \vee \mathbb{S}^{2n}$ and $H(f) = 0$. Also, $H(f)$ is always zero for odd n since $\alpha^2 = -\alpha^2$ in this case.

Example 4.102. **Case** $n = 1$ In this case it is the covering map, viewed as a fibration $\mathbb{S}^0 \to \mathbb{S}^1 \to \mathbb{R}P^1$. It is measured by its degree, which is 2.

Case $n = 2$ We use $\mathbb{S}^2 = \mathbb{C}P^1$ and view $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$. The map $\mathbb{S}^3 \to \mathbb{S}^2$ is defined as

$$(z_0, z_1) \mapsto [z_0 : z_1].$$

From the definition, this is a fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$, which is called the *Hopf fibration*. It is easy to see that $C(f) = \mathbb{C}P^2$. Thus $H(f) = 1$ since $H^*(\mathbb{C}P^2) = \frac{\mathbb{Z}[\alpha]}{\alpha^3}$.

Case $n = 4$ Replacing the field $\mathbb{C}$ by Hamilton's quaternion $\mathbb{H}$, with some construction work yields the fibration $\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4$. And $C(f) = \mathbb{H}P^2$ and $H(f) = 1$.

Case $n = 8$ Using Cayley octonion, we have Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^{15} \to \mathbb{S}^8$. And $C(f) = \mathbb{O}P^2$ and $H(f) = 1$.

It is a fundamental theorem of Adams that maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$ of Hopf invariant 1 exists only when $n = 2$, 4, and 8: the original proof, John F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. Math. 72 1 (1960) 20-104, is not quite accessible. a much simpler proof using K-theory is given by, J. F. Adams and M. F. Atiyah *K-theory and the Hopf invarant*, The Quarterly Journal of Mathematics, Volume 17, Issue 1, 1966, Pages 31-38.

It has many interesting corollaries:

1. $\mathbb{R}^n$ has a structure of division algebra (over $\mathbb{R}$) only for $n = 1$, 2, 4, and 8.
2. $\mathbb{S}^n$ is "parallelisable" i.e, has n linearly independent tangent vector fields only for $n = 0$, 1, 3, and 7.
3. The fibrations $\mathbb{S}^p \to \mathbb{S}^q \to \mathbb{S}^r$ exist only for triples $(p, q, r) = (0, 1, 1)$, $(1, 3, 2)$, $(3, 7, 4)$, and $(7, 15, 8)$.

One can also define the Hopf invariant in terms of degree. Let y, z be two different regular values for a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, then the manifolds $f^{-1}(y)$ and $f^{-1}(z)$ can be oriented and the linking number is defined as the degree of a function: Let M and N be two manifolds of dimension $n-1$ in $\mathbb{S}^{2n-1}$. Choose a point $p \in \mathbb{S}^{2n-1}$ which is not in M or N, and think $\mathbb{S}^{2n-1} \setminus p$ as $\mathbb{R}^{2n-1}$. Then the linking number $\operatorname{link}(M, N)$ of M and N is defined as the degree of the map

$$g : M \times N \to \mathbb{S}^{2n-2}, \quad (x, y) \mapsto \frac{x - y}{\|x - y\|}.$$

Let us understand this definition in terms of low dimensional examples. First is the toy example: the linking of two $\mathbb{S}^0$ in $\mathbb{S}^1$ or $\mathbb{R}^1$. Let the coordinate of $\mathbb{S}^0$ be $\{a, b\}$ and $\{c, d\}$ respectively. Then the map g is determined by the order of these numbers. For example, if $a < c < b < d$, then two $\mathbb{S}^0$ are linked both from our common sense and from the formula since g maps to 1 once and -1 thrice, so degree is one. If $a < b < c < d$, then degree is zero since maps to -1 four times. And if $a < c < d < b$, it maps to 1 and -1 both twice. But the degree is 0 as well since $g(a, c)$ and $g(a, d)$ are considered as oppositely orientated because that of c and d are.

A more realistic example is for two $\mathbb{S}^1$ in $\mathbb{S}^3$ or $\mathbb{R}^3$. So $g : T^2 \to \mathbb{S}^2$. For a point $\boldsymbol{v}$ in the unit sphere, the orthogonal projection of the link to the plane perpendicular to $\boldsymbol{v}$ gives a link diagram on plane. A point in T^2 sent to $\boldsymbol{v}$ corresponds to a crossing in the link diagramme where γ_1 is over γ_2. A neighbourhood of it is mapped to a neighbourhood of $\boldsymbol{v}$ preserving or reversing the orientation depending on the sign of the crossing. Thus it is just a signed counting of the number of times g covers $\boldsymbol{v}$.

There is a more concrete formula (Gauss formula) which can be generalisd to higher dimensions

$$\text{link}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^{T_1} \int_0^{T_2} \frac{(\dot{\gamma_1}, \dot{\gamma_2}, \gamma_1 - \gamma_2)}{|\gamma_1 - \gamma_2\|^3} dt_1 dt_2.$$

This is an integration interpretation of degree.

Now, the Hopf invariant $H(f) = \text{link}(f^{-1}1(y), f^{-1}(z))$ for any two regular values y and z. To understand the equivalence of these definitions, we understand the cup product as the intersection of the Poincaré dual of cocycles. For the toy model above, $\mathbb{S}^1$ bounds a $\mathbb{D}^2$. Pairs of two points in $\mathbb{S}^1$ are linked if and only if two semi circle in $\mathbb{D}^2$ with the pairs as end points intersect. By gluing the boundary $\mathbb{S}^1$ by a double covering to another $\mathbb{S}^1$, we get $\mathbb{R}P^2$, and two semi-circle above become two $\mathbb{S}^1$.

For two $\mathbb{S}^1$'s, which are considered as the inverse image of a regular value of map $\mathbb{S}^3 \to \mathbb{S}^2$, we have a similar story. But now $\mathbb{S}^3$ bounds $\mathbb{D}^4$ where one can consider the picture in $\mathbb{C}^2$, and $\mathbb{S}^1$ bounds an immersed disk. Then the intersection number of these two surfaces in $\mathbb{D}^4$ is exactly the same as the linking number of two $\mathbb{S}^1$'s in $\mathbb{S}^3$. One can prove this fact by pulling two circles until they touch. So the intersection is at the boundary and easy to look at. Finally, since $\mathbb{S}^1$ are fibres of the map, after gluing they will become a point and the original surfaces will become a closed one.

With an explicit example bearing in mind, one can consider the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$ where the fibre is (z_1, z_2) with a fixed ratio. For example, $\mathbb{S}^1 \times 0$, $0 \times \mathbb{S}^1 \subset \mathbb{S}^3 \subset \mathbb{C} \times \mathbb{C}$ are two of them.

4.4.4 Alexander duality

Let us remind the reduced homology and reduced cohomology. $\widetilde{H}_i(M) = H_i(M, x)$ and $\widetilde{H}^i(M) = H^i(M, x)$. So reduced ones differ from the originals only for $i = 0$.

Theorem 4.103. *If K is a compact, locally contractible, non-empty, proper subset of $\mathbb{S}^n$, then $\widetilde{H}_i(\mathbb{S}^n \setminus K; \mathbb{Z}) = \widetilde{H}^{n-i-1}(K; \mathbb{Z})$ for all i.*

Proof. We first handle the case $i \neq 0$. By Poincaré duality, we have

$$H_i(\mathbb{S}^n \setminus K; \mathbb{Z}) = H_c^{n-i}(\mathbb{S}^n \setminus K).$$

And by the definition of cohomology with compact supports,

$$H_c^{n-i}(\mathbb{S}^n \setminus K) = \varinjlim H^{n-i}(\mathbb{S}^n \setminus K, U \setminus K)$$

where U is taken as open neighbourhoods of K. By excision, $H^{n-i}(\mathbb{S}^n \setminus K, U \setminus K) = H^{n-i}(\mathbb{S}^n, U)$. And by the long exact sequence for pairs, $H^{n-i}(\mathbb{S}^n, U) = \widetilde{H}^{n-i-1}(U)$ when $i \neq 0$. Now if we can show

$$\varinjlim \widetilde{H}^{n-i-1}(U) = \widetilde{H}^{n-i-1}(K),$$

the case for $i \neq 0$ is complete.

To show this, we use the fact that K is a retract of some neighbourhood U_0 in $\mathbb{S}^n$ since it is locally contractible. Thus in the direct limit we can only choose those open neighbourhood $U \subset U_0$ that can be retracted to K. This implies the surjectivity of the map $\varinjlim H^*(U) \to H^*(K)$ since we can pull back the cohomology of K to that of U. To prove the injectivity, note that any $U \subset U_0$ can be regarded as a subspace of $\mathbb{R}^n \subset \mathbb{S}^n$. The linear homotopy $U \times I \to \mathbb{R}^n$ from the identity to the retraction $U \to K$ takes $K \times I$ to K, hence takes $V \times I$ to U for some (small) neighbourhood V of K by compactness of I. Hence the inclusion $V \hookrightarrow U$ is homotopic to the retraction $V \to K \subset U$. Thus the restriction $H^*(U) \to H^*(V)$ factors through $H^*(K)$. Therefore if an element of $H^*(U)$ restricts to 0 in $H^*(K)$, it will be zero in $H^*(V)$ and thus in $\varinjlim H^*(U)$.

For the case $i = 0$, $H^n(\mathbb{S}^n, U) = \widetilde{H}^{n-1}(U)$ does not hold. Instead we have the short exact sequence

$$0 \to \widetilde{H}^{n-1}(U) \to H^n(\mathbb{S}^n, U) \to H^n(\mathbb{S}^n) \to 0.$$

By taking the direct limit, we see the first term becomes $\varinjlim \widetilde{H}^{n-1}(U) = \widetilde{H}^{n-1}(K)$. By Poincaré duality, the middle term is $H_0(\mathbb{S}^n \setminus K)$ and the last term is $H_0(\mathbb{S}^n) = \mathbb{Z}$. So this sequence tells us $\widetilde{H}_0(\mathbb{S}^n \setminus K) = \widetilde{H}^{n-1}(K)$. □

In the proof, the local contractibility is used to guarantee $\varinjlim \widetilde{H}^{n-i-1}(U) = \widetilde{H}^{n-i-1}(K)$. Without this condition, a pathological phenomenon may occur: look at the the following example.

Example 4.104. Let K denote the subset of the graph of the function $y = \sin(\frac{1}{x})$ for $x \neq 0$ and y-axis with $|x|, |y| \leq 1$. Since there are three path components,

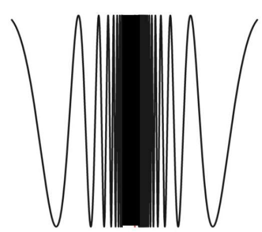


Figure 4.3: Two topologist's sine curves juxtaposed back-to-back

$H^0(K;\mathbb{Z})$ is free abelian of rank 3. However, for the direct limit $\varinjlim H^0(U)$, we only need calculate it for open path connected neighbourhoods of K, and thus $\varinjlim H^0(U) = \mathbb{Z}$. Notice that K is not locally contractible at the origin. Hence, it is not a cellular space, either. This is a juxtaposition of two topologist's sine curves. Note that it is connected but not path connected. We also notice that the Alexander duality fails for this space.

This theorem has many interesting applications. Let us start with the lowest non-trivial dimension $n = 2$.

Corollary 4.105 (Jordan curve theorem)**.** *Let $K \subset \mathbb{S}^2$ be a simple closed curve, then $\mathbb{S}^2 \setminus K$ has two components.*

Proof. Alexander duality says that $\widetilde{H}^0(\mathbb{S}^2 \setminus K) = H^1(\mathbb{S}^1) = \mathbb{Z}$. So $H_0(\mathbb{S}^2 \setminus K) = \mathbb{Z}/2\mathbb{Z}$. □

For $n = 3$, if we take K as knots. The Alexander duality simply tells us that we cannot distinguish different knots by their homology groups. A result of Gordon-Luecke tells us that the fundamental group of the knot complement determines the knot (*Knots are determined by their complements.* J. Amer. Math. Soc. 2 (1989), no. 2, 371-415.). Actually, if we choose $K \subset \mathbb{S}^n$ as a space homeomorphic to $\mathbb{S}^m$, we will have the Alexander duality as well and the proof is a delightful use of Mayer-Vietoris sequence. Especially, this works for the Alexander horned sphere. It is an example homeomorphic to $\mathbb{D}^3$ with its

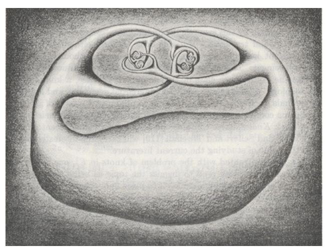


Figure 4.4: Alexander horned sphere

boundary homeomorphic to $\mathbb{S}^2$, but its complement is not simply connected. However, by Alexander duality, its complement has trivial first homology.

Example 4.106. A non-orientable closed surface N cannot be embedded in $\mathbb{S}^3$ as a submanifold. This is because $H^2(N;\mathbb{Z}) \approx \mathbb{Z}/2\mathbb{Z}$ is not free.

For the convenience for readers, we want to add that if we work on Čech cohomology instead of singular cohomology, then the local contractibility constraint can be removed. This is because, by definiton of Čech cohomology, we have $\check{H}^q(K) = \varinjlim H^q(U)$ for all neighbourhoods U of K. Especially, for Čech cohomology H^0 detects the connected components instead of path connected components. Notice that it does not contradict to the Eilenberg-Steenrod uniqueness axiom since all cellular spaces are locally contractible, hence they are path connected if connected.

A remark on the definition of Čech cohomology. It is defined as follows: let X be the space for which we would like to define "cohomology".

1. For each open cover $\mathfrak{U} = \{U_\alpha\}$ of X, associate a simplicial complex $N(\mathfrak{U})$ called its *nerve*. This associates a vertex to each U_α and a set of $k+1$ vertices are considered to span a k-simplex if the corresponding U_α's have a non-empty intersection.

2. For a refinement $\mathfrak{V} = \{V_\beta\}$ of $\mathfrak{U}$ (i.e, each V_β is contained in some U_α), the inclusions induce a simplicial map $N(\mathfrak{V}) \to N(\mathfrak{U})$.

3. The Čech cohomology $\check{H}^q(X)$ is defined as $\varinjlim H^q(N(\mathfrak{U}))$.

4.4.5 Manifolds with boundary

An *n-manifold with boundary* is a Hausdorff space M in which each point has an open neighbourhood homeomorphic either to $\mathbb{R}^n$ (such a point is called an *interior point*) or to the half space $\mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n | x_n \geq 0\}$ (resp. a *boundary point*). An interior point $x \in M$ has $H_n(M, M \setminus x) = \mathbb{Z}$. A boundary point x corresponds to a point $(x_1,\dots,x_n) \in \mathbb{R}^n_+$ with $x_n = 0$. By excision, we have $H_n(M, M \setminus x) = H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ \setminus 0) = 0$.

If M is a compact manifold with boundary, then ∂M has a collar neighbourhood, i.e. an open neighouborhood homeomorphic to $\partial M \times [0,1)$ by a homeomorphism sending ∂M to $\partial M \times 0$. (Morton Brown, *Locally flat imbeddings of topological manifolds*, Annals of Mathematics, Vol. 75 (1962), p. 331-341, or Robert Connelly, *A new proof of Brown's collaring theorem*, Proceedings of the American Mathematical Society 27 (1971), 180-182.)

A compact manifold M with boundary is called orientable if $\mathring{M} := M \setminus \partial M$ is orientable. If $\partial M \times [0,1)$ is a collar neighbourhood of ∂M, then $H_i(M, \partial M) = H_i(M \setminus \partial M, \partial M \times (0, \frac{1}{2}))$. So Lemma 4.79 gives a relative fundamental class, denoted as $[M, \partial M]$ restricting to a given orientation at each point of $M \setminus \partial M$. The following tells how to relate a relative fundamental class to $\mu_{\partial M}$. Later, for simplicity, we will write it as $[\partial M]$.

Proposition 4.107. *An orientation of M determines an orientation of ∂M.*

Proof. Consider an open neighbourhood U of a point $x \in \partial M$ which is homeomorphic to an open half disk in $\mathbb{R}^n_+$. Let $V = \partial U = U \cap \partial M$ and let $y \in \mathring{U} = U \setminus V$. We have the following isomorphisms

$$\begin{aligned}
H_n(\mathring{M};, \mathring{M} \setminus \mathring{U}) &= H_n(\mathring{M}, \mathring{M} \setminus y) \\
&= H_n(M, M \setminus y) \\
&= H_n(M, M \setminus \mathring{U}) \\
&\xrightarrow{\partial} H_{n-1}(M \setminus \mathring{U}, M \setminus \mathring{U}) \\
&= H_{n-1}(M \setminus \mathring{U}, (M \setminus \mathring{U}) \setminus x) \\
&= H_{n-1}(\partial M, \partial M \setminus x) \\
&= H_{n-1}(\partial M, \partial M \setminus V)
\end{aligned}$$

The connecting homomorphism is that of the triple $(M, M \setminus \mathring{U}, M \setminus U)$ which is an isomorphism since $H_*(M, M \setminus U) = H_*(M, M) = 0$. The isomorphism that follows comes from the observation that the inclusion $(M \setminus \mathring{U}) \setminus x \to M \setminus U$ is a homotopy equivalence. The next to last isomorphism is given by excision of $\mathring{M} \setminus \mathring{U}$. □

In particular, $\partial[M, \partial M]$ restricts to a generator of $H_{n-1}(\partial M, \partial M \setminus x)$ for all $x \in \partial M$ and thus is the fundamental class $[\partial M]$ determined by the orientation of ∂M which is induced from that of M.

Theorem 4.108 (Lefschetz duality)**.** *Suppose M is a compact oriented n-manifold with boundary. Then the homomorphisms*

$$\begin{aligned} D :& H^p(M) \to H_{n-p}(M, \partial M), \quad \alpha \mapsto \alpha \cap [M, \partial M] \\ D :& H^p(M, \partial M) \to H_{n-p}(M), \quad \alpha \mapsto \alpha \cap [M, \partial M] \end{aligned}$$

are isomorphisms. And the following diagramme is commutative.

$$\begin{array}{ccccccc} H^{q-1}(M) & \longrightarrow & H^{q-1}(\partial M) & \xrightarrow{\delta} & H^q(M, \partial M) & \longrightarrow & H^q(M) \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\ H_{n-q-1}(M, \partial M) & \xrightarrow[\partial]{} & H_{n-q}(\partial M) & \longrightarrow & H_{n-q}(M) & \longrightarrow & H_{n-q}(M, \partial M) \end{array}$$

Proof. We apply Theorem 4.83 to $M \setminus \partial M$. Via a collar neighbourhood, we have $H^p(M, \partial M) = H^p_c(M \setminus M)$. And obviously, $H_{n-p}(M) = H_{n-p}(M \setminus M)$. Hence $D : H^p(M, \partial M) \to H_{n-p}(M)$ is an isomorphism. The commutativity can be checked by inspecting the definition and using the boundary formula for cap product. (for details, see F. E. A. Johnson, *Lefschetz duality and topological tubular neighbourhoods*, Transactions of the American Mathematical Society, Volume 172, October 1972.)

Finally by Five lemma, $D : H^p(M) \to H_{n-p}(M, \partial M)$ is an isomorphism as well. □

For general manifolds with boundary, we also have $H^p_c(M) \cong H_{n-p}(M, \partial M)$, and $H^p_c(M, \partial M) \cong H_{n-p}(M)$ if we define $H^p_c(M, \partial M) := \varinjlim H^p(M, (M \setminus K) \cup \partial M)$.

Next, we want to know what kind of n-manifolds can be the boundary of an $n + 1$-dimensional manifold with boundary. From the classification of surfaces[1], we know that each orientable closed surface is the boundary of certain 3-manifold. What about non-orientable ones?

Theorem 4.109. *Let an n-manifold $M^n = partial W^{n+1}$. Then $\chi(M)$ is even.*

Proof. When n is odd, it follows from Proposition 4.90. When n is even, take the double of W, which is obtained by take two copies W^+ and W^- of W and glue them along the boundary. We denote it by $2W$. So $W^+ \cup W^- = 2W$ and $W^+ \cap W^- = M$. Hence, we have

$$\chi(2W) + \chi(M) = \chi(W^+) + \chi(W^-) = 2\chi(W)$$

Since $2W$ is an odd dimensional manifold, $\chi(2W) = 0$. Hence $\chi(M) = 2\chi(W)$ which is an even number. □

Example 4.110. The double of a Möbius band is a Klein bottle. See Figure 4.5. The double of an annulus of dimension 2 is a torus. The double of a disk is a sphere.

[1]For a simple proof, see E. C. Zeeman, *An Introduction to Topology*, https://webhomes.maths.ed.ac.uk/ v1ranick/surgery/zeeman.pdf

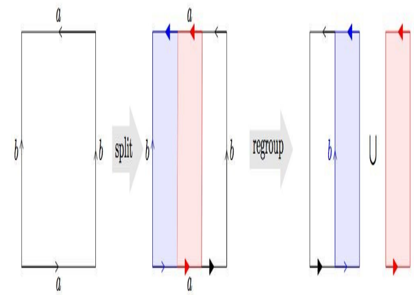


Figure 4.5: A Klein bottle cut into two Möbius bands

For non-orientable surfaces, $\mathbb{R}P^2$ is not a boundary, but one can check that the Klein bottle is a boundary. More generally, one can check that $\mathbb{R}P^{2k}$ and $\mathbb{C}P^{2k}$ are not boundaries.

The next result relates the signature with the boundaries.

Theorem 4.111. *Let $M^{4k} = \partial W^{4k+1}$ where W is a compact oriented $4k+1$-manifold, then $\sigma(M) = 0$.*

Proof. We use $\mathbb{R}$ as coefficient for (co)homology. We denote $[M] = \mu_M$.

The inclusion $i : M \to W$ induces a homomorphism $i^* : H^{2k}(W) \to H^{2k}(M)$. Let $U = \operatorname{im} i^*$.

For $u = i^*(w) \in U$, we have

$$\langle u, u\rangle = \langle i^*(w) \cup i^*(w), [M]\rangle = \langle i^*(w \cup w), [M]\rangle = \langle w \cup w, i_*[M]\rangle = 0$$

The last equality holds since $i_*\partial = 0$ in the long exact sequence of the pair (W, M) and $[M] = \partial[W, M]$. We can see that the following diagramme is commutative.

$$\begin{array}{ccccc} H^{2k}(W) & \xrightarrow{i^*} & H^{2k}(M) & \xrightarrow{\delta} & H^{2k+1}(W,M) \\ \downarrow D & & \downarrow D & & \downarrow D \\ H_{2k+1}(W,M) & \xrightarrow[\partial]{} & H_{2k}(M) & \xrightarrow[i_*]{} & H_{2k}(W) \end{array}$$

So $\operatorname{rank} \operatorname{im} i^* = \operatorname{rank} \ker \delta = \operatorname{rank} \ker i_*$. Since i^* and i_* are dual homomorphisms of each other (this is why we use $\mathbb{R}$ as the coefficient), so $\operatorname{rank} \operatorname{coker} i^* = \operatorname{rank} \ker i_*$. Hence

$$\operatorname{rank} H^{2k}(M) = \operatorname{rank} \operatorname{im} i^* + \operatorname{rank} \operatorname{coker} i^* = 2 \operatorname{rank} U.$$

Let $V = H^{2k}(M)$ and the positive/negative eigenspace of the intersection form $\langle v, v\rangle$ would decompose it as $V^{\pm}$. The intersection form is 0 on the linear subspace U, so $V^+ \cap U = 0$. Hence $\operatorname{rank} V^+ + \operatorname{rank} U \le \operatorname{rank} V$. Similarly $\operatorname{rank} V^- + \operatorname{rank} U \le \operatorname{rank} V$. However, the intersection is non-singular, so $\operatorname{rank} V^+ + \operatorname{rank} V^- = \operatorname{rank} V$. Thus $\operatorname{rank} V^{\pm} = \operatorname{rank} U$ and hence $\sigma(M) = 0$. □

This was proved by Thom, as a part of his cobordism theory (René Thom, *Quelques propriétés globales des variétés différentiables*, Commentarii Mathematici Helvetici 28, 17-86 (1954))

Linking number, Massey product

We can interpret the linking number of two cycles in Euclidean space by cup product in the complementary space. For example, suppose that $\mathbb{S}^p$ and $\mathbb{S}^q$ are disjoint spheres in $\mathbb{S}^n$ where $n = p + q + 1$, and $1 \leq p < q \leq n - 2$. By Alexander duality theorem, the complementary space $\mathbb{S}^n \setminus (\mathbb{S}^p \cup \mathbb{S}^q)$ has cohomology group $\mathbb{Z}$ in dimensions p, q and $p + q$. Therefore the cup product of the generators in dimensions p and q will be a certain multiple of the generator of cohomology class in dimension $p + q$. It can be shown that this multiple is just the linking number of $\mathbb{S}^p$ and $\mathbb{S}^q$.

Another slightly different but easier to generalise definition is the following: we consider open neighbourhood of the linked spheres and let them be U_1, U_2, and M is the complement of U_1, U_2 in $\mathbb{S}^n$ and let the boundary be $B = B_1 \cup B_2$. Let w and v be the generators of H^p and H^q respectively. And let the generators $\mu \in H^n(M, B)$, $\mu_i \in H^{n-1}(B_i)$. These can be chosen compatible with the orientations induced from $\mathbb{S}^n$. Then there is an inclusion $g : B \to M$. We have the exact sequence

$$H^{n-1}(M) \xrightarrow{g^*} H^{n-1}(B) \xrightarrow{\delta} H^n(M, B) \to 0$$

Since μ_i are mapped to μ, the linking number is just the number m such that $g^*(w \cup v) = m(\mu_1 - \mu_2)$.

To understand the situation, let us consider the (simplest, hopefully) case of intersections oof two $\mathbb{S}^1$ in $\mathbb{S}^3$. Then the generators w and v correspond to singular discs $\mathbb{D}_1$ and $\mathbb{D}_2$ bounded by the two $\mathbb{S}^1$'s. And the generator of $H^2(\mathbb{S}^3 \setminus (\mathbb{S}^1 \cup \mathbb{S}^1))$ corresponds to a path connecting B_1 and B_2. Hence the intersection, assuming that they are in a general position, is a signed count of these paths. Hopefully the reader can see the second viewpoint from this interpretation.

A similar idea can be applied to understand the Massey product. Let us have three spheres in $\mathbb{S}^n$, any two of them have linking number 0. The generators of cohomology in dimensions between 0 and $n - 1$ are denoted by w_1, w_2, w_3. We can similarly define μ_i and the same exact sequence. Notice that g^* is injective since the 3 cohomology groups are *free* abelian groups of dimensions 2, 3 and 1. Another way to see this is via the naturality of Alexander duality, namely, the following commutative diagramme:

$$\begin{array}{ccc} H^q(X) & \longrightarrow & H^q(Y) \\ \downarrow D & & \downarrow D \\ H_{n-q-1}(\mathbb{S}^n \setminus X) & \longrightarrow & H_{n-q-1}(\mathbb{S}^n \setminus Y) \end{array}$$

Now triple product can be understood as a higher linking number of links, like the ones for Borromean ring.

Theorem 4.112. *There exists an integer m_{13} such that*

$$g^*\langle w_1, w_2, w_3\rangle = m_{13}(\mu_1 - \mu_3).$$

For Borromean ring, this integer is 1. Here is a sketch of the proof: If $x \in H^{n-1}(M)$ and $g^*(x) = a_1\mu_1 + a_2\mu_2 + a_3\mu_3$, it follows from the exactness that $a_1 + a_2 + a_3 = 0$. Hence we only need to prove the coefficient of μ_2 in $g^*\langle w_1, w_2, w_3\rangle$ is 0, or $g_2^*\langle w_1, w_2, w_3\rangle = 0$ for $g_2 : B_2 \to M$. Actually, we can show that $g_2^* w_1 = g_2^* w_3 = 0$, which follows from the naturality of Alexander duality and the fact that pairwise linking numbers are 0. Thus we understand g_2^* on the q_1-th cohomology as

$$H_{p_1}(U_1) \oplus H_{p_1}(U_2) \oplus H_{p_1}(U_3) \to H_{p_1}(\mathbb{S}^n \setminus B_2) = H_{p_1}(U_2) \oplus H_{p_1}(\mathbb{S}^n \setminus \overline{U}_2)$$

Then w_1 corresponds to the generator of $H_{p_1}(U_1)$ where $U_1 \subset \mathbb{S}^n \setminus \overline{U}_2$ and the degree of

$$H_{p_1}(U_1) \to H_{p_1}(\mathbb{S}^n \setminus \overline{U}_2)$$

is just the linking number of $\mathbb{S}^1_1$ and $\mathbb{S}^1_2$.

We do not go into more details since Massey products can't distinguish Brunnian links other than the simplest ones, namely the Borromean rings. For this fact, see Truls Bakkejord Ræder, *Massey products and Linking*, Institutt for matematiske fag, NTNU.

4.4.6 Thom isomorphism

Let B be a manifold. A *vector bundle* $\pi : E \to B$ of rank n is a family of n-dimensional real vector spaces $\{E_x\}_{x\in B}$, with $E := \sqcup_{x\in B} E_x$ and $\pi : E \to B$ mapping E_x to x, equipped with a topology for E such that π is continuous and the following *local triviality* condition holds:

For each $x \in B$ there exists a neighbourhood U of x and a homeomorphism

$$t : E|_U := \pi^{-1}(U) \to U \times \mathbb{R}^n$$

which is fibre preserving in the sense that for all $x \in B$ the restriction of t on E_x is a vector space isomorphism onto $\mathbb{R}^n$.

The space E is called the *total space* and B is called the *base space*. The map π is called the *projection*. A continuous map $s : B \to E$ such that $\pi \circ s = \mathrm{id}$ is called a *section* of E. We can view B as a subset of E via the zero section $x \mapsto 0 \in E_x$. We denote $E_0 = E \setminus B$.

Typical examples of vector bundles include the trivial bundle $B\times\mathbb{R}^n$ and the tangent bundle $T_M = \sqcup_{x\in M} T_x M$ where M is a smooth manifold.

Example 4.113. A vector bundle $\zeta = (E, \pi, B)$ of rank n is trivial if and only if it has n sections which are linearly independent on $\pi^{-1}(x)$ for all $x \in B$.

Indeed, if ζ has n sections $\sigma_1, \dots, \sigma_n$ which are linearly independent, then we can define a bundle isomorphism $\Psi : B \times \mathbb{R}^n \to E$ by

$$\Psi(b, (x_1, \dots, x_n)) = x_1\sigma_1(b) + \dots x_n\sigma_n, \quad (b \in B, (x_1, \dots, xn) \in \mathbb{R}n)$$

It is obvious that every trivial bundle has n linearly independent sections.

A section of T_M is called a *vector field* on M. Thus, $T_{\mathbb{R}^n}$ is trivial but, by Hairly ball theorem, $T_{\mathbb{S}^2}$ is not trivial.

Let us define the *normal bundle* for a smooth manifold M with an atlas $\{(U_i, \phi_i)\}$. Each diffeomorphism $\phi_i \to \mathbb{R}^n$ induces a map

$$(\phi_i)_* : T_{U_i} \to T_{\mathbb{R}^n} = \mathbb{R}^n \times \mathbb{R}^n.$$

The normal bundle $N = N_{S/M}$ of a submanifold S in M is defined by the exact sequence

$$0 \to T_S \to (T_M)|_S \to N \to 0.$$

As with the case of a manifold, a *local orientation* of a vector bundle at $x \in B$ is a preferred generator $\mu_x \in H^n(E_x, E_x \setminus 0)$. A vector bundle is called *orientable* if for every point $xinB$, there is a neighbourhood $(x \in)U \subset B$ such that the there is a cohomology class $\mu_U \in H^n(\pi^{-1}(U), \pi^{-1}(U)_0)$ such that $\mu_U|_{E_x} = \mu_x$.

Theorem 4.114. *Let $\pi : E \to B$ be an oriented vector bundle of rank n. Then*

1. $H^m(E, E_0) = 0$ *for* $m < n$.
2. *There exists a unique cohomology class $u \in H^n(E, E_0)$, called the* Thom class *such that for all $x \in B$, the restriction of u to $H^n(E_x, E_x \setminus 0) = \mathbb{Z}$ is the preferred generator determined by the orientation.*
3. *The map $T : H^m(B) \to H^{m+n}(E, E_0)$, $\alpha \mapsto \pi^*\alpha \cup u$ is an isomorphism.*

Proof. For the sake of simplicity, we assume that B is compact. Thus we can choose a finite covering $\{U_i\}$ such that on each U_i, E is a trivial bundle.
1. First consider the case of trivial bundle $E = B \times \mathbb{R}^n$. By Künneth formula, we have

$$H^*(B) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n \setminus 0) = H^*(E, E_0),$$

and hence

$$H^m(E, E_0) = H^{m-n}(B) \otimes \mathbb{Z} = H^{m-n}(B).$$

So $H^n(E, E_0) = \mathbb{Z}$, and we choose u as the generator corresponding to the orientation. Then the theorem is verified to hold in this case.
2. We proceed by induction to construct u. Suppose $B = V \cup W$ where the assertions of the theorem hold for $E|_V$, $E|_W$ and $E|_{V\cap W}$. Considering the long exact sequence of the pair (E, E_0) we have

$$H^{m-1}(E|_{V\cap W}, E_0|_{V\cap W}) \to H^m(E, E_0) \to H^m(E|_V, E_0|_V) \oplus H^m(E|_W, E_0|_W)$$

The firs assertion follows from the assumption on V and W. For $m = n$, we have

$$0 \to H^n(E, E_0) \to H^n(E|_V, E_0|_V) \oplus H^n(E|_W, E_0|_W) \to H^n(E|_{V\cap W}, E_0|_{V\cap W}) \to \cdots$$

By assumption, the Thom classes u_V and u_W exist and are unique. By uniqueness, they have the same image in $E_{V\cap W}$, namely $u_{V\cap W}$. Thus they form a cohomology class $u \in H^n(E, E_0)$ which is uniquely defined since $H^{n-1}(E, E_0) = 0$.

To show the last assertion, we consider the following diagramme

$$\begin{array}{ccccc} H^m(V) \oplus H^m(W) & \longrightarrow & H^m(V\cap W) & \xrightarrow{\delta} & H^{m+1}(B) \\ \downarrow T & & \downarrow T & & \downarrow T \\ H^{m+n}(E|_V, E_0|_V) \oplus H^{m+n}(E|_W, E_0|_W) & \to & H^{m+n}(E|_{V\cap W}, E_0|_{V\cap W}) & \xrightarrow{\delta} & H^{m+n+1}(E, E_0) \end{array}$$

If we can show the diagamme commutes, then the 5-lemma will give the right T which is also isomorphism. Again, the point is to show the second square commutes. Choose a representative $\phi \in \mathbb{S}^{m+n}(E, E_0)$ of u. Then the restrictions ϕ_V, ϕ_W and $\phi_{V\cap W}$ represent the Thom classes u_V, u_W and $u_{V\cap W}$ respectively. Now take $a \in H^k(V\cap W)$ and a representative ψ. Suppose $\delta a = b$ and if we write $\psi = \psi_V - \psi_W$ where $\psi_V \in S^k(V)$ and $\psi_W \in S^k(W)$, we have $[\delta\psi_V] = b$. Hence

$$T\delta(a) = \pi^*(b) \cup u = \pi^*[\delta\psi_V] \cup u.$$

Next

$$\delta T(a) = \delta(\pi^*(a) \cup u_{V\cap W}) = [\delta\pi^*(\psi V) \cup \phi V] = T\delta.$$

The second equality holds because ϕ_V is closed, the last because π^* commutes with δ since it is a cochain map.

3. Suppose B is covered by finitely many open sets $B_1, \ldots, B_k$ such that the bundle E_{B_i} is trivial for each B_i. We proceed by induction on k: suppose the theorem holds for E. The case $k = 1$ being trivial, assume that the assertions holds for $k-1$. Then the theorem holds for $E|_{B_1\cup\cdots\cup B_{k-1}}$ and for $E|_{(B_1\cup\cdots\cup B_{k-1})\cap B_k}$ a well. Hence by *2.* the theorem holds for k. □

Now let M^n be a closed smooth manifold and S be a codimension k closed submanifold which is cooriented, i.e. the normal bundle N_S is an oriented vector bundle. Then a natural coorientation would be induced from orientations of TS and TM. The tubular neighbourhood theorem states that every submanifold S in M has a tubular neighbourhood which is diffeomorphic to the normal bundle. Then we can indentify such a tubular neighbourhood with our normal bundle $N_{S/M}$. And the Thom isomorphism applied to the normal bundle gives

$$H^*(S) \xrightarrow{T} H^{*+k}(N_S, N_S \setminus S) \to H^{*+k}(M, M \setminus S) \to H^{*+k}(M).$$

Without confusion, we denote the image of $1 \in H^0(S)$ in this sequence by Φ as the image of $u \in H^k(M, M\setminus S)$ in $H^k(M)$, and call the *Thom class* of S. Actually, this is the inverse of Poincaré with respect to S, i.e

Proposition 4.115. $i_*[S] = \Phi \cap [M]$.

Proof. We denote the inclusions $\kappa : N_S \to M$, $i : S \to M$ and the retraction $r : N_S \to S$. Then $i_* \circ r_* = \kappa_*$ on homology. We have

$$\Phi \cap [M] = u \cap [M, M \setminus S] = u \cap k_*[N, N \setminus S] = i_* \circ r_*(k^*u \cap [N, N \setminus S]).$$

Thus $\Phi \cap [M]$ is an image of some element of $H_{n-k}(S)$ by i_*. In other words, $\Phi \cap [M] = t \cdot i_*[S]$. We must show that $t = 1$. We only need to prove it in N. For any point $p \in S$, we choose a neighbourhood $U \subset S$, such that $(V, U) = (\mathbb{R}^n, \mathbb{R}^{n-k})$, where $V = \pi^{-1}(U)$ and $\pi : N \to S$ is the normal bundle. Then by definition of fundamental class, $[M]$ and $[S]$ restrict to fundamental classes μ_0 and ν_0 of $\mathbb{R}^n$ and $\mathbb{R}^{n-k}$ respectively. And u is pulled back to the Thom class u_0 of $\mathbb{R}^{n-k}$ in $\mathbb{R}^n$. Let $g : V \to N$ be the inclusion. Then reading off the relation in $H_n(N, N \setminus Q)$ where $Q \subset V$ is the set corresponding to $\mathbb{R}^k$, we have

$$i_*[S] = g_*\nu_0 = g_*(g^*u \cap \mu_0) = u \cap g_*(\mu_0) = u \cap [M] = t \cdot i_*[S].$$

The second inequality holds because of the elementary relation $u_0 \cap \mu_0 = nu_0$ in the case of trivial bundles (see Step 1 of the proof of Thom isomorphism.) Hence $t = 1$. □

Now we can complete the proof of Equation (4.87) resorting to Proposition 4.115. $\Phi_x \cap [M]$ is just $\epsilon(x)$ for any $x \in X \cap Y$ where the coorientation is induced from that of X and Y. (Note that we use normal bundle for the first and use tangent bundle for the second.) Hence the proof is reduced to that of

Proposition 4.116. $\Phi_{X\cap Y} = \Phi_X \cup \Phi Y$. *Equivalently,* $i_*[X \cap Y] = i_*[X] \cdot i_*[Y]$.

Proof.

$$\begin{aligned}
Phi_{X\cap Y} \cup [M] &= (i^M_{X\cap Y})_*[X \cap Y] \\
&= (i^M_Y)_*(i^Y_{X\cap Y})_*[X \cap Y] \\
&= (i^M_Y)_*(u^Y_{X\cap Y} \cap [Y]) \\
&= (i^M_Y)_*((i^M_Y)^*u^M_X \cap [Y]) \\
&= \Phi_X \cap (i^M_Y)_*[Y] \\
&= \Phi_X \cap (\Phi_Y \cap [M]) \\
&= (\Phi_X \cup \Phi_Y) \cap [M]
\end{aligned}$$

□

Thom isomorphism has lots of interesting applications, e.g. the Lefschetz fixed point theorem. Here, we mention another application, namely the Euler class.

Definition 4.117. Let $E \to M$ be a vector bundle and $o : M \to E$ be the zero section. Then the Euler class $e(E)$ is the image of Thom class u_E under the composition

$$H^n(E, E_0) \to H^n(E) \xrightarrow{o^*} H^n(M).$$

For the special case when S is a submanifold of M and E is taken as its normal bundle, the Euler class $e(S)$ is the pull back of the Thom class through

$$i : S \to (M, M \setminus S).$$

In this case, Euler class can be viewed as the obstruction of deforming S into $M \setminus S$.

Proposition 4.118. *If the inclusion $S \subset M$ is homotopic to a map $f : S \to M$ whose image is contained in $M \setminus S$, then $e(S) = 0$.*

Proof. By assumption, the inclusion $i : S \to (M, M \setminus S)$ is homotopic to a map $\phi : S \to (M, M \setminus S)$ which is factored through $(M \setminus S, M \setminus S)$. Hence in cohomology $i^* = \phi^*$ factors through $H^*(M \setminus S, M \setminus S) = 0$. So $i^* = 0$ and hence $e(N) = 0$. □

Proposition 4.119. *$T(e(S)) = u \cup u$ where $u \in H^k(M, M \setminus S)$ is the Thom class.*

Proof. $T(e) = T(i^*(u)) = \pi^* i^*(u) \cup u = u \cup u$. □

Corollary 4.120. *If the codimension k of S in M is odd, then $2e(S) = 0$.*

The name "Euler class" comes from the following

Theorem 4.121. *Let M be an oriented n-manifold, $\Delta : M \to M \times M$ be the diagonal map. Then*

1. *$\langle e(M), [\Delta_M] \rangle = \chi(M)$ where χ is the Euler characteristic.*
2. *The normal bundle of the diagonal $(Delta_M)$ is isomorphic to TM.*

The Euler class is a characteristic class in the following sense.

Definition 4.122. A cohomology class $c(E) \in H^*(M)$ associated to any vector bundle $E \to M$ is called a *characteristic class* if it is natural with respect to pull-backs, that is, $c(f^*E) = f^*(c(E))$.