Homotopy Theory

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Preface

A word from the transcriber

This is a study memo of “Homotopietheorie” by tom Dieck, Kamps and Puppe.

The preface by Puppe

This paper goes back to a lecture that I gave in the fall and winter of 1966/67 at the University of Minnesota, Minneapolis, Minn. USA, and the aim of which was to build up the basics of homotopy theory without gaps, without using other parts of algebraic topology (such as homology theory) and to arrive at interesting results (such as the suspension theorems and James’ theorem on the loop space of a suspension). In the winter semester of 1967/68 I read about the same topic again at the University of Saarland, Saarbrücken and tried to improve the presentation. Two listeners of this lecture have written the present paper: K.H. Kamps the §§0-7 and the appendix, T.tom Dieck the §§8-17.

In §§1-9, the theory of cofibrations and fibrations is discussed in detail. The results and methods are mostly known, but are not found elsewhere in a systematic compilation and seem to me to be fundamental.

§10 on the operation of the fundamental groupoid on the homotopy sets was expanded by tom Dieck based on his own ideas. (In the lecture only the case $K=$ pt space was discussed.)

In §§11-13, the homotopy groups are introduced in connection with the functors “suspension”, “loop space” and ”H-space”, “Co-H-space”.

§14 contains the fibre sequences from which the exact Homotopy sequences for pairs and for fibrations result as corollaries. Dual to this is the “Cofibre sequences”. We have omitted its description because it can be developed quite analogously and because it is discussed in detail in [19] (under the name “mapping sequences”). (The representation in [19] is at some points more complicated than is possible today by proceeding exactly dual to §14 with the help of the results of §§1, 2.)

The §§15-17 bring the homotopy excision theorem of Blakers-Massey, suspension theorems and a generalisation of James on the theorem on the loop space of a suspension. This theorem is proved using purely homotopy-theoretical means, and one obtains a true homotopy equivalence, where the other methods used only yield a weak homotopy equivalence. After completion of this manuscript have I realised that the proof given in §17 can still be simplifiied somewhat while maintaining the basic ideas [28].

Originally, I used James' theorem to prove the suspension theorems. Only later did I find the elementary proof of the homotopy excision theorem, which is given here in §15. (I received an important idea about this through an oral communication from J. M. Boardman.) It provides a simpler way to the suspension theorems and thus to the first interesting statements about the homotopy groups of spheres (cf.16.3) than the theorem of James and than all other methods known to us. Therefore, we have changed it accordingly. What remains of the previous structure is that the homotopy groups appear relatively late, although this is no longer necessary. Only a small part of the previous §§1-12 is needed for them. For James' theorem, the theory of §§1-12 on the other hand, is used decisively (see in particular 17.8 Auxiliary sentence 14).

I would like to thank my two co-authors for their cooperation. I would like to thank Mr Ulrich Mayr for a critical review and Mrs Marianne Karl for writing the manuscript.

Heidelberg, the 5th October, 1970 D. Puppe

Contents

Chapter 0

Fundamentals

0.1 Category theoretical foundations

We adopt the point of view of Brinkmann-Puppe [4] and build the theory of categories on a set theory with universes on (Brinkmann-Puppe [4], 1.1.5, 1.1.6). We henceforth assume the basic concepts of category theory (category, functor, natural transformation, dual category, diagramme, etc.), such as in Brinkmann-Puppe [4], 0., 2. are defined as known.
If $\mathfrak{C}$ is a category, then denote $|\mathfrak{C}|$ the set of objects of $\mathfrak{C}$, $\mathfrak{C}(X,Y)$ the set of morphisms from X to Y $(X, Y \in |\mathfrak{C}|)$, id_X the unit of $\mathfrak{C}(X,Y)$ $(X \in \mathfrak{C})$.
$f : X \to Y$ stands for $f \in \mathfrak{C}(X,Y)$. For the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$, we write gf or $g \circ f$.

Notation 0.1. In the following, we will mainly deal with the top category of topological spaces and continuous maps.
We assume the basics of set-theoretic topology to be known.
We use the following designations.

- $\mathbb{N}$ denotes the set of natural numbers: $\mathbb{N} = \{0, 1, 2, \dots\}$.
- Let $\mathbb{R}$ be the topological space of real numbers. The following two subsets of $\mathbb{R}$ will be encountered frequently: the closed unit interval $[0,1]$ of the real numbers - we denote it by I - and the subspace of the non-negative real numbers $\{x \in \mathbb{R} | x > 0\}$ - we denote it by $\mathbb{R}^+$.
- Let $\mathbb{R}^n$ be the n-dimensional Euclidean space ($n \in \mathbb{N}$, $n \geq 1$), $\mathbb{E}^n$ be the n-dimensional solid sphere of Radius 1 ($n \in \mathbb{N}$, $n \geq 1$), $\mathbb{S}^n$ be the n-sphere ($n \in \mathbb{N}$).
- If X, Y are topological spaces, then let $\mathrm{proj}_1 : X \times Y \to X$ be the projection of the topological product $X \times Y$ onto the first factor, $\mathrm{proj}_2 : X \times Y \to Y$ be the projection onto the second factor. Let X be a topological space, A

a subset of X, then let X/A be the topological space that arises from X when A is identified to a point.[1]

Definition 0.2. Let $\mathfrak{C}$ be a category.

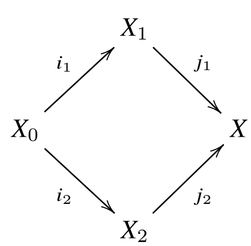
- If $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ are morphisms of $\mathfrak{C}$ with $gf = \mathrm{id}_X$, then g is called a *left inverse* to f, f a *right inverse* to g.
- A morphism of $\mathfrak{C}$ is called a *section* if it has a left inverse, a morphism is called a *retraction* if it has a right inverse.
- A morphism f of $\mathfrak{C}$ is called an *isomorphism* if there exists a morphism g which is inverse (i.e. left inverse and right inverse) to f. Such a g is uniquely determined by f. We write $g =: f^{-1}$.

Definition 0.3. Let $\mathfrak{C}$ be the category $\mathcal{top}$ of topological spaces. Let X be a topological space, A be a subspace of X, $i : A \subset X$ be the inclusion, then especially interesting is a retraction $r : X \longrightarrow A$, for which $ri = \mathrm{id}_A$.
We call such a retraction a *retraction from X to A*.
The subspace A is called the *retract* of X if a retraction from X to A exists.

Definition 0.4. A *natural equivalence relation* "$\sim$" in a category $\mathfrak{C}$ consists of one equivalence relation each "$\sim_{(X,Y)}$=: "$\sim$" in each set of morphisms $\mathfrak{C}(X, Y)$ $(X, Y \in |\mathfrak{C}|)$, so that for all $f, g : X \longrightarrow Y$, $f', g' : Y \longrightarrow Z$ the following applies: $(f \sim g$ and $f' \sim g') \Rightarrow (f' f \sim g' g)$.

If "$\sim$" is a natural equivalence relation in $\mathfrak{C}$, then one can form the *factor category* $\mathfrak{C}/(\sim)$ (Mitchell [17], 1.3). $\mathfrak{C}/(\sim)$ has the same objects as $\mathfrak{C}$.
The morphisms of $\mathfrak{C}/(\sim)$ are the equivalence classes $[f]$ with respect to "$\sim$" of the morphisms f of $\mathfrak{C}$. The composition in $\mathfrak{C}/(\sim)$ is given by the equation $[g][f] = [gf]$.
The units of $\mathfrak{C}/(\sim)$ are the equivalence classes with respect to "$\sim$" of the units of $\mathfrak{C}$.

Definition 0.5. Let $\mathfrak{C}$ be a category. A diagramme in $\mathfrak{C}$



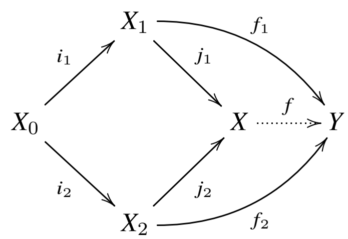
(0.6)

is called a *cocartesian square* if the conditions (1) and (2) are satisfied:

(1) $j_1 i_1 = j_2 i_2$ (i.e., the diagramme is commutative),

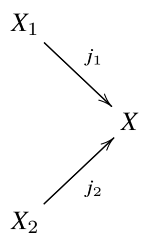
[1] If A is empty, then X/A is the topological sum of X and a space that has exactly one point.

(2) For every two morphisms $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ of $\mathfrak{C}$ with $f_1 i_1 = f_2 i_2$ there is exactly one morphism $f : X \to Y$ of $\mathfrak{C}$ with $f h_\nu = f_\nu$ $(\nu = 1, 2)$.

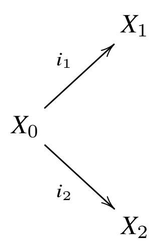


In (2), the requirement of unambiguity is waived from f, one obtains the term "weak cocartesian square" (Freyd).

Remark 0.7. In a cocartesian square (0.6)



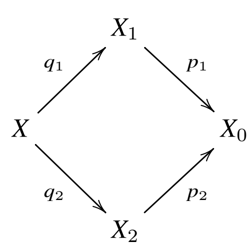
is clearly determined by



up to isomorphism.

Definition 0.8. Dual[2] to the term "cocartesian square" is the term 'cartesian square".
Let $\mathfrak{C}$ be a category.
A diagramme in $\mathfrak{C}$

(0.9)

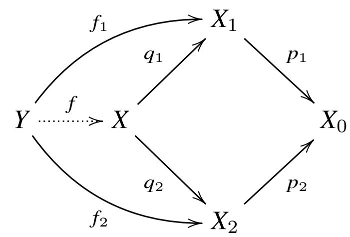


[2]Exactly: $({}^*\mathfrak{C}|\mathfrak{C})$ - dual in the sense of Brinkmann-Puppe [4], 2.2 (Transition from $\mathfrak{C}$ to the dual category ${}^*\mathfrak{C}$)

is called a *cartesian square* if (1) and (2) are satisfied:

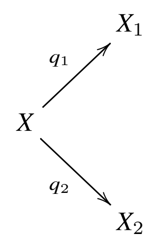
(1) $p_1 q_1 = p_2 q_2$,

(2) For every two morphisms $f_1 : Y \to X_1$, $f_2 : Y \to X_2$ of $\mathfrak{C}$ with $p_1 f_1 = p_2 f_2$ there is exactly one morphism $f : Y \to X$ of $\mathfrak{C}$ with $q_\nu f = f_\nu$ $(\nu = 1, 2)$.

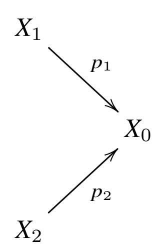


In (2), the requirement of unambiguity is waived from f, one recovers the term “weak cartesian square”.

Remark 0.10. In a cartesian square (0.9)

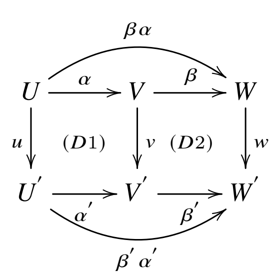


is clearly determined by



up to isomorphism.

Theorem 0.11. *Let $\mathfrak{C}$ be a category. Given the diagrammes (D1), (D2), and the outer box (D3) in $\mathfrak{C}$:*



(0.12)

Claim.

(a) If (D1) and (D2) are weakly cocartesian squares, (D3) is a weakly cocartesian square, too.

(b) If (D1) and (D2) are cocartesian squares, (D3) is a cocartesian square, too.

(c) If (D1) and (D2) are weakly cartesian squares, then (D3) is a weakly cartesian square.

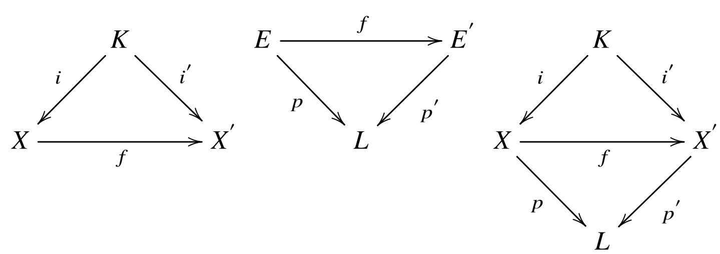
(d) If (D1) and (D2) are cartesian squares, then (D3) is a cartesian square.

Proof. The proof of the theorem is simple and is left to the reader (see Brown [5], 6.6.5, Kamps [15], 0.10).
Note: (c) is dual to (a), (d) is dual to (b). □

Definition 0.13. In addition to the category of topological spaces, we will deal with some other categories derived from the category $\mathcal{T}op$.
For this we carry out the following general category-theoretic constructions.
Let $\mathfrak{C}$ be a category and let K, L be objects of $\mathfrak{C}$.
We define categories $\mathfrak{C}^K$, $\mathfrak{C}_L$, $\mathfrak{C}^K_L$.
Let the *objects* of $\mathfrak{C}^K$ be the morphisms of $\mathfrak{C}$ that have K as the source, let the objects of $\mathfrak{C}_L$ be the morphisms of $\mathfrak{C}$, which have L as their target, let the objects of $\mathfrak{C}^K_L$ be the diagrammes ξ in $\mathfrak{C}$ of the form

$$K \xrightarrow{i} X \xrightarrow{p} L.$$

Let i, i' (p, p'; ξ, ξ') be objects of $\mathfrak{C}^K$ ($\mathfrak{C}_L$, $\mathfrak{C}^K_L$ respectively).
The *morphisms* $i \to i'$ ($p \to p'$, $\xi \to \xi'$) of $\mathfrak{C}^K$ ($\mathfrak{C}_L$, $\mathfrak{C}^K_L$ resp.) be the commutative diagrammes in $\mathfrak{C}$ of the form



We then write (abuse of language):

$$f : i \to i', \quad f : p \to p', \quad f : \xi \to \xi'.$$

The composition of the morphisms in the new categories is induced by the composition in $\mathfrak{C}$. Units are

$$\mathrm{id}_X : i \to i, \quad \mathrm{id}_E : p \to p, \quad \mathrm{id}_X : \xi \to \xi.$$

$\mathfrak{C}^K$ means the category of objects under K, $\mathfrak{L}$ the category of objects over L, $\mathfrak{C}^K_L$ the category of objects under K and over L.

We note: A morphism f of $\mathfrak{C}^K$ ($\mathfrak{C}_L$, $\mathfrak{C}^K_L$) is an isomorphism of $\mathfrak{C}^K$ ($\mathfrak{C}_L$, $\mathfrak{C}^K_L$) resp.) if and only if f is an isomorphism of $\mathfrak{C}$.

Remark 0.14. If K is a *copoint* of the category $\mathfrak{C}$, i.e. if $\mathfrak{C}(K, X)$ has exactly one element for all $X \in |\mathfrak{C}|$, then $\mathfrak{C}^K$ can be canonically identified with $\mathfrak{C}$ and $\mathfrak{C}^K_L$ with $\mathfrak{C}_L$. If L is a *point* of the category $\mathfrak{C}$, i.e. $\mathfrak{C}(X, L)$ have exactly one element for all $X \in |\mathfrak{C}|$, then $\mathfrak{C}_L$ can be canonically identified with $\mathfrak{C}$ and $\mathfrak{C}^K_L$ with $\mathfrak{C}^K$.

Definition 0.15. In the case $\mathfrak{C} = \mathcal{T}op$, the objects of $\mathfrak{C}^K$ are called *spaces under K*, the objects of $\mathfrak{C}_L$ are called *spaces over L*, the objects of $\mathfrak{C}^K_L$ are called *spaces under K and over L*, the morphisms of $\mathfrak{C}^K$ ($\mathfrak{C}_L$, $\mathfrak{C}^K_L$) the *maps under K* (*over L*, *under K and over L* resp.).

Instead of map over L, one also says *fibrewise map*, since a map over L $f : p \to p'$ for each $b \in L$ maps $p^{-1}b$ the *fibre over b* into $p'b$ the fibre over b.

The empty topological space $\emptyset$ is a copoint in the category $\mathcal{T}op$. Therefore, the following applies: $\mathcal{T}op^{\emptyset} = \mathcal{T}op$, $\mathcal{T}op^{\emptyset}_L = \mathcal{T}op_L$ $(L \in |\mathcal{T}op|)$.
Each point space P, i.e., every topological space P whose underlying set has exactly one element, is a point in the category $\mathcal{T}op$. Therefore, the following applies: $\mathcal{T}op_P = \mathcal{T}op$, $\mathcal{T}op^K_P = \mathcal{T}op^K$ $(K \in |\mathcal{T}op|)$.
We also use the term $\mathcal{T}op^o := \mathcal{T}op^P$. We also call $\mathcal{T}op^o$ the *category of pointed topological spaces.*
We can consider the objects of $\mathcal{T}op^o$ as pairs (X, o) where X is a topological space and $o \in X$ is the base point. The morphisms $(X, o) \to (X', o')$ of $\mathcal{T}op^o$ are the basepoint-preserving (pointed) continuous maps, i.e. the continuous maps $f : X \to X'$ with $f(o) = o'$.

Definition 0.16. If $\mathfrak{C}$ is a category, we also have the category of pairs $\mathfrak{C}(2)$.
The objects of $\mathfrak{C}(2)$ are the morphisms of $\mathfrak{C}$.
Let u, u' be objects of $\mathfrak{C}(2)$.
The morphisms $u \to u'$ of $\mathfrak{C}(2)$ are the commutative diagrams in $\mathfrak{C}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

We write $(f, g) : u \to u'$.
The composition in $\mathfrak{C}(2)$) is induced from the composition in $\mathfrak{C}$. Unit $u \to u$ is the morphism $(\mathrm{id}_X, \mathrm{id}_Y)$.
A morphism (f, g) of $\mathfrak{C}(2)$ is exactly then a isomorphism of $\mathfrak{C}(2)$ if f and g are isomorphisms of $\mathfrak{C}$.

At the end of 0.1 we ask the reader to familiarise himself with the concept of adjoint functors (see Mitchell [17], V.).

0.2 Fundamentals of homotopy theory

Definition 0.17. A continuous map of the form $\varphi : X \times I \to Y$ where X and Y are topological spaces is called *homotopy.*

A homotopy $\varphi : X \times I \to Y$ gives, by $\varphi_t(x) := \varphi(x,t)$ for $x \in X$, a family of continuous maps $\varphi_t : X \to Y$, $t \in I$.
If $j_t : X \to X \times I$ is the continuous map $x \mapsto (x,t)$, we obtain $\varphi_t = \varphi \circ j_t$.

Definition 0.18. Let X, Y be topological spaces, $f, g : X \to Y$ be continuous maps.
f is called *homotopic to* g if a homotopy $\varphi : X \times I \to Y$ exists with $\varphi_0 = f$ and $\varphi_1 = g$, so if there is a continuous map $\varphi : X \times I \to Y$, so that for all $x \in X$

$$\varphi(x,0) = f(x), \quad \varphi(x,1) = g(x).$$

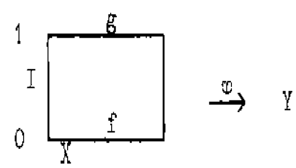


Figure 1:

Such a φ is called a *homotopy from f to g.*
We write: $f \simeq g$, if f is homotopic to g, and $\varphi : f \simeq g$, if φ is a homotopy from f to g.

Theorem 0.19. *"$\simeq$" is a natural equivalence relation in $\mathcal{T}op$ (see Definition 0.4).*

Proof. 1. [Reflexivity] Let $f : X \to Y$ be a continuous mapping. By $\varphi(x,t) := f(x)$ for $(x,t) \in X \times I$ we get a homotopy $\varphi : X \times I \to Y$ from f to f.

2. [Symmetry] Let $f, g \in \mathcal{T}op(X,Y)$ and $\varphi : f \simeq g$, then

$$\varphi'(x,t) := \varphi(x, 1-t), \quad \text{for} \quad (x,t) \in X \times I$$

delivers a homotopy from g to f.

3. [Transitivity] Let $f, g, h \in \mathcal{T}op(X,Y)$, $\varphi : f \simeq g$, $\psi : g \simeq h$, then $\chi : X \times I \to Y$ is a continuous (!) map (!), given by

$$\chi(x,t) = \begin{cases} \varphi(x, 2t), & \text{if} \quad 0 \le t \le \frac{1}{2}, x \in X, \\ \psi(x, 2t-1), & \text{if} \quad \frac{1}{2} \le t \le 1, x \in X, \end{cases}$$

which is a homotopy from f to h.

4. [Naturality] Let $f, g : X \to Y$, $f', g' : Y \to Z$ be continuous maps with $f \simeq g$, $f' \simeq g'$.
   Assertion: $f'f \simeq g'g$.

   Proof. Let $\varphi : f \simeq g$, $\varphi' : f' \simeq g'$. Then the following

$$f' \circ \varphi : f'f \simeq f'g, \quad \varphi' \circ (g \times \mathrm{id}_I) : f'g \simeq g'g, \quad \text{thus} f'f \simeq g'g,$$

   applies since "$\simeq$", as already shown, is transitive. □

□

Remark 0.20. Since "$\simeq$" is a natural equivalence relation in $\mathcal{Top}$, we can form the factor (residue) category $\mathcal{Top}/(\simeq)$ (cf. Definiton 0.4).
We note them $\mathcal{Top}h$ and call them the *homotopy category associated to* $\mathcal{Top}$.
Thus, for $X, Y \in |\mathcal{Top}|(= |\mathcal{Toph}|)$, $\mathcal{Top}h(X, Y)$ consists of the homotopy classes of the continuous maps from X to Y.
We shorten: $\mathcal{Top}h(X, Y) =: [X, Y]$.
If f is a continuous mapping, then $[f]$ denote the homoopy class of f.

Definition 0.21. A continuous map $f : X \to Y$ is called a *homotopy equivalence* (an *h-equivalence*for short) if $[f]$ is an isomorphism in $\mathcal{Toph}$, that is, if there exists a continuous map $g : Y \to X$ with $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$.
Such a g is called the *homotopy inverse* (*h-inverse* for short) to f.
If $f : X \to Y$, $g : Y \to X$ are continuous maps with $gf \simeq \mathrm{id}_X$, then g is called a *homotopy left inverse* (*h-left inverse*) to f, f a *homotopy right inverse* (*h-right inverse*) to g.

Definition 0.22. The terms "null homotopic" and "contractible" are derived from the homotopy concept.

(1) A continuous map $\chi : X \to Y$ is called *constant* if $y_0 \in Y$ exists with $\chi(X) = \{y_0\}$.

(2) A continuous map $f : X \to Y$ is called *null homotopic* if it is homotopic to a constant map.

(3) A topological space X is called *contractible* if id_X is null homotopic.

Remark 0.23. Let a, b be real numbers with $a < b$. If one replaces the interval $[0, 1]$ with the interval $[a, b]$ in the definition of the term "homotopy", one obtains, as one might easily think, an equivalent term.
A corresponding remark is always applicable when a definition is based on the homotopy concept for example, in the definition of the homotopy extension property (cf. Definition 1.5), the covering homotopy property (cf. Definition 2.21, Theorem 2.23) and the definition of the terms "cofibration" and "fibration" (cf. Definition 1.6 and Definition 2.26).
In consistency with Definition 0.17 we call continuous maps $\varphi : X \times [a, b] \to Y$ ($X, Y \in |\mathcal{Top}|$) *homotopies* and define for $t \in [a, b]$ a continuous mapping

$j_t : X \to X \times [a,b]$ by $j_t(x) := (x,t)$ for $x \in X$.
If $\varphi : X \times [a,b] \to Y$ is a homotopy, we assume for $t \in [a,b]$, $\varphi_t := \varphi \circ j_t : X \to Y$.
So we have $\varphi_t(x) = \varphi(x,t)$ for $x \in X$, $t \in [a,b]$,

Let K and L be topological spaces.
We define homotopy terms in the category $\mathcal{Top}^K$ of the topological spaces under K, in the category $\mathcal{Top}_L$ of the topological spaces over L, in the category $\mathcal{Top}_L^K$ and in the category of pairs $\mathcal{Top}(2)$ (cf.(Definitions 0.13, 0.15 and 0.16.)

Definition 0.24. Let $f, g : i \to i'$ be morphisms of $\mathcal{Top}^K$:

$$\begin{array}{ccc} & K & \\ {}^{i}\swarrow & & \searrow^{i'} \\ X & \xrightarrow[f,g]{} & X' \end{array}$$

f is called a *homotopy under K to g* if a homotopy $\varphi : X \times I \to X'$ exists with $\varphi : f \simeq g$ and $\varphi(i \times \mathrm{id}_I) = i' \circ \mathrm{proj}_1$.

$$\begin{array}{ccc} X \times I & \xrightarrow{\varphi} & X' \\ {\scriptstyle i\times \mathrm{id}_I}\uparrow & & \uparrow{\scriptstyle i'} \\ K \times I & \xrightarrow[\mathrm{proj}_1]{} & K \end{array}$$

where φ is a *homotopy under K from f to g*. We write $f \overset{K}{\sim} g$ if f is homotopic under K to g, and $\varphi : f \overset{K}{\sim} g$ if φ is a homotopy under K from f to G.
The condition $\varphi(i \times \mathrm{id}_I) = i' \circ \mathrm{proj}_1$ states: for all $t \in I$, $\varphi_t \circ i = i'$ holds, i.e. for all $t \in I$, $\varphi : X \to X'$ is a morphism of $\mathcal{Top}^K$, $\varphi_t : i \to i'$.

Special cases.

(1) K is a subspace of X, i is the inclusion $K \subset X$. In addition to "homotopic under K" and "homotopy under K" are then also the terms "homotopic relative to K", and "homotopy relative to K" are common.
Then we also write

$$\text{“}f \simeq g \operatorname{rel} K\text{''} \quad \text{instead of} \quad \text{“}f \overset{K}{\simeq} g\text{''} \quad \text{and}$$
$$\text{“}\varphi : f \simeq g \operatorname{rel} K\text{''} \quad \text{instead of} \quad \text{“}\varphi : f \overset{K}{\simeq} g\text{''}.$$

A homotopy φ relative to K has the property: for each $a \in K$, $\varphi(a,t)$ is independent of $t \in I$.

(2) If K is a one-point space, i.e. $\mathcal{Top}^K = \mathcal{Top}^o$ (cf. Definition 0.15), thus, the terms "pointed homotopic" and "pointed homotopy" is common.

Definition 0.25. Homotopies under K can be expressed as morphisms of $\mathcal{T}op^K$. Let us first consider the situation in the category $\mathcal{T}op$.
If X is a topological space, then we have the *cylinder on X* $IX := X \times I$[3] Homotopies in $\mathcal{T}op$ are now morphisms of $\mathcal{T}op$ of the form $IX \to Y$, where Y is another topological space. We are now transferring the cylinder construction from $\mathcal{T}op$ to $\mathcal{T}op^K$.

If $i : K \to X$ is a space under K, then let $I^K X$ be the topological space that is created from $X \times I$ when $(ia, t) \in X \times I$ for each $(a, t) \in K \times I$ is identified with $(ia, 0) \in X \times I$.

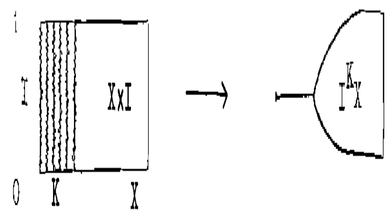


Figure 2:

If we connect the natural projection $X \times I \to I^K X$ to $K \xrightarrow{i} X \xrightarrow{j_0} X \times I$, we get a space under K, $I^K i : K \to I^K X$.
If $\overline{\varphi} : I^K i \to i'$ a morphism of $\mathcal{T}op^K$, where $i' : K \to X'$ is another space under K, then one can obtain a homotopy under K, $\varphi :: X \times I \to X'$ by composing the natural projection from $X \times I$ to $I^K X$ with $\overline{\varphi}$.
The assignment $\overline{\varphi} \to \varphi$ provides a bijection between the morphisms of $\mathcal{T}op^K$ of the form $I^K i \to i'$ and the homotopies under K.

Definition 0.26. Let $f, g : p \to p'$ be morphisms of $\mathcal{T}op_L$:

$$\begin{array}{ccc} E & \xrightarrow{f,g} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & L & \end{array}$$

f is called a *homotopy over L* to g ($f \underset{\simeq}{L} g$) if there exists a homotopy $\varphi : E \times I \to E'$ with $\varphi : f \simeq g$ and $p' \circ \varphi = p \circ \mathrm{proj}_1$.

$$\begin{array}{ccc} E \times I & \xrightarrow{\varphi} & E' \\ & {\scriptstyle p \circ \mathrm{proj}_1}\searrow \quad \swarrow{\scriptstyle p'} & \\ & L & \end{array}$$

[3]The transcriber believes $\mathrm{Cyl}\,X$ would be a better notation than IX.

Where φ is a *homotopy over L from f to g* ($\varphi : f \underset{\simeq}{L} g$).
The condition $p^{'}\varphi = p \circ \mathrm{proj}_1$ states: for all $t \in I$, $p^{'}\varphi_+ = p$, i.e. for all $t \in I$, $\varphi_+ : E \to E^{'}$ is a morphism of $\mathcal{T\!op}_L$, $\varphi_t : p\ top^{'}$.
Further, the equation $p^{'}\varphi = p \circ \mathrm{proj}_1$ means: the homotopy φ over L are exactly the morphism $\varphi : p \circ \mathrm{proj}_1 \to p^{'}$ of $\mathcal{T\!op}_L$. The cylinder construction in $\mathcal{T\!op}$ thus corresponds in $\mathcal{T\!op}_L$ to the transition from the space $p : E \to L$ over L to the space $L_L p := p \circ \mathrm{proj}_1 : E \times I \to L$ over L. If φ is a homotopy over L, then for all $t \in I$ and $b \in L$

$$\varphi_t(p^{-1}b) \subset p^{'-1}b,$$

that is, the fibre $p^{-1}b$ over b are mapped over b during the entire homotopy φ in the fibre $p^{'-1}b$ over b. Therefore, in addition to "homotopic over L" and "homotopy over L" we also have useful designations "vertically homotopic", "fibre-wise homotopic", "vertical homotopy", "fibre-wise homotopy".

Using the same formulae as in the proof of Theorem (0.19) show you:

Theorem 0.27. *"$\overset{K}{\simeq}$" and "$\underset{L}{\simeq}$" are natural equivalence relations in $\mathcal{T\!op}^K$ and $\mathcal{T\!op}_L$ respectively.*

Definition 0.28. Thus one has factor categories

$$\mathcal{T\!op}^K/(\overset{K}{\simeq}) =: \mathcal{T\!op}^K h \quad \text{and} \quad \mathcal{T\!op}_L/(\underset{L}{\simeq}) =: \mathcal{T\!op}_L h.$$

Let $i : K \to X$, $i^{'} : K \to X^{'}$ be spaces under K, $p : E \to L$, $p^{'} : E^{'} \to L$ be spaces over L, so we write instead of $\mathcal{T\!op}^K h(i, i^{'})$ also (inaccurately) $[X, X^{'}]^K$, instead of $\mathcal{T\!op}_L h(p, p^{'})$ also (inaccurately) $[E, E^{'}]_L$.
If K is a one-point space, we use the designation $[X, X^{'}]^o$.
If f is a morphism of $\mathcal{T\!op}^K$ or $\mathcal{T\!op}_L$, then denote $[f]^K$ resp. $[f]_L$ the equivalence class of f with respect to "$\overset{K}{\simeq}$" resp. "$\underset{L}{\simeq}$".
If K is a one-point space, we use the designation $[f]^0$. A morphism f of $\mathcal{T\!op}^K$ or $\mathcal{T\!op}_L$, is called *homotopy equivalence* (*h-equivalence*) *under K* or *homotopy equivalence over L* if $[f]^K$ resp. $[f]_L$ is an isomorphism in $\mathcal{T\!op}^K h$ and $\mathcal{T\!op}_L h$, respectively.

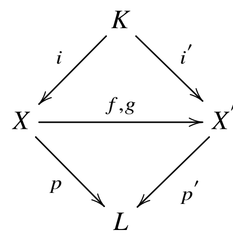
Remark 0.29. If a morphism f of $\mathcal{T\!op}^K$ (or $\mathcal{T\!op}_L$) an h-equivalence under K (over L), so is f, interpreted as morphism of $\mathcal{T\!op}$, an h-equivalence.

Definition 0.30. (1) Let p and $p^{'}$ be spaces over L. p is called *h-equivalent over L to $p^{'}$* if p and $p^{'}$ are isomorphic objects of $\mathcal{T\!op}_L h$, that is, if there exists an h-equivalence over L $p \to p^{'}$.

(2) Let i and $i^{'}$ be spaces under K. i is called *h-equivalent under K to $i^{'}$* if i and $i^{'}$ are isomorphic objects of $\mathcal{T\!op}^K h$.

The definition of a homotopy concept in $\mathcal{T\!op}^K_L$ is now clear.

Definition 0.31. Let $f, g : \xi \to \xi'$ be morphisms of $\mathcal{T}\!op_L^K$:



A *homotopy under K and over L from f to g* is a homotopy $\varphi : X \times I \to X'$, such that $\varphi : f \simeq g$ and φ_t for all $t \in I$ is a morphism of $\mathcal{T}\!op_L^K$ $\xi \to \xi'$.

The homotopy relation defined in this way is a natural equivalence relation in $\mathcal{T}\!op_L^K$. One therefore has a factor category $\mathcal{T}\!op_L^K h$.

Remark 0.32. In the special case $K = \emptyset$ (or L is a one-point space), the homotopy concept of (0.31) coincides with the homotopy concept of (0.26) (or (0.24)) in $\mathcal{T}\!op_L$ (or $\mathcal{T}\!op^K$).

Remark 0.33. Homotopies under K and over L can be understood as morphisms of $\mathcal{T}\!op_L^K$.

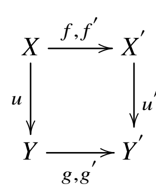
If $\xi = (K \xrightarrow{i} X \xrightarrow{p} L)$ is a space under K and over L, then first, by Definition (0.25), we obtain a space under K $I^K i : K \to I^K X$.
$I^K X$ is created from $X \times I$ by identifying $(ia, t) \in X \times I$ for each $(a, t) \in K \times I$ with $(ia, 0) \in X \times I$.
The continuous map $p \circ \mathrm{proj}_1 : X \times I \to L$ is compatible with the identifications made in $X \times I$, thus induces a continuous map $I^K X \to L$.
We thus obtain a space below K and above L

$$I_L^K \xi = (K \to I^K X \to L).$$

Let $\overline{\varphi} : I_L^K \xi \to \xi'$ be a morphism of $\mathcal{T}\!op_L^K$, where $\xi' = (K \xrightarrow{i'} X' \xrightarrow{p'} L)$ is another object of $\mathcal{T}\!op_L^K$, then one obtains a homotopy under K and above L $\varphi : X \times I \to X'$ by composing the natural projection on $X \times I$ to $I^K X$ with $\overline{\varphi}$.
The assignment $\overline{\varphi} \to \varphi$ provides a bijection between the morphisms of $\mathcal{T}\!op_L^K$ of the form $I_L^K \xi \to \xi'$ and the homotopies under K and over L.

Finally, we have the following homotopy notion in $\mathcal{T}\!op(2)$.

Definition 0.34. Let $(f, g), (f', g') : U \to u'$ be morphisms of $\mathcal{T}\!op(2)$:



A *homotopy of pairs from* (f, g) *to* $(f^{'}, g^{'})$ is a pair (φ, ψ) of homotopies $\varphi : X{\times}I \to X^{'}$, $\psi : Y{\times}I \to Y^{'}$ such that $\varphi : f \simeq f^{'}$, $\psi : g \simeq g^{'}$ and $u^{'} \circ \varphi = \psi \circ (u{\times}\mathrm{id}_I)$.

$$\begin{array}{ccc} X \times I & \xrightarrow{\varphi : f \simeq f^{'}} & X^{'} \\ {\scriptstyle u\times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle u^{'}} \\ Y \times I & \xrightarrow[\psi : g \simeq g^{'}]{} & Y^{'} \end{array}$$

The last condition says: For all $t \in I$, (φ_t, ψ_r) is a morphism of $\mathcal{T}op(2)$ $u \to u^{'}$. The homotopy relation defined in this way is a natural equivalence relation in $\mathcal{T}op(2)$. We therefore have a factor category $\mathcal{T}op(2)h$.

Chapter 1

Cofibrations

1.1 Homotopy extensions and cofibrations

1.1.1 The extension problem

Let $i : A \to X$, $g : A \to Y$ be continuous maps. We ask: Does a continuous map $f : X \to Y$ exist with $fi = g$, i.e. is the diagramme

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle g}\downarrow & & \\ Y & & \end{array} \tag{1.1}$$

completed by a continuous map $f : X \to Y$ to the commutative triangle below?

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle f} & \\ Y & & \end{array} \tag{1.2}$$

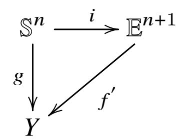
If i is specifically an inclusion $A \subset X$, the problem is whether a continuous constant defined on the subspace A of X can be expanded to a continuous map defined on X.
This problem is generally not soluble.

Example 1.3. let i be the inclusive of the n-Sphere $\mathbb{S}^n$ in the $(n+1)$-ball $\mathbb{E}^{n+1}$. Since $\mathbb{S}^n$ is not a retract of $\mathbb{E}^{n+1}$ (Eilenberg-Steenrod [9], XI. Theorem 3.2, Hurewicz-Wallman [13] , IT. 1 . B)), $g = \mathrm{id}_{\mathbb{S}^n}$ cannot be extended to $\mathbb{E}^{n+1}$.

However, it applies:

Theorem 1.4. *If i is the inclusion $\mathbb{S}^n \subset \mathbb{E}^{n+1}$, then Diagramme (1.1) always becomes a commutative triangle (1.2) if there is a continuous map $f' : \mathbb{E}^{n+1} \to Y$*

with $f^{'}i \simeq g$ *exists.*



Proof. Let $\varphi : \mathbb{S}^n \times I \to Y$, $\varphi : f^{'}i \simeq g$.
We define $\Phi^{'} : (\mathbb{E}^{n+1} \times \{0\}) \cup (\mathbb{S}^n \times I) \to Y$ by

$$\begin{cases} (x,0) \mapsto f^{'}(x), & x \in \mathbb{E}^{n+1}, \\ (a,t) \mapsto \varphi(a,t), & (a,t) \in \mathbb{S}^n \times I. \end{cases}$$

The definition makes sense, since $\varphi_0 = f^{'}i$, and returns a continuous map, since $\mathbb{E}^{n+1} \times \{0\}$ and $\mathbb{S}^n \times I$ are completed in $(\mathbb{E}^{n+1} \times \{0\}) \cup (\mathbb{S}^n \times I)$. By projecting from the point $(0, \dots, 0, 2) \in \mathbb{R}^{n+2}$ we get a retraction[1]

$$r : \mathbb{S}^n \times I \to (\mathbb{E}^{n+1} \times \{0\}) \cup (\mathbb{S}^n \times I).$$

Then $\Phi := \Phi^{'}r : E^{n+1} \times I \to Y$ is an extension of $\Phi^{'}$ and for the continuous map

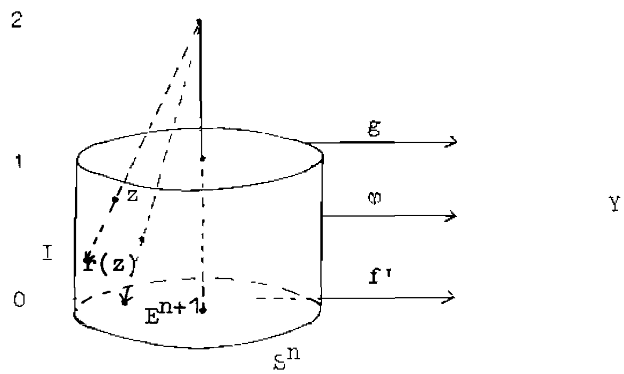


Figure 1.1:

$f := \Phi_1 : \mathbb{E}^{n+1} \to Y$ (i.e. $f(x) = \Phi(x,1)$ for $x \in \mathbb{E}^{n+1}$) satisfies $fi = g$. □

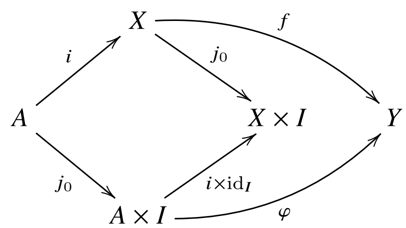
1.1.2 The homotopy extension property (HEP). Cofibrations

The essential step in the proof of Theorem (1.4) was the extension of the homotopy $\varphi : \mathbb{S}^n \times I \to Y$ to the homotopy $\Phi : \mathbb{E}^{n+1} \times I \to Y$, such that Φ_0 is a given extension $(f^{'})$ of φ_0. This leads us to the following definition.

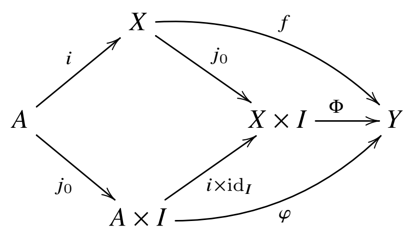
[1] An explicit formula for r can be found in Hilton [ll], p.11.

Definition 1.5. Let $i : A \to X$ be a continuous map, let Y be a topological space.
We say *i has the homotopy extension property* (*HEP* for short) *for Y*, if for all continuous maps $f : X \to Y$ and $\varphi : A \times I \to Y$, such that $\varphi(a, 0) = fia$ for all $a \in A$ (i.e. $\varphi_0 = fi$), a continuous map $\Phi : X \times I \to Y$ exists, such hat $\Phi(i \times \mathrm{id}_I) = \varphi$[2] and $\Phi(x, 0) = fx$ for all $x \in X$ (i.e. $\Phi_0 = f$).

So i has the HEP for Y if and only if every commutative diagramme in $\mathcal{T}op$ of the form



can be supplemented by a continuous map $\Phi : X \times I \to Y$ to the commutative diagramme



We illustrate the definition for the special case an inclusion $i : A \subset X$ through a sketch.

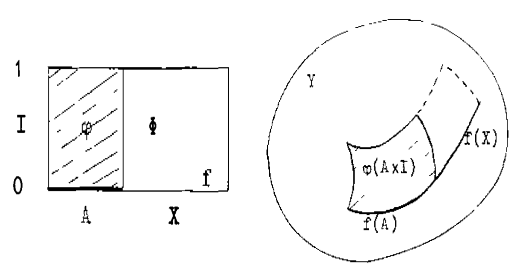
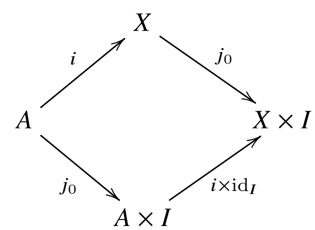


Figure 1.2:

[2]We then say, even if i is not an inclusion, Φ is an *extension* of φ.

Definition 1.6. A continuous map $i : A \to X$ is called a *cofibration* if i has the HEP for all topological spaces.

So i is a cofibration if and only if the diagramme in $\mathscr{T}\!op$



(1.7)

is a weakly cocartesian square (cf. Definition (0.5)).
From the proof of Theorem (1.4) we see:

Example 1.8. $i : \mathbb{S}^n \subset \mathbb{E}^{n+1}$ is a cofibration.

Exercise 1.9. Every homeomorphism is a cofibration.

Exercise 1.10. Let $i : A \to B$, $j : B \to C$ be continuous maps, Y be a topological space. If i and j have the HEP for Y, then ji also has the HEP for Y.

From (1.10) it follows (see also Theorem (0.11) Claim (a)):

Corollary 1.11. *The composition of two cofibrations is a cofibration.*

1.1.3 The mapping cylinder of a continuous mapping

Let $f : A \to X$ be a continuous map.

Definition 1.12. The *mapping cylinder* Z_f *of* f is the quotient space that arises from the topological sum $X + (A \times I)$ if $(a, 0) \in A \times I$ for each $a \in A$ is identified with $fa \in X$.

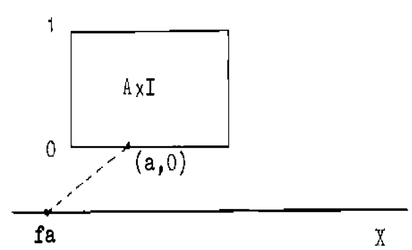


Figure 1.3:

Let p be the projection of $X + (A \times I)$ onto the quotient space Z_f.
Let $j : X \to Z_f$, $k : A \times I \to Z_f$ be the continuous maps obtained by taking the injections of X respectively. $A \times I$ is composed into the topological sum $X + (A \times I)$ with p.
We use the following (inaccurate) abbreviations:

$j(x) = p(x) := x$ for $x \in X$,

$k(a,t) = p(a,t) := (a,t)$ for $(a,t) \in A \times I$.

$k_1 : A \to Z_f$ be a continuous map $a \in A \mapsto k(a,1) = (a,1) \in Z_f$, i.e.,

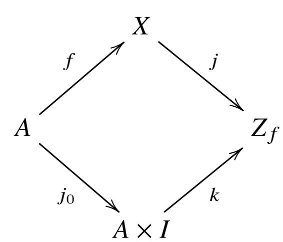
$k_1 = k \circ j_1$ (See Definitions 0.17 and 0.18.)

Theorem 1.13. *$j : X \to Z_f$ and $k_1 : A \to Z_f$ are closed embeddings.*

Proof. k_1 is a cloesd embedding, since k_1 is the composition of homeomorphisms $a \in A \mapsto (a,1) \in A \times \{1\}$ with the closed embedding(!) $k|_{A\times\{1\}} : A \times \{1\} \to Z_f$ is just injective.
j is closed: if F is a closed subset of X, then it follows from the continuity of f: $p^{-1}j(F) = F + (f^{-1}F \times \{0\})$ is cloesed in $X + (A \times I)$, i. e., $j(F)$ is closed in Z_f, because p is an identification. □

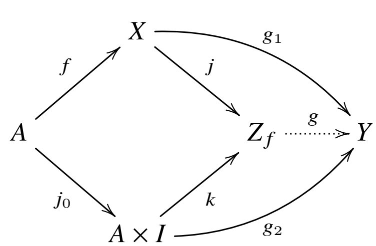
Theorem 1.14. *The following diagramme in $\mathcal{Top}$*

(1.15)



is a cocartesian square (See Definition0.5).

Proof. The diagramme (1.15) is commutative by definition of Z_f, j and k.
There are continuous maps $g_1 : X \to Y$, $g_2 : A \times I \to Y$ with $g_1 f = g_2 j_0$.
We have to show that there is exactly one continuous map $g : Z_f \to Y$ with $gj = g_1$ and $gk = g_2$.

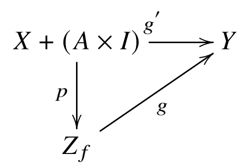


Uniqueness: g is uniquely determined by g_1 and g_2, since $Z_f = j(X) \cup k(A \times I)$.
Existence: g_1 and g_2 together define a continuous map $g' : X + (A \times I) \to Y$.

Since $g_1 f = g_2 j_0$, $g^{'}$ is contracted with the identifications we made in $X+(A\times I)$ when constructing the mapping cylinder of f:

$$g^{'}(a,0) = g_2(a,0) = g_2 j_0 a = g_1 f a = g^{'}(fa)$$

for all $a \in A$.
$g^{'}$ therefore induces a continuous map $g : Z_f \to Y$, such that the diagramme

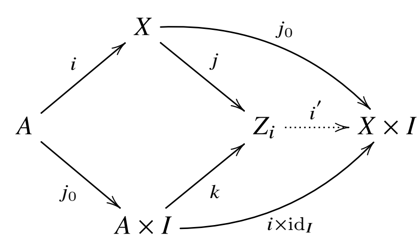


is commutative, then g is the continuous map we are looking for. □

1.1.4 Different characterisations of the cofibration concept

The following theorem characterises cofibrations with the help of the mapping cylinder and shows that a continuous map i is already a cofibration if it has the HEP for the mapping cylinder Z_i.

Remark 1.16. Let $i : A \to X$ be a continuous map. Since $j_0 i = (i \times \mathrm{id}_I) j_0$ and since (1.15) is a cocartesian square, there exists a continuous map $i^{'} : Z_i \to X\times I$ with $i^{'} j = j_0$ and $i^{'} k = i \times \mathrm{id}_I$.



Theorem 1.17. *For a continuous map $i : A \to X$, the following statements are equivalent:*

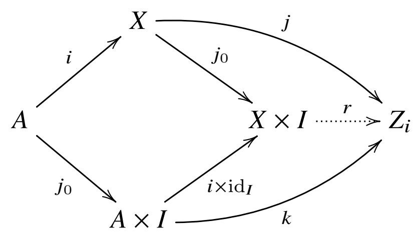
(a) i is a cofibration.

(b) i has the HEP for the mapping cylinder Z_i.

(c) $i^{'} : Z_i \to X \times I$ is a section in the category of the topological spaces (i.e. there exists a continuous map $r : X \times I \to Z_i$ with $ri^{'} = \mathrm{id}_{Z_i}$).

Proof. (thm:1-1-16a) $\Rightarrow$ *(thm:1-1-16b)* is trivial.
(thm:1-1-16b) $\Rightarrow$ *(thm:1-1-16c)* i have the HEP for Z_i. Since $Ji = kj0$, then

there exists a continuous map $r : X \times I \to Z_i$ with $rj_0 = j$ and $r(i \times \mathrm{id}_I) = k$.



We claim: $ri' = \mathrm{id}_{Z_i}$. Since (1.15) is a cocartesian square, this follows from the equations

$$(ri')j = rj_0 = j = \mathrm{id}_{Z_i} \circ j,$$
$$(ri')k = r(i \times \mathrm{id}_I) = k = \mathrm{id}_{Z_i} \circ k.$$

(thm:1-1-16c) $\Rightarrow$ *(thm:1-1-16a)* Let $r : X \times I \to Z_i$ be a continuous map with $ri' = \mathrm{id}_{Z_i}$.
Claim: i is a cofibration.

Proof. (of the claim) Given continuous maps $g : X \to Y$ and $\varphi : A \times I \to Y$ with $gi = \varphi j_0$. Since (1.15) is a cocartesian square, there (exactly) exists a continuous map $\Phi' : Z_i \to Y$ with $\Phi' j = g$ and $\Phi' k = \varphi$.
Set $\Phi := \Phi' r : X \times I \to Y$.
Then the following holds:

$$\Phi j_0 = \Phi' r j_0 = \Phi' r i' j = \Phi' j = g,$$
$$\Phi(i \times \mathrm{id}_I) = \Phi' r (i \times \mathrm{id}_I) = \Phi' r i' k = \Phi' k = \varphi.$$

□

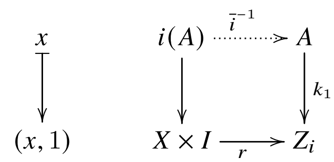
□

Corollary 1.18. *If a continuous map $i : A \to X$ is a cofibration, then i is an embedding.*
Moreover, if X is Hausdorff, then $i(A)$ is closed in X.

Proof. Since $i : A \to X$ is a cofibration, we can, according to Theorem (1.17) select a continuous map $r : X \times I \to Z_i$ with $ri' = \mathrm{id}_{Z_i}$.
For $a \in A$ we have

$$r(ia, 1) = ri'k(a, 1) = k(a, 1) = (a, 1) \in A \times 1 \subset Z_i.$$

i is thus injective and induces a bijective continuous map $\bar{i} : A \to i(A)$.
The inverse $\bar{i}^{-1} : i(A) \to A$ is continuous, since in the commutative diagramme



the extended arrows are continuous maps and k_1 according to (1.13) is an embedding.
i is also an embedding.
If we set $r' := i'r : X \times I \to X \times I$, then the following holds

$$i(A) = \{x \in X | r'(x, 1) = (x, 1)\}.$$

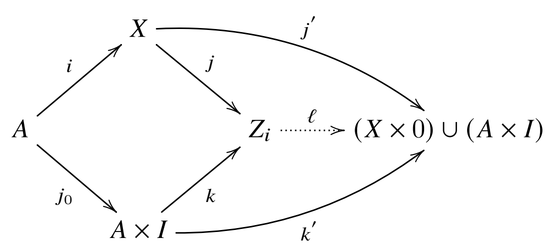
If X is Hausdorff, so is $X \times I$ Hausdorff and the diagonal of $(X \times I) \times (X \times I)$ is therefore a closed subset of the product. Since $i(A)$ is the preimage of this diagonal in the continuous map

$$X \to (X \times I) \times (X \times I), \quad x \mapsto (r'(x, 1), (x, 1)),$$

then it follows that $i(A)$ is closed in X. □

Remark 1.19. Corollary (1.18) shows in particular that one can limit oneself to inclusions $i : A \subset X$ in the definition of the term "cofibration".

Remark 1.20. Let $i : A \subset X$ be an inclusion. We compare the mapping cylinder of i with the subspace $(X \times 0) \cup (A \times I)$ of the product $X \times I$.
Consider the diagramme



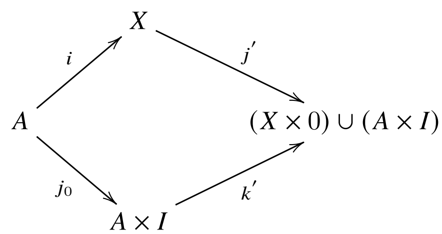
where j' is the map $x \in X \mapsto (x, 0) \in (X \times 0) \cup (A \times I)$, and k' is the inclusion.
Since $j'i = k'J_0$ and since (1.15) is a cocartesian square, exactly one continuous map $\ell : Z_i \to (X \times 0) \cup (A \times I)$ is induced with $\ell j = j'$ and $\ell k = k'$. Thus ℓ is bijective.

Theorem 1.21. *ell is a homeomorphism if one of the following conditions is satisfied:*

(a) A is closed in X.

(b) $(X \times 0) \cup (A \times I)$ *is a retract of* $X \times I$.

Proof. The proof for the condition (b) is given in Appendix A.
At this point we prove the theorem under the Prerequisite (a).
We show that the following diagramme is a cocartesian square.



(1.22)

The assertion then follows from Theorem (1.14), since i and j_0 are, in a cocartesian square, unique up to isomorphism. (see Definition (0.5)).
We have already established that (1.22) is commutative.
Given continuous maps $g_1 : X \to Y$, $g_2 : A \times I \to Y$ with $g_1 i = g_2 j_0$. Then there is a uniquely determined map of sets $g : (X \times 0) \cup (A \times I) \to Y$ with $gj^{'} = g_1$, and $gk^{'} = g_2$.
The restrictions of g on $X \times 0$ and $A \times I$ are continuus, since g_1 and g_2 are continuous.
$X \times 0$ and, since A is closed in X, $A \times I$ are closed in $X \times I$, that is, in $(X \times 0) \cup (A \times I)$.
Therefore, g is continuous.
(1.22) is thus a cocartesian square. □

Remark 1.23. If $i : A \subset X$ is an inclusion, we can calculate the amount that the mapping cylinder of i is based on the bijective map ℓ of (1.20) with $(X \times 0) \cup (A \times I)$ identified. The continuous map $\ell : Z_i \to (X \times 0) \cup (A \times I)$ is then the identity on the underlying sets.
The topology of the mapping cylinder of i on the set $(X \times 0) \cup (A \times I)$ is thus finer than the induced subspace topology defined by the product $X \times I$. According to Theorem (1.21) the topologies coincide if A is closed in X or $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$.

In general, however, the topologies are different.

Example 1.24. Let $X := [0, 1] = I$, $A :=]0, 1]$.
In $(X \times 0) \cup (A \times I)$ consider the sequence $an := (\frac{1}{n}, \frac{1}{n})$, $(n = 1, 2, 3, \dots)$. This sequence converges to $(0, 0)$ if one takes $(X \times 0) \cup (A \times I)$ the induced subspace topology by the product $X \times I$.
However, if $(X \times 0) \cup (A \times I)$ has the topology of the mapping cylinder of i, the sequence a_n does not converge to $(0, 0)$, since the point $(0, 0)$ has neighbourhoods with respect to the topology of the mapping cylinder that do not meet a point of the diagonal of $A \times I$.

Theorem 1.25. *(see Strøm [27], 2. Theorem 2) An inclusion* $i : A \subset X$ *is a cofibration if and only if the subspace* $(X \times 0) \cup (A \times I)$ *of* $X \times I$ *is retract of* $X \times I$.

Proof. We use the characterisation of the term "cofibration" of Theorem (1.17) (c).
($\Rightarrow$): If $i : A \subset X$ is a cofibration, then there exists a continuous map $r : X \times I \to Z_i$ with $ri^{'} = \mathrm{id}_{Z_i}$, where $^{'} : Z_i \to Z_i$ which is defined in (1.16) is a continuous map. If we set r with the continuous map $\ell : Z_i \to (X \times 0) \cup (A \times I)$ of (1.20) together, we get a retraction from $X \times I$ to $(X \times 0) \cup (A \times I)$.
($\Leftarrow$): Conversely, if $r^{'}$ is a retraction from $X \times I$ to $(X \times 0) \cup (A \times I)$ then $r := \ell^{-1} r^{'} : X \times I \to Z_i$ is a map with $ri^{'} = \mathrm{id}_{Z_i}$. r is continuous, since ℓ^{-1} after Theorem (1.21) (b) is continuous. □

Remark 1.26. The proof of Theorem (1.25) is based on the fact that a continuous map $\ell : Z_i \to (X \times 0) \cup (A \times I)$ defined in (1.20) is a homeomorphism under certain conditions. For this we have Theorem (1.21) (b), which we only prove in the Appendix. However, if you put it in advance Theorem (1.25) assuming that A is closed in X, one can refer to the already proved Theorem (1.21) (a).

Example 1.27. We give an example of a closed inclusion $i : A \subset X$, which is not a cofibration, and an example of a cofibration $i : A \subset X$, where A is not closed in X.

Example 1 Let $X := \{0\} \cup \{\frac{1}{n} | n = 1, 2, 3, \dots\} \subset \mathbb{R}$, $A := \{0\}$. A is a closed subspace of X.

Claim: The inclusion $i : A \subset X$ is not a cofibration.

Proof. If $i : A \subset X$ were a cofibration, then according to Theorem (1.25) a retraction $r : X \times I \to (X \times 0) \cup (A \times I)$ of $X \times I$ would exist on $(X \times 0) \cup (A \times I)$. For $n = 1, 2, 3, \dots$ there is a path component of the point $(\frac{1}{n}, 0)$

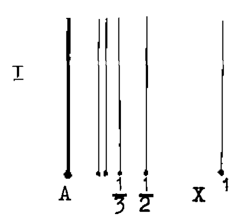


Figure 1.4:

in $(X \times 0) \cup (A \times I)$ just from this point.
Since r is continuous and fixes the point $(\frac{1}{n}, 0)$, r must therefore map the distance $\{\frac{1}{n}\} \times I$ to the point $(\frac{1}{n}, 0)$ $(n = 1, 2, 3, \dots)$.
On the other hand, r fixes the range $\{0\} \times I$ point-wise. But this is a contradiction to the continuity of r at the point $(0, 1)$. □

Example 2 Let $X := \{a, b\}$, where $a \neq b$. We give X the topology whose open sets are $\emptyset$, $\{a\}$, X. Let A be the subspace $\{a\}$ of X. A is not closed in X.

Claim: The inclusion $i : A \subset X$ is a cofibration.

Proof. We use the characterisation of Theorem (1.17) (c). We define $r : X \times I \to Z_i$ by

$$(x,t) \mapsto \begin{cases} (x,t), & \text{if} \quad \mathrm{x} = \mathrm{a} \quad \text{or} \quad t = 0, \\ (a,t), & \text{if} \quad t > 0. \end{cases}$$

The reader should convince himself that r is continuous. Since $ri' = \mathrm{id}_{Z_i}$

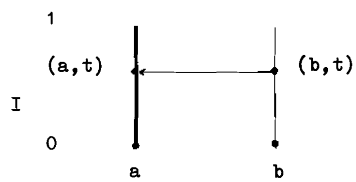


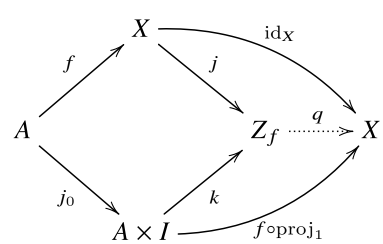
Figure 1.5:

(i' as in (1.16)), the assertion follows. □

1.1.5 Decomposition of a continuous map into a cofibration and a homotopy equivalence

With the help of the mapping cylinder we show that each continuous map can replaced up to homotopy equivalence by a (closed) cofibration.

Remark 1.28. Let $f : A \to X$ be a continuous map and Z_f be the mapping cylinder of f.
Then continuous maps $j : X \to Z_f$, $k : A \times I \to Z_f$, $k_1 : A \to Z$ are defined as in Definition (1.12).
Since $f = f \circ \mathrm{proj}_1 \circ j_0 : A \to X$ and since (1.15) is a cocartesian square, there exists exactly one continuous map $q : Z_f \to X$ with $qj = \mathrm{id}_X$ and $qk = f \circ \mathrm{proj}_1$.



q is described by the formulae

$$qx = x \quad \text{for} \quad x \in X,$$
$$q(a,t) = fa \quad \text{for} \quad (a,t) \in A \times I.$$

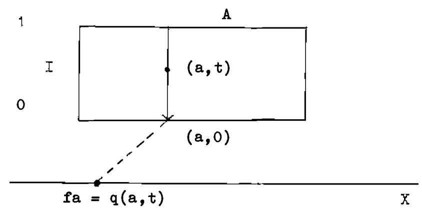


Figure 1.6:

Theorem 1.29. *(a) The diagramme*

$$\begin{array}{ccc} & A & \\ {\scriptstyle k}\swarrow & & \searrow{\scriptstyle f} \\ Z_f & \xrightarrow[q]{} & X \end{array}$$

is commutative.

(b) k_1 *and* j *are cofibrations.*

(c)

$$qj = \mathrm{id}_X,$$
$$jq \simeq \mathrm{id}_{Z_f} \operatorname{rel} j(X).$$

Since k_1 is a closed embedding by Theorem (1.13), from Theorem (1.29) follows, :

Corollary 1.30. *Every continuous map f can be factored into the form $f = u \circ v$, where v is a (closed) cofibration and u is a homotopy equivalence.*

Proof. (of (1.29 (a)) $qk_1 = qkj_1 = f \circ \mathrm{proj}_1 \circ j_1 = f$. □

For the proof of parts (b) and (c) of Proposition (1.29) we need:

Theorem 1.31. *Let $f : A \to B$ be a continuous map, C be a topological space. If f is an identification and is C locally compact, then*

$$f \times \mathrm{id}_C : A \times C \to B \times C$$

is an identification, too.

We prove (1.31) in (2.13) with the aid of mapping spaces (cf. also Schubert [23], I, 7.9, Theorem 5)[3].

[3] A direct proof for $C = I$ can be found in Hilton [11], VII, Lemma 3.4.

Proof. (of (1.29 (b)) We first identify (cf. (1.13)

$$A = k_1(A) = A \times 1,$$
$$X = j(X).$$

k_1 and j are then the inclusions

$$A \times 1 \subset Z_f, \quad X \subset Z_f.$$

To prove that these inclusions are cofibrations, let's apply (1.25).
Note Remark (1.26) ($A \times 1$ and X are closed in Z_f after (1.13)).

So we have to show:

(1) $(Z_f \times 0) \cup (A \times 1 \times I)$ is a retract of $Z_f \times I$,

(2) $(Z_f \times 0) \cup (X \times I)$ is a retract of $Z_f \times I$.

First, we show (1) holds:

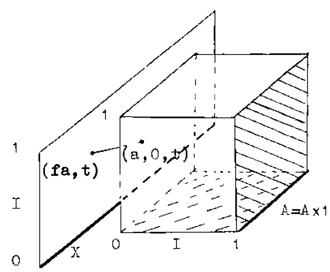


Figure 1.7:

The projection from the point $(0,2) \in \mathbb{R} \times \mathbb{R}$ provides a continuous map

$$\lambda : I \times I \to (I \times 0) \cup (1 \times I)$$

By

$$(x,t) \mapsto (x,0) \quad \text{for} \quad x \in X,$$
$$(a,s,t) \mapsto (a,\lambda(s,t)) \quad \text{for} a \in A, \; s,t \in I$$

one obtains a continuous map

$$\overline{r} : (X + (A \times I)) \times I \to (Z_f \times O) \cup (A \times 1 \times I).$$

Since $\lambda(0,t) = (0,0)$ for all $t \in I$, the following holds for $a \in A$ and $t \in I$

$$\overline{r}(a,0,t) = (a,\lambda(0,t)) = (a,0,0) = (fa,0) = \overline{r}(fa,0).$$

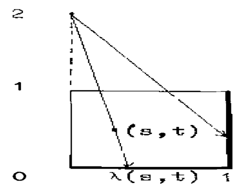


Figure 1.8:

There is therefore exactly one map

$$r : Z_f \times I \to (Z_f \times O) \cup (A \times 1 \times I)$$

with $r(p \times \mathrm{id}_I) = \overline{f}$.

$$\begin{array}{ccc} (X + (A \times I)) \times I & \xrightarrow{\overline{r}} & (Z_f \times O) \cup (A \times 1 \times I) \\ {\scriptstyle p\times \mathrm{id}_I}\downarrow & \nearrow_{r} & \\ Z_f \times I & & \end{array}$$

r is continuous since $\overline{r}$ is continuous and since $p \times \mathrm{id}_I$ after Theorem (1.31) is an identification (I is locally compact.) Since $\lambda|_{(I\times 0)\cup(1\times I)} = \mathrm{id}_{(I\times 0)\cup(1\times I)}$, it follows that

$$r|_{(Z_f\times 0)\cup(A\times 1\times I)} = \mathrm{id}_{(Z_f\times 0)\cup(A\times 1\times I)}\,.$$

Thus (1) is proved.

Next, we show (2) holds:

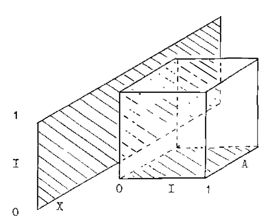


Figure 1.9:

The projection from the point $(1, 2) \in \mathbb{R} \times \mathbb{R}$ gives a continuous map

$$\lambda' : I \times I \to (I \times 0) \cup (0 \times I).$$

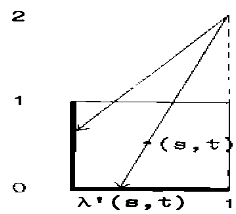


Figure 1.10:

By

$$(x,t) \mapsto (x,t) \quad \text{for} \quad x \in X,$$
$$(a,s,t) \mapsto (a,\lambda^{'}(s,t)) \quad \text{for} a \in A,\ s,t \in I$$

one obtains a continuous map

$$\overline{r}^{'} : (X + (A \times I)) \times I \to (Z_f \times 0) \cup (X \times I).$$

Since $\lambda^{'}(0,t) = (0,t)$ for all $t \in I$, for $a \in A$ and $t \in I$ we have

$$\overline{r}^{'}(a,0,t) = (a,\lambda^{'}(0,t)) = (a,0,t) = (fa,t) = \overline{r}^{'}(fa,t).$$

There is therefore exactly one map

$$\overline{r}^{'} : (Z_f \times I) \to (Z_f \times 0) \cup (X \times I).$$

with $r^{'}(p \times \mathrm{id}_I) = \overline{r}^{'}$. $r^{'}$ is continuous, since $\overline{r}^{'}$ is continuous and since $p \times \mathrm{id}_I$is an identification (1.31). Since $\lambda(s,0) = (s,0)$ for all $s \in I$, it follows that

$$r^{'}|_{(Z_f \times 0) \cup (X \times I)} = \mathrm{id}_{(Z_f \times 0) \cup (X \times I)}\,.$$

Thus (2) is proved. □

Proof. (of (1.29 (c)) Note that $qj = \mathrm{id}_X$ results from the definition of q.
We define $\varphi : X_f \times I \to X_f$ by

$$\varphi(x,t) := x \quad \text{for} \quad x \in X,\ t \in I,$$
$$\varphi(a,s,t) := (a,st) \quad \text{for} \quad a \in A,\ s,t \in I$$

φ is well-defined since

$$\varphi(a,0,t) = (a,0) = fa = \varphi(fa,t) \quad \text{for} \quad a \in A.$$

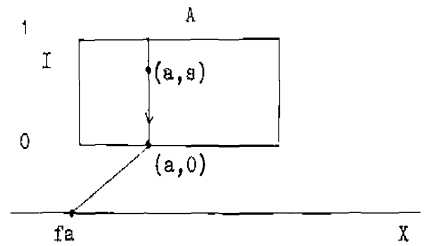


Figure 1.11:

With the help of (1.31) it is easy to see that φ is continuous.
The following applies

$$\begin{aligned} \varphi(x,0) = x = jq(x) \quad &\text{for} \quad x \in X, \\ \varphi(a,s,0) = (a,0) = fa = jq(a,s) \quad &\text{for} \quad a \in A,\ s \in I, \\ \text{thus} \quad \varphi_0 = jq, \quad \varphi_1 = \mathrm{id}_{Z_f}. \end{aligned}$$

Since $\varphi(x,t) = x$ for all $x \in X$, $t \in I$, φ is a homotopy $\operatorname{rel} X$ $(= j(X))$, so

$$\varphi : jq \simeq \mathrm{id}_{Z_f} \operatorname{rel} X.$$

□

1.1.6 Mapping cylinder of a pair (Double Mapping cylinder)

We generalise the notion of the mapping cylinder of a continuous map.

Definition 1.32. Let $f : A \to X$, $g : A \to Y$ be continuous maps.
The *mapping cylinder* $Z_{(f,g)}$ *of the pair* (f,g) is the coset space that results from the topological sum $X + (A \times I) + Y$ if $(a,0) \in A \times I$ is identified with $fa \in X$ for each $a \in A$ and $(a,1) \in A \times I$ is identified with $ga \in Y$ for each $a \in A$.

By composing the injection of X or Y into the topological sum $X + (A \times I) + Y$ with the projection on $Z_{(f,g)}$ one obtains injective continuous maps

$$j_X : X \to Z_{(f,g)}, \quad j_Y : Y \to Z_{(f,g)}.$$

Theorem 1.33. *j_X, j_Y are closed embeddings and cofibrations.*

Proof. The proof is analogous to the proof of the corresponding parts of Theorem (1.13) and Theorem (1.29).
We leave the exact implementation to the reader.

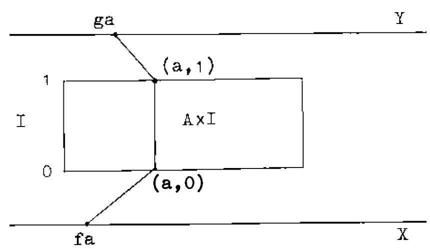


Figure 1.12:

When proving that j_X is a cofibration, one uses the fact that for the map introduced in the proof of (1.29)

$$\lambda' : I \times I \to (I \times 0) \cup (0 \times I), \quad \lambda'(1, t) = (1, 0) \quad \text{for all} \quad t \in I.$$

□

In particular, Theorem (1.33) allows us to regard X and Y as (closed) subspaces of $Z_{(f,g)}$:

$$X \subset Z_{(f,g)}, \quad Y \subset Z_{(f,g)}.$$

Example 1.34. 1. If $g = \mathrm{id}_A$, then $Z_{(f,g)}$ is (essentially) the mapping cylinder Z_f of f.

2. If Y has exactly one point, i.e. g is the only map $A \to Y$, then $Z_{(f,g)}$ is called the *mapping cone* of f.

   We then use the notation $C_f := Z_{(f,g)}$.

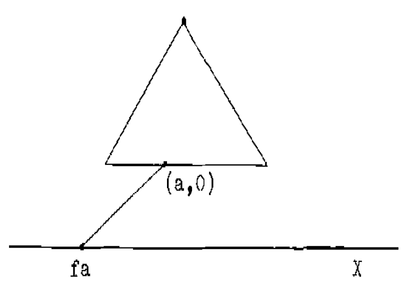


Figure 1.13:

Remark 1.35. C_f is created from the mapping cylinder Zf of f, by identifying $A \times 1 \subset Z_f$ to a point.

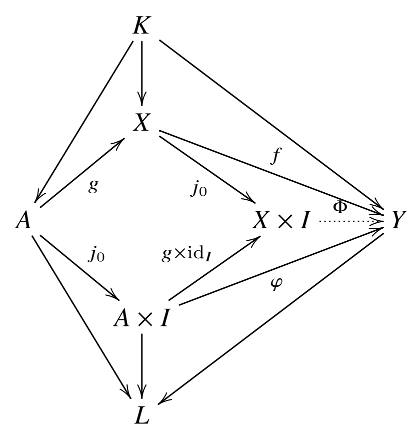
From Theorem (1.33) follows:

Theorem 1.36. *(1 .32) Theorem: If $f : A \to X$ is a continuous map, then the inclusion $X \subset C_f$ is a (closed) cofibration.*

1.1.7 Transition to other categories

Let K, L be topological spaces. With the help of the homotopy term defined in (0.32) in the category $\mathcal{Top}_L^K$, the definition of the term cofibration can be transferred from $\mathcal{Top}$ to $\mathcal{Top}_L^K$.

Definition 1.37. Let $\alpha = (K \to A \to L)$, $\xi = (K \to X \to L)$ be spaces under K and over L, let $g : \alpha \to \xi$ be a map under K and over L.
g is called a *cofibration in* $\mathcal{Top}_L^K$, precisely if for all spaces under K and over L, $\eta = (K \to Y \to L)$, for all maps under K and over L, $f : \xi \to \eta$ and all homotopies under K and over L, $\varphi : A \times I \to Y$ with $\varphi_0 = fg$, a homotopy under K and over L, $\Phi : X \times I \to Y$ exists with $\Phi(g \times \mathrm{id}_I) = \varphi$ and $\Phi_0 = f$.



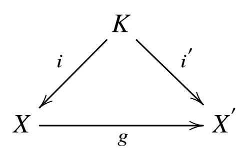
The theorems of this paragraph on cofibrations can be transferred from $\mathcal{Top}$ to $\mathcal{Top}_L^K$. We leave the exact execution to the reader. Note in particular the special cases $K = \emptyset$, $L = \mathrm{pt}$ and make clear the concept of cofibration in $\mathcal{Top}^0$ (*pointed cofibration*).
At this point, only the construction in the category $\mathcal{Top}_L^K$ will be mentioned, which corresponds to the construction of the mapping cylinder in $\mathcal{Top}$.

Definition 1.38. Let $\xi = (K \xrightarrow{i} X \xrightarrow{p} L)$ and $\xi' = (K \xrightarrow{i'} X \xrightarrow{p'} L)$ be objects of $\mathcal{Top}_L^K$ and $f \in \mathcal{Top}_L^K(\xi, \xi')$. We first have the topological space $I^K X$ (cf. (0.25)).
Let Z_f^K be the topological space that arises from the topological sum $X' + I^K X$ if for each $x \in X$, $fx \in X'$ is identified with the image of $(x, 0) \in X \times I$ under the natural projection $X \times I \to I^K X$. We put $i' : K \to X'$ together with the injection of X' into the topological sum $X' + I^K X$ and the natural projection $X' + I^K X \to Z_f^K$ and obtain a continuous map $K \to Z_f^K$.

$p' : X' \to L$ and $p \circ \mathrm{proj}_1 : X \times I \to L$ induce a continuous map $Z_f^K \to L$) (!). We thus obtain an object $K \to Z_f^K \to L$ of $\mathcal{T}op_L^K$, the *mapping cylinder of f in $\mathcal{T}op_L^K$*.

The reader may want to prove

Theorem 1.39. *Let*



be a commutative diagramme in $\mathcal{T}op$. Then $g : i \to i'$ is a cofibration in $\mathcal{T}op^K$ if $g : X \to X'$ is a cofibration in $\mathcal{T}op$.

1.2 Homotopy cofibrations

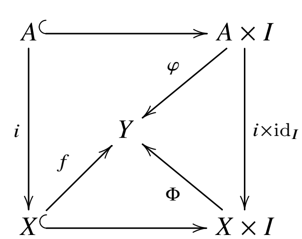
1.2.1 The homotopy extension property up to homotopy. h-cofibrations

We generalise the concept of cofibration.

Definition 1.40. Let $i : A \to X$ be a continuous map, Y a topological space. i has the *homotopy extension property (HEP) up to homotomy* for Y if for all continuous maps $f : X \to Y$ and all homotopies $\varphi : A \times I \to Y$ with $\varphi_0 = fi$ a homotopy $\Phi : X \times I \to Y$ exists with

(1) $\Phi(i \times \mathrm{id}_I) = \varphi$ and

(2) $\Phi_0 \overset{A}{\simeq} f$.



(We regard Φ_0 and f as morphisms of $\mathcal{T}op^A$, with $\Phi_0, f \in \mathcal{T}op^A(i, fi)$; because of (1) $\Phi_0 i = \varphi_0 = fi$ holds.)

Definition 1.41. A continuous map $i : A \to X$ is called a *homotopy cofibration* (*h-cofibration* for short) if i has the HEP up to homotopy for all topological spaces Y. In addition to the term "homotopy cofibration", the term "*weak cofibration*" is also used.

Remark 1.42. Every cofibration is an h-cofibration. In particular, every homeomorphism is an h-cofibration (cf. (1.9)).

Theorem 1.43. *The composition of two h-cofibrations is an h-cofibration.*

The proof of the theorem is left to the reader.

Definition 1.44. Let $iA \to X$, $i' : A \to X'$ be spaces under A.
i is said to be *dominated by* i' *in* $\mathcal{Top}^A$ if one of the following equivalent (!) statements is satisfied:

(a) there exist morphisms of $\mathcal{Top}^A$, $g : i \to i'$, $g' : i' \to i$ with $g'g \overset{A}{\simeq} \mathrm{id}_X$,

(b) there exists a section in $\mathcal{Top}^A$, $g : i \to i'$,

(c) there exists a retraction in $\mathcal{Top}^A$, $g' : i' \to i$.

Remark 1.45. In the case $A = \emptyset$, this notion goes back to J. H. C. Whitehead.

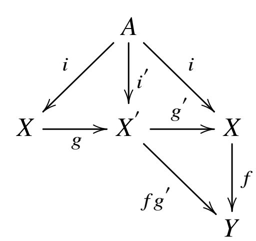
Theorem 1.46. Assumption*:* $i : A \to X$, $i' : A \to X'$ *are spaces under* A. *Let* i *be dominated by* i' *in* $\mathcal{Top}^A$.
Assertion.

(a) If Y *is a topological space and* i' *has the HEP up to homotopy for* Y*, then so is* i.

(b) If i' *is an h-cofibration, then so is* i .

Proof. (b) immediately follows from (a).

(a): By assumption there are $g \in \mathcal{Top}^A(i, i')$, $g' \in \mathcal{Top}^A(i', i)$ with $g'g \overset{A}{\simeq} \mathrm{id}_X$.
Given are continuous maps $f : X \to Y$, $\varphi : A \times I \to Y$ with $\varphi_0 = fi$.



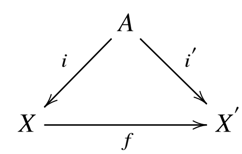
Since $g'i' = i$, it follows that $\varphi_0 = fg'i'$. Since i' has the HEP up to homotopy for Y, there exists a homotopy $Phi' : X' \times I \to Y$ with $\Phi'(i' \times \mathrm{id}_I) = \varphi$ and $\Phi'_0 \overset{A}{\simeq} fg'$. Set $\Phi := \Phi'(g \times \mathrm{id}_I) : X \times I \to Y$. Then $\Phi(i \times \mathrm{id}_I) = \Phi'(gi \times \mathrm{id}_I) = \Phi'(i' \times \mathrm{id}_I) = \varphi$ and $\Phi_0 = \Phi'_0 g \overset{A}{\simeq} fg'g \overset{A}{\simeq} f$, because $g'g \overset{A}{\simeq} \mathrm{id}_X$.
Therefore i has the HEP up to homotopy for Y. □

Specifically, theorem (1.46) yields:

Corollary 1.47. *"HEP up to homotopy" and "h-cofibration" are invariant under isomorphism in $\mathcal{Top}^A h$, i.e. under homotopy equivalence under A.*

Remark 1.48. Theorem (1.46) becomes false if one replaces "HEP up to homotopy" by "HEP" in (a) or "h-cofibration" by "cofibration" in (b).
We give an example of this in (1.100) (cf. (1.103)).

Theorem 1.49. *Let the diagramme in $\mathcal{Top}$*



be commutative up to homotopy, i.e. $fi \simeq i'$.

If i is an h-cofibration or has at least the HEP up to homotopy for X', then there is a continuous map $g : X \to X'$ with $g \simeq f$ and $gi = i'$.
(Compare the problem in (1.1), (1.2) and Theorem (1.4).

Proof. Let $\varphi : fi \simeq i'$, $\varphi : A \times I \to X'$.
Since $\varphi_0 = fi$ and since i has the HEP up to homotopy for X' there exists a homotopy $\Phi : X \times I \to X'$ with $\Phi(i \times \mathrm{id}_I) = \varphi$ and $\Phi_0 \overset{A}{\simeq} f$.
Set $g := Phi_1 : X \to X'$. Then $gi(a) = \Phi(ia, 1) = \varphi(a, 1) = i'(a)$ for all $a \in A$, i.e.

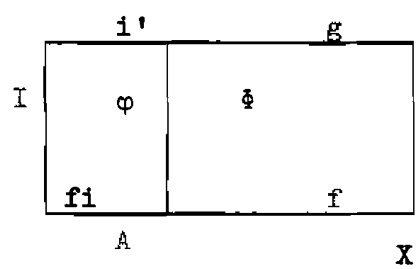


Figure 1.14:

$gi = i'$.
Furthermore, we have $g = \Phi_1 \simeq \Phi_0 \simeq f$. □

1.2.2 Different characterisations of the term "h-cofibration"

Theorem 1.50. *Let ε be a real number with $0 < \varepsilon < 1$, Y a topological space, $i : A \to X$ a continuous map.*
Then the following are equivalent:

(a) i has the HEP up to homotopy for Y.

(b) For all continuous maps $: X \to Y$ *and all homotopies* $\varphi : A \times I \to Y$ *such that* $\varphi(a,t) = fi(a)$ *for all* $\in A$ *and all* $t \in [0,1]$ *with* $t \leq \varepsilon$[4]*, there exists a homotopy* $\Phi : X \times I \to Y$ *with* $\Phi(i \times \mathrm{id}_I) = \varphi$ *and* $\Phi_0 = f$.

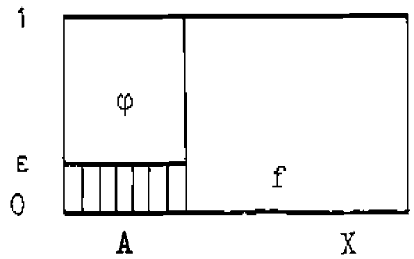


Figure 1.15:

As a corollary, Theorem (1.50) provides a characterisation of the concept of "h-cofibration".

Proof. (of Theorem 1.50) (a) ⇒ (b): Given $f : X \to Y$ and $\varphi : A \times I \to Y$ with $\varphi(a,t) = fi(a)$ for all $a \in A$ and $t \in [0,1]$ with $t < \varepsilon$.
Since $\varphi_\varepsilon = fi$ (Recall: $\varphi_\varepsilon = \varphi j_\varepsilon$ (0.23)), and since i has the HEP up to homotopy for Y, there exists $\Phi^{'} X \times [\varepsilon, 1] \to Y$ with $\Phi^{'}(i \times \mathrm{id}_{[\varepsilon,1]}) = \varphi|_{A \times [\varepsilon,1]}$ and $\Phi^{'}_\varepsilon \simeq f$ (cf. (0.23)).
Let $\Phi^{''} : X \times [0,\varepsilon] \to Y$ be a homotopy under A with $\Phi^{''}_0 = f$ and $\Phi^{''}_\varepsilon = \Phi^{'}_\varepsilon$. $\Phi^{'}$ and $\Phi^{''}$ together define the desired homotopy $\Phi : X \times I \to Y$.

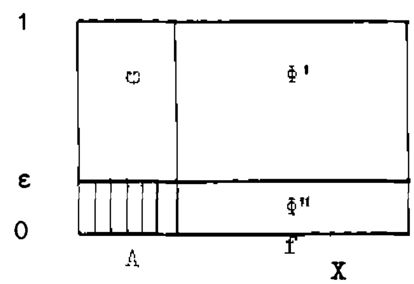


Figure 1.16:

(b) ⇒ (a): Given continuous maps $f : X \to Y$, $\varphi : A \times I \to Y$ with $\varphi_0 = fi$. We extend φ to $\varphi^{'} : A \times [-1,+1] \to Y$ by $\varphi^{'}(a,t) := \varphi(a, \max(t,0))$. Then $\varphi^{'}(a,t) = fi(a)$ for $a \in A$, $- \leq t \leq 0$. By assumption (we replace $(0,\varepsilon,1)$

[4]We say: "φ is somewhat constant ".

with $(-1,0,1)$.) there exists a continuous map $\Phi' : X \times [-1,1] \to Y$ with $\Phi'(i \times \mathrm{id}_{[-1,1]}) = \varphi'$ and $\Phi'_{-1} = f$.

For $\Phi := \Phi'|_{X \times I} : X \times I \to Y$ then $\Phi(i \times id_I) = \varphi$, and $\Phi_0 = \Phi'_0 \overset{A}{\simeq} \Phi'_{-1} = f$. □

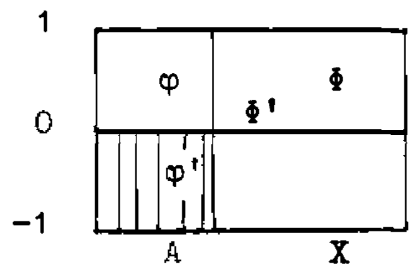


Figure 1.17:

Theorem 1.51. *Let ε be a real number with $0 < \varepsilon < 1$, $iA \to X$ be a continuous map.*
Then the following are equivalent:

(a) i is a h-cofibration.

(b) There exists a continuous map $r : X \times I \to Z$ with the following property $(E(i,\varepsilon))$:

$$(E(i,\varepsilon)) \begin{cases} r(x,0) & = x \quad for \quad xinX \\ r(ia,t) & = \begin{cases} (a,0), & a \in A,\ 0 \le t \le \varepsilon, \\ \left(a, \frac{t-\varepsilon}{1-\varepsilon}\right), & a \in A,\ \varepsilon \le t \le 1. \end{cases} \end{cases}$$

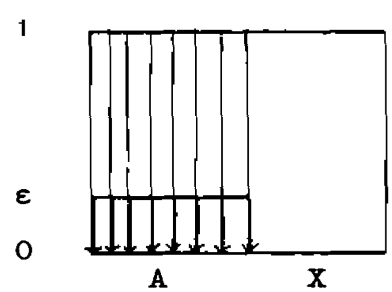


Figure 1.18:

Proof. We carry out the proof using Theorem (1.50).
We can obviously restrict ourselves to the case $\varepsilon = \frac{1}{2}$.

(a) $\Rightarrow$ (b): We assume that i is an h-cofibration.
We considered the embedding $j : X \to Z_i$ (cf. (1.12), (1.13)) and the homotopy $\varphi : A \times I \to Z_i$, which is given by

$$(a,t) \mapsto \begin{cases} (a,0), & 0 \le t \le \frac{1}{2}, \\ (a, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Then for $0 \le t \le \frac{1}{2}$ $\varphi(a,t) = (a,0) = ia = jia$.
Since i has the HEP up to homotopy for Z_i, according to (1.50) there exists a homotopy $\Phi : X \times I \to Z_i$ with $\Phi(i \times id_I) = \varphi$ and $\Phi_0 = j$. $r := \Phi$ is the sought continuous map.

(b) $\Rightarrow$ (a): We now assume the existence of $r : X \times I \to Z_i$ with $r(x,0) = x$ $(x \in X)$ and

$$r(ia,t) = \begin{cases} (a,0), & a \in A,\ 0 \le t \le \frac{1}{2}, \\ (a, 2t-1), & a \in A, \frac{1}{2} \le t \le 1. \end{cases}$$

in advance.
Given are continuous maps $f : X \to Y$, $\varphi : A \times I \to Y$ with $\varphi(a,t) = fia$ for $0 \le t \le \frac{1}{2}$, $a \in A$.
We define $\Phi' : Z_i \to Y$ by

$$x \mapsto f(x) \quad \text{for} \quad x \in X,$$
$$(a,t) \mapsto \varphi\left(a, \frac{1+t}{2}\right) \quad \text{for} \quad (a,t) \in A \times I.$$

Since $(a,0) \mapsto \varphi(a, \frac{1+t}{2}) = fia$, Φ' is a well-defined continuous map. Set $\Phi := \Phi' r : X \times I \to Y$. Then

$$\Phi(ia,t) = \begin{cases} \Phi'(a,0) = \varphi\,(a,0) = \varphi(a,t), & a \in A,\ 0 \le t \le \frac{1}{2}, \\ \Phi'(a, 2t-1) = \varphi(a,x), & a \in A, \frac{1}{2} \le t \le 1. \end{cases}$$
$$\Phi(x,0) = \Phi'(x) = f(x) \quad \text{for} \quad x \in X, \text{ i. e, } \Phi(i \times \mathrm{id}_I) = \varphi,\ \Phi_0 = f.$$

According to Theorem (1.50), i is therefore an h-cofibration. □

Remark 1.52. In "(a) $\Rightarrow$ (b)" we have only used that I has the HEP up to homotopy for the mapping cylinder Z_i.

Addition to Theorem (1.51). From theorem (1.51) and Remark (1.52) it follows:

Theorem 1.53. *A continuous map i is an h-cofibration if and only if it has the HEP up to homotopy for the mapping cylinder Z_i.*

Corollary 1.54. *If a continuous map $i : A \to X$ is an h-cofibration, then i is an embedding. Furthermore, if X is Hausdorff, then $i(A)$ is closed in X.*

Proof. Corollary (1.54) follows from theorem (1.51) in a similar way as Corollary (1.18) from theorem (1.17). The proof of (1.18) can be adopted almost word for word. □

Remark 1.55. Corollary (1.54) shows that the definition of the concept of "h-cofibration" can be restricted to inclusions $i : A \subset X$. We now prove that in the characterisation of the concept of "h-cofibration" of Theorem (1.51) the mapping cylinder Z_i can be replaced by $(X\times 0)\cup(A\times I) \subset X\times I$ if $i : A \subset X$ is an inclusion.

Theorem 1.56. *Let ε be a real number with $0 < \varepsilon < 1$. An inclusion $i : A \subset X$ is an h-cofibration if and only if there exists a continuous map $r' : X \times I \to (X \times 0) \cup (A \times I)$ with the following property $(E'(i,\varepsilon))$:*

$$(E'(i,\varepsilon)) \begin{cases} r'(x,0) & = (x,0) \quad \textit{for} \quad x \in X \\ r'(a,t) & = \begin{cases} (a,0), & a \in A,\ 0 \le t \le \varepsilon \\ (a, \frac{t-\varepsilon}{1-\varepsilon}) & a \in A,\ \varepsilon \le t \le 1. \end{cases} \end{cases}$$

Proof. $(\Rightarrow)$: We assume that i is an h-cofibration. According to Theorem (refthm:1-2-11) there exists a continuous map $r : X \times I \to Z_i$ with the property $(E'(i.\varepsilon))$. If we set $r' := \ell r : X\times I \to (X\times 0)\cup(A\times I)$, where ℓ is the continuous map defined in (1.20), we obtain a continuous map with the property $(E'(i,\varepsilon))$.
$(\Leftarrow)$: We assume the existence of a continuous map $r' : X\times I \to (X\times 0)\cup(A\times I)$ with the property $(E'(i,\varepsilon))$.

We choose a real number δ with $0 < \delta < 1$ and define a map $s : (X \times 0) \cup (A \times I) \to Z_i$ by

$$(x,0) \mapsto x \quad \text{for} \quad x \in X,$$
$$(a,t) \mapsto \begin{cases} (a,0), & a \in A,\ 0 \le t \le \delta \\ (a, \frac{t-\delta}{1-\delta}) & a \in A,\ \delta \le t \le 1. \end{cases}$$

Lemma 1.57. *s is continuous*

Proof. $(X \times 0) \cup (A \times [0,\delta])$ and, since $\delta > 0$, $A \times [\delta, 1]$ are closed subsets of $(X \times 0) \cup (A \times I)$. It is therefore sufficient to show that the restrictions of s to these subsets are continuous.

The restriction $s|_{A\times[\delta,1]}$ is continuous, since it is the composition of the continuous map

$$A \times [\delta,1] \to A \times I, \quad (a,t) \mapsto (a, \frac{t-\delta}{1-\delta}),$$

with the injection of $A \times I$ into the direct sum $X + (A \times I)$ and the projection onto Z_i. If one combines the projection onto the first factor $\text{proj}_1 : X \times I \to X$ with the injection of X into the direct sum $X + (A \times I)$ and the projection onto Z_i, one obtains a continuous map $X \times I \to Z_i$. $s|_{(X\times 0)\cup(A\times[0,\delta])}$ is continuous as a restriction of a continuous map to $(X \times 0) \cup (A \times [0,\delta]$. This proves the lemma. □

According to the lemma just proved, we obtain a continuous map $X \times I \to Z_i$ by $r := sr'$. We set $\varepsilon' := \varepsilon + (1 - \varepsilon)\delta$. Then $0 < \varepsilon' < 1$. A simple calculation shows that r' satisfies the property $(E(i, \varepsilon'))$. According to Theorem (1.51), i is therefore a cofibration. This proves Theorem (1.56). □

Remark 1.58. If A is closed in X , then Theorem (1.56) follows immediately from (1.51) by Theorem (1.21) (a).

Theorem 1.59. *If $i : A \to X$ is an h-cofibration and Y is an arbitrary topological space, then $\mathrm{id}_Y \times i : Y \times A \to Y \times X$ is also an h-cofibration.*

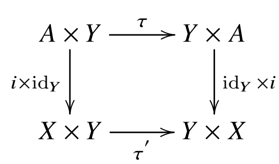
Proof. According to (1.54), we can assume without significant restriction that i is an inclusion, $i : A \subset X$. Let ε be a real number with $0 < \varepsilon < 1$. By Theorem (1.56), there exists a continuous map $r' : X \times I \to (X \times 0) \cup (A \times I)$ with the property $(E'(i, \varepsilon))$. The continuous map $id_Y \times r' : Y \times X \times I \to (Y \times X \times 0) \cup (Y \times A \times I)$ then has the property $(E'(\mathrm{id}_Y \times i, \varepsilon))$. Thus, according to (1.56), $\mathrm{id}_Y \times i$ is an h-cofibration. □

Remark 1.60. If Y is locally compact, then Theorem (1.59) follows from Theorem (1.51) using a similar inference as in the proof just presented.

If Y is locally compact, then the mapping cylinder $Z_{\mathrm{id}_Y \times i}$ is homeomorphic to $Y \times Z_i$ by Theorem (1.31).

Corollary 1.61. *If $i : A \to X$ is an h-cofibration and Y is an arbitrary topological space, then $i \times \mathrm{id}_Y : A \times Y \to X \times Y$ is also an h-cofibration.*

Proof. Let $\tau : A \times X \to Y \times X$ and $\tau' : X \times Y \to Y \times X$ be the commutation of the factors, τ and tau' are homeomorphisms that make the following diagramme commutative.

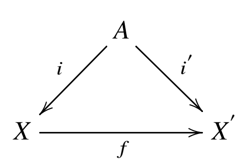


This means, however, that (τ, τ') is an isomorphism of $\mathcal{T}op(2)$ $i \times \mathrm{id}_Y \to \mathrm{id}_Y \times i$. It is easy to see that the property of being an h-cofibration is invariant under isomorphism in $\mathcal{T}op(2)$. According to Theorem (1.59), $\mathrm{id}\, Y \times i$ is an h-cofibration, and so is $i \times \mathrm{id}_Y$. □

1.2.3 h-equivalences and h-equivalences under A

The following theorem plays a central role in the construction of homotopy theory.

Theorem 1.62 (cf. Dold [7], 3.6). *Let*

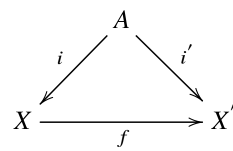


be a commutative diagramme in $\mathcal{T}op$*. Let* i *and* i' *be h-cofibrations,* f *be a homotopy equivalence.*

Claim*:* f*, conceived as a morphism of* $\mathcal{T}op^A$*,* $f :\rightarrow i'$*, is a homotopy equivalence* under A.

Theorem (1.62) follows as a consequence of

Theorem 1.63. *Let*



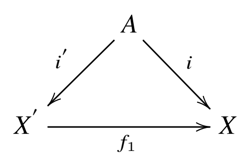
be a commutative diagramme in $\mathcal{T}op$*. Let* i *and* i' *be h-cofibrations.*

Claim*: If* $[f]$ *has a left inverse in* $\mathcal{T}oph$*, then* $[f]^A$ *has a left inverse in* $\mathcal{T}op^Ah$*. (In the first case, we consider* f *as a morphism of* $\mathcal{T}op$ *onto (*$f \in \mathcal{T}op(X, X')$*), in the second as a morphism of* $\mathcal{T}op^A$ *onto (*$f \in \mathcal{T}op^A(i, i')$*).)*

Proof. ((1.63) $\Rightarrow$ (1.62)) f is, by assumption, an h-equivalence, i.e., $[f]$ is an isomorphism in $\mathcal{T}oph$. In particular, $[f]$ has a left inverse in $\mathcal{T}oph$. By Theorem (1.63), therefore, there exists $f_1 \in \mathcal{T}op^A$ such that

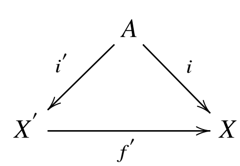
$$[f_1]^A[f]^A = [\mathrm{id}_X]^A. \tag{1.64}$$

In particular, in $\mathcal{T}oph$, $[f_1][f] = [\mathrm{id}_X]$ holds. Since $[f]$ is an isomorphism in $\mathcal{T}oph$, the last equation implies that $[f_1]$ is an isomorphism in $\mathcal{T}oph$. Therefore, $[f_1]$ has a left inverse in $\mathcal{T}oph$. Applying Theorem (1.63) to the commutative diagramme in $\mathcal{T}op$



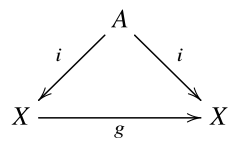
so it follows: $[f_1]^A$ has a left inverse in $\mathcal{T}op^Ah$. Furthermore, since $[f_1]^A$ according to (1.64) has a right inverse, $[f_1]^A$ is an isomorphism in $\mathcal{T}oph$. Therefore, according to (1.64), $[f]^A$ is an isomorphism in $\mathcal{T}op^Ah$, i.e., f is an h-equivalence under A. □

Proof. (of Theorem (1.63)) Let $f' : X' \rightarrow X$ be homotopy left inverse of f, i.e., $f'f \simeq \mathrm{id}_X$. Then $f'i' = f'fi \simeq i$, so the diagramme



is commutative up to homotopy. Since i' is an h-cofibration, we can assume, by Theorem (1.49), that this diagramme is even commutative. If we set $g := f'f$, then the assumptions of the following Lemma are satisfied.

Lemma 1.65. *If the following is a commutative diagramme in $\mathcal{Top}$, i is an h-cofibration, and $g \simeq \mathrm{id}_X$,*



then there exists a morphism $g' : i \to i$ of $\mathcal{Top}^A$ such that $g'g \overset{A}{\simeq} \mathrm{id}_X$. Thus, there exists a morphism $g : i \to i$ of $\mathcal{Top}^A$ with $gg' \overset{A}{\simeq} \mathrm{id}_X$, i.e., $g'f' \overset{A}{\simeq} \mathrm{id}_X$. But this means: $g'' : i' \to i$ is a morphism of $\mathcal{Top}^A$ such that $[g'f']^A$ is left inverse to $[f]^A$. Therefore, Lemma (1.65) remains to be proven.

Before proceeding to the proof (which is rather involved), we record a remark required in the proof.

Remark 1.66. If $\alpha, \alpha' : U \to V$ are homotopic continuous maps and $\chi : U \times I \to V$ is a homotopy $\alpha \simeq \alpha'$, then one obtains

$$(u,t) \mapsto \chi(u, \max(2t-1, 0)) \quad \text{for} \quad (u,t) \in U \times I \tag{1.67}$$

a homotopy $\alpha \simeq \alpha'$, which is somewhat constant.

Proof. (of 1.65) Let $\varphi : g \simeq \mathrm{id}_X$, $\varphi : X \times I \to X$. By Remark 1.66, we can choose the homotopy φ such that it is somewhat constant, for example $\varphi(x,t) = g(x)$ for $x \in X$ and $0 \le t \le \frac{1}{2}$. For $\varphi' = \varphi(i \times \mathrm{id}_I) : A \times I \to X$ then $\varphi' : i \simeq i$ (since $gi = i$) and $\varphi'(a,t) = ia$ for $a \in A$ and $0 \le t \le \frac{1}{2}$. Since i is an h-cofibration, according to Theorem (1.50) there exists a homotopy $\psi : X \times I \to X$ with $\psi_0 = \mathrm{id}_X$ and $\psi(i \times \mathrm{id}_I) = \varphi' = \varphi(i \times \mathrm{id}_I)$. Set $g' := \psi_1 : X \to X$. Then $g'i = i$. We define $F : X \times I \to X$ by

$$(x,s) \mapsto \begin{cases} \psi(gx, 1-2s), & 0 \le s \le \frac{1}{2}, \\ \varphi(x, 2s-1), & \frac{1}{2} \le s \le 1, \end{cases} \quad x \in X.$$

The definition makes sense since

$$\psi(gx, 0) = gx = \varphi(x, 0) \quad \text{for} x \in X,$$

and yields a homotopy $F : g'g \simeq \mathrm{id}_X$. For $a \in A$ and $0 \le s \le \frac{1}{2}$, the following holds:

$$\psi(gia, 1-2s) = \psi(ia, 1-2s) = \varphi(ia, 1-2s).$$

We have thus achieved that the points from A under F traverse null-homotopic paths in X. We take advantage of this and define $\Phi : A \times I \times I \to X$ by

$$(a, s, , t) \mapsto \begin{cases} \varphi(ia, 1-2s(1-t)), & 0 \le s \le \frac{1}{2}, \\ \varphi(ia, 1-2(1-s)(1-t)), & \frac{1}{2} \le s \le 1, \end{cases} \quad a \in A, t \in I.$$

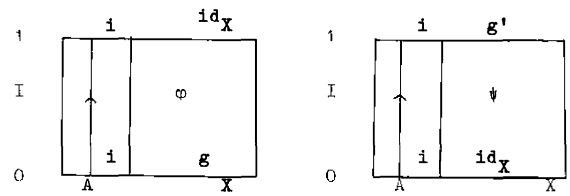


Figure 1.19:

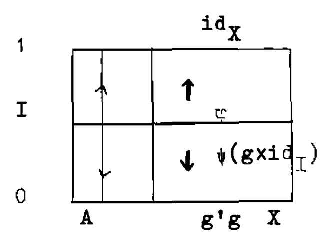


Figure 1.20:

Φ is a well-defined continuous map with the properties

$$\Phi(a,s,0) = F(ia,s) \quad a \in A, s \in I,$$
$$\Phi(a,0,t) = \Phi(a,s,1) = \Phi(a,1,t) = ia \quad a \in A, t, s \in I.$$

These properties are retained if one modifies Φ as in (1.67) to $\Phi' : A \times I \times I \to X$ such that $\Phi'(a,s,t)$ is independent of t for $0 \le t \le \frac{1}{2}$. Since i is an h-cofibration, $i \times \mathrm{id}$ is an h-cofibration by Corollary (1.61). Therefore, there exists a continuous map $\widetilde{\Phi} : X \times I \times I \to X$ with $\widetilde{\Phi}(i \times \mathrm{id}_I \times \mathrm{id}_I) = \Phi'$ and $\widetilde{\Phi}(x,s,0) = F(x,s)$ for $x \in X$, $s \in I$. We're done with that, because if we define $\widetilde{\Phi}_{(s,t)} : X \to X$ for $s,t \in I$ by $\widetilde{\Phi}_{(s,t)}(x) := \widetilde{\Phi}(x,s,t)$, $x \in X$, then

$$g'g = F_0 = \widetilde{\Phi}_{(0,0)} \overset{A}{\simeq} \widetilde{\Phi}_{(0,1)} \overset{A}{\simeq} \widetilde{\Phi}_{(1,1)} \overset{A}{\simeq} \widetilde{\Phi}_{(1,0)} = F_1 = \mathrm{id}_X .$$

Thus Lemma (1.65 is proven, □

hence also proven is Theorem (1.63). □

Remark 1.68. Theorem (1.62) is essentially a formal theorem and also holds if one replaces the category $\mathcal{T}\!op$ by the category $\mathcal{T}\!op_L^K$ (K and L are topological

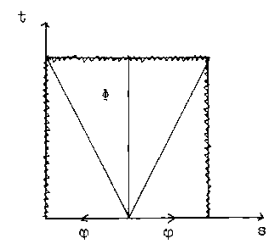


Figure 1.21:

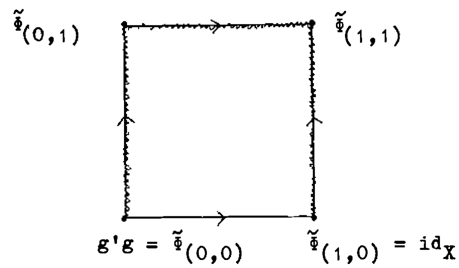


Figure 1.22:

spaces), i.e., if one assumes a commutative triangle in $\mathcal{T}op_L^K$ (cf. Kamps [15], 6.2).

1.2.4 Applications

The following theorems contain applications of Theorem (1.49) and Theorem (1.62). Let X be a topological space, A be a subspace of X, and $i : A \subset X$ be the inclusion. We have the following notions.

Definition 1.69. (a) A is a *weak retract* of X if and only if (a continuous map) $r : X \longrightarrow A$ with $ri \simeq \mathrm{id}_A$ exists.

(b) A is a *retract* of X if and only if there exists an $r : X \longrightarrow A$ with $ri = \mathrm{id}_A$.

(c) A is a *weak deformation retract* of X, if and only if $r : X \longrightarrow A$ with $ri \simeq \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$ exists, i.e., if i is a homotopy equivalence.

(d) A is *deformation retract* of X if and only if $r : X \longrightarrow A$ exists with $ri = \mathrm{id}_A$ and $ir = \mathrm{id}_X$.

(e) A is a *strong deformation retract* of X if and only if $r : X \longrightarrow A$ exists with $ri = \mathrm{id}_A$ and $ir \overset{A}{\simeq} \mathrm{id}_X$.

Note that here we consider i, id_X, r as morphisms of $\mathcal{T}op^A$, $i : \mathrm{id}_A \longrightarrow i$, $\mathrm{id}_X : i \longrightarrow i$, $r : i \longrightarrow \mathrm{id}_A$. This is possible because $ri = \mathrm{id}_A$.

Remark 1.70. Then trivially:

A is a retract of $X \Rightarrow A$ is a weak retract of X.

A is a strong deformation retract of $X \Rightarrow A$ is a deformation retract of X.

A is a deformation retract of $X \Rightarrow A$ is a weak deformation retract of X.

The converses of these statements are false in all three cases (cf. Spanier [24], 1.4.1, 1.4.8, 1.4.7).

However, we have the following theorem (cf. Spanier [24], 1.4.10, 1.4.11):

Theorem 1.71. *If i is an h-cofibration, then:*

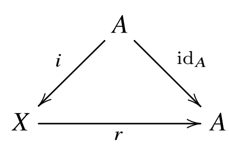
(1) If A is a weak retract of X, then A is a retract of X.

(2) If A is such a deformation retract of X, then A is a deformation retract of X.

(3) If A is a deformation retract of X, then A is a strong deformation retract of X.

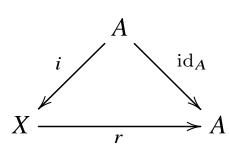
Proof. (1) and (2) are consequences of Theorem (1.49). (3) is a consequence of Theorem (1.62).

Regarding (1) Assuming that a continuous mapping exists $r : X \to A$ with $ri \simeq \mathrm{id}_A$. Theorem (1), applied to the diagramme



yields a continuous map $r' : X \to A$ with $r'i = \mathrm{id}_A$.

Regarding (2) By assumption, there exists $r : X \to A$ with $ri \simeq \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$. Since $ri \simeq \mathrm{id}_A$, the diagramme

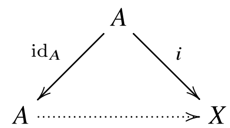


(1.72)

is commutative up to homotopy. Since i is an h-cofibration, by Theorem (1.49) there exists a continuous map $r' : X \to A$ with $r' \simeq r$ and $' = \mathrm{id}_A$. Since $ir \simeq \mathrm{id}_X$ and $r' \simeq r$, we have $ir' \simeq \mathrm{id}_X$. So A is the deformation retract of X.

Regarding (3) We prove the further statement: If $r : X \to A$ is a continuous map with $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$, then $ir \overset{A}{\simeq} \mathrm{id}_X$.

Proof. The diagram (1.72) is commutative in our situation, r is a homotopy equivalence. Since i is an h-cofibration, r is a homotopy equivalence under A according to Theorem (1.62). Therefore, there is a morphism from $\mathcal{Top}^A\ \mathrm{id}_A \to i$ that is homotopy inverse under A to r. Since $i : A \to X$ is the only continuous map that makes the diagramme

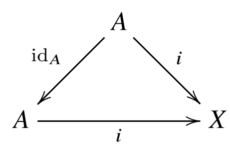


commutative, i is the only morphism from $\mathcal{Top}^A\ \mathrm{id}_A \to i$. So i is homotopy inverse under A to r. So $ir \overset{A}{\simeq} \mathrm{id}_X$. □

Thus Theorem 1.71 is completely proved. □

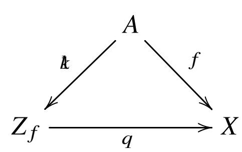
Theorem 1.73. *An inclusion $i : A \subset X$ is an h-cofibration and an h-equivalence if and only if A is a strong deformation retract of X.*

Proof. ⇒: follows as a consequence of Theorem (1.71) (2), (3) or directly from Theorem (1.62), applied to the diagramme



⇐: Let A be a strong deformation retract of X. As one immediately sees, this is equivalent to the inclusion $i : A \subset X$ being h-equivalent under A to id_A (cf. (0.30)). Since id_A is an h-cofibration, it follows from Theorem (1.46) i is an h-cofibration. Furthermore, i is an h-equivalence, since every strong deformation retract is a weak deformation retract (cf. (1.70)). □

Theorem 1.74. *Let $f : A \to X$ be a continuous map. We considered the commutative diagramme*



of Theorem (1.29, (a). The following statements are equivalent:

(a) f is an h-cofibration.

(b) $[q]$ is an isomorphism in $\mathcal{Top}^A h$ (i.e., q is a homotopy equivalence under A , $q \in \mathcal{Top}^A(k_1, f)$)

(c) $[q]^A$ *is a retraction in* $\mathscr{T\!op}^A h$.

Proof. (a) $\Rightarrow$ (b): follows from Theorem (1.62), since q is a homotopy equivalence according to Theorem (1.29) (c) and k_1 is a cofibration according to Theorem (1.29) (b).
(b) $\Rightarrow$ (c): is trivial.
(c) $\Rightarrow$ (a): The assumption is that $[q]$ is a retraction in $\mathscr{T\!op}^A h$. So f is dominated by k_1 in $\mathscr{T\!op}^A$. Since k_1 is a cofibration according to Theorem (1.29) (b), i.e., an h-cofibration, (a) follows from Theorem (1.46). □

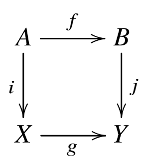
Since k_1 is a *closed* cofibration (cf. (1.13)), we obtain:

Corollary 1.75 (cf. Puppe [21], 7. Corollary 2)**.** *For every h-cofibration* $i : A \to X$ *there exists a closed cofibration* $i' : A \to X'$ *that is h-equivalent under* A *to* i *(cf. (0.30)).*

1.2.5 h-equivalences and h-equivalences of pairs

From Theorem (1.62) we can derive a corresponding theorem for the category of pairs $\mathscr{T\!op}(2)$ instead of $\mathscr{T\!op}^A$.

Theorem 1.76. *Let*



be a commutative diagram in $\mathscr{T\!op}$*. Let* i *and* j *be h-cofibrations. Let* f *and* g *be h-equivalences.*

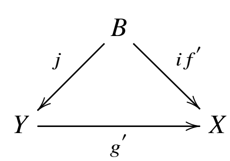
Claim*: The morphism* $(f, g) : i \to j$ *of* $\mathscr{T\!op}(2)$ *is an h-equivalence of pairs.*

Proof. We prove:

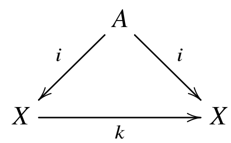
Proposition 1.77. Claim*: The morphism* $[(f, g)]$ *of* $\mathscr{T\!op}(2)h$ *has a left inverse.*

Applying (1.77) twice, it follows: $[(f, g)]$ is an isomorphism of $\mathscr{T\!op}(2)h$, i.e., (f, g) is an h-equivalence of pairs.

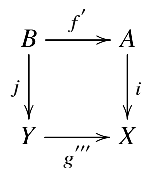
Proof. (of 1.77) Let $f' : B \to A$, $g' : YtoX$ be h-inverse to f and g, respectively. Consider the following diagramme:



We have $g'j = if'$, for $g' \simeq' jf'$ (since $ff' \simeq \mathrm{id}_B$) $= g'gif'$ (since $jf = gi$) $\simeq if'$ (since $g' \simeq \mathrm{id}_X$). Since j is an h-cofibration, according to Theorem (1.49) there exists a continuous map $g'' : Y \to X$ with $g' \simeq g''$ and $g'' j = if$. We note: g'' is h-inverse to g. We choose a homotopy $\varphi : f' f \simeq \mathrm{id}_A$, $\varphi : A \times I \to A$, such that φ is somewhat constant (cf. (1.66)). Since $i\varphi = if' f = g'' jf = g'' gi$ and since i is an h-cofibration, there exists a homotopy $\overline{\varphi} : X \times I \to X$ with $\overline{\varphi}(i \times \mathrm{id}_I) = i\varphi$ and $\overline{\varphi}_0 = g'' g$. Set $k := \overline{\varphi}_1 : X \to X$. Then the following diagramme is commutative, since $ki = \overline{\varphi}_1 i = i\varphi_1 = i$.



k is an h-equivalence, since $k = \overline{\varphi}_1 \simeq \overline{\varphi}_0 = g'' g \simeq \mathrm{id}_X$. According to Theorem (1.62) there exists a continuous map $k' : X \to X$ with $k' i = i$ and $k' k \overset{A}{\simeq} \mathrm{id}_X$. Set $g''' := k' g''$. Since $g''' j = k' g'' j = k' if' = if'$, (f', g''') is a morphism of $\mathcal{Top}(2)$, $(f' g''') : j \to i$.



□

Proposition 1.78. Claim*: $[(f, g''')]$ is left inverse to $[(f, g)]$ in $\mathcal{Top}(2)h$.*

Proof. We first choose a homotopy $\psi : k' k \overset{A}{\simeq} \mathrm{id}_X$, $\psi : X \times I \to X$, and define $\chi : X \times I \to X$ by

$$\chi(x, t) = \begin{cases} k' \overline{\varphi}(x, 2t) & \text{for} 0 \leq t \leq \frac{1}{2} \\ \psi(x, 2t - 1) & \text{for} \frac{1}{2} \leq t \leq 1. \end{cases}$$

The definition makes sense and yields a homotopy $\chi : g''' g \simeq \mathrm{id}_X$. Since ψ is a homotopy under A and since $k' \overline{\varphi}(i \times \mathrm{id}_I) = i_\varphi$, for $(a, t) \in A \times I$ we have

$$\chi(ia, t) = i\varphi(a, \min(2t, 1)).$$

If we define $\varphi' : A \times I \to A$ by $\varphi'(a, t) := \varphi(a, \min(2t, 1)$ for $(a, t) \in A \times I$, we obtain a homotopy $\varphi' ; f' f \simeq \mathrm{id}_A d$ with $\chi(i \times \mathrm{id}_I) = i\varphi'$. □

This proves (1.78) and therefore (prop:1-2-33). □

Remark 1.79. Theorem (1.76) is essentially a formal theorem and also holds if one replaces the category $\mathcal{Top}$ by $\mathcal{Top}^K_L$, where K and L are topological spaces (cf. Kamps [15], 6.4).

We conclude the section with a proposition.

Theorem 1.80. *Let $i : A \subset X$ be an h-cofibration and let A be contractible.*[5] *Then the natural projection $p : X \to X/A$ is an h-equivalence.*

Proof. Since A is contractible, we can choose a homotopy $\varphi : A \times I \to A$ between id_A and a constant map φ_1. We can assume (cf. (1.66)) that $\varphi(a, t) = a$ for $a \in A$ and $0 \leq t \leq \frac{1}{2}$. Since i is an h-cofibration, according to Theorem (1.50), there exists an extension of $i\varphi : A \times I \to X$ to a homotopy $\Phi : X \times I \to X$ with $\Phi_0 = \mathrm{id}_X$. Since $\Phi_1|_A$ is constant, Φ_1 induces a uniquely determined continuous map $f : X/A \to X$ with $fp = \Phi_1$. We show that f is h-inverse to p. Furthermore, $\mathrm{id}_A = \Phi_0 \simeq \Phi_1 = fp$. Since $\Phi(A \times I) \subset A$ and therefore $p\Phi|_{A \times I}$ is constant, Φ induces exactly one map $\overline{\Phi} : (X/A) \times I \to X/A$, which makes the following diagramme commutative.

$$\begin{array}{ccc} (X/A) \times I & \xrightarrow{\overline{\Phi}} & X/A \\ \uparrow{\scriptstyle p \times \mathrm{id}_I} & & \uparrow{\scriptstyle p} \\ Y & \xrightarrow[\Phi]{} & X \end{array}$$

$\overline{\Phi}$ is continuous, since $p \times \mathrm{id}_I$ is an identification according to (1.31). Now $\mathrm{id}_{XA} = \overline{\Phi}_0 \simeq \overline{\Phi}_1$. Since $\overline{\Phi}_1 \circ p = p \circ \Phi_1 = pfp$ and since p is surjective, we have $\overline{\Phi}_1 = pf$ and hence $\mathrm{id}_{X/A} \simeq pf$. □

1.3 Local characterisations of cofibrations and h-cofibrations

The following paragraph, which characterises cofibration and h-cofibration locally, is based on studies by D. Puppe (cf. [21]) and A. Strøm (cf. [27]).

1.3.1 Haloes

Definition 1.81. Let A, V be subspaces of a topological space X with $A \subset V \subset X$. V is called a *halo* of A in X[6] if there exists a continuous map $v : X \to I$ such that

$$A \subset v^1(0) \quad \text{and} \quad X \setminus V \subset v^{-1}(1). \tag{1.82}$$

A continuous v with (1.82) is called a *halo function of V*.

Remark 1.83. Let $A \subset X$. Then X is a halo of A in X, since $v = 0 : X \to I$ is a halo function of X.

Lemma 1.84. *Let $A \subset V \subset X$.*

(a) If V is a halo of A in X, then V is a neighbourhood of $\overline{A}$ in X.[7]

[5] Then in particular $A \neq \emptyset$.

[6] We briefly say: V is the *halo of* A if the context indicates which space X is meant.

[7] $\overline{A}$ denotes the closed hull of A in X.

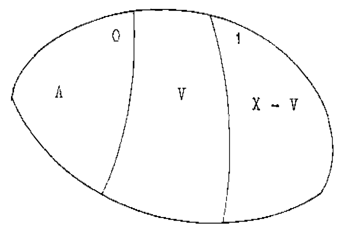


Figure 1.23:

(b) If X is normal and V is a neighbourhood of $\overline{A}$, then V is a halo of A.

(c) Let $\mathbb{R}^+$ be the subspace $\{x \in \mathbb{R} | x \geq 0\}$ of $\mathbb{R}$. If $u : X \to \mathbb{R}^+$ is a continuous map such that $A \subset u^{-1}[0, \alpha_1]$ for some real number $\alpha_1 \geq 0$, then $u^{-1}[0, \alpha_2[$ and $u^{-1}[0, \alpha_2]$ for every real number $\alpha_2 > \alpha_1$ are haloes of A and even of $u^{-1}[0, \alpha_1]$.

Proof. (a): Let v be a halo function of V. Then

$$A \subset v^{-1}(0) \subset v^{-1}[0,1[\subset V.$$

Since $v^{-1}(0)$ is closed in X, it follows that $\overline{A} \subset v^{-1}(0)$ follows. Since $v^{-1}[0,1[$ is open in X, we obtain the claim.
(b): is a direct consequence of Urysohn's theorem (cf. Sohubert [23], 1.8.4 Theorem 1).
(c): $v : X \to I$, defined by

$$v(x) := \min\left(1, \max\left(0, \frac{u(x) - \alpha_1}{\alpha_2 - \alpha_1}\right)\right) \quad \text{for} \quad x \in X,$$

is a halo function of $u^{-1}[0, \alpha_2[$ and $u^{-1}[0, \alpha_2]$. □

Lemma 1.85. *Let $A \subset X$.*

(a) Every superset of a halo of A is a halo of A.

(b) The intersection of finitely many haloes of A is a halo of A.

Proof. (a): Let $A \subset V \subset V' \subset X$. If V is a halo of A and $v : X \to I$ is a halo function of V, then v is also a halo function of V.
(b): It suffices to consider the intersection of two haloes. If V and W are haloes of A, $v : X \to I$ and $w : X \to I$ are halo functions of V and W, respectively, then $u : X \to I$, defined by

$$u(x) := \max(v(x), w(x)) \quad \text{for} \quad x \in X,$$

is a halo function of $V \cap W$. □

Corollary 1.86. *Let $A \subset V \subset X$. If V is a halo of A in X, then there exists a closed subset U of X, $A \subset U \subset V$, such that U is a halo of A in X and V is a halo of U in X. In particular, every halo V of A contains a closed halo U of A.*

Proof. Let V be a halo of A in X, and $\nu : X \to I$ be a halo function of V. By $U := \nu[0, \frac{1}{2}]$ we obtain a closed subset of X with $A \subset U \subset V$. U is a halo of A by (1.84) (c), and V is a halo of U by (1.84) (c) and (1.85) (a). □

Definition 1.87. Let $A \subset V \subset X$. V *can be contracted in X to A* $\operatorname{rel} A$[8] if and only if there exists a continuous map $r : V \to A$ such that $r|_A = \mathrm{id}_A$ and $(V \subset X) \stackrel{A}{\simeq} (V \stackrel{r}{\to} A \subset X)$. What is clear is that V can be contracted in X onto

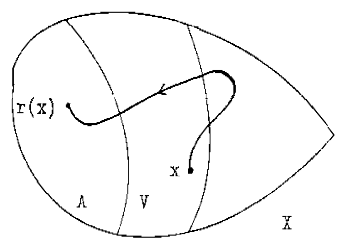


Figure 1.24:

A if and only if a homotopy $\varphi : Y \times I \to X$ exists such that $\varphi_1(V) \subset A$ and $\varphi : (V \subset X) \stackrel{A}{\simeq} \varphi_1$. Such a φ is called a *contraction* of V in X onto A.

It is also clear:

Remark 1.88. If $A \subset V' \subset V \subset X$ and V can be contracted to A in X, then so can V'.

Theorem 1.89. *Let $A \subset X$. Then the following statements are equivalent:*

1. *A has a halo V in X that can be contracted to A.*
2. *For every halo U of A in X, there exists a halo W of A in X with $W \subset U$ that can be contracted to A in U.*
3. *There exists a halo V of A in X and a homotopy $\psi : X \times I \to X$ such that $\psi_1(V) \subset A$ and $\psi : \mathrm{id}_X \stackrel{A}{\simeq} \psi_1$.*

Proof. (2) ⇒ (1): follows from Remark (1.83).
((1) ⇒ (3): Let V' be a halo contractible to A. By (1.86) and (1.88), we can assume V' is closed in X. Let $\nu' : X \to$ be a halo function of V' and $\varphi : V' \times I \to X$ be a contraction of V' in X to A. We set $V := \nu'^{-1}[0, \frac{1}{2}]$. By (1.84) (c), V is a

[8] To avoid any misunderstandings, let us say briefly: *V can be contracted to A.*

halo of A. Let $u : X \to I$ be given by $u(x) := \min(2 - 2v'(x), 1)$. We now define $\psi : X \times I \to X$ by

$$\psi(x,t) := \begin{cases} \varphi(x, t \cdot u(x)), & \text{if } \; x \in V' \\ x, & \text{if } \; x \in v'^{-1}(1). \end{cases}$$

ψ is well-defined. π is continuous since $v'^{-1}(1)$ and V' are closed in X. ψ is a homotopy $\operatorname{rel} A$ since φ is a homotopy $\operatorname{rel} A$. $\psi_0 = \mathrm{id}_X$, since $\varphi_0 = (V' \subset X)$. $\psi_1(V) \subset A$ since $V \subset V'$ since $u(x) = 1$ for $x \in V$ and since $\varphi_1(V') \subset A$.
(3) $\Rightarrow$ (2): Let U be a halo of A in X, $u : X \to I$ be a halo function of U. Let V, ψ be as in (3)), $v : X \to I$ be a halo function of V. We define

$$\begin{aligned} &w' : X \to I \quad \text{by} \quad w'(x) := \max_{t \in I} u(\psi(x,t)), \\ &w : X \to I \quad \text{by} \quad w(x) := \max(v(x), w'(x)), \\ &W := w^{-1}[0,1[. \end{aligned}$$

Then $W \subset U$ holds. W is a halo of A in X. A contraction of W in U onto A is obtained by $W \times I \to U$, where $(x,t) \mapsto \psi(x,t)$. □

Remark 1.90. . The continuity of w' follows from the following lemma, the proof of which we leave to the reader (cf. Brown [5], 7.3.8).

Lemma 1.91. *Let X, C be topological spaces. Let C be compact. If $\gamma : X \times C \to \mathbb{R}$ is a continuous map, then the map $g : X \to \mathbb{R}$ defined by*

$$g(x) := \max_{c \in C} \gamma(x,c)$$

is continuous.

Definition 1.92. We now discuss the connection between the concepts defined so far in §3 and some other concepts of set-theoretic topology.
1: A topological space X is called *completely regular* if for every point $x \in X$ and every neighbourhood U of x there exists a continuous map $f : X \to [0,1]$ with $f(x) = 0$ and $X \setminus U \subset f^{-1}(1)$.[9]
2: A topological space X is called *locally point contractible* in $x_0 \in X$ if for every neighbourhood V of x_0 there exists a neighbourhood U of x_0 and a homotopy $\varphi : U \times I \to V$ such that $U \subset V$, $\varphi_1(U) = \{x_0\}$ and $\varphi : (U \subset V) \overset{\{x_0\}}{\simeq} \varphi_1$.

In a completely regular space X, for each $x_0 \in X$, the concepts “neighbourhood of x_0 in X” and “halo of $\{x_0\}$ in X” coincide. The equivalence (1) $\Leftrightarrow$ (2) of Theorem (1.89) therefore yields:

Theorem 1.93. *Let X be completely regular, $x_0 \in X$. Then X is locally point-contractible at x_0 if and only if $\{x_0\}$ has a halo contractible on $\{x_0\}$.*

[9]In contrast to Schubert [23], 1.9.1, we do not require that X is Hausdorff.

1.3.2 Local characterisations of h-cofibrations

We can now characterise h-cofibrations locally.

Theorem 1.94. *Let $i : A \subset X$ be an inclusion. Then the following statements are equivalent:*

(a) i is an h-cofibration.

(b) A has a contractible halo on A in X.

Proof. (a) $\Rightarrow$ (b): Let i be an h-cofibration. Then, by the characterization of the notion of "h-cofibration" in Theorem (1.56) ($\varepsilon = \frac{1}{2}$), there exists a continuous map $r : X \times I \to (X \times 0) \cup (A \times I)$ with $r(x, 0) = x$ for $x \in X$ and

$$r(a,t) = \begin{cases} (a,0) & a \in A,\ 0 \leq t \leq \frac{1}{2} \\ (a, 2t-1) & a \in A,\ \frac{1}{2} \leq t \leq 1. \end{cases}$$

We define $v : X \to I$ by $v(x) := 1 - \mathrm{proj}_2 \circ r(x, 1)$. v is continuous, $A \subset v(0)$. Using $V := v[0, 1[$, we therefore obtain a halo of A in X (cf. (1.84) (c)). We define $\psi : X \times I \to X$ by

$$\psi(x,t) := \mathrm{proj}_1 \circ r(x,t).$$

ψ is continuous and $\psi(a, t) = a$ for $a \in A$, $t \in I$, $\psi(x, 0) = x$ for $x \in X$, thus

$$\psi : \mathrm{id}_X \overset{A}{\simeq} \psi_1.$$

Furthermore, $\psi_1(V) \subset A$ holds: if $v(x) < 1$, i.e. $\mathrm{proj}_2\, r(x, 1) > 0$, then $r(x, 1) \in A \times I$ and hence

$$\psi(x,1) \in A.$$

$\psi|_{V \times I}$ therefore yields a contraction of V in X onto A. This proves (b).
(b) $\Rightarrow$ (a): We assume (b. By Theorem (1.89) (3), there then exists a halo V of A in X and a homotopy $\psi : X \times I \to X$ with $\psi_1(V) \subset A$ and $\psi : \mathrm{id}_X \overset{A}{\simeq} \psi_1$. Since property (1.89) (3) is preserved when passing to a smaller halo, we can assume, by (1.86), that V is closed in X. Let v be a halo function for V. We want to prove that i is an h-cofibration. Let $f : X \to Y$, $\varphi : A \times I \to Y$ be continuous maps with $\varphi(a, 0) = f(a)$ for $a \in A$. We define $\Phi : X \times I \to Y$ by

$$\Phi(x,t) := \begin{cases} \varphi(\psi_1(x), t(1 - v(x))) & x \in V \\ f\psi_1(x) & x \in v^{-1}(1). \end{cases}$$

Φ is well-defined. Φ is continuous since V is closed in X. $\Phi(a, t) = \varphi(a, t)$ for $a \in A$.

$$\Phi_0 = f\psi_1 \overset{A}{\simeq} f \circ \mathrm{id}_X = f.$$

This proves (a). ∎

Example 1.95. *Note by the transcriber*: These are actually counterexamples.

Example 1. $X := \{0\} \cup \{\frac{1}{n} | n = 1, 2, 3, \dots\} \in \mathbb{R}, A := \{0\}$.

In (1.27) we saw that $i : A \subset X$ is not a cofibration. From Theorem (1.94) it follows that i is not an h-cofibration either: since none of the points $\frac{1}{n}, (n = 1, 2, \dots)$ can be connected to 0 by a path, 0 has no contractible halo in X.

Example 2. Let $X_n := \{(x, y) \in \mathbb{R}^2 | (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\} \subset \mathbb{R}^2$, $n = 1, 2, 3, \dots$ $X := \cup_{n=1}^{\infty} X_n \subset \mathbb{R}^2$,[10] $A := \{(0, 0)\}$.

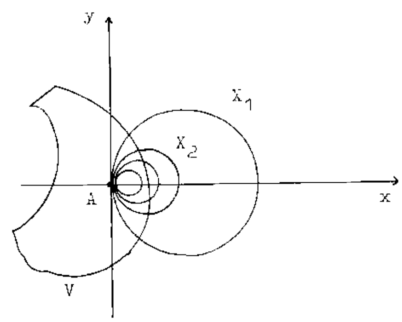


Figure 1.25:

Proposition 1.96. Claim*: $i : A \subset X$ is not an h-cofibration.*

Proof. We prove this indirectly (= proceed by contradiction) and assume that i is an h-cofibration. By Theorem (1.94), there is then a halo V in X that is contractible to A. Let $\varphi : V \times I \longrightarrow X$ be a contraction of V in X to A. V is a neighbourhood of $(0, 0)$ in X. Therefore, there exists a natural number n_0 such that $X_{n_0} \subset V$. We define a retraction $r : X \longrightarrow X_{n_0}$ by

$$r(x) := \begin{cases} x, & x \in X_{n_0} \\ (0, 0) & \text{otherwise.} \end{cases}$$

The composition

$$X_{n_0} \times I \subset V \times I \longrightarrow (\varphi) X \xrightarrow{r} X_{n_0}$$

is then a contraction of X onto $\{(0, 0)\}$. Since the 1-sphere $\mathbb{S}^1$ is not contractible (Eilenberg-Steenrod [9], XI. Theorem 3.1), such a contraction cannot exist. □

Example 3. $X := \mathbb{R}^1$ (or $X := I$), $A := \{0\} \cup \{\frac{1}{n} | n = 1, 2, 3, \dots\}$.

Proposition 1.97. Claim*: $i : A \subset X$ is not an h-cofibration.*

[10]Note by the transcriber: X is called a *Hawaiian earring*.

Proof. Suppose (on the contrary) that i is an h-cofibration. Then there exists a halo V of A in X and a contraction $\varphi : V \times I \to X$ of V in X on A. V is a neighbourhood of A in X, and thus contains, in particular, an interval of the form $[0, \frac{1}{n}]$ $(n \geq 1)$. Since $\{\frac{1}{n}\} \subset \varphi_1([0, \frac{1}{n}]) \subset A$, $\varphi_1([0, \frac{1}{n}])$ would not be connected. But this contradicts the continuity of φ_1. □

1.3.3 Local characterisations of cofibrations

The next theorem characterises cofibrations (cf. Strøm [27], 2. Lemma 4).

Theorem 1.98. *Let $i : A \subset X$ be an inclusion. Then the following statements are equivalent:*

(a) i is a cofibration.

(b) There exists a continuous map

$$u : X \to \mathbb{R}^+$$

and a homotopy $\varphi : X \times I \to X$ such that

(1) $A \subset u^{-1}(0)$,

(2) $\varphi(x, 0) = x$ for all $x \in X$,

(3) $\varphi(a, t) = a$ for all $(a, t) \in A\times$,

(4) $\varphi(x, t) \in A$ for all $(x, t) \in X \times I$ with $t > u(x)$.

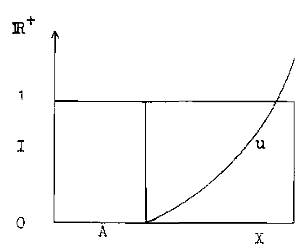


Figure 1.26:

Remark 1.99. If A is *closed* in X, then the conditions imposed on u and φ in (b) imply that:

$$\varphi(x, u(x)) \in A, \quad \text{if} \quad u(x) < 1.$$

(Consider a sequence $t_n \in I$ with $u(x) < t_n$ that converges to $u(x)$.) In particular, if $u(x) = 0$, then $x = \varphi(x, 0) = \varphi(x, u(x)) \in A$ and therefore $A = u^{-1}(0)$.

Proof. (of 1.98) By Theorem (1.25) an inclusion $i : A \subset X$ is a cofibration if and only if $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$.

(a) $\Rightarrow$ (b): If i is a cofibration, then there exists a retraction $r : X \times I \to (X) \cup (A \times I)$ (from $X \times I$ to $(X \times 0) \cup (A \times I)$). We define

$$u : X \to \mathbb{R}^+ \quad \text{by} \quad u(x) := \max_{t \in I}(t - \operatorname{proj}_2 r(x,t))$$
$$\varphi : X \times I \to X \quad \text{by} \quad \varphi(x,t) := \operatorname{proj}_1 r(x,t).$$

u is continuous since I is compact (cf. (1.91)). u and φ satisfy the conditions of (b).

(If $t > u(x)$, then $\operatorname{proj}_2 r(x,t) > 0$, therefore $r(x,t) \in A \times I$ and therefore $\varphi(x,t) = \operatorname{proj}_1 r(x,t) \in A$.)

(b) $\Rightarrow$ (a): If continuous mappings $u : X \to \mathbb{R}^+$, $\varphi : X \times I \to X$ with the properties of (b) are given, one obtains a retraction $r' : X \times I \to (X) \cup (A \times I)$ (from $X \times I$ to $(X \times 0) \cup (A \times I)$) by

$$r'(x) := \begin{cases} (\varphi(x,t)), 0), & t \leq u(x) \\ (\varphi(x,t), t - (x)) & t \geq u(x). \end{cases}$$

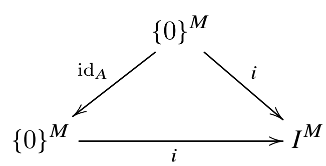
□

Not every h-cofiberation is a cofibration:

Example 1.100. Let M be an uncountable set. We define $X := I^M$ (product topology), $A := \{0\}^M$.

Proposition 1.101. Claim *: $i : A \subset X$ is an h-cofibration, but not a cofibration.*

Proof. In the commutative diagramme



i is a homotopy equivalence under A, since $\{0\}$ is a strong deformation retract of I. id is an h-cofibration. i is therefore an h-cofibration by Theorem (1.46). Assuming that i is a cofibration, then by (1.98) and (1.99) (A is closed in X) there exists a continuous map $u : I^M \to \mathbb{R}^+$ such that

$$u^{-1}(0) = \{0\}^M. \tag{1.102}$$

Since $\{0\} = \cap_{n=1}^{\infty}[0, \frac{1}{n}[$, it follows that $u^{-1}(0) = \cap_{n=1}^{\infty} u^{-1}[0, \frac{1}{n}[$. $[0, \frac{1}{n}[$ is the neighbourhood of 0 in $\mathbb{R}^+$. Therefore, $u^{-1}[0, \frac{1}{n}[$ is the neighbourhood of 0^M in I^M (u is continuous). By definition of the product topology, there exists a finite set $E_n \subset M$ such that

$$u^{-1}[0, \frac{1}{n}[\supset \{0\}^{E_n} \times I^{M \setminus E_n}.$$

(We identify: $I^M = I_n^E \times I^{M \setminus E_n}$.) So

$$\cap_{n=1}^{\infty} u^{-1}[0, \frac{1}{n}[\supset \{0\}^{M'} \times I^{M \setminus M'}.$$

where $M^{'}(= cup_{n=1}^{\infty} E_n)$ is a countable set. But this contradicts (1.102), because $M \setminus M^{'} \neq \varnothing$, since M is uncountable. □

Remark 1.103. (Cf. (1.48)). Example (1.100) also shows that the concept of "cofibration" is not invariant under homotopy equivalence under A:

id_A and i are isomorphic objects of $\mathcal{T}op^A$.
While id_A is a cofibration, i is not.

1.3.4 The product theorem for cofibrations

Theorem 1.104 (Product theorem for cofibrations)**.** *(cf. Strøm* [27]*, 2. Theorem 6). If $i : A \subset X$, $j : B \subset Y$ are cofibrations and A is closed in X, then*

$$(X \times B) \cup (A \times Y) \subset X \times Y$$

is a cofibration.

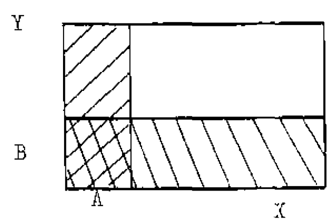


Figure 1.27:

Since $\varnothing \subset X$ is a closed cofibration (this follows from (1.25), since $X \times 0$ is retract of $X \times I$), one obtains from (1.104):

Corollary 1.105. *(Corollary 1 to 1.104.) If $j : B \subset Y$ is a cofibration and X is an arbitrary topological space, then $\mathrm{id}_X \times j : X \times B \to X \times Y$ is also a cofibration.*

Corollary 1.106. *(Corollary 2 to 1.104.) If $i : A \subset X$, $j : B \subset Y$ are cofibrations, then $i \times j : A \times B \subset X \times Y$ is also a cofibration.*

Proof. (of 1.104) For the cofibration $A \subset X$, we choose continuous maps $u : X \to \mathbb{R}^+$, $\varphi : X \times I \to X$ with the properties (b1)-(b4) of (1.98) (b). $v : Y \to \mathbb{R}^+$, $\psi : Y \times I \to Y$ be corresponding maps for the cofibration $B \subset Y$. We define continuous maps

$$w : X \times Y \to \mathbb{R}^+, \quad w(x, y) := \min(u(x), v(y)),$$
$$\chi : X \times Y \times I \to X \times Y, \quad \chi(x, y, t) := (\varphi(x, \min(t, v(y))), \psi(y, \min(t, u(x)))).$$

For w and χ we verify conditions (b1)-(b4) of (1.98) (b). We set $C := (X \times B) \cup (A \times Y)$. Then:

(1) $w(c) = 0$ if $c \in C$,
Then $\chi(a, y, t) = (a, \psi(y, 0)) = (a, y)$.

(2) $\chi(x, y, 0) = (\varphi(x, 0), \psi(y, 0)) = (x, y)$ for all $(x, y) \in X \times Y$.
Then $\chi(x, b, t) = (\varphi(x, 0), b) = (x, b)$.

(3) We claim: $\chi(c, t) = c$ for all $(c, t) \in C$.

Case 1 : $c = (a, y)$ with $a \in A$, $y \in Y$.
Then $\chi(a, y, t) = (a, \psi(y, 0)) = (a, y)$.

Case 2 : $c = (x, b)$ with $x \in X$, $b \in B$.
Then $\chi(x, b, t) = (\varphi(x, 0), b) = (x, b)$.

(4) We claim: $\chi(x, y, t) \in C$ for all $(x, y, t) \in X \times Y \times I$ with $t > w(x, y)$.

Case 1 : $u(x) \leq v(y)$.
It follows $u(x) = w(x, y) < t \leq 1$, so $u(x) < 1$ and $u(x) \leq \min(t, v(y))$.

Case $\min(t, v(y)) > u(x)$:
then $\varphi(x, \min(t, v(y))) \in A$ holds because of (1.98) (b) (b4) for u and φ.

Case $\min(t, v(y)) = u(x)$:
then from (1.99) it follows that $\varphi(x, \min(t, v(y))) \in A$, since $u(x) < 1$ and since A is closed in X. So $\chi(x, y, t) \in A \times Y \subset C$.

Case 2 : $u(x) > v(y)$
Then $\min(t, u(x)) > v(y)$ and hence $\psi(y, \min(t, u(x))) \in B$ because of (1.98) (b) (b4) for u and and ψ. So $\chi(x, y, t) \in X \times B \subset C$.

So $(X \times B) \cup (A \times Y) \subset X \times Y$ is a cofibration by Theorem (1.98). □

The following example shows that the assumption "A is closed in X" in Theorem (1.104) cannot be omitted:

Example 1.107. Let M be an uncountable set. According to (1.100), $\{0\}^M \subset I^M$ is a closed h-cofibration. $\{0\} \subset I$ is a closed cofibration. (This follows from (1.25), since $(I \times 0) \cup (O \times I)$ is a retract of $I \times I$.)

Proposition 1.108. Claim *:* $C := (I^M \times 0) \cup (0^M \times I) \subset (I^M \times I$ *is not a h-cofibration.*

Proof. We set $C := (I^M \times 0) \cup (0^M \times I)$ and assume that $C \subset I^M \times I$ is an h-cofibration. Then, by (1.92), there is a halo V of C in $I^M \times I$ and a retraction $r : V \to C$. Since $0^M \times I \subset V$ and since I is compact, there is a neighbourhood U of 0^M in I^M with $U \times I \subset V$. By the definition of the product topology, there exists a finite set $E \subset M$ such that $U \supset 0^E \times I^{M \setminus E}$. (We identify $I^M = I^E \times I^{M \setminus E}$.) We define $\alpha : I^{M \setminus E} \times I \to I^M \times I$ by $\alpha(x, t) := (0^E, x, t)$ for $x \in I^{M \setminus E}$, $t \in I$. Then

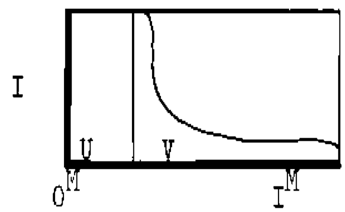


Figure 1.28:

$\alpha(I^{M\setminus E} \times I) \subset V$ holds. Let $\beta : I^M \times I \to I^{M\setminus E} \times I$ be given by $\beta(y, x, t) := (x, t)$ for $y \in I^E$, $x \in I^{M\setminus E}$, $t \in I$. In the diagramme

$$\begin{array}{ccc} I^M \times I & \xrightarrow{r'} & (I^M \times 0) \cup (0^{M\setminus E} \times I) \\ \downarrow{\scriptstyle \alpha'} & & \uparrow{\scriptstyle \beta'} \\ V & \xrightarrow[r]{} & C \\ \cap & & \\ I^M \times I & & \end{array}$$

let α' and β' arise by restricting α and β, respectively. By $r' := \beta' r \alpha'$ we obtain a retraction

$$I^M \times I \to (I^M \times 0) \cup (0^{M\setminus E} \times I).$$

According to (1.25), $\{0\}^{M\setminus E} \subset I^{M\setminus E}$ would then be a cofibration. This contradicts (1.100), since $M \setminus E$ is still uncountable. □

We note that (1.106) also holds for h-cofibrations.

Theorem 1.109. *If $i : A \to X$, $j : B \to Y$ are h-cofibrations, then $i \times j : A \times B \to X \times Y$ is an h-cofibration.*

Proof. Note that $i \times j = (i \times \mathrm{id}_Y)(\mathrm{id}_A \times j)$. $i \times \mathrm{id}_Y$ and $\mathrm{id}_A \times j$ are h-cofibrations, by (1.59) and (1.61). The composition of h-cofibrations is again an h-cofibration (cf. (1.43)). □

1.3.5 A characterisation of closed cofibrations

At the end of this section, we provide a characterisation of *closed* cofibrations.

Theorem 1.110 (Puppe [21], 7th Corollary 3)**.** *Let $i : A \subset X$ be an inclusion. Then the following statements are equivalent:*

(a) i is a cofibration and A is closed in X.

(b) i is an h-cofibration and A is a set of zeros (i.e., there exists a continuous map $u : X \to \mathbb{R}^+$ with $A = u^{-1}(0)$).

Proof. (a) $\Rightarrow$ (b): follows from (1.98) and (1.99).
(b) $\Rightarrow$ (a): Let $f : X \to Y$, $\varphi : A \times I \to Y$ be continuous maps with $\varphi(a,0) = f(a)$ for all $a \in A$. Since i is an h-cofibration by assumption, there exists an extension $\Phi^{'} : X \times I \to Y$ of φ and a homotopy $\Phi^{''} : X \times I \to Y : f \overset{A}{\simeq} \Phi^{'}_0$. Let $u : X \to \mathbb{R}^+$ be a continuous map with $A = u^{-1}(0)$. We can assume that $u(X) \subset [O, \frac{1}{2}]$ (If necessary, replace $u(x)$ for $x \in X$ by $\min(u(x), \frac{1}{2})$). We define a map

$$\Phi : X \times I \to Y$$

by

$$\Phi(x,t) := \begin{cases} \Phi^{'}(x, \frac{t-u(x)}{1-u(x)}), & \text{if } t \geq u(x) \\ \Phi^{''}(x, \frac{t}{u(x)}), & \text{if } t \leq u(x) \text{ and } u(x) > 0 \\ f(x), & \text{if } t \leq u(x) \text{ and } u(x) = 0, \text{ i.e, } (x,t) \in A \times 0. \end{cases}$$

i is well-defined:

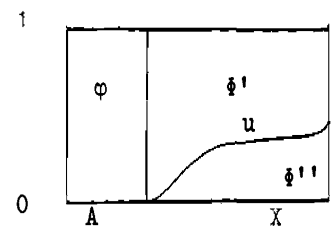


Figure 1.29:

Let $u(x) = t$. If $u(x) > 0$, then $Phi^{'}(x,0) =$
$Phi^{''}(x,1)$; if $u(x) = 0$, i.e., $x \in A$, then $\Phi^{'}(x,0) = \varphi(x,0) = f(x)$. Φ is an extension of φ, since $\Phi^{'}$ is an extension of φ. $\Phi_0 = f$, since
$Phi^{''}_0 = f$.
What remains to be proven is the continuity of Φ. The inequalities $t \geq u(x)$ and $t \leq u(x)$ describe closed subsets F and G of $X \times I$. $\Phi|_F$ is continuous since $\Phi^{'}$ is continuous. We are finished when we show that $\Phi|_G$ is continuous.
Since $\Phi^{''}$ is continuous, we first have to prove that $\Phi|_G$ is continuous at the points of the open subset of G, which is described by $u(x) > 0$. It therefore remains to verify the continuity of $\Phi|_G$ at the points of $A \times 0$.
Let $a \in A$. Then $(\Phi|_G)(a,0) = f(a)$. Let V be a neighbourhood of $f(a)$ in Y, $t \in I$. Since $\Phi^{''}$ is a homotopy under A, $\Phi^{''}(a,t) = \Phi^{''}(a,0) = f(a)$. Since $\Phi^{''}$ is continuous in (a,t), there exist neighbourhoods U_t of a in X, R_t of t in I such that $\Phi^{''}(U_t \times R_t) \subset V$. By the compactness of I, there exist finitely many points $t_0, \ldots, t_m$ such that $I = \cup_{k=0}^m R_{t_k}$. Set $U := \cap_{k0}^m U_{t_k}$. U is a neighbourhood of a in X such that $\Phi^{''}(U \times I) \subset V$. But then $(\Phi^{''}|_G)((U \times I) \cap G) \subset V$ holds. So $\Phi|_G$ is continuous in $(a,0)$. □

Chapter 2

Fibrations

2.1 Mapping Spaces

2.1.1 The compact-open topology

Let X, Y be topological spaces. On the set $\mathscr{Top}(X,Y)$ of *continuous* maps $X \to Y$ we define a topology called the *compact open topology.* If $K \subset X$, $Q \subset Y$ are subsets, then $T(K,Q) \subset \mathscr{Top}(X,Y)$ is defined by

$$T(K,Q) := \{u \in \mathscr{Top}(X,Y) | u(K) \subset Q\}.$$

Definition 2.1. Let the *compact-open topology* on $\mathscr{Top}(X,Y)$ be the topology generated by the sets of the form $T(K,Q)$, where K is a compact subset of X and Q is an open subset of Y.

The elements of the compact-open topology on $\mathscr{Top}(X,Y)$ are thus precisely those subsets of $\mathscr{Top}(X,Y)$ that are arbitrary unions of finite intersections of sets of the form $T(K,Q)$ where K is a compact subset of X, Q is an open subset of Y.

Special case If X is a discrete topological space, then $\mathscr{Top}(X,Y)$ is the set of all maps $X \to Y$. Since the compact subsets of X are precisely the finite subsets of X, it is easy to see that the compact-open topology on $\mathscr{Top}(X,Y)$ coincides with the product topology.

Convention If X, Y are topological spaces, we will, henceforth, always consider the set $\mathscr{Top}(X,Y)$ to have the compact open topology We denote the topological space thus obtained by Y^X.

We now summarise the most important properties of the compact open topology. We will generally refrain from proofs, since these, if they are not already very simple, are detailed in Bourbaki [3], §3, n° 4 (p. 43 ff).

Remark 2.2. Bourbaki works with the following two concepts: A topological space X is *compact* if X is Hausdorff and every open covering of X contains a finite subcover. A topological space X is *locally compact* if X is Hausdorff and every point of X has a neighbourhood basis of compact sets. When studying Bourbaki's proofs, one finds that the following theorems are also true if one abandons the "Hausdorff" requirement for both concepts.

2.1.2 The exponential law

Definition 2.3. If X, Y are sets, then let Y^X denote the set of all maps $X \to Y$. Let X, Y, Z be sets. To a map $f : X \times Y \to Z$ we associate a map $\overline{f} : X \to Z^Y$, namely, for $x \in X$ let $\overline{f}(x) : Y \to Z$ be the map that transforms $y \in Y$ into $f(x, y) \in Z$. $\overline{f}$ is characterised by the equation

$$(\overline{f}(x))(y) = f(x, y) \quad \text{for} \quad x \in X, y \in Y. \tag{2.4}$$

$f \mapsto \overline{f}$ yields a bijection $Z^{X\times Y} \xrightarrow{\cong} (Z^Y)^X$ (*exponential law*). f and $\overline{f}$ are said to be *adjoint* to each other.

Remark 2.5. Now let X, Y, Z be topological spaces, and let $f : X \times Y \to Z$ be a map. For $x \in X$, f induces a map $\overline{f}(x) : Y \to Z$ by (2.4).

Theorem 2.6. Premise*: f is continuous.*

Claim 1 *$\overline{f}(x)$ is continuous for all $x \in X$. f therefore induces a map $\overline{f} : X \to Z^Y$, where Z^Y again denotes the set of continuous maps $Y \to Z$, provided with the compact-open topology.*

Claim 2 *$\overline{f}$ is continuous.*

Theorem (2.6) can be reversed if Y is locally compact.

Theorem 2.7. *If $f : X \times Y \to Z$ is a map that induces a continuous map $\overline{f} : X \to Z^Y$ by (2.4), then f is continuous if Y is locally compact.*

By Theorem (2.6), the map $f \mapsto \overline{f}$ defines a map

$$\vartheta : Z^{X\times Y} \to (Z^Y)^X.$$

ϑ is injective. Theorem (2.7) states:

Theorem 2.8. *ϑ is surjective, hence bijective, if Y is locally compact.*

Corollary 2.9. *ϑ is topological if X and Y are Hausdorff and Y is locally compact (*exponential law for mapping spaces*).*

2.1.3 Composition of maps

Let X, Y, Z be topological spaces. Let $\varkappa : Y^X \times Z^Y \to Z^X$ be the *composition map*, i.e., $\varkappa(u, v) := v \circ u$ for $u \in Y^X$, $v \in Z^Y$.

Theorem 2.10. *We have*

(a) $\varkappa(u_0, v)$ is continuous in v for all $u_0 \in Y^X$.

(b) $\varkappa(u, v_0)$ is continuous in u for all $v_0 \in Z^Y$.

(c) $\varkappa$ is continuo if Y is locally compact.

2.1.4 Applications

Definition 2.11. Let X, Y be topological spaces. Define $f : Y^X \times X \to Y$ by

$$f(u,x) : u(x) \quad u \in Y^X,\ x \in X.$$

f is called an *evaluation map*. (See Hu [12], p. 74).

Theorem 2.12. *If X i locally compact, then the evaluation map is continuous.*

Proof. According to (2.4) the map induced by the evaluation map is id_{Y^X}. Since this map is continuous and since X is locally compact, the assertion of the theorem follows from Theorem (2.7). □

Theorem 2.13 (See (1.31))**.** *Let X, X', Y be topological spaces. If $p : X \to X'$ is an identification and Y is locally compact, then*

$$p \times \mathrm{id}_Y : X \times Y \to X' \times Y$$

is an identification.

Proof. Let Z be another topological space. Let $f : X \times Y \to Z$ and $f' : X' \times Y \to Z$ be maps that make the following diagramme commutative.

$$\begin{array}{ccc} X \times Y & \xrightarrow{f} & Z \\ {\scriptstyle p\times \mathrm{id}_Y}\downarrow & \nearrow{\scriptstyle f'} & \\ X' \times Y & & \end{array}$$

We assume that f is continuous and have to prove that f' is continuous. It is easy to see that (2.4) induces a commutative diagramme.

$$\begin{array}{ccc} X & \xrightarrow{\overline{f}} & Z^Y \\ {\scriptstyle p}\downarrow & \nearrow{\scriptstyle \overline{f'}} & \\ X' & & \end{array}$$

We use the continuity of f and Theorem (2.6), Claim 1. The continuity of f implies, by Theorem (2.6), Claim 2, the continuity of $\overline{f}$. Thus, $\overline{f'}$ is continuous, because p is an identification. Since Y is locally compact, the continuity of f' follows from Theorem (2.7). □

2.1.5 Mapping spaces and adjoint functors

We conclude this section with a category-theoretical consideration.

Let C be a fixed locally compact topological space. We define two covariant functors $S, T : \mathcal{T}op \to \mathcal{T}op$.

Definition 2.14 (of T)**.** If Y is a topological space, then we set

$$TY := Y^C .$$

If $g : Y \to Y'$ is a continuous map, then let

$$Tg : Y^C \to Y'^C$$

be the map $g^C : u \in Y^C \mapsto g \circ u \in Y'^C$. Note that g is continuous by Theorem (2.10) (b).

Definition 2.15 (of S)**.** If X is a topological space, then we set

$$SX := X \times C.$$

If g is a continuous map, then let

$$Sg : g \times \mathrm{id}_C .$$

Remark 2.16. For the functor T of (2.14) we also use the notation $-^C$, and for the functor S of (2.15) we use the notation $- \times C$.

Remark 2.17. Since C is locally compact, for any two topological spaces X, Y we have a bijective map

$$\mathcal{Top}(SX, Y) \to \mathcal{Top}(X, TY),$$

namely the map

$$\vartheta : \mathcal{Top}(X \times C, Y) \to \mathcal{Top}(X, Y^C),$$

which transforms $f : X \times C \to Y$ into $\overline{f} : X \to Y^C$ (cf. (2.4), (2.8)).

ϑ is natural. If $g : X' \to X$ and $h : Y \to Y'$ are continuous maps, then, as the reader immediately calculates, the following diagram is commutative:

$$
\begin{array}{ccccc}
f & \mathcal{Top}(X \times C, Y) & \xrightarrow{\vartheta} & \mathcal{Top}(X, Y^C) & \overline{f} \\
\downarrow & \downarrow{\scriptstyle \mathcal{Top}(g \times \mathrm{id}_C, h)} & & \downarrow{\scriptstyle \mathcal{Top}(g, h^C)} & \downarrow \\
h \circ f \circ g \times \mathrm{id}_C & \mathcal{Top}(X' \times C, Y') & \xrightarrow[\vartheta]{} & \mathcal{Top}(X' \times C, Y') & h^C \circ \overline{f} \circ g
\end{array}
$$

But this means:

Theorem 2.18. *S and T are adjoint functors, more precisely: T is adjoint to S , S is coadjoint to T (cf. Mitchell* [17]*, V.1).*

2.2 Fibrations

In this section we introduce the concept of fibration, which is dual to the concept of cofibration.

2.2.1 The covering homotopy property (CHP). Fibrations

We first review the concept of homotopy. In (0.17) and (0.18), we defined the concept of homotopy using the functor $- \times I$ (cf. (2.15), (2.16)) and the natural transformations

$$j_\nu : \mathrm{id}_{\mathcal{Top}} \to - \times I \quad (\nu = 0, 1)$$

given by the continuous maps

$$j_\nu : X \to X \times I, \quad x \mapsto (x, \nu).$$

If $f, g : X \to Y$ are continuous maps, then f is homotopic to g if and only if there exists a homotopy $\varphi : X \times I \to Y$ such that $f = \varphi \circ j_0$ and $g = \varphi \circ j_1$. The following theorem shows that the concept of homotopy can also be introduced using the functor $-^I$ (cf. (2.14), (2.16)) and two natural transformations

$$q_0, q_1 : -^I \to \mathrm{id}_{\mathcal{Top}}$$

defined as follows:

If Y is a topological space, then let $q_0 : Y^I \to Y$ be the map that assigns a (*normalised*) *path* u *in* Y, i.e., a continuous map $u : I \to Y$, to the *starting point*, i.e., the point $u_0 \in Y$, and let $q_1 : Y^I \to Y$ be the map that transforms a path u in Y to the *end point*, i.e., the point $u_1 \in Y$. Note that $q_0, q_1 : Y^I \to Y$ are continuous since I is locally compact.

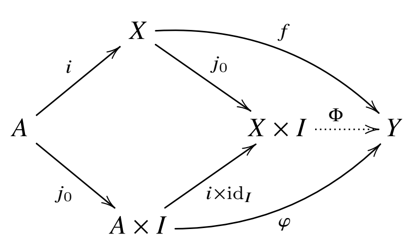
Theorem 2.19. *Let $f, g : X \to Y$ be continuous maps. f is homotopic to g if and only if there exists a continuous map $\overline{\varphi} : X \to Y^I$ with $q_0\overline{\varphi} = f$ and $q_1\overline{\varphi} = g$.*

Proof. Since I is locally compact, the transition $\varphi \mapsto \overline{\varphi}$ of (2.4) yields a bijection between the homotopies $X \times I \to Y$ and the continuous maps $X \to Y^I$ ($- \times I$ and $-^I$ are adjoint functors). If $\varphi : X \times I \to Y$ is a homotopy, then, as one immediately calculates,

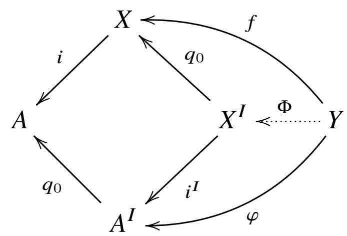
$$\varphi j_\nu = q_\nu \overline{\varphi} \quad (\nu = 0, 1). \tag{2.20}$$

This, however, immediately leads to the assertion of the theorem. □

Theorem (2.19) shows that the morphisms $j_\nu : X \to X \times I$ in $\mathcal{Top}$ and $q_\nu : Y \to Y^I$ in ${}^*\mathcal{Top}$, the category dual to $\mathcal{Top}$, play a formally analogous role. We take this opportunity to show in the diagramme

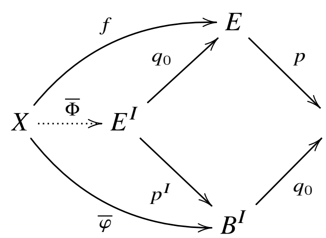


with which we introduced the homotopy extension property (of a continuous map i for a topological space Y), replacing $X \times I$ by X^I, j_0 by q_0, and reversing the arrows:



We change the notations, write $p : E \to B$ for $i : X \to A$, X for Y, $\overline{\varphi}$, $\overline{\Phi}$ for φ, Φ and are led to the following definition.

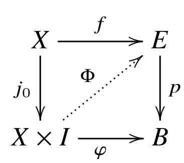
Definition 2.21. Let $p : E \to B$ be a continuous map and X a topological space. We say *p has the covering homotopy property* (short: *CHP*) for X if and only if for all continuous maps $f : X \to E$, $\overline{\varphi} : X \to B^I$ with $q_0\overline{\varphi} = pf$ there exists a continuous map $\overline{\Phi} : X \to E^I$ with $p^I \circ \overline{\Phi} = \overline{\varphi}$ and $q_0\overline{\Phi} = f$.



(2.22)

We take advantage of the fact that $- \times I$ and $-^I$ are adjoint functors, and as in (2.4) we go from $\overline{\varphi}$ to φ, from $\overline{\Phi}$ to Φ and obtain, as one immediately confirms (cf. also equation (2.20)):

Theorem 2.23. *A continuous map $p : E \to B$ has the CHP for a topological space X if and only if for all continuous maps $f : X \to E$ and all homotopies $\varphi : X \times I \to B$ with $\varphi j_0 = pf$ there exists a homotopy $\Phi : X \times II \to E$ with $p\Phi = \varphi$[1] and $\Phi j_0 = f$.*



(2.24)

Remark 2.25. The fact that $p : E \to B$ has the CHP for X geometrically means that one can raise homotopies $\varphi : X \times I \to B$ to homotopies $\Phi : X \times I \to E$ with a given initial position $f : X \to E$ over φj_0.

[1] We then also say: *Φ is above φ.*

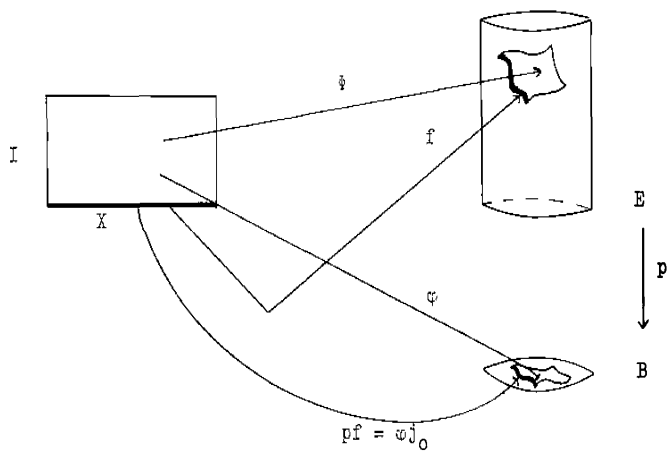


Figure 2.1:

Definition 2.26. A continuous map $p : E \to B$ is called a *fibration*[2] if p has the CHP for all topological spaces X.

Remark 2.27. E is called the *total space*, B is the *base space* of the fibration p.

Clearly wwe have:

Theorem 2.28. *A continuous map $p : E \to B$ is a fibration if and only if the following diagramme in $\mathscr{T}\!op$ is a weak Cartesian square (cf. (0.8)).*

$$\begin{array}{ccccc} & & E & & \\ & \overset{q_0}{\nearrow} & & \overset{p}{\searrow} & \\ E^I & & & & B \\ & \underset{p^I}{\searrow} & & \underset{q_0}{\nearrow} & \\ & & B^I & & \end{array} \tag{2.29}$$

2.2.2 Examples

Definition 2.30. A continuous map $p : E \to B$ is called *trivial* if there exists a topological space F and a homeomorphism $\psi : E \to B \times F$ that makes the following diagramme commutative,

$$\begin{array}{ccc} E & \xrightarrow{\psi} & B \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}_1} & \\ & B & \end{array}$$

[2]In the literature the term *Hurewicz fibration* is also common.

i.e., a continuous map $p : E \to B$ is trivial if it is isomorphic to a projection in the category $\mathcal{T}op_B$ of topological spaces over B.

Theorem 2.31. *A trivial mapp* $: E \to B$ *is a fibration.*

Proof. We can assume without significant restriction that p is a projection: $p = \mathrm{proj}_1 : B \times F \to B$. For $f : X \to B \times F$ and $\varphi : X \times I \to B$ with $\varphi j_0 = \mathrm{proj}_1 \circ f$, one obtains $\Phi : X \times I \to B \times F$ with $p\Phi = \phi$ and $\Phi j_0 = f$ by the definition

$$\Phi(x,t) := (\varphi(x,t), \mathrm{proj}_2 \circ f(x)) \quad \text{for} \quad x \in X, t \in I.$$

$$\begin{array}{ccc} X & \xrightarrow{f} & E = B \times F \\ {\scriptstyle j_0}\downarrow & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle p=\mathrm{proj}_1} \\ X \times I & \xrightarrow[\varphi]{} & B \end{array}$$

□

Definition 2.32. If $p : E \to B$ is a map, $U \subset B$ a subset, then denote by

$$P_U : p^{-1}(U) \to U$$

the restriction of p to the source $p^{-1}(U)$ and destination the U.

Definition 2.33. A continuous map $p : E \to B$ is called *locally trivial* if every point $b \in B$ has a neighbourhood U such that p_U is trivial.

Example 2.34. The tangent bundle $TM \to M$ of a C^r-manifold M $(r \geq 1)$ is a locally trivial map. (For the definition of the terms C^r-manifold and tangent bundle, see Lang [16], II. §1, 111.§2.)

In §9 we prove:

Theorem 2.35. *If* $p : E \to B$ *is locally trivial and* B *is paracompact, then* p *is a fibration. In particular, the tangent bundle* $TM \to M$ *of a paracompact* C^r*-manifold* M $(r \geq 1)$ *is a fibration.*

Special locally trivial mappings are the coverings.

Definition 2.36. A continuous map $p : E \to B$ is called a *covering* if for every point $b \in B$ there exists a neighbourhood U of b in B and a discrete topological space D such that p_U in $\mathcal{T}op_U$ is isomorphic to $\mathrm{proj}_1 : U \times D \to U$.

Theorem 2.37. *Every covering is a fibration.*

Proof. Spanier [24], 2.2 Theorem 3. □

Remark 2.38. If p is a covering, then Φ in (2.24) is even uniquely determined by φ and f.

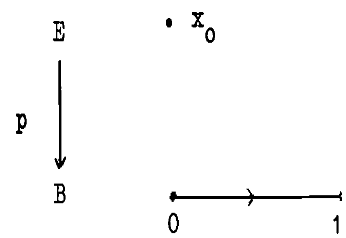


Figure 2.2:

Example 2.39. 1. Let E be a topological space that has exactly one point x_0, $B := I$. Map $p : E \to B$ to the point 0. p is not a fibration because p does not have the CHP for the space E: For $f := \mathrm{id}_E$ and $\varphi := \mathrm{proj}_2 : E \times I \to I$, there does not even exist a (set) map $\Phi : E \times I \to E$ with $p\Phi = \varphi$.

The map p in Example 1 is not surjective. The next example presents a surjective map that is not a fibration.

2. Let E be the topological sum $\{x_0 + I$, $B := I$. Let $p : E \to B$ be given by $p(x_0) := 0$, $p(t) := t$ for $t \in I$. p is not a fibration because p does not have the

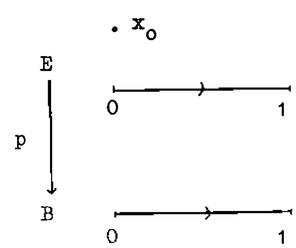


Figure 2.3:

CHP for $X = \{x_0\}$. For $f : X \to E$ with $f(x_0) := x_0$ and $\varphi := \mathrm{proj}_2 : X \times I \to I$, there exists no continuous map $\Phi : X \times I \to E$ with $p\Phi = \varphi$ and $\Phi(x_0, 0) = x_0$.

3. Let E be the factor space obtained from $I \times I$ by identifying $(0, t)$ with $(1, 1-t)$ for each $t \in I$. Let B be derived from I by identifying the points 0 and 1. B is homeomorphic to $\mathbb{S}^1$. E is called a *Möbius strip*. Let $p : E \to B$ be the continuous map induced by $\mathrm{proj}_1 : I \times I \to I$. p is locally trivial because $p_{B \setminus \{b\}}$ is trivial for all $b \in B$. Thus, by Theorem (2.35), p is a fibration (B is compact!). However, p is not trivial because E is not homeomorphic to $\mathbb{S}^1 \times I$.

(Justification: The boundary of $\mathbb{S}^1 \times I$ is homeomorphic to $\mathbb{S}^1 \times \dot{I}$, where $\dot{I} := \{0\} \cup \{1\} \subset I$, the boundary of E is homeomorphic to $\mathbb{S}^1$. $\mathbb{S}^1$ is connected, but $\mathbb{S}^1 \times \dot{I}$ is not. A point x of E or $\mathbb{S}^1 \times \dot{I}$ is called a *boundary point* if, for every neighbourhood of x, there exists a smaller neighbourhood of x that is

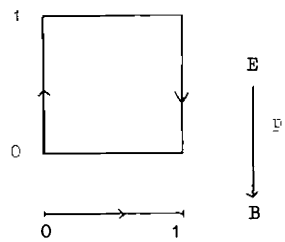


Figure 2.4:

simply connected after removing x (For the term "simply connected"', see Schubert [23], III.5.3.).)

We mention in passing that the following concept of fibration also plays an important role in the literature.

Definition 2.40. A continuous map p is called a *Serre fibration* if p has the CHP for I^n, $n = 0, 1, 2, \ldots$. (Let I^0 be a topological space with exactly one point.)

Remark 2.41. Spanier uses the term *weak fibration* instead of Serre fibration ([24], p. 374).

Proposition 2.42. *A continuous map p is a Serre fibration if and only if p satisfies the CHE for all CW complexes.*

Proof. Puppe [20], Theorem 4.6. □

Remark 2.43. Like the notion of "fibration", the notion of "cofibration" can be characterised in terms of both $-\times I$ and $-^I$ due to the adjointness of the functors $-\times I$ and $-^I$.

Theorem 2.44. *A continuous map $i : A \to X$ is a cofibration if and only if for all continuous maps $\overline{\varphi} : A \to Y^I$ and $f : X \to Y$ with $q_0\overline{\varphi} = fi$ there exists a continuous map $\overline{\Phi} : X \to Y$ with $\overline{\Phi}i = \overline{\varphi}$ and $q_0\overline{\varphi} = f$.*

$$\begin{array}{ccc} A & \xrightarrow{\overline{\varphi}} & Y^I \\ {\scriptstyle i}\downarrow & \overline{\Phi}\nearrow & \downarrow{\scriptstyle q_0} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{2.45}$$

2.2.3 The mapping path space of a continuous map

The role played by the mapping cylinder of a continuous image in the area of cofibrations is taken over by the mapping path space of a continuous image in the case of fibrations.

Let $p : E \to B$ be a continuous map.

Definition 2.46. The path space

$$W_p := \{(e,u) \in E \times B^I | p(e) = u(0)\}$$

of the product $E \times B^I$ is called the *mapping path space of* p.

The elements of W_p are therefore the pairs (e,u) consisting of a point e of E and a (normalized) path u in B that starts at $p(e)$ (cf. (2.53)).

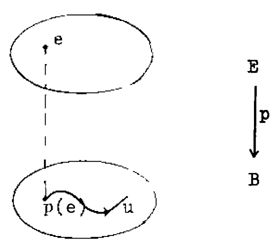
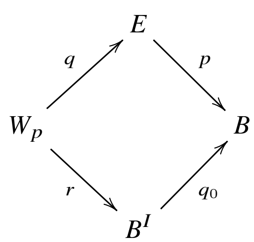


Figure 2.5:

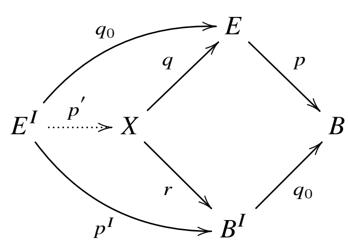
Theorem 2.47. *The diagram in* $\mathscr{T}op$



(2.48)

is a Cartesian square (cf. (0.8)). Let $q(r)$ *be the restriction to* W_p *of the projection of the product* $E \times B$ *onto the first (second) factor. We leave the (simple) proof to the reader.*

Consider the diagramme



Since $pq_0 = q_0P^I$ and since (2.48) is a Cartesian square, there exists exactly one continuous map $p' : E^I \to W$ with $q \circ p' = q_0$ and $r \circ p' = p^I$.

Theorem 2.49. *The following statements are equivalent:*

(a) p is a fibration.

(b) p has the CHP for the mapping path space W.

(c) p' is a retraction (i.e., there exists a continuous map $s : W_p \to E$ with $p's = \mathrm{id}_{W_p}$.[3]

Proof. The proof of Theorem (2.49) is dual to the proof of Theorem (1.17) and is left to the reader as an exercise. □

2.2.4 Decomposition of a continuous map into a homotopy equivalence and a fibration

We now prove the dual of theorem (1.29).

Definition 2.50. Let $g : Y \to B$ be a continuous map. Let $q : W_g \to Y$, $r : W_g \to B^I$ be the continuous maps that induce the projections of $Y \times B$ onto the individual factors, as in (2.48). We set $r_1 := q_1 \circ r : W_g \to B$, i.e., $r_1(y, u) = u(1) \in B$ for $(y, u) \in W_g$, and define a continuous map $j : Y \to W_g$ by $j(y) := (y, g(y))$ for $y \in Y$. Let $g(y)$ denote the constant path $I \to B$ that maps each $t \in I$ into $g(y) \in B$.

Theorem 2.51. *We have*

(a) The diagramme

$$\begin{array}{ccc} Y & \xrightarrow{\ j\ } & W_g \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle r_1} & \\ & B & \end{array}$$

is commutative.

(b) r and q are fibrations.

(c) $qj = \mathrm{id}_Y$.

(d) $jq \underset{Y}{\simeq} \mathrm{id}_{W_g}$.

We consider jq and id_{W_g} as morphisms $q \to q$ of $\mathcal{T}op_Y$. This is possible because by (c) $qjq = q$. In particular, Theorem (2.51) implies that any continuous map up to homotopy equivalence can be replaced by a fibration:

Corollary 2.52. *Every continuous map g can be factorised in the form $g = v \circ u$, where v is a fibration and u is a homotopy equivalence.*

Before proving Theorem (2.51), we make some remarks about paths.

Definition 2.53. Let X be a topological space. A *path in X* is a continuous map $w : [0, a] \to X$, where $a \in [0, \infty[$. $w(0)$ is called the *starting point*, and $w(a)$ the *end point* of w. If $a = 1$, we speak of a *normalised path.*

[3] One then also says: p' *has a section.*

Definition 2.54. If $w_1 : [0, a_1] \to X$, $w_2 : [0, a_2] \to X$ are paths with $w_1(a_1) = w_2(0)$, then let $w_2 + w_1 : [0, a_1 + a_2] \to X$ be the path defined by

$$w_2 + w_1(t) := \begin{cases} w_1(t), & 0 \le t \le a_1 \\ w_2(t - a_1) & a_1 \le t \le a_1 + a_2 \end{cases}$$

Definition 2.55. If $w : [0, a] \to X$ is a path, then let $(-w) : [0, a] \to X$ be the path defined by $(-w)(t) := w(a - t)$ for $0 \le t \le a$.

Definition 2.56. If $w_1 : [0, a_1] \to X$, $w_2 : [0, a_2] \to X$ are paths with $w_1(0) = w_2(0)$, then we set

$$w_2 - w_1 := w_2 + (-w_1).$$

Definition 2.57. If $w : [0, a] \to X$ is a path, then let $w_I : I \to X$ be the normalised path given by $w_T(t) := w(a - t)$ for $t \in I$.

Definition 2.58. If $\varphi : X \times [0, a] \to Y$ is a homotopy ($a \in [0, \infty[$), then for $x \in X$ let

$$\varphi^x : [0, a] \to Y$$

be the path defined by $\varphi^x(t) := \varphi(x, t)$.

Proof. (of Theorem 2.51) (a) and (c) are clear.

Regarding (b): We first show that r_1 is a fibration. If

$$\begin{array}{ccc} X & \xrightarrow{f} & W_g \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle r_1} \\ X \times I & \xrightarrow[\varphi]{} & B \end{array} \tag{2.59}$$

is a commutative diagramme in $\mathcal{T}\!op$, then we have to construct a continuous map $\Phi : X \times I \to W_g$ with $r_1\Phi = \varphi$ and $\Phi j_0 = f$. Let $x \in X$. $f(x) \in W_g$ is a pair (y, u) with $y \in Y$, $u : I \to B$, such that $g(y) = u(0)$. Since (2.59) is commutative, we have

$$\varphi^x(0) = \varphi(x, 0) = r_1 f(x) = u(1).$$

The last equation allows us to define for $t \in I$

$$\Phi(x, t) := (y, ((\varphi^x|[0, t]) + u)_I) \in W_g(!).$$

One immediately verifies $\Phi j_0 = f$ and $r_1\Phi = \varphi$. The remaining proof of the continuity of Φ is left to the reader as an exercise.

Next, we want to prove that q is a fibration. To do so, we start with a commutative diagramme in $\mathcal{T}\!op$ of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & W_g \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle q} \\ X \times I & \xrightarrow[\varphi]{} & Y \end{array}$$

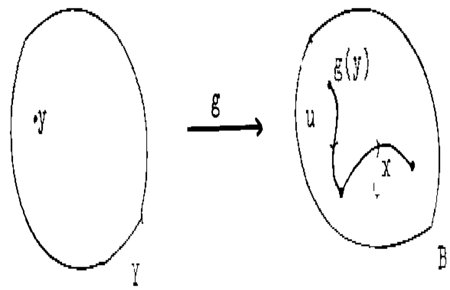


Figure 2.6:

and have to construct $\Phi : X \times I \to W_g$ with $q\Phi = \varphi$ and $\Phi j_0 = f$. For $x \in X$, $f(x)$ is a pair

$$(y,u), \quad y \in Y, u : I \to B \text{ with } g(y) = u(0).$$

Since $qf = \varphi j_0$, it follows that $\varphi^x(0) = \varphi(x,0) = qf(x) = y$ and hence $g\varphi^x(0) = g(y) = u(0)$. We can therefore define for $t \in I$

$$\Phi(x,t) := (\varphi(x,t), (u - g\varphi^x|[0,t])_I) \in W_g(!).$$

The reader should verify that $\Phi : X \times I \to W_g$ is the desired continuous (!!)

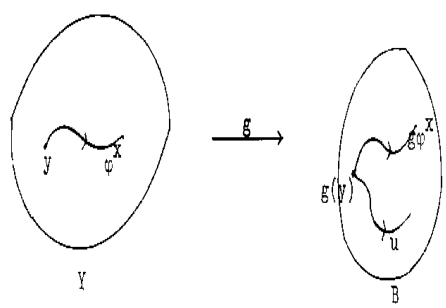


Figure 2.7:

map.

Regarding (d): We define a homotopy $\varphi : W_g \times I \to W_g$ by $\varphi(y,u,t) := (y, (u|[0,t])_I)$ for $(y,u) \in W_g$, $t \in I$. Then

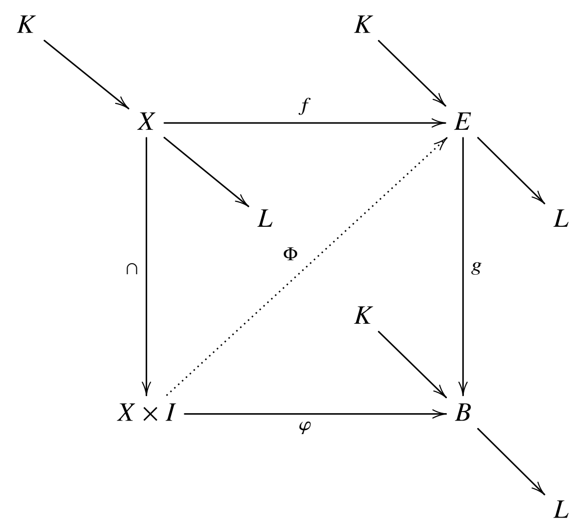
$$\varphi : jq \underset{Y}{\simeq} \mathrm{id}_{W_g}$$

□

2.2.5 Transition to other categories

Let K, L be topological spaces. By Theorem (2.23), the definition of the notion of fibration can be transferred from $\mathcal{Top}$ to $\mathcal{Top}_L^K$ using the homotopy notion defined in (0.31) in the category $\mathcal{Top}_L^K$.

Definition 2.60. Let $\varepsilon = (K \to E \to L)$, $\beta = (K \to B \to L)$ be spaces under K and over L, and $g : \varepsilon \to \beta$ be a map under K and over L. g is called a *fibration in* $\mathcal{Top}_L^K$ if and only if for all spaces under K and over L $\xi = (K \to X \to L)$, for all maps under K and over L $f : \xi \to \varepsilon$ and for all homotopies under K and over L $\varphi : X \times I \to B$ with $\varphi_0 = gf$ there exists a homotopy under K and over L $\Phi : X \times I \to E$ with $g\Phi = \varphi$ and $\Phi_0 = f$.



Of particular importance in the following will be fibrations in $\mathcal{Top}^0$ (*pointed fibrations*) and fibrations in $\mathcal{Top}_L$ (*fibrations above* L).

Definition 2.61. In (0.33) we transferred the construction of the cylinder from $\mathcal{Top}$ to $\mathcal{Top}_L^K$. We now give the construction in $\mathcal{Top}_L^K$, which corresponds to the construction of the path space Y^I of a topological space Y in $\mathcal{Top}$. If $\eta = (K \xrightarrow{i} Y \xrightarrow{p} L)$ is a space under K and over L, then let $Y_L^I L$ be the subspace of Y^I defined by

$$Y_L^I := \{u \in Y | pu \text{ constant}\}$$

To a point $k \in K$ we assign the constant path $I \to> Y$ that maps each $t \in I$ into $i(k) \in Y$. This yields a continuous map $K \to Y_L^I$. By $u \in Y_L^I \mapsto pu(0) \in L$ we obtain a continuous map $Y_L^I \to L$. We denote the object of $\mathcal{Top}_L^K$ $K \to Y_L^I L$ thus obtained by $W_K \eta$.

If ξ is another space under K and over L, then one has a bijection

$$\mathcal{Top}_L^K(I_L^K \xi, \eta) \cong \mathcal{Top}_L^K(\xi,_L^K \eta),$$

where $I_L^K \xi$ is defined as in (0.33). The definition of I_L^K or W_L^K can be extended in an obvious way to morphisms of $\mathcal{T\!op}_L^K$. One then obtains adjoint functors

$$I_L^K, W_L^K : \mathcal{T\!op}_L^K \to \mathcal{T\!op}_L^K.$$

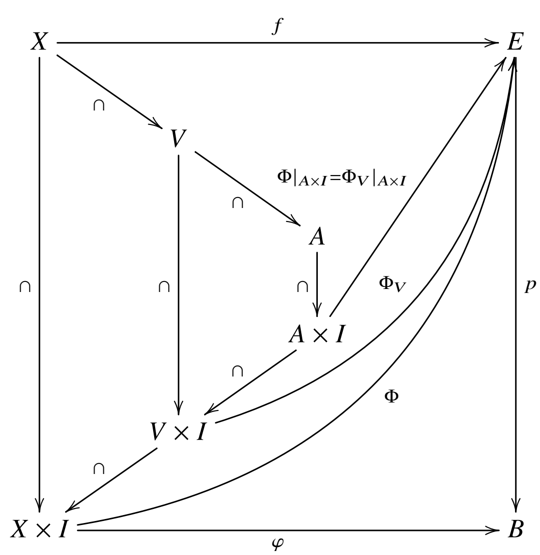
2.2.6 A certain relative covering homotopy property

At the end of this section, we prove a proposition that deals with a certain relative covering homotopy property. We will need this proposition in §2.6.

Theorem 2.62. *Let $p : E \to B$ be a fibration, X a topological space, $A \subset V \subset X$, V a halo of A in X (cf. (1.81)), and let $f : X \to E$, $\varphi : X \times \to B$, $\Phi_V : V \times I \to E$ be continuous maps such that*

$$\begin{aligned} \varphi(x,0) &= pf(x) \quad \textit{for} \quad x \in X, \\ \Phi_V(x,0) &= f(x) \quad \textit{for} \quad x \in V, \\ p \circ \Phi_V &= \varphi|_{V\times I}. \end{aligned}$$

Then there exists a homotopy $\Phi : X \times I \to E$ such that $p\Phi = \varphi$, $\Phi(x,0) = f(x)$ for $x \in X$ and $\Phi|_{A\times I} = \Phi_V|_{A\times I}$.



Proof. Since every halo of A contains a closed halo according to (1.86), we can assume that V is closed in X. Since V is a halo of A in X, we can choose a continuous map $v : X \to I$ such that

$$A \subset v^{-1}(1), \quad X \setminus V \subset v^{-1}(0)$$

(cf. (1.81), (1.82)). We define $\overline{\varphi} : X \times \to B$ by

$$\overline{\varphi}(x,t) := \varphi(x, \min(v(x)+t, 1)) \quad \text{for} \quad (x,t) \in X \times I$$

and $\Phi'_V : (X \times 0) \cup (V \times I) \to E$ by

$$\Phi'_V(x,0) := f(x) \quad \text{for} x \in X,$$
$$\Phi'_V(x,t) := \Phi_V(x,t) \quad \text{for}(x,t) \in V \times I.$$

Note that Φ'_V is well-defined since $\Phi_V(x,0) = f(x)$ for $x \in V$, and continuous since V is closed in X. Since $X \setminus V \subset v^{-1}(0)$,

$$\overline{f}(x) := \Phi'_V(x, v(x)) \quad \text{for} \quad x \in X$$

provides a continuous map $\overline{f} : X \to E$. Verify $p\overline{f}(x) = \overline{\varphi}(x,0)$ for $x \in X$. Since p is a fibration, there exists a homotopy $\overline{\Phi} : X \times I \to E$ over $\overline{\varphi}$ (i.e., $p\overline{\Phi} = \overline{\varphi}$) with $\overline{\Phi}(x,0) = \overline{f}(x)$ for $x \in X$. We define $\Phi : X \times I \to E$ by

$$\Phi(x,t) := \begin{cases} \Phi'_V(x,t), & \text{if} \quad 0 \le t \le v(x) \\ \overline{\Phi}(x, t - v(x), & \text{if} \quad v(x) \le t \le 1. \end{cases}$$

The reader easily confirms that Φ is a well-defined continuous map with the desired properties. □

2.3 Homotopy Fibrations

2.3.1 The covering homotopy property (CHP) up to homotopy. h-Fibrations

The concept of homotopy cofibration corresponds to the concept of homotopy fibration.

Definition 2.63. Let $p : E \to B$ be a continuous map, X a topological space. p has the *covering homotopy property (CHP) up to homotopy for* X if and only if for all continuous maps $f : X \to E$, $\varphi : X \times I \to B$ with $\varphi_0 = pf$ there exists a homotopy $\Phi : X \times I \to E$ over φ (i.e., $p\Phi = \varphi$) such that $\Phi_0 \underset{B}{\simeq} f$ (cf. diagramme (2.24)). We consider Φ_0 and f as morphisms $pf \to p$ of $\mathcal{T}op_B$.

Example 2.64. Let $E := I \times \{0\} \cup \{0\} \times I \subset I \times I$, $B := I$, $p : E \to B$ be the projection onto the first factor. Let X be a topological space that has exactly one point. p has the CHP up to homotopy for X, but p does not have the CHP for X.

0From the adjointness of the functors $- \times I$ and $-^I$ it follows that:

Theorem 2.65. *A continuous map $p : E \to B$ has CHP up to homotopy for a topological space X if and only if for all continuous maps $f : X \to E$, $\overline{\varphi} : X \to B^I$ there exists a continuous map $\overline{\Phi} : X \to E^I$ with $p^I\overline{\Phi} = \overline{\varphi}$ and $q_0\overline{\Phi}\varphi \underset{B}{\simeq} f$ (cf.*

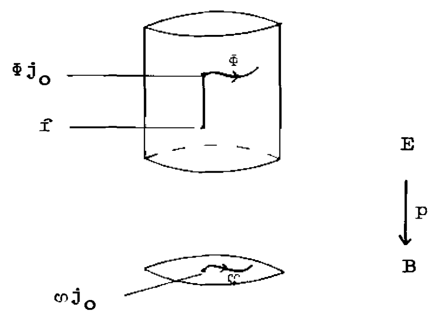


Figure 2.8:

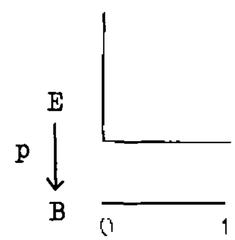
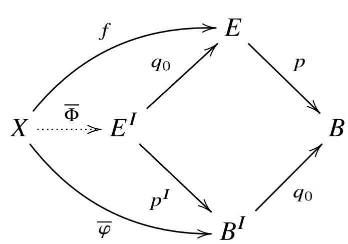


Figure 2.9:

diagramme (2.22)).



Definition 2.66. A continuous map $p : E \twoheadrightarrow B$ is called a *homotopy fibration* (*h-fibration* for short) if p satisfies the CHP up to homotopy for all topological spaces X. In addition to the term "homotopy fibration," the term "*weak fibration*" is also commonly used.
Note: Every fibration is an h-fibration.

Definition 2.67. Let $p : E \to B$, $p' : E' \to B$ be spaces over B. *p is dominated by p' (in $\mathcal{T}op_B$)* if one of the following equivalent (!) statements holds:

(a) there exist morphisms of $\mathcal{T}op_B$ $g : p \to p'$, $g' : p' \to p$ such that $g'g \underset{B}{\simeq} \mathrm{id}_E$,

(b) there exists a section in $\mathcal{T}op_Bh$ $g : p \to p'$,

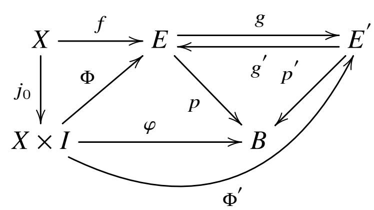
(c) there exists a retraction in $\mathcal{T}op_Bh$ $g' : p' \to p$.

Theorem 2.68. *Let $p : E \to B$, $p' : E' \to B$ be spaces over B. p is dominated by p' in $\mathcal{T}op_B$.*
Claim:

(a) If X is a topological space and p' has CHP up to homotopy for X, then so does p.

(b) If p' is an h-fibration, then so is p.

Proof. (b) is a consequence of (a).
Regarding (a): By assumption, there exist morphisms of $\mathcal{T}op_B$ $g : p \to p'$, $g' : p' \to p$ with $g'g \underset{B}{\simeq} \mathrm{id}_E$. Given are continuous maps $f : X \to E$, $\varphi : X \times I \to B$ with $\varphi j_0 = pf$. From $p'g = p$, it follows $p'(gf) = \varphi j_0$. Since p' has the CHP up to homotopy for X, there exists $\Phi' : X \times I \to E'$ with $p'\Phi' = \varphi$ and $\Phi' j_0 \underset{B}{\simeq} gf$.

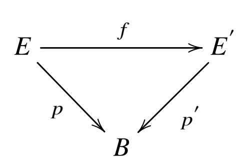


Set $\Phi := g'\Phi' : X \times I \to E$. Then $p\Phi = pg'\Phi' = p'\Phi' = \varphi$, since $pg' = p'$, and $\Phi j_0 = g'\Phi' j_0 \underset{B}{\simeq} g'gf \underset{B}{\simeq} f$, because $g' \underset{B}{\simeq} \mathrm{id}_E$. So p has the CHP up to homotopy for X. □

Corollary 2.69. *"CHP up to homotopy" and "h-fibration" are invariant under homotopy equivalence over B.*

Remark 2.70. The continuous map p of Example (2.64) is dominated by id_B. id_B is a fibration, i.e., an h-fibration. By Theorem (2.68), p is therefore an h-fibration. Since p is not a fibration, this example also shows that Theorem (2.68) becomes false if one replaces "CHP up to homotopy" with "CHP" in (a) or "h-fibration" with "fibration" in (2.68).

Theorem 2.71. *Let the diagramme in $\mathcal{T}op$*

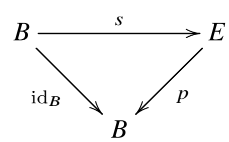


be commutative up to homotopy, i.e., $p'f \simeq p$. If p' is an h-fibration or if p' at least has the CHP up to homotopy for E, then there exists a continuous map $g : E \to E'$ with $g \simeq f$ and $p'g = p$.

Proof. Since $p^{'}$ has the CHP up to homotopy for E, there exists for $\varphi : p^{'} f \simeq p : E \times I \to B$ a homotopy $\Phi : E \times I \to E^{'}$ over φ with $\Phi j_0 \underset{B}{\simeq} f$. For $g := \Phi j_1 : E \to E^{'}$ then $g = \Phi j_1 \simeq \Phi j_0 \underset{B}{\simeq} f$, i.e. $g \simeq f$, and $p^{'} g = p^{'} \Phi j_1 = \varphi j_1 = p$. □

Corollary 2.72. *If an h-fibration $p : E \to B$ has a section up to homotopy, then it has a setion.*

Proof. By assumption, $s : B \to E$ such that $ps \simeq \mathrm{id}_B$ exists. Theorem (2.71), applied to the diagramme



proves the existence of a continuous map $s^{'} : B \to E$ such that $ps^{'} = \mathrm{id}_B$. □

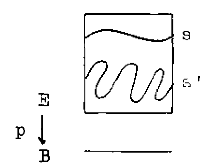


Figure 2.10:

Remark 2.73. Corollary (2.72) states that an h-fibration that has no section also has no section up to homotopy. This remark is important because not every fibration has a section.

Example 2.74. Let p be the restriction of the tangent bundle $T(\mathbb{S}^2) \to \mathbb{S}^2$ of the 2-sphere to the tangent vectors that are nonzero. p is a fibration because p is locally trivial and $\mathbb{S}^2$ is compact. p has no section (cf. [25], II. Theorem 27.8), and thus no section up to homotopy.

2.3.2 Different characterisations of the term "h-fibration"

We now provide various characterisations of the term "h-fibration."

Theorem 2.75. *Let ε be a real number with $0 < \varepsilon < 1$, X a topological space, and $p : E \to B$ a continuous map. Then the following two statements are equivalent:*

(a) p has the CHP up to homotopy for X.

(b) For all continuous maps $f : X \to E$, $\varphi : X \times I \to B$ such that $\varphi(x, t) = pf(x)$ for all $x \in X$ and all $t \in [0, 1]$ with $t \leq \varepsilon$, there exists a homotopy $\Phi : X \times I \to E$ over φ with $\Phi_0 = f$.

Proof. (a) ⇒ (b): Given continuous maps $f : X \to E$, $\varphi : X \times I \to B$ with $\varphi(x,t) = pf(x)$ for $x \in X$ and $0 \le t \le \varepsilon$. Then the following diagramme is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ \downarrow{\scriptstyle j_\varepsilon} & & \downarrow{\scriptstyle p} \\ X \times [\varepsilon,1] & \xrightarrow[\varphi|_{X\times[\varepsilon,1]}]{} & B \end{array}$$

Since p has the CHP up to homotopy for X ($[0,1]$ is replaced by $[\varepsilon,1]$), there exists a homotopy $\Phi' : X \times [\varepsilon,1] \to E$ with $p\Phi' = \varphi|_{X\times[\varepsilon,1]}$ and $\Phi'_\varepsilon \underset{B}{\simeq} f$. Choose a homotopy $\Phi'' : X \times [0,\varepsilon] \to E$ over B with $\Phi''_0 = f$ and $\Phi''_\varepsilon = \Phi'_\varepsilon$. Φ' and Φ'' together then define a continuous map $\Phi : X \times I \to E$ with $p\Phi = \varphi$ and $\Phi_0 = f$.
(b) ⇒ (a): Given continuous maps $f : X \to E$, $\varphi : X \times I \to B$ with $\varphi_0 = pf$. We define $\varphi' : X \times [-1,1] \to B$ by

$$\varphi'(x,t) := \varphi(x, \max(t,0)).$$

Then $\varphi'(x,t) = pf(x)$ for $x \in X$, $-1 \le t \le 0$. By assumption (we replace $(0,\varepsilon,1)$ with $(-1,0,1)$.) there exists a continuous map $\Phi' : X \times [-1,1] \to$ over φ' with $\Phi'_{-1} = f$. For $\Phi := \Phi'|_{X\times I} : X \times I \to E$, then $p\Phi = \varphi$ and $\Phi_0 = \Phi'_0 \underset{B}{\simeq} \Phi'_{-1} = f$. □

Theorem 2.76. *Let ε be a real number with $0 < \varepsilon < 1$, X a topological space, and $p : E \to B$ a continuous map. Then the following two statements are equivalent:*

(a) p is an h-firation.

(b) For all topological spaces X and all continuous maps $\Phi' : X \times [0,\varepsilon] \to E$, $\varphi : X \times I \to B$ with $p\Phi' = \varphi|_{X\times[0,\varepsilon]}$, there exists a homotopy $\Phi : X \times I \to E$ over φ with $\Phi_0 = \Phi'_0$.

$$\begin{array}{ccc} X \times [0,\varepsilon] & \xrightarrow{\Phi'} & E \\ \cap & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\varphi]{} & B \end{array}$$

Proof. (b) ⇒ (a): To prove that p is an h-fibration, we use the characterisation of the notion of h-fibration given by Theorem (2.75). Given continuous maps $f : X \to E$, $\varphi : X \times I \to B$ with $\varphi(x,t) = pf(x)$ for $x \in X$, $0 \le t \le \varepsilon$. We define $\Phi' : X \times [0,\varepsilon] \to E$ by $\Phi'(x,t) := f(x)$ for $x \in X$, $0 \le t \le \varepsilon$. Then $p\Phi' = \varphi|_{X\times[0,\varepsilon]}$. We apply (b) and obtain a homotopy $\Phi : X \times I \to E$ over φ with $\Phi_0 = f$.
(a) ⇒ (b): We can assume $\varepsilon = \frac{1}{2}$ without loss of generality. By assumption, there are continuous maps $\Phi' : X \times [0,\frac{1}{2}] \to E$, $\varphi : X \times I \to B$ with $p\Phi' = \varphi|_{X\times[0,\frac{1}{2}]}$. We define $\widetilde{\varphi} : X \times [0,\frac{1}{2}] \to B$ by $\widetilde{\varphi}(x,s,t) := \varphi(x, 1-(1-s)(1-t))$ for $x \in X$,

$0 \le s \le \frac{1}{2}$, $0 \le t \le 1$. Then $\widetilde{\varphi}(x,s,0) = p\Phi'(x,s)$ for $(x,s) \in X \times [0,\frac{1}{2}]$. Since p is an h-fibration by assumption and thus has the CHP up to homotopy for $[0,\frac{1}{2}]$, there exists a continuous map $\widetilde{\Phi} : X \times [0,\frac{1}{2}] \times I \to E$ over $\widetilde{\varphi}$ with $\widetilde{\Phi}_0 \underset{B}{\simeq} \Phi'$. Therefore, there exists a continuous map $\Psi : X \times [0,\frac{1}{2}] \times I \to E$ such that $p\Psi(x,s,t)$ is independent of t and $\Psi_0 = \Phi'$, $\Psi_1 = \widetilde{\Phi}_0$. We define $\Phi : X \times I \to E$ by

$$\Phi(X,t) := \begin{cases} \Psi(x,t,2t), & x \in X, 0 \le t \le \frac{1}{2} \\ \widetilde{\Phi}(x,\frac{1}{2},2t-1), & x \in X, \frac{1}{2} \le t \le 1. \end{cases}$$

It is easy to verify that Φ is a continuous map with $\Phi_0 = \Phi'_0$ and $p\Phi = \varphi$. □

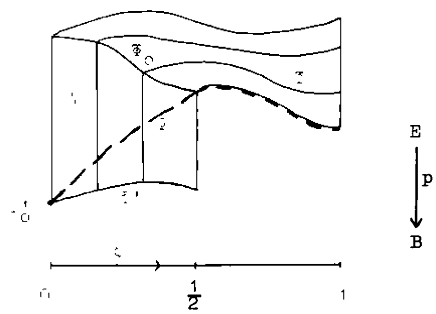


Figure 2.11:

The following theorem characterises the property of a continuous map p to be an h-fibration, using the mapping path space W_p .

Theorem 2.77. *Let ε be a real number with $0 < \varepsilon < 1$, and $p : E \to B$ a continuous map. Then the following are equivalent:*

(a) p is an h-fibration.

(b) There exists a continuous map $s : W_p \to E$ such that $s(e,u)(0) = e$ for $(e,u) \in W_p$,

$$p(s(e,u)(t)) = \begin{cases} u(0), & 0 \le t \le \varepsilon \\ u\left(\frac{t-\varepsilon}{1-\varepsilon}\right), & \varepsilon \le t \le 1. \end{cases} \quad (e,u) \in W_p$$

Proof. We can assume $\varepsilon = \frac{1}{2}$ without loss of generality. In the proof we make several use of the adjointness of $\times I$ and $-^I$ and go from a continuous map $\varphi : X \times I \to$ according to (2.4) to $\overline{\varphi} : X \to Y^I$ and vice versa reverts from $\overline{\varphi}$ to φ.

(a) $\Rightarrow$ (b): We assume that p is an h-fibration. We consider the projection $q : W_p \to E$ onto the first factor and the continuous (!) map $\varphi : W_p \times I \to B$, which is given by

$$\varphi(e,u,t) := \begin{cases} u(0), & 0 \le t \le \frac{1}{2} \\ u(2t-1), & \frac{1}{2} \le t \le 1. \end{cases} \quad (e,u) \in W_p$$

For $0 \le t \le \frac{1}{2}$, then

$$\varphi(e,u,t) = u(0) = p(e) = pq(e,u).$$

Since p has the CHP up to homotopy for W_p, by Theorem (2.75) there exists a homotopy $\Phi : W_p \times \to E$ with $p\Phi = \varphi$ and $\Phi j_0 = q$. We define $s := \overline{\Phi} : W_p \to E^I$ and obtain the desired continuous map.

Remark 2.78. In "(a) ⇒ (b)" we only used the fact that p has the CHP up to homotopy for the mapping path space W_p.

(b) ⇒ (a): We assume for $\varepsilon = \frac{1}{2}$ the existence of a continuous map $s : W_p \to E^I$ as described in (b). Given continuous maps $f : X \to E$, $\varphi : X \times I \to B$, such that $\varphi p(x,t) = pf(x)$ for $x \in X$, $0 \le t \le \frac{1}{2}$. We define $\varphi' : X \times I \to B$ by $\varphi'(x,t) := \varphi\left(x, \frac{1+t}{2}\right)$ and proceed to $\overline{\varphi}' : X \to B^I$. We define $\Phi' : X \to W_p$ by $\Phi'(x) := (f(x), \overline{\varphi}' x)) \in W_p$ (!). We set $\overline{\Phi} := s\Phi' : X \to E^I$, proceed to $\Phi : X \times I \to E$ and obtain a continuous map with $p\Phi = \varphi$ and $\Phi j_0 = f$ (!). Therefore, by Theorem (2.75), p is an h-fibration. □

Corollary to Theorem (2.77). From Theorem (2.77) and Remark (2.78) it follows:

Theorem 2.79. *A continuous map p is an h-fibration if and only if it satisfies the CHP up to homotopy for the mapping path space W_p.*

Remark 2.80. There is no dual counterpart to Corollary (1.54).

1. Not every fibration is surjective, because for every topological space B, the only mapping $\emptyset \to B$ is a fibration (!). However, the following holds:

   Theorem 2.81. *If $p : E \to B$ is an h-fibration and $p(E)$ meets every path component of B, then p is surjective.*

   Proof. For $b \in B$, by assumption, there exists a point $b_0 \in p(E)$ that lies in the same path component of B as b. We can therefore choose a path $w : I \to B$ with $w(0) = b_0$, $w(1) = b$. Here, we can assume: $w(t) = w(0)$ for $0 \le t \le \frac{1}{2}$. Choose $e_0 \in E$ with $p(e_0) = b_0$. Since p is an h-fibration and therefore the CHP has a homotopy for $X = \text{pt}$, by Theorem (2.75) there exists a path $v : I \to E$ with $pv = w$ and $v(0) = e_0$. Then $pv(1) = w(1) = b$, hence $b \in p(E)$. □

2. Not every surjective fibration is an identification.

   Example 2.82. Let E be the set $\mathbb{Q}$ of rational numbers, endowed with the discrete topology, B be the set of rational numbers, endowed with the subspace topology induced by $\mathbb{R}$, $p := \mathrm{id}_{\mathbb{Q}}$. Then p is a bijective continuous map, but p is not an identification.

Proof. Let $f : X \to E$, $\varphi : X \times I \to B$ be continuous maps with $\varphi j_0 = pf$. Since B only admits constant paths, for $x \in X$ we have $\varphi^x(1) = \{pf(x)\}$, where $\varphi^x : I \to B$ is defined as in (2.58). $\Phi := f \circ \mathrm{proj}_1 : X \times I \to E$ is therefore a continuous map with $p\Phi = \varphi$ and $\Phi j_0 = f$. □

However, the following applies:

Theorem 2.83. *If $p : E \to B$ is a surjective fibration and B is locally pathwise connected, i.e., every point of B has a neighbourhood basis of pathwise connected subsets of B (cf. Schubert* [23]*, III.1.2, Definition 2), then p is an identification.*

Proof. Strøm [27] I. Theorem 1. □

Remark 2.84. Fibrations are generally not closed, as the example $\mathrm{proj}_1 : \mathbb{R}^2 \to \mathbb{R}$ shows. h-fibrations are generally not open, as can be seen from example (2.64) (cf. also (2.70)). A fibration p is certainly open if p is locally trivial (cf. (2.33)).

2.3.3 Homotopy equivalences and fibrewise homotopy equivalences

The following fundamental theorem of homotopy theory is by A. Dold ([6], Theorem 6.1). This theorem is dual to Theorem (1.62).

Theorem 2.85. Premise*: Let*

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

be a commutative diagramme in $\mathcal{T}op$. Let p and p' be h-fibrations, and f be an h-equivalence.
Claim*: f, conceived as a morphism of $\mathcal{T}op_B$, $f : p \to p'$, is an h-equivalence over B.*

Proof. We present the reader with the task of reducing the proof of Theorem (2.85), dual to the proof of Theorem (1.62), to the following Lemma using Theorem (2.71). □

Lemma 2.86. *If*

$$\begin{array}{ccc} E & \xrightarrow{\ g\ } & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

is a commutative diagramme in $\mathcal{T}op$, p is an h-fibration, and if $g \simeq \mathrm{id}_E$ then there exists a morphism $g' : p \to p$ of $\mathcal{T}op_B$ with $gg' \underset{B}{\simeq} \mathrm{id}_E$.

Proof. We choose a homotopy $\varphi : g \simeq \mathrm{id}_E : E \times I \to E$ such that $\varphi(e,t) = g(e)$ for $e \in E$ and $0 \le t \le \frac{1}{2}$. Since $pg = g$, then for $p\varphi : E \times I \to B$ we have $p\varphi : p \simeq p$ and $p\varphi(e,t) = p(e)$ for $e \in E$ and $0 \le t \le \frac{1}{2}$. Since p is an h-fibration, by Theorem (2.75) there exists a homotopy $\psi : E \times I \to E$ over $p\varphi$ with $\psi_0 = \psi j_0 = \mathrm{id}_E$. Set $g' := \pi_1 : E \to E$. Then $pg' = p$. We claim $gg' \underset{B}{\simeq} \mathrm{id}_E$. We define a homotopy $F : E \times I \to E$ by

$$(e,s,t) \mapsto \begin{cases} p\varphi(e, 12s(1-t)), & 0 \le s \le \frac{1}{2} \\ p\varphi(e, 1-2(1-s)(1-t)), & \frac{1}{2} \le s \le 1. \end{cases} \quad e \in E, \quad t \in I,$$

Then $(e,s,0) = pF(e,s)$ $(e \in E,\ s \in I)$, $\Phi(e,0,t) = \Phi(e,s,1) = \Phi(e,1,t) = p(e)$ $(e \in E,\ t,s \in I)$. We can modify Φ (cf. (1.66)) so that additionally

$$\Phi(e,s,t) = pF(e,s) \quad \text{for} \quad 0 \le t \le \frac{1}{2}.$$

Since p is an h-fibration, by Theorem (2.75) there exists a continuous map $\widetilde{\Phi} : E \times I \times I \to E$ with $p\widetilde{\Phi} = \Phi$ and $\widetilde{\Phi}(e,s,0) = F(e,s)$ for $e \in E$, $s \in I$. We define $\widetilde{\Phi}_{(s,t)} : E \to E$ for $s,t \in I$ by $\widetilde{\Phi}_{(s,t)}(e) := \widetilde{\Phi}(e,s,t)$ $(e \in E)$. Then $g' = F_0 = \widetilde{\Phi}_{(0,0)} \underset{B}{\simeq} \widetilde{\Phi}_{(0,0)} \underset{B}{\simeq} (\widetilde{\Phi}_{(1,1)} \underset{B}{\simeq} \widetilde{\Phi}_{(1,0)} = F_1 = \mathrm{id}_E$. □

Remark 2.87. Theorem (refthm:2-6-21) is also essentially a formal theorem and holds even if one replaces the category $\mathcal{T}\!op$ with the category $\mathcal{T}\!op_L^K$ (K and L topological spaces), i.e., if one assumes a commutative triangle in $\mathcal{T}\!op_L^K$ (cf. Kamps [15], 5.2).

The following notion is dual to the notion of "strong deformation retract".

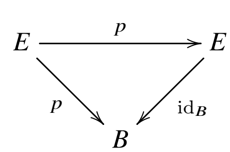
Definition 2.88 (Dold [6])**.** A continuous map $p : E \to B$ is called *shrinkable* if there exists a continuous map $s : B \to E$ such that $ps = \mathrm{id}_B$ and $sp \underset{B}{\simeq} \mathrm{id}_E$.

We consider p, id_E, s as morphisms of $\mathcal{T}\!op_B$, $p : p \to \mathrm{id}_B$, $id_E : p \to p$, $s : \mathrm{id}_B \to p$. This is possible because $ps = \mathrm{id}_B$. One immediately considers:

Lemma 2.89. *A continuous map $p : E \to B$ is shrinkable if and only if p is h-equivalent over B to* id_B*.*

Theorem 2.90. *A continuous map $p : E \to B$ is an h-fibration and an h-equivalence if and only if it is shrinkable.*

Proof. ($\Rightarrow$): follows from Theorem (2.85) applied to the diagramme



($\Leftarrow$): If p is shrinkable, then p is in particular an h-equivalence. If p is shrinkable, then by Lemma (2.89) p is h-equivalent to id_B over B. Since id_B is an h-fibration, it follows from Theorem (2.68) (b), p is an h-fibration. □

Theorem 2.91. *Let $g : Y \to B$ be a continuous map. Consider the commutative diagram of Theorem (2.51) (a).*

$$\begin{array}{ccc} Y & \xrightarrow{\ j\ } & W_g \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle r_1} & \\ & B & \end{array}$$

The following statements are equivalent:

(a) g is an h-fibration.

(b) j is an h-equivalence over B (i.e., $[j]_B$ is an isomorphism in $\mathscr{T\!op}h$).

(c) $[j]_B$ is a section in $\mathscr{T\!op}h$.

Proof. (a) $\Rightarrow$ (b: By Theorem (2.51), j is an h-equivalence and r_1 is a fibration. If g is an h-fibration, then j is an h-equivalence over B by Theorem (2.85).
(b) $\Rightarrow$ (c: Trivial.
(c) $\Rightarrow$ (a: By assumption, g is dominated by r_1 in $\mathscr{T\!op}_B$. Since r_1 is a fibration, i.e., hence an h-fibration, by Theorem (2.51), g is an h-fibration, by Theorem (2.68) (b). □

Since r_1 is a fibration, we obtain:

Corollary 2.92. *For every h-fibration $p : E \to B$, there exists a fibration $p^{'} : E^{'} \to B$ that is h-equivalent over B to p.*

Theorem 2.93. *Let $p : E \to B$ be an h-fibration, X a topological space, $A \subset V \subset X$, V a halo of A in X (cf. (1.81)). Let ε be a real number with $0 < \varepsilon < 1$ and let*

$$\begin{array}{ccc} (V \times I \cup (X \times [0,\varepsilon]) & \xrightarrow{\ \Phi^{'}\ } & E \\ \cap & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\ \varphi\ ]{} & B \end{array}$$

be a commutative diagramme in $\mathscr{T\!op}$. Then there exists a homotopy $\Phi : X \times I \to E$ over φ (i.e., $p\Phi = \varphi$) such that

$\Phi|_{(A\times I)\cup(X\times 0)} = \Phi^{'}|_{(A\times I)\cup(X\times 0)}$, *i.e, the following diagamme is commutative:*

$$\begin{array}{ccc} (A \times I \cup (X \times [0,\varepsilon]) & \xrightarrow{\ \Phi^{'}|_{(A\times I)\cup(X\times 0)},\ \Phi|_{(A\times I)\cup(X\times 0)}\ } & E \\ \cap & & \\ (V \times I \cup (X \times [0,\varepsilon] & \xrightarrow{\ \Phi^{'}\ } & E \\ \cap & & \\ X \times I & \xrightarrow{\ \Phi\ } & E \\ X \times I & \xrightarrow[\ \varphi\ ]{} & B \end{array} \qquad E \xrightarrow{\ p\ } B$$

Proof. Proof. Since V is a halo of A in X, there exists a continuous map $v : X \to I$ with $A \subset v^{-1}(1)$ and $X \setminus V \subset v^{-1}(0)$. We define

$$\begin{aligned} \overline{\varphi} : X \times I \to B \quad &\text{by} \quad \overline{\varphi}(x,t) := \varphi(x, \min(v(x)+t, 1)) \\ \overline{\Phi'} : X \times [0,\varepsilon] \to E \quad &\text{by} \quad \overline{\Phi'}(x,t) := \Phi'(x, \min(v(x)+t, 1)). \end{aligned}$$

The definition of $\overline{\Phi'}$ makes sense since $X \setminus V \subset v^{-1}(0)$. It holds that $p\overline{\Phi'} = \overline{\varphi}|_{X\times[0,\varepsilon]}$. Since p is an h-fibration, by Theorem (2.76) there exists a homotopy $\overline{\Phi} : X \times I \to E$ over φ with $\overline{\Phi}|_{X\times 0} = \overline{\Phi'}|_{X\times 0}$. We define $\Phi : X \times I \to$ E by

$$\Phi(x,t) := \begin{cases} \Phi'(x,t), & \text{if} \quad 0 \le t \le v(x) \\ \overline{\Phi}(x, t - v(x)), & \text{if} \quad v(x) \le t \le 1. \end{cases}$$

and obtain a homotopy with the desired properties(!). □

2.4 Induced Fibrations

2.4.1 Induced Fibrations

Definition 2.94. Let

$$\begin{CD} \overline{E} @>{\overline{\alpha}}>> E \\ @V{\overline{p}}VV @VV{p}V \\ \overline{B} @>>{\alpha}> B \end{CD} \tag{2.95}$$

be a diagramme in the category $\mathcal{T}\!op$ of topological spaces. $\overline{p}$ is said to be *induced from* p *by* α if (2.95) is a Cartesian square.

Theorem 2.96. *For a diagram*

$$\begin{CD} @. E \\ @. @VV{p}V \\ \overline{B} @>>{\alpha}> B \end{CD}$$

in $\mathcal{T}\!op$ *there exists a diagram*

$$\begin{CD} \overline{E} @>{\overline{\alpha}}>> E \\ @V{\overline{p}}VV \\ \overline{B} \end{CD}$$

in $\mathcal{T}\!op$ *that is unique up to isomorphism, such that (2.95) is a Cartesian square.*

Proof. Uniqueness follows purely from category-theoretic reasons (cf. (0.8)). Existence: One immediately realises that the following definition yields a Cartesian square (2.95). □

Definition 2.97. Let $\overline{E}$ be the subspace

$$\overline{E} := \{(\overline{b}, e) \in \overline{B} \times E | \alpha\overline{b} = pe\}$$

of the product $\overline{B} \times E$. Let $\overline{p} : \overline{E} \to \overline{B}$ be the projection onto the first factor, and $\overline{\alpha} : \overline{E} \to E$ be the projection onto the second factor.[4].

We have already encountered the construction of (2.97) in a special case, namely in the definition of the path space W_p of a continuous map p.

Example 2.98. If $p : E \to B$ is a continuous map, then we have the Cartesian square (2.48)

$$\begin{array}{ccc} W_p & \xrightarrow{q} & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ B^I & \xrightarrow[q_0]{} & B \end{array}$$

$r : W_p \to B$ is thus induced from p by $q_0 : B^I \to B$.

Example 2.99. Let $p : E \to B$ be a continuous map, and $\alpha : \overline{B} \subset B$ be the inclusion of a subspace $\overline{B}$ of B. Then one can define a special Cartesian square (2.95) as follows:

$$\overline{E} := p^{-1}(\overline{B}) \subset E.$$

Let α be the inclusion $p^{-1}(\overline{B}) \subset E$, and $\overline{p}$ be the restriction of p. Using the notations of (2.32), we have $\overline{p} = p_{\overline{B}}$.

Theorem 2.100. *In the diagramme in $\mathcal{T\!op}$*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\overline{\alpha}} & E \\ {\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle p} \\ \overline{B} & \xrightarrow[\alpha]{} & B \end{array} \tag{2.101}$$

let p_α be induced from p by α (cf. (2.94)). Let X be a topological space. Claim*:*

(a) If p has the CHP for X , then so does p_α.

(b) If p has the CHP up to homotopy for X , then so does p_α.

Before proving Theorem (2.100), we note an immediate consequence.

Corollary 2.102. *In (2.101), let p_α be induced from p by α. Then:*

[4]Remark by the transcriber: this construction is called a *pull-back*.

(a) If p is a fibration, then so is p_α.

(b) If p is an h-fibration, then so is p_α.

p_α is then called "the" (h-)fibration induced from p by α.

Proof. (of Theorem (2.100)) *(a)*: We want to prove that p has the CHP for X. Let $f : X \to E_\alpha$, $\varphi : X \times I \to E$ be continuous maps with $p_\alpha f = \varphi j_0$.

$$\begin{array}{ccccc} X & \xrightarrow{f} & E_\alpha & \xrightarrow{\overline{\alpha}} & E \\ \downarrow{\scriptstyle j_0} & \overset{\Phi}{\nearrow} \;\; \overset{\Phi'}{\nearrow} & \downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\varphi]{} & \overline{B} & \xrightarrow[\alpha]{} & B \end{array}$$

Set $f^{'} := \widetilde{\alpha} f$, $\varphi^{'} := \alpha\varphi$. Then $pf^{'} = \varphi^{'} j_0$. Since p has the CHP for X, there exists a continuous map $\Phi^{'} : X \times I \to E$ with $p\Phi^{'} = \varphi^{'}$ and
$Phi^{'} j_0 = f^{'}$. Since $p\Phi^{'} = \alpha\varphi$ and since (2.101) is a Cartesian square by assumption, there exists exactly one continuous map $\Phi : X \times I \to E_\alpha$ with $\alpha\Phi = \Phi^{'}$ and $p_\alpha\Phi = \varphi$. We are finished when we show $\Phi j_0 = f$. Since (2.101) is a Cartesian square, this follows from the equations $\widetilde{\alpha}(\Phi^{'} j_0) = \Phi^{'} j_0 = f^{'} = \widetilde{\alpha} f$ and $p_\alpha = (\Phi j_0) = \varphi j_0 = p_\alpha f$.
(b): We use the characterisation of the "CHP up to homotopy for X" of Theorem (2.75) with $\varepsilon = \frac{1}{2}$ and assume a continuous map $f : X \to E_\alpha$ and a homotopy $\varphi : X \times I \to \overline{B}$ such that $\varphi(x,t) = p_\alpha f(x)$ for $x \in X$ and $0 \le t \le \frac{1}{2}$. Then $\alpha\varphi(x,t) = \alpha p_\alpha f(x) = p\alpha f(x)$ for $x \in X$ and $0 \le t \le \frac{1}{2}$. We can then construct $\Phi' : X \times I \to E_\alpha$ with $p_\alpha\Phi = \varphi$ and $\Phi j_0 = f$ as in the proof of (a). □

2.4.2 The homotopy theorem for h-fibrations

Remark 2.103. Let $\alpha : A \to B$ be a continuous map.
We want to define a covariant functor

$$\alpha^* : \mathscr{T}\!op_B \to \mathscr{T}\!op_A.$$

Definition 2.104. For each object $p : E \to B$ of $\mathscr{T}\!op_B$, we choose a Cartesian square

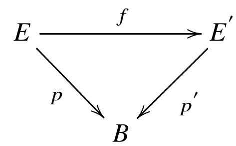
$$\begin{array}{ccc} E_\alpha & \xrightarrow{\widetilde{\alpha}} & E \\ \downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p} \\ \overline{B} & \xrightarrow[\alpha]{} & B \end{array} \tag{2.105}$$

and set $\alpha^*(p) := p_\alpha, \alpha^*(p) \in \mathrm{Obj}(\mathscr{T}\!op_A)$.

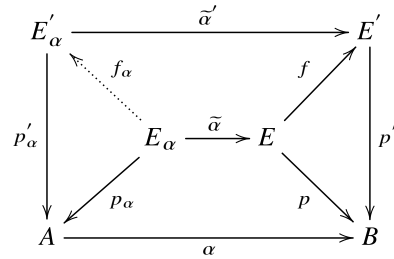
Definition 2.106. Let $p^{'} : E^{'} \to B$ be another object of $\mathscr{T}\!op_B$,

$$\begin{array}{ccc} E^{'}_\alpha & \xrightarrow{\widetilde{\alpha}^{'}} & E^{'} \\ \downarrow{\scriptstyle p^{'}_\alpha} & & \downarrow{\scriptstyle p^{'}} \\ A & \xrightarrow[\alpha]{} & B \end{array} \tag{2.107}$$

let $f : p \to p^{'}$ be a Cartesian square chosen for $p^{'}$, and let $f : p \to p^{'}$ be a morphism of $\mathscr{T}op_B$. We therefore have a commutative diagramme in $\mathscr{T}op$



Consider



Since $\alpha \circ p_\alpha = p \circ \widetilde{\alpha} = p^{'} \circ f \circ \widetilde{\alpha}$ and since (2.107) is a Cartesian square, there exists exactly one continuous map $f_\alpha : E_\alpha \to E^{'}_\alpha$ with $\widetilde{\alpha}^{'} \circ f_\alpha = f \circ \widetilde{\alpha}$ and $p^{'}_\alpha \circ f_\alpha = p_\alpha$.

The last equation allows us to consider f_α as a morphism of $\mathscr{T}op_A$, $f_\alpha : p_\alpha \to p^{'}_\alpha$. We define $\alpha^*(f) := f_\alpha$, $\alpha^*(f) : \alpha^*(p) \to \alpha^*(p^{'})$. It is easy to see: α^* is a covariant functor $\mathscr{T}op_B \to \mathscr{T}op_A$.

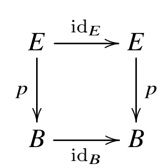
Remark 2.108. (on (2.106)) If one chooses the Cartesian squares of p and $p^{'}$ as in (2.97), i.e.

$$E_\alpha = \{(a, e) | \alpha a = pe\} \subset A \times E,$$
$$E^{'}_\alpha = \{(a, e^{'}) | \alpha a = p' e^{'}\} \subset A \times E^{'},$$

then $f_\alpha(a, e) = (a, fe) \in E^{'}_\alpha$ for $(a, e) \in E_\alpha$.

Remark 2.109. (on (2.103)) The definition of α^* depends on the choice of Cartesian squares (2.105). However, different choices yield equivalent functors (!).

Since for every continuous map $p : E \to B$ the diagramme



is a Cartesian square, the following remark follows:

Conclusion 1 (id_B) is equivalent to $\mathrm{id}_{\mathscr{T}op_B}$.

If in the diagramme in $\mathcal{T}op$

$$\begin{array}{ccccc} (E_\alpha)_\beta & \xrightarrow{\widetilde{\beta}} & E_\alpha & \xrightarrow{\widetilde{\alpha}} & E \\ {\scriptstyle (p_\alpha)_\beta}\downarrow & (1) & {\scriptstyle p_\alpha}\downarrow & (2) & \downarrow{\scriptstyle p} \\ C & \xrightarrow[\beta]{} & A & \xrightarrow[\alpha]{} & B \end{array}$$

the two squares (1) and (2) are Cartesian, then the outer rectangle is Cartesian (cf. (0.12) (d)).

We therefore obtain from the remark:

Conclusion 2 $(\alpha\beta)^*$ is equivalent to $\beta^*\alpha^*$.

We now consider the situation of (2.103) again. The following theorem shows that the functor α^* is compatible with fibrewise homotopies.

Theorem 2.110. *If $p : E \to B$, $p^{'} : E^{'} \to B$ are continuous maps, $f_0, f_1 : p \to p^{'}$ are morphisms of $\mathcal{T}op_B$, then the following holds*

$$(f_0 \underset{B}{\simeq} f_1) \Rightarrow (f_{0\alpha} \underset{B}{\simeq} f_{1\alpha}).$$

Proof. We choose $\varphi : f_0 \underset{B}{\simeq} f_1 : E \times I \to E^{'}$. We can consider φ as a morphism of $\mathcal{T}op_B$, $\varphi : p \circ \mathrm{proj}_1 \to p^{'}$. We apply α^* and obtain a morphism of $\mathcal{T}op_A$

$$\alpha{*}(\varphi) : \alpha^*(p \circ \mathrm{proj}_1) \to \alpha^*(p^{'}) = p^{'}_{\alpha}.$$

However, we can identify $\alpha^*(p \circ \mathrm{proj}_1)$ with $p_\alpha \circ \mathrm{proj}_1 : E_\alpha \times I \to A$. This is immediately apparent if we choose the Cartesian squares (2.105) as in (2.97). Then we have: $\alpha^*(\varphi) : f_{0\alpha} \underset{B}{\simeq} f_{1\alpha}$. □

Remark 2.111. The functor $\alpha^* : \mathcal{T}op_B \to \mathcal{T}op_A$ defined in (2.103) for a continuous map $\alpha : A \to B$ (after selecting Cartesian squares) thus induces, by Theorem (2.110), a functor of the factor categories $\mathcal{T}op_Bh \to \mathcal{T}op_Ah$. We also denote this functor by α^*.

Proposition 2.112. *In the diagram in $\mathcal{T}op$*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\widetilde{\alpha}} & E \\ {\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[\alpha]{} & B \end{array}$$

let p be induced by p by a.
Claim*: If p is shrinkable, then so is p (cf. (2.88)).*

Proof. Let p be shrinkable, i.e., p and id_B are isomorphic objects of $\mathcal{T}op_B h$ (2.89). Since $\alpha^* : \mathcal{T}op_B h \to \mathcal{T}op_A$ is a functor, $p_\alpha = \alpha^*(p)$ and $\alpha^*(\mathrm{id}_B)$ are isomorphic objects of $\mathcal{T}op_A$. But $\alpha^*(\mathrm{id}_B)$ is isomorphic to id_A. Hence, p_α is isomorphic to id_A in $\mathcal{T}op_A$, i.e., p_α is shrinkable. □

Definition 2.113. If B is a topological space, then let $\mathcal{F}ib_B h$ denote the full subcategory (cf. Mitchell [17] I.3) of $\mathcal{T}op_B h$ whose objects are the h-fibrations $p : E \to B$.

Remark 2.114. Let $\alpha : A \to B$ be a continuous map. α induces (after selecting Cartesian squares) by (2.103) and (2.111) a functor

$$\alpha^* : \mathcal{T}op_B h \to \mathcal{T}op_A h.$$

By Corollary (2.102) (b), this functor can be restricted to a functor

$$\mathcal{F}ib_B h \to \mathcal{F}ib_A.$$

We again denote the new functor by α^*

Theorem 2.115 (Homotopy theorem for h-fibrations)**.** *Let $\alpha, \beta : A \to B$ be continuous maps. If $\alpha \simeq \beta$, there is a natural equivalence (cf. Mitchell* [17]*, II.9))*

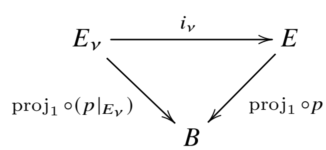
$$\Lambda : \alpha^* \to \beta^* : \mathcal{F}ib_B h \to \mathcal{F}ib_A h.$$

To prove the homotopy theorem for h-fibrations, we need a lemma.

Remark 2.116. For a continuous map $p : E \to B \times I$ and $\nu = 0, 1$, we set

$$E_\nu := p^{-1}(B \times \nu) \subset E.$$

Let $i_\nu : E_\nu \to E$ be the inclusion $E_\nu \subset E$. Then the following diagramme is commutative.



We can therefore regard i as a morphism of $\mathcal{T}op_B$,

$$i_\nu : \mathrm{proj}_1 \circ (p|_{E_\nu}) \to \mathrm{proj}_1 \circ p.$$

With these notations, we formulate:

Lemma 2.117. *If $p : E \to B \times I$ is an h-fibration, then i_ν is an h-equivalence over B ($\nu = 0, 1$).*

Proof. It suffices to prove (2.117) for $\nu = 0$. We define $\varphi : B \times I \times I \to B \times I$ by

$$\varphi(b, s, t) := \begin{cases} (b, s), & 0 \le t \le \frac{1}{2}, \\ (b, s(2-2t), & \frac{1}{2} \le t \le 1. \end{cases} \quad b \in B, s \in I$$

For $\psi := \varphi \circ (p \times \mathrm{id}_I) : E \times I \to B \times I$ then $\psi(e,t) = p(e)$ holds for $e \in E$ and $0 \le t \le \frac{1}{2}$. Since p is an h-fibration, there exists a homotopy $\Phi : E \times I \to E$ with $p\Phi = \psi$ and $\Phi_0 = \mathrm{id}_E$. For $e \in E$, $\mathrm{proj}_2 \circ p \circ \Phi(e,1) = 0$, where $\mathrm{proj}\, 2 : B \times I \to I$ is the projection onto the second factor, i.e., $\Phi_1(E) \subset E_0$. Φ_1 thus induces a continuous map $r : E \to E_0$. For $e \in E$, $\mathrm{proj}_1 \circ p \circ r(e) = \mathrm{proj}_1 \circ p \circ \Phi(e,1) = \mathrm{proj}_1 \circ p(e)$. We can therefore regard r as a morphism of $\mathcal{T}op_B$, $r : \mathrm{proj}_1 \circ p \to \mathrm{proj}_1 \circ (p|_{E_0})$. One immediately verifies

$$\mathrm{id}_E = \Phi_0 \underset{B}{\simeq} \Phi_1 = i_0 r.$$

Since $\Phi(E \times I) \subset E$ (!), Φ induces a homotopy $\overline{\Phi} : E_0 \times I \to E_0$. Then we have the folloing homotopy (!)

$$\mathrm{id}_{E_0} = \overline{\Phi}_0 \underset{B}{\simeq} \overline{\Phi}_1 = r i_0.$$

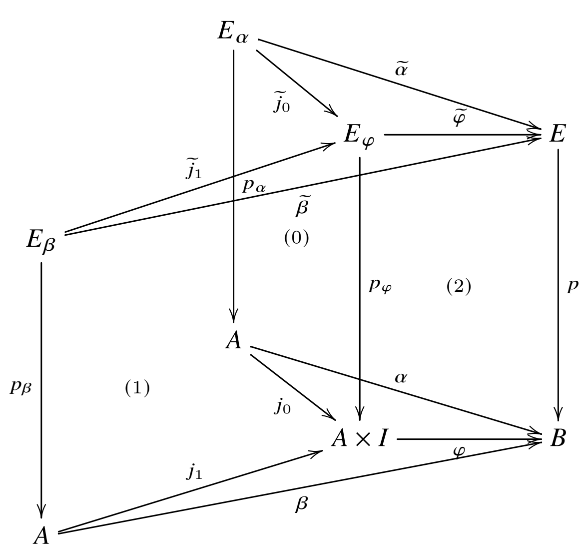
So $[r]_B$ is inverse to $[i_0]_B$ in $\mathcal{T}op_B h$. This proves the claim. □

We are now in a position to prove the homotopy theorem for h-fibrations.

Proof. (of Theorem 2.115) We choose a homotopy $\varphi : \alpha \simeq \beta : A \times I \to B$. By selecting Cartesian squares, we obtain functors

$$j_\nu^* : \mathcal{T}op_{A \times I} h \to \mathcal{T}op_A h \quad (\nu = 0, 1),$$
$$\varphi^* : \mathcal{T}op_B h \to \mathcal{T}op_{A \times I} h.$$

We can assume that in the definition of $\alpha^* = (\varphi j_0)^*$ and $\beta^* = (\varphi j_1)*$, $\alpha^*, \beta^* : \mathcal{T}op_B h \to \mathcal{T}op_A h$, those Cartesian squares were selected that result from juxtaposing the Cartesian squares selected in the definition of φ^* and j_0^* or j_1^* (cf. (2.109)). If $p : E \to B$ is a continuous map, we have a commutative diagramme with Cartesian squares (0), (1), (2):



We consider $j_\nu : A \to A \times I$ as inclusion $A = A \times \nu \subset A \times I$. We can then assume (cf. (2.99)): $E_\alpha = p_\varphi^{-1}(A \times 0)$, $E_\beta = p_\varphi^{-1}(A \times 1)$, $\widetilde{j}_0$, $\widetilde{j}_1$, are the inclusions $E_\alpha \subset E_\varphi$ and $E_\beta \subset E_\varphi$, respectively, p_α, p_β are restrictions of p_φ. We now assume that $p \in \mathrm{Obj}(\mathcal{F}ib_B h)$, i.e., p is an h-fibration. By Corollary (2.102) (b), $p_\varphi : E_\varphi \to A \times I$ is then an h-fibration. From Lemma (2.117), applied to p_φ, now follows:

$$\widetilde{j}_0 : p_\alpha \to \mathrm{proj}_1 \circ p_\varphi, \quad \widetilde{j}_1 : p_\beta \to \mathrm{proj}_1 \circ p_\varphi$$

are h-equivalences over A , i.e., $[\widetilde{j}_0]_A$ and $[\widetilde{j}_1]_A$ are isomorphisms of $\mathcal{T}op_A h$. We set $\Lambda_p := [\widetilde{j}_1]_A^{-1} \circ [\widetilde{j}_0]_A$. $\Lambda_p : p_\alpha \to p_\beta$ is an isomorphism of $\mathcal{T}op_A h$. The reader should convince himself that $\Lambda := (\Lambda_p|_{p \in \mathrm{Obj}(\mathcal{F}ib_B h)})$ is a natural transformation. $\Lambda : \alpha^* \to \beta^* : \mathcal{F}ib_B h \to \mathcal{F}ib_A h$ is thus a natural equivalence.

□

Remark 2.118. If we consider the just defined morphism of $\mathcal{T}op_A h$ $\Lambda_p : p_\alpha \to p_\beta$ as a morphism of $\mathcal{T}op h$, $\Lambda_p : E_\alpha \to E_\beta$, then in $\mathcal{T}op h$ $[\widetilde{\beta}] \circ \Lambda_p = [\widetilde{\alpha}]$.

So we have proven exactly the following:

Theorem 2.119. *If $\alpha \simeq \beta : A \to B$, there exists a natural equivalence $\Lambda : \alpha^* \to \beta* : \mathcal{F}ib_B h \to \mathcal{F}ib_A h$ such that for all h-fibrations $p : E \to B$ the following diagramme in $\mathcal{T}op h$ is commutative.*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\Lambda_p} & E_\beta \\ & {\scriptstyle [\widetilde{\alpha}]} \searrow \quad \swarrow {\scriptstyle [\widetilde{\beta}]} & \\ & E & \end{array}$$

From the homotopy theorem for h-fbration we obtain two corollaries.

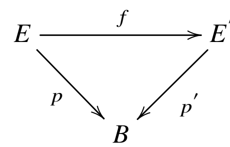
Definition 2.120. (1) If $p : E \to B$ is a space over B, $U \subset B$, then we have the space over U $p_U : p^{-1}(U) \to U$ (cf. (2.32)). We use the notation $E_U := p^{-1}(U)$. If $b_0 \in B$, then we abbreviate $E_{b_0} := E_{\{b_0\}}$.

(2) If $p : E \to B$, $p' : E' \to B$ are spaces over B, and if $f : p \to p'$ is a map over B, $U \subset B$, then $f(E_U) \subset E'_U = p'^{-1}(U)$. So f can be restricted to a continuous map $f_U : E_U \to E'_U$. f_U is a map over U, $f_U : p_U \to p'_U$. If $b \in B$, then we abbreviate $f_{b_0} := f_{\{b_0\}}$.

Corollary 2.121 (The first corollary to Theorem 2.119)**.** Assumption*: Let $\alpha : A \to B$ be homotopic to $\varkappa : A \to B$ with $\varkappa(A) = \{b_0\}$ for some $b_0 \in B$.*
Claim*:*

(a) If $p : E \to B$ is an h-fibration and $p_\alpha : E_\alpha \to A$ is induced from p by a, then p_α is h-equivalent over A to the projection onto the first factor $\mathrm{proj}_1 : A \times E_{b_0} \to A$.

(b) Let the following be a commutative diagramme in $\mathcal{T}op$.

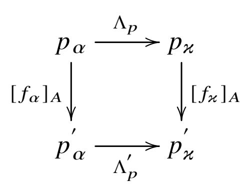


Let p_α and p'_α be induced by p and p', respectively, by α. If p and p' are h-fibrations and $f_{b_0} : E_{b_0} \to E_{b'_0}$, is an h-equivalence, then the morphism of $\mathcal{T}op$ $f_\alpha : p_\alpha \to p'_\alpha$ defined in (2.106) is an h-equivalence over A.

Proof. (a): Let $p_\varkappa : E_\varkappa \to A$ be induced from p by $\varkappa$. By Theorem (2.115) p_α is h-equivalent over A to $p_\varkappa$. The claim now follows, since one can make the following special choice for $p_\varkappa$ according to (2.97):

$$E_\varkappa = \{(a,e)|b_0 = \varkappa(a) = p(e)\} = A \times E_{b_0}, \quad p_\varkappa(a,e) = a \quad \text{for} \quad (a,e) \in E_\varkappa.$$

(b): If $p_\varkappa$ and $p'_\varkappa$ are induced from p and p' by $\varkappa$, respectively, then by Theorem (2.115) there re isomorphisms of $\mathcal{T}op_A h$ $\Lambda_p : p_\alpha \to p_\varkappa$, $\Lambda_{p'} : p'_\alpha \to p'_\varkappa$ such that the following diagramme in $\mathcal{T}op_A h$ is commutative:



To show that $[f_\alpha]_A$ is an isomorphism of $\mathcal{T}op_A h$, we have to show that $[f_\varkappa]_A[f_\varkappa]_A$ is an isomorphism of $\mathcal{T}op_A h$. Let us choose $p_\varkappa$ and $p'_\varkappa$ as in (2.97), i.e., $p_\varkappa = \text{proj}_1 : A \times E_{b_0} \to A$, $p'_\varkappa = \text{proj}_1 : A \times E'_{b_0} \to A$, so (cf. (2.108)) $f_\varkappa = \text{id}_A \times f_{b_0}$. By assumption, $f b_0$ is an h-equivalence. If $g =: E'_{b_0} \to E_{b_0}$ is homotopy inverse to f_{b_0}, then, as one immediately sees, $[\text{id}_A \times g]_A : p'_\varkappa \to p_\varkappa$ is inverse to $[f_\varkappa]_A$ in $\mathcal{T}op_A h$. $[f_\varkappa]_A$ is therefore an isomorphism in $\mathcal{T}op_A h$, which was to be shown. □

Definition 2.122. A topological space X is called *locally contractible* if every point $x \in X$ has a neighbourhood $U \subset X$ such that the inclusion $U \subset X$ is null-homotopic (cf. (0.22)).

Corollary 2.123 (The second corollary to Theorem 2.119)**.** *Every h-fibration $p : E \to B$ over a locally contractible space B is locally trivial up to fibrewise homotopy equivalence.*

Proof. By assumption, for $b \in B$ there exists a neighbourhood $U \subset B$ and a point $b_0 \in B$ such that $(U \subset B) \simeq \varkappa : U \to B$, where $\varkappa(U) = \{b_0\}$. The restriction of p $p_U : p^{-1}U \to U$ is induced from p by $U \subset B$. Thus, by (2.121) (a), p_U is h-equivalent over U to $\text{proj}_1 : U \times E_{b_0} \to U$. □

Theorem 2.124. *Let*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\widetilde{\alpha}} & E \\ {\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[\alpha]{} & B \end{array}$$

be a Cartesian square in $\mathscr{T\!op}$. Claim*: If p is an h-fibration and α is an h-equivalence, then $\widetilde{\alpha}$ is an h-equivalence.*

Proof. Let $\beta : B \to A$ be the h-inverse of α. We choose a Cartesian square

$$\begin{array}{ccc} E_{\alpha\beta} & \xrightarrow{\widetilde{\beta}} & E_\alpha \\ {\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ B & \xrightarrow[\beta]{} & A \end{array}$$

and then we have the Cartesian squares

$$\begin{array}{ccc} E_{\alpha\beta} & \xrightarrow{\widetilde{\alpha}\widetilde{\beta}} & E \\ {\scriptstyle p_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\alpha\beta]{} & B \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\mathrm{id}_E} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\mathrm{id}_B]{} & B \end{array}$$

Since $\alpha\beta \simeq \mathrm{id}_B$, by Theorem (2.119) there exists an h-equivalence $\lambda_p : E_{\alpha\beta} \to E$ such that the diagramme

$$\begin{array}{ccc} E_{\alpha\beta} & \xrightarrow{\lambda_p} & E \\ & {\scriptstyle \widetilde{\alpha}\widetilde{\beta}}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_E} & \\ & E & \end{array}$$

is commutative up to homotopy: $\lambda_p \simeq \widetilde{\alpha}\widetilde{\beta}$. Hence $\widetilde{\alpha}$ has an h-right inverse and $\widetilde{\beta}$ an h-left inverse. We exchange the roles of α and β and an analogous conclusion then yields: $\widetilde{\beta}$ has an h-right inverse. Therefore $\widetilde{\beta}$ is an h-equivalence, and therefore $\widetilde{\alpha}$ is an h-equivalence. □

2.4.3 Induced cofibrations

The definitions and theorems of §§2.4.1 and 2.4.2 can be dualised. We leave the proofs to the reader.

Definition 2.125. Let

$$\begin{array}{ccc} A & \xrightarrow{\xi} & \overline{A} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow[\widetilde{\xi}]{} & \overline{X} \end{array} \tag{2.126}$$

be a Cartesian square in $\mathcal{T}\!op$. $\bar{i}$ is called *induced from i by ξ* if (2.126) is a co-Cartesian square.

Theorem 2.127. *For a diagramme*

$$\begin{array}{ccc} A & \xrightarrow{\xi} & \overline{A} \\ {\scriptstyle i}\downarrow & & \\ X & & \end{array}$$

in $\mathcal{T}\!op$, there exists a diagramme

$$\begin{array}{ccc} & & \overline{A} \\ & & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow[\bar{\xi}]{} & \overline{X} \end{array}$$

in $\mathcal{T}\!op$ that is unique up to isomorphism, such that (2.126) is a co-Cartesian square.

Proof. Uniqueness: follows purely from category-theoretic reasons.
Existence: It is easy to see that the following definition yields a co-Cartesian square (2.126). □

Definition 2.128. Let $\overline{X}$ be the factor space resulting from the topological sum $X + \overline{A}$ if, for each $a \in A$, $ia \in X$ is identified with $\xi a \in \overline{A}$. Let $\bar{i} : \overline{A} \to \overline{X}$ and $\bar{\xi} : X \to \overline{X}$ be the continuous maps obtained by combining the injection of $\overline{A}$ and X into the topological sum $X + \overline{A}$ with the projection of $X + \overline{A}$ onto the factor space $\overline{X}$. If i is an inclusion $A \subset X$, then we use the notation $\overline{A} \cup_{\xi} X$ for $\overline{X}$.

We have already encountered the construction of (2.128) in a special case, namely in the definition of the mapping cylinder of a continuous map.

Example 2.129. If $f : A \to X$ is a continuous map, then we have the co-Cartesian square.

$$\begin{array}{ccc} A & \xrightarrow{j_0} & A \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[j]{} & Z_f \end{array} \qquad \text{(Diagramme 1.15 reposted)}$$

$k : A \times I \to Z_f$ is thus induced from f by j_0.

Theorem (2.100) corresponds to the following theorem:

Theorem 2.130. *In the diagramme in $\mathcal{Top}$*

$$\begin{array}{ccc} A & \xrightarrow{\xi} & \overline{A} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow[\bar{\xi}]{} & \overline{X} \end{array} \tag{2.131}$$

let $\bar{i}$ be induced by i by ξ. Let Y be a topological space.
Claim*:*

(a) If i has the HEP for Y, then so does $\bar{i}$.

(b) If i has the HEP up to homotopy for Y, then so does $\bar{i}$.

Corollary 2.132. *In (2.131), let $\bar{i}$ be induced by i by ξ. Then:*

(a) If i is a cofibration, then so is $\bar{i}$.

(b) If i is an h-cofibration, then so is $\bar{i}$.

$\bar{i}$ is then called "the" (h-)cofibration induced from i by ξ.

Example 2.133 ((Attaching cells)). Let i be the inclusion $\mathbb{S}^{n-1} \subset \mathbb{E}^n$ of the $(n-1)$-sphere $\mathbb{S}^{n-1}$ into the n-ball $\mathbb{E}^n$. Let $\xi : \mathbb{S}^{n-1} \to X$ be a continuous map. The co-Cartesian square

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \xrightarrow{\xi} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ \mathbb{E}^n & \xrightarrow[\bar{\xi}]{} & X \cup_\xi \mathbb{E}^n \end{array}$$

be defined as in (2.128). We say: $X \cup_\xi \mathbb{E}^n$ arises from X by attaching the n-cell $e^n = X \cup_\xi \mathbb{E}^n \setminus \mathbb{S}^{n-1}$ by means of ξ. Since $\mathbb{S}^{n-1} \subset \mathbb{E}^n$ is a cofibration (1.8), it follows from Corollary (2.132) (a) $\bar{i} : X \to X \cup_\xi \mathbb{E}^n$ is a cofibration.

Definition 2.134. Let $\xi : A \to \overline{A}$ be a continuous map. We define a functor $\xi_* : \mathcal{Top}^A \to \mathcal{Top}^{\overline{A}}$. For each object $i : A \to X$ of $\mathcal{Top}^A$, we choose a co-Cartesian square (2.126) and set

$$\xi_*(i) := \bar{i}, \quad \xi_*(i) \in \mathrm{Mor}_{\mathcal{Top}^A}.$$

If g is a morphism of $\mathcal{Top}^A$,

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle i}\swarrow & & \searrow{\scriptstyle i'} & \\ X & & \xrightarrow[g]{} & & X' \end{array}$$

and the following diagramme

$$\begin{array}{ccc} A & \xrightarrow{\xi} & \overline{A} \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \overline{i}'} \\ X' & \xrightarrow[\overline{\xi}']{} & \overline{X}' \end{array}$$

is the co-Cartesian square chosen for i', then there exists exactly one continuous map $\overline{g} : \overline{X} \to \overline{X}'$ with $\overline{g} \circ \overline{\xi}' = \overline{\xi}' \circ g$ and $\overline{g} \circ \overline{i} = \overline{i}'$. We set $\xi_*(g) := \overline{g}$, $\xi_*(g) : \xi_*(i) \to \xi_*(i')$.

Note: The definition of ξ_* depends on the choice of co-Cartesian squares. Different choices yield equivalent functors.
$(\mathrm{id}_A)_*$ is equivalent to $\mathrm{id}_{\mathscr{T}\mathit{op}^A}$.
$(\eta\xi)_*$ is equivalent to $\eta_*\xi_*$ if $\eta : \overline{A} \to \overline{\overline{A}}$ is another continuous map.

Definition 2.135. If A is a topological space, let $\mathscr{C}\mathit{of}^A h$ denote the full subcategory of $\mathscr{T}\mathit{op}^A h$ whose objects are the h-cofibrations $i : A \to X$. If $\xi : A \to \overline{A}$ is a continuous map, then the functor $\xi_* : \mathscr{T}\mathit{op}^A \to \mathscr{T}\mathit{op}^{\overline{A}}$, defined by choosing co-Cartesian squares, first induces a functor $\mathscr{T}\mathit{op}^A h \to \mathscr{T}\mathit{op}^{\overline{A}} h$. This functor, in turn, induces a functor

$$\mathscr{C}\mathit{of}^A h \to \mathscr{C}\mathit{of}^{\overline{A}} h,$$

which we also denote by ξ_*.

Then we have:

Theorem 2.136 (Homotopy thorem for hhcofibrations)**.** *Let $\xi\eta : A \to \overline{A}$ be continuous maps. If $\xi \simeq \eta$, a natural equivalence exists*

$$\xi_* \to \eta_* : \mathscr{C}\mathit{of}^A h \to \mathscr{C}\mathit{of}^{\overline{A}} h.$$

Finally, we mention the theorem corresponding to (2.124).

Theorem 2.137. *If (2.126) is a co-Cartesian square, i is an h-cofibration, and ξ is an h-equivalence, then $\widetilde{\xi}$ is also an h-equivalence.*

2.5 Extension of sections

2.5.1 Numerable coverings

Definition 2.138. Let X be a topological space. A *partition of unity* is a family $\mathfrak{U} = (u_j : X \to I | j \in J)$ of continuous maps u with the properties:

(a) For every $x \in X$ there exists a neighbourhood W such that $u_j(W) = \{0\}$ except for finitely many $j \in J$.

(b) For all $x \in X$, $\sum_{j\in J} u_j(x) = 1$.

Note that because of (a), (b) is essentially a finite sum.

Let $\mathfrak{V} = (V_j | j \in J)$, $V_j \subset X$. $\mathfrak{V}$ is called:

a cover of X if and only if $\cup_{j\in J} V_j = X$;

open if and only if every V_j is open;

locally finite if and only if for every $x \in X$ there exists a neighbourhood W such that $W \cap V_j = \emptyset$ except for finitely many $j \in J$.

A family $(u_j : X \to [0, \infty[| j \in J)$ is called *locally finite* if $(u_j^{-1}]0, \infty[| j \in J)$ is locally finite. Let $u = (u_j | j \in J)$ be a partition of unity and $\mathfrak{V} = (V_j | j \in J)$. We say: $\mathfrak{V}$ is *numbered* by u if for every $j \in J$ the inclusion $u_j^{-1}]0, 1] \subset V$ holds. $\mathfrak{V}$ is called *enumberable* if there exists a partition of unity that numbers $\mathfrak{V}$ (then $\mathfrak{V}$ is a cover, and we say it is a *enumberable* cover).

Numerable covers are a fundamental tool for many proofs below. The next theorem tells us something about when numerable covers exist.

Theorem 2.139. *Let X be a Hausdorff space.*

(a) X is paracompact if and only if every open covering is enumberable.

(b) X is normal if and only if every locally finite open covering is enumberable.

For the proof, see Bourbaki [2], §4, n° 3, 4.

The following theorem is important for the application of numerable coverings in homotopy theory.

Theorem 2.140. *Let $\mathfrak{U} = (U_j | j \in J)$ be an numerable cover of $X \times I$. There exists a numerable cover $(V_k | k \in K)$ of X and a family $(\varepsilon_k \in K)$ of positive real numbers such that for $t_1, t_2 \in I$ and $|t_1 - t_2| < \varepsilon_k$ there exists a $j \in J$ with $V_K \times [t_1, t_2] \subset U_j$.*

Proof. We can assume that $\mathfrak{U}$ is given by a partition of unity $(u_j | j \in J)$, i.e. $U_j. = u_j^{-1}]0, 1], j \in J$. For each r-tuple $k = (j_1, \ldots, j_r) \in J^r$ we define a continuous map $v_k : X \to I$ by

$$v_k(x) := \sum_{i=1}^{r} \min(u_{j_i}(x, t) | t \in [\frac{i-1}{r+1}, \frac{i+1}{r+1}]).$$

Let $K = \cup_{r1}^{\infty} j^r$. We show that $\mathfrak{V} = (v_k^{-1}]0, 1] | k \in K)$ is a numerable cover of X. Every point $(x, t) \in X \times I$ has an open product neighbourhood $U(x, t) \times V(x, t)$ that is contained in a suitable $_i$ and only meets finitely many U_i. $V(x, t_1), \ldots, V(x, t_n)$ covers I, let $\frac{2}{r+1}$ be a Lebesgue number of this cover, and let U be $U(x, t_1) \cap \cdots \cap U(x, t_n)$. Every set $U \times [\frac{i-1}{r+1}, \frac{i+1}{r+1}]$ is then contained in a suitable Uj_i, so x lies in $v_k^{-1}]0, 1]$, $k = (j_1, \ldots, j_r)$; $\mathfrak{V}$ is therefore a cover.

Furthermore, there are only finitely many $j \in J$ for which $U_j \cap (U \times I)$ is not empty. Since $v_k(x) \neq 0$ implies the relation $U_{j_i} \cap \{x\} \times I \neq \emptyset$, $(v_k | k \in K_r)$, $K_r = J \cup J^2 \cup \cdots \cup J^r$ is locally finite for every r. Therefore, a continuous function w_r is defined by

$$w_r(x) = \sum_{k \in K_{r-1}} v_k(x) \quad \text{for} \quad r > 1$$

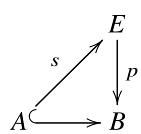
and $w_1(x) = 0$. Let

$$z_k(x) = \max(0, v_k(x) - r w_r(x)) \quad \text{for} \quad k = (j_1, \dots, j_r) \in K.$$

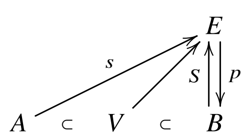
For $x \in X$, we choose $k' = (j_1, \dots, j_r) \in K$ with minimal r such that $v_{k'}(x) > 0$. Then $w_r(x) = 0$, $z_{k'}(x) = v_{k'}(x)$, and we see that the $z_k^{-1}]0,1]$ cover X. If we choose $m > r$ such that $v_{k'}(x) > \frac{1}{m}$, then $w_m(x) > \frac{1}{m}$ and consequently $m w_m(y) > 1$ for all y in a suitable neighbourhood of x. In this neighbourhood, z_k vanishes for all $k = (j_1, \dots, j_s)$ with $s \geq m$. Therefore, $(z_k | k \in K)$ is locally finite; $(z_k / \sum_{k \in K} z_k | k \in K)$ is numbered $(v_k^{-1}]0,1])$. $V_k = v_k^{-1}]0,1]$ and $\varepsilon_k = \frac{1}{2r}$ for $k = (j_1, \dots, j_r)$ satisfy the requirements of the theorem. □

2.5.2 The section extension property (SEP)

Definition 2.141. Let $p : E \to B$ be a continuous map and $A \subset B$. A *section of p over A* is a continuous map $s : A \to E$ with $ps(a) = a$ for all $a \in A$. A section of p over B is called a section of p for short.



p has the *section extension property* (the *SEP*) if: for every $A \subset B$ and every section s over A that extends to a Halo V of A (in B), there exists a section $S : B \to E$ of p with $S|_A = s$.



In particular, there is then a section of p; set $A = V = \emptyset$.

Theorem 2.142. *If $p : E \to B$ is dominated by $p' : E' \to B$ and p' has the SEP, then so does p.*

Proof. Since p is dominated by p', there are maps over B $f : E \to E'$ and $g : E' \to E$ and a homotopy $\varphi : E \times I \to E$ over B, $\varphi : \mathrm{id}_E \underset{B}{\simeq} gf$.
Let $A \subset B$, s be a cut of p over A, s_V a section of p over a halo V of A such that $s_V|_A = s$. Then $f s_V$ is a section of p' over V. By Corollary (1.86), we can

choose a closed halo U of A such that $U \subset V$ and V is a halo of U. Since p' has the SEP, there exists a section $S' : B \to E'$ of p' with $S'|_U = fs_V|_U$. We choose a halo function u of U and define $S : B \to E$ by

$$S(b) := \begin{cases} gS'(b), & b \in u^{-1}(1) \\ \varphi(s_V(b), u(b)), & b \in U. \end{cases}$$

S is well-defined, continuous, and a section of p that extends s. □

Corollary 2.143. *If $p : E \to B$ is shrinkable, then p has the SEP (see (2.88), (2.89).*

Example 2.144. The SEP does not generally extend to induced objects.
The projection $\mathrm{proj}_1 : \mathbb{E}^2 \times \mathbb{S}^1 \to \mathbb{E}^2$ is induced by the map $p : \mathbb{S}^1 \to P$, P a one-point space. p clearly has the SEP, but proj_1 does not. Let $A = \mathbb{S}^1 \subset E^2$ and $s : A \to \mathbb{E}^2 \times \mathbb{S}^1$ be given by $s(z) = (z, z)$. s can be extended to a halo of A in $\mathbb{E}^2$, but not to $\mathbb{E}^2$.

If $p : E \to B$ is shrinkable, so is every induced object (see (2.112)). In this case, the SEP is transferred to induced objects. For a converse, see Dold [6], Proposition 3.1.

Theorem 2.145. *If $p : E \to B$ has the SEP and $A \subset B$ is an open subset for which there exists a function $v : B \to I$ with $v^{-1}[0, 1[= A$, then the restriction $p_A : p^{-1}A \to A$ (see (2.32)) has the SEP.*

Proof. Let $u : A \to [0, 1]$ and a cut s of p over $u^{-1}[0, 1[$ be given. We have to construct a section over A that coincides with s on $u^{-1}(0)$. For this purpose, we construct a sequence $S_n : B \to E$, $n = 2, 3, \ldots$ of sections with the properties:

(1) For $v(b) < 1 - \frac{1}{n}$, $S_{n+1}(b) = S_n(b)$.

(2) For $b \in A$ with $u(b) < \frac{1}{n}$, $v(b) < 1 - \frac{1}{n+1}$, $S_n(b) = s(b)$.

First, we choose continuous functions

$$\mu_n, \lambda_n : [0, 1] \to [0, 1]$$

as follows:

$$\mu_n(x) := \begin{cases} 1 - \frac{1}{n} & \text{for} \quad x \geq \frac{1}{n} \\ 1 - \frac{1}{n+2} & \text{for} \quad x \leq \frac{1}{n+1} \end{cases}$$

and $\mu_n(x) \geq 1 - \frac{1}{n}$ for all $x \in [0, 1]$;

$$\lambda_n(x) \begin{cases} := 1 - \frac{1}{n+1} & \text{for} \quad x \geq \frac{1}{n} \\ > 1 - \frac{1}{n+1} & \text{for} \quad x \leq \frac{1}{n} \end{cases}$$

and $1 - \varepsilon\lambda_n(x) > \mu_n(x)$ for all $x \in [0, 1]$ and for some $\varepsilon > 0$. By $w(b) = (1 - u(b)/(1 - v(b))$ for $v(b) < 1$ and $w(b) = 0$ otherwise a continuous function $w : B \to I$ is described. $w^{-1}]0, 1]$ is a halo of $w^{-1}[\frac{1}{6}, 1]$. s is defined on

$w^{-1}]0,1] \subset u^{-1}[0,1[$. Because p has the SEP, there is a section $S_2 : B \to E$ that coincides with s on $w^{-1}[\frac{1}{6},1]$, and therefore also on $\{b \in B | v(b) < \frac{2}{3}, u(b) < \frac{1}{2}$.

This provides the beginning of the induction. Pursuant to the step from n to $n+1$, we define a section s_n over $V_n = \{b \in A | v(b) < \lambda_n(u(b))\}$ by

$$s_n(b) := \begin{cases} S_n(b) & \text{for} \quad v(b) < 1 - \frac{1}{n+1} \\ s(b) & \text{for} \quad u(b) < \frac{1}{n}. \end{cases}$$

By the induction hypothesis (2), $s_n(b)$ is well-defined. In V_n, one of the two inequalities holds. V_n is a halo of $A_n = \{b \in A | v(b) < \mu_n(u(b))\}$ in B.

A halo function h_n is given by

$$h_n(b) := \begin{cases} 0 & \text{for} \quad v(b) \leq \mu_n(u(b)), b \in A \\ \frac{\mu_n(u(b) - v(b)}{\mu_n(u(b) - \lambda_n(u(b))} & \text{for} \quad \mu_n(u(b)) \leq v(b) \leq \lambda_n(u(b)), b \in A \\ 1 & \text{for} \quad v(b) \geq \lambda_n(u(b)), b \in B \setminus A \text{ or } b \in A \end{cases}$$

Note that the three parts of the domain are closed in B. From the SEP for p, we conclude that there exists a section $S_{n+1} : B \to E$ that coincides with s_n on A_n.

Thus (1) and (2) hold. From $v(b) < 1 - \frac{1}{n}$ it follows: $v(b) < \mu_n(u(b))$, $b \in A_n$, $S_{n+1}(b) = s_n(b) = S_n(b)$. From $u(b) < \frac{1}{n+1}$, $v(b) < 1 - \frac{1}{n+2}$, it follows : $v(b) < \mu_n(u(b))$, $b \in A_n$, $S_{n+1}(b) = s_n(b) = s(b)$. □

2.5.3 The section extension theorem

Let $p : E \to B$ and $A \subset B$ be given. We say p has the *SEP over*) A if the restriction $p_A : p^{-1}A \to A$ has the SEP.

Theorem 2.146. *Let $p : E \to B$ be a space over B. If there exists a numerable covering $(V_j | j \in J)$ of B such that p has the SEP over every set V_j, then p has the SEP.*

Proof. Let $V_j | j \in J)$ be a numerable covering of B such that p has the SEP over every set V_j. Let $A \subset B$, s be a section of p over A, and s_A be an extension of s to a halo V of A with halo function u.

Let $(u'_j | j \in J)$ be a numbering of (V_j). We assume that $0 \notin J$ and set $J' = J \cup \{0\}$. By $u_0 = 1 - u$, $u_j = u \cdot u'_j$ for $j \in J$, a partition of unity $(u_j | j \in J')$ is defined. For $K \subset J'$ we set

$$u_K = \sum_{j \in K} u_j : B \to I$$

and $U_K = u_k^{-1}]0,1]$ ($u_\varnothing = 0$, $U_\varnothing = \varnothing$). u_K is continuous; A lies in U_K if $0 \in K$. We consider the set of pairs

$$\mathfrak{S} = \{(K, s) | 0 \in K \subset J', s \text{ intersection over } U_K, s|_A = s_A\}.$$

$\mathfrak{S}$ is not empty, since $(\{0\}, s_V|_{U_{\{0\}}})$ lies in $\mathfrak{S}$. On $\mathfrak{S}$ we introduce an ordering: $(K, s) \leq (K', s')$ if and only if

(1) $K \subset K^{'}$;

(2) from $s(b) \neq s^{'}(b)$ follows $b \in U_{K' \setminus K}$.

We want to apply Zorn's Lemma to the ordered set $(\mathfrak{S}, \leq)$. Therefore, we show that

Proposition 2.147. *every chain in $\mathfrak{S}$ has an upper bound.*

Proof. Let $\mathfrak{T} \subset \mathfrak{S}$ be a chain, $\mathfrak{T} \neq \emptyset$. We set $L = \cup_{(K,s) \in \mathfrak{T}} K$ and want to define a section $t : U_L \to E$. Let $b \in U_L$. We choose a neighbourhood W of b such that

$$P_W = \{j \in J^{'} | W \cap u_j^{-1}]0,1] \neq \emptyset\}$$

is finite. We consider

$$\mathfrak{T}_W = \{(K,s) \in \mathfrak{T} | (L \setminus K) \cap P_W \neq \emptyset\}.$$

$\mathfrak{T}_W$ is not empty because P_W is finite and $\mathfrak{T}$ is a chain. For $(K,s) \in \mathfrak{T}_W$, $b \in U_L \cap W \subset U_K$; and by condition (2) in the definition of $\leq$, for $(K,s), (K^{'}, s^{'}) \in \mathfrak{T}_W$

$$s(c) = s^{'}(c), \quad c \in U_L \cap W.$$

By $t(b) = s(b)$, $(K,s) \in \mathfrak{T}_W$, $t(b)$ is therefore uniquely defined, and because $t|_{U_L \cap W} = s|_{U_L \cap W}$, $t : U_L \to E$ is also continuous; thus (L,t) lies in $\mathfrak{S}$. For $(K,s) \in \mathfrak{T}$, $(K,s) \leq (L,t) : K \subset L$ is clear; and from $s(b) \neq t(b)$, $(L \setminus K) \cap P_W \neq \emptyset$ follows, i.e. there exists a $j \in L \setminus K$ with $u_j(b) > 0$, hence $b \in U_{L \setminus K}$. Thus we have shown that Zorn's theorem can be applied. □

Therefore, let (K,s) be maximal in $(\mathfrak{S}, \leq)$. We show that $K = J^{'}$. Then $U_K = U_{J'} = B$ and s is a section over B that extends s; thus, the theorem is proven. Suppose $K \neq= J^{'}$. We then choose $j \in J^{'} \setminus K$.

The continuous function

$$: u_j^{-1}]0,1] \to I, \quad w(b) = \min\left(1, \frac{u_K(b)}{u_j(b)}\right), \quad b \in u_j^{-1}]0,1],$$

provides a halo $w^{-1}]0,1]$ of $w^1(1)$. Let $w^1(1) \subset U_K$ and $s|_{w^{-1}(1)}$ has an extension $s^{'}$ over $u_j^1]0,1]$, since p has the SEP over $u_j^1]0,1] \subset V_j$ by Theorem (2.145) and $s|_{w^{-1}}(1)$ can be extended to $w^{-1}]0,1]$ by $s|_{w^{-1}}]0,1]$. Let $t : U_K \cup U_{\{j\}} \to E$ defined by

$$t(b) := \begin{cases} s(b) & \text{for} \quad u_j(b) \leq u_K(b) \\ s^{'}(b) & \text{for} \quad u_j(b) \geq u_K(b). \end{cases}$$

Then $(K,s) \leq (K \cup \{j\}, t)$ and this contradicts the maximality of (K,s). □

Remark 2.148. In the proof of the section extension theorem, Theorem (2.145) can be avoided by making the following stricter assumption: There exists a numerable cover (V_j) of B such that $p_U : p^{-1}U \to U$ has the SEP for every open subset U that lies in some V_j. This property is easily seen in many applications that we will make later.

Finally, we mention an immediate consequence of the proven theorems.

Theorem 2.149. *If $p : E \to B$ is numerable locally trivial with a contractible fibre, then p has the SEP and hence also a section. The assumption on p should explicitly state: there exists a numerable cover $(V_j | j \in J)$ of B and a family $(F_j | j \in J)$ of contractible topological spaces F_j such that for all $j \in J$, $p_{V_j} : p^{-1}V_j \to V_j$ in $\mathcal{T}op_{V_j}$ is isomorphic to $\mathrm{proj}_1 : V_j \times F_j \to V_j$*

For the proof, see Dold [6].

2.6 The “local-global” transition in the case of fibrations

2.6.1 The "local-global" transition for fibrewise homotopy equivalences

Theorem 2.150. *Let $p' : E' \to B$ and $p : E \to B$ be spaces over B, let $f : E' \to E$ be a map over B (i.e., $pf = p'$), and let $(V(j) | j \in J)$ be a numerable covering of B. For every $j \in J$, we have an induced map*

$$f_j := f_{V(j)} : p'_{V(j)} \to p_{V(j)}$$

(cf. (2.120)).

If f_j is a fibrewise homotopy equivalence for every $j \in J$, then f is also a fibreise homotopy equivalence.

Proof. We transfer the construction of the mapping path space (cf. 2.23) to the category $\mathcal{T}op_B$ and consider the space

$$W = W_{f,B} = \{(e, w) | f(e) = w(0), pw = \text{const}\} \subset E' \times E^I$$

together with the mappings

$$k : E' \to W, \quad k(e) = (e, f(e))$$
$$r : W \to E, \quad r(e, w) = w(1)$$

(we identify points in E with the corresponding constant paths in E^I). W is a space over B by the map $(e, w) \mapsto p'(e)$; k and r thus become maps over B. Theorem (2.51) can be extended to the category $\mathcal{T}op_B$. Therefore:

(a) k is an h-equivalence over B.

(b) r is a fibration over B.

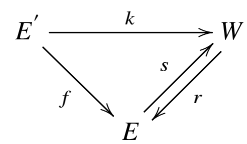
Because $rk = f$, it follows from (a) that r is an h-equivalence over B if and only if f is an h-equivalence over B. The above construction can, of course, be applied to any fibre-wise mapping. If we assume f_j, $V(j)$ instead of f, B, then will resul the fibration over $V(j)$

$$r_j : W_{f_j, V(j)} := W_j \to p^{-1}(V(j)) =: U(j).$$

The reader should convince himself that the fibration r_j is equal to the fibration induced by r over $U(j)$

$$r_{U(j)} : W_{U(j)} \to U(j).$$

By assumption, f is an h-equivalence over $V(j)$, hence r_j is an h-equivalence over $V(j)$. By remark (b above and by Theorem (2.90), applied to the category $\mathcal{T}op_V(j)$, r_j is shrinkable in $\mathcal{T}op_V(j)$, hence shrinkable in $\mathcal{T}op$. By (2.143), r_j therefore has the SEP. Since $r_j = r_{U(j)}$ and $(U(j)|j \in J)$ is an enumberable covering of E (if $(v_j|j \in J)$ is a numbering of $(V(j)|j \in J$ then $(v_j p|j \in J)$ is a numbering of $(V(j)|j \in J)$.), r has the SEP by Theorem (2.146). Therefore there is a section $s : E \to W$ of r. s is itself a map over B. From the commutative diagramme



in $\mathcal{T}op_B$, we see that f has an h-right inverse f' over B. (We have a projection proj : $W \to E'$ and can choose $f' = \text{proj} \circ s$.) The proof now ends according to the familiar pattern: $f'_{V(j)}$ is h-right inverse over $V(j)$ to f_j , thus an h-equivalence over $V(j)$. Consequently, f' has an h-right inverse over B and therefore f' and hence f are h-equivalences over B. □

Definition 2.151. Let $\mathfrak{V} = (V_j|j \in J)$ be a covering of the space B. We say $\mathfrak{V}$ is *null-homotopic* if and only if every inclusion $V_j \subset B$ is null-homotopic (cf. (0.22)).

Theorem 2.152. *Let $p : E \to B$ and $p' : E' \to B$ be h-fibrations and let $f : E \to E'$ be a map over B. Let B have a numerable, null-homotopic covering $(V(j)|j \in J)$. If in every path component of B there is a point b for which $f_b : E_b \to E'_b$ is an h-equivalence, then f is an h-equivalence over B.*

Proof. Let the inclusion $V(j) \subset B$ be homotopic to the constant map k_j. By assumption, we can assume that $f_{b(j)}$ for $k_j(V(j)) = \{b(j)\}$ is an h-equivalence. From Corollary (2.121) (b) we see that $f_{V(j)}$ is an h-equivalence over $V(j)$. The claim follows from (2.150). □

2.6.2 The "local-global" transition for fibrations and h-fibrations

Theorem 2.153. *Let $p : E \to B$ be a continuous map and let $(V(j)|j \in J)$ be a numerable covering of B. If $p_{V(j)}$ is a fibration for all $j \in J$, then p is a fibration.*

Corollary 2.154. *If p is trivial over every set $V(j)$, then p is a fibration.*

Remark 2.155 (*Additional*). If $(V(j)|j \in J)$ is an open covering and $p_{V(i)}$ is a fibration for $j \in J$, then p has the CHP for paracompact spaces X.

Proof. (of Theorem 2.153) We prove the theorem and point out the changes that are necessary to prove the additional remark. We assume the following situation:

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle j_0}\downarrow & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\varphi]{} & B \end{array}$$

The proof of the theorem now proceeds as follows: $(U(j) = \varphi^{-1}V(i)|j \in J)$ is a numerable cover of $X \times I$ (valid for theorem and the corollary). By Theorem (2.140), there exists a numerable cover $(X_k|k \in K)$ of X and a family of positive real numbers $(\varepsilon_k|k \in K)$ such that for $|t_1 - t_2| < \varepsilon_k$, there exists a $j \in J$ with $X_k \times [t_1, t_2] \subset U(j)$. We show that for $Z \subset X_k$ q_Z has the SEP. By the section extension theorem (2.146), q has a section corresponding to a homotopy $\Phi : X \times I \to E$ over φ with initial f.

Let $Z \subset X_k$. We want to show that q_Z has the SEP. By the correspondence between sections and homotopies explained above, we must show: Let V be a halo of A in Z; let $\Phi_V : V \times I \to E$ be a homotopy with $\Phi_V(x, 0) = f(x)$, $p\Phi_V(x, t) = \varphi(x, t)$ for $x \in V$, $t \in I$; then there exists a homotopy $\Phi : Z \times I \to E$ with $p\Phi = \varphi|_{Z\times I}$, $\Phi|_{A\times I} = \Phi_V|_{A\times I}$ and $\Phi(z, 0) = f(z)$ for $z \in Z$. If p is a fibration over $\varphi(Z \times I)$, this follows from Theorem (2.62). We only know that for

$$0 = t_0 < t_1 < \cdots < t_n = 1 \quad \text{with} \quad t_i - t_{i-i} < \varepsilon_k$$

$\varphi(Z \times [t_{i-1}, t_i]) \subset V(j)$.

We can therefore apply Theorem (2.62) to $\varphi|_{Z\times[t_{i-1},t_i]}$. More precisely: Let $w : Z \to I$ be a function with $A \subset w^{-1}(1)$, $Z \setminus V \subset w^{-1}(0)$. Let $W_i = w^{-1}[t_i, 1]$, $i = 1, 2, \ldots, n$. Then W_i is a halo of W_{i+1} in Z, $i = 1, 2, \ldots, n-1$, and V is a halo of W_1. Using Theorem (2.62), we construct in sequence

$$\Phi_i : Z \times [t_{i-1}, t_i] \to E, \quad i = 1, 2, \ldots, n,$$

with

$$\begin{aligned} &p\Phi = \varphi|_{Z\times[t_{i-1},t_i]}, \\ &\Phi_i(z, t_{i-1}) = \Phi_{i-1}(z, t_{i-1}) \quad \text{for} \quad z \in Z,\ i > 1, \\ &\Phi_1(z, 0) = f(z) \quad \text{for} \quad z \in Z, \\ &\Phi_i|_{W_i\times[t_{i-1},t_i]} = \Phi_V|_{W_i\times[t_{i-1},t_i]}. \end{aligned}$$

All Φ_i together yield $\Phi : Z \times I \to E$ with $p\Phi = \varphi|_{zZ\times I}$, $\Phi(z, 0) = f(z)$ for $z \in Z$, $\Phi|_{A\times I} = \Phi_V|_{A\times I}$. □

Theorem 2.156. *Let $p : E \to B$ be a continuous map and let $(V(j)|j \in J)$ be a numerable covering of p. If $p_{V(j)}$ is an h-fibration for all $j \in J$, then p is an h-fibration.*

Corollary 2.157. *If p is trivial over every set $V(j)$, then p is an h-fibration.*

Remark 2.158 (*Additional*). If $(V(j)|j \in J)$ is an open covering and $p_{V(i)}$ is an h-fibration for $j \in J$, then p has the CHP up to homotopy for paracompact spaces X.

Proof. (of Theorem 2.156) The proof is analogous to the proof of Theorem (2.153). We again start from the situation

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle j_0}\downarrow & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\varphi]{} & B \end{array}$$

only we now assume that $\varphi(x,t) = \varphi(x,0)$ for $t \leq 1/2$, because we are dealing with h-fibrations (see (2.75) ($\varepsilon = 1/2$)).

As in the proof of Theorem (2.153), we consider the map $q : W \to X$ and have to show that q has a section. As there, we choose the cover (X_k) of X and the family (ε_k). It suffices again to show that for $Z \subset X_k$, q_Z has the SEP. So let $A \subset Z$ and V be a closed halo of A in Z. A section s_V of q_Z over V corresponds to a homotopy $\Phi_V : V \times I \to E$ over φ with $\Phi_V(z,0) = f(z)$ for $z \in V$. We construct $\Phi_0 : V \times I \cup Z \times [0, t_1] \to E$ over φ with $\Phi_0|_{A\times I} = \Phi_V|_{A\times I}$ and $\Phi_0(z,0) = f(z)$ for $z \in Z$, if $t_1 < 1/2$.

To this end, let $w : Z \to I$ be a function with $A \subset w^{-1}(1)$, $Z \setminus Y \subset w^{1}(0)$. Let $\tau_Z : I \to I$ for $t_1 > w(z)$ be the piecewise affine function that maps $(0, w(z), t_1, 1/2, 1)$ in sequence to $(0, w(z), w(z), w(z), 1/2, 1)$ and is affine in the intermediate intervals; $\tau_Z : I \to I$ is equal to id_I for $t_1 < w(z)$. $\tau_Z(t)$ depends continuously on $(z,t) \in Z \times I$.

We define

$$\Phi_0(z,t) := \begin{cases} (\Phi_V, \tau_Z(t), & \text{for} \quad z \in V \\ f(z) & \text{for} \quad z \in w^{-1}(0), 0 \leq t \leq t_1. \end{cases}$$

Φ_0 has the desired properties; in particular, it lies above φ because for $t < 1/2$, $\varphi(x,t) = \varphi(x,0)$. We now choose $0 = t_0 < t_1 < \cdots < t_n = 1$, such that $t_1 < 1/2$ and $t_{i+1} - t_i < \varepsilon_k$, $(i = 1, \ldots, n-1)$. Let $W_i = w^{-1}[t_i, 1]$ for $i = 1, \ldots, n$ and let $W_0 = V$. Then, for $1 \leq i \leq n-1$, we inductively construct maps $\Phi_i : Z \times [t_{i-1}, t_{i+1}] \to E$ over φ with $\Phi_i|_{W_i \times [t_{i-1}, t_{i+1}]} = \Phi_0|_{W_i \times [t_{i-1}, t_{i+1}]}$ and $\Phi_i(z, t_{i-1}) = \Phi_{i-1}(z, t_{i-1})$ for $z \in Z$, by applying Theorem (2.93). (The theorem applies to: W_i instead of A, W_{i-1} instead of V, Z instead of X, $[t_{i-1}, t_{i+1}]$ instead of I, $[t_{i-1}, t_i]$ instead of $[0, \varepsilon]$.) We define $\Phi : Z \times I \to E$ by $\Phi(z,t) = \Phi_i(z,t)$ for $t_{i-1} \leq t \leq t_i$, $i < n-1$ and $\Phi(z,t) = \Phi_{n-1}(z,t)$ for $t_{n-2} \leq t \leq t_n$. Φ lies over φ and has the starting point $f|_Z$; the corresponding section of q_Z extends $s_V|_A$, because $\Phi|_{A\times I} = \Phi V|_{A\times I}$. □

See Dold [6] for details.

Chapter 3

Homotopy sets and homotopy groups

3.1 Action of fundamental groupoids

3.1.1 Fundamental groupoid

Let K and X be topological spaces. We define a *category* $\mathscr{P}^K X$ as follows: Objects are continuous maps $f : K \to X$, also written as (X, f), since we want to consider K as fixed. *Morphisms* from (X, f) to (X, g) are continuous maps

$$u : K \times [0, p_u] \to X, \quad p_u \in \mathbb{R}^+$$

with

$$u(k, 0) = f(k), \quad u(k, p_u) = g(k)$$

for all $k \in K$. A *composition*, written $(u, v) \mapsto v + u$, is defined as

$$(v + u))(k, t) := \begin{cases} u(k, t), & 0 \le t \le p_u, \\ v(k, t - p_u), & p_u \le t \le p_u + p_v, \end{cases} \qquad k \in K.$$

(so $p_{u+v} = p_u + p_v$).

$\mathscr{P}^K X$ is called the *category of paths of* X *under* K and if K is a point space, the *category of paths in* X. We now define a natural equivalence relation (cf. (0.4)) in $\mathscr{P}^K X$ (essentially the homotopy relative to the endpoints). Let $u : K \times [0, p_u] \to X$ and $v : K \times [0, p_v] \to X$ be morphisms from $\mathscr{P}^K X$ of (X, f) to (X, g). u is called *equivalent to* v if there exist constant morphisms u' and v' from (X, g) to (X, g) (i.e. $u'(k, t) = u'(k, 0)$ and correspondingly for v'), such that $u' + u$ and $v' + v$ have the same domain $K \times [0, p]$ and (considered as maps $K \times [0, p] \to X$) are homotopic relative to $K \times \{0, p\}$.

The reader confirms that this gives an equivalence relation on the morphism sets that is compatible with "+". We can therefore move on to the factor category, which we denote by $\Pi^K X$. In $\Pi^K X$, every morphism is an isomorphism ($\Pi^K X$ is a groupoid).

$\Pi^K X$ is called the *fundamental groupoid of* X *under* K. The case where K is a point space is particularly important. We then speak of the *fundamental groupoid* ΠX of X. The objects of ΠX "are" then simply the points of X. If we pick a point $x \in X$, then the morphisms in ΠX from x to x form a group with respect to composition, the *fundamental group* $\pi_1(X, x)$ of X at the point x.

3.1.2 Functoriality of fundamental groupoids

Let $i : K \hookrightarrow A$ be a space under K. We first assume that i is a closed cofibration; later (3.1.7), we weaken to h-cofibration.

We want to assign a map

$$\widehat{u} : [A, (X, u_0)]^K \longrightarrow [A, (X, u_p)]^K$$

of the homotopy sets under K (= morphism sets in $\mathcal{T}\!op^K h$) to a morphism $u : K \times [0, p] \longrightarrow X$ from (X, u_0) to (X, u_p) in $\mathcal{P}^K X$. (Let A denote the object $i : K \longrightarrow A$.)

Let $f : A \longrightarrow X$ be given with $fi = u_0$. A *translation* of f along u is a map $\varphi : A \times [0, p] \longrightarrow X$ with $\varphi \circ (i \times \mathrm{id}) = u$ and $\varphi_0 = f$. There do exist translations of f along u: This is clear for $p = 0$ and follows for $p > 0$, because i is a cofibration. We want to set

$$\widehat{u}[f]^K = [\varphi_p]^K.$$

Some preparations for this.

Proposition 3.1. *Let* $\psi : f \overset{K}{\simeq} f'$. *Let* $\chi : u \simeq u' \operatorname{rel} K \times \{0, p\}$. *Let* φ' *be a translation of* f' *along* u'. *Then* $[\varphi_p]^K = [\varphi']^K$

Proof. ψ, χ, φ, and φ' together define a map from $K \times [0, p] \times I \cup U A \times [0, p] \times \{0, p\}$ to X, which we want to extend to $A \times [0, p] \times I$. This is a homotopy extension problem for

$$j : K \times I \cup A \times \{0, 1\} \subset A \times I$$

($[0, p]$ is the homotopy interval); j is a cofibration by the product theorem (1.104). The end of the extended homotopy yields a homotopy $\varphi_p \simeq \varphi'$ under K. (Illustrated by drawing!) □

In particular, we have shown with (3.1) that by setting $\widehat{u}[f]^K = [\varphi_p]^K$ a *map* $\widehat{u}$ is induced.

Proposition 3.2. *Let* u *be the constant homotopy. Then* $\widehat{u}$ *is the identity because a constant homotopy can be used to translate along* u.

Proposition 3.3. *Let* u, v *be in* $\mathcal{P}^K X$ *and let* $v + u$ *be defined. Then*

$$\widehat{v + u} = \widehat{v}\widehat{u}.$$

Proof. If one shifts f with φ along u and φ_p with ψ along v, then $\psi + \varphi$ is a shift of f along $v + u$. □

From (3.1) it follows that $\widehat{u}$ depends only on the class $[u]$ of u in $\Pi^K X$. We set $\widehat{[u]} := \widehat{u}$. By (3.2) and (3.3) we obtain:

Theorem 3.4. *The assignment* $(g : K \to X) \mapsto [A, (X, g)]^K$ *and* $[u] \mapsto \widehat{[u]}$ *defines a covariant functor*

$$\Pi^K X \to \mathcal{Sets}$$

from $\Pi^K X$ *into the category of sets.*

Corollary 3.5. *For every* u, $\widehat{u}$ *is bijective since* $[u]$ *is an isomorphism in* $\Pi^K X$.

Remark 3.6. If K is a one-point space, then $[A, (X, g)]$ is canonically a ponted set. $\widehat{u}$ is a pointed map. As a proof, note that a constant map $f : A \to \{u(0)\} \subset X$ can be translated along a path

$$u : [0, p] = K \times [0, p] \to X$$

by

$$\varphi : (a, t) \mapsto u(t)$$

3.1.3 Action of π_1

The functor just constructed measures the difference between "homotopic in $\mathcal{Top}^K$" and "homotopic in $\mathcal{Top}$".

Theorem 3.7. *Let* $f : A \to (X, g)$ *and* $f^{'} : Ato(X, g^{'}$ *be morphisms in* $\mathcal{Top}$. *Then* $[f] = [f^{'}]$ *if and only if there exists a* $u \in \mathcal{P}^K X$ *from* (X, g) *to* $(X, g^{'})$ *such that* $[f^{'}]^K = \widehat{u}[f]^K$.

Proof. If $[f^{'}]^K = \widehat{u}[f]^K$, then $[f^{'}]^K = [\varphi_p]^K$, where φ is a translation of f along u. Thus $[f] = [\varphi_o] = [\varphi_p] = [f^{'}]$.
Conversely, if φ is a homotopy from f to $f^{'}$, then f' results from f by translation along $u = \varphi \circ (i \times \mathrm{id}_I)$. □

In particular, if K is a one-point space and $X \in \mathrm{Obj}(\mathcal{Top})$, then we consider the map $v : [A, X]^o \to [A, X]$, $v[f]^o = [f]$. Then (3.1.2) specifically yields an action of the fundamental group $\pi_1(X, 0)$ on $[A, X]^o$.

Recall that an *action* of a group G on a set M is a map $a : G \times M \to M$ with the properties $a(\mathrm{id}, m) = m$, $a(g, a(h, m)) = a(gh, m)$.

$$\begin{array}{ccc} G \times G \times M & \xrightarrow{\mathrm{id}_G \times a} & G \times M \\ \downarrow & & \downarrow a \\ G \times M & \xrightarrow[a]{} & M \end{array} \qquad \begin{array}{ccc} (g, h, m) & \longmapsto & (g, a(h, m)) \\ \downarrow & & \downarrow \\ (gh, m) & \longmapsto & a(gh, m) = a(g, (a(h, m)) \end{array}$$

Theorem 3.8. v *is*

(a) injective $\Leftrightarrow \pi_1(X, o)$ *acts trivially on* $[A, X]^o$.

(b) surjective $\Leftrightarrow$ *X is path-connected.*

Proof. (b): ($\Leftarrow$) Let X be path-connected, and let $f : A \to X$ be given. There is a path u from f_o to o. If we shift f along u, the result is a pointed map $f^{'} : A \to X$ with $[f] = [f^{'}]$.
($\Rightarrow$) Let ν be surjective, $f_x : A \to \{x\} \subset X$ and $\nu[f^{'}]^o = [f_x]$. If we restrict the homotopy from $f^{'}$ to f_x to $o \times I$, we obtain a path from o to x.
(a): ($\Rightarrow$) Let ν be injective. Then, by the previous theorem, for $z \in \pi_1(X, o)$, $x \in [A, X]^o$, $\nu(\widehat{z}x) = \nu(x)$, hence $\widehat{z}x = x$; i.e., $\widehat{z}$ acts trivially for every $z \in \pi_1(X, o)$.
($\Leftarrow$) Let $\nu x = \nu y$. Then there exists some $z \in \pi_1(X, o)$ with $\widehat{z}x = y$. If $\pi_1(X, o)$ acts trivially, then $x = y$; and hence ν is injective. □

Definition 3.9. X is called *n-simple* if X is $\mathbb{S}^n$-simple. X is called *simple* if X is A-simple for every well-pointed space A. Here, $A \in \mathrm{Obj}(\mathcal{T\!op}^o)$ is called *well-pointed* if $\{o\} \in A$ is a closed cofiberation.

In this context, the following notion should also be mentioned: $A \in \mathrm{Obj}(\mathcal{T\!op}^o)$ is called *h-well-pointed* if $\{o\} \in A$ is an h-cofibration.

3.1.4 Example

Every element of $\pi_1(X, o)$ can be represented by a map $u : [0, 1] \to X$ with $u(0) = u(1) = o$. If we combine maps $\mathbb{S}^1 \to X$ with the map $q : I \to \mathbb{S}^1$, $q(t) = (\cos 2\pi t, \sin 2\pi t)$, this induces a bijective map

$$[\mathbb{S}^1, (X, o)]^o \cong \pi_1(X, o)$$

If we identify with this map, the action of $\pi_1(X, o)$ on $[\mathbb{S}^1, (X, o)]^o$ takes the form

$$\widehat{u}[f] = [u] + [f] - [u]$$

(Proof is left as an exercise). As a consequence we get: A space X is 1-simple if and only if $\pi_1(X, o)$ is abelian and X is path-connected.

3.1.5 An h-equivalence induces a bijection.

Lemma 3.10. *Let* $\psi : X \times I \to Y$ *be a homotopy,* $\psi : \xi \simeq \eta$*, and let* $g : K \to X$ *be an object in* $\mathcal{T\!op}^K$*. Then the following diagramme is commutative:*

$$\begin{array}{ccc} & [A, (X, g)]^K & \\ {}^{\xi_*}\swarrow & & \searrow^{\eta_*} \\ [A, (Y, \xi g)]^K & \xrightarrow[\psi\circ(g\times \mathrm{id})]{} & [A, (Y, \eta g)]^K \end{array}$$

Here, ξ_* *(corresponding to* η_**) is defined by* $\xi_*[f]^K := [\xi f]^K$ *for* $f \in [A, (X, g)]^K$.

Proof. Let $[f]^K \in [A, (X, g)]^K$. Then $\psi \circ (f \times \mathrm{id})$ is a translation of ξf along $\psi \circ (g \times \mathrm{id})$. □

Theorem 3.11. *Let $\xi : X \to Y$ be an ordinary h-equivalence. Then $\xi_* : [A,(X,g)]^K \to [A,(Y,\xi g)]^K$ is bijective.*

Corollary 3.12. *If A, X, Y and ξ are pointed and ξ is a (not necessarily pointed) h-equivalence, then $\xi_* : [A,X]^o \to [A,Y]^o$ is bijective.*

Proof. Let ξ' be h-inverse to ξ and let ψ be a homotopy from $\xi'\xi$ to id_X. Then, by the lemma, the following diagramme is commutative.

$$\begin{array}{ccccc}
 & & [A,(Y,\xi g)]^K & & \\
 & \overset{\xi_*}{\nearrow} & & \overset{\xi'_*}{\searrow} & \\
[A,(X,g)]^K & & & & [A,(X,\xi'\xi g)]^K \\
 & \underset{\mathrm{id}_*}{\overset{\cong}{\searrow}} & & \underset{\psi\circ(g\times id_*)}{\overset{\cong}{\swarrow}} & \\
 & & [A,(X,g)]^K & &
\end{array}$$

So ξ'_* has a right inverse. Similarly, we see that ξ'_* has a left inverse. Hence, ξ'_* and ξ_* are bijective. □

3.1.6 Naturality of induced maps

Theorem 3.13. *Let $K \to B$ and $K \to A$ be closed cofibrations. Given $\alpha : BtoA$ in $\mathscr{T\!op}^K$, $\xi : X \to Y$ in $\mathscr{T\!op}$, and $u : K \times [0,p] \to X$ in $\mathscr{P}^K X$. Then the following diagramme is commutative.*

$$\begin{array}{ccc}
[A,(X,u_0)]^K & \xrightarrow{\ \widehat{u}\ } & [A,(X,u_p)]^K \\
{\scriptstyle [\alpha,\xi]^K}\downarrow & & \downarrow{\scriptstyle [\alpha,\xi]^K} \\
[B,(Y,\xi u_0)]^K & \xrightarrow[\ \widehat{\xi u}\ ]{} & [B,(Y,\xi u_p)]^K
\end{array}$$

Here $[\alpha,\xi]^K : [f]^K \mapsto [\xi f \alpha]^K$

Proof. If f is translated along u with φ, then $\xi f \alpha$ can be translated along ξu with $\xi\varphi(\alpha \times \mathrm{id}_{[0,p]})$. □

3.1.7 Case $i : K \to A$ s a h-cofibration.

We want to generalise §§(3.1.2) - (3.1.6) to this case. By Corollary (1.75) there exists a closed cofibration $j : K \to B$ and an h-equivalence $\alpha : B \to A$ under K. We define

$$\widehat{u} : [A,(X,u_0)]^K \to [A,(X,u_p)]^K$$

by stating that

$$\begin{array}{ccc} [A,(X,u_0)]^K & \overset{\widehat{u}}{\dashrightarrow} & [A,(X,u_p)]^K \\ \cong \big\downarrow \alpha^* & & \cong \big\downarrow \alpha^* \\ [B,(X,_0)]^K & \underset{\widehat{\xi u}}{\longrightarrow} & [B,(X,u_p)]^K \end{array}$$

is commutative (α^* is given by $\alpha^*[f]^K := [f\alpha]^K$.) By (3.1.6) it follows that this definition is independent of the choice of $j : K \to B$ and α.

The propositions, theorems, and corollaries from (3.1.2) - (3.1.6) can now be applied to the more general case. Similarly for the definition of A-simplicity: if a space is A-simple for every well-pointed space, then it is also A-simple for every h-well-pointed pace; thus, the definition of "simplicity" does not change its content.

Remark 3.14. Theorem (1.62) can be viewed as a special case of the theory presented here:
i.e, a comparison in $\mathcal{T\!op}h$ and $\mathcal{T\!op}^K h$ between

isomorphisms in Theorem (1.62), vs

morphisms here.

3.1.8 Category of pairs

For a category $\mathfrak{C}$ we have formed in (0.16) the category of pairs $\mathfrak{C}(2)$. $\mathfrak{C}(2)$ has as objects the morphisms $a : A \to A'$, $g : X \to X'$, ... of $\mathfrak{C}$ and as morphisms $a \to g$ for the pairs $(f : A \to X,\ f' : A' \to X')$ with $gf = f'a$.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ a \big\downarrow & & \big\downarrow g \\ A' & \underset{f'}{\longrightarrow} & X' \end{array}$$

We want to consider the category of pairs in particular for $\mathfrak{C} = \mathcal{T\!op}^K$. In $\mathcal{T\!op}^K(2)$ we have a notion of homotopy: A homotopy is a set (f_t, f_t'), $t \in I$, of morphisms from $\mathcal{T\!op}^K(2)$ such that f_t and f_t' are homotopies in $\mathcal{T\!op}^K$.

These concepts can obviously be generalised. We will also use the category $\mathcal{T\!op}^K(n)$ $(n \geq 1)$: objects are $(f_1, \ldots, fn-1, i)$

$$K \xrightarrow{i} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$$

and morphisms are commutative diagrammes

$$\begin{array}{ccccccccc} K & \xrightarrow{i} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & X_n \\ \big\| & & \big\downarrow & & \big\downarrow & & & & \big\downarrow \\ K & \underset{j}{\longrightarrow} & Y_1 & \underset{g_1}{\longrightarrow} & Y_2 & \underset{g_2}{\longrightarrow} & \cdots & \underset{g_{n-1}}{\longrightarrow} & Y_n \end{array}$$

The definition of a homotopy concept in $\mathcal{Top}^K(n)$ is clear.

3.1.9 Action of fundamental groupoids generalised

The action of the fundamental groupoid can be generalised to $\mathcal{Top}^K(2)$ or $\mathcal{Top}^K(n)$. We sketch this for $\mathcal{Top}^K(2)$.K

Let $K \xrightarrow{i} A \xrightarrow{j} A'$ be closed cofibrations, let $g : X \to X'$ and $u : K \times [0, p] \to X$ be continuous maps. We want to define a map

$$\widehat{u} : [(j, i), (g, u_0)]^K \to [(j, i), (g, u_p)]^K$$

between homotopy sets in $\mathcal{Top}^K(2)$.

Let $[(f, f')]^K$ be given (with $gf = f'j$, $fi = u_0$). Since i and j are cofibrations, one can first find a homotopy $\varphi : A \times [0, p] \to X$ with $\varphi_0 = f$ and $\varphi \circ (i \times \mathrm{id}) = u$ and then a homotopy $\varphi' : A' \times [0, p] \to X'$ with $\varphi'_0 = f'$ and $\varphi' \circ (j \times \mathrm{id}) = g \circ \varphi$. (φ, φ') can be called a *translation of* (f, f') *along* u. We set

$$\widehat{u}[(f, f')]^K = [(\varphi, \varphi')]^K.$$

One can be convinced that this induces a well-defined map $\widehat{u}$. Earlier theorems can also be transferred.

Theorem 3.15. *The map* $(h : K \to X) \mapsto [(j, i), (g, h)]^K$ *and* $[u] \mapsto \widehat{u} =: \widehat{[u]}$ *defines a functor*

$$\Pi^K X \to \mathcal{Sets}.$$

Theorem 3.16. *Let* $(f_0, f_0') \to (j, i) \to (g, u_0)$ *and* $(f_1, f_1') : (j, i) \to (g, u_1)$ *be morphisms in* $\mathcal{Top}^K(2)$*. Then* $[(f_0, f_0')] = [(f_1, f_1')]$ *in* $\mathcal{Top}(2)h$ *if and only if there exists some* $u \in \mathcal{P}^K X$ *from* $(X, u_0$ *to* (X, u_1) *such that*

$$[(f_1, f_1')]^K = \widehat{u}[(f_0, f_0')]^K.$$

Specifically, if K is a one-point space and j and g are pointed maps, then we are interested in

$$[j, g]^o \to [j, g].$$

If this map is bijective, then g is called *j-simple.* If j is the inclusion $\mathbb{S}^{n-1} \subset \mathbb{E}^n$, we say *$n$-simple* instead of j-simple.

(3.1.6) also has its counterpart here, which, as in (3.1.7), can be used to replace "closed cofibration" in the assumptions with "h-cofibration" (see the following section).

3.1.10 (3.1.5) and (3.1.7) generalised

While (3.1.9) almost automatically regenerates what was said earlier and we could therefore be brief, we need to go into more detail about generaliations of (3.1.5) and (3.1.7).

Theorem 3.17. *Let*

$$K \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \tag{3.18}$$

be a sequence of h-cofibrations $f <$, regarded as an object in $\mathcal{T}op^K(n)$. Then there exists an object

$$K \xrightarrow{g_0} B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} B_n \tag{3.19}$$

in $\mathcal{T}op^K(n)$ with closed cofibrations g_i that is h-equivalent to (3.18) in $\mathcal{T}op^K(n)$.

Proof. We proceed by induction on n. For $n = 1$, see 1.75.
Let the following diagramme represent an h-equivalence in $\mathcal{T}op^K(n-1)$, where $n \geq 2$:

$$\begin{array}{ccccccc}
K & \xrightarrow{f_0} & A_1 & \longrightarrow & \cdots & \longrightarrow & A_{n-1} \\
\Vert & & \downarrow{\scriptstyle h_1} & & & & \downarrow{\scriptstyle h_{n-1}} \\
K & \xrightarrow[g_0]{} & B_2 & \longrightarrow & \cdots & \longrightarrow & B_{n-1}
\end{array}$$

Let the following diagramme be a Cartesian square:

$$\begin{array}{ccc}
A_{n-1} & \xrightarrow{f_{n-1}} & A_n \\
{\scriptstyle h_{n-1}}\downarrow & & \downarrow{\scriptstyle h} \\
B_{n-1} & \xrightarrow[i]{} & B
\end{array}$$

Then i is an h-cofibration and h is an h-equivalence (see (2.132) (b), (2.137)). We replace i by a closed cofibration g_{n-1} such that k is an h-equivalence under B_{n-1}:

$$\begin{array}{ccc}
B_{n-1} & \xrightarrow{i} & B \\
& {\scriptstyle g_{n-1}}\searrow & \downarrow{\scriptstyle k} \\
& & B_n
\end{array}$$

Let $h_n = kh$. We show that $(h_1, \ldots, h_n)$ is the equivalence we are looking for. Let $(h_1', \ldots, h_{n-1}')$ be h-inverse in $\mathcal{T}op^K(n-1)$ to $(h_1, \ldots, h_{n-1})$, and let φ_{n-1} be a homotopy from $h_{n-1}' h_{n-1}$ to $\mathrm{id}_{A_{n-1}}$, which is somewhat constant. Then there is $h_n' : B_n \to A_n$ and $\varphi_n : h_n' h_n \simeq \mathrm{id}_{A_n}$ with

$$h_n' g_{n-1} = f_{n-1} h_{n-1}'$$

and

$$\varphi_n(f_{n-1}(a), t) = f_{n-1}\varphi_{n-1}(a, \min(2t, 1)).$$

From this it easily follows that $(h_1', \ldots, h_n')$ has an h-right inverse $(h_1, \ldots, h_n)$ (assume a constant homotopy at homotopies $\varphi_1, \ldots, \varphi_{n-1}$). The existence of h_n' and φ_n with the properties mentioned can be seen from the proof of theorem (1.76). □

3.1.11 The category $\mathcal{T}op(n)$

From the category $\mathcal{T}op(n)$, let the objects

$$(i_\nu): \quad A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \cdots \xrightarrow{i_{n-1}} A_n$$
$$(f_\nu): \quad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$$
$$(g_\nu): \quad Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} Y_n$$

be given and the morphism $(\xi_\nu) : (f_\nu) \to (g_\nu)$. Let i_ν be h-cofibrations, and $\xi_\nu : X_\nu \to Y_\nu$ be h-equivalences. (ξ_ν) induces a map between homotopy sets

$$(\xi_\nu)_* : [(i_\nu), (f_\nu)] \to [(i_\nu), (g_\nu)].$$

Theorem 3.20. *$(\xi_\nu)_*$ is bijective.*

Proposition 3.21 (Addition)**.** *If f_ν and g_ν are h-cofibrations, then (ξ_ν) is an h-equivalence in $\mathcal{T}op(n)$.*

We prove the theorem by induction on n. The induction step is based on the following Lemma 3.23 and 3.24. Because of (3.1.10), we can restrict ourselves to the case that the i_ν are closed cofibrations.

Lemma 3.22. *In the commutative digramme bellow let j be an h–cofibrartion and ξ be an h-equivalence.*

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\xi]{} & Y \end{array}$$

Then there is a map $F : B \to X$ with $Fj = f$ and $\xi F \simeq g$ rel A.

Proof. We consider $f : A \to X$ and $gj : A \to Y$ as objects of $\mathcal{T}op^A$. ξ induces

$$\xi_* : [(B, j), (X, f)]^A \to [(B, j), (Y, gj)]^A.$$

By (3.1.5), ξ_* is bijective. Hence, there exists F such that $\xi_* F^A = [g]^A$, Q.E.D. □

Lemma 3.23. *Given a commutative diagramme below in which i is a closed cofibration, ξ and ξ' are h-equivalences and $(\xi a_0, \xi' a_0')$ is homotopic to $(\xi a_1, \xi' a_1')$ by a homotopy (φ, φ').*

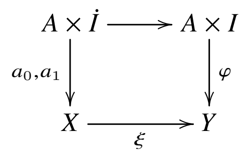
$$\begin{array}{ccccc} A & \xrightarrow{a_0, a_1} & X & \xrightarrow{\xi} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A' & \xrightarrow[a_0', a_1']{} & X' & \xrightarrow[\xi']{} & Y' \end{array}$$

Then

(a) There exists a homotopy $\Psi : a_0 \simeq a_1$ *with* $\xi\Psi \simeq \varphi \operatorname{rel} A \times I$.

(b) For every homotopy Ψ *with the property mentioned in (a) there exists a homotopy* $\Psi' : a_0' \simeq a_1'$ *with* $\xi'\Psi' \simeq \varphi' \operatorname{rel} A' \times \dot{I}$ *and* $f\Psi = \Psi' \circ (i \times \mathrm{id}_I)$.

Proof. (a): We apply Lemma 3.22 to the diagramme below:



(b): Let $\Psi : a_0 \simeq a_1$ be given and a homotopy $\Phi : \varphi \simeq \xi\Psi \operatorname{rel} A \times \dot{I}$. By assumption, furthermore, $g\varphi = \varphi'(i \times \mathrm{id}_I)$. One sees that

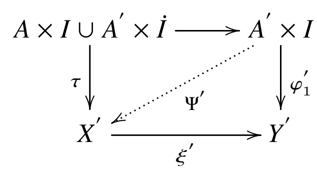
$$g\Phi, \quad \xi' a_0', \quad \xi' a_1', \quad \varphi'$$

together define a map from

$$A \times I \times I \cup A' \times 0 \times I \cup A' \times 1 \times I \cup A' \times I \times 0$$

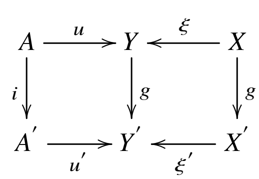
to Y' (to the middle summand independent of the I-coordinate), which can be extended to $A' \times I \times I$. Restricting this extension to $A' \times I \times 1$ gives a homotopy $\varphi_1' : \xi' a_0' \simeq \xi' a_1'$ with $\varphi_1|_{A \times I} = g\xi\Psi$ and $\varphi_1' \simeq \varphi' \operatorname{rel} A' \times I$.

We now determine, by Lemma 3.22, a Ψ' in the following diagramme



where $\tau|_{A \times I} = f\Psi$, $\tau|_{A' \times 0} = a_0'$, $\tau|_{A' \times 1} = a_1'$. Then Ψ' has the claimed properties. □

Lemma 3.24. *Let the following commutative diagrame given:*



Let i *be a closed cofibration,* ξ *and* ξ' *be h-equivalences,* $v : A \to X$ *be a map, and* $\varphi : \xi v \simeq u$ *be a homotopy. Then there exists a map* $v' : A' \to X'$ *and a homotopy* $\varphi' : \xi' v' \simeq u'$ *with* $v' i = f v$ *and* $\varphi'(ia, t) = g\varphi(a, \min(2t, 1))$.

Proof. We have bijective maps

$$[A', (X', fv)]^A \xrightarrow{\xi'_*} [A', (Y', \xi' fv)]^A \xrightarrow{g\varphi} [A', (Y', u'i)]^A.$$

Let $[v']^A$ be chosen such that $\widehat{g\varphi}\xi'_*[v']^A = [u'*]^A$. This means:

(1) $v'i = fv$.

(2) $\xi' v'$ can be translated along $g\varphi$ to a map that is homotopic to u' under A; this yields a φ'.

□

Proof. (of Theorem 3.20) We prove by Lemma 3.23 that $(\xi_\nu)_*$ is injective, and by Lemma 3.24 that $(\xi_\nu)_*$ is surjective. For injectivity, we prove the sharper claim by induction on n: If a homotopy $(\varphi_\nu) : (\xi_\nu a_\nu \simeq (\xi_\nu b_\nu)$ is given in $\mathcal{Top}(n)$, then there exists a homotopy $(\Psi_\nu) : (a_\nu) \simeq (b_\nu)$ with $\xi_\nu \Psi_\nu \simeq \varphi_\nu \operatorname{rel} X_\nu \times \dot{I}$. Lemma 3.23 clearly provides the induction start and induction step. Lemma 3.24 provides the induction step for the proof of surjectivity. □

Proof. (of the additional proposition 3.21) Since $(\xi_\nu)_* : [(g_\nu), (f_\nu)] \to [(g_\nu), (g_\nu)]$ is bijective, there exists $(\eta\nu)$ such that $(\xi_\nu \eta_\nu) \simeq \mathrm{id}$. From $(\xi_\nu \eta_\nu \xi_\nu) \simeq (\mathrm{id} \circ \xi_\nu) = (\xi_\nu \circ \mathrm{id})$ and the bijectivity of $(\xi_\nu)_*$, it follows that $(\eta_\nu \xi_\nu) \simeq \mathrm{id}$. □

3.1.12 Suggestion for further studies

We mention extensions of the theory that the reader may carry out for their own benefit.
First, the generalisation of (3.1.10) and (3.1.11) to infinite sequences.
Second, the dual situation: Let $p : E \to B$ be an h-fibration. Let $u : f \simeq g$ be a homotopy from $f : X \to B$ to $g : X \to B$. Define

$$\widehat{u} : [(X, f), E]_B \to [(X, g), E]_B.$$

Develop properties analogous to (3.1.1) - (3.1.11).

3.2 Suspension and loop space

3.2.1 Suspension

Definition 3.25. Let $X \in \mathrm{Obj}(\mathcal{Top})$. In $X \times I$, we identify both $X \times 0$ and $X \times 1$ as a point, respectively. Let $\Sigma' X$ be the resulting factor space (intuitively: a double cone over X). $\Sigma' X$ is called a *suspension* of X. Let $[x, t]$ be the image of (x, t) in $\Sigma' X$. If $f : X \to Y$ is a continuous map, then $(\Sigma' f)[x, t] = [fx, t]$ induces a well-defined continuous map $\Sigma' f : \Sigma' X \to \Sigma' Y$. We thus have a functor

$$\Sigma' : \mathcal{Top} \to \mathcal{Top}, \quad X \mapsto \Sigma' X, \quad f \mapsto \Sigma' f.$$

Σ' is compatible with homotopies and therefore induces a functor $\mathcal{T}op h \to \mathcal{T}op h$, which will again be denoted by Σ'.

Let $X \in \mathrm{Obj}(\mathcal{T}op^o)$. The factor space

$$\Sigma X = X \times I/(X \times 0 \cup X \times 1 \cup o \times I)$$

is called the *(reduced) suspension* of X. Let $[x,t]$ be the image of (x,t) in ΣX. The canonical projection $p : \Sigma' X \to \Sigma X$ is an identification. We again have functors

$$\Sigma : \mathcal{T}op^o \to \mathcal{T}op^o, \quad \Sigma : \mathcal{T}op^o h \to \mathcal{T}op^o h.$$

(The set identified to a point becomes the base point of ΣX.)

Theorem 3.26. *Let $X \in \mathrm{Obj}(\mathcal{T}op^o)$ be well-pointed (i.e., $o \to X$ be a cofibration). Then $p : \Sigma' X \to \Sigma X$ is an h-equivalence and ΣX is well-pointed.*

Proof. We consider the two cocartesian squares:

$$\begin{array}{ccccc} X \times 0 \cup X \times 1 \cup o \times I & \xrightarrow{q} & o \times I & \xrightarrow{r} & o \\ \downarrow a & & \downarrow b & & \downarrow c \\ X \times I & \longrightarrow & \Sigma' X & \xrightarrow[p]{} & \Sigma X \end{array}$$

Since a is a cofibration, so also b and then c (see (2.132) (a)). And since r is an h-equivalence, so also p (see (2.137)). □

Example 3.27. (1) $\Sigma'\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$. A homeomorphism is given by $[\boldsymbol{x}, t] \mapsto (\sin \pi t \cdot \boldsymbol{x}, \cos \pi t)$.

(2) Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be the canonical basis of $\mathbb{R}^n$. Let $\boldsymbol{e}_1$ be the base point of $\mathbb{S}^{n-1}$. A pointed homeomorphism

$$h_n : \Sigma\mathbb{S}^{n-1} \to \mathbb{S}^n$$

is described by

$$h_n[x,t] = \frac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{x}) + \cos 2\pi t \cdot \frac{\boldsymbol{e}_1 - \boldsymbol{x}}{2} + \sin 2\pi t \left| \frac{\boldsymbol{e}_1 - \boldsymbol{x}}{2} \right| \boldsymbol{e}_{n+1}$$

(We regard $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ by $\boldsymbol{e}_i \mapsto \boldsymbol{e}_i$, $i \le n$).

3.2.2 Loop space

Let $X \in \mathrm{Obj}(\mathcal{T}op^o)$. The space

$$\Omega X := \{w : I \to X | w(0) = w(1) = o\} \subset X^I$$

with the topology induced by X^I is called the *loop space* of X.

Let $PX = \{w : [0, e_w] \to X | 0 \le e_w < \infty\}$ and $\mathbb{R}^+ = [0, \infty[$. $w \mapsto (e_w, \widetilde{w} : \mathbb{R}^+ \to X)$, $\widetilde{w}(t) = w(\min(t, e_w))$ yields an injection $PX \to \mathbb{R}^+ \times X^{\mathbb{R}^+}$. Let PX preserve the induced topology. We consider the subspace

$$\Omega' X = \{w : [0, e_w] \to X | w(0) = w(e_w) = 0\}$$

of PX. $\Omega' X$ is also a kind of loop space of X; unlike ΩX, the parameter interval of a loop in $\Omega' X$ can have any length.

We define a "+" operation in $\Omega' X$ as follows: if $u : [0, e_u] \to X$ and $v : [0, e_v] \to X$ are in $\Omega' X$, then

$$v + u : [0, e_u + e_v] \to X$$

is defined by

$$(v + u)t := \begin{cases} u(t) & \text{for} \quad t \le e_u \\ v(t - e_u) & \text{for} \quad t \ge e_u \end{cases}$$

Theorem 3.28. $(\Omega' X, +)$ *is a topological monoid.*

Proof. The operation "+" is associative, and $k : [0, 0] \to X$, $k0 = o$, is a neutral element. It remains to prove the continuity of $(u, v) \mapsto v + u$. $\Omega' X$ was defined as a subspace of $\mathbb{R}^+ \times X^{\mathbb{R}^+}$.

We therefore have to prove the continuity of the two component maps in

$$((e_u, \widetilde{u}), (e_v, \widetilde{v})) \mapsto (e_u + e_v, \widetilde{v + u})$$

This is clear for the first component, because $((e_u, \widetilde{u}) \mapsto e_u$ is continuous. The second component is continuous by Theorem (2.6) if the adjoint map

$$\mathbb{R}^+ \times \Omega' X \times \Omega' X \to X \quad (t, (e_u, \widetilde{u}), (e_v, \widetilde{v})) \mapsto (\widetilde{v + u})(t)$$

is continuous. However, by definition of +, this is continuous on the closed parts $t \le e_u$ (or $t \ge e_u$) of the domain, since the evaluation map $X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X$ is continuous because $\mathbb{R}^+$ is locally compact (cf. (2.11), (2.12)). ◻

3.2.3 Comparison of ΩX and $\Omega' X$

We compare ΩX and $\Omega' X$. The inclusion $\Omega X \subset \Omega' X$ of the sets is a topological embedding; for $X^I \to \mathbb{R}^+ \times X^{\mathbb{R}^+}$, $w \mapsto (1, \widetilde{w})$, is an embedding because a left inverse exists.

Theorem 3.29. ΩX *is a deformation retract of* $\Omega' X$.

Proof. We define a homotopy $\varphi : \Omega' X \times I \to \Omega' X$ by $e_{\varphi(w,t)} = (1-t)e_w + t$ where

$$\varphi(w, t)(s) = w\left(\frac{e_w}{(1-t)e_{\lfloor w+t}} \cdot s\right), \quad e_w > 0,$$

$$\varphi(k, t)(s) = o \quad (k \text{ is the neutral element of } \Omega' X).$$

If φ is continuous, it is a homotopy with the desired properties. For the continuity of φ, the continuity of

$$\mathbb{R}^+ \times \Omega' X \times I \to X \quad (s, w, t) \mapsto \varphi(w, t)(s)$$

is crucial (cf. the continuity consideration in (3.2.2)).

This map is certainly continuous for $(s, w, t) \neq (0, k, 0)$, since $(s, e, t) \mapsto se/((1-t)e+t)$ is continuous for $e > 0$, $t \geq 0$ or $e \geq 0$, $t > 0$. For the point $(0, k, 0)$, we conclude as follows: Because of the continuity of $b : \Omega' X \times \mathbb{R}^+$, $b(u, t) = \widetilde{u}(t)$, for a neighbourhood W of $o \in X$ there exists a neighbourhood $U \times V$ of $(k, 0)$ with $b(U \times V) \subset W$. We can choose U so small that for $w \in U$, $e_w \geq 1$, and we can assume that V has the form $[0, a[$. For $(s, w, t) \in V \times U \times I$ then $\varphi(w, t)(1) \in W$. □

Remark 3.30. ΩX has as its base point the constant path $I \to \{o\} \subset X$. $\Omega X \subset \Omega' X$ is therefore not pointed. $\xi : \Omega' X \to \Omega X$, $\xi(w) = \varphi(w, 1)$, on the other hand, is pointd and an ordinary h-equivalence. If X is h-well-punctured, ξ is an h-equivalence in $\mathcal{T\!op}^o$, as follows from the next theorem.

Theorem 3.31. *If X is h-well-pointed, so are ΩX and $\Omega' X$.*

Proof. We use the local characterisation of h-cofibrations (see (1.94)). Let V be a halo of o that can be contracted to o in X. Then ΩV and $\Omega' V$ are haloes of o in ΩX and $\Omega' X$, respectively, with the same property. (If v is a halo function for V , then v', $v'(w) = \max_{t \in I} v(w(t))$, is a halo function for ΩV.) □

3.2.4 Adjointness of functors Σ and Ω

We recall the adjunction

$$\mathcal{T\!op}(X \times I, Y) \cong \mathcal{T\!op}(X, Y^I),$$

where a map $f : X \times I \to Y$ is associated with the map $\overline{f}$ defined by $\overline{f}(x)(t) = f(x, t)$ (see (2.17)). If X and Y are pointed, then $f(X \times \{0, 1\}) = \{o\}$ is equivalent to $\overline{f}(X) \subset \Omega Y$ and $f(\{o\} \times I) = \{o\}$ is equivalent to $\overline{f}(o) = o$. Thus, canonical bijections

$$\mathcal{T\!op}^o(\Sigma X, Y) \cong \mathcal{T\!op}^o(X, \Omega Y)$$
$$[\Sigma X, Y]^o \cong [X, \Omega Y]^o$$

are induced. For $Y = \Sigma X$ the identity of ΣX corresponds to a map $k : X \to \Omega \Sigma X$.

The following diagramme is commutative:

$$\begin{array}{ccc} [A, X]^o & \xrightarrow{\Sigma} & [\Sigma A, \Sigma X]^o \\ & {\scriptstyle k_*} \searrow & \downarrow {\scriptstyle \cong} \\ & & [A, \Omega \Sigma X]^o \end{array}$$

The study of Σ is thus reduced to the study of the map k.

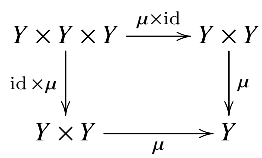
3.3 H-spaces and co-H-Spaces

3.3.1 H-spaces

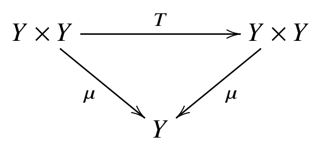
Let Y be a topological space. A continuous map

$$\mu : Y \times Y \to Y$$

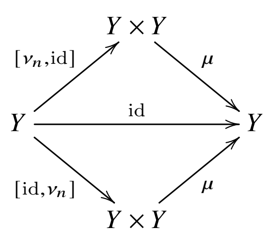
is called a *connexion* in Y. μ is called h-*associative* if the following diagramme is commutative up to homotopy:



Let $T : Y{\times}Y \to Y{\times}Y$ be the interchange of the factors, $T(x, y) = (y, x)$. μ is called h-*commutative* if the following diagramme is commutative up to homotopy:



Let $n \in Y$ and $\nu_n : Y \to Y$be the map with constant value n. n is called an h-*neutral element for* μ if the following diagramme is commutative up to homotopy:



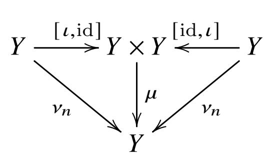
In the above diagramme, we denote by $[f, g] : A \to B \times C$ the map with the components $f : A \to B$, $g : A \to C$.)

With n, every element of the path component of n is also h-neutral for μ. If n and m are h-neutral for μ, then with $\nu_m \nu_n = \nu_m$, $\nu_n \nu_m = \nu_n$ we have

$$\nu_m \simeq \mu[\nu_n, \mathrm{id}]\nu_m = \mu[\nu_n \nu_m, \nu_m] = \mu[\nu_n, \nu_m] = \mu[\nu_n, \nu_m \nu_n] = \mu[\mathrm{id}, \nu_m] \simeq \nu_n;$$

So n and m lie in the same path component of Y.

Let μ be a conjunction with h-neutral element n. A map $\iota : Y \to Y$ is called an h-*inverse for* μ if the following diagramme is commutative up to homotopy:



If only the right (resp. left) triangle is h-commutative, then ι is called the h-right (resp. h-left) inverse of μ.

We can formulate analogous notions in the category $\mathcal{T}\!op^o$. For a connexion, $\mu(o,o) = o$ holds, and an h-neutral element is necessarily the constant mapto the base point o. The above diagrammes must be commutative up to pointed homotopy.

Definition 3.32. A pair (Y,μ) consisting of $Y \in \mathrm{Obj}(\mathcal{T}\!op)$ and a connexion μ in Y that has an h-neutral element is called an *H-space*. The term *pointed H-space* is defined analogously.

Remark 3.33. For the category-theoretical aspect of these conceptual formations see Brinkmann-Puppe [4], 7. We are primarily concerned here with the geometric side of the theory.

Example 3.34. (1) Topological groups.

(2) Topological monoids. In particular, $\Omega' X$ with the connexion $\mu(u,v) = v + u$ (see (3.2.2)). $(\Omega' X, \mu)$ is even a pointed H-space with a strictly associative connexion. Furthermore, the pointed map

$$\iota : \Omega' X \to \Omega' X, \quad \iota w = -w, \quad -w(t) := w(e_w - t),$$

is a pointed h-inverse for μ. The homotopy

$$\varphi : \Omega' X \times I \to \Omega' X, \quad \varphi(w,t) = -(w|_{[0,te_w]}) + w|_{[0,te_w]},$$

for example, shows that ι is an h-right inverse.

(3) ΩX with the operation $\mu(u,v) = (v+u)$ is an h-associative H-space with an h-inverse (in $\mathcal{T}\!op^o$). Proof is left as an exercise.

3.3.2 Induced connexions

A connexion μ in Y induces a connexion

$$\mu_* : [A,Y] \times [A,Y] \to [A,Y]$$

for every A : we set

$$\mu_*([f],[g]) = [\mu \circ [f,g]].$$

If μ is h-associative (resp. h-commutative), then μ_* is associative (resp. commutative).

If n is h-neutral for μ, then the class of the constant map $\nu_A : A \to Y$ with value n is a neutral element for μ_*. If ι is an h-inverse for μ, then $[\iota f]$ is the inverse of $[f] \in [A,Y]$ with respect to μ_*. All of these statements are easily verified (see Brinkmann-Puppe [4], 7.6). The same applies to pointed connexions and homotopy sets.

Examples (2) and (3) from (3.3.1 give the following conclusion: Let $A \in \mathrm{Obj}(\mathcal{T\!op})$, $B \in \mathrm{Obj}(\mathcal{T\!op}^o)$, $X \in \mathrm{Obj}(\mathcal{T\!op}^o)$. Then

$$[A, \Omega Z], \quad [A, \Omega' X], \quad [B, \Omega X]^o, \quad [B, \Omega' X]^o$$

"are" groups.

The connexion μ_* is *natural*, i.e., if $\alpha : B \to A$ is a continuous map, then

$$\alpha^* : [A, Y] \to [B, Y]$$

is a homomorphism with respect to μ_*. If μ has an h-neutral element, then $\alpha*$ preserves the neutral elements, $\alpha^*[\nu_A] = [\nu_B]$.

3.3.3 Loop space morphisms

Let μ be a connective in Y and μ' be a connexiion in Y'. For $\xi : Y \to Y'$ let the following diagramme be h-commutative.

$$\begin{array}{ccc} Y \times Y & \xrightarrow{\mu} & Y \\ {\scriptstyle \xi\times\xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ Y' \times Y' & \xrightarrow[\mu']{} & Y' \end{array}$$

We then say that ξ is a *homomorphism up to homotopy* from (Y, μ) to (Y', μ').

The induced morphism

$$\xi_* : ([A, Y], \mu_*) \to ([A, Y'], \mu'_*)$$

is a homomorphism. Neutral elements are not preserved without an additional condition, but they are preserved in the pointed case.

Theorem 3.35. *The map $\xi : \Omega' X \to \Omega X$, $\xi(w) = w_I$, defined in (3.2.3), is a pointed homomorphism up to homotopy.*

Proof. We have to show that the maps $(u, v) \mapsto (v + u)_I$ and $(u, v) \mapsto (v_I + u_I)_I$ are pointed homotopic. A homotopy

$$\varphi : \Omega' X \times \Omega' X \times I \to \Omega X$$

is given by

$$\varphi(u, v, t) = (v_t + u_t)_I$$

where u_t is the path

$$u_t(s) = u\left(\frac{s e_u}{1 - t + t e_u}\right), \quad 0 \le s \le 1 - t + t e_u := e_{u_t}.$$

□

Corollary 3.36. *(1)) $\xi_* : [A, \Omega' X] \to [A, \Omega X]$ is an isomorphism for every A.*

(2)) $\xi_* : [B, \Omega' X]^o \to [B, \Omega X]^o$ is an isomorphism if B or X is h-well-pointed.

Proof. ξ is a homotopy equivalence. ξ is a pointd h-equivalence if X is h-well-pointed (see (3.2.3)). If B is h-well-pointed, (3.1.7) applies. □

3.3.4 Well-pointed H-spaces

Theorem 3.37. *Let (Y, μ) be an H-space with h-neutral element n. Let $\{n\} \to Y$ be an h-cofibration. Then there is a map μ' homotopic to μ such that (Y, μ') is a pointed H-space. $[\mu']^o$ is uniquely determined by μ. If μ is associative (resp. commutative) up to homotopy, then μ' is associative (resp. commutative) up to pointed homotopy. If $\iota : Y \to Y$ is h-neutral for μ', then there exists a pointed $\iota' : Y \to Y$ that is h-neutral for μ'; $[\iota']^o$ is uniquely determined by $[\iota'] = [\iota]$.*

Proposition 3.38. *If, moreover, $Y \vee Y := Y \times \{n\} \cup \{n\} \times Y \subset Y \times Y$ is an h-cofibration, then there exists μ' with*

$$\mu'(n, y) = \mu'(y, n) = y \quad \text{for all} \quad y \in Y.$$

Then n is a (strictly) neutral element for μ'. The condition on $Y \vee Y \subset Y \times Y$ is satisfied, for example, if $\{n\} \to Y$ is a closed cofibration (see (1.104)).

Proof. Let $\alpha, \beta : Y \to Y$ be defined by $\alpha(y) = u(y, n)$, $\beta(y) = u(n, y)$. Since n is h-neutral for μ, there exist homotopies $c\varphi : \alpha \simeq \mathrm{id}$, $\psi : \beta \simeq \mathrm{id}$. We use the paths u and v, which are defined by $u(t) = \varphi(n, t)$, $v(t) = \psi(n, t)$.

Using the notations from §3.1, then

$$[\alpha]^o = (-u)^{[\mathrm{id}]^o}, \quad [\beta]^o = (-v)^{[\mathrm{id}]^o}$$

(It is clear which points are to be considered base points.). □

3.3.5 Action of π_1 on pointed H-spaces

Theorem 3.39. *Let (Y, μ) be an H-space with neutral element n. Let Y be pointed by a base point that lies in the path component of n. Let A be h-well-pointed. Then $\pi_1(Y)$ acts trivially on $[A, Y]^o$.*

Proof. We choose a pointed map $\xi : Y_1 \to Y$ that is an ordinary h-equivalence with inverse n and such that $o \to Y$ is a closed cofibration. Y_1 is an H-space with the cconnexion $\mu_1 = \eta \circ mu \circ (\xi \times \xi)$ and the h-neutral element o. $\xi_* : [A, Y_1]^o \to [A, Y]^o$ is bijective (3.1.7), in particular also for $A = \mathbb{S}^1$. Because of (3.1.7), it suffices to prove the claim for $[A, Y_1]^o$. By the additional proposition (3.38), we can replace μ_1 by a connexion μ_1' that has a strictly neutral element o. Now let $f : A \to Y_1$ and $u : [0, p] \to Y$ with $u(0) = u(p) = o$ be given, $\varphi = \mu_1' \circ (f \times \mu)$ is a translation of f along u, $\varphi_p = f$. □

Corollary 3.40. *$\pi_1(Y)$ is abelian. (See (3.1.4))*

Corollary 3.41. *The maps*

$$[A, \Omega x]^o \to [A, \Omega X], \quad [A, \Omega' X]^o \to [A, \Omega' X]$$

are injective (see (3.1.3)). For h-well-pointed A and path-connected ΩX, the above four groups thus coincide.

3.3.6 Proof of Theorem 3.37

Proof. (of Theorem (3.37)): We know from (3.3.5) that for an h-well-pointed space A

$$[A, Y]^o \to [A, Y]$$

is injective. We apply this for $A = Y \times Y$ if μ is h-commutative. Then $[\mu' T]^o$ and $[\mu']^o$ have the same image $[\mu T] = [\mu]$, and are thus equal. We treat h-associative μ in a similar way. The uniqueness of $[\mu']^o$ also follows from this.

If ι is an h-inverse for μ, then

$$\iota\nu_n \simeq \mu[\mathrm{id}, \nu_n] \circ \iota\nu_n = \mu[\iota\nu_n, \nu_n] = \mu[\iota, \mathrm{id}]\nu_n \simeq \nu_n\nu_n = \nu_n.$$

Therefore, there is a path w in Y from $\iota(n)$ to n. $\iota' \in \widehat{w}[\iota]$ is a map with $\iota'(n) = n$ and $\iota'] = [\iota]$. ι' is a pointed inverse, since, for example, $[\mu'[\iota', \mathrm{id}]]^o$ and $[\nu_n]^o$ have the same image $[\nu_n]$ in $[Y, Y]$. □

3.3.7 H-spaces with an h-inverse

The following theorem states, among other things, that H-spaces in many cases have an h-inverse.

Theorem 3.42. *(a) Let (Y, μ) be an H-space and let Y have a numerable null-homotopic cover. Then the following are equivalent:*

(I) For every $x \in Y$, $\ell_x : Y \to Y$, $\ell_x(y) = \mu(x, y)$ (resp. $r_x : Y \to Y$, $r_x(y) = \mu(y, x)$) is an h-equivalence.

(II) In $([A, Y], \mu_*)$, for every $a \in [A, Y]$, the left translation (resp. right translation) is bijective.

(b) If Y is path-connected, then ℓ_x and r_x are h-equivalences.

(c) If μ is h-associative and $([\mathrm{pt}, Y], \mu_*)$ is a group, then ℓ and r are h-equivalences.

Remark 3.43 (Conclusive remarks). *(a)*: If the left translation is bijective, then there is, in particular, a right inverse. From (aII) it follows that μ has an h-right inverse (resp. h-left inverse).
(b): If μ is h-associative (resp. h-commutative and satisfies *(aII)* and there exists an h-right inverse ι_r and an h-left inverse ι_ℓ for μ, then $\iota_\ell \simeq \iota_r$ and μ has an h-inverse.
(c): If μ is h-associative, $[\mathrm{pt}, Y]$ is a group, and if Y has a numerable null-homotopic cover, then for every A, $([A, Y], \mu_*)$ is a group.

Proof. (of the theorem) *(a)*: Let $f : Y \times Y \to Y \times Y$ be the map $f(x, y) = (x, \mu(x, y))$. Note that $\mathrm{proj}_1 : Y \times Y \to Y$ is a fibration and f is a fibrewise map $\mathrm{proj}_1 \to \mathrm{proj}_1$. The map f_* induced by f

$$f_* : [A, Y \times Y] = [A, Y] \times [A, Y] \to [A, Y] \times [A, Y]$$

has the form $(a, b) \mapsto (a, \mu_*(a, b))$.
(aII) $\to$ *(aI)*: From the assumption it follows that f_* is bijective. This holds for every A. Hence f is an h-equivalence and then even an h-equivalence over Y (see (2.85)); in particular, ℓ_x is an h-equivalence.
(aI) $\to$ *(aII)*: From the assumption and theorem (2.152) it follows that f is an h-equivalence, hence f_* is bijective, hence $b \mapsto \mu_*(a, b)$ is bijective.
(b): Let w be a path from x to n. Let ℓ_{w_t} be a homotopy from ℓ_x to ℓ_n. As n is h-neutral, we have $\ell_n \simeq \mathrm{id}$.
(c): Since $[\mathrm{pt}, Y]$ is a group, for every $x \in Y$ there is an x' such that $\mu(x', x)$ lies in the path component of n. Since μ is h-associative, $\ell_{x'} \circ \ell_x \simeq \ell_{\mu(x',x}$ holds. Taken together:

$$\ell_{x'} \circ \ell_x \simeq \ell_{\mu(x',x} \simeq \ell_n \simeq \mathrm{id}\,.$$

ℓ_x has an h-left inverse. Similarly, the existence of an h-right inverse follows. Thus ℓ_x is an h-equivalence. □

3.3.8 A group structure in $[\Sigma A, X]^o$

Let A and X be from $\mathrm{Obj}(\mathcal{T}\!op^o)$. We have a canonical bijection (see (3.2.4))

$$[\Sigma A, X]^o \cong [A, \Omega X]^o.$$

In (3.3.2), we introduced a group structure in $[A, \Omega X]^o$, which we can transfer to $[\Sigma A, X]^o$ using the bijection. This connexion in $[\Sigma A, X]^o$ can be described explicitly as follows:
Let $f, g : \Sigma A \to X$ be given. We define $g + f : \Sigma A \to X$ by

$$(g+f)[a,t] = \begin{cases} f[a,2t], & t \le \frac{1}{2} \\ g[a,2t-1] & t \ge \frac{1}{2}. \end{cases}$$

The connexion is given by $[g]^o + [f]^o = [g + f]^o$. We give another description of the connexion in $[\Sigma A, X]^o$. If $X, Y \in \mathrm{Obj}(\mathcal{T}\!op^o)$, we denote by $X \vee Y$ their sum (= their coproduct) in $\mathcal{T}\!op^o$.

Given $f : X \to Z$, $g : Y \to Z$ from $\mathcal{T}\!op^o$, let $\langle f, g\rangle : X \vee Y \to Z$ be the map that is equal to f on X and g on Y. Let $i_1, i_2 : \Sigma A \to \Sigma A \vee \Sigma A$ be the injections of the summands. With the map

$$\gamma : \Sigma A \to \Sigma A \vee \Sigma A,$$

defined by

$$(g+f)[a,t] = \begin{cases} f[a,2t], & t \le \frac{1}{2} \\ g[a,2t-1] & t \ge \frac{1}{2}. \end{cases}$$

we have $[g]^o + [f]^o = [g + f]^o = [\langle f, g\rangle \circ \gamma]^o$.

3.3.9 Co H-spaces

The relationships from (3.3.8) lead to the following definitions, which are dual to those from (3.3.1). Let C be a pointed space. A continuous map (from $\mathcal{T}op^o$)

$$\gamma : C \to C \vee C$$

is called a *co-connexion* in C. γ is called *h-associative* if the following diagramme is commutative up to pointed homotopy.

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C \vee C \\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\gamma\vee\mathrm{id}} \\ C \vee C & \xrightarrow[id\vee\gamma]{} & (C \vee C) \vee C = C \vee (C \vee C) \end{array}$$

γ has an h-*neutral element* if, with the constant map $\nu : C \to C$, the following diagram is commutative up to pointed homotopy.

$$\begin{array}{ccccc} & & C \vee C & & \\ & \overset{\gamma}{\nearrow} & & \overset{\langle\nu,\mathrm{id}\rangle}{\searrow} & \\ C & & \xrightarrow{\quad\mathrm{id}\quad} & & C \\ & \underset{\gamma}{\searrow} & & \underset{\langle\mathrm{id},\nu\rangle}{\nearrow} & \\ & & C \vee C & & \end{array}$$

The reader should formulate when γ is called *h-commutative* and when an *h-inverse* exists.

Definition 3.44. A pair (C, γ) consisting of $C \in \mathrm{Obj}(\mathcal{T}op^o)$ and a co-connection γ in C with an h-neutral element is called a *(pointed) co-H-space*.

From (3.3.8) we see that $(\Sigma A, \gamma)$ is an h-associative co-H-space. γ has an h-inverse.

Other concepts that refer to H-spaces can also be transferred to Co-H-spaces. Thus, $\alpha : BtoA$ from $\mathcal{T}op^o$ induces a homomorphism $\Sigma\alpha : \Sigma B \to \Sigma A$ of Co-H-spacems, and $\Sigma\alpha$ induces a homomorphism for every $X \in \mathrm{Obj}(\mathcal{T}op^o)$

$$(\Sigma\alpha)^* : [\Sigma A, X]^o \to [\Sigma B, X]^o, \quad (\Sigma\alpha)^*[f]^o = [f \circ \Sigma\alpha]^o.$$

This homomorphism is "natural in X". We further have a commutative diagramme of homomorphisms

$$\begin{array}{ccc} [A, \Omega X]^o & \xrightarrow{\alpha^*} & [B, \Omega X]^o \\ {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\ [\Sigma A, X]^o & \xrightarrow[(\Sigma\alpha)^*]{} & [\Sigma B, X]^o \end{array}$$

Similarly, $\xi : X \to Y$ from $\mathcal{T}op^o$ yields a commutative diagramme of homomorphisms

$$\begin{array}{ccc} [\Sigma A, X]^o & \xrightarrow{\xi_*} & [\Sigma A, Y]^o \\ \cong\big\downarrow & & \big\downarrow\cong \\ [A, \Omega X]^o & \xrightarrow[(\Omega\xi)_*]{} & [A, \Omega Y]^o \end{array}$$

$\Omega\xi$ is a homomorphism of H-spaces.

3.3.10 Pointed co H-spaces

Theorem 3.45. *Let (C, γ) be a pointed co-H-space and (M, μ) a pointed H-space. Then the connectives induced by γ and μ in $[C, M]^o$ are identical and are commutative and associative.*

Proof. We will resort to the Eckmann-Hilton argument.
We write the connexions in $[C, M]^o$ as $+_\gamma$ and $+_\mu$. We work in $\mathcal{T}op^o$. $0 : C \to M$, represented by the constant map, is neutral for both connexions. We have the projection $p_k : M \times M \to M$ onto the factors and the injections $i_\ell : C \to C \vee C$ of the summands $(k, \ell = 1, 2)$. For $f : C \vee C \to M \times M$, we set $f_{k\ell} = p_k \circ f \circ i_\ell$. It is $\mu f = p_1 f +_\mu p_2 f$ and $f\gamma = fi_1 +_\gamma fi_2$. It follows that

$$(f_{11} +_\gamma f_{12}) +_\mu (f_{21} +_\gamma f_{22}) = (\mu f)\gamma = \mu(f\gamma) = (f_{11} +_\mu f_{21}) +_\gamma (f_{12} +_\mu f_{22}).$$

If we set $f_{12} = f_{21} = 0$, the coonnexions are equal. This yields commutativity if we set $f_{11} = f_{22} = 0$. $f_{12} = 0$ demonstrates associativity. □

Corollary 3.46. *The two group structures in $[\Sigma A, \Omega X]^o$ are equal and abelian.*

From this, we obtain, by the adjunction $[\Sigma A, \Omega X]^o \cong [\Sigma^2 A, X]^o$ ($\Sigma^2 A = \Sigma(\Sigma A)$), that the two connexions in $[\Sigma^2 A, X]^o$, which we will describe shortly, are equal. Namely: Given $f, g : \Sigma^2 A \to X$, we can form

$$(g +_1 f)[a, s, t] = \begin{cases} f[a, 2s, t], & \text{for} \quad s \leq \frac{1}{2} \\ g[a, 2s - 1, t] & \text{for} \quad s \geq \frac{1}{2} \end{cases}$$

(comes from ΣA) and

$$(g +_{22} f)[a, s, t] = \begin{cases} f[a, s, 2t], & \text{for} \quad t \leq \frac{1}{2} \\ g[a, s, 2t - 1] & \text{for} \quad t \geq \frac{1}{2} \end{cases}$$

(comes from ΩX).

Corollary 3.47. $\Sigma_* : [\Sigma A, X]^o \to [\Sigma^2 A, \Sigma X]^o$ *is a homomorphism.*

Proof. $\Sigma(g +_1 f) = \Sigma g +_1 \Sigma f$. Following the previous remark, we can use $+_1$ as a connexion. □

Corollary 3.48. *The two connections in $\Omega^2 X$ are homotopic and h-commutative. The same applies to the co-connections in $\Sigma^2 A$.*

A proof is based on the remark that a natural operation in $[Y,Z]^o$ "in Y" induces an H-space structure in Z that is h-commutative (resp. h-associative, resp. ...) if the connexion in $[Y,Z]^o$ is commutative (resp. associative, resp. ...). For a detailed treatment of these questions: see Brinkmann-Puppe [4], 7.8.

3.3.11 Well-pointed co H-spaces

Let (C,γ) be a Co-H-space. Let C be h-wellpointed. Let X be a topological space and $u : [0,p] \to X$ be a path.

Theorem 3.49. $\widehat{u} : [C, X, (u(0))]^o \to [C, (X, u(p))]^o$ *is a homomorphism.*

Proof. First, let $\{o\} \subset C$ be a closed cofibration. Let $f, g : C \to (X, u(0))$ be given and let φ (resp. ψ) be a translation of f (resp. g) along u. Then $\langle \varphi_t, \psi_t \rangle \circ \gamma$ defines a translation of $g + f$ along u with the end $\varphi_p + \psi_p *$.

If C is only assumed to be h-well-pointed, then we can find a co-H-space (C', Y') and a pointed h-equivalence $C' \to C$, which is a homomorphism of co-H-spaces up to homotopy (cf. proof of (3.39), beginning) and where further $\{o\} \subset C$ is a closed cofibration. The claim in this case follows from (3.1.7). □

Reference Brinkmann-Puppe [4].

3.4 Homotopy groups

3.4.1 Definition of homotopy groups

In (3.2.1) we have specified a homeomorphism

$$h_n : \Sigma \mathbb{S}^{n-1} \to \mathbb{S}^n$$

Thus, we define a homeomorphism

$$\Sigma^k \mathbb{S}^{n-k} \cong \mathbb{S}^n$$

by $h_n \circ (\Sigma h_{n-1}) \circ \cdots \circ (\Sigma^{k-1} h_{n-k+1})$.
With these fixed homeomorphisms, we have isomorphisms for $X \in \mathrm{Obj}(\mathcal{T}\!op^o)$

$$[\mathbb{S}^n, X]^o \cong [\Sigma \mathbb{S}^{n-1}, X]^o \cong [\Sigma^k \mathbb{S}^{n-k}, X]^o.$$

More precisely:
For $n > 1$, we define a group structure on $[\mathbb{S}^n, X]^o$ by the first bijection. By (3.3.10), we can use any of the k "attachment coordinates" in $[\Sigma^k \mathbb{S}^{n-k}, X]^o$ to define addition. $\Sigma^i h_{n-i}$ induces a homomorphism for $i \geq 1$.
For $n = 0$, we can identify $[\mathbb{S}^0, X]^o$ with the set of path components of X; $[\mathbb{S}^0, X]^o$ is a pointed set, with the component of the base point as the base point.

Definition 3.50. We set $\pi_n(X) = [\mathbb{S}^n, X]^o$. $\pi_n(X)$ is called the n-th *homotopy group* of the (pointed) space X.

$\pi_n(X)$ is a group for $n \geq 1$ and an abelian group for $n \geq 2$. The group structure was initially defined with a fixed homeomorphism $h_n : \Sigma\mathbb{S}^{n-1} \cong \mathbb{S}^n$. We will see later to what extent this structure is independent of the choice of a homeomorphism $\Sigma\mathbb{S}^{n-1} \cong \mathbb{S}^n$ (cf. (3.7.3)).

$\pi_1(X)$ is canonically isomorphic with the fundamental group $\pi_1(X, o)$ defined in (3.1.1) (cf. (3.1.4)).

The chain of isomorphisms

$$[\mathbb{S}^n, X]^o \cong [\Sigma^k\mathbb{S}^{n-k}, X]^o \cong [\Sigma\mathbb{S}^{n-k-1}, \Omega^k X]^o \cong [\mathbb{S}^{n-k}, \Omega^k X]^o$$

yields an isomorphism for $n > k$ (using (3.45))

$$\pi_n(X) \cong \pi_{n-k}(\Omega^k X).$$

For $n = k > 0$, one can of course define a group structure in $\pi_0(\Omega^n X)$ using the H-space structure of $\Omega^n X$; then the latter isomorphism also holds for $k = n$.

It should be clear how, for $n \geq 1$, π_n can be considered as a functor from $\mathcal{T}op^o h$ into the category of groups.

3.4.2 A modified description for the homotopy groups

We now give a modified description for the homotopy groups of a space X.

For $n \geq 1$, let

$$\begin{aligned} I^n &= \{(t_1, \ldots, t_n) | t \in I\} \\ \partial I^n &= \{(t_1, \ldots, t_n) | t_i = 0 \text{ or } 1 \text{ for at least one } i\} \subset I^n. \end{aligned}$$

We consider I^0 as the one-point space $\{z\}$, ∂I^0 as the empty set, and $I^0/\partial I^0$ as $\{o, z\}$. By the rule $o \mapsto 1$, $z \mapsto -1$, we identify $I^0/\partial I^0$ with $\mathbb{S}^0$. For $n \geq 1$, let ∂I^n be the base point of $I^n/\partial I^n$. We apply the definition of suspension (see (3.2.1)) and obtain canonical homeomorphisms

$$\Sigma(I^n/\partial I^n) \cong I^n \times I/\partial I^n \times I \cup I^n \times \partial I \cong I^{n+1}/\partial I^{n+1}.$$

Combining these homeomorphisms with those given at the beginning of the paragraph, we get (canonically)

$$I^n/\partial I^n \cong \Sigma^n(I^0/\partial I^0) \cong \Sigma^n(\mathbb{S}^0) \cong \mathbb{S}^n.$$

Elements of $\pi_n(X)$ can be represented in this way by maps

$$f : (I^n, \partial I^n) \to (X, o)$$

The group structure in $\pi_n(X)$ is induced by the rule

$$(g + f)(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, t_{i-1}, 2t_i, t_{i+1}, \ldots, t_n), & t_i \leq \frac{1}{2} \\ g(t_1, \ldots, t_{i-1}, 2t_i - 1, t_{i+1}, \ldots, t_n), & t_i \geq \frac{1}{2} \end{cases}$$

(for any i with $1 \leq i \leq n$).

3.4.3 Relative homotopy groups

Let $g : X^{'} \to X$ be from $\mathcal{T}op^o$ and $A \in \mathrm{Obj}(\mathcal{T}op^o)$. We set

$$CA = A \times I/(A \times 1 \cup o \times I)$$

and have an embedding (!)

$$i : A \cong A \times 0 \subset CA.$$

We consider the homotopy set $[i, g]^o$, i.e., in the category $\mathcal{T}op^o(2)$ (cf. (3.1.8)), homotopy classes of pairs $(f^{'}, f)$ that make the following diagramme commutative.

$$\begin{array}{ccc} A & \xrightarrow{f^{'}} & X^{'} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ CA & \xrightarrow[f]{} & X \end{array}$$

We look at the auxiliary space

$$F_g = \{(x^{'}, u) | u(0) = g(x^{'}), u(1) = o\} \subset X^{'} \times X^I.$$

To a pair $(f^{'}, f)$, we assign the map $\overline{f} : A \to F_g$, which is defined by $\overline{f}(a) = (f^{'}(a), u)$, $u(t) = f[a, t]$. It is confirmed that this induces a bijective map

$$[i, g]^o \cong [A, F_g]^o$$

If A is a suspension, $A = \Sigma A^{'}$, then we can impose a group structure on these sets. $[a, t] \mapsto (1 - t)a + te_1$ defines a homeomorphism $C\mathbb{S}^{n-1} \cong \mathbb{E}^n$ that makes the following diagramme commutative.

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \xrightarrow{\subset} & \mathbb{E}^n \\ {\scriptstyle i}\downarrow & \swarrow{\scriptstyle \cong} & \\ C\mathbb{S}^{n-1} & & \end{array}$$

We use this homeomorphism in the following definition.

Definition 3.51. We set

$$\pi_n(g) = [\mathbb{S}^{n-1} \subset \mathbb{E}^n, g]^o \cong [\mathbb{S}^{n-1} \xrightarrow{i} \mathbb{E}^n, g]^o \cong \pi_{n-1}(F_g).$$

Specifically, if $g : X^{'} \subset X$, then we also write $pi_n(X, X^{'}$ for $\pi_n(g)$ and denote $\pi_n(X, X^{'})$ as the *n-th (relative) homotopy group of the pair* $(X, X^{'})$. $\pi_n(g)$ is defined for $n \geq 1$ and "is" a group for $n \geq 2$ (abelian for $n \geq 3$).

3.4.4 Another description for the relative homotopy groups

We now give another description for the relative homotopy groups of a pair (X, X'). We have canonical homeomorphisms

$$\mathbb{S}^{n-1} \cong I^{n-1}/\partial I^{n-1}$$
$$C\mathbb{S}^{n-1} \cong I^{n-1} \times I/(\partial I^{n-1} \times I \cup I^{n-1} \times 1).$$

We put

$$J^{n-1} = \partial I^{n-1} \times I \cup I^{n-1} \times 1.$$

Then we can represent elements of $\pi_n(X, X')$ by maps

$$f : (I^n, \partial I^n, J^{n-1}) \to (X, X', o)$$

The group structure is induced by the following rule:

$$(g + f)(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, t_{i-1}, 2t_i, t_{i+1}, \ldots, t_n), & t_i \le \frac{1}{2} \\ g(t_1, \ldots, t_{i-1}, 2t_i - 1, t_{i+1}, \ldots, t_n), & t_i \ge \frac{1}{2} \end{cases}$$

for any i with $1 \le i \le n-1$. The reader should follow the path from the definition in (3.4.3) to this rule.

Proposition 3.52. *The canonical projection*

$$p : (I^n, \partial I^n, o) \to (I^n/J^{n-1}, \partial I^n/J^{n-1}, o),$$

with any $o \in J^{n-1}$ on the left, is an h-equivalence of pointed pairs.

Proof. J^{n-1} is pointedly contractible. Thus, by Theorem (1.80), applied to the category $\mathcal{Top}^o$, there are individual h-equivalences. Now apply the analogue to Theorem (1.76) for $\mathcal{Top}^o$. □

From this, one derives an isomorphism induced by p

$$\pi_n(X, X') \cong [I^n, \partial I^n; X, X']^o.$$

3.5 The fibre sequences

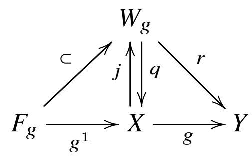
In this section let $g : X \to Y$ be a pointed map.

3.5.1

We have already considered the spaces

$$W_g = \{(x, u) | u(0) = g(x)\} \subset X \times Y^I$$
$$F_g = \{(x, u) | u(1) = o, u(0) = g(x)\} \subset W_g$$

previously ((2.46); (3.4.3)). They appear in the following diagram:



In this equation, the maps are defined by

$$r(x,u) = u(1)$$
$$q(x,u) = x$$
$$g^1 = q|_{F_g}$$
$$j(x) = (x, g(x))$$

$g(x)$ being a constant path with image $\{g(x)\}$.

The following statements hold (Theorem (2.51), applied to the category $\mathcal{T}op^o$):

$g = rj$, $qj = \mathrm{id}_X$, $jq \simeq \mathrm{id}_{W_g}$, j is an h-equivalence, r is a fibration in $\mathcal{T}op^o$.

3.5.2

Theorem 3.53. *Let $A \in \mathrm{Obj}(\mathcal{T}op^o)$ and let g, g^1 be as in (3.5.1). Then the sequence of pointed sets*

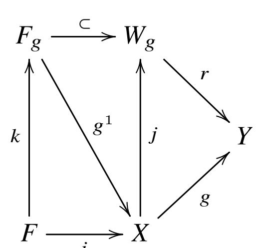
$$[A, F_g]^o \xrightarrow{g^1_*} [A, X]^o \xrightarrow{g_*} [A, Y]^o$$

is exact (i.e., $\ker g_* = \mathrm{im}\, g^1_*$, where $\ker g_* = g_*^{-1}(o)$).

Proof. $\mathrm{im}\, g^1_* \subset \ker g_*$: This holds because gg^1 is pointed null homotopic. A null homotopy $\varphi_t : F_g \times II \to Y$ is given by $\varphi_t(x,u) = u(t)$.
$\mathrm{im}\, g^1_* \supset \ker g_*$: Let $f : A \to X$ be given such that gf is pointed null homotopic. Let $\varphi : A \times I \to Y$ be a pointed null homotopy of gf. We define $f' : A \to F$ by $f'(a) = (f(a), \varphi^a)$, with the path φ^a given by $\varphi^a(t) = \varphi(a,t)$, f' is continuous and $g^1 f' = f$. □

3.5.3

Let $g : X \to Y$ be a pointed h-fibration, let $F = g^{-1}(o)$, and let $i : F \to X$ be the inclusion. In the diagramme bellow, let k be induced by j (note that $j(F) \subset F_g$ since $rj = g$).



Now $g^1 k = i.\ j$ is a pointed h-equivalence, so by Theorem (2.85), transferred to $\mathcal{Top}^o$, it is even a pointed h-equivalence over Y. Consequently, k is a pointed h-equivalence.

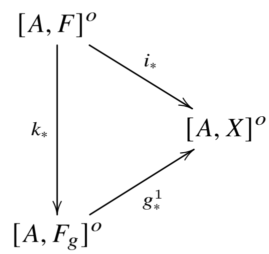
Conclusion The sequence

$$[A, F]^o \xrightarrow{i_*} [A, X]^o \xrightarrow{g_*} [A, Y]^o$$

is exact.

Remark 3.54. If one only assumes that g is an h-fibration in $\mathcal{Top}$, one can only conclude that k is an ordinary h-equivalence.

However, according to (3.1.5 and (3.1.7), $k*$ is bijective in the diagramme if A is h-well-pointed.



For this A, the sequence from the above conclusion is exact.

3.5.4

Theorem 3.55. *The map $g^1 : F_g \to X$ (defned in 3.5.1) is a pointed fibration.*

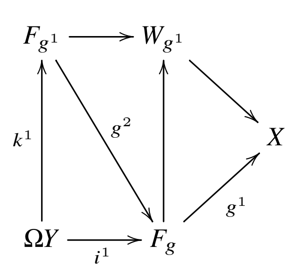
Proof. Let $WY \subset Y^I$ be the subspace of paths ending at the base point. Let $t : WY \to Y$ be the projection $tu = u(0)$. Then t is a pointed fibration and g^1 is induced from t by g. □

The space

$$(g^1)^{-1}(o) = \{(x, u) | x = o, u(0) = g(x) = o, u(1) = o\} = o \times \Omega Y$$

can be identified with ΩY. Let $i^1 : \Omega Y \to F_g$ be the embedding. Applying the construction from (3.5.3) to g^1 instead of g and then the last theorem, we obtain that

Corollary 3.56. *In the following diagramme, k is a pointed h-equivalence.*



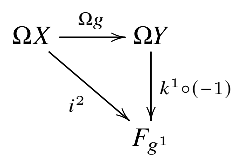
k^1 is formed analogously as k, and g^2 is formed analogously as g^1.

3.5.5

The fibre of $g^2 : F_{g^1} \longrightarrow F_g$ over the base point can be identified with ΩX; let $i^2 : \Omega X \longrightarrow F_{g^1}$ be the embedding.

In general, we want to understand $(-1) : \Omega Z \to \Omega Z$ as the map that transforms every path into its negative.

Theorem 3.57. *The following diagramme is commutative up to pointed homotopy.*



Proof. By definition we have

$$F_{g^1} = \{((x,v),u) | (x,v) \in F_g, u \in X^I, u(0) = g^1(x,v), u(1) = o\}.$$

Now, $g^1(x,v) = x$ and, by definition of F_g, $v(1) = o$ and $v(0) = g(x)$. Therefore, we can also identify F_{g^1} with the space

$$\{(v,u) | v(0) = g(u(0)), v(1) = o, u(1) = o\} \subset Y^I \times X^I$$

(by $(v,u) \mapsto ((u(0),v),u)$). The map i^2 then has the form $i^2(u) = (o,u)$ and the map $k^1 \circ (-1) \circ \Omega g$ has the form $u \mapsto (-(gu), o)$. A pointed homotopy

$$\varphi : \Omega X \times I \longrightarrow G_{g^1}$$

with $\varphi_0 = i^2$ and $\varphi_1 = k^1 \circ (-1) \circ \Omega g$ can be defined by

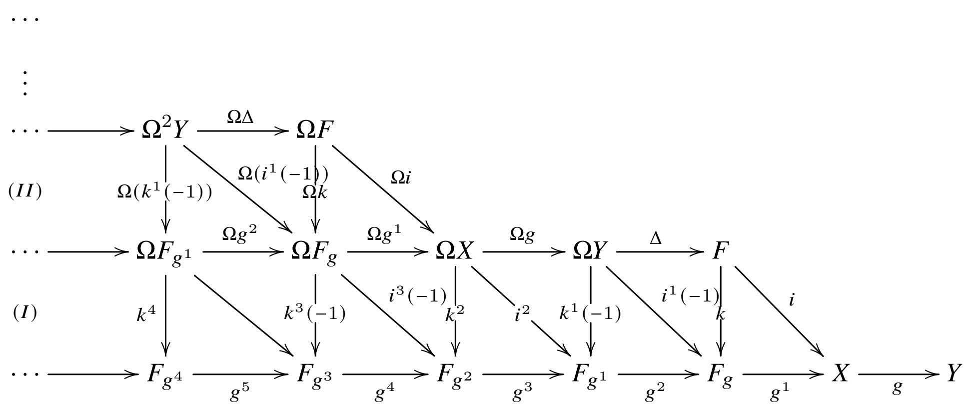
$$(u,t) \mapsto (-(gu|_{[0,t]})_I, (u|_{[t,1]})_I$$

The lower index I here again means: normalisation of the parameter interval to I. □

3.5.6

We iterate the processes described so far and obtain the following large diagramme which is h-commutative ((3.5.5), Theorem). Stage (II) follows from stage (I) by applying the functor Ω. The terms with $F, \Omega F, \ldots$ only occur if g

is an h-fibration.



The vertical maps are pointed h-equivalences (if g is a fibration in $\mathcal{T}op^o$; if g is only a fibration in $\mathcal{T}op$, then $k, \Omega k, \ldots$ are pointed maps and h-equivalences in $\mathcal{T}op$). The map Δ is chosen, perhaps by composing $i^1(-1)$ with an h-inverse of k.

If g is only an h-fibrttion in $\mathcal{T}op$, Δ will in general not be pointed, but it maps the base point back into the path component of the base point; i.e., one can choose Δ to be pointed in any case if Y and thus ΩY is h-well-pointed (cf. (3.1.2), (3.1.7), (3.2.3)).

The above discussion results in the following theorem.

Theorem 3.58. *The sequence*

$$Y \xleftarrow{g} X \xleftarrow{g^1} F_g \xleftarrow{g^2} F_{g^1} \leftarrow \cdots$$

is pointed h-equivalent to the sequence

$$Y \xleftarrow{g} X \xleftarrow{g^1} F_g \leftarrow \Omega Y \xleftarrow{\Omega g} \cdots$$

and, if g is a pointed h-fibration with fibre F, also to

$$Y \xleftarrow{g} X \xleftarrow{i} F \xleftarrow{\Delta} \Omega Y \xleftarrow{\Omega g} \cdots$$

By h-equivalent we do not mean "h-equivalent in the category of sequences" but only mean "equivalent in the category of sequences above $\mathcal{T}op^o h$".

Corollary 3.59. *For each $A \in \mathrm{Obj}(\mathcal{T}op^o)$, the following are exact:*

$$\begin{array}{ccccccccccc}
[A,Y]^o & \longleftarrow & [A,X]^o & \longleftarrow & [A,F_g]^o & \longleftarrow & [A,\Omega Y]^o & \longleftarrow & [A,\Omega X]^o & \longleftarrow & \cdots \\
 & & & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 & & & & [A \subset CA, F_g]^o & \longleftarrow & [\Sigma A, Y]^o & \longleftarrow & [\Sigma A, X]^o & \longleftarrow & \cdots
\end{array}$$

Especially, for $A = \mathbb{S}^0$ and $g : X \subset Y$ we obtain the exact sequence

$$\pi_0(Y) \leftarrow \pi_0(X) \leftarrow \pi_1(Y,X) \leftarrow \pi_1(Y) \leftarrow \pi_1(X) \leftarrow \cdots$$

From the fourth position onwards, the sequences consist of groups and homomorphisms.

Corollary 3.60. *Let either g be a pointed h-fibration or g be an h-fibration and A be h-well-punctured. Then the following are exact:*

$$\begin{array}{ccccccccccc} [A,Y]^o & \longleftarrow & [A,X]^o & \longleftarrow & [A,F]^o & \longleftarrow & [A,\Omega Y]^o & \longleftarrow & [A,\Omega X]^o & \longleftarrow & \cdots \\ & & & & & & \cong\downarrow & & \cong\downarrow & & \\ & & & & & & [\Sigma A,Y]^o & \longleftarrow & [\Sigma A,X]^o & \longleftarrow & \cdots \end{array}$$

Especially for $A = \mathbb{S}^o$ we obtain the exact sequence

$$\pi_0(Y) \leftarrow \pi_0(X) \leftarrow \pi_0(F) \leftarrow \pi_1(Y) \leftarrow \pi_1(X) \leftarrow \cdots$$

Remark 3.61. $\partial : [A,\Omega^n Y]^o \to [A,\Omega^{n-1}F]$ is induced by the map $\Omega^{n-1}\Delta$ if g is assumed to be a pointed h-fibration. In general, however, we must define ∂ by

$$[A,\Omega^n Y]^o \to [A,\Omega^{n-1}F_g]^o \overset{\cong}{\leftarrow} [A,\Omega^{n-1}F]^o$$

Remark 3.62. If one only wants to have the exact sequence of homotopy groups, one need not require that $X \to Y$ be an h-fibration. It suffices to assume that $X \to Y$ has the CHP (= covering homotopy property) for all cubes I^n, $n \geq 0$, i.e., that $X \to Y$ is a Serre fibration (cf. (2.40)).

3.5.7

Let $p : E \to B$ be a fibration and a pointed map, and let $f : B' \to B$ be a pointed map. We assume that the following diagramme is Cartesian.

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[f]{} & B \end{array}$$

E' and p' are pointed, and p' is a fibration. Let $i : A \to CA$ as in (3.4.3). The pair (p', p) induces a map

$$(p',p)_* : [i,g]^o \to [i,f]^o$$

Theorem 3.63. $(p',p)_*$ *is bijective if A is h-well-pointed.*

Proof. From (3.4.3) we see that it suffices to examine the map

$$q_* : [A, F_g]^o \to [A, F_f]^o$$

Here, $q : F_g \to F_f$ is the map $(x, u) \mapsto (p' x, pu)$. The claim follows from the following theorem.

Theorem 3.64. *If p is a fibration, then q is shrinkable. In particular, q is an h-equivalence.*

Proof. The map $p : E \to B$ yields a map $W_p : WE \to WB$ (see (3.5.4)). Let $f_1 : F_f \to WB$ be defined by $f_1(y, w) = w$ and $g_1 : F_g \to WE$ accordingly. Then the following diagramme is Cartesian.

$$\begin{array}{ccc} F_g & \xrightarrow{g_1} & WE \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle W_p} \\ F_f & \xrightarrow[f_1]{} & WB \end{array}$$

Since p is a fibration, we can construct a section s of W_p as follows: The homotopy $\varphi : WB \times I \to B$, $\varphi(w, t) = w(t)$ can be raised to a homotopy $\Phi : WB \times I \to E$ with $\Phi(w, 1) = o$. Let the adjoint map to Φ be $s : WB \to WE$. s is a section of W_p. Now, W_p is a fibration. (To prove this, consider adjoint maps and apply the following theorem of Strøm to the closed cofibration $X \times 1 \subset X \times I$.) □

Theorem 3.65 (Strøm [26], Theorem 4)**.** *Let $p : E \to B$ be a fibration and $A \subset X$ a closed cofibration. hen every commutative diagramme in $\mathcal{T\!op}$ of the form*

$$\begin{array}{ccc} (X \times 0) \cup (A \times I) & \xrightarrow{\varphi} & E \\ {\scriptstyle \subset}\downarrow & \overset{\overline{\Phi}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\Phi]{} & B \end{array}$$

can be supplemented by a homotopy $\overline{\Phi} : X \times I \to E$ such that $p\overline{\Phi} = \Phi$ and $\overline{\Phi}|_{(X\times 0)\cup(A\times I)} = \varphi$.

Furthermore, $s \circ W_p$ is an h-equivalence because WE is contractible. By Theorem (2.85), $s \circ W_p$ has an h-left inverse t over WB. In other words: ts is a section of W_p and $ts \circ W_p$ is homotopic to the identity in $\mathcal{T\!op}_{WB}$; but this just means: W_p is shrinkable. By (2.112), the shrinkability of the induced object q follows.

□

(End of proof for Theorem 3.63)

We mention the following special case.

Theorem 3.66. *Let $p : E \to B$ be a fibration and pointed, let $o \in B' \subset B$ and let $E' = p^{-1}B'$. Then p induces an isomorphism of homotopy groups*

$$\pi_n(E, E') \cong \pi_n(B, B').$$

Remark 3.67. This property of the map p above essentially serves to define the notion of "quasi-fibration." See Dold-Thom [8].

3.5.8

The dual cofibre sequence is presented in detail in Puppe [19]. There, one also finds statements about additional algebraic structures at the beginning of the sequence. For relationships between cofibre and fibre sequences, see [18].

Reference Dold-Thom [8], Nomura [18], Puppe [19].

3.6 The Blakers-Massey excision theorem

Let Y be a topological space. Let Y_1 and Y_2 be open subspaces of Y that cover Y, $Y = Y_1 \cup Y_2$. We set $Y_0 = Y_1 \cap Y_2$.

Let

$$\pi_i(Y_1, Y_0) = 0 \quad \text{for} \quad 0 < i < p, p \geq 1,$$
$$\pi_i(Y_2, Y_0) = 0 \quad \text{for} \quad 0 < i < q, q \geq 1,$$

for every choice of the base point in Y_0.

Under these conditions, we have

Theorem 3.68 (Excision Theorem)**.** *The map induced by inclusion*

$$\iota : \pi_n(Y_2, Y_0) \to \pi_n(Y, Y_1)$$

is an isomorphism for $1 \leq n < p+q-2$ and an epimorphism for $1 \leq n \leq p+q-2$.

A theorem of this kind was proved by Blakers and Massey in [1]. See also Spanier [24], p. 484.

We prove the theorem in (3.6.3). Sections (3.6.1) and (3.6.2) provide preparatory lemmata.

3.6.1

Let pairs $A' \subset A$ and $X' \subset X$ be given. A map $f : (A, A') \to (X, X')$ is called *compressible* if f is homotopic relative to A' to a map g with $g(A) \subset X'$. f is called *null-homotopic* ($f \simeq 0$) if f is homotopic as a map of pairs to a constant map k with $k(A) \subset X'$.

Lemma 3.69. *(a) Let f be compressible and A contractible. Then f is null-homotopic.*

(b) Let f be null-homotopic and $A' \subset A$ be a cofibration. Then f is compressible.

Proof. (a): Easy.
(b): By assumption, there exists a homotopy $\varphi : (A \times I, A' \times I) \to (X, X')$ from f to a constant map k. Since $A' \subset A$ is a cofibration, there exists a homotopy $\psi : A \times I \to X'$ with $\psi(a,t) = \varphi(a, 1-t)$ for $a \in A'$ and $\psi(a, 0) = k(a)$ for $a \in A$. Let $g = \psi_1$. We define $F : A \times I \to X$ by

$$F(a,t) = \begin{cases} \varphi(a, 2t) & \text{for} \quad t \leq \frac{1}{2} \\ \psi(a, 2t-1) & \text{for} \quad t \geq \frac{1}{2} \end{cases}$$

and apply the HEP to the pair (this is possible by the product theorem (1.104))

$$A' \times I \cup A \times \dot{I} \subset A \times I$$

to obtain a deformation of F to $\widetilde{F} : f \simeq g \operatorname{rel} A'$ (cf. proof of Theorem (1.62)). □

3.6.2

By an *axis-parallel cube in* $\mathbb{R}^n$, $n \geq 1$, we mean in the following a point set of the form

$$W(a, \delta, L) = W = \{x \in \mathbb{R}^n | a_i \leq x_i \leq a_i + \delta \text{ for } i \in L, a_i = x_i \text{ for } i \notin L\}$$

for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $\delta > 0$, $L \subset \{1, \ldots, n\}$ (L may be empty). A *face* of W is a set of points of the form

$$W' = \{x \in W | x_i = a_i \text{ for } i \in L_0, x_j = a_j + \delta \text{ for } j \in L_1\}$$

for some $L_0 \subset L$, $L_1 \subset L$ (W' may be empty).

By ∂W we denote the union of all proper faces of W. The following subsets of a cube W will be significant:

$$K_p(W) = \{x \in W | x_i < a_i + \frac{\delta}{2} \text{ for at least } p \text{ values of } i \in L\},$$
$$G_p(W) = \{x \in W | X_i > a_i + \frac{\delta}{2} \text{ for at least } p \text{ values of } i \in L\},$$

where $1 \leq p \leq n$. (Intuitively speaking, $K_p(W)$ is the subset of W of points for which at least p coordinates are “small”.) For $p > \dim W$, we understand $K_p(W)$ and $G_p(W)$ to be the empty set.

Lemma 3.70. *Given $A \subset Y$, $f : W \to Y$, and $p \leq \dim W$. Let*

$$f^{-1}(A) \cap W' \subset K_p(W') \text{ for all } w' \subset \partial W.$$

Then there exists a map g homotopic to f relative to ∂W with

$$g^{-1}(A) \subset K_p(W).$$

(An analogous theorem holds with G_p instead of K_p.)

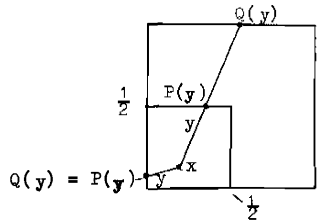


Figure 3.1:

Proof. We can assume $W = I^n$, $n \geq 1$. Let $h : I^n \to I^n$ be the following map: Let $x = (\frac{1}{4}, \ldots, \frac{1}{4})$. For a half-line y beginning at x, consider its intersection points $P(y)$ with the boundary of $[0, \frac{1}{2}]^n$ and $Q(y)$ with the boundary of I^n. h maps the segment from $P(y)$ for $Q(y)$ to the point $Q(y)$, and the segment from x to $P(y)$ affinely to the segment from x to $Q(y)$. (See Figure 3.1.)

Note that $h \simeq \mathrm{id}_I \operatorname{rel} \partial I^n$. We set $g = fh$. Let $x \in I^n$ and $g(x) \in A$. If $x_i < \frac{1}{2}$ for all i , then $x \in K_n(I^n) \subset K_p(I^n)$. If $x_1 \geq \frac{1}{2}$ for at least one i, then $h(x) \in \partial I^n$ and thus $h(x) \in W'$ with $\dim W' = n - 1$. Since $h(x) \in f^{-1}(A)$ also holds, by assumption $h(x) \in K_p(W')$. Thus, for at least p coordinates, $\frac{1}{2} > h(x)_i = \frac{1}{4} + t(x_i - \frac{1}{4})$. However, by definition of h, $t \geq 1$ (since there exists an i with $x_i \geq \frac{1}{2}$). It follows that $h(x)_i \geq x_i$; and for at least p coordinates, $\frac{1}{2} > x_i$. □

3.6.3 Proof of the Excision Theorem

We show the epimorphism for $n \leq p + q - 2$. First, we convince ourselves that it suffices to deform a map

$$f : (I^n, \partial I^n, J^{n-1}) \to (Y, Y_1, o)$$

into a map g such that

$$\operatorname{proj} g^{-1}(Y \setminus Y_2) \cap \operatorname{proj} g^{-1}(Y \setminus Y_1) = \emptyset. \tag{3.71}$$

where $\operatorname{proj} : I^n \to I^{n-1}$, $\operatorname{proj}(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$.)

If a map g with this property is given, then we choose (by Urysohn's theorem) a continuous function $\tau : I^{n-1} \to [0, 1]$, which takes the value 1 on the closed set $\operatorname{proj} g^{-1}(Y \setminus Y_2)$ and the value zero on $\partial I^{n-1} \cup \operatorname{proj} g^{-1}(Y \setminus Y_1)$. (This is possible because $g^{-1}(Y \setminus Y_2) = g^{-1}(Y_1 \setminus Y_0)$ with J^{n-1} has empty intersection.)

Let $\varphi : I^n \to I^n$ be defined by

$$\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, \tau + (1 - \tau)x_n),$$

where $\tau = \tau(x_1, \ldots, x_{n-1})$, and g_0 by $g_0 = g \circ \varphi$. Then g_0 can be regarded as a map

$$g_0 : (I^n, \partial I^n, J^{n-1}) \to (Y_2, Y_0, o)$$

The reader may want to verify $\iota[g_0] = [g]$.

We now show that there exists a map g homotopic to f with the above property. We partition I^n into subcubes W such that either $f(W) \subset Y_1$ or $f(W) \subset Y_2$ holds. Let $W_1, W_2, \dots, W_r$ be those cubes W for which $f(W) \subset Y_1$ but $f(W) \not\subset Y_2$. Similarly, let $W_1^{'}, W2^{'}, \dots, W_g^{'}$ be those W with $f(W) \subset Y_2$ but $f(W) \not\subset Y_1$. The indexing is chosen so that $\dim W_i \leq \dim W_{i+1}$ and $\dim W_i^{'} \leq \dim W_{i+1}^{'}$. We also set

$$U_i = \cup_{f(W)\in Y_i} W, \quad i = 0, 1, 2.$$

We now construct a family of maps $f_k : I^n \to Y$, $k = 0, 1, \dots, r$, with the properties:

(a) $f(W) \subset Y_i \to f_k(W) \subset Y_i$.

(b) $f_k^{-1}(Y_1 \setminus Y_0) \cap W_j \subset K_p(W_j)$ for all $j \leq k$.

(c) $f_k \simeq f$ as a map of triples.

We set $f_0 = f$. Let f_{k-1} be already constructed. For every proper side W of W_k, we have $f_{k-1}^{-1}(Y_1 \setminus Y_0) \cap W \subset Kp(W)$ by the induction hypothesis (b).

Proposition 3.72 (Intermediate claim). *There is a homotopy $\psi : W_k \times I \to Y_1 \operatorname{rel} \partial W_k$ with $\psi_0 = f_{k-1}|_{W_k}$ and*

$$\psi_1^{-1}(Y_1 \setminus Y_0) \subset K_p(W_k).$$

Proof. **Case 1:** $\dim W_k = 0$ We must connect $f_{k-1}(W_k)$ within Y_1 to a point in Y_0 (since $K_p(W_k) = \varnothing$). Since $n > 0$, there is a path in I^n from W_k to a point in J^{n-1}. The image of this path at f_{k-1} connects $f_{k-1}(W_k)$ to a point in Y_0; a suitable initial segment runs entirely in Y_1 and ends in Y_0.

Case 2: $0 < \dim W_k < p$ For every side W of W_k, $K_p(W) = 0$ and consequently, by induction hypothesis (b), $f_{k-1}(W) \subset Y_0$. We therefore obtain from f_{k-1} a map

$$(W_k, \partial W_k) \to (Y_1, Y_0).$$

Since $\pi_i(Y_1, Y_0) = 0$ for $i = \dim W_k$ (and any choice of the base point), the lemma in (3.6.1) can be applied. It yields the desired homotopy ψ.

Case 3: $\dim W_k \geq p$ We apply the lemma from (3.6.2). This proves the intermediate claim. We extend the obtained homotopy ψ to a homotopy $\Psi : I^n \times I \to Y$ of f_{k-1}, namely constant on $U_2 U W_1 \cup \cdots \cup W_{k-1}$ (this is possible because this set contains no interior points of W_k) and then recursively on $W_{k+1}, \dots, W_r$ with values in Y_1 (this is possible because $\partial W_j \subset W_j$ is a cofibration). Let $\Psi_1 = f_k$. Ψ is a homotopy $\operatorname{rel} U_2$, and since $J^{n-1} \subset U_2 j$, so is $\operatorname{rel} J^{n-1}$. $\Psi(\partial I^n \times I) \subset Y_1$. Hence Ψ is a homotopy in the category of space triples and (a), (b) and (c) are satisfied for f_k.

We set $g_0 = f_r$ and recursively construct a family $g_0, \dots, g_s$ of maps $I^n \to Y$ with the properties:

(a′) $g_0(W) \subset Y_i \to g_1(W) \subset Y_i$.

(b′) $g_1^{-1}(Y_2 \setminus Y_0) \cap W_j' \subset G_p(W_j')$ for all $j \leq 1$.

(c′) $g_1 \simeq g_0 \operatorname{rel} U_1$.

(Note that $U_1 \supset \partial I^n \supset J^{n-1}$). We define $g = g_s$. Then $g \simeq f$ is a map of triples. It remains to prove the statement (3.71) for g.

Let $y \in \operatorname{proj} g^{-1}(Y_1 \setminus Y_0)$ and say $y = \operatorname{proj}(x)$, $x \in g^{-1}(Y_1 \setminus Y_0)$, $x \in W$. Then $x \in K_p(W)$, $y \in K_{p-1}(\operatorname{proj}(W))$, i.e., y has at least $p-1$ small coordinates. Similarly, $y \in \operatorname{proj} g^{-1}(Y_2 \setminus Y_0)$ implies that y has at least $q-1$ large coordinates. Since $n-1 < p-1+q-1$, both relations cannot exist simultaneously.

We show injectivity for $n < p+q-2$. Let f and g be two maps $(I^n, \partial I^n, J^{n-1}) \to (Y_2, Y_0, o)$. Let their composition with the inclusion $u : (Y_2, Y_0, o) \to (Y, Y_1, o)$ be homotopic. We choose a homotopy

$$\varphi : (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (Y, Y_1, o)$$

between $\varphi_0 = uf$ and $\varphi_1 = ug$. It suffices to deform ϖ relative to J^n into a map Ψ that satisfies

$$t\Psi^{-1}(Y \setminus Y_2) \cap t\Psi^{-1}(Y \setminus Y_1) = \emptyset$$

where $t = \operatorname{proj} \times \operatorname{id} : I^n \times I \to I^{n-1} \times I)$. If we choose a function $\tau : I^{n-1} \times I \to [0,1]$ that is zero on $\partial(I^{n-1} \times I) \cup t\Psi^{-1}(Y \setminus Y_1)$ and equal to one on $t\psi^{-1}(Y \setminus Y_2)$, then we can consider the composition of Ψ with

$$(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_{n-1}, \tau + (1-\tau)x_n, x_{n+1})$$

as a homotopy from f to g. The deformation of φ into Ψ occurs as in the proof of epimorphism. Here we must assume $n + 1 \leq p + q - 2$.

□

3.6.4

Let $A \subset X$ be an h-cofibration, $A \neq \emptyset$. We choose a base point in A and consider the map

$$p : (X, A) \to (X/A, o)$$

that identifies A to a point, as a map of pointed space pairs.

Theorem 3.73. *Let $\pi_i(A) = 0$ for $0 \leq i \leq m$, $m \geq 1$. Let $\pi_i(X, A) = 0$ for $0 < i \leq n$. Then*

$$p_* : \pi_i(X, A) \to \pi_i(X/A, o)$$

is an isomorphism for $i \leq n + m$ and an epimorphism for $i = n + m + 1$.

Proof. We denote by $C^{'}A$ the cone $(A \times I)/(A \times 0)$ and by $X \cup C^{'}A$ the factor space $(X + C^{'}A/\sim$, by $a \sim (a, 1)$. In the co-Cartesian diagram, let P be a point space and c be the canonical homeomorphism.

$$\begin{array}{ccccc} X \cup C^{'}A & \xrightarrow{q^{'}} & (X \cup C^{'}A)/C^{'}A & \xrightarrow[c]{\cong} & X/A \\ \cup\uparrow a & & \uparrow & & \\ C^{'}A & \xrightarrow[b]{} & P & & \end{array}$$

a is an h-cofibration, b an h-equivalence, hence q' an h-equivalence and hence also $q := cq'$. (Cf. (2.137)). According to (3.20),

$$q_* : \pi_i(X \cup C^{'}A), C^{'}) \to \pi_i(X/A, o)$$

is an isomorphism (for any choice of the base point in $C^{'}A$). We set $Y = X \cup C^{'}A$, Q the vertex of the cone, $Y_2 = Y \setminus Q$, A_1 the base of the cone, and $Y_1 = C^{'}A$. Then we have the maps induced by inclusions as follows.

$$\begin{array}{ccccc} \pi_i(X, A) & \xrightarrow[\cong]{\alpha} & \pi_i(Y_2, Y_1 \cap Y_2) & \xrightarrow{\beta} & \pi_i(Y, Y_1) \\ & & \cong\uparrow & & \uparrow\cong \\ & & \pi_i(Y_2, Y_2 \cap (Y_2 \setminus A_1)) & \xrightarrow[e]{} & \pi_i(Y, Y_2 \setminus A_1) \end{array}$$

The isomorphisms shown arise from the fact that the inclusions are h-equivalences; e is isomorphic for $i \le n + m$ and epimorphic for $i = n + m + 1$ by the excision theorem. (Since A is path-connected, one can choose any point of $C^{'}A$ as the base point, in such a way that the inclusions are pointed maps.)

With the commutative diagramme below, the claim now follows.

$$\begin{array}{ccc} \pi_i(X, A) & \xrightarrow{\beta\alpha} & \pi_i(Y, C^{'}A) \\ & {\scriptstyle p_*}\searrow \quad \swarrow{\scriptstyle q_*} & \\ & \pi_i(X/A, o) & \end{array}$$

□

3.7 Sandwich theorem

In this section, let Y be an h-well-pointed space with

$$\pi_i(Y) = 0 \quad \text{for} \quad 0 \le i \le n, \ n \ge 0.$$

We investigate the suspension map (cf. (3.2.4))

$$\Sigma(X, Y) = \Sigma : [X, Y]^o \to [\Sigma X, \Sigma Y]^o.$$

3.7.1 Sphere bouquets

Let $(\mathbb{S}^j_k, k \in K)$ be a family of pointed j-spheres and $B = \vee_{k\in K}\mathbb{S}^j_k$ be their sum (coproduct) in $\mathcal{T\!op}^o$.

Theorem 3.74. *$\Sigma(B,Y)$ is bijective for $0 \le j \le 2n$ and surjective for $j = 2n+1$.*

Proof. We derive this theorem from the excision theorem in §3.6. First, we can assume that B is a sphere, since Σ and $[-,Y]^o$ are compatible with the summation. Recall that $\Sigma^{'}Y$ was the double cone of Y (3.25). We consider the subsets

$$C_-Y = \{[y,t] | t < 1\}$$
$$C_+Y = \{[y,t] | t > 0\}$$

of $\Sigma^{'}Y$. Let $p : \Sigma^{'}Y \to \Sigma Y$ be the canonical projection (3.25).

We denote by σ the composition

$$\begin{array}{ccccc}
\pi_j(Y) & \xrightarrow[c]{\cong} & \pi_j(C_-Y \cap C_+Y) & \xrightarrow[\Delta^{-1}]{\cong} & \pi_{j+1}(C_-Y, C_-Y \cap C_+Y) \\
\downarrow{\scriptstyle\sigma} & & & & \downarrow{\scriptstyle a} \\
\pi_{j+1}(\Sigma Y) & \xleftarrow[p_*]{\cong} & \pi_{j+1}(\Sigma^{'}Y) & \xleftarrow[b^{-1}]{\cong} & \pi_{j+1}(\Sigma^{'}Y, C_+Y)
\end{array}$$

where a and b are induced by inclusions, c by the map $y \mapsto [y, \frac{1}{2}]$. From the exact homotopy sequence (3.5.6) we see that a is an isomorphism since C_Y and C_Y are contractible (with direct proof for $j = 0$).

b is an isomorphism since the inclusion $(\Sigma^{'}Y, o) \to (\Sigma^{'}Y, C_+Y)$ consists individually of h-equivalences, p_* is an isomorphism since p is an h-equivalence ((3.2.1), (3.1.5)). From the excision theorem, a is an isomorphism for $j+1 < (n+2)+(n+2)-2$ and an epimorphism for $j+1 = (n+2)+(n+2)-2$.

It remains to show the equality $\sigma = \Sigma(\mathbb{S}^j, Y)$. We have the spaces $C^{'}\mathbb{S}^j = (\mathbb{S}^j \times I)/(\mathbb{S}^j \times 0)$, $C\mathbb{S}^j$, $C\mathbb{S}^j/\mathbb{S}^j = \Sigma\mathbb{S}^j$. We use the fact that we can describe elements of the homotopy groups of (A,B) by maps

$$(C^{'}\mathbb{S}^j, \mathbb{S}^j) \to (A,B)$$

or

$$(C\mathbb{S}^j, \mathbb{S}^j) \to (A,B)$$

(see §3.4). Let $f : \mathbb{S}^j \to Y$ be given. The element $c[f]$ is equal to $\Delta[g]$, where $g : C^{'}\mathbb{S}^j \to C_-Y$ is defined by $g[s,t] = [f(s), \frac{t}{2}]$. Further: $a[g]$ is represented by h, $h[s,t] = [f(s),t]$. Finally, $p_*b^{-1}[h]$ is represented by $\ell : (C\mathbb{S}^j, \mathbb{S}^j) \to (\Sigma Y, o)$, where ℓ is the composition of Σf with the projection $(C\mathbb{S}^j, \mathbb{S}^j) \twoheadrightarrow (C\mathbb{S}^j/\mathbb{S}^j, o)$. However, ℓ and Σf represent the same element. ◻

3.7.2 Cellular spaces

If $(Y_k | k \in K)$ is a family of topological spaces, then we denote by $\oplus_{k\in K} Y_k$ the topological sum (= "disjoint union") of Y_k's. We say: A space X arises from the space A by attaching n-cells ($n > 1$) if there exists a cocartesian square of the form

$$\begin{array}{ccc} \oplus_{k\in K}\mathbb{S}_k^{n-1} & \xrightarrow{f} & A \\ \cap & & \downarrow j \\ \oplus_{k\in K}\mathbb{E}_k^{n} & \xrightarrow[g]{} & X \end{array}$$

where $\mathbb{S}_k^{n-1}$ is a $(n-1)$-sphere and $\mathbb{E}_k^n$ is a n-ball.

We say: The space X has a *cellular decomposition* of dimension n if there exists a sequence

$$X^0 \subset X^1 \subset \cdots \subset X^n = X$$

of spaces such that

(a) X^0 is discrete, and

(b) X^{i+1} arises from X^i by attaching $(i+l)$ cells, $0 \le i < n$.

Theorem 3.75. *Suppose X has a cellular decomposition of dimension j. Then $\Sigma(X,Y)$ is bijective for $j \le 2n$ and surjective for $j < 2n+1$.*

Proof. We can reduce the suspension to the map $Y \to \Omega\Sigma Y$. Because of (3.7.1), the theorem to be proved follows from the following. □

Theorem 3.76. *Let $f : A \to B$ be a pointed map such that $f_* : \pi_j(A) \to \pi_j(B)$ is an isomorphism for $0 \le j < n$ and an epimorphism for $j = n$. Then $f_* : [X,A]^o \to [X,B]^o$ is bijective if X has a cellular decomposition of dimension $< n$, and surjective if X has a cellular decomposition of dimension n.*

A simple proof can be found in Spanier [24], 7.6.23, p. 405.

3.7.3 Homotopy groups of spheres

The theorems proved so far yield the following statements about the homotopy groups of spheres.

Theorem 3.77. *(a) $\pi_i(\mathbb{S}^n) = 0$ for $0 \le i < n$.*

(b) $\Sigma : \pi_i(\mathbb{S}^n) \cong \pi_{i+1}(\mathbb{S}^{n+1})$ for $i \le 2n-2$.

(c) $pi_i(\mathbb{S}^n) \cong \mathbb{Z}$, $n \ge 1$. The identity map $\mathrm{id}_{\mathbb{S}^n}$ is a generator of $\pi_n(\mathbb{S}^n)$.

Proof. (a): follows from (3.7.1) since $\pi_0(\mathbb{S}^n) = 0$ for $n > 0$.
(b): follows from (3.7.1) and (a).
(c): We see from (3.7.3) that in

$$\pi_1(\mathbb{S}^1) \xrightarrow{\Sigma} \pi_2(\mathbb{S}^2) \xrightarrow{\Sigma} \pi_3(\mathbb{S}^3) \xrightarrow{\Sigma} \cdots$$

the first map is surjective and all subsequent ones are bijective. For $n = 1$, the statement (c) can be easily derived from the consideration of the covering $\mathbb{R} \to \mathbb{S}^1$ (see, for example, Spanier [24], 1.8.12, p. 54). For the case $n = 2$, we consider the Hopf fibration $H : \mathbb{S}^3 \to \mathbb{S}^2$ with fibre $\mathbb{S}^1$ (see, for example, Steenrod [25], 20.1, p. 105): We consider $\mathbb{S}^3$ to be the set $\{(z_0, z_1)||z_0|^2 + |z_1|^2 = 1\}$ of pairs (z_0, z_1) of complex numbers and $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ to be the complex number sphere. Then H is defined by $H(z_0, z_1) = z_0/z_1$. H is locally trivial with a fibre homeomorphic to $\mathbb{S}^1$. From the piece of the exact fibre sequence

$$\pi_2(\mathbb{S}^3) \to \pi_2(\mathbb{S}^2) \to \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^3)$$

and $\pi_2(\mathbb{S}^3) = 0$, $pi_1(\mathbb{S}^3) = 0$, we see that $\pi_2(\mathbb{S}^2)$ is isomorphic to $\mathbb{Z}$. But then $\Sigma : \pi_1(\mathbb{S}^1) \to \pi_2(\mathbb{S}^2)$ must be an isomorphism. With $\Sigma[\mathrm{id}] = [\mathrm{id}]$, the claims follow. This proves (c). □

The inverse isomorphism of $\mathbb{Z} \to \pi_n(\mathbb{S}^n)$, $k \mapsto k[\mathrm{id}]$ is called *degree*, and written deg. $\deg[f] \cdot \deg[g] = \deg[f \cdot g]$ holds. In other words: If $f : \mathbb{S}^n \to \mathbb{S}^n$ has deg $= k$, then $f_* : \pi_n(\mathbb{S}^n) \to \mathbb{S}^n$ is a multiplication by k. Furthermore, the following holds:

$$f^* : \pi_n[\mathbb{S}^n, X]^o \to pi_n[\mathbb{S}^n, X]^o$$

is (in the additive group $[\mathbb{S}^n, X]^o$) the map $f^*z = kz$ if f has deg $= k$. It follows that the group structure in $[\mathbb{S}^n, X]^o$ defined in (3.4.1) is independent of the chosen homeomorphism $\mathbb{S}^n \cong \Sigma\mathbb{S}^{n-1}$ for $n \geq 2$.

Remark 3.78. Statement (b) of the last theorem is the suspension theorem proved by H. Freudenthal in 1937 (see [10]). References: Spanier [24], Steenrod [25].

3.8 The theorem of James

In this section, a space JX is constructed for a pointed space X, which (under certain conditions on X) is h-equivalent to $\Omega\Sigma X$. The space JX has the advantage over $\Omega\Sigma X$ that it can be easily and clearly constructed from X. For example, a cell decomposition of X directly yields a cell decomposition of JX.

3.8.1 The James construction

Let $X \in \mathrm{Obj}(\mathcal{T}\!op^o)$. We consider the set of finite words $x_1 \cdots x_n$ of points $x_i \in X$. On this set, we introduce the equivalence relation generated by

$$x_1 \cdots x_{i-1n} o x_i \cdots x_n \sim x_1 \cdots x_{i-1} x_i \cdots x_n$$

(where $x_1 \cdots x_n$ is arbitrary and o is the base point of X) and call the factor set JX. Let X^n be the n-fold Cartesian product of X with itself. We have a surjective map

$$p : \oplus_{n=1}^{\infty} X^n \to JX,$$

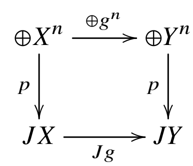
that assigns the class of the word $x_1 \cdots x_n$ to a point $(x_1 \cdots x_n)$ of the topological sum. Let JX be given the factor topology.

The set JX carries a connexion (multiplication) that is expressed on representatives by writing the words one after the other. The connexion is associative and has the class of the base point as its neutral element. JX thus becomes a monoid.

If $g : X \to Y$ is a continuous punctured map and $g^n : X^n \to Y^n$ is their n-fold product, a continuous map

$$Jg : JX \to JY$$

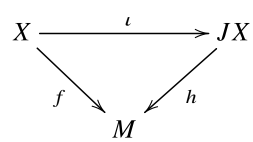
which makes the following diagramme commutative.



Jg respects monoid structures. If $\varphi_t : X \to Y$ is a pointed homotopy from φ_0 to φ_1, then $J(\varphi_t)$ is a pointed homotopy from $J(\varphi_0)$ to $J(\varphi_1)$.

Let $\iota : X \to JX$ be the map that assigns the class of the word x to the point $x \in X$. Let M be a topological monoid, i.e., a pointed topological space together with an associative continuous multiplication that has the base point as a neutral element.

Theorem 3.79 (Universal property of J)**.** *If $f : X \to M$ is a pointed continuous map, then there exists exactly one continuous map $h : JX \to M$ that makes the following diagrame commutative and respects the monoid structures.*



Proof. The uniqueness of h follows since JX is generated by image ι. The existence of h follows from the fact that the continuous map $H : \oplus X^n \to M$, $H(x_1, \ldots, x_n) = fx_1 \cdots fx_n$, can be factored over $p : \oplus X^n \to JX$. □

We now prove some topological properties of the construction J. Let $J_m(X) = p(X^m)$. The map p induces $p_m : X^m \to J_m X$. We first give JX the identification topology by p.

Lemma 3.80. *Let $a = a_1 \cdots a_k \in J_m X$. Let $a_i \neq o$ for $i = 1, \ldots, k$ and let $L_o = \{i | a_i \notin \overline{\{o\}}\}$. A neighbourhood basis of a is given by the sets $U_1 \cdots U_k U^{m-k}$ defined immediately. Let U_i be an open neighbourhood of a_i, U be an open neighbourhood of o, and let $o \notin U_i$ for $i \in L_o$. We set*

$$U_1 \cdots U_k U^{m-k} = p_m \cup_\lambda (U_1^\lambda \times \cdots \times U_1^\lambda),$$

where

$$L_o \subset L \subset \{1, \ldots, k\},$$

and $\lambda : L \longrightarrow \{1, \ldots, m\}$ *is monotone increasing and*

$$U_j^{\lambda} = \begin{cases} U_i & \textit{for} \quad j = \lambda(i) \\ U & \textit{for} \quad j \notin \operatorname{im} \lambda. \end{cases}$$

Proof. Let $a \in U_1 \cdots U_k U^{m-k}$. This set is open since

$$\cup_{\lambda}(U_1^{\lambda} \times \cdots \times U_m^{\lambda})$$

is open and saturated. Let W be an open neighbourhood of a in $J_m X$. Then $p_m^{-1} W$ is open in X^m and saturated. Let us set

$$aj^{\lambda} = \begin{cases} a_i, & \text{for} \quad j = \lambda(i) \\ o, & \text{for} \quad j \notin \operatorname{im} \lambda, \end{cases}$$

then $a_1^{\lambda} \cdots a_m^{\lambda} \in p_m^{-1} W$. There are open neighbourhoods V_j^{λ} of a_j^{λ} such that

$$V_1^{\lambda} \times \cdots \times V_m^{\lambda} \subset p_m^{-1} W.$$

Let

$$\begin{aligned} U_i' &= \cap_{\lambda} V_{\lambda(i)}^{\lambda}, \\ U &= \cap_{j \notin \operatorname{im} \lambda} V_j^{\lambda}, \\ U_i &= \begin{cases} U_i' \cap (X \setminus \overline{\{o\}}) & \text{for} \quad i \in L_o \\ U_i' & \text{for} \quad i \notin L_o. \end{cases} \end{aligned}$$

Then $a_i \in U_i$, $o \in U$ and $U_1 \cdots U_k U^{m-k} \subset W$. □

Lemma 3.81. $J_m X \subset JX$ *is a topological embedding. It is closed if* o *is closed in* X.

Proof. The following commutative diagramme shows that $J_m X \subset JX$ is continuous.

$$\begin{array}{ccc} X^m & \subset & \oplus_{n=1}^{\infty} X^n \\ \downarrow{\scriptstyle p_m} & & \downarrow{\scriptstyle p} \\ J_m X & \subset & JX \end{array}$$

Now let A be an open neighbourhood of a in $J_m X$. We construct an open neighbourhood B of a in JX such that

$$B \cap J_m X \subset A.$$

Let $a = a_1 \cdots a_k$, $a_i \neq o$ for $i = 1, \ldots, k$. By Lemma 3.80, there exists a neighbourhood of a contained in A of the form $U_1 \cdots U_k U^{m-k}$. Let

$$B' = \oplus_{n=|L_o|}^{\infty} \cup_\lambda U_1^\lambda \times \cdots \times U_n^\lambda$$

(where $L_o \subset L \subset \{1, \ldots, k\}$, $\lambda : L \to \{1, \ldots, n\}$ is monotonically increasing, $|L_o|$ is the number of elements of L_o, and U_j^λ is defined as before). B' is open and saturated in $\oplus_{n=1}^{\infty} X^n$. $B = p(B')$ is an open neighbourhood of a in JX with $B \cap J_m X \subset A$. If $\ell < k$ and o is closed in X , then $B \cap J_\ell X \neq \emptyset$. It follows that in this case $JX \setminus J_\ell X$ is open. □

Lemma 3.82. *JX is a topological direct limit of the subspaces $J_1 X \subset J_2 X \subset \cdots$.*

Proof. p can be written as

$$\oplus_{n=1}^{\infty} X^n \xrightarrow{\oplus p_n} \oplus_{n=1}^{\infty} J_n X \to JX.$$

Since p is an identification, the second map $\oplus_{n=1}^{\infty} J_n X \to JX$ is an identification. □

Lemma 3.83. *(a) For every topological space Y,*

$$P_m \times \mathrm{id}_Y : X^m \times Y \to J_m X \times Y$$

is an identification.

(b) The monoid structure on JX induces a continuous map

$$\mu : J_m \times J_n X \to J_{m+n} X.$$

Proof. (a): Let $B \subset J_m X \times Y$ and $A = (p_m \times \mathrm{id})^{-1} B$ be open in $X^m \times Y$. We need to show that B is open. Let $(a, y) \in B$ and let $a = a_1 \cdots a_k$ with $a_i \neq o$ for $i = 1, \ldots, k$. We define a_j^λ as in the proof of Lemma 3.80. Then $(a_1^\lambda, \ldots, a_m^\lambda, y) \in A$. There are open neighbourhoods V_j^λ of a_j^λ and W of y such that

$$V_1^\lambda \times \cdots \times V_m^\lambda \subset A.$$

We define U_i and U as in the proof of Lemma 3.80 and set $W = \cap W^\lambda$. Then

$$(a, y) \in U_1 \cdots U_k U^{m-k} \times W \subset B.$$

(b): From (a) it follows that

$$(p_m \times p_n) = (p_m \times \mathrm{id}) \circ (\mathrm{id} \times p_n)$$

is an identification. We have a commutative diagramme

$$\begin{array}{ccc} X^m \times X^n & \longrightarrow & X^{m+n} \\ \downarrow{\scriptstyle p_m \times p_n} & & \downarrow{\scriptstyle p_{m+n}} \\ J_m X \times J_n X & \dashrightarrow & J_{m+n} X \end{array}$$

The continuity of the dotted map follows. □

Remark 3.84. 1) $JX \to JX$, $x \mapsto xa$, $a \in JX$, is continuous.

2) $X \times JX \to JX$, $(x, y) \mapsto xy$, is *not* continuous if X is the space of rational numbers.

Lemma 3.85. *If Z is h-well-pointed, so is JZ.*

Proof. Since J is compatible with pointed homotopies, we can assume without significant restriction by (1.75) that $\{o\} \subset Z$ is a closed cofibration. We have a filtration of JZ

$$\{o\} =: J_0Z \subset J_1Z \subset J_2Z \subset \cdots$$

We first show that $J_{n-1}Z \subset J_nZ$ is a cofibration. Consider the following diagramme where q' and q are restrictions of the map p from (3.8.1).

$$\begin{array}{ccc} \cup_{i=1}^{n} Z^{i-1} \times \{o\} \times X^{n-i} & \subset & Z^n \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ J_{n-1}Z & \subset & J_nZ \end{array}$$

The upper inclusion is $(Z, \{o\})^n$, the n-fold product of the (closed) cofibration $\{0\} \to Z$ with itself, thus, according to Theorem (1.104), it is a cofibration. By returning to the definition of cofibration, one shows that the below inclusion is also a cofibration, considering:

(1) q is an identification;

(2) if two points at q have the same image, then so does the ones at q'.

$\oplus_{n=1}^{\infty} J_nZ \to JZ$ is an identification (Lemma 3.82). We again return to the definition of cofibration and conclude that $\{o\} \to JZ$ is a cofibration. □

3.8.2 The natural transformation $J \to \Omega'\Sigma$

Let $u : X \to I$ be a continuous function with $u^{-1}(0) = \{o\}$. Using u, we define a pointed map

$$f_u : X \to \Omega'\Sigma X$$

by

$$f_u(x) : [0, u(x)] \to \Sigma X$$
$$f_u(x)(t) = \begin{cases} \left[x, \frac{t}{u(x)}\right], & x \neq o, \\ o & x = o. \end{cases}$$

(See the definition of Ω' and Σ in (subsect:3-11-2), (3.2.1).) One has to convince oneself that f_u is continuous (see the proof of the next lemma). The universal property of JX (3.8.1) provides us with a map

$$h_u : JX \to \Omega'\Sigma X,$$

which makes the following diagramme commutative.

$$\begin{array}{ccc} X & \xrightarrow{\iota} & JX \\ & {\scriptstyle f_u}\searrow \quad \swarrow{\scriptstyle h_u} & \\ & \Omega^{'}\Sigma X & \end{array}$$

Lemma 3.86. *Let $g : X \to Y$ be a pointed map. Let $u : X \to I$ (resp. $v : Y \to I$) be a function with $u(0) = \{o\}$ (resp. $v(0) = \{o\}$). Then the following diagramme is commutative up to pointed homotopy (of homomorphisms).*

$$\begin{array}{ccc} JX & \xrightarrow{Jg} & JY \\ {\scriptstyle h_u}\downarrow & & \downarrow{\scriptstyle h_u} \\ \Omega^{'}\Sigma X & \xrightarrow[\Omega^{'}\Sigma g]{} & \Omega^{'}\Sigma Y \end{array}$$

Proof. We first define a map

$$\varphi : X \times I \to \Omega^{'}\Sigma Y$$

by

$$\varphi(x, s) : [0, (1 - s)u(x) + sv(g(x))] \to \Sigma Y$$

$$\varphi(x, s)(t) = \begin{cases} [gx, t/((1 - s)u(x) + sv(g(x))], & \text{for} \quad gx \neq o, \\ o & \text{for} \quad gx = o. \end{cases}$$

Claim: φ is continuous. Let the reader repeat the definition of the topology of $\Omega^{'}\Sigma Y$ (see (3.2.2)), to realise that the important thing is to prove the continuity of $X \times I \times \mathbb{R}^{+} \to \Sigma Y$, $(x, s, t) \mapsto [gx, t/((1 - s)ux + s \cdot vgx)]$. The only question is continuity at points (x, s, t) with $gx = o$. Let U be a neighbourhood of the base point of ΣY. Then there exists a neighbourhood V of o in Y such that $[v, t] \in U$ for all $(v, t) \in V \times I$. Let $W = g^{-1}V$. Then $W \times I \times \mathbb{R}^{+}$ is a neighbourhood of (x, s, t) and $\varphi(W \times I \times \mathbb{R}^{+}) \subset U$.

For the adjoint map $\overline{\varphi} : X \to (\Omega^{'}\Sigma)^{I}$ of φ, we can find a map $\overline{\psi}$ that makes the following diagramme commutative.

$$\begin{array}{ccc} X & \xrightarrow{\iota} & JX \\ & {\scriptstyle \overline{\varphi}}\searrow \quad \swarrow{\scriptstyle \overline{\psi}} & \\ & (\Omega^{'}\Sigma X^{I} & \end{array}$$

We can give $(\Omega^{'}\Sigma Y)$ the structure of a topological monoid: the product of two elements w_1, w_2 is the path defined by $t \mapsto w_2(t) + w_1(t)$. (Show that the multiplication is continuous.) The map $\overline{\psi}$ adjoint to $\psi : JX \times I \to \Omega^{'}\Sigma Y$ is a homotopy of the desired kind. $\square$

Now let $\mathfrak{U}$ be the full subcategory of $\mathscr{T}op h$ with the objects: $X \in \mathrm{Obj}(\mathfrak{U})$ if and only if there exists an isomorphism $i : X \to X^{'}$ in $\mathscr{T}op h$ and a function $u : X^{'} \to I$ exists with $u^{-1}(0) = \{o\}$.

For $X \in \mathrm{Obj}(\mathfrak{U})$ we define a pointed homotopy class

$$\eta_X : JX \to \Omega^{'}\Sigma X$$

by the commutative diagramme

$$\begin{array}{ccc} JX & \xrightarrow{\eta X} & \Omega^{'}\Sigma X \\ {\scriptstyle Ji}\downarrow & & \downarrow{\scriptstyle \Omega^{'}\Sigma i} \\ JX^{'} & \xrightarrow[{[h_u]^o}]{} & \Omega^{'}\Sigma X^{'} \end{array}$$

Here we consider J and $\Omega^{'}\Sigma$ as functors $\mathscr{T}op^o h \to \mathscr{T}op^o h$.

Lemma 3.87. *η_X does not depend on the choice of $X^{\text{‘}'}$, i, and u. The η_X, $X \in \mathrm{Obj}(\mathfrak{U})$, yield a natural transformation*

$$\eta : J|_{\mathfrak{U}} \to \Omega^{'}\Sigma|_{\mathfrak{U}}$$

of functors $\mathfrak{U} \to \mathscr{T}op^o h$.

Proof. A formal consequence of Lemma 3.86. □

3.8.3 James' theorem

We formulate *James' theorem.*

Theorem 3.88. *Let $X \in \mathrm{Obj}(\mathscr{T}op^o)$ be a space with the properties:*

(a) X is path connected.

(b) X has a numerable null-homotopic cover (see (2.5.1), (2.6.2)).

(c) X is h-well-pointed (see (3.1.3), the last paragaph).

Then $\eta_X : JX \to \Omega^{'}\Sigma X$ is an isomorphism in $\mathscr{T}op^o h$.

Remark 3.89. The theorem was originally proven by I. M. James in the following form: Let X be a countable cellular space with exactly one zero cell. Then η_X induces an isomorphism of all homotopy groups. See James [14]. A detailed comparison of our version of James's theorem and the version in which James proved it can be found in Puppe [22], pp. 52-53.

The proof of the theorem is long and is divided into several steps (3.8.3) - (3.8.9). First (in (3.8.3)), we make preparations that facilitate the actual proof: First, it suffices to show that η_X is an isomorphism in $\mathscr{T}op h$; second, we replace X by a space with a "nice" neighbourhood of the base point; in (3.8.4), we construct a larger diagramme and formulate some lemmata about the objects that appear. In (3.8.5), we prove James's theorem, assuming the lemmata from (3.8.4). These lemmata are then proved in (3.8.6) - (3.8.9).

Now for the announced preparations. With X, JX and $\Omega'\Sigma X$ are also h-well-pointed. We proved this for JX in Lemma 3.85. For $\Omega'\Sigma X$, we conclude as follows: first, with X, ΣX is also h-well-pointed. (Replace X by a well-pointed space that is isomorphic to X in $\mathscr{T\!op}^o h$ and apply the theorem from (3.2.1) (thm:3-11-1) and then $\Omega'\Sigma X$ by a theorem in (3.2.3) (3.31). If η_X is an isomorphism in $\mathscr{T\!op}^o h$, then by Theorem (1.62) it follows that η_X is an isomorphism in $\mathscr{T\!op}^o h$.

Instead of X, we now consider the space

$$X' = (I + X)/\{1, o\}.$$

("X with a *whisker* at the base point".) We choose $0 \in I$ as the new base point. If X has properties (a) to (c), as assumed in James's theorem, then so does X'. This is clear for (a) and (c). By (c), in particular, X and X' are pointed h-equivalent. Therefore, (b) for X' follows from the next Lemma.

Lemma 3.90. *If X is dominated by Y (in $\mathscr{T\!op}$) and Y has a numerable null-homotopic cover, then so is X.*

Proof. Let (V_λ) be a numerable null-homotopic cover of Y. Let $f : X \to Y$, $g : Y \to X$ be maps with $gf \simeq \mathrm{id}_X$. Then $U_\lambda = f^{-1}(V_\lambda)$ form a numerable cover of X. It is also null-homotopic. Since $U_\lambda \xrightarrow{f} V_\lambda \xrightarrow{\subset} Y \xrightarrow{g} X$ is both null-homotopic and homotopic to the inclusion $U_\lambda \subset X$. □

The space X' has a canonical function $u : X' \to I$ with $u^{-1}0 = \{o\}$, defined as the identity on the summand I and as a constant map on the summand X. From now on, we will write X instead of X'' and always understand the function $u : X \to I$ to be the one just given.

3.8.4

We first define some objects that will later appear in a large diagramme. For $Z \in \mathrm{Obj}(\mathscr{T\!op}^o)$, let the space

$$W'Z = \{w | w(0) = o\} \subset PZ$$

(cf. (3.2.2)) be the space of paths with an arbitrary parameter interval and the base point as the starting point. Let

$$r : W'Z \to Z, \quad r(w) = w(e_w),$$

be the map that assigns each path w its endpoint. Let $C'X = X \times I/((X \times 0) \cup (o \times I))$. For $JX \times C'X$, we introduce the equivalence relation defined by

$$(z, x, 1) \sim (zx, o)$$

Let the resulting factor space be Y. We consider $h = h_u : JX \to Omega^{'}\Sigma X$ and $k : C^{'}X \to W^{'}\Sigma X$; k is defined by

$$k(x,t) : [0, tu(x)] \to \Sigma X, \quad k(x,t)(s) = \left[x, \frac{s}{u(x)}\right].$$

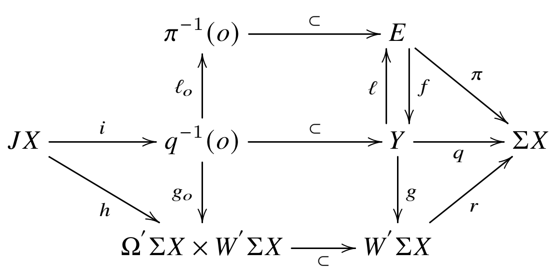
We define g by the commutative diagramme

$$\begin{CD} JX \times C^{'}X @>{h\times k}>> \Omega^{'}\Sigma X \times W^{'}\Sigma X \\ @V{\alpha}VV @VV{\beta}V \\ Y @>>{g}> W^{'}\Sigma X \end{CD}$$

Here, α is the identification just explained and β is the map

$$\beta(u,w) = w + u.$$

The proof of James' theorem is based on the following diagrammr, the individual parts of which will be explained immediately if not already done so.



Thus $q = rg$, so $q[z,x,t] = [x,t]$. And $i(z) = [z,o]$.

Lemma 3.91. *r is a fibration. $W^{'}\Sigma X$ is contractible.*

Lemma 3.92. *The map i is an h-equivalence.*

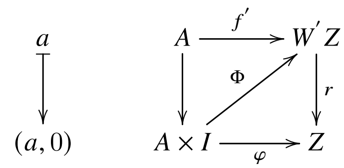
Lemma 3.93. *Y is contractible.*

Lemma 3.94. *There exists an h-fibration $\pi : E \to \Sigma X$, inclusions $\ell : Y \subset E$, $\ell_o : q^{-1}(o) \subset \pi^{-1}(o)$ with $pi\ell = q$, such that Y is a strong deformation retract of E and $q^{-1}(o)$ is a strong deformation retract of $\pi^{-1}(o)$.*

In the large diagramme, let f be a mapping with $f\ell = \mathrm{id}_Y$, $\ell f \simeq \mathrm{id}_E \,\mathrm{rel}\, Y$. We conclude this section with the

Proof. (of Lemma 3.91) Let $Z = \Sigma X$. In the diagramme, we define Φ by

$$\Phi(a,t) = \varphi^a|_{[0,t]} + f^{'}(a) \qquad (see(2.58)).$$



A contraction $\varphi : W' \times I \to W'Z$ is described by

$$\varphi(w,t) = w|_{[0,te_w]}$$

□

3.8.5 Proof of James' theorems

Proof. We return to the diagram in (3.8.4) and Lemmata 3.91 to 3.94. Since

$$\ell f \simeq \mathrm{id}_E \operatorname{rel} Y$$

we have

$$rgf = qf = \pi\ell f \simeq \pi : \operatorname{rel} Y.$$

Let $\varphi : E \times I \to \Sigma X$ be a homotopy $rgf \simeq \pi \operatorname{rel} Y$. Since r is a fibration, we can lift φ to $\Phi : E \times I \to W'\Sigma X$ with $\Phi_0 = gf$. We have $r\Phi_1 = \pi$. Since j is an h-equivalence and Y is contractible (Lemma 3.93), E is also contractible. Therefore, Φ_1 is an h-equivalence, as a map between contractible spaces, and consequently, by Theorem (2.85), an h-equivalence over ΣX. Hence, Φ_1 induces an h-equivalence

$$\psi : \pi^{-1}(o) \to \Omega'\Sigma X.$$

Since φ was a homotopy relative to Y, in particular,

$$r\Phi(q^{-1}(o) \times I) = \varphi(q^{-1}(o) \times I) = \{o\},$$

thus $\Phi(q^{-1}(o) \times I) \subset \Omega'\Sigma X$, i.e., Φ induces a homotopy

$$\Phi' : q^{-1}(o) \times I \to \Omega'\Sigma X.$$

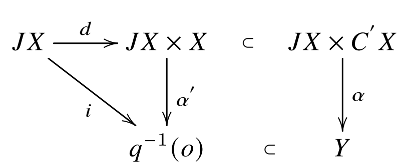
Then $\Phi_0 = g_0$, since $\Phi_0 = gf$ and $f\ell = \mathrm{id}_Y$. Consequently,

$$g_0 = \Phi'_0 \simeq \Phi'_1 = \psi\ell_0.$$

Since ψ and ℓ_0 are h-equivalences, g_0 is also an h-equivalence; and since i is also an h-equivalence (Lemma 3.92), so is h. □

3.8.6 Proof of lemma 3.92

Proof. We consider the diagramme below where d is defined by $d(z) = (z, o)$.



The map α' induced by α is an identification since $JX \times X$ is closed and saturated in $JX \times C'X$. Because of the special shape of the space X, there is an open neighbourhood U of o and a map $\rho : XtoX$ with $\rho \simeq \mathrm{id}_X \,\mathrm{rel}\, o$ and $\rho(U) = o$.

The map

$$j : JX \times X \to JX, \quad j(z, x) = (J\rho)z \circ \rho x = (J\rho)(zx)$$

is continuous since

$$V_m = \{x_1 \cdots x_n | \text{all } x \text{ up to at most } m \text{ are in } U\}$$

is open in JX because $JX = \cup_m V_m$ and because $J|_{V_m \times X}$ is continuous, as follows from the following diagramme with Lemma 3.83 (b)

$$\begin{array}{ccc} V_m \times X & \xrightarrow{j} & JX \\ {\scriptstyle J_\rho \times \rho}\downarrow & & \uparrow{\scriptstyle \mu} \\ J_m X \times X & \xrightarrow[\tau]{} & X \times J_m X \end{array}$$

where τ is the swapping of the factors. j induces j' with $j = j' \circ \alpha'$. The map j' is h-inverse to i. In fact,

$$j' i = J\rho \simeq J(\mathrm{id}_X) = \mathrm{id}_{JX}$$

and from the diagramme below, we see that the homotopy $\rho \simeq \mathrm{id}$ induces a homotopy $J\rho \times \rho \simeq \mathrm{id}$ and then $ij' \simeq \mathrm{id}$

$$\begin{array}{ccc} JX \times X & \xrightarrow{J\rho \times \rho} & JX \times X \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ q^{-1}(o) & \xrightarrow[ij']{} & q^{-1}(o). \end{array}$$

□

3.8.7 Proof of lemma 3.93

Proof. Let $Z_m = J_m X \times (C'X \setminus (X \setminus o)) \cup J_{m-1}X \times C'X$. Here, we consider $X \simeq X \times 1 \subset C'X$ as a subspace of $C'X$ (cf. (3.4.3)). We consider the diagramme in which i_m is injective and α_m is induced by α.

$$\begin{array}{ccc} Z_m & \subset & JX \times C'X \\ {\scriptstyle \alpha_m}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y_m & \xrightarrow[i_m]{} & Y \\ & {\scriptstyle g_m}\searrow & \downarrow{\scriptstyle g} \\ & & W'\Sigma X \end{array}$$

We give Y_m the identification topology by means of α_m. We define $g_m = gi_m$.

Lemma 3.95. *g_m is an embedding.*

Corollary 3.96. *i_m is an embedding.*

We first assume Lemma 3.95 and then prove Lemma 3.93.

For i_m, the contraction given in the proof of Lemma 3.91

$$\varphi : W^{'}\Sigma X \times I \to W^{'}\Sigma X$$

induces a contraction of $g(Y)$ and consequently a (perhaps non-continuous) map

$$\psi : Y \times I \to Y$$

with $g\psi(z, s) = \varphi(gz, s)$ (g is injective).

$\psi(Y_m \times I) \subset Y_m$ and the map $\psi_m : Y_m \times I \to Y_m$ induced by ψ is continuous, as follows from the diagramme

$$\begin{array}{ccc} Y_m \times I & \xrightarrow{r_m \times \mathrm{id}} & W^{'}\Sigma X \times I \\ \downarrow{\scriptstyle \psi_m} & & \downarrow{\scriptstyle \varphi} \\ Y_m & \xrightarrow[g_m]{} & W^{'}\Sigma X \end{array}$$

using Lemma 3.95. Let $\rho : X \to X$ be a map as in (3.8.6). The following diagrame uniquely defines a continuous map $rho^{'}$, and a homotopy $\rho \simeq \mathrm{id}_X$ induces a homotopy $\rho^{'} \simeq \mathrm{id}_Y$.

$$\begin{array}{ccc} JX \times C^{'}X & \xrightarrow{J\rho \times C^{'}\rho} & JX \times C^{'}X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow[\rho^{'}]{} & Y \end{array}$$

We show that $\psi^{'} = \psi(\rho^{'} \times \mathrm{id}_I) : Y \times I \to Y$ is continuous. Let $V_m \subset JX$ be as defined in (3.8.6). Then the sets $V_m \times C^{'}X \times I$ form an open cover of $JX \times C^{'}X \times I$.

It suffices to show that in the following diagramme the map $\widetilde{\psi}_m = \psi \circ (\alpha \times \mathrm{id}_I|_{V_m \times C^{'}X \times I}$ is continuous.

$$\begin{array}{ccc} V_m \times C^{'}X \times I & \subset & JX \times C^{'}X \times I \\ \downarrow{\scriptstyle \widetilde{\psi}_m} & \swarrow & \downarrow{\scriptstyle \alpha \times \mathrm{id}_I} \\ Y & \xleftarrow[\psi^{'}]{} & Y \times I \end{array}$$

$\alpha \times \mathrm{id}_I$ is an identification: this follows from the following commutative diagramme.

$$\begin{array}{ccc} V_{m-1} \times C^{'}X \times I & \xrightarrow{J\rho \times C^{'}\rho \times \mathrm{id}} & Z_m \times I \\ \downarrow{\scriptstyle \widetilde{\psi}_{m-1}} & & \downarrow{\scriptstyle \alpha \times \mathrm{id}} \\ Y \supset Y_m & \xleftarrow[\psi_m]{} & Y_m \times I \end{array}$$

It follows that

$$\mathrm{id}_Y \simeq \rho' = \psi_0' \simeq psi_1' = 0,$$

and thus Lemma 3.93 is proven. □

Proof. (of Lemma 3.95) Let $\gamma_m = g_m\alpha_m$. We have to show: If $(a, b, t_0) \in Z$ and A is an open saturated neighbourhood of (a, b, t_0), then there exists an open set B in $W'\Sigma X$ such that $(a, b, t_0) \in \gamma_m^{-1}(B) \subset A$.

To construct B, we distinguish between different cases.

Case 1 $a = a_1 \cdots a_n$, $a_i \neq o$ for $i = 1, \ldots, n$, $(b, t_0) \in C'X \setminus X$. Let $a_{n+1} = b$. There are open neighbourhoods U_i of a_i in X and U of o in X and ε with $0 < 4\varepsilon < \min(t_0, 1 - t_0$ such that $U \cap U_i = \varnothing$ and

$$U_1 \cdots U_n U^{m-n} \times U_{n+1} \times]t_0 - \varepsilon, t_0 + \varepsilon[\subset A$$

(see Lemma 3.80). Let $t_1 = t_0 + 2\varepsilon$. We define $B \subset W'\Sigma X$ as the set of all paths w with

(a) $|e_w - u(a, b, t_0)| < \delta$

(b) $w(u(a_1 \cdots a_i)) \in U \times I \cup X \times (I \setminus [\varepsilon, 1 - \varepsilon])$, $\quad i \leq n$,

(c) $w(u(a_1 \cdots a_{i-1}) + t_1 u(a_i)) \in U_i \times]t_1 - \varepsilon, t_1 + \varepsilon[$, $\quad i \leq n$,

(d) $w(u(a, b, t_0) + \delta) \in U_{n+1} \times]t_0 - \varepsilon, t_0 + \varepsilon[$.

Note that: $(a, b, t_0) \mapsto u(a, b, t_0) = \sum_{i=1}^n u(a_i) + t_0 u(b)$ defines a continuous function $u : JX \times C'X \to \mathbb{R}^+$. We understand $a_1 \cdots a_i$ for $i = 0$ to be the base point. We extend $w : [0, e_w] \to \Sigma X$ to $w : \mathbb{R}^+ \to \Sigma X$ by $w(t) = w(e_w)$ for $t \geq e_w$. Suppose $\gamma_m(a, b, t_0) \in B$ and B is open in $W'\Sigma X$ (see the definition of the topology of $W'\Sigma X$ in (3.8.4), (3.2.2) and the definition of the compact-open topology in (2.1)).

Claim: For sufficiently small U_i, ε and δ, $\gamma_m^{-1} B \subset A$.

Proof. Let $(x, y, t) \in Z_m$, $x = x_1 \cdots x_m$ and $\gamma_m(x, y, t) = w \in B$. Then by (a) $e_w < u(a, b, t_0) + \delta$ and consequently by (d) $(y, t) = w(e_w) = w(u(a, b, t_0) + \delta) \in U_{n+1} \times]t_0 - \varepsilon, t_0 + \varepsilon[$.

Because of (c), w meets the sets

$$U_1 \times]t_1 - \varepsilon, t_1 + \varepsilon[, \ldots, U_n \times]t_1 - \varepsilon, t_1 + \varepsilon[$$

in this order, and because of (b), w must run from $U_i \times]t_1 - \varepsilon, t_1 + \varepsilon[$ to $U_{i+1} \times]t_1 - \varepsilon, t_1 + \varepsilon[$ via the base point. (Note the special shape of the paths in gY!) Then there exist j_i, $1 \leq j_1 < \cdots < j_n \leq n$, such that $x_{j_i} \in U_i$. Let $j \neq j_1 < \ldots, j_n$. Then

$$\begin{aligned} u(x_j) &\leq u(x, y, t) - \sum_{i=1}^n u(x_{j_i}) - tu(y) \\ &\leq u(a, b, t_0) - \sum_{i=1}^n (a_i) - t_0 u(b) + \varepsilon' = \varepsilon' \end{aligned}$$

for given $\varepsilon' > 0$, if only δ, U_i, and ε are sufficiently small. Let ε' be chosen such that $u^{-1}[0, \varepsilon'[\subset U$. Then

$$(x, y, t) \in U_1 \cdots U_n U^{m-n} \times U_{n+1} \times]t_0 - \varepsilon, t_0 + \varepsilon[.$$

The above discussion also applies for $n = 0$, i.e., $a = o$. □

Case 2 $(b, t_0) \in X$. We have the equivalences

$$(a, b, t_0) \sim (a, o), \quad \text{if} \quad t_0 = 0 \text{ or } b = o,$$
$$(a, b, t_0) \sim (ab, o), \quad \text{if} \quad t_0 = 1.$$

We therefore only consider (a, o).

First, let $a \neq o$, so $n \geq 1$. Then $(a, o) \sim (a_1 \cdots a_{n-1}, (a_n, 1))$, where again $a_1 \cdots a_{n-1} = o$ for $n = 1$. There are open neighbourhoods U_i of a_i in X , U of o in X, and V of o in $C'X$ and an ε with $0 < \varepsilon < \frac{1}{2}$", such that

$$U_i \cap U = \emptyset, \quad Z_m \cap U_1 \cdots U_n U^{m-n} \times V \subset A, \quad U_1 \cdots U_n U^{m-n} \times]1 - \varepsilon, 1[\subset A$$

(use Lemma 3.80; for $n = 1$, set $U_1 \cdots U_n U^{m-n} = U^{m-n}$).

In the following, we do not distinguish between subsets of $X \times I$, $C'X$, and ΣX. We can assume that V has the form $V = U \times I \cup V_0$ with

$$X \times 0 \subset V_0 \subset X \times [0, \varepsilon[$$

Let

$$V_1 = \{(x, t) | (x, 1 - t) \in V_0\}, \quad \text{and} \quad V' = V \cup V_1.$$

Let $B \subset W'\Sigma X$ be the set of path w with

(a) $|e_w - u(a)| < \delta$

(b) $w(u(a_1 \cdots a_i)) \in V', \quad i \leq n,$

(c) $w(u(a_1 \cdots a_{i-1}) + \frac{1}{2}u(a_i)) \in U_i \times]\varepsilon, 1 - \varepsilon[, \quad i \leq n,$

(d) $w(u(a) + \delta) \in V'$.

Then B is open in $W'\sigma X$ and $\gamma_m(a, o) \in B$. We verify: For sufficiently small U_i, ε, δ, we have $\gamma_m^{-1}(B) \subset A$.

Proof. Let $(x, y, t) \in Z_m$, $x = x_1 \cdots x_m$, and $\gamma_m(x, y, t) = w$ be from B. Then $e_w < u(a) + \delta$ because of (a) and consequently $(y, t) = w(e_w) \in V$ because of (d). Let $x_{m+1} = y$. Similar to the first case, from (b) and (c) it follows that there exist j_i, $1 \leq j_1 < \cdots < j_n \leq m + 1$, such that $x_{j_i} \in U_i$ and either

(1) $j_n \leq m$, or

(2) $j = m + 1$ and $t > \varepsilon$

Case (1): Let $j \neq j_1, \ldots, j_n, m+1$. Then, for a given $\varepsilon' > 0$,

$$u(x_j) \leq u(x, y, t) - \sum_{i=1}^{n} u(x_{j_i}) \leq u(a) - \sum_{i=1}^{n} u(a_i) + \varepsilon' = \varepsilon',$$

if δ and U_i are small enough. We can choose ε' so small that $x_j \in U$ follows. Likewise, one can achieve: $tu(y) \leq \varepsilon'$. We choose ε' so small that it follows: $y \in U$ or $t < \frac{1}{2}$ and therefore $(y, t) \in V' \setminus V_1 \subset V$. Overall: $(x, y, t) \in U_1 \cdots U_n U^{m-n} \times V$.
Case (2): Since $U \cap U_n = \varnothing$ and $t > \varepsilon$, we have

$$(y, t) \in V' \setminus ((U \times I) \cup V_0) \subset V_1$$

and hence even $t > 1 - \varepsilon$. Let $j \neq j_1, \ldots, j_n$. Then, for a given $\varepsilon' > 0$,

$$u(x_j) \leq u(x, y, t) - \sum_{i=1}^{n} u(x_{j_i}) - tu(y) \leq u(a) - \sum_{i=1}^{n} u(a_i) + \varepsilon' = \varepsilon',$$

if only δ, U_i, and ε are small enough. One reaches $x_j \in U$ and consequently

$$(x, y, t) \in U_1 \cdots U_{n-1} U^{m-n} \times U_n \times]1 - \varepsilon, 1].$$

Finally, let $a = o$. Then there exist U and V as above with $Z_m \cap (U^m \times V) \subset A$. We define B as the set of paths w with

(a) $e_w < \delta$,

(b) $w(\delta) \in V'$.

Then $\gamma_m(x, y, t) \in B$, $x_i \in U$ and $y \in U$ or $t < \frac{1}{2}$ if δ is sufficiently small. Furthermore, $(y, t) = w(e_w) = w(\delta) \in V'$, hence $(y, t) \in V$ and hence $(x, y, t) \in U^m \times V$. □

□

3.8.8

The following lemma is used to prove Lemma 3.94 in the next subsection.

Lemma 3.97. Prerequisite*: Let $q : Y \to B$ be a map, $A \subset B$, V be a halo of A in B. Let the restrictions of q $q_A : Y_A \to A$ and $q_{B \setminus A} : Y_{B \setminus A} \to B \setminus A$ (cf. (2.120), (2.32)) be h-fibrations. Let $V \setminus A$ have a numerable null-homotopic covering. Let there be a commutative diagramme*

$$\begin{array}{ccc} Y_V & \xrightarrow{r} & Y_A \\ {\scriptstyle q_V}\downarrow & & \downarrow{\scriptstyle q_A} \\ V & \xrightarrow[\rho]{} & A \end{array}$$

In it, let ρ be a deformation retraction, r an h-equivalence, and $r_A = r|_{Y_A}$ an h-equivalence. For $b \in V \setminus A$, let $r_b : Y_b \to Y_{\rho(b)}$ be an h-equivalence.
Claim*: There exists a commutative diagramme*

$$\begin{array}{ccccc} Y & & \subset & & E \\ & {\scriptstyle q}\searrow & & \swarrow{\scriptstyle p} & \\ & & B & & \end{array}$$

such that:

(a) ρ is an h-fibration;

(b) Y is a strong deformation retract of E;

(c) Y_A is a strong deformation retract of E_A over A;

(d) $Y_{B\setminus A}$ is a strong deformation retract of $E_{B\setminus A}$ over $B \setminus A$.

Proof. We start with the following diagramme

$$\begin{array}{ccccc} Y_V & \overset{r'}{\dashrightarrow} & Y' & \xrightarrow{\widetilde{\rho}} & Y_A \\ & {\scriptstyle q_V}\searrow & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q_A} \\ & & V & \xrightarrow[\rho]{} & A \end{array}$$

where q' is the h-fibration induced by ρ of q_A and r' is determined by $q'r' = q_V$ and $\widetilde{\rho}r' = r$. According to (2.124), $\widetilde{\rho}$ is an h-equivalence since ρ is an h-equivalence and q_A is an h-fibration. r' is an h-equivalence because $\widetilde{\rho}$ and r are h-equivalences.

Construction of E: In the topological sum $Y + Y_V \times I + Y'$, we identify y and $(y, 0)$ as well as $(y, 1)$ and $r'(y)$, for $y \in Y_V$. The maps q, $q_V \circ \mathrm{proj}_1$, and q' on the three summands are consistent with these identifications and induce $p : E \to B$. We clearly have an embedding $Y \subset E$.

We prove (b), (c), (d) and (a), in that order.
(b): According to (1.29) and (1.73), Y_V is a strong deformation retract of E_V, because E_V is the mapping cylinder of r' and r' is an h-equivalence. Consequently, Y is a strong deformation retract of E.
(c): We can regard E_A as a mapping cylinder of r'_A. We have a commutative diagramme

$$\begin{array}{ccccc} Y_A & & \xrightarrow{r_A} & & Y_A \\ & {\scriptstyle r'_A}\searrow & & \nearrow{\scriptstyle \widetilde{\rho}_A} & \\ & & Y'_A & & \end{array}$$

$\widetilde{\rho}_A$ is a homeomorphism because $\rho|_A$ is the identity of A. Since r_A is an h-equivalence by assumption, r'_A is also an h-equivalence, and even over A by

(2.85), as q_A and $q_A^{'}$ are h-fibrations. But then, by (1.73), applied to the category $\mathscr{T}\!op_A$ of spaces over A, Y_A is a strong deformation retract of E_A over A since $Y_A \subset E_A$ is a cofibration over A ((1.29) (b) applied to $\mathscr{T}\!op_A$).
(d): We have a commutative diagramme

$$\begin{array}{ccc} Y_b & \xrightarrow{r_b} & Y_{\rho(b)} \\ & {\scriptstyle r_b^{'}}\searrow \quad \nearrow{\scriptstyle \widetilde{\rho}_b} & \\ & Y_b^{'} & \end{array}$$

in which $\widetilde{\rho}_b$ is a homeomorphism. Consequently, $r_b^{'}$ is an h-equivalence for every $b \in V \setminus A$. Since $q_{V\setminus A}$ and $q_{V\setminus A}^{'}$ are h-cover and $V \setminus A$ has a numerable null-homotopic cover, $r_{V\setminus A}^{'}$ is an h-equivalence over $V \setminus A$ (see (2.152)). Analogous to (c), $Y_{V\setminus A}$ is then a strong deformation retract of $E_{V\setminus A}$ over $V \setminus A$. Consequently, $Y_{B\setminus A}$ is a strong deformation retract of $E_{B\setminus A}$ over $B\setminus$.
(a): Since in the following diagramme

$$\begin{array}{ccc} Y_{B\setminus A} & \subset & E_{B\setminus A} \\ & {\scriptstyle q_{B\setminus A}}\searrow \quad \swarrow{\scriptstyle p_{B\setminus A}} & \\ & B \setminus A & \end{array}$$

by (d), the inclusion is an h-equivalence over $B \setminus A$, and since $q_{B\setminus A}$ is an h-fibration, then by (2.68) $p_{B\setminus A}$ is also an h-fibration. $Y^{'}$ is a strong deformation retract of E_V over V. Consequently, by (2.68), P_V is an h-fibration because $q^{'}$ is an h-fibration. Finally, since $(V, B \setminus A)$ is a numerable cover of B (if v is a halo function of V, then $(1 - v, v)$ is a numeration of $(V, B \setminus A)$), (2.156) tells us that p is an h-fibration. □

3.8.9 Proof of lemma 3.94

We verify the assumptions of Lemma 3.97 for the map $q : Y \to B$, $B = \Sigma X$, from (3.97) and $A = \{o\}$]. We recall that X has the special form $X = X_o \vee I$ given at the end of (3.95). The map q_A is certainly an h-fibration.

The identification $JX \times C^{'}X \to Y$ induces the following commutative diagramme

$$\begin{array}{ccc} Y_{\Sigma X\setminus\{o\}} & & JX \times (C^{'}X \setminus X) \\ & {\scriptstyle q_{\Sigma X\setminus\{o\}}}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}_2} & \\ & \Sigma X \setminus \{o\} & \end{array}$$

in which Q is a homeomorphism since $JX \times (C^{'}X \setminus X)$ is open and saturated in $JX \times C^{'}X$. Hence $q_{\Sigma X\setminus\{o\}}$ is also an h-fibration (namely isomorphic to a trivial fibration).

Because of the special shape of X, we can describe a neighbourhood of o in ΣX by

$$V = \{[x,t] | x \in [0, \frac{1}{2}[\quad \text{or} \quad t \in [0, \frac{1}{4}[\cup]\frac{3}{4}, 1]\}$$

V is a halo with halo function $v(x,t) = \min(1, 2u(x), 4t, 4(1-t))$, with u as at the end of (3.95).

$V \setminus \{o\}$ has a numerable null-homotopic cover.

Proof. We write $V \setminus \{o\} = V_0 \cup V_1$ with

$$V_0 = \{[x,t] | x \in]0, \frac{1}{2}[\quad \text{or} \quad t \in]0, \frac{1}{4}[\}$$
$$V_1 = \{[x,t] | x \in]0, \frac{1}{2}[\quad \text{or} \quad t \in]\frac{3}{4}, 1[\}.$$

$(X \setminus o) \times \frac{1}{8}$ is the deformation retract of V_0 and $(X \setminus o) \times \frac{7}{8}$ is the deformation retract of V_1. Furthermore, $X \setminus o$ is h-equivalent to X_0. From Lemma 3.90, we see that V_0 and V_1 have a numerable null-homotopic cover. (V_0, V_1) is a numerable cover of $V \setminus \{o\}$: if v_0 is the map $[x,t] \mapsto \min(2\max(t - \frac{1}{4}, 0), 1)$, then $(1 - v_0, v_0)$ is a numeration of (V_0, V_1). Our claim on $V \setminus \{o\}$ now follows from the following simple remark: Let (V_λ) be a numerable cover of a space X. If every V_λ has a numerable null-homotopic cover, then so does X.

We construct a homotopy $\varphi : V \times I \to V$ from $\varphi_0 = \mathrm{id}_V$ to $\varphi_1 : V \xrightarrow{\rho} A \subset V$, as illustrated in the figure 3.2. In particular, φ shows that ρ is a deformation

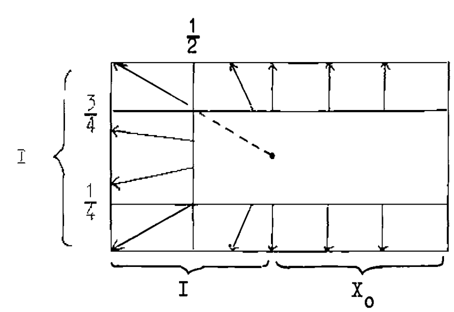


Figure 3.2:

retraction.

Let $\widetilde{V} \subset C'$ be the preimage of V under the canonical projection $C'X \to \Sigma X$. The homotopy φ is induced by the homotopy $\widetilde{\varphi} : V \times I \to V$, as the figure 3.2 shows. We define a homotopy $\varphi' : Y_V \times I \to Y_V$ by the following commutative diagramme

$$\begin{array}{ccc} JX & \xrightarrow{i_b} & Y_b \\ {\scriptstyle R_z}\downarrow & & \downarrow{\scriptstyle r_b} \\ JX & \xrightarrow[i]{} & Y_0 \end{array}$$

where $i_b(x) = (x, b)$, $i(x) = (x, 0)$, $R_z(x) = xz$ with an element z determined by $\widetilde{\varphi}(b, 1) = (z, 1) \in C^{'}X$. i_b is a homeomorphism. i is an h-equivaience (Lemma 3.92). The map R_z is homotopic to id_{JX}, since X is path-connected. This proves Lemma 3.94. □

3.8.10 Counterexamples to (3.8.3)

We show by examples that η_X is in general *not* an h-equivalence if one of the assumptions (a) - (c) on X in (3.8.3) is dropped.
(a) Path-connectedness: Let X be the topological sum of its path components. A homomorphism $h_u : JX \to \Omega^{'}\Sigma X$ induces a homomorphism

$$\pi_0(JX) \to \pi_0(\Omega^{'}\Sigma X).$$

$\pi_0(JX)$ is the free monoid over the pointed set $\pi_0 X$, while $\pi_0(\Omega^{'}\Sigma X) \cong \pi_0(\Omega\Sigma X) \cong pi_1(\Sigma X)$ is a group.
(b) Numerable null-homotopic cover: Let X be the subspace of the plane $\mathbb{R}^2$, illustrated by the following figure 3.3. Let $A = \{a_0, a_1, a_2, \dots\}X$, where $a_0 =$

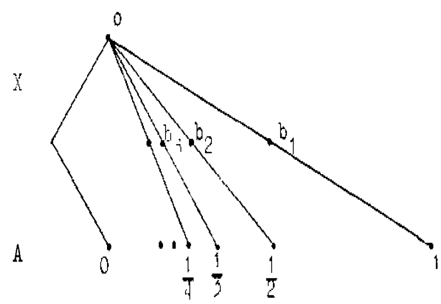


Figure 3.3:

$(0, 0)$, $a_n = (\frac{1}{n}, 0$ $(n = 1, 2, 3, \dots)$, and let $U_n = X \setminus \{b_n, b_{n+1}, b_{n+2}, \dots\}$. Let h be a pointed h-inverse of $h = h_u$. Let $i : \Omega^{'}\Sigma X \to \Omega^{'}\Sigma X$ be an h-inverse for the connexion in the pointed H-space $\Omega^{'}\Sigma X$ (see (3.3.1)). We set $i^{'} = h^{'}ih$. The map $A \to JX$, $a \mapsto i^{'}(a) \cdot a$, is null-homotopic since

$$a \mapsto h(i^{'}(a) \cdot a) = hi^{'}(a) \cdot h(a)$$

is homotopic to

$$a \mapsto ih(a) \cdot h(a),$$

which is null-homotopic. Let $\varphi : A \times I \to JX$ be a null-homotopy. JU_n is open in $JX \setminus \cup JU_n$. Consequently, the compact set $\varphi(A \times I)$ is contained in some JU_n. It follows that $i^{'}a_n$ is a left inverse of a_n in $\pi_0(JU_n)$. On the other hand, $a_n \neq 0$ in $\pi_0(JU_n)$, a contradiction.
(c) h-well-pointedness: Let X be the subspace of the plane $\mathbb{R}^2$, illustrated by

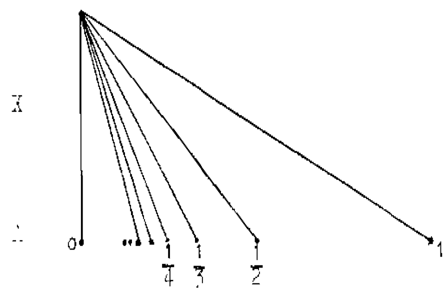


Figure 3.4:

the following figure 3.4. Let A be declared as in the last example. JX is path-connected since X is path-connected. $\Omega'\Sigma X$ is not path-connected. We show that $\pi_1(\Sigma X) \neq 0$. We use singular homology. In the exact sequence

$$H_2(\Sigma X, \Sigma A) \to H_1(\Sigma A) \to H_1(\Sigma X)$$

$H_1(\Sigma A) \cong \prod_1^\infty \mathbb{Z}$ is uncountable, while $H_2(\Sigma X, \Sigma A) \cong H_2(\Sigma(X/A)) \cong H_1(X/A) \cong \widetilde{H}_0(A)$ is countable. (Note: X/A is well-pointed.) Consequently, $H_1(\Sigma X)$ is uncountable and therefore so is $\pi_1 \Sigma X)$ (cf. Hu [12], Theorem 6.1).

Appendix A

In this appendix, we provide the proof of Theorem (1.21) (b).

Theorem A.1 (cf. Puppe [21], footnote 1) on p. 81, Strøm [27], 2. Lemma 3)**.** *Let X be a topological space, A a subspace of X, and $i : A \subset X$ the inclusion.*
Claim*: If $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$, then the bijective continuous map $\ell : Z_i \to (X \times 0) \cup (A \times I)$ defined in (1.20) is a homeomorphism.*

Proof. We follow Strøm [27], 2. Lemma 3. We need to prove the continuity of ℓ^{-1}. First, we identify the set underlying the mapping cylinder of i under the bijective mapping ℓ with $(X \times 0) \cup (A \times I)$. We then have to show that the subspace topology induced by the product $X \times I$ on $(X \times 0) \cup (A \times I)$ is finer than the topology of the mapping cylinder.

So let C be a subset of $(X \times 0) \cup (A \times I)$ such that $C \cap (X \times 0)$ is open in $X \times 0$ and $C \cap (A \times I)$ is open in $A \times I$.
Claim: C is open in the subspace $(X \times 0) \cup (A \times I)$ of $X \times I$.

Proof. We define $U \subset X$ by

$$U := \{x \in X | (x, 0) \in C\}.$$

U is open in X since $C \cap (X \times 0)$ is open in $X \times 0$. We further define open subsets $U_1, u_2, U, \dots$ of X by

$$U_n := \cup \{V | V \text{ is an open subset of } X \text{ and } (V \cap A) \times [0, \frac{1}{n}[\subset C\}.$$

We set

$$B := U \times 0 \cup \cup_{n=1}^{\infty} ((A \cap U_n) \times [0, \frac{1}{n}[).$$

Claim:

$$C = (C \cap (A \times]0, 1])) \cup B. \tag{A.2}$$

Proof. (of A.2) *LHS* $\subset$ *RHS*: Let $c \in C$.

Case 1 $c = (x, 0)$ for some $x \in X$. Then $c \in U \times 0 \subset B$.

Case 2 $c = (a, t)$ for some $a \in A$, $t \in]0, 1]$. Then $c \in C \cap (A \times]0, 1])$.

LHS $\supset$ *RHS*: We show $B \subset C$. Let $b \in B$.

Case 1 $b \in U \times 0$. Then $b \in C$ by definition of U.

Case 2 $b \in (A \cap U_n) \times [0, \frac{1}{n}[$ for a natural number $n > 1$, i.e., $b = (a, t)$ for some $a \in A \cap U_n$ and some $t \in [0, \frac{1}{n}[$. Since $a \in A \cap U_n$, there exists an open subset V of X with $a \in A \cap V$ and $(V \cap A) \times [0, \frac{1}{n}[\subset C$. Hence $b = (a, t) \in C$.

□

We consider the equation (A.2). By assumption, $C \cap (A \times I)$ is open in $A \times I$. Therefore, $C \cap (A \times]0, 1])$ is open in $A \times]0, 1]$. $A \times]0, 1]$ is open in $(X \times 0) \cup (A \times I)$. Therefore, $C \cap (A \times]0, 1])$ is open in $(X \times 0) \cup (A \times I)$.

If we show:

$$B \text{ is open in } (X \times 0) \cup (A \times I), \tag{A.3}$$

we have proven that C is open in $(X \times 0) \cup (A \times I)$.

First, we prove

$$A \cap U = A \cap \cup_{n=1}^{\infty} U_n \tag{A.4}$$

$$\text{If } V \text{ is an open subset of } X \text{ with } V \cap A \subset U_n, \text{ then } V \subset U_n. \tag{A.5}$$

Proof. (of A.4) *LHS* $\supset$ *RHS*: Let $x \in A \cap \cup_{n=1}^{\infty} U_n$. Then there exists n_0 with $x \in A \cap U_{n_0}$. Therefore, there exists an open subset V of X with $x \in A \cap V$ and $(V \cap A) \times [0, \frac{1}{n_0}[\subset C$. So $(x, 0) \in C$, i.e. $x \in U$ and therefore $x \in A \cap U$.
LHS $\subset$ *RHS*: Let $x \in A \cap U$. So $(x, 0) \in C$ and $x \in A$, i.e. $(x, 0) \in C \cap (A \times I)$. Since $C \cap (A \times I)$ is open in $A \times I$ by assumption, there exists an open subset V' of A and a natural number $n_0 \geq 1$ with $(x, 0) \in V \times [0, \frac{1}{n_0}[\subset C$. Since V' is open in A, there exists an open subset V of X with $V' = V \cap A$.

So $(x, 0) \in V' \times [0, \frac{1}{n_0}[= (V \cap A) \times [0, \frac{1}{n_0}[\subset C$, so $x \in U_{n_0}$, so $x \in A \cap \cup_{n=1}^{\infty} U_n$. □

Proof. (of A.5) We show: $(V \cap A) \times [0, \frac{1}{n}[\subset C$. Let $v \in V \cap A$, then $v \in U_n$ since $V \cap A \subset U_n$. Thus there is an open subset W of X with $v \in W \cap A$ and $(W \cap A) \times [0, \frac{1}{n}[\subset C$. So $\{v\} \times [0, \frac{1}{n}[\subset C$. □

In particular, from (A.5) it follows: an open subset of X that does not meet A is a subset of U_n for all n. This immediately yields:

$$X \setminus \cup_{n=1}^{\infty} U_n \subset \overline{A}, \text{ where } \overline{A} \text{ denotes the closed closure of } A \text{ in } X. \tag{A.6}$$

We now exploit the assumption "$(X \times 0) \cup (A \times I)$ is a retract of $X \times I$" and prove

$$U \subset \cup_{n=1}^{\infty} U_n. \tag{A.7}$$

Let $r : X \times I \to (X \times 0) \cup (A \times I)$ be a retraction. If $t \in]0, 1]$, then $A \times t$ is the closed closure of $A \times t$ in $(X \times 0) \cup (A \times I)$. Since r is continuous and fixes the points of $A \times I$, for $t \in]0, 1]$, we have:

$$r(\overline{A} \times t) = A \times t. \tag{A.8}$$

We claim:

$$r((X \setminus \cup_{n=1}^{\infty} U_n) \times I) \subset (X \setminus U_n) \times I \text{ for all } n. \tag{A.9}$$

Proof. (of (A.9)) Let $x \in X \setminus \cup_{n=1}^{\infty} U_n$, $t \in I$. We assume: there exists a natural number $n \geq 1$ with $r(x,t) \in U_n \times I$. Since r is continuous and U_n is open in X, there would then be open neighbourhoods V and M of x and t, respectively, in X and I, respectively, with $r(V \times M) \subset U_n \times I$. It would follow:

$$(V \cap A) \times t = r((V \cap A) \times t) \subset U_n \times I,$$

so $V \cap A \subset U_n$, hence by (A.5) $V \subset U$ and thus $x \in U_n \subset \cup_{n=1}^{\infty} U_n$. Our assumption therefore leads to a contradiction, i.e., (A.9) is proven. □

Proof. (of (A.7)) Now let $x \in X \setminus \cup_{n=1}^{\infty} U_n$. From (A.6), (A.8), (A.9), and (A.4), for all $t \in]0,1]$, it follows:

$$r(x,t) \in (A \cap (X \setminus \cup_{n=1}^{\infty} U_n)) \times I = (A \cap (X \setminus U)) \times I \subset (X \setminus U) \times I$$

and therefore, since r is continuous and $X \setminus U$ is closed in X:

$$(x,0) = r(x,0) \in (X \setminus U) \times I, \quad \text{so} \quad x \in X \setminus U.$$

This shows: $X \setminus \cup_{n=1}^{\infty} U_n \subset X \setminus U$, i.e., (A.7) is proven. □

We now define: $Vn := U \cap U_n$, $n = 1, 2, 3, \dots$ Then we have:

$$U = \cup_{n=1}^{\infty} V_n, \tag{A.10}$$

since $\cup_{n=1}^{\infty} V_n = \cup_{n=1}^{\infty} (U \cap U_n) = U \cap \cup_{n=1}^{\infty} U_n = U$, and by (A.7) $U \subset \cup_{n=1}^{\infty} U_n$.

We claim:

$$A \cap U_n = A \cap V_n. \tag{A.11}$$

Proof. (of (A.11)) $A \cap U_n \supset A \cap V_n$, since $V_n \subset U_n$. Let $x \in A \cap U_n$. Since $x \in U_n$, there exists an open subset W of X with $x \in W$ and $(W \cap A) \times [0, \frac{1}{n}[\subset C$. Since $x \in W \cap A$, it follows that $(x,0) \in C$, i.e., $x \in U$. Hence $x \in A \cap U_n \cap U = A \cap V_n$. □

Using (A.10) and (A.11), it is easy to prove:

$$B = ((X \times 0) \cup (A \times I)) \cap \cup_{n=1}^{\infty} (V_n \times [0, \frac{1}{n}[). \tag{A.12}$$

Since V_n is open in X, it follows from (
refeq:a-9) that B is open in $(X \times 0) \cup (A \times I)$. □

So we have proven (A.3) and are done. □

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