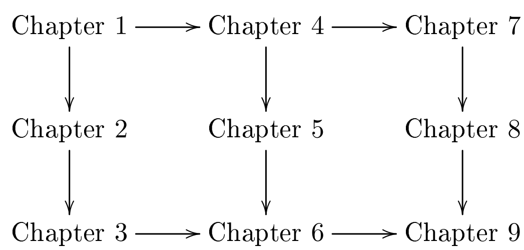
Morse Theory

M. M. Postnikov

Contents

Chapter dependency scheme

(Non-essential connections of a random nature are not reflected in the diagramme.)



Preface

A word from the transcriber

This is an English translation of "Morse Theory" by M. M. Postnikov originally written in Russian.

The preface by the original authour

When studying Morse theory, the main difficulty for beginners is the "synthetic" nature of this theory, i.e. the fact that it is located at the junction of at least three areas of mathematics (topology, analysis and geometry). The purpose of this book is to help the beginner overcome this difficulty. Actually, only a smaller part of the book is devoted to Morse theory: most of it is occupied with presenting the necessary information from topology and geometry.

The first three chapters are devoted to topology.

Since general topology is described in many Russian textbooks, the first chapter (general methodological issues) is written rather concisely and with special emphasis on facts that are usually not covered in textbooks. Some of the issues discussed in this chapter may also be of interest to a specialist.

The second chapter is entirely devoted to homotopy equivalence of topological spaces. As far as the author knows, this material (which is well known to specialists) has not been put together by anyone yet.

The third chapter discusses cellular decompositions. It seems surprising that, despite the main role that cellular decompositions play in modern topology, a coherent presentation of their theory has not yet been published anywhere.

The next two chapters are devoted to the theory of smooth manifolds. Although there are a number of expositions of this theory in Russian, none of them is suitable for our purposes. We construct a theory of smooth manifolds, following Chevalley, in a non-coordinate form; local coordinates are used only when it seems appropriate. With great regret, the author had to limit himself to finite-dimensional manifolds only - the coverage of infinite-dimensional manifolds would violate the elementary nature of the presentation. This is all the more regrettable because (as it has become clear recently) it is infinite-dimensional manifolds that represent the natural field of construction of Morse theory.

The sixth chapter, devoted to the theory of critical points of smooth functions, also belongs to the theory of smooth manifolds. The novelty here is the systematic consideration of not only non-degenerate critical points, but also non-degenerate critical manifolds. Morse numbers connecting the numbers of critical points of a given index with the Betti numbers of the manifold. This is the only place in the book (except for the directly related first half of paragraph 6 of Chapter 9) where we go beyond the topological material described in the first three chapters. However, all the necessary properties of Betty numbers in this supplement are clearly formulated.

The seventh chapter is devoted to the geometry of affine connectivity spaces-and Riemannian geometry. The presentation here is conducted mainly in the "classical" spirit, but in compliance with all modern requirements of rigour and from a global point of view. We are dealing here only with the very basics of Riemannian geometry, so much that is usually given in geometry courses remains outside the scope of our presentation. On the other hand, we had to include here some essential facts of Riemannian geometry (Whitehead's theorem on the existence of neighbourhoods normal with respect to any point, and the Hopf-Rinov theorem on complete Riemannian spaces), which are usually not considered in standard Riemannian geometry courses. Although we conduct the entire presentation of Riemannian geometry in the spirit of Cartan's ideas, nevertheless, due to the limitations of the tasks set, we managed to do without external differentiation (although in one place, namely, when deriving Cartan's basic equations, the concept of the external differential, although implicitly, still essentially appears).

The eighth chapter outlines the theory of the so-called "index form". We present here, adhering mainly to the original Morse construction, and only in the last paragraph we turn to a more modern interpretation related to the replacement of the length functional with the action functional. This, of course, somewhat lengthens and complicates the construction, but at the same time it is possible to preserve both the historical perspective and geometric clarity. In Appendix to this chapter, the "moving end" problem is treated in a similar way. Here, Morse's initial reasoning was significantly simplified.

In the ninth and final chapter, the main theorem of Morse theory is proved, describing the structure of the space of curves connecting two given points of the complete Riemannian space. In essence, this theorem is a fairly simple reworking of the main results of the previous chapter, with the aim of giving these results a more invariant appearance. In conclusion of this chapter, we give the simplest applications of Morse theory to the topology and geometry of Riemannian spaces.

A more detailed understanding of the contents of the book can be obtained from the table of contents and short summaries given to each chapter.

Formally, the reader is not required to have any knowledge beyond the first-year program of mathematical departments of universities and pedagogical colleges, although, of course, a certain level of mathematical culture and the ability to work with the book is assumed.

When working on the manuscript of this book, the author allowed himself to make extensive use of his previous book (see the list of references of Postnikov [9]). In particular, chapters seven and eight reproduce the corresponding chapters of this book almost verbatim.

The list of references at the end of the book is provided with a "Historical and literary commentary" aimed at helping the reader to navigate the literature more easily. The list of references does not claim to be complete.

Addition during proofreading. The manuscript of this book was completed in 1965, and its printing was delayed. At present, the author would present many things in a completely different way, from a more modern perspective. But this would have postponed the publication of the book indefinitely, and therefore it was decided to leave it in its original form.

The Authour

Chapter 1

Necessary information from general topology

In this introductory chapter we present the information we need based on general set-theoretic topology. As a rule, we do not seek to exhaust a particular issue raised. Only questions for which there is no coherent presentation in the literature known to the authour are considered in detail.

§1.1 sets out the definition of topological spaces, introducing the classes of spaces distinguished by the axioms of countability and separability are defined, and the most important operations on topological spaces (free associations, topological sums and topological products) are described.

In §1.2, compact and closed spaces are considered (paracompact, finally compact, locally compact, etc.). The results of this point are mostly known, but so far were not collected together anywhere.

§1.3 presents classical theorems on continuous functions (the theorems of Urysohn and Tietze and the theorem on the existence of a partition of unity).

In §1.4, the presentation is concentrated mainly around Stone's theorem on the paracompactness of metric spaces. As a consequence of this theorem, in particular, we prove (based on the general results of §1.2 the well-known theorem of P. S. Aleksandrov on the separability of connected locally compact metric spaces.

In §1.5, the main attention is paid to the issue of constructing continuous maps. In this regard, identification maps and, in particular, their topological products are considered in detail.

§1.6 contains rather heterogeneous material, grouped around the concept of topology of identification. In particular, this section describes the construction of a cylinder of an arbitrary continuous map.

1.1 Topological spaces

Since the basic facts of general topology are widely known, in this chapter we only recall some concepts and clarify terminology.

A set X is said to be defined as a *topological space* or to have a *topology* introduced if some of its subsets are declared open and the following axioms hold:

1) the set X itself is open;

2) the empty set $\varnothing$ is open;

3) the union of any number of open sets is open;

4) the intersection of any finite number of open sets is open.

Let A be an arbitrary subset of the topological space X. Every open set U containing A is called its *neighbourhood*. A point a of a set A is called its *inner point* if some of its neighbourhood is contained in the set A. The set $\operatorname{int} A$ of all interior points of a set A is called its *interior*. The interior of $\operatorname{int} A$ is the largest open set contained in the set A. Therefore $\operatorname{int}(\operatorname{int} A) = \operatorname{int} A$. A set A is open if and only if $\operatorname{int} A = A$, that is, when any of its points is its interior point. If $A \subset B$, then $\operatorname{int} A \subset \operatorname{int} B$. For any family $\{A_\alpha\}$ of subsets of the space X the following inclusions occur

$$\operatorname{int} \cup_\alpha A_\alpha \supset \cup_\alpha \operatorname{int} A_\alpha, \quad \operatorname{int} \cap_\alpha A_\alpha \subset \cap_\alpha \operatorname{int} A_\alpha,$$

and for a finite family $\{A_\alpha\}$ equality takes place in the second of these relations.

A point x of a topological space X is called *isolated* if it, considered as a subset of the space X, is its open subset. A topological space in which all points are isolated is called *discrete*. The space X is discrete if and only if each of its subsets is open.

A family $\{U_\alpha\}$ of neighbourhoods U_α of a set A is called its *fundamental system of neighbourhoods* if for an arbitrary neighbourhood U of the set A there exists an α such that $U_\alpha \subset U$. A topological space X is called a *space of countable local weight*[1] if any of its points has a countable fundamental system of neighbourhoods. In this case we also say that the space X satisfies the *first axiom of countability*.

A *base* (or a *base of open sets*) of a topological space X is any family of its open sets that has the property that the unions of the sets of this family exhaust all the open sets of the space X. In order for some family of open sets of the space X to be a base, it is necessary and sufficient that for any point of the space X from this family it is possible to choose a fundamental system of neighbourhoods of this point. A family of open sets of a space X that has the property that all their possible finite intersections form a base of the space X is called its *prebase*. The assignment of some pre-base completely determines the

[1]Comment by the transcriber: The *weight* of a topological space is the smallest cardinality of an open base.

topology of the space X, and any family of subsets of the set X can be taken as a pre-base of some topology defined on this set. A space X is called a *space with a countable base* (or a *space of countable weight*) if it has at least one base (or, equivalently, a pre-base) containing at most a countable number of sets. In this case, we also say that the space X satisfies the *second axiom of countability.* Every space with a countable base has a countable local weight.

An example of a topological space with a countable base is the number line $\mathbb{R}$, considered in the usual *Euclidean topology.* By definition, the base of this topology is the family of all possible open intervals (a, b), $a < b$. To obtain a countable base, it is sufficient to restrict ourselves to intervals (a, b) with rational a and b.

The complements (in X) of open sets of a topological space X are called its *closed* subsets. A set X is completely defined as a topological space if it is specified which of its subsets are closed. Closed sets have the following basic properties, dual properties of open sets

1) the empty set $\varnothing$ is closed;

2) the entire set X is closed;

3) the intersection of any number of closed sets is closed;

4) the union of any finite number of closed sets is closed.

A point $x \in X$ is called a *contact point* (or an *adherent point*) of the set $a \subset X$ if any of its neighbourhoods intersects with A. The set $\overline{A}$ of all contact points of the set A is called its *closure*. It coincides with $X \subset \operatorname{int}(X \subset A)$ and therefore is the smallest closed set of the space X containing the set A. In particular, the set A is closed if and only if $\overline{A} = A$, that is, when it contains all its contact points. In addition, $\overline{\overline{A}} = A$ and $\overline{A} \subset \overline{B}$ if $A \subset B$. For any family $\{A_\alpha\}$ of subsets of the space X there are inclusions

$$\overline{\cup_\alpha A_\alpha} \supset \cup_\alpha \overline{A}_\alpha; \quad \overline{\cap_\alpha A_\alpha} \subset \cap_\alpha \overline{A}_\alpha,$$

and for a finite family $\{A_\alpha\}$ the first of these relations holds equality.

If $\overline{A} = X$ then the set A is called *everywhere dense* (in X). A space X is called *separable* (or *countable dense*) if there exists a countable everywhere dense set in it. Any space with a countable base is separable.

The set $A \subset X$ is called *nowhere dense* (in X) if $\operatorname{int}\overline{A} = \varnothing$, i.e. if the complement of its closure is everywhere dense ($\overline{X \setminus \overline{A}} = X$). A set $A \subset X$ is nowhere dense if and only if any non-empty open set $U \subset X$ contains a non-empty open subset disjoint from A (or, in other words, when any non-empty open subset X the closed set $F \subset X$ is contained in a closed set different from X that contains the set A). The closure of a nowhere dense set is nowhere dense, and an arbitrary subset of a nowhere dense set is also nowhere dense. The union of a finite number of nowhere dense sets is nowhere dense. A space X is called a *space of the first category* (in the sense of Baire-Hausdorff) if it is decomposable into the union of a countable number of nowhere dense sets (here, as elsewhere in the future, by a countable set we mean a finite or countable set). Otherwise, the space X is called a *space of the second category.*

A point a of a space X is called the *limit* of the sequence $\{x_n\}$ of points $x_n \in X$, $i = 1, 2, \dots$ if for each its neighbourhood U there is an integer N such that $x_n \in U$ for any $n \geq N$. In this case, the sequence $\{x_n\}$ is called *convergent.* Generally speaking, a convergent sequence can have several different limits. If all points x_n of the convergent sequence $\{x_n\}$ belong to the set $A \subset X$, then each limit a of this sequence belongs to the closure $\overline{A}$ of the set A. If the space X has a countable local weight, then the converse is also true, i.e.,

Proposition 1.1. *for an arbitrary subset $A \subset X$, each point $a \in A$ is the limit of a certain sequence $\{x_n\}$ of points from A.*

Proof. Indeed, let $\{U_n\}, n \geq 1$ be a countable fundamental system of neighbourhoods of a point a. Let us construct the sequence $\{x_n\}$, taking as a point x_n, $i = 1, 2, \dots$ an arbitrary point of the set A belonging to he intersection $V_n = \cap_{n=1}^{n} U_i$ (such a point necessarily exists, because $a \in \overline{A}$, and the intersection V_n is a neighbourhood of the point a). It is clear that the sequence $\{x_n\}$ constructed in this way converges to the point a. □

Each subset A of the topological space X is defined as a topological space if its open sets are taken to be intersections with A of the open sets of the space X. This topology of the set A is called the *induced topology*, and the set A itself, equipped with an induced topology, is a subspace of the space X. Any subspace B of the subspace A is a subspace of the space X. The closed sets of the subspace A are the intersections with A of the closed sets of the space X. A subspace A is open (resp. closed) in the space X if and only if any open (resp. closed) subset of it is open (resp. closed) in the space X. Each open subspace of a space of the first category is itself a space of the first category. Therefore, if the space X contains an open subspace that is a space of the second category, then it is itself a space of the second category. Each subspace of a space with a countable base (or a space of countable local weight) is a space with a countable base (or, respectively, a space of countable local weight). Every open subspace of a separable space is separable.

In particular, each subset of the number line $\mathbb{R}$, for example the unit segment $I = [0, 1]$, is a topological space with a countable base.

A topological space X is called *Hausdorff* if any two of its different points have disjoint neighbourhoods, and *completely Hausdorff* if any two of its different points have neighbourhoods with disjoint closures. Thus, every completely Hausdorff space is Hausdorff. Any point of a Hausdorff space is closed, i.e. it is a closed subset of it. Each convergent sequence of points in a Hausdorff space has a single limit. Any subspace of a Hausdorff (completely Hausdorff) space is Hausdorff (completely Hausdorff).

A Hausdorff space is called *regular* if any of its closed subsets and any point not belonging to this subset have disjoint neighbourhoods, and *normal* if any of its two disjoint closed subsets have disjoint neighbourhoods. Formally, a stronger condition for the existence of neighbourhoods with disjoint closures leads - in contrast to the case of Hausdorff property- to the same class of spaces. Thus, for example, in normal space, any two disjoint closed sets have neighbourhoods with disjoint closures. Any normal space is regular. Any regular space is completely Hausdorff. A Hausdorff space is regular (respectively normal) if and only if for any neighbourhood U of its arbitrary point (an arbitrary closed subset) there exists a neighbourhood V such that $\overline{V} \subset U$. Any subspace of a regular space is regular and any closed subspace of a normal space is normal. Every discrete space is normal. Any subspace of the number line $\mathbb{R}$, in particular, the unit segment I, is a normal space.

Let $\{X_\alpha; \alpha \in A\}$ be a family of subspaces X_α of the topological space X such that

$$X = \cup_{\alpha \in A} X_\alpha.$$

According to the definition of a subspace, for every closed (open) set $A \subset X$ and any $\alpha \in A$, the intersection $A \cap X_\alpha$ is closed (open) in the subspace X_α. In the case when the converse is true, i.e., when any set $A \subset X$ for which all sets $A \cap X_\alpha$ are closed (open) in the corresponding subspaces X_α, is itself closed (open) in the space X, we will say that the space x is a *free union of subspace* X_α. As is easy to see,

Proposition 1.2. *any closed or open subspace A of a free union X of subspaces X_α is a free union of subspaces $A \cap X_\alpha$.*

In addition,

Proposition 1.3. *the space $X = \cup_{\alpha \in A} X_\alpha$ is a free union of its subspaces X_α, $\alpha \in A$ if*

1) all subspaces X_α are open; or

2) all subspaces X_α are closed and their number is finite.

Proof. Indeed, if all subspaces X_α are open (closed), then every set that is open (closed) in one or another of the subspaces X_α is open (closed) throughout the space X. In particular, if for a set $A \subset X$ any of the intersections $A \cap X_\alpha)$ is open (closed) in the corresponding subspace X_α, then all these intersections are open (closed) in X, and therefore the set

$$A = \cup_{\alpha \in A}(A \cap X_\alpha)$$

is the union of open closed sets $A \cap X_\alpha$ in X. Consequently, A itself is open (closed, because the number of terms, by condition, is finite). □

A family $\{A_\alpha; \alpha \in A\}$ of subsets A_α of a space X will be called *locally finite* if any point $x \in X$ has a neighbourhood intersecting only with a finite number of subsets A_α. It turns out that in condition 2) of the previous proposition, the requirement of finiteness of the family $\{X_\alpha; \alpha \in A\}$ can be weakened to the requirement of local finiteness, i.e.

Proposition 1.4. *the space $X = \cup_\alpha X_\alpha$ is a free union of its subspaces X_α, $\alpha \in A$ if all these subspaces are closed and the family $\{X_\alpha; \alpha \in A\}$ is locally finite.*

Since the family of $\{A \cap X_\alpha; \alpha \in A\}$ is locally finite together with the family $\{X_\alpha; \alpha \in A\}$, it immediately follows from the above reasoning that it is enough for us to prove only the following (which has an independent interest) proposition:

Proposition 1.5. *The union $F = \cup_\alpha F_\alpha$ of any locally finite family $\{F_\alpha; \alpha \in A\}$ of closed sets $F_\alpha \subset X$ is closed.*

We will prove even more, namely that

Proposition 1.6. *for any locally finite family $\{A_\alpha; \alpha \in A\}$ of subsets of an arbitrary space X the foollowing equality holds*

$$\overline{\cup_\alpha A_\alpha} = \cup_\alpha \overline{A}_\alpha.$$

Proof. Indeed, since $\overline{\cup_\alpha A_\alpha} \supset \cup_\alpha \overline{A}_\alpha$, proof requires only the reverse inclusion

$$\overline{\cup_\alpha A_\alpha} \subset \cup_\alpha \overline{A}_\alpha. \tag{1.7}$$

Let $A = \cup_\alpha A_\alpha$ and let $x/in\overline{A}$. Due to the local finiteness of the family $\{A_\alpha$, the point x has a neighbourhood U_0 intersecting only with a finite number of sets $A_{\alpha_1}, \ldots, A_{\alpha n}$ of this family. Therefore, for any neighbourhood U of the point x, its neighbourhood $V = U \cap U_0$ does not intersect with the sets A_α for $\alpha \neq \alpha_1, \ldots, \alpha_n$. But $x \in \overline{A}$ and therefore $V \cap A \neq \emptyset$. Therefore, $V \cap \cup_{i=1}^n A_{\alpha_i} \neq \emptyset$, i.e, $U \cap \underline{\cup_{i=1}^n A_{\alpha_i}} \neq \emptyset$. Due to the arbitrariness of the neighbourhood U, it follows that $x \in \overline{\cup_i A_{\alpha_i}} = \cup_{i=0}^n \overline{A}_{\alpha_i} \subset \cup_\alpha \overline{A}_\alpha$. Thus, the inclusion (1.7) is proved. □

In connection with the proven proposition, it is useful to note that

Proposition 1.8. *a family $\{A_\alpha\}$ of arbitrary sets is locally finite if and only if the family $\{\overline{A}_\alpha\}$ consisting of the closures $\overline{A}_\alpha$ of sets A_α is locally finite.*

Proof. Indeed, it is clear that an open set U intersects some set A if and only if it intersects the closure $\overline{A}$ of this set. □

The concept of free union that we have considered had, so to speak, an "internal" character: it related to subspaces of a given, "already existing," topological space X. It can also be considered when the space X is not given in advance.

Let an arbitrary set X be represented as a union

$$X = \cup_{\alpha \in A} X_\alpha$$

of sets X_α, each of which is a topological space. It is easy to see that, by declaring closed (open) sets those and only those sets $A \subset X$ for which, for any $\alpha \in A$, the intersection $A \cap X_\alpha$ is closed (open) in the space X_α, we define the set X as a topological space. However, generally speaking, the spaces X_α will not be subspaces of this space. (To get an appropriate example, it is enough to consider the case when all the spaces X_α coincide as sets with the space X, but are equipped with different topologies.) In the case when each space X_α is a subspace of the constructed space X, we will say that the spaces X_α are *freely united*, and we will call the space X a *free union of the spaces* X_α. It will be a free union in the previously defined sense of its subspaces X_α.

It is clear that

Proposition 1.9. *if all intersections* $X_{\alpha_1} \cap X_{\alpha_2}$, $\alpha_1, \alpha_2 \in A$, *are closed (open) spaces of each spaces* X_{α_1} *and* X_{α_2}, *then the spaces* X_α *are freely united and are closed (respectively, open) spaces of their free union* X.

In particular, the spaces X_α are freely combined if they do not intersect in pairs. We will call the free union of disjoint topological spaces a *topological sum*. Each term X_α of the topological sum X is closed and simultaneously open in this sum. Conversely, if

$$X = \cup_{\alpha \in A} X_\alpha$$

and if each subspace of X_α is both closed and open in X, then the space X is the topological sum of the subspaces of X_α. If each term of a topological sum is Hausdorff, completely Hausdorff, regular or normal, then the topological sum also has the same property. The same remark applies to topological sums, the terms of which have a countable local weight. For spaces with a countable base and separable, a similar statement is true only when the number of terms of the topological sum is countable.

Along with the topological sum, we will also consider the *topological product* $X \times Y$ of any two (not necessarily disjoint) topological spaces X and Y. The points of this product are, by definition, all possible pairs of the form (x, y), where $x \in X$, $y \in Y$, and the base of its open sets is the collection of all subsets of the form $U \times V$, where U is an arbitrary open subset of the space X, and V is an arbitrary open subset of the space Y (the subset $U \times V$ consists, as its notation suggests, of all pairs (x, y), where $x \in U$, $y \in V$). The topological product of any (possibly infinite) number of topological spaces X_α, $a \in A$ is defined similarly (in this case, in the case of an infinite number of factors, when constructing open sets of the base of the product, one should choose open sets that coincide with the entire space in all but a finite number of factors). The topological product of any number of Hausdorff, completely Hausdorff, or regular spaces is, respectively, Hausdorff, completely Hausdorff, and regular. For spaces of countable local weight of spaces with a countable base or separable spaces, the corresponding statement holds if the number of factors is countable. The topological product of normal spaces is not, generally speaking, a normal space.

For example, there is a normal space X whose topological product $X \times I$ on the segment $I = [0, 1]$ of the number line $\mathbb{R}$ is no longer a normal space. The normal space X, for which the product $X \times I$ is also normal, we will call it *stably normal*.

1.2 Compact and some other similar spaces

A family $\{A_\alpha; \alpha \in A\}$ of subsets A_α of a topological space X is called a *covering* of the set $B \subset X$ if

$$B \subset \cup_{\alpha \in A} A_\alpha.$$

In particular (for $B = X$), a family $\{A_\alpha\}$ is called a *covering of the space* X if

$$X = \cup_{\alpha \in A} A_\alpha.$$

A covering $\{A_\alpha; \alpha \in A\}$ is called *open* if all sets A_α are open, and *closed* if all sets A_α are closed. In the future, we usually consider only open coverings.

The covering $\{A_\alpha; \alpha \in A\}$ of the space X is called *point finite* if any point $x \in X$ belongs to only a finite number of sets A_α, *locally finite* if any point $x \in X$ has a neighbourhood intersecting only with a finite number of sets A_α (cf. §refsect:1-1), and is *stellar finite* if any set A_α intersects only a finite number of other sets of covering. Any locally finite cover is point finite and any open stellar-finite cover is locally finite.

We will call the covering $\{B_\beta; \beta \in B\}$ a *subcovering* of the covering $\{A_\alpha; \alpha \in A\}$ if $B \subset A$ and $B_\beta = A_\beta$ for any $\beta \in B$. We will say that the cover $\{B_\beta; \beta \in B\}$ is a *refinement* of the cover $\{A_\alpha; \alpha \in A\}$ if for any $\beta \in B$ there exists an $\alpha \in A$ such that $B_\beta \subset A\alpha$. It is clear that any subcovering of the covering $\{A_\alpha; \alpha \in A\}$ is a refinement of this covering.

A space X is called *paracompact* if any of its open coverings can be refined into a locally finite open covering. Every closed subspace of a paracompact space is paracompact. The topological sum of any number of paracompact spaces is paracompact. Each discrete space is paracompact.

To prove deeper properties of paracompact spaces, it is useful to first prove that:

Proposition 1.10. *for a subset A and a closed subset F of a paracompact space X that does not intersect with it to have disjoint open neighbourhoods, it is sufficient that the set F have an open cover $\{U_\alpha; \alpha \in A\}$ such that $\overline{U}_\alpha \cap A = \varnothing$ for each $\alpha \in A$.*

Proof. Indeed, let us consider the covering $\{X \setminus F, U_\alpha\}$ of the entire space X. Let $\{W_\beta; \beta \in B\}$ be a locally finite covering refining the covering $\{X \setminus F, U_\alpha\}$, and let $B^{'}$ be the set of all indices $\beta \in B$ for which $W_\beta \cap F \neq \varnothing$. Since $W_\beta \not\subset X \setminus F$, for each index $\beta \in B^{'}$ there exists an index $\alpha \in A$ such that $W_\beta \subset U_\alpha$ and therefore $\overline{W}_\beta \cap A = \varnothing$ (since $\overline{W}_\beta \subset \overline{U}_\alpha$). Therefore, the set $\cup_{\beta \in B'} \overline{W}_\beta$ does not intersect with the set A. But $\cup_{\beta \in B'} \overline{W}_\beta = \overline{\cup_{\beta \in B'} W_\beta}$ due to the local finiteness of the family $\{W_\beta\}$. Therefore, the set $U = X \setminus \cup_{\beta \in B'} \overline{W}_\beta$ is open and contains the set A, i.e. it is an open neighbourhood of the set A. To complete the proof, it remains to note that the set $W = \cup_{\beta \in B'} W_\beta$ is open, contains the set F and does not intersect with the set U. □

It follows easily from this proposition, in particular, that

Proposition 1.11. *any Hausdorff paracompact space X is normal.*

Proof. Indeed, let F_1 and F_2 be disjoint closed subsets of the space X. Since the space X is Hausdorff, for any two points $x_1 \in F_1$ and $x_2 \in F_2$ there are disjoint neighbourhoods $U(x_1)$ and $U(x_2)$. In particular, the point x_2 will not belong to the closure $\overline{U(x_1)}$ of the neighbourhood $U(x_1)$ of thee point x_1. In other words, the covering of the set F_1, consisting of the neighbourhoods $U(x_1$, $x_1 \in F_1$, satisfies the conditions of the proposition just proved (for $F = F_1$ and A consisting of the point x_2). Consequently, there are disjoint open sets V and $V(x_2$, respectively containing the set F_1 and the point x_2. In particular, the closure $\overline{V}(x_2)$ of the set $V(x_2)$ does not intersect with the set F_1. Therefore, the covering of the set F_2, consisting of the sets $V(X_2)$, $x_2 \in F_2$, also satisfies the conditions of our proposition (for $F = F_2$ and $A = F_1$). Consequently, the sets F_1 and F_2 can be separated by disjoint neighbourhoods. □

A space X (or, more generally, some subset of it) is called *compact* (respectively, *Lindelöf*) if any of its open covers can be refined with a finite (respectively, countable) open cover, or, equivalently, if any of its open covers can be subdivided with a finite (respectively, countable) cover.

Each compact space is Lindelöf. Any discrete compact (Lindelöf) space is finite (countable). Any finite (countable) space is compact (Lindelöf). A subset of an arbitrary space is compact (Lindelöf) if and only if it is compact (Lindelöf) in the induced topology. In a Hausdorff space X, the set of points of some convergent sequence together with the limit of this sequence constitutes a compact subset of the space X. The closed subspace A of a compact (Lindelöf) space X is compact (Lindelöf). In particular, the intersection of a compact set with a closed one is compact.

The classical Heine-Borel lemma states that any closed bounded subset of the real line $\mathbb{R}$ is compact. In particular, the interval $I = [0, 1]$ is compact.

Below we prove that in regular (and even Hausdorff) spaces all compact sets are closed. In this regard, it is useful to note that compact sets of regular spaces have a property that any closed sets have in a normal space, namely, for any neighbourhood U of a compact subset C of a regular space X, there exists a neighbourhood V such that $\overline{V} \subset U$.

Proof. Indeed, since the space X is regular, each point $x \in C$ has a neighbourhood V_x such that $\overline{V}_x \subset U$. Since the set C is compact, from its covering $\{V_x, x \in C\}$ one can choose a finite subcovering $V_{x_1}, \ldots, V_{x_n}$. The set

$$V = \cup_{i=1}^{n} V_{x_i}$$

is open and its closure $\overline{V}$ (which is the union of the closures $\overline{V}_{x_i}$ of the sets V_{x_i}) is contained in the neighbourhood U. □

A topological space X will be called *sequentially compact* if each sequence of its points contains a convergent subsequence. It is easy to see that

Proposition 1.12. *any Hausdorff compact space of countable weight is sequentially compact.*

Proof. Indeed, let this not be so, i.e. let there exist a compact space of countable local weight, containing a sequence $\{x_n\}$, with no subsequence of which converges. It is easy to see that the set of points of this sequence is, firstly, infinite, secondly, discrete, and, thirdly, closed. On the other hand, any closed discrete subset of a compact space is necessarily finite. □

It is equally easy to prove that

Proposition 1.13. *any Lindelöf and sequentially compact space is compact.*

Proof. To prove this statement, it suffices to show that from any countable open covering $\{U_i\}$ of a sequentially compact space X one can choose a finite subcovering. Consider the sets $V_n = U_1 \cup \cdots \cup U_n$, $n = 1, 2, \ldots$. If it is impossible to choose a finite subcovering from the covering $\{U_i\}$, then we can obviously assume that all sets V_n are distinct. Let $x_n \in VV_{n+1} \setminus V_n$, $n = 1, 2, \ldots$. Since the point x_n can belong to the set U_i only when $n < i$, then none of the sets U_i contains the limit of any convergent subsequence of the sequence $\{x_n\}$. But this is impossible, because the sets Ui cover, according to the condition, the entire space X. The resulting contradiction shows that from the covering $\{U_i\}$ one can choose a finite subcovering. □

In connection with this statement, it is useful to keep in mind that

Proposition 1.14. *any space X with a countable base is Lindelöf.*

Proof. Indeed, by selecting for an arbitrary open covering Γ of the space X in a countable base of this space a subfamily of all its elements contained in the elements of the covering Γ, we obviously obtain a countable covering of the space X as a refinement in the covering Γ. □

The compactness of sequentially compact spaces follows not only from Lindelöf property, but also from paracompactness, i.e.

Proposition 1.15. *a paracompact and sequentially compact space is compact.*

To prove this statement it suffices to show that

Proposition 1.16. *any locally finite covering $\{U_\alpha\}$ of a sequentially compact space X is finite.*

Proof. Assume the contrary, i.e. assume that the covering $\{U_\alpha\}$ is infinite, and consider some of its countable subfamily $\{U_i\}$. Let, as above, $V_n = U_1 \cup \cdots \cup U_n$. From the local finiteness of the covering $\{U_\alpha\}$ it follows directly that the family $\{V_n\}$ contains infinitely many different sets Therefore, passing to some of its subfamily if necessary, we can assume that all sets V_n are distinct. Let $x_n \in V_n \setminus V_{n-1}$ and let x be the limit of some convergent subsequence of the sequence $\{x_n\}$. By definition, any neighbourhood of a point x contains infinitely many points of the sequence $\{x_n\}$ and therefore intersects with an infinite number of elements of the covering $\{U_\alpha\}$. Since this contradicts the local finiteness of the covering $\{U_\alpha\}$, the assumption that the covering $\{U_\alpha\}$ is infinite is false. □

The union of a finite (countable) number of compact (Lindelöf) subspaces of an arbitrary space is compact (Lindelöf). In particular, the union of countably many compact sets is Lindelöf. The topological sum of a finite (countable) number of compact (Lindelöf) spaces is a compact (Lindelöf) space. The topological product of any number of compact spaces is compact.

For two (and therefore any finite number of factors) this statement is obvious, since in each open covering of the product $X \times Y$ of factors one can inscribe a covering consisting of "rectangles" $U_\alpha \times V_\beta$, where U_α and V_β are elements of some coverings of the spaces X and Y, respectively.

For paracompact and Lindelöf spaces, the corresponding statement is generally false (even for the case of two factors). However, it can be shown that

Proposition 1.17. *the topological product of a paracompact and a compact space is paracompact.*

Proof. Indeed, let Γ be an arbitrary open covering of the product $X \times Y$ of a paracompact space X and a compact space Y. Without loss of generality, we can assume that the elements of the covering Γ have the form $U_\alpha \times V_\beta$ where u_α and V_β are some open coverings of the spaces X and Y, respectively. Since for each point $x \in X$ the "layer" $x \times Y$ of the product $X \times Y$ is obviously compact, from the covering Γ one can choose a finite subfamily $\{U_{\alpha_i,x} \times V_{\beta_i,x} | i = 1, 2, \ldots, n_x\}$ such that

$$x \times Y \subset \cup_{i=1}^{n_x} (U_{\alpha_i,x} \times V_{\beta_i,x}).$$

In this case, we can obviously assume that $x \in U_{\alpha_i,x}$ for all $i = 1, 2, \ldots, nx$. Let

$$U_x = \cap_{i=1}^{n_x} U_{\alpha_i,x}.$$

It is clear that the sets $U_x \times V_{\beta_i,x}$, $i = 1, 2, \ldots, n_x$, are open in $X \times Y$ and still cover the layer $x \times Y$:

$$x \times Y \subset \cup_{i=1}^{n_x} (U_x \times V_{\beta_i,x}).$$

Moreover, $x \in U_x$, so that the family $\{U_x | x \in X\}$ is an open covering of the space X. Let $\{W_\delta | \delta \in \Delta\}$ be a refined locally finite open covering of the covering $\{U_x | x \in X\}$. Having chosen for each $\delta \in \Delta$ a point $x_\delta \in X$ such that $W_\delta \in U_{x_\delta}$, we consider the sets $W_\delta \times^{V}_{\beta_i,x_\delta}$, $i = 1, 2, \ldots, n_{x_\delta}$. It is clear that

$$W_\delta \times Y \subset \cup_{i=1}^{n_{x_\delta}} (W_\delta \times V_{\beta_i,x_\delta}).$$

Consequently, the family $\{W\delta \times V_{\beta_i,x_\delta} | \delta \in \Delta, i = 1, 2, \ldots, n_{x_\delta}\}$ is an open covering of the space $X \times Y$, obviously a refinement the covering Γ. Therefore, to complete the proof, it only remains to show that this covering is locally finite. Let (x, y) be an arbitrary point of the space $X \times Y$. By the paracompactness of the space X, the point x has a neighbourhood $U(x)$ in X that intersects only a finite number of open sets W_δ. Consider a neighbourhood $U(x) \times V_{\beta_{i_0},x_{\delta_0}}$ of a point (x, y) in the space $X \times Y$, where $V_{\beta_{i_0},x_{\delta_0}}$ is one of the sets of the form V_{β_i,x_δ}, containing the point y. If this neighbourhood intersects some set $W_\delta \times V_{\beta_i,x_\delta}$ then $U(x) \cap W_\delta \neq \varnothing$ and therefore the number of such sets is finite. Consequently, the covering $W_\delta \times V_{\beta_{i_0},x_{\delta_0}}$, $i = 1, 2, \ldots, n_{x_\delta}$ is locally finite. □

It immediately follows from the proved proposition that the topological product of a Hausdorff paracompact and a Hausdorff compact space is a Hausdorff paracompact and therefore a normal space. Since the segment $I = [0, 1]$ is a Hausdorff compact space, it follows, in particular, that

Proposition 1.18. *any Hausdorff paracompact space is stably normal.*

It is clear that any compact space is paracompact. Therefore,

Proposition 1.19. *any Hausdorff compact space is stably normal,*

Unlike compactness, Lindelöf-ness, generally speaking, does not ensure paracompactness. However,

Proposition 1.20. *any regular Lindelöf space X is paracompact (and hence normal).*

Proof. We will begin the proof of this statement by considering two disjoint closed sets F_1 and F_2 of the space X. Since the space X is regular, for any point $x \in X$, there exists a neighbourhood $U(x)$ such that $\overline{U}(x) \cap F_i \neq \varnothing$ if $x \notin F_i$. Since the set F_i, $i = 1, 2$, is a closed subset of the Lindelöf space X, it itself is Lindelöf and therefore from its covering $\{U(x)|x \in F_i\}$ one can choose a countable subcovering, i.e. in the set F_i there exists a countable system of points $x_{i,1}, \ldots, x_{i,n}, \ldots$ such that

$$F \subset \cup_{n=1}^{\infty} U(x_{i,n}), \quad i = 1, 2.$$

In this case $\overline{U}(x_{1,n}) \cap F_2 = \varnothing$ and similarly $\overline{U}(x_{22,n}) \cap F_1 = \varnothing$. Now we define by induction for any $n \geq 1$ the sets $V_{1,n}$ and $V_{2,n}$, setting

$$V_{1,n} = \overline{U}(x_{1,n}) \setminus \cup_{k=1}^{n} \overline{U}(x_{2,k}),$$
$$V_{2,n} = \overline{U}(x_{2,n}) \setminus \cup_{k=1}^{n} \overline{U}(x_{1,k}).$$

It is easy to see that $V_{1,n}$ and $V_{2,m}$ do not intersect. Indeed, if $n \leq m$, then

$$V_{1,n} \cap V_{2,m} \subset U(X_{1,n}) \cap (U(x_{2,m}) \setminus \overline{U}(x_{1,n}) = \varnothing,$$

and if $n > m$, then

$$V_{1,n} \cap V_{2,m} \subset (U(X_{1,n}) \setminus \overline{U}(x_{2,m}) \cap (U(x_{2,m}) = \varnothing.,$$

Consequently, the sets

$$V_1 = \cup_{n=1}^{\infty} V_{1,n}, \quad V_2 = \cup_{n=1}^{\infty} V_{2,n}$$

also do not intersect. On the other hand, they obviously contain the sets F_1 and F_2, respectively. Thus, we have proved that any two disjoint closed sets F_1 and F_2 of the space X can be enclosed in disjoint open sets V_1 and v_2. In other words, we have proved that the space X is normal. □

Now let Γ be an arbitrary open covering of X. Since X is regular by assumption, each point $x \in X$ has a neighbourhood $U(x)$ whose closure $\overline{U}(x)$ is contained in some element of Γ. Since X is also Lindelöf, there exists a countable system of points $x_1, \dots, x_k, \dots$ in it such that

$$X = \cup_{n=1}^{\infty} U(x_k).$$

Let U_k be an element of the covering Γ containing the set $\overline{U}(x_k)$. Using the already proven normality of the space X, for any $n \geq 1$ we can construct by induction an open set U_k^n such that

$$\overline{U}(x_k) \subset U_k^n \subset U_k \quad \text{and} \quad \overline{U}_k^n \subset U_k^{n+1}, \quad n, k = 1, 2, \dots$$

For each $n \geq 1$ we now put

$$V^n = \cup_{k=1}^{n} U_k^n.$$

It is clear that $\overline{V}^n \subset V^{n+1}$ and that the sets V^n form an open covering of the space X. Consider the sets

$$H^n = V^n \setminus \overline{V}^{n-2}, \quad n = 1, 2, \dots$$

(for $m \leq 0$ we conditionally assume that $V^m = \varnothing$). It is easy to see that the (obviously open) sets H^n form a covering of the space X. (Indeed, for any point $x \in X$ there exists an n such that $x \in V^n$ and $x \notin V^{n-1}$. But then $x \notin \overline{V}^{n-2}$ and therefore $x \in H^n$.) In addition, it is clear that this covering is star-finite (since $H^{n_1} \cap H^{n_2} = \varnothing$ for $n_2 \geq n_1 + 2$).

Let $k = 1, \dots, n$. Put

$$W_k^n = H^n \cap U_k^n = U_k^n \setminus \overline{V}^{n-2}, \quad n = 1, 2, \dots$$

Since $H^n \subset V^n = \cup_{k=1}^{n} U_k^n$, then $\cup_{k=1}^{n} W_k^n = H^n$. Consequently, the sets W_k^n constitute an (open) covering of the space X. This covering is a refinement of the covering Γ (since $W_k^n \subset U_k^n \subset U_k$) and is star-finite (since it is obtained from the star-finite covering $\{H^n\}$ by decomposing each of its elements into a finite number of sets). Since any star-finite open covering is locally finite, it is thus proved that any open covering of X can be refined into a locally finite open covering, i.e., that X is paracompact.

Let us now prove the fact mentioned above, namely, that

Proposition 1.21. *any compact subset C of a Hausdorff space X is closed,*

Proof. To this end, for any point $x \in X \setminus C$ and any point $c \in C$, we choose some disjoint neighbourhoods $U_c(x)$ and $U_x(c)$. For each fixed point $x \in X \setminus C$, the sets $U_x(c)$ $c \in C$ obviously form an open covering of the subset C. Therefore, there exists a finite system of points $c_1, \ldots, c_n \in C$ such that

$$C \subset \cup_{i=1}^{n} U_x(c_i).$$

Let

$$U(x) = \cup_{i=1}^{n} U_{c_i}(x).$$

The set $U(x)$ is open, contains the point x and does not intersect the set C. Thus, each point x of the set $X \setminus C$ is its interior point, i.e., this set is open. Therefore, the set C itself is closed. □

A subset A of a topological space X will be called *compactly closed* if its intersection with any compact subset $C \subset X$ is closed (in X). From the proposition just proved it follows immediately that

Proposition 1.22. *any closed subset of a Hausdorff space X is compactly closed.*

Thus, for Hausdorff spaces, the classes of compact, closed, and compact-closed subsets are related by a simple inclusion relation: each of these classes is wider than the previous one.

We will call a space X a *compactly generated*[2] space if it is a free union of all its compact subspaces, i.e. if its subset A is closed if and only if for any compact subset $C \subset X$ the intersection $A \cap C$ is closed in C. In a compactly generated space, any compactly closed set is closed. According to what was said above, if X is a Hausdorff space, then the converse is also true. Thus,

Proposition 1.23. *A Hausdorff space X is said to be a compactly generated space if and only if any of its compactly closed subsets is closed.*

Further, it is easy to see that

Proposition 1.24. *any closed (resp. open) subspace Y of a compactly generated (resp. compactly generated and regular) space X is also a compactly generated space.*

Proof. Indeed, let the set $A \subset Y$ have the property that for any compact set $C \subset Y$ the intersection $A \cap C$ is closed in C. We need to prove that then A is closed in Y. If Y is closed, then instead we will prove that A is closed in X. Since X is a compactly generated space, it suffices to prove that for any compact set $C \subset X$ the intersection $A \cap C$ is still closed in C. But this intersection coincides with the intersection $A \cap (Y \cap C)$, and the set $Y \cap C$, being the intersection of a compact and closed set, is compact. Therefore, by hypothesis, the intersection $A \cap (Y \cap C)$ is closed in C. □

[2]Transcriber's note: Postnikov has coined the word "kaonic."

Now let the subspace Y be open. Let us consider an arbitrary contact point a of the set A belonging to the subspace Y. Since the space X is, by assumption, regular, then this point has a neighbourhood U such that $\overline{U} \subset Y$. Since for any compact set $C \subset X$ the intersection $\overline{U} \cap C \subset Y$ is compact, the intersection $\overline{U} \cap C \cap A$ is closed in $\overline{U} \cap C$ and therefore in C. Consequently, the intersection $\overline{U} \cap A$ is closed in the space X. But it is clear that $a \in \overline{\overline{U} \cap A}$. Thus, $a \in \overline{U} \cap A$ and, therefore, $a \in A$. We have thus proved that each contact point of the set A belonging to the subspace Y lies in A. But this means that the set A is closed in the subspace Y.

It is clear that all compact spaces are compactly generated spaces. Moreover, it is easy to see that

Proposition 1.25. *any Hausdorff space X with countable local weight is compactly generated.*

Proof. Indeed, let a be an arbitrary contact point of some compactly closed subset A of the space X and let $\{a_n\}$ be an arbitrary sequence of points of the set A converging to the point a. Since this sequence together with its limit a forms a compact set, the intersection of this set with the set A is closed. But for $a \notin A$ this intersection would coincide with the sequence $\{a_n\}$ and would be obviously not closed (since the limit of this sequence $a \notin A$ would not belong to it). Consequently, $a \in A$. Thus, $\overline{A} = A$, i.e. the set A is closed. □

We will call a space X *countably compact* if it is a free union of some countable family $\{X_n, n \geq 0\}$ of its compact subspaces X_n. Without loss of generality, we can assume that $X_n \subset X_{n+1}$ for all $n \geq 0$. Every countably compact space is a compactly generated space and Lindelöf space. Any closed subspace of a countably compact space is countably compact. The topological sum of a countable number of countably compact spaces is a countably compact space. Moreover,

Proposition 1.26. *any Hausdorff countably compact space X is paracompact and, in particular, normal.*

Since the space X is Lindelöf, it suffices to establish its regularity. However, it will be more convenient for us to immediately prove its normality.

Let $F^{(1)}$ and $F^{(2)}$ be arbitrary disjoint closed subsets of X. Let us consider compact (and therefore closed and normal) subspaces

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots,$$

whose free union is the space X. It turns out that for any $n \geq 0$ there exist open (in X_n) sets $U_n^{(i)}$, $i = 1, 2$ such that

$$X_n \cap F^{(i)} \subset U_n^{(i)}, \quad i = 1, 2,$$
$$\overline{U}_n^{(1)} \cap \overline{U}_n^{(2)} = \varnothing \quad \text{and (for } n > 0) \quad U_n^{(i)} \cap X_{n-1} = U_{n-1}^{(i)}.$$

For $n = 0$, the existence of such sets is ensured by the normality of the space X_0. Let for some $n \geq 0$ the sets $U_n^{(i)}$, $i = 1, 2$ have already been constructed. Let us consider in the subspace X_{n+1} the sets $(X_n \cap F^{(i)}) \cup \overline{U}_n^{(i)}$ $i = 1, 2$. These sets are closed and their intersection is empty. Therefore, they have neighbourhoods $V^{(1)}$ and $V^{(2)}$ with disjoint closures $\overline{V}^{(1)}$ and $\overline{V}^{(2)}$. On the other hand, since the sets $U_n^{(1)}$ and $U_n^{(2)}$ are open in the subspace X_n of the space X_{n+1}, then in the space X_{n+1} there exist open sets $W^{(1)}$ and $W^{(2)}$ such that

$$W^{(i)} \cap X_n = U_n^{(i)}, \quad i = 1, 2.$$

We will put

$$U_{n+1}^{(i)} = V^{(i)} \cap (W^{(i)} \cup (X_{n+1} \setminus X_n)), \quad i = 1, 2.$$

The sets $U_{n+1}^{(i)}$, $i = 1, 2$ constructed in this way obviously possess all the necessary properties.

Let us now consider the sets

$$U^{(i))} = \cup_{n=0}^{\infty} U_n^{(i)}, \quad i = 1, 2.$$

Since for any $n \geq 0$ the sets $U_n^{(i)} = U^{(i)} \cap X_n$ are open in the subspace X_n, the sets $U^{(i)}$, $i = 1, 2$ are open in the space X. In addition, they do not intersect and contain the sets $F^{(1)}$ and $F^{(2)}$, respectively. Thus, any two non-intersecting closed sets $F^{(1)}$ and $F^{(2)}$ of the space X have non-intersecting neighbourhoods $U^{(1)}$ and $U^{(2)}$. But this means that the space X is normal.

Remark 1.27. In the above proof we used only the normality and closedness of the subspaces X_n.

Therefore,

Proposition 1.28. *any space that is a free union of a skeletal increasing sequence of closed normal subspaces is itself a normal space.*

A space X is called *locally compact* (resp. emphlocally countably compact) if each of its points has a neighbourhood whose closure is compact (resp. countably compact). Any compact (resp. countably compact) space is locally compact (resp. locally countably compact). Any discrete space is locally compact. For each compact subset of a locally compact (resp. locally countably compact) space, there exists a neighbourhood whose closure is compact (resp. countably compact). Every Hausdorff locally countably compact and, in particular, locally compact space is regular (since a Hausdorff countably compact space is normal). Any closed subspace of a locally compact (resp. locally countably compact) space X is locally compact (resp. locally countably compact). The corresponding assertion for open subspaces is also true only if X is Hausdorff. In particular, any open subspace of a Hausdorff compact space is locally compact.

Theorem 1.29. *Conversely, any Hausdorff locally compact space X is an open subspace of some Hausdorff compact space X' and the space X' can be constructed so that the "growth" of $X \setminus X'$ consists of only one point (P. S. Alexandrov's theorem on one-point compactification).*

Proof. Indeed, let ω be an arbitrary point not belonging to the space X. Let us introduce a topology into the set $X^{'} = X \cup \omega$, considering its open sets to be those and only those sets $A^{'} \subset X^{'}$, for which the set $A = A^{'} \setminus \omega$ is open in the space X and - in the case when to $\omega \in A^{'}$ - is, in addition, the complement (in X) of some compact set. It is easy to verify that the space $X^{'}$ is compact, Hausdorff, and the topology induced on X by the topology of the space $X^{'}$ coincides with the original topology of the space X. □

From the theorem on one-point compactification, in particular, it follows directly that

Proposition 1.30. *any Hausdorff locally compact space X is a compactly generated space.*

This statement can easily be proved directly. Indeed, let A be an arbitrary compactly closed subset of a locally compact space X. Let us prove that A is closed.

Proof. Let $x \in X \setminus A$. Since X is locally compact, x has a neighbourhood U whose closure $\overline{U}$ is compact. Since A is compactly closed, $A \cap \overline{U}$ is closed, and hence its complement $V = X \setminus (A \cap \overline{U})$ is open. Thus, x has a neighbourhood V that does not intersect $A \cap \overline{U}$. But then $x \in U \cap V \subset U \cap V \subset X \setminus A$, i.e. $x \in \mathrm{int}(X \setminus A)$. Thus, $X \setminus A$ is open, and hence A is closed. □

The last argument is almost literally preserved for locally countable compactly generated spaces. (The closedness of the set $A \cap \overline{U}$ follows in this case from the easily proven fact that the intersection of a compactly closed set with an arbitrary countably compact subspace is closed in this subspace.) Thus,

Proposition 1.31. *any Hausdorff locally countable compact space is a compactly generated space.*

Let us now deal exclusively with locally compact spaces (mainly Hausdorff).
First of all, we will prove that

Proposition 1.32. *any Hausdorff locally compact space X is a space of the second category.*

Moreover,

Proposition 1.33. *any subspace of the first category of a Hausdorff locally compact space X has the property that its complement is everywhere dense in the space X.*

Proof. Indeed, let $\{A_n; n \geq 1\}$ be an arbitrary countable family of nowhere dense subsets of X. We must prove that in any neighbourhood U of an arbitrary point x of X there exist points that do not belong to any of the sets A_n. For this purpose, we construct in X a family $\{U_n; n \geq 0\}$ of nonempty open sets U_n such that

1) the closure of the set $\overline{U}_0$ is compact and is contained in a neighbourhood of U;

2) for any $n \geq 1$ the relations

$$\overline{U}_n \subset U_{n-1}, \quad U_n \cap A_n = \varnothing$$

hold.

The existence of the set U_0 follows directly from the local compactness and regularity of the space X. Let for some $n \geq 1$ the set U_{n-1} has already been constructed. Since the set A_n is nowhere dense, in the open set U_{n-1} there exists a non-empty open set V_n such that $V_n \cap A_n = \varnothing$. Since the space X is regular, there exists a non-empty open set $U_n \subset V_n$ such that $\overline{U}_n \subset V_n$. It is clear that $\overline{U}_n \subset U_{n-1}$ and $\overline{U}_n \cap A_n = \varnothing$. Thus, the sets U_n have been constructed by induction for all $n \geq 0$.

Let us now consider the sets open in the subspace $\overline{U}_0$

$$W_n = \overline{U}_0 \setminus \overline{U}_n, \quad n \geq 1.$$

If the intersection of the sets $\overline{U}_n$, $n \geq 1$, is empty, then the sets W_n form a covering of the subspace $\overline{U}_0$. Since this subspace is compact by construction, a finite subcovering $\{W_{n_1}, \ldots, W_{n_k}\}$ can be chosen from this covering. The corresponding sets $\overline{U}_{n_1}, \ldots, \overline{U}_{n_k}$ will then have an empty intersection. The latter is impossible, since this intersection is equal to $\overline{U}_m$, where $m = \max(n_1, \ldots, n_k)$. Consequently, the intersection of sets $\overline{U}_n$, $n \geq 1$ is not empty. Since each point of the last intersection does not belong to any of the sets A_n, the above proposition is completely proved. □

Now recall that any Hausdorff locally compact space is regular. Therefore, if it is also Lindelöf, then it is also paracompact. Since the topological sum of paracompact spaces is a paracompact space, it follows immediately that

Proposition 1.34. *any Hausdorff locally compact space that is a topological sum of Linelöf subspaces is paracompact.*

It turns out that the converse is also true (and even without the Hausdorff proposition), i.e.

Proposition 1.35. *any locally compact paracompact space X is a topological sum of Lindelöf subspaces.*

Proof. Indeed, due to the local compactness of the space X, any point of it has a neighbourhood with compact closure. All these neighbourhoods form a certain covering Σ of the space X. Since the space X is paracompact, a locally finite covering Γ can be inscribed in the covering Σ. This covering (like the covering Σ) has the property that the closure of each of its elements U is compact. On the other hand, due to the local finiteness of the covering Γ, each point $x \in \overline{U}$ has a neighbourhood $V(x)$ that intersects only a finite number of elements of this covering. But, since the set $\overline{U}$ is compact, there exists a finite system of points $x_1, \dots, x_n \in \overline{U}$ such that

$$\overline{U} \subset \cup_{i=1}^{n} V(x_i).$$

Consequently, the set $\overline{U}$, and therefore the set U, intersects only a finite number of elements of the covering Γ, i.e. the covering Γ is star-finite.

Having now chosen some element $U0$ of the covering Γ, we consider for any $n \geq 1$ the subset X_n of X consisting of all points $xinX$ for which there exist n sets $U_1, \dots, U_n \in \Gamma$ such that $x \in U_n$ and

$$U_0 \cap U_1 \neq \emptyset, \quad U_1 \cap U_2 \neq \emptyset, \dots, U_{n-1} \cap U_n \neq \emptyset.$$

It is clear that the subset X_n is not empty, open and - by the star-like finiteness of the covering Γ - is the union of some finite number of elements of this covering. Therefore, the set

$$X_\infty = \cup_{n=1}^{\infty} X_n$$

is also open and is the union of a countable number of elements of the covering Γ. The union X_∞^* of their closures is obviously contained in the closure $\overline{X}_\infty$ of the subspace X_∞ (in fact, it coincides with this closure). Being the union of a countable number of compact sets, the subspace X_∞^* is Lindelöf.

Since any element of the covering Γ is either contained in X_∞ or does not intersect X_∞, the complement $X \setminus X_\infty$ is a union of some elements of $U \in \Gamma$ and is therefore open. Consequently, the subspace X_∞ is not only open but also closed, and therefore $X_\infty^* = X_\infty$. Thus, the subspace X_∞ is Lindeöf.

To complete the proof, it remains to note that the subspaces X_∞ corresponding to different elements U_0 of the covering Γ either coincide or do not intersect and that the union of X is the entire space X. □

Connectedness

A topological space X is called *connected* if each of its non-empty closed and simultaneously open subspaces coincides with the entire space X. In other words, a space X is connected if it cannot be decomposed into a union of two (or more) non-empty disjoint open (or closed) sets. Any space X is a union, generally speaking, not free, of disjoint closed maximal connected subspaces, called the *connected components* of X. A connected Hausdorff space has no isolated points. In particular, a discrete space is connected if and only if it consists of a single point.

A set obtained from a connected set by adding some of its points of contact is connected. In particular, the closure of a connected set is connected. The union of connected sets with non-empty intersection is connected. For example, a space X is connected if any two of its points belong to a connected set. Any interval (a, b) or segment $[a, b]$ of the number line $\mathbb{R}$ is connected. In particular, the unit segment $I = [0, 1]$ is connected.

From the definition of a connected space and the theorem proved above it immediately follows that

Proposition 1.36. *any connected paracompact locally compact space is Lindelöf.*

1.3 Continuous functions

A numerical function f defined on a topological space X is called *continuous* if for any point $x_0 \in X$ and any $\varepsilon > 0$ there exists a neighbourhood U of the point x_0 that

$$|f(x) - f(x_0)| < \varepsilon$$

for all points $x \in U$. Such functions have many properties of continuous functions of a numerical argument. For example, any continuous function defined on a compact space X is bounded and takes its largest and smallest values. Any continuous function defined on a connected space X takes all intermediate values. The restriction $f|_A$ to an arbitrary subspace $A \subset X$ of any continuous function f on X is a function continuous on A.

For a numerical function f defined on a space X and an arbitrary number a, we will denote by the symbol $[f \leq a]$ the set of all points $x \in X$ for which $f(x) \leq a$. The notations $[f < a]$, $[f = a]$, $[a < f \leq b]$, etc. have a similar meaning. A function f is continuous if and only if for any $a \in \mathbb{R}$ the set $[f \leq a]$ is closed or, equivalently, if for any $a \in \mathbb{R}$ the set $[f < a]$ is open. From this, in particular, it follows that for any continuous function f and any number a the set $[f = a]$ is closed and the set $[f \neq a]$ is open. We will call the closure $\overline{[f \neq 0]}$ of the set $[f \neq 0]$ the *support* of the function f.

The sets $[f = a]$ are a special case of the *coincidence sets* $[f = g]$, defined for any pair of continuous (on the space X) functions f and g and consisting, by definition, of all points $x \in X$ for which $f(x) = g(x)$. It is easy to see that

Proposition 1.37. *for any two continuous functions f and g the set $[f = g]$ is closed.*

Proof. Indeed, if $x \notin [f = g]$ then $f(x) \neq g(x)$ and therefore the points $f(x)$ and $g(x)$ have disjoint neighbourhoods U and V. The set $f^{-1}(U) \cap g^{-1}(V)$ is open, contains the point x and does not intersect the set $[f = g]$. It is thus proved that the complement $X \setminus [f = g]$ of the set $[f = g]$ is open. Consequently, the set $[f = g]$ itself is closed. □

Continuous functions exist on any topological space X. Indeed, any constant function, i.e. a function that takes the same value at all points $x \in X$, is obviously continuous. However, non-constant continuous functions, generally speaking, may not exist. For their existence, it is sufficient that the space X be normal (and contain more than one point). Namely, it can be shown that for any two distinct points x_0 and x_1 of a normal space X, there exists a continuous function f on X such that $f(x_0) = 0$ and $f(x_1) = 1$. Moreover, as P. S. Uryson first proved,

Proposition 1.38. *for any closed set A of a normal space X and any of its neighbourhoods U, there exists on the space X a continuous function f that takes values from the interval $[0,1]$, is equal to one on A and zero outside U, i.e. such that*

$$A \subset [f = 1], \quad U \supset [f \neq 0].$$

This statement is known as Urysohn's lemma. Note that here we do not exclude cases when $A = \varnothing$ or $U = X$ (however, in these cases the lemma is trivial). We will call each function f satisfying the conditions of the Urysohn lemma a *Urysohn function* of the pair (U, A).

Proof. To prove Urysohn's lemma, we construct in the space X a family $\{V_r\}$ of neighbourhoods V_r of the set A contained in a neighbourhood U, numbered by binary-rational numbers $r \in (0,1)$ (i.e., numbers of the form $\frac{m}{2^n}$, where $n = 1, 2, \dots$ and $m = 1, 2, \dots, 2^n - 1$) and possessing the property that

$$\overline{V}_r \subset V_{r'}, \quad \text{if} \quad r < r'.$$

Such neighbourhoods V_r are easily constructed by induction on n. For $n = 1$, for the neighbourhood $V_{1/2}$ we should take an arbitrary neighbourhood V of the set A for which $\overline{V} \subset U$ (such a neighbourhood exists by the normality of the space X). Let for some $n \geq 1$ the neighbourhoods $V_{m/2^n}$ have already been constructed $(m = 1, 2, \dots, 2^n - 1)$. Let us take for each number $r = (2s+1)/2^{n+1}$, $s = 0, 1, \dots, 2^n - 1$, the neighbourhood of V_r to be an open set V (existing due to the normality of the space X) for which $\overline{V}_{s/2^n} \subset V$ and $\overline{V} \subset \overline{V}_{(s+1)/2^n}$ (for $s = 0$, the set $\overline{V}_{s/2^n}$ means the set A, and for $s = 2^n - 1$, the neighbourhood of $V_{(s+1)/2^n}$ means the neighbourhood of U). Thus, we obtain neighbourhoods of V_r for $n + 1$ as well.

Having constructed the neighbourhoods of V_r, we define the function g on the space X by the formula

$$g(x) = \sup_{x \notin V_r} r, \quad x \in X.$$

In other words, the value $g(x)$ of the function g at a point of $x \in X$ is equal to the upper line of all numbers r, for which $x \notin V_r$. The function g is unequivocally defined and continuous (for for any $a \in \mathbb{R}$ there are a lot of $[g < a]$ coincides with the many $\cup_{r<a} V_r$ and therefore open). Its values belong to a segment of $[0,1]$ and it is equal to zero on a lot of A and to unity outside the vicinity of U. Therefore, the function $f = 1 - g$ is the desired function of Uryson. □

Remark 1.39. Urysohn's lemma does not assert that $A = [f = 1]$ or that $U = [f \neq 0]$. Generally speaking, a function f for which at least one of these equalities holds may not exist. Closed sets of the form $[f = \text{const}]$ are sometimes called *functionally closed*. Similarly, open sets of the form $[f \neq \text{const}]$ are called *functionally open*. It can be proved that a closed (resp. open) set of a normal space is functionally closed (resp. open) if and only if it can be represented as the intersection (resp. union) of a countable family of open (resp. closed) sets.

We will not need this fact and will leave it without proof.

Let us now show that

Proposition 1.40. *for any continuous function g defined on a closed subspace A of a normal space X, there exists on the space X a continuous function f such that*

$$f(x) = g(x)$$

for each point $x \in A$.

This statement is known as Tietze's theorem (sometimes it is also called the Brouwer-Urysohn theorem). The function f provided by this theorem we will call the extension of the function g (from the subspace A to the entire space X).

Proof. First, we prove Tietze's theorem under the additional assumption that

$$|g(x)| \leq 1 \tag{1.41}$$

for all $x \in A$. We define by induction on the set A the sequence $\{g_n\}$ of continuous functions, setting

$$g_0 = g,$$
$$g_{n+1} = g_n + \frac{2^n}{3^{n+1}}(2h_n - 1), \quad n \geq 0,$$

where h_n is the Urysohn function on the space X, constructed for the closed (in A and therefore in X) set $[g_n \leq -\frac{2^n}{3^{n+1}}]$ and its neighbourhood $X \setminus [g_n \geq \frac{2^n}{3^{n+1}}]$. By induction, for any point $x \in A$, the estimate

$$|g_n(x)| \leq \left(\frac{2}{3}\right)^n$$

(it should be borne in mind that $g_{n+1}(x) = g_n(x) + \frac{2^n}{3^{n+1}}$ when $g_n(x) \leq -\frac{2^n}{3^{n+1}}$ and $g_{n+1}(x) = g_n(x) - \frac{2^n}{3^{n+1}}$ for $g_n(x) \geq \frac{2^n}{3^{n+1}}$.) Therefore

$$\lim_{n\to\infty} g_n x) = 0$$

for any point $x \in A$.

Now we compose from functions continuous on the space X

$$f_n(x) = -\frac{2^n}{3^{n+1}}(2h_n(x) - 1)$$

an infinite series

$$f_0(x) + f_1(x) + \cdots + f_n(x) + \cdots \tag{1.42}$$

Since for any point $x \in X$ the inequality $|f_n(x)| \leq \frac{2^n}{3^{n+1}}$ holds and the numerical series

$$\frac{1}{3} + \frac{2}{9} + \cdots + \frac{2^n}{3^{n+1}} + \cdots \tag{1.43}$$

converges, the functional series (1.42) also converges (at all points $x \in X$ and its sum $f(x)$ is continuous on X. Moreover, for $x \in A$, for the n-th partial sum $s_n(x) = f_0(x) + \cdots ; f_n(x)$ of Series (1.42), the equality

$$s_n(x) = g_0(x) - g_{n+1}(x)$$

holds. Passing to the limit in this relation (for $n \to \infty$), we immediately obtain that for any point $x \in A$, the equality

$$f(x) = g(x)$$

holds. Thus, Tietze's theorem under the additional assumption (1.41) is completely proved. In this case, the constructed function f satisfies the inequality

$$|f(x)| \leq 1, \quad x \in X$$

(since the sum of series (1.43) is equal to unity).

Let us now consider the case when for all points $x \in A$ the strict inequality

$$|g(x)| < 1, \tag{1.44}$$

is satisfied and we will show that then there exists an extension f of the function g for which a similar strict inequality

$$|f(x)| < 1, \tag{1.45}$$

is satisfied at all points $x \in X$.

Indeed, as has been proved, there exists an extension f_0 of the function g that satisfies the inequality $|f_0(x)| \leq 1$ at any point $x \in X$. Let $A_1 = A \cup [f_0 = 1] \cup [f_0 = 1]$. Let us define a numerical function g_1 on the set A_1, assuming

$$g_1(x) = \begin{cases} g(x), & \text{if} \quad x \in A, \\ 1, & \text{if} \quad x \in [f_0 = -1], \\ -1, & \text{if} \quad x \in [f_0 = 1]. \end{cases}$$

Since the sets A, $[f_0 = -1]$ and $[f_0 = 1]$ are closed and pairwise disjoint, the function g_1 is uniquely defined and continuous. Moreover,

$$|g_1(x)| \leq 1$$

for any point $x \in A$. Consequently, as proved above, on the space X there exists an extension f_1 of the function g_1 such that

$$|f_1(x)| \leq 1$$

for any point $x \in X$. But then the function

$$f(x) = \frac{f_0(x) + f_1(x)}{2}$$

will be an extension on X of the function g_1 satisfying the condition (1.45).

To prove Tietze's theorem in the general case, it is now sufficient to note that for any function $g(x)$ continuous on A, the function $g*(x) = \frac{2}{\pi} \arctan g(x)$ is continuous on A and satisfies the condition (1.44). Let $f_*(x)$ be its extension satisfying condition (1.45). It is clear that the function $f(x) = \tan\{\frac{\pi}{2} f(x)\}$ is continuous on X and is an extension of the function $g(x)$.

Thus, Tietze's theorem is completely proven. □

In addition to Tietze's theorem, it is useful to note that

Proposition 1.46. *if for any point $x \in A$ the inequalities*

$$a \le g(x) \le b$$

(resp. the inequalities $a < g(x) < b$) hold, then the extension f of the function g can be constructed so that for any point $x \in X$ the inequalities

$$a \le f(x) \le b$$

(resp. the inequalities $a < f(x) < b$) hold.

Proof. Indeed, for $a = -1$, $b = 1$ this statement has in fact already been proven above. The general case is reduced to this special one by an obvious linear transformation. □

Now let $\Gamma = \{U_\alpha; \alpha \in A\}$ be an arbitrary open locally finite covering of X. A family $\{f_\alpha; \alpha \in A\}$ of continuous functions f_α on X will be called a *partition of unity subordinate to the covering* Γ if

1) for any $\alpha \in A$ the function f_α is non-negative and its support $[\overline{f_\alpha \neq 0}]$ is contained in the element U_α of the covering Γ;

2) for any point $x \in X$ the equality

$$\sum_{\alpha \in A} f_\alpha(x) = 1$$

holds.

(due to the local, and therefore point, finiteness of the covering Γ, in the last sum for any point $x \in X$ only a finite number of terms are non-zero).

It turns out that

Proposition 1.47. *for any locally finite covering $\Gamma = \{U_\alpha; \alpha \in A\}$ of a normal space X there exists a subordinate partition of unity $\{f_\alpha; \alpha \in A\}$.*

To prove this statement, it suffices to prove that

Proposition 1.48. *for any locally finite covering $\Gamma = \{U_\alpha; \alpha \in A\}$ of a normal space X, there exists a covering $\Delta = \{V_\alpha; \alpha \in A\}$ such that*

$$\overline{V}_\alpha \subset U_\alpha$$

for any $\alpha \in A$.

Proof. Indeed, since the space X is normal, for any element V_α of the covering Δ there exists an open set W_α such that

$$\overline{V}_\alpha \subset W_\alpha, \quad \overline{W}_\alpha \subset U_\alpha$$

But then, as is easy to see, the functions

$$f_\alpha(x) = \frac{g_\alpha(x)}{\sum_{\alpha\in A} g_\alpha(x)}, \quad \alpha \in A, \quad x \in X,$$

where g_α is the Urysohn function of the pair $(W_\alpha, \overline{V}_\alpha)$, are uniquely defined, continuous (due to the local finiteness of the covering Γ) and constitute a partition of unity subordinate to the covering Γ.

All that remains for us, therefore, is to construct the covering Δ. For this purpose, we consider the set $\mathfrak{G}$ of all (open) coverings $\{G_\alpha; \alpha \in A\}$ of the space X for which for any $\alpha \in A$ either $\overline{G}_\alpha \subset U_\alpha$ or $G_\alpha = U_\alpha$. This set is not empty (for example, it contains the given covering Γ). For any such covering $\{G_\alpha; \alpha \in A\}$ we will denote by the symbol A_0 the set of all $\alpha \in A$ for which $\overline{G}_\alpha \subset U_\alpha$.

Let us introduce a partial ordering into the set $\mathfrak{G}$, assuming that $\{G_\alpha\} < \{G'_\alpha\}$ if $A_0 \subset A'_0$ and if $G_\alpha = G'_\alpha$ for any $\alpha \in A$. Roughly speaking, to obtain a "larger" covering $\{G'_\alpha\}$ from a covering $\{G_\alpha\}Oa$, we need, without touching the already constructed sets G_α, to replace some U_α with smaller sets G_α.

It is clear that this relation is indeed a partial ordering relation (i.e. it has the property of transitivity).

Let us now consider an arbitrary chain (= linearly ordered subset) in the set $\mathfrak{G}$. Denoting the elements of this subset by the symbols Γ^β, where β runs through some set of indices B, and setting $\Gamma^\beta = \{G^\beta_\alpha; \alpha \in A\}$, we define for each $\alpha \in A$ the set $G_\alpha \subset X$ by the formula

$$G_\alpha = \cap_{\beta\in B} G^\beta_\alpha.$$

It is clear that for any $\alpha \in A$ all sets $G^\beta_\alpha \neq U_\alpha$ (if such sets exist) coincide with each other. Therefore, there exists an index $\beta_\alpha \in B$ such that $G_\alpha = G^{\beta_\alpha}_\alpha$. Consequently, the set G_α is open and has the property that either $G_\alpha = U_\alpha$ or $\overline{G}_\alpha \subset U_\alpha$. We will show that the family $\{G_\alpha; \alpha \in A\}$ is a covering of the space X.

Let x be an arbitrary point in X and let A_x be a subset of A consisting of all indices $\alpha \in A$ for which $x \in U_\alpha$. Since the set A_x is finite due to the local finiteness of the covering Γ, among the indices β_α, $\alpha \in A_x$, there exists the largest (with respect to the ordering of the set B induced by the ordering of the set $\mathfrak{G}$) index β_{α_0}. Since the family $\Gamma^\beta_{\alpha_0} = \{G^{\beta_{\alpha_0}}_\alpha \alpha \in A\}$ is a covering, there exists an index $\alpha_* \in A$ such that $x \in \Gamma^{\beta_{\alpha_0}}_{\alpha_*}$. Since $G^{\beta_{\alpha_0}}_{\alpha_*} \subset U_{\alpha_*}$, the index α_* belongs to the subset A_x. But it is clear that for any index $\alpha \in A_x$ the equality $G^{\beta_{\alpha_0}}_\alpha = G^{\beta_\alpha}_\alpha = G_\alpha$ holds. Consequently, $x \in G_{\alpha_*}$.

The constructed covering $\{G_\alpha; \alpha \in A\}$ belongs, according to what was said above, to the set $\mathfrak{G}$ and is, as is easy to see, its smallest element, following all the elements of Γ^β, i.e., it represents the upper bound of the considered chain (= linearly ordered subset.)

Thus, we have proved that any chain (= linearly ordered subset) of the set $\mathfrak{G}$ has an upper bound. Consequently, according to Zorn's lemma[3], this set contains at least one maximal element $\{V_\alpha; \alpha \in A\}$.

To complete the proof, it remains to show that the covering $\Delta = \{V_\alpha; \alpha \in A\}$ has the property that for any of its elements V_α the inclusion holds

$$\overline{V}_\alpha \subset U_\alpha.$$

Suppose not. Then there exists an index $\alpha_0 \in A$ such that $V_{\alpha_0} = U_{\alpha_0}$. Consider the closed set

$$F = X \setminus \cup_{\alpha \neq \alpha_0} U_\alpha.$$

It is clear that $F \subset U_{\alpha_0}$. Therefore, since the space X is normal, there exists an open set V such that

$$F \subset V, \quad \overline{V} \subset U_{\alpha_0}.$$

It is easy to see that the family $\Delta' = \{V, V_\alpha; \alpha \in A \setminus \alpha_0\}$ is a covering of the space X, belongs to the set $\mathfrak{G}$, and is distinct from the covering Δ and has the property that $\Delta < \Delta'$. But in view of the maximality of the covering Δ this is impossible. □

Thus, the theorem formulated above is completely proven.

Remark 1.49. We needed the local finiteness of the covering Γ to ensure that all sets A_x were finite. Consequently, the covering Δ also exists for point-finite coverings Γ.

1.4 Metric spaces

A set X is said to be defined as a *metric space* or to have a *metric* introduced into it if any two of its points $x, y \in X$ are assigned a non-negative real number $\rho(x, y)$ (called the *distance* between these points), and the following axioms are satisfied:

1) the equality $\rho(x, y) = 0$ holds if and only if $x = y$;

2) for any two points $x, y \in X$ the equality holds

$$\rho(x, y) = \rho(y, x);$$

3) for any three points $x, y, z \in X$ the inequality holds

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

[3]This lemma states that a partially ordered set in which every chain (= linearly ordered subset) has an upper bound contains a maximal element.

Axiom 1) is called the *axiom of non-degeneracy* of the metric, axiom 2) is called the *axiom of symmetry*, and axiom 3) is called the *triangle axiom*.

An example of a metric space is the n-dimensional arithmetic space $\mathbb{R}^n$ with the usual Euclidean metric (in this metric, the distance $\rho(\boldsymbol{u},\boldsymbol{v})$ between the points $\boldsymbol{u} = (u_1,\dots,u_n) \in \mathbb{R}^n$ and $\boldsymbol{v} = (v_1,\dots,v_n) \in \mathbb{R}^n$ is taken to be the length $|\boldsymbol{u}-\boldsymbol{v}| = \sqrt{(u_1-v_1)^2+\cdots+(u_n-v_n)^2}$ of the vector $\boldsymbol{u}-\boldsymbol{v}$).

The *distance* $\rho(A,B)$ between two subsets A and B of a metric space X is the greatest lower bound $\inf \rho(x,y$ of the distances $\rho(x,y)$, where x and y are arbitrary points of the subsets A and B, respectively. This distance can be equal to zero even without the sets A and B intersecting. For any $\varepsilon > 0$, the set $S_\varepsilon(A)$ of all points $x \in X$ for which $\rho(x,A) < \varepsilon$ is called a *spherical* ε*-neighbourhood* of the subset $A \subset X$. In any metric space, one can introduce one and only one topology in which all spherical neighbourhoods of each of its points are open and constitute a fundamental system of its neighbourhoods. In this topology, a set $A \subset X$ is closed if and only if any point $x \in X$ for which $\rho(x,A) = 0$ belongs to this set. We will call this topology the *natural topology* of the metric space X and in what follows we will consider each metric space (for example, the Euclidean space $\mathbb{R}^n$) as a topological space with this natural topology.

The topology of the space $\mathbb{R}^n$ can be described in another way, noting that it is a topological product of n copies of the number line $\mathbb{R}$. Therefore, the base of this space is, for example, the family of all parallelepipeds

$$a^1 < t^1 < b^1, \dots, a^n < t^n < b^n, \quad (t^1,\dots,t^n) \in \mathbb{R}^n,$$

where a^i and $b^i > a^i$, $i = 1,\dots,n$ are arbitrary real numbers. The parallelepipeds for which the weight of the number a^i and b^i are rational obviously also form the base of the space $\mathbb{R}^n$.

By definition, the natural topology of a metric space X has the property that for any point $x \in X$, the system of all possible spherical neighbourhoods $S_\varepsilon(x)$ is a fundamental system of neighbourhoods of the point x. A similar statement also holds for compact subsets, i.e., for any compact subset $C \subset X$, the neighbourhoods $S_\varepsilon(C)$ constitute a fundamental system of its neighbourhoods. In other words, for any open set $U \supset C$, there exists $\varepsilon > 0$ such that $S_\varepsilon(c) \subset U$. For arbitrary closed subsets, the corresponding statement is, generally speaking, false.

In the natural topology of X, the metric $\rho(x,y)$ is a continuous function on the product $X \times X$. Moreover, for any point $x \in X$ and any closed set $A \subset X$, the function $\rho(x,A)$ is continuous on X. For any closed set $A \subset X$ and any neighbourhood U, the function

$$f(x) = \frac{\rho(x, X \setminus U)}{\rho(x,A) + \rho(x, X \setminus U)}$$

is the Urysohn function of the pair (U,A). For this function, the set $[f = 1]$ exactly coincides with the set A (and the set $[f \neq 0]$ with the set U). The set $[f > \frac{1}{2}]$ is a neighbourhood of the set A whose closure $[f \geq \frac{1}{2}]$ is contained in the neighbourhood of U. Thus, any metric space, being obviously Hausdorff, is normal. Moreover, unlike arbitrary normal spaces, in the metric space X any closed set is functionally closed and any open set is functionally open.

A topological space X is called *metrisable* if there exists on it (generally speaking, not a unique) metric that is consistent with the topology of the space X, i.e. such that the natural topology defined by it coincides with the topology of the space X.

Proposition 1.50. *This metric can always be chosen so that the resulting metric space is bounded, i.e. so that the distance between any two of its points does not exceed some fixed number $K > 0$.*

Proof. Indeed, any metric ρ can be transformed into a bounded (by a number K) metric ρ', defining the same topology, by setting

$$\rho'(x, y) = K\frac{\rho(x, y)}{1 + \rho(x, y)}$$

for any points $x, y \in X$. □

Proposition 1.51. *Every discrete space is metrisable. The topological sum of any number and the topological product of a countable number of metrisable spaces are metrisable.*

Proof. The metric in the topological product of a countable number of metrisable spaces is introduced by means of an obvious infinite series. In order for this series to converge, it is sufficient to choose a metric in the n-th factor, bounded, say, by the number $1/n^2$. Verifying that the metric constructed in this way is consistent with the topology of the product does not present any difficulties, and we will omit it. □

From the metrisability of the product of metric spaces it immediately follows, in particular, that any metric space is stably normal.

Each subset A of a metric space X is naturally defined as a metric space, and the metric of the subset A is consistent with its topology induced by the natural topology of the entire space X.

In particular, any subset of the Euclidean space $\mathbb{R}^n$ is a metric space. Among these subsets, the unit ball $\mathbb{E}^n$ of the space $\mathbb{R}^n$, consisting of all points $\boldsymbol{v} \in \mathbb{R}^n$ for which $|\boldsymbol{v}| \leq 1$, will play a special role for us in what follows (here and below we identify points of the space $\mathbb{R}^n$ with their radius vectors). The ball $\mathbb{E}^n$ is closed in the space $\mathbb{R}^n$ and is the *closure of the open unit ball E^n* consisting of points $\boldsymbol{v} \in \mathbb{R}^n$ for which $|\boldsymbol{v}| < 1$. The boundary of the ball $\mathbb{E}^n$ is the unit sphere $\mathbb{S}^{n-1}$ consisting of points $\boldsymbol{u} \in \mathbb{R}^n$ for which $|\boldsymbol{u}| = 1$. Each point of the ball $\mathbb{E}^n$ has the form $v\boldsymbol{u}$, where $0 \leq v \leq 1$ and $\boldsymbol{u} \in \mathbb{S}^{n-1}$, and for $v \neq 0$ this representation is unambiguous.

In the definitions presented it was assumed that $n > 0$. Sometimes it will be convenient for us to extend them to the case $n = 0$, assuming that the ball $\mathbb{E}^0$ consists of one point. In this case we will assume that $E^0 = \mathbb{E}^0$ and that $\mathbb{S}^{-1} = \varnothing$.

It is easy to show that the ball $\mathbb{E}^n$ (and also the open ball E^n) and the sphere $\mathbb{S}^{n-1}$ (for $n > 1$) are connected. Moreover, the ball $\mathbb{E}^n$ and the sphere $\mathbb{S}^{n-1}$ are compact, and the open ball E^n is locally compact. Finally, the space $\mathbb{R}^n$ (and therefore all its subspaces) is a space with a countable base and, therefore, has countable local weight, is separable and Lindelöf.

It is easy, however, to see that

Proposition 1.52. *any metric space X is a space of countable local weight.*

Proof. Indeed, for each point $x \in X$ the neighbourhoods $S_{1/n}(x)$ obviously constitute a fundamental system of neighbourhoods. □

In particular,

Proposition 1.53. *every metric space is a compactly generated space.*

Moreover,

Proposition 1.54. *for any metric space X the following properties are equivalent:*

1) the space X has a countable base;

2) the space X is separable;

3) the space X is Lindelöf.

Proof. Indeed, implications 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) hold for any topological spaces. To prove implication 2) $\Rightarrow$ 1) it suffices to note that for any countable everywhere dense set $\{x_n\}$ the open sets of the form $S_{1/m}(x_n)$, where $n, m = 1, 2, \dots,$ form a base of the space X.
To prove implication 3 $\Rightarrow$ 2), we, having chosen an arbitrary $n > 0$, consider the family of all subsets of the space X, each of which has the property that the pairwise distances between any two of its points are not less than $1/n$. It is clear that this family (partially ordered by inclusion) satisfies the conditions of Zorn's lemma and, therefore, it contains a maximal subset $C_{1/n}$. This subset is discrete and therefore (by the Lindelöf property) countable. Consequently, the set

$$C = \cup_{n=1}^{\infty} C_{1/n}$$

is also countable. To complete the proof, it remains to note that the set C is everywhere dense (since for any point $x \in X$ and any $n > 0$ in the set $C_{1/n}$, and - due to its maximality - therefore in the set C, there exists a point y such that $\rho(x, y) < 1/n$). □

Since every compact space is finally compact, this theorem implies that

Proposition 1.55. *every compact metric space is separable.*

Local compactness, however, is no longer sufficient for separability of a metric space. Nevertheless,

Proposition 1.56. *any connected locally compact metric space is separable.*

This theorem is due to P. S. Alexandrov. It follows immediately from the result of 2) above and the fact that

Proposition 1.57. *any metric space X is paracompact.*

The last proposition was first proven by Stone.

Proof. To prove it, we must establish that any covering $\{U_\alpha; \alpha \in A\}$ of X can be refined into a locally finite covering. For this purpose, we assume that the set of indices A of the covering under consideration is well-ordered. Let α_0 be its first element. Denoting for any set $A \subset X$ and any $n > 0$, by the symbol $[A]_n$, the set (obviously closed) of all points $x \in X$ for which $S_{2^{-n}}(x) \subset A$, we construct by induction for each $\alpha \in A$ and any $n > 0$ the closed set F^n_α, by

$$F^n_{\alpha_0} = [U_{\alpha_0}]_n, \quad F^n_\alpha = [U_\alpha \setminus \cup_{\beta<\alpha} F^n_\beta]_n.$$

Let x be an arbitrary point in X and let α be the smallest index in A for which $x \in U_\alpha$, and n be a number such that $S_{2^{-n}}(x)) \subset U_\alpha$. If $x \notin F^n_\alpha$, then $S_{2^{-n}}(x) \not\subset U_\alpha \setminus \cup_{\beta<\alpha} F^n_\beta$, and therefore there exists an index $\alpha_1 < \alpha$ such that $S_{2^{-n}}(x) \cap F^n_{\alpha_1} \neq \varnothing$. But then

$$x \in S_{2^{-n}}(F^n_{\alpha_1}) = S_{2^{-n}}([U_{\alpha_1} \setminus \cup_{\beta<\alpha} F^n_\beta]_n) \subset U_{\alpha_1} \setminus \cup_{\beta<\alpha} F^n_\beta \subset U_{\alpha_1},$$

which is impossible due to the choice of index α. Consequently, $x \in F^n_\alpha$. It is thus proved that

$$X = \cup_{n,\alpha} F^n_\alpha.$$

Let us further consider the sets

$$\Phi^n_\alpha = \overline{S}_{2^{-(n+3}}(F^n_\alpha), \quad G^n_\alpha S_{2^{-(n+2)}}(F^n_\alpha).$$

It is clear that the closed set Φ^n_α is contained in the open set G^n_α. Further, since

$$\overline{S}_{2^{-n}}(F^n_\alpha) \subset U_\alpha \setminus \cup_{\beta<\alpha} F^n_\alpha,$$

then for any $\beta < \alpha$ the intersection $S_{2^{-n}}(F^n_\alpha) \cap F^n_\beta$ is empty because $\rho(F^n_\alpha, F^n_\beta) \geq 2^{-n}$. Therefore, $\rho(G^n_\alpha, G^n_\beta) \geq 2^{-(n+1)}$. Since $\Phi^n_\alpha \subset G^n_\alpha$, it follows that for any $n > 0$ the set

$$\Phi^n = \cup_\alpha \Phi^n_\alpha$$

is closed.

Now we define by induction the open sets V^n_α, setting

$$V^1_\alpha = G^1_\alpha, \quad V^n_\alpha = G^n_\alpha \setminus \cup_{m<n} \Phi^m.$$

Let $x \in X$. Since $\cup_{n,\alpha} F^n_\alpha = X$ and, therefore, $\cup_{n,\alpha} \Phi^n_\alpha = X$, there exist $n \geq 0$ and $\alpha \in A$ such that $x \in \Phi^n_\alpha$. If n is chosen (for a given α) to be the smallest possible, then

$$x \in \Phi^n_\alpha \setminus \cup_{m<n} \Phi^m_\alpha = \Phi^n_\alpha \setminus \cup_{m<n} \Phi^m$$

(as we know, $\Phi_\alpha^n \subset G_\alpha^n$ and therefore $\Phi_\alpha^m \cap \Phi_\beta^m \neq \varnothing$, if $\alpha \neq \beta$). Hence

$$x \in G_\alpha^n \setminus_{m<n} \Phi^m = V_\alpha^n.$$

This shows that the sets V_α^n form a covering of the space X.

Since
$$V_\alpha^n \subset G_\alpha^n \subset S_{2^{-n}}(F_\alpha^n) \subset U_\alpha \setminus \cap_{\beta<\alpha} F_\beta^n \subset U_\alpha$$
then the covering $\{V_\alpha^n\}$ is a refinement of the covering $\{U_\alpha\}$.

Finally, let $x \in X$ and let $x \in F_{\alpha'}^n$. Since

$$S_{2^{-(n+3)}}(x) \subset \Phi_{\alpha'}^n \subset \Phi^n$$

then
$$S_{2^{-(n+3)}}(x) \cap V_\alpha^m = \varnothing$$
for any $\alpha \in A$ and any $m > n$. On the other hand, since $\rho(G_\alpha^m, G_\beta^m) \geq 2^{-(m+1)}$ for $\alpha \neq \beta$, and $2 \cdot 2^{-(m+3)} < 2^{-(n+1)}$ for $m \leq n$, then for each $m \leq n$ there is at most one index $\alpha \in A$ for which $S_{2^{-(n+3)}}(x) \cap G_\alpha^m \neq \varnothing$, and hence at most one index $\alpha \in A$ for which $S_{2^{-(n+3)}}(x) \cap V_\alpha^m = \varnothing$. Thus, the neighbourhood $S_{2^{-(n+3)}}(x)$ of the point x intersects at most n elements of the covering $\{V_\alpha^n\}$. Consequently, this covering is locally finite.

Thus, Stone's theorem is completely proved. □

From Stone's theorem and the results of §1.2 it immediately follows, in particular, that

Proposition 1.58. *a metric space is compact if and only if it is sequentially compact.*

All the properties of metric spaces considered above are related not so much to metric spaces as to metrisable spaces and therefore had, in essence, a topological character. Let us now consider some "metric" properties of metric spaces, i.e. properties that essentially depend on the metric given in the space.

A sequence $\{x_n\}$ of points of a metric space X is called *fundamental* if for any $\varepsilon > 0$ there exists $N > 0$ such that $\rho(x_n, x_m) < \varepsilon$ when $n, m > N$. It is clear that any convergent sequence is fundamental. If the converse is true, i.e. if any fundamental sequence of points of X converges, then this space is called a *complete space*. Obviously, every closed subspace of a complete space is itself a complete space.

Proposition 1.59. *If any closed bounded subspace of a metric space X is compact, then X is complete.*

Proof. Indeed, let $\{x_n\}$ be an arbitrary fundamental sequence of points in X. If among its points there are only a finite number of distinct ones, then this sequence obviously converges. Let among the points x_n there be infinitely many distinct ones. The set of these points, being obviously discrete and bounded, cannot be closed (for otherwise it would be compact, and any discrete compact set is finite). Therefore, for this set there exists an adherent point a, that does not belong to it. This point is the limit of some convergent subsequence of the sequence $\{x_n\}$, and therefore of the entire sequence $\{x_n\}$ (since this sequence, by assumption, is fundamental). □

The conditions of this theorem are satisfied, in particular, by the Euclidean space $\mathbb{R}^n$. Therefore, the space $\mathbb{R}^n$ (and consequently any of its closed subspaces) is complete.

On the other hand, every compact metric space also satisfies these conditions. Consequently,

Proposition 1.60. *any compact metric space is complete.*

The requirement of local compactness for completeness is no longer sufficient, even if we additionally assume the existence of a countable base. However, in any locally compact topological space X with a countable base, one can introduce a metric consistent with the topology of this space, with respect to which the space X is a complete metric space. We will not need this result, and therefore we will leave it without proof.

It can be shown that any complete metric space is a space of the second category. The proof of this statement essentially repeats the proof of a similar proposition for Hausdorff locally compact spaces (see §1.2). It is only necessary to require that instead of the compactness of the sets $\overline{U}_n$, their diameters tend to zero. Since we will not need this statement either, we will not give this proof in detail.

1.5 Continuous maps

Let X and Y be arbitrary topological spaces. It is easy to see that

Proposition 1.61. *for any single-valued map*

$$f : X \to Y$$

of X into Y the following properties are equivalent:

1) *the complete preimage $f^{-1}(B)$ under the map f of an arbitrary closed set $B \subset Y$ is closed in X;*

2) *the complete preimage $f^{-1}(V)$ under the map f of an arbitrary open set $V \subset Y$ is open in X;*

2') *the complete preimage $f^{-1}(V)$ under the map f of an arbitrary element V of some prebase of open sets of Y is open in X;*

3) *for any point $x \in X$ and any neighbourhood V of $f(x))$ in Y there exists in X a neighbourhood U of x such that $f(U) \subset V$;*

3') *for any point $x \in x$ and any neighbourhood V of $f(x)$ in the space Y, belonging to some fundamental system of neighbourhoods of this point, there exists in the space X a neighbourhood U of the point x such that $f(U) \in V$;*

4) *for any set $A \subset X$ the inclusion holds*

$$f(\overline{A}) \subset \overline{f(A)}$$

Maps $f : X \to Y$ that have these properties are called *continuous*. These include, in particular, all constant maps, i.e. maps that take the entire space X to some fixed point y_0 of the space Y.

The continuous functions considered in §1.3 are nothing more than continuous maps of the space X into the real numbers $\mathbb{R}$.

The definition of the *coincidence set* $[f = g]$ is literally transferred from continuous functions to any continuous maps $f, g : X \to Y$. However, unlike the case of numerical functions, this set may not be closed. We can only assert that

Proposition 1.62. *if the space Y is Hausdorff, then for any two continuous maps $f, g : X \to Y$ the set $[f = g]$ is closed.*

Indeed, in the proof for continuous functions given in §1.3, only the Hausdorff property of the real line was used.

A special case of a coincidence set is the set $[f = \mathrm{id}_X]$ of all fixed points of the map $f : X \to X$ (i.e., the points $x \in X$ for which $f(x) = x$). According to what has just been said,

Proposition 1.63. *the set of fixed points of an arbitrary continuous map $f : X \to X$ of a Hausdorff space X is closed in itself in this space.*

The composition[4]

$$g \circ f : X \to Z$$

of any two continuous maps $f : X \to Y$, $g : Y \to Z$ is also a continuous map. Moreover, for any space X the identity map

$$\mathrm{id}_X : X \to X$$

(defined by the formula $\mathrm{id}_X(x) = x$) is continuous. In the language of category theory, these statements mean that the totality $\mathscr{X}$ of all topological spaces and all their continuous maps forms a category. Isomorphisms of this category, i.e. bijective[5] continuous maps $f : X \to Y$ for which the inverse map $f^{-1} : Y \to X$ is also continuous, are called *homeomorphisms*. Spaces X and Y for which there exists at least one homeomorphism $X \to Y$ are called *homeomorphic*. As a rule, we will further consider homeomorphic spaces as identical.

[4]A map $h : X \to Z$ is called the *composition* of the maps $f : X \to Y$ and $g : Y \to Z$ if $h(x) = g(f(x))$ for any point $x \in X$. The composition of the maps f and g is denoted by the symbol $g \circ f$.

[5]A map $f : X \to Y$ is called *injective* if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$, *sujective* if $f(X) = Y$, and bijective if it is both injective and surjective.

Remark 1.64. In §1.1 we defined the concept of the topological sum of any non-intersecting spaces. The convention just introduced allows us to define the *topological sum* of arbitrary spaces, including intersecting ones, as the topological sum of spaces homeomorphic to them, but non-intersecting. The topological sum constructed in this way is defined up to homeomorphism.

A space Y is called a *continuous image* of a space X if there exists at least one continuous surjective map $X \to Y$. It is easy to see that a continuous image of any compact (resp. Lindelöf or countably compact) space is compact (resp. Lindelöf or countably compact). Similarly, a continuous image of a connected space is connected.

Continuous mappings

$$u : I \to X$$

into the space X of the unit segment $I = [0, 1]$ we will call *paths* of the space X. We will call the points $x_0 = u(0)$ and $x_1 = u(1)$ respectively the *beginning* and *end* of the path u and we will say that the path u *connects* the point x_0 with the point x_1. We will call the space X *path-connected* if any two of its points can be connected by at least one path.

Proposition 1.65. *Any path-connected space X is connected.*

Proof. Indeed, for each path $u : I \to X$ the set $u(II) \subset X$ is connected (since the segment I is connected). Thus, any two points of a linearly connected space X belong to a connected set. Therefore, the space X is connected. □

An arbitrary topological space X decomposes into a union of disjoint linearly connected subspaces, called the *components of linear connectivity* of the space X. We will denote the set of all components of linear connectivity of the space X by the symbol $\pi_0(X)$.

Let the space X be represented as the union of some family $\{X_\mu;$
$mu \in M\}$ of its subspaces:

$$X = \cup_{\mu \in M} X_\mu$$

Then for any continuous map

$$f : X \to Y$$

and each $\mu \in M$ the partial map

$$f|_{X_\mu} : X_\mu \to Y,$$

as we know, is continuous. The converse, generally speaking, is not true, i.e. the continuity of the partial mappings $f|_{X_\mu}$ does not imply the continuity of the map f. However, it is easy to see that

Proposition 1.66. *if the space X is a free union of its subspaces X_μ, $\mu \in M$, then for any space Y the map*

$$f : X \to Y$$

is continuous if and only if all partial maps

$$f|_{X_\mu} : X_\mu \to Y,$$

are continuous.

This proposition allows us to construct continuous maps $X \to Y$ from continuous maps

$$f_\mu : X_\mu \to Y$$

of subspaces X_μ, $x \in M$, of the space X, of which it is a free union. In this case, it is only required that the maps f_μ be
emphconsistent, i.e. that

$$f_{\mu_1}|_{X_{\mu_1} \cap X_{\mu_2}} = f_{\mu_2}|_{X_{\mu_1} \cap X_{\mu_2}}$$

for any $\mu_1, \mu 2 \in M$. Indeed, by putting

$$f(x) = f_\mu(x), \quad \text{if} \quad x \in X_\mu,$$

where X is an arbitrary point in the space X, we obtain a single-valued map

$$f : X \to Y$$

for which

$$f|_{X_\mu} = f_\mu$$

for any $\mu \in M$. According to the previous proposition, the map f constructed in this way is continuous.

As a rule, we will use this construction only in the case when the number of subspaces X_μ is finite and each of them is closed. As we know, in this case the space X is automatically a free union of subspaces X_μ.

Another (even more important) method of constructing continuous maps uses the map α^{-1} (generally speaking, multi-valued), the inverse of a given surjective continuous map

$$\alpha : P \to X.$$

Let $g : P \to Y$ be a continuous map such that the composite map

$$f = g \circ \alpha^{-1} : X \to Y$$

is single-valued. Under what conditions is the map f continuous?

In order to give (even if only an incomplete) answer to this question, we shall call a continuous surjective map $\alpha : P \to X$ an identification map if each subset $A \subset X$ whose complete preimage $\alpha^{-1}(A)$ under the map α is closed (resp. open) in P is itself closed (resp. open) in X. It is clear that

Proposition 1.67. *if the map α is an identification map, then the map $f = g \circ \alpha^{-1}$ (assumed to be single-valued) is continuous if and only if the map $g = f \circ \alpha$ is contnuous.*

We will call a subset $S \subset P$ *saturated* with respect to the map $\alpha :\to X$ if it is a complete preimage of its image, i.e. if

$$S = \alpha^{-1}(\alpha(S)).$$

For the set S to be saturated, it is sufficient to require that $S \supset \alpha^{-1}(\alpha(S))$, since always $S \subset \alpha^{-1}(\alpha(S))$. The union and intersection of saturated sets are saturated. In addition, the complete preimage $\alpha^{-1}(A)$ of an arbitrary set $A \subset X$ is saturated with respect to the map $\alpha : P \to X$.

For any continuous surjective map $\alpha : P \to X$, each closed (esp. open) set $A \subset X$ is the image of some closed (resp. open) saturated set $S \subset P$ (for example, the set $S = \alpha^{-1}(A)$). It turns out that the inverse property characterises identification maps, i.e.

Proposition 1.68. *a continuous surjective map $\alpha : P \to X$ is an identification map if and only if for any closed (resp. open) saturated set $S \subset P$ the set $\alpha(S)$ is closed (resp. open) in X.*

Proof. Indeed, if α is an identification map, then for any saturated closed (resp. open) set S the set $\alpha(S)$ is closed (resp. open), since the set $\alpha^{-1}(\alpha(S)) = S$ is closed (open). Conversely, let for any closed (resp. open) saturated set S the set $\alpha(S)$ be closed (resp. open). Consider an arbitrary set $S \subset X$ for which the set $S = \alpha^{-1}(A)$ is closed (resp. open) in P. Then, since the set S is saturated, the set $A = \alpha(S)$ will, by assumption, be closed (resp. open), and therefore the map α will be an identification map. □

It follows directly from the proved statement that

Proposition 1.69. *any continuous map α of a compact space P onto a Hausdorff space X is an identification map.*

Proof. Indeed, let the closed saturated set $S \subset P$ be the complete preimage of the subset $A \subset x$. Being a closed subset of the compact space P, the set S is compact and therefore its image $\alpha(S) = A$ is also compact, and therefore closed (since the space X is Hausdorff). □

In general, a continuous bijective map may not be a homeomorphism. However,

Proposition 1.70. *any bijective identification map $\alpha \to X$ is a homeomorphism.*

Proof. Indeed, any set $T \subset P$ is saturated with respect to the bijective map α and therefore, if it is closed (in P), then the set $(\alpha^{-1})^{-1}T = \alpha(T)$ is also closed (in X). But this also means that the inverse map $\alpha^{-1} : X \to P$ is continuous. □

In particular,

Proposition 1.71. *any bijective continuous map α of a compact space P onto a Hausdorff space X is a homeomorphism.*

Let, for example, P be an arbitrary bounded convex body lying in the Euclidean space $\mathbb{R}^n$, i.e., an arbitrary bounded closed (and therefore compact) convex subset of the space $\mathbb{R}^n$ containing interior points. It is easy to see that any such body is stellar with respect to any of its internal points $\boldsymbol{x}_0$, i.e., each ray

$$\boldsymbol{x}_0 + \boldsymbol{u}t, \quad \boldsymbol{u} \in \mathbb{S}^{n-1}, 0 \le t < \infty$$

starting from the point $\boldsymbol{x}_0$ intersects the boundary of the body P at one point. In other words, for any vector $\boldsymbol{u} \in \mathbb{S}^{n-1}$ there exists a number $\varphi(\boldsymbol{u}) > 0$ such that $\boldsymbol{x}_0 + \boldsymbol{u}t \in P$ for $0 \le t \le \varphi(\boldsymbol{u})$ and $\boldsymbol{x}_0 + \boldsymbol{u}t \notin P$ for $t > \varphi(\boldsymbol{u})$. Moreover, elementary geometric considerations show that the function $\varphi(\boldsymbol{u})$ of the vector $\boldsymbol{u} \in \mathbb{S}^{n-1}$ is continuous on the sphere $\mathbb{S}^{n-1}$. Since, in addition, any point $\boldsymbol{x} \in P$ has the form $\boldsymbol{x}_0 + \boldsymbol{u}t$, where $\boldsymbol{u} \in \mathbb{S}^{n-1}$, $0 \le t \le \varphi(\boldsymbol{u})$, and for $\boldsymbol{x} \ne \boldsymbol{x}_0$ this representation is single-valued, then the formula

$$\alpha(\boldsymbol{x}_0 + \boldsymbol{u}t) = \frac{t}{\varphi(\boldsymbol{u})}\boldsymbol{u}$$

defines a bijective continuous map

$$\alpha : P \to \mathbb{E}^n$$

of the body P onto the unit ball $\mathbb{E}^n$. Since the body P is compact and the ball $\mathbb{E}^n$ is Hausdorff, this map is homeomorphic. Thus, we have proved that

Proposition 1.72. *any bounded convex body $P \subset \mathbb{R}^n$ is homeomorphic to the ball $\mathbb{E}^n$.*

From this, in particular, it follows that

Proposition 1.73. *for any $n \ge 0$ and $m \ge 0$ the product $\mathbb{E}^n \times \mathbb{E}^m$ of the balls $\mathbb{E}^n$ and $\mathbb{E}^m$ is homeomorphic to the ball $\mathbb{E}^{n+m}$.*

Proof. Indeed, the product $\mathbb{E}^n \times \mathbb{E}^m$ is obviously a bounded convex body of the space $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$. □

Note, by the way, that

Proposition 1.74. *for $n \ne m$ the balls $\mathbb{E}^n$ and $\mathbb{E}^m$ are not homeomorphic.*

The proof of this statement, despite its obvious "obviousness", is not at all simple and requires a deep study of the topology of Euclidean spaces. Since we essentially do not need this statement, we will not prove it here.

We will call a continuous map $f : P \to X$ a *map of compact character* if the preimage $f^{-1}(C)$ of any compact set $C \subset X$ is a compact subset of the space P. It turns out that

Proposition 1.75. *if the space X is Hausdorff and compactly generated, then for any map $f : P \to X$ of compact character the subset $f(P)$ is closed in the space X, and if, in addition, the space P is also Hausdorff, then the map f, considered as a map onto the subset $f(P)$, is an identification map.*

Proof. Indeed, let C be an arbitrary compact subset of X. Since the set $f^{-1}(C)$ is compact by assumption, the set $f(f^{-1}(C))$ is also compact and therefore closed. On the other hand, since

$$f^{-1}(C) = f^{-1}(C \cap f(P)),$$

then $f(f^{-1}(C)) = C \cap f(P)$. Thus, the subset $f(P)$ is compactly closed. Consequently, it is closed. □

Similarly, if for some subset $A \subset f(P)$ the subset $f^{-1}(A) \subset P$ is closed, then for any compact set $C \subset S$ the set $ff^{-1}(A \cap C) = f^{-1}(A) \cap f^{-1}(C)$ is compact, and therefore the set $A \cap C = f(f^{-1}(A \cap C))$ is also compact. Thus, the set A is compactly closed and, therefore, closed.

In particular, we see that

Proposition 1.76. *an injective map of a compact character of a Hausdorff space to a Hausdorff compactly generated space is a homeomorphism onto a closed subspace.*

Let P, Q, X, Y be arbitrary topological spaces and

$$\alpha : P \to X, \quad \beta : Q \to Y$$

be arbitrary continuous maps. It is easy to see that the formula

$$(\alpha \times \beta)(p, q) = (\alpha(p), \beta(q)), \quad p \in P, q \in Q,$$

uniquely determines some continuous map

$$\alpha \times \beta : P \times Q \to X \times Y.$$

We will call this map the *topological product* of maps α and β. It is surjective (esp. injective) if maps α and β are surjective (resp. injective).

Along with the product $\alpha \times \beta$, we will also consider the topological sum

$$\alpha \cup \beta : P \cup Q \to X \cup Y$$

of the mappings α and β. This sum is defined when the spaces $P \cup Q$ and $X \cup Y$ are defined (i.e. when $P \cap Q = \varnothing$ and $X \cap Y = \varnothing$), and, by definition, is a map that coincides on P with the map α, and on Q with the map β. It is also surjective (resp. injective) when the maps α and β are surjective (resp. injective). Furthermore, if the maps α and β are identification maps, then their sum $\alpha \cup \beta$ is also an identification map.

For the map $\alpha \times \beta$ the analogue of the last statement, generally speaking, does not apply. Having in mind to indicate sufficient conditions under which the product $\alpha \times \beta$ of two identification maps is also an identification map, we will call a continuous surjective map $\alpha : P \to X$ *locally compact* (resp. *locally countably compact*) if any point $p \in P$ has a neighbourhood saturated with respect to the map α, the closure of which is contained in a closed, saturated and compact (resp. countably compact) set $C \subset P$. If the space X is locally compact, then any surjective map $P \to X$ of compact character is locally compact. If a locally compact (resp. locally countable-compact) map$P \to X$ exists, then the space P is necessarily locally compact (resp. locally countable-compact). On the other hand, if the space P is compact (resp. countably compact), then for every space X any continuous sujective map $P \to X$ is locally compact (resp. locally countably compact).

It is easy to see that

Proposition 1.77. *if the space X is regular and the map $\alpha : P \to X$ is locally compact (resp. locally countably compact), then for any point $p \in P$ and any of its neighbourhoods S^* saturated with respect to the map α, there exists a saturated neighbourhood S of the point p whose closure $\overline{S}$ is compact (resp. countably compact) and is contained in the neighbourhood S^*.*

Proof. Indeed, by hypothesis, the point p has a saturated neighbourhood T, whose closure $\overline{T}$ is contained in a compact (resp. countably compact) set and is therefore itself compact (resp. countably compact). Let us consider the set $U = \alpha(T \cap S^*)$. Since the set $T \cap S^*$ is saturated with respect to the map α, the set U is open, i.e. it is a neighbourhood of the point $\alpha(p)$ in the space X. But then, due to the regularity of the space X, the point $\alpha(p)$ has a neighbourhood V such that $\overline{V} \subset U$. Let $S = \alpha^{-1}(V)$. The set S is saturated and is a neighbourhood of the point p. Moreover, since $\overline{S} \subset \alpha^{-1}(\overline{V}) \subset \alpha^{-1}(U) = T \cap S^*$, then, firstly, $\overline{S} \subset S^*$ and, secondly, $\overline{S} \subset T$, so that the set $\overline{S}$ is compact (resp. countably compact) and is contained in S^*. □

Let us now show that

Proposition 1.78. *if the identification map $\alpha : P \to X$ is locally compact and the space X is regular, then for any identification map $\beta : Q \to Y$ the map*

$$\alpha \times \beta : P \times Q \to X \times Y$$

is also an identification map.

Proof. Indeed, let W be an arbitrary subset of the product $X \times Y$ whose preimage $(\alpha \times \beta)^{-1}(W)$ under the map $\alpha \times \beta$ is open in the space $P \times Q$. We must prove that each such set W is open in the space $X \times Y$, i.e., that any of its points (x_0, y_0) is its interior point. In other words, we must prove that in the spaces X and Y there exist neighbourhoods U and V of the points x_0 and y_0, respectively, such that

$$(x_0, y_0) \in U \times V \subset W.$$

Let p_0 and q_0 be points in the spaces P and Q, respectively, such that $\alpha(p_0) = x_0$, $\beta(q_0) = y_0$. It is clear that to prove the existence of neighbourhoods U and V it is sufficient to prove that the points p_0 and q_0 have (in the spaces P and Q, respectively) saturated (with respect to the maps α and β, respectively) neighbourhoods S and T such that

$$(p_0, q_0) \in S \times T \subset (\alpha \times \beta)^{-1} W.$$

For this purpose, we consider the set S^* of all points $p \in P$ for which the following holds
$$(p, q_0) \in (\alpha \times \beta))^{-1}(W).$$
It is clear that the set S^* contains the point p_0, is open in the space P and is saturated with respect to the map α. Therefore, by the proposition proved above, the point p_0 has a neighbourhood S that is saturated with respect to the map α, whose closure $\overline{S}$ is compact and is contained in the neighbourhood S^*.

Let, further, T be the set of all points $q \in Q$ with the property that for any point $p \in \overline{S}$ the following inclusion holds
$$(p, q) \in (\alpha \times \beta)^{-1}(W).$$
In other words, T is the maximal subset of the space Q for which the following inclusion holds
$$\overline{S} \times T \subset (\alpha \times \beta)^{-1}(W).$$
In particular, we have
$$S \times T \subset (\alpha \times \beta)^{-1}(W).$$
Moreover, $q_0 \in T$ (since $\overline{S} \subset S^*$). Further,
$$\overline{S} \times \beta^{-1}(\beta(T)) \subset \alpha^{-1}(\alpha(\overline{S})) \times \beta^{-1}(\beta(T)) = (\alpha \times \beta)^{-1}(\alpha \times \beta)(\overline{S} \times T) \subset (\alpha \times \beta)^{-1}(W),$$
whence, in view of the maximality of the set T, it follows that $\beta^{-1}(\beta(T)) \subset T$, i.e., that the set T is saturated with respect to the map β.

Therefore, to complete the proof, we only need to prove that the set T is open in the space Q.

To this end, we note that since the set $(\alpha \times \beta)^{-1}(W)$ is, by assumption, open, each of its points is its interior point and therefore for points p and q there exist (in the spaces P and Q, respectively) neighbourhoods $S_q(p)$ and $T_p(q)$ such that
$$(p, q) \in S_q(p) \times T_p(q) \subset (\alpha \times \beta)^{-1}(W).$$
For each point $q \in T$, all possible sets of the form $S_q(P) \cap \overline{S}$, $p \in \overline{S}$, obviously constitute an open covering of the subspace $\overline{S} \subset P$. Therefore, since this subspace is, by construction, compact, there exists a finite system of points $p_1, \dots, p_n \in \overline{S}$, such that
$$\overline{S} \subset \cup_{i=1}^n S_q(p_i).$$
Let
$$T(q) = \cap_{i=1}^n T_{p_i}(q).$$
It is clear that the set $T(q)$ is open (in Q), contains the point q and has the property
$$p \times T(q) \subset (\alpha \times \beta)^{-1}(W)$$
for any point $p \in \cup_{i1}^n S_q(p_i)$, and therefore, in particular, for any point $p \in \overline{S}$. Therefore, $T(q) \subset T$, i.e. $q \in \operatorname{int} T$. Thus, the set T, as stated, is open. Thus, the above statement is completely proven. □

It is possible to specify other conditions under which the product of identification maps is an identification map. For example,

Proposition 1.79. *if the identification maps $\alpha : P \to X$ and $\beta : Q \to Y$ are locally countably compact, the spaces P and Q are Hausdorff, and the spaces X and Y are regular, then the map*

$$\alpha \times \beta : P \times Q \to X \times Y$$

is an identification map.

Proof. Indeed, in this case the points p_0 and q_0 have (in the spaces P and Q, respectively) saturated neighbourhoods S^* and T^* with respect to the maps α and β, whose closures $\overline{S}^*$ and $\overline{T}^*$ are contained in the closed saturated countably compact sets $C \subset P$ and $D \subset Q$. Let

$$C_0 \subset C_1 \subset \cdots \subset C_n \subset \cdots .$$
$$D_0 \subset D_1 \subset \cdots \subset D_n \subset \cdots$$

be increasing sequences of compact subsets $C_n \subset P$ and $D_n \subset Q$, whose free unions are the sets C and D, respectively. Without loss of generality we can assume that

$$p_0 \in C_0, \quad q_0 \in D_0.$$

Note also that, since the spaces P and Q are, by assumption, Hausdorff, all sets C_n and D_n are closed.

First of all, for any $n \geq 0$ we will construct open sets $S_n \subset C_n$ and $T_n \subset D_n$ saturated with respect to the maps $\alpha|_{C_n}$ and $\beta|_{D_n}$ (in the spaces C_n and D_n, respectively) such that

$$p_0 \in S_n, \quad q_0 \in T_n, \quad \overline{S}_n \subset S_{n+1}, \quad \overline{T}_n \subset T_{n+1}$$

and

$$\overline{S}_n \times \overline{T}_n \subset W_n, \tag{1.80}$$

where

$$W_n = (\alpha|_{C_n} \times \beta|_{D_n})^{-1} = (C_n \times D_n) \cap (\alpha \times \beta)^{-1}(W).$$

To this end, we note that since the space C_n is compact, the map $\alpha|_{C_n}$ is locally compact. Therefore, the maps $\alpha|_{C_n} : C_n \to X$ and $\beta|_{D_n} : D_n \to Y$ satisfy the conditions of the previous proposition. Therefore, for any point $(p, q) \in W_n$ in the spaces C_n and D_n there exist saturated (with respect to the maps $\alpha|_{C_n}$ and $\beta|_{D_n}$) neighbourhoods $S^*_{q,n}(p)$ and $T^*_{p,n}(q)$ of the points p and q such that

$$(p, q) \in S^*_{q,n}(p) \times T^*_{p,n}(q) \subset W_n.$$

Furthermore, since the spaces X and Y are regular by assumption, the points p and q have in the spaces C_n and D_n saturated neighbourhoods $S_{q,n}(p)$ and $T_{p,n}(q)$ such that $\overline{S}_{q,n}(p) \subset S^*_{q,n}(p)$ and $\overline{T}_{p,n}(q) \subset T^*_{p,n}(q)$ (here we mean closures in the spaces C_n and D_n; however, since the subspaces C_n and D_n are closed, these closures coincide with the closures in the spaces P and Q). Thus, for any point $(p,q) \in W_n$ we have constructed in the spaces C_n and D_n saturated neighbourhoods $S_{q,n}(p)$ and $T_{p,n}(q)$ of the points p and q with respect to the maps $\alpha|_{C_n}$ and $\beta|_{D_n}$ such that

$$(p,q) \in \overline{S}_{q,n}(p) \times \overline{T}_{p,n}(q) \subset W_n.$$

In particular, for $n = 0$, $p = p_0$, $q = q_0$ we obtain neighbourhoods

$$S_0 = S_{q,0}(p), \quad T_0 = T_{p,0}(q)$$

of the points p_0 and q_0 (in the spaces C_0 and D_0) that have property (1.80).

Reasoning by induction, we assume that for some $n > 0$ neighbourhoods S_{n-1} and T_{n-1} possessing property (1.80) have already been constructed. It is clear that for any point $q \in \overline{T}_{n-1}$ all sets of the form $\overline{S}_{n-1} \cap S_{q,n}(p)$, $p \in \overline{S}_{n-1}$, form an open covering of the space $\overline{S}_{n-1}$. Since this space, being a closed subspace of the compact space C_{n-1}, is compact, there exists a finite system of points $p_1, \dots, p_n \in \overline{S}_{n-1}$ such that

$$\overline{S}_{n-1} \subset S_{q,n},$$

where

$$S_{q,n}, = \cup_{i=1}^{s} S_{q,n}(p_i).$$

Let

$$T_n(q) = \cap_{i=1}^{s} T_{p_i,n}(q).$$

It is clear that the sets $S_{q,n}$ and $T_n(q)$ are saturated (with respect to the maps $\alpha|_{C_n}$ and $\beta|_{D_n}$), open (in the spaces C_n and D_n) and have the property that

$$\overline{S}_{q,n} \times \overline{T}_n(q) \subset W_n.$$

Moreover, $q \in T_n(q)$, so that all possible sets of the form $\overline{T}_{n-1} \cap \overline{T}_n(q)$, $q \in \overline{T}_{n-1}$ form an open covering of the space $\overline{T}_{n-1}$. Since the space $\overline{T}_{n-1}$ (similar to the space $\overline{S}_{n-1}$) is compact, there exists a finite system of points $q_1, \dots, q_n \in \overline{T}_{n-1}$ such that

$$\overline{T}_{n-1} \subset T_n,$$

where

$$T_n = \cup_{i=1}^{t} T_n(q_i).$$

It is clear that the set T_n together with the set

$$S_n = \cap_{i=1}^{n} S_{q_i},$$

satisfies all the conditions imposed on the neighbourhoods of S_n and T_n.

Thus, the neighbourhoods S_n and T_n are constructed for all $n \geq 0$.

Now let

$$S_\infty = \cup_{i=1}^{\infty} S_n, \quad T_\infty = \cup_{i=1}^{\infty} T_n.$$

It is obvious that the sets S_∞ and T_∞ are saturated (with respect to the maps α and β, respectively), since the sets C and D are saturated by assumption. In addition, $p_0 \in S_\infty$, $q_0 \in T_\infty$, and $S_\infty \times T_\infty \subset (\alpha \times \beta)^{-1}(W)$. Therefore, the sets

$$S = S^* \cap S_\infty, \quad T = T^* \cap T_\infty$$

are also saturated and have the property that

$$(p_0, q_0) \in S \times T \subset (\alpha \times \beta)^{-1} W_n.$$

Therefore, to complete the proof, it remains only for us to show that the sets S and T are open (in the spaces P and Q, respectively).

mLet us first consider the set S_∞. Since the set S_n is open in the subspace C_n, then for any $m \leq n$ the set $S_n \cap C_m$ is open in the subspace $C_m \subset C_n$. On the other hand, $S_n \cap Cm = S_n \subset S_m \cap C_m$ for $m > n$. Therefore,

$$S_\infty \cap C_m = \cup_{n=m}^{\infty}(S_n \cap C_m)$$

for any $m \geq 0$ and this set is open in the subspace C_m. Since the space C is a free union of subspaces C_m, it follows that the set S_∞ is open in the space C and, therefore, the set $S = S^* \cap S_\infty$ is open in the subspace $S^* \subset C$. But the last subspace is open in the space P. Consequently, the set S is also open in the space P.

The fact that the set T is open in the space Q is proved similarly.

Thus, the statement formulated above is completely proven. □

1.6 Topologies of identification, glued spaces, relative homeomorphisms

Let P be an arbitrary topological space and let

$$\alpha : P \longrightarrow X$$

be an arbitrary map of the space P onto some set X. We introduce a topology into the set X, considering a subset $A \subset X$ to be open (resp. closed) if and only if its complete preimage $\alpha^{-1}(A)$ is open (resp. closed). We will call this topology the *identification topology* (defined by the map α). It is the weakest (i.e. containing the largest number of closed sets) topology of the space X in which the map α is continuous.

It is clear that

Proposition 1.81. *if the space X is equipped with the identification topology, then the map α is an identification map in the sense of §1.5.*

Conversely,

Proposition 1.82. *if a topological space X has the property that a given surjective map $\alpha : P \longrightarrow X$ is an identification map then the topology of this space is the identification topology defined by the map α.*

In other words, the identification topology of the space X is uniquely determined by the requirement that the given map α be an identification map.

In most applications, the set X is the set of all classes under some equivalence relation defined in the space P, and the map $\alpha : P \to X$ is the natural projection that associates with each point $p \in P$ its equivalence class. In this case, the set x, equipped with the identification topology, is called the *factor space* of the space P with respect to the given equivalence relation. However, the difference between factor spaces and any spaces equipped with the identification topology is essentially purely formal, since for any identification map $\alpha : P \to X$ there exists on P an equivalence relation such that the corresponding factor space is naturally homeomorphic to the space X. In this equivalence relation, the points $p_1, p_2 \in P$ are equivalent if and only if

$$\alpha(p_1) = \alpha(p_2).$$

An important example of a factor space arises when considering an arbitrary continuous map

$$f : A \to Y$$

of a closed subspace A of some topological space Z into a given space Y. Assuming that the spaces X and Y do not intersect, we introduce in their topological sum

$$P = X \cup Y$$

an equivalence relation, considering that

1) points $x_1, x_2 \in X$ are equivalent if and only if either $x_1 = x_2$ or $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$;

2) points $x \in X$ and $y \in Y$ are equivalent if and only if $y = f(x)$;

3) points $y_1, y_2 \in Y$ are equivalent if and only if $y_1 = y_2$.

We will denote the corresponding factor space of the space P by the symbol $X \cup_f Y$ and we will say that *it is obtained by gluing the space X along the subspace A to the space Y by means of the map F.*

The natural projection

$$\alpha : P \to X \cup_f Y$$

is a homeomorphism on the subspace $Y \subset P$. Therefore, in the future we will, as a rule, identify the space Y with its image $\alpha(Y)$, i.e. we will consider the space Y as a subspace of the space $X \cup_f Y$:

$$Y \subset X \cup_f Y.$$

It is easy to verify that the space Y is closed in the space $X \cup_f Y$.

Further, as is easy to see, the natural projection a is a homeomorphism on the open set $X \setminus A$. Therefore, we can also assume that

$$X \setminus A \subset X \cup_f Y.$$

In this case

$$X \setminus A = (X \cup_f Y) \setminus Y,$$

so that the space $X \setminus A$ is open in the space $X \cup_f Y$.

Thus, the space $X \cup_f Y$ can be considered as the union

$$X \cup_f Y = (X \setminus A) \cup Y$$

of two mutually complementary spaces $X \setminus A$ and Y, the first of which is open and the second is closed. In this connection, we will sometimes say that the space $X \cup_f Y$ is obtained *by gluing* the spaces $X \setminus A$ and Y.

On the subspace A the natural projection α coincides with the map f.

It is customary to call a *pair of spaces* (or simply a *pair*) an arbitrary pair (X, A) consisting of some topological space X and some of its subspace A. We will call a pair (X, A) *Hausdorff* if the space X is Hausdorff, and *compact* if the space X is compact and its subspace A is closed.

Let (X, A) and (Z, Y) be arbitrary pairs. By a *map*

$$g : (X, A) \to (Z, Y) \tag{1.83}$$

of a pair (X, A) into a pair (Z, Y) we mean an arbitrary continuous map g of the space X into the space Y for which $g(A) \subset Y$. We will call the map (1.83) a *relative homeomorphism* if it homeomorphically maps the subspace $X \setminus A$ onto the subspace $Z \setminus Y$. From the above properties of the natural projection

$$\alpha : P \to X \cup fY$$

it immediately follows that

Proposition 1.84. *the restriction*

$$\alpha|_X : X \to X \cup_f Y$$

of the projection α on the space X is a relative homeomorphism of the pair (X, A) onto the pair $(X \cup_f Y, Y)$.

Relative homeomorphisms of the form $\alpha|_X$ are identification maps, and the subspace Y is closed for them. It turns out that in the class of all relative homeomorphisms, maps of the form $\alpha|_X$ are uniquely characterised by these properties up to homeomorphism, i.e.

Proposition 1.85. *for every relative homeomorphism*

$$g : (X, A) - \to (Z, Y),$$

for which the subspace Y is closed in the space Z and the map $g : X \to Z$ is an identification map, there exists a homeomorphism

$$h : X \cup_f Y \to Z, \quad f = g|_A,$$

identical on the subspace Y such that

$$g = h \circ \alpha|_X.$$

Thus, if the specified conditions are met, the space Z can be considered as the result of gluing the space X to the space Y along the subspace A by means of the map $f = g|_A$.

Remark 1.86. This formulation implies that $X \cap Y = \varnothing$. Otherwise, the space X should be replaced by a homeomorphic space that already has this property.

To prove the formulated proposition, we first note that the subspace A, being the prreimae under a continuous map of the closed subspace Y, is itself closed. Therefore (in view of the remark made above), the space $X \cup_f Y$ is defined. Let's consider the map

$$h : X \cup_f Y \to Z,$$

coinciding on $X \setminus A$ with the map g and identical on Y. It is clear that this map is bijective and has the property that the map $h \circ \alpha$ coincides on X with the map g (and is the identity map on Y). From this, firstly, it follows that the map h is continuous (since the map $h \circ \alpha$ is continuous, and the projection α is an identification map). Secondly, since for any closed set $C \subset X \cup_f Y$ the intersection of the closed set $(h \circ \alpha)^{-1}(h(C)) = \alpha^{-1}(C) \subset C \cup Y$ with the space X coincides with the set $g^{-1}(h(C)$, then, since the map g is an identification map, the set $g(C)$ is closed in Z. Consequently, the map h^{-1} is also continuous. Thus, the map h, as stated, is homeomorphic.

As we know, the condition on the map g is automatically satisfied if the space X is compact and the space Z is Hausdorff. Moreover, it is clear that in this case, for the space Y to be closed, it is sufficient that the space A is closed. Thus,

Proposition 1.87. *for any Hausdorff pair (Z, Y) relatively homeomorphic to a compact pair (X, A), the space Z is homeomorphic to the space obtained by gluing the space X to the space Y along the subspace A.*

Generally speaking, the space $X \cup_f Y$ may not be Hausdorff, even if the spaces X and Y are Hausdorff. However,

Proposition 1.88. *the space $X \cup_f Y$ is Hausdorff if*

1) the space $X \setminus A$ is Hausdorff;

2) each point of the subspace $X \setminus A$ has a neighbourhood whose closure does not intersect the subspace A;
and either

3) the space Y is Hausdorff and any two disjoint open (in A) subsets of the subspace A that are saturated with respect to the map f are cut out on A by disjoint open subsets of the space X;
or

4) the space Y is completely Hausdorff and any two disjoint closed subsets of the subspace A have disjoint neighbourhoods in X.

Proof. Indeed, condition 1) obviously ensures the Hausdorff property (the existence of disjoint neighbourhoods) for any pair of distinct points of the space $X \cup_f Y$ belonging to the subspace $X \setminus A$, and condition 2) ensures the Hausdorff property for any two points $x \in X \setminus A$ and $y \in Y$. Therefore, we need to check the Hausdorff property only for (distinct) points $y_1, y_2 \in Y$.

Let condition 3) be satisfied. Then the points y_1 and y_2 have in Y non-intersecting neighbourhoods V_1 and V_2. The preimages $f^{-1}(V_1)$ and $f^{-1}(V_2)$ (possibly empty) of these neighbourhoods also do not intersect and are open sets in A, saturated with respect to the map f. Therefore, according to the condition, in X there exist open disjoint sets U_1 and U_2 such that

$$U_1 \cap A = f^{-1}(V_1), \quad U_2 \cap A = f^{-1}(V_2).$$

It is clear that the sets

$$(U_1 \setminus A) \cup V_1, \quad (U_2 \setminus A) \cup V_2$$

are open in the space $X \cup_f Y$ and do not intersect (since they serve as images under the natural projection $\alpha : X \cup Y \to X \cup_f Y$ of non-intersecting, open, and saturated sets with respect to the map α of $U_1 \cup V_1$ and $U2 \cup V_2$). Since these sets contain the points y_1 and y_2, the Hausdorff property for these points is thus completely proved.

Let condition 4) be satisfied. Then the points y_1 and y_2 have neighbourhoods V_1 and V_2 in Y, the closures of which $\overline{V}_1$ and $\overline{V}_2$ do not intersect. Let us consider the sets $V_1' = f^{-1}(V_1)$ and $V_2' = f^{-1}(V_2)$. Since $\overline{V}_1' \subset f^{-1}(\overline{V}_1)$, $\overline{V}_2' \subset f^{-1}(\overline{V}_2)$ and $f^{-1}(\overline{V}_1) \cap f^{-1}(\overline{V}_2 = \varnothing$, then we have

$$\overline{V}_1' \cap \overline{V}_2' = \varnothing.$$

Thus, the sets $\overline{V}_1'$ and $\overline{V}_2'$ are disjoint closed subsets of the subspace A (recall that A is assumed to be closed). Therefore, according to the condition, these sets have disjoint neighbourhoods U_1 and U_2 in X. It is clear that the sets

$$(U_1 \setminus A) \cup V_1, \quad (U_2 \setminus A) \cup V_2$$

are open in the space $X \cup_f Y$ and do not intersect (since they serve as images under the natural projection α of non-intersecting, open and saturated sets $(U_1 \setminus A) \cup V_1' \cup V_1$ and $(U_2 \setminus A) \cup V_2' \cup V_2$ with respect to the map α.) Since these sets contain the points y_1 and y_2, the Hausdorff property is thus proved in this case as well. □

It is clear that conditions 1), 2) and 4) (in the part concerning the subspace A) are automatically satisfied if the space X is normal. Therefore,

Proposition 1.89. *if the space X is normal and the space Y is completely Hausdorff, then the space $X \cup_f Y$ is Hausdorff.*

Remark 1.90. It can be shown that if the spaces X and Y are normal, then the space $X \cup_f Y$ is also normal. We will not need this fact.

In the special case when the space Y consists of only one point y_0 (and, consequently, the map $f : A \to Y$ automatically turns out to be constant), the space $X \cup_f Y$ is denoted by the symbol X/A and is called the result of the *contraction of the subspace A to the point y_0*. In this case, condition 4) is obviously satisfied. As for conditions 1) and 2), they are certainly satisfied if the space X is regular. Thus,

Proposition 1.91. *if the space X is regular, then for any of its closed subspaces A the space X/A is Hausdorff.*

Another important special case of gluing arises when considering an arbitrary continuous map

$$f : X \to Y.$$

Let $X \times 1$ be a subspace of the product $X \times I$, where I is the unit segment $[0, 1]$ consisting of all points of the form $(x, 1)$, $x \in X$, and let

$$f_1 : X \times 1 \to Y$$

be the map into the space Y, defined by the formula

$$f_1(x, 1) = f(x).$$

Let us consider the space

$$Z_f = (X \times I) \cup_{f_1} Y.$$

This space is called the *mapping cylinder by f* and, as we shall see later, plays a fundamental role in the study of the homotopy properties of this map. Each of its points either has the form (x, t), where $x \in X$, $0 \leq t < 1$, or is a point y of the space Y.

It is clear that condition 2) for the space Z_f is always satisfied (a neighbourhood of the point (x, t), $x \in X$, $0 \leq t < 1$, the closure of which does not intersect the subspace $X \times 1$ is, for example, any neighbourhood of the form $X \times [0, t + \varepsilon)$, where ε is any positive number less than $1 - t$), and condition 1) is satisfied if the space X is Hausdorff. Moreover, condition 4) (in the part not related to the space Y) is also obviously always satisfied (open sets of the subspace $X \times 1$ have the form $G \times 1$, where G are open sets of the space X; they are cut out from $X \times 1$ by open sets $G \times I$ in $X \times I$; if the sets $G_1 \times 1$ and $G_2 \times 1$ do not intersect, then the sets $G_1 \times 1$ and $G_2 \times I$ also do not intersect). Consequently,

Proposition 1.92. *if the spaces X and Y are Hausdorff, then for any map $f : X \to Y$ the space Z_f is also Hausdorff.*

Chapter 2

Homotopy equivalences

This chapter mainly sets out various criteria that allow, in some cases, to judge whether a given continuous map will be a homotopy equivalence.

In the introductory §2.1, the basic concepts of the homotopy theory of continuous maps are presented and the simplest connections between these concepts are established. In particular, a simple but useful lemma is proved here, establishing conditions under which two homotopic maps that coincide on some subspace are homotopic relattive to this subspace. At the end of this section, the concept of a m-connected space is considered and some elementary properties of such spaces are proved.

In §2.2, after a number of simple remarks on homotopy equivalences, a well-known characteristic property of their cylinders is proved.

In §2.3 it is proved that (under certain conditions) the homotopy type of the glued space $X \cup_f Y$ depends only on the homotopy type of the space Y and the homotopy equivalence class of the map f.

In §2.4 the concept of weak homotopy equivalence is introduced and in connection with this a number of properties of homotopy groups are presented. However, the detailed theory of homotopy groups remains almost completely outside the scope of our exposition (it is enough to say that we do not even use their group operation here).

In §2.5 the concept of homotopy limit is considered (in both the "weak" and "strong" versions) and it is proved that the limit of homotopy equivalences is also a homotopy equivalence.

2.1 Homotopies and extensions of continuous maps

Each family

$$f_t : X \to Y, \quad 0 \le t \le 1, \tag{2.1}$$

of continuous maps of a topological space X into a topological space Y defines by the formula

$$F(x,t) = f_t(x), \quad x \in X, t \in I,$$

a certain map

$$F : X \times I \to Y \tag{2.2}$$

into the space Y of the product $X \times I$ of the space X and the unit segment $I = [0, 1]$. We will call the family (2.1) a *homotopy* of maps of the space X into the space Y if the corresponding map (2.2) is continuous. It is clear that, conversely, any continuous map (2.2) defines some homotopy (2.1).

We will call the map f_0 the *initial* map, and the map f_1 the *final* map of homotopy (2.1).

In what follows, we will often have to consider not separate spaces X and Y, but pairs (X, A) and (Y, B), where A and B are some subspaces of the spaces X and Y respectively. In this case, we will be interested, as a rule, only in homotopies of the form

$$f_t : (X, A) \to (Y, B),$$

i.e., homotopies (2.1) for which

$$f_t(A) \subset B$$

for any $t \in I$. We will call such homotopies *homotopies of pair maps.*

Maps

$$f, g : X \to Y$$

(or maps $f, g : (X, A) \to (Y, B)$) we will call *homotopic* (notation, $f \sim g$) if there exists a homotopy

$$f_t : X \to Y, \quad 0 \le t \le 1,$$

(or, correspondingly, a homotopy $f_t : (X, A) \to (Y, B)$ such that

$$f_0 = f, \quad f_1 = g$$

i.e., that the map f is its initial, and the map g is its final map. In this case we will also say that the maps f and g are related by a homotopy f_t (notation $f_t : f \sim g$).

Each continuous map

$$f : X \to Y$$

defines a certain homotopy

$$(1_f)_t : X \to Y, \quad 0 \le t \le 1,$$

for which

$$(1_f)_t(x) = f(x)$$

for any $x \in X$ and $t \in I$. We will call this homotopy a *stationary homotopy* of the map f.

The homotopy relation is, as is easy to see, an equivalence relation, i.e. it is

reflexive since $(1_f)_t : f \sim f$,

symmetric if $f_t f \sim g$, then $f_{1-t} : g \sim f$,

transitive if $f_t : f \sim g$ and $g_t : g \sim h$, then $h_t : f \sim h$, where

$$h_t = \begin{cases} f_{2t} & \text{for} \quad 0 \le t \le 1/2, \\ g_{2t-1} & \text{for} \quad 1/2 \le t \le 1. \end{cases}$$

Therefore, the set of all continuous maps $X \to Y$ (respectively, the set of all continuous maps $(X, A) \to (Y, B)$) splits into disjoint homotopy classes consisting of pairwise homotopic maps. We will denote the set of all homotopy classes of maps $X \to Y$ by the symbol $[X, Y]$, and maps $(X, A) \to (Y, B)$ by the symbol $[(X, A), (Y, B)]$. We will denote the class containing a given map f by the symbol $[f]$.

For any two homotopies

$$f_t : X \to Y, \quad g_t : Y \to Z, \qquad 0 \le t \le 1,$$

the family of maps

$$h_t = g_t \circ f_t : X \to Z, \quad 0 \le t \le 1,$$

is also, obviously, a homotopy. It follows that

Proposition 2.3. *if the maps $f_0 : X \to Y$ and $g_0 : Y \to Z$ are homotopic, respectively, to the maps $f_1 : X \to Y$ and $g_1 : Y \to Z$, then the map $g_0 \circ f_0 : X \to Z$ is homotopic to the map $g_1 : X \to Z$.*

In other words,

Proposition 2.4. *for any maps*

$$f : X \to Y, \quad g : Y \to Z$$

the homotopy class $[g \circ f] \in [X, Z]$ of the map

$$g \circ f : X \to Z$$

depends only on the homotopy classes $[f] \in [X, Y]$ and $[g] \in [Y, Z]$ of the maps f and g, respectively.

In particular, for any space Z and any continuous map

$$f : X \to Y$$

the formula

$$f_*[\varphi] = [f \circ \varphi], \quad \varphi : Z \to X,$$

uniquely determines some map of sets

$$f_* : [Z, X] \to [Z, Y].$$

We will say that the map f_* *is induced* by the continuous map f.

Similar statements hold, of course, for maps of pairs.

Recall that the map
$$f : X \to Y$$
is called the *extension* of the map
$$g : A \to Y,$$
where A is some subspace of the space X, if
$$f(a) = g(a)$$
for any point $a \in A$. In this case, we also say that the map g is a *restriction* of the map f on the subspace A and write
$$g = f|_A.$$
Otherwise, we can say that
$$g = f \circ i,$$
where
$$i : A \to X$$
is an inclusion map, i.e. a map defined by the formula
$$i(a) = a$$
for any point $a \in A$. In what follows, we will indicate that some map $i : A \to X$ is an inlusion map by replacing the symbol "$\to$" with the symbol "$\subset$", i.e. instead of $i : A \to X$ we will write
$$i : A \subset X.$$

A map
$$r : X \to A$$
of a space X onto its subspace A is called *retractive* if it is an extension of the identity map
$$1_A : A \to A,$$
i.e., if
$$r \circ i = 1_A,$$
or, in other words, if
$$r(a) = a$$
for any point $a \in A$. In this case, we will write
$$r : X \supset A.$$

Subspaces A of X for which retractive map $r : X \supset A$ exist are called its *retracts*. For a retractive map (or *retraction*, for short) $r : X \to A$ the set A coincides with the set of all fixed points of the map $i \circ r$. Therefore,

Proposition 2.5. *every retract A of a Hausdorff space X is closed in X.*

It is easy to see that

Proposition 2.6. *a space A of a space X is a retract of it if and only if for any space Y each map $g : A \to Y$ admits an extension $f : X \to Y$.*

Proof. Indeed, if any map $g : A \to Y$ can be extended, then, in particular, the identity map $1_A : A \to A$ can also be extended. Conversely, if there exists a retractive map $r : X \to A$, then for any map $g : A \to Y$ the composition $g \circ r : X \to Y$ will be an extension of the map g. □

The concept of the extension of maps is closely related to the concept of a homotopy of maps, since for any homotopy $f_t : X \to Y$ connecting the map $f : X \to Y$ with the map $g : X \to Y$, the corresponding map $F : X \times I \to Y$ is an extension to the space $X \times I$ of the map $(X \times 0) \cup (X \times 1) \to Y$, which maps each point $(x, 0)$, $s \in X$, to the point $f(x)$, and each point $(x, 1)$, $x \in X$, to the point $g(x)$.

A pair (X, A) is said to satisfy the *axiom of homotopy extension* if for any space Y, any map $f : X \to Y$ and any homotopy $g_t : A \to Y$ of the map $g = f|_A$ there exists a homotopy $f_t : X \to Y$ such that $f_0 = f$ and $f_t|_A = g_t$ for any $t \in I$. The significance of this axiom is primarily that for pairs (X, A) subject to it, the property of the map $g : A \to Y$ to allow the extension $f : X \to Y$ depends only on the homotopy class of the map g, i.e., together with the map g, each homotopic map $g' : A \to Y$ can be extended to the entire space X.

Examples of pairs satisfying the axiom of homotopy extension are given below.

It is clear that

Proposition 2.7. *if the pair (X, A) satisfies the axiom of homotopy extension, then the subspace $(X \times 0) \cup (A \times I)$ of the product $X \times I$ is its retract.*

Proof. Indeed, the homotopy f_t constructed for the identity map $1_X : X \to X$ and the stationary homotopy $(1_i)_t : S \to X$ of the inclusion map $i : A \subset X$ obviously defines a retracting map $X \times I \to (X \times 0) \cup (A \times I)$. □

If the subspace A is closed in the space X, then the converse is also true, i.e.

Proposition 2.8. *if the subspace $(X \times 0) \cup (A \times I)$ of the product $X \times I$ is its retract, and the subspace A is closed in the space X, then the pair (X, A) satisfies the axiom of homotopy extension.*

Proof. Indeed, the problem of constructing a homotopy f_t is equivalent to the problem of extending to the entire space $X \times I$ the map

$$G : (X \times 0) \cup (A \times I) \to Y,$$

defined by the formula

$$G(x, t) = \begin{cases} f(x), & \text{if} \quad t = 0, \\ g_t(x), & \text{if} \quad x \in A \end{cases}$$

(for closed A the space $(X\times 0)\cup(A\times I)$ is obviously a free union of the spaces $X\times 0$ and $A\times I$ and therefore this map is continuous). Therefore, if the space $(X\times 0)\cup(A\times I)$ is a retract of the space $X\times I$, then the homotopy f_t exists. □

In connection with the last statement, it is useful to keep in mind that

Proposition 2.9. *if the space X is Hausdorff and the subspace $(X\times 0)\cup(A\times I)$ of the product $X\times I$ is its retract, then the subspace S is closed in the space X.*

Proof. Indeed, then the subspace $(X\times 0)\cup(A\times I)$ is closed in the product $X\times I$ and therefore the set

$$A\times 1=[(X\times 0)\cup(A\times I)]\cap(X\times 1)$$

is closed in the space $X\times 1$. □

Thus,

Proposition 2.10. *if the space X is Hausdorff or if the subspace A is closed, then the pair (X,A) satisfies the axiom of homotopy extension if and only if the subspace $(X\times 0)\cup(A\times I)$ is a retract of the product $X\times I$.*

Remark 2.11. The property of a pair (X,A) to satisfy the axiom of homotopy propagation is mainly local in nature, i.e., it is essentially determined by the structure of the space X in some neighborhood of the subspace A. The precise meaning of this statement can be given in many different ways. For example, it is easy to prove that

Proposition 2.12. *a pair (X,A) (with closed A) satisfies the axiom of homotopy extension if and only if the subspace A is functionally closed and there exists a homotopy $f_t: X\to Y$ and a function $\varphi: X\to I$ equal to zero on the subspace A such that*

$$\begin{aligned} &f_0(x)=x, && x\in X,\\ &f_t(a)=a, && (a,t)\in A\times I,\\ &f_1(x)\in A, && \text{if}\quad \varphi(x)<1. \end{aligned}$$

It can also be shown that

Proposition 2.13. *a pair (X,A) (with closed A) satisfies the axiom of homotopy extension if and only if there exists on the space X a continuous non-negative function φ equal to zero on the subspace A, and a map F into the space X of the subspace of the product $X\times I$ consisting of all points (x,t) for which*

$$0\le t\le\varphi(x),$$

having the following properties:

$$\begin{aligned} F(x,0)&=x, \quad \text{for any point}\quad x\in X,\\ F(x,\varphi(x))&\in A, \quad \text{if}\quad \varphi(x)\le 1. \end{aligned}$$

We will not need these statements and therefore we will not prove them here.

A homotopy

$$f_t : X \to Y, \quad 0 \le t \le 1,$$

will be called a *homotopy relative to a subspace* $A \subset X$ if this homotopy is stationary on A, i.e. if

$$f_t(a) = f_0(a)$$

for any point $a \in A$ and any $t \in I$. Accordingly, we will call the two maps

$$f, g : X \to Y$$

homotopic relative to A (notation $f \sim g \operatorname{rel} A$) if they are related by some homotopy relative to A. Of course, for this it is necessary that

$$f|_A = g|_A,$$

i.e. that the maps f and g coincide on A. It is clear that homotopy relative to A is also an equivalence relation, and therefore the set of all continuous maps $X \to Y$ that coincide on A splits into disjoint *homotopy classes relative to* A. The class containing the given map $f : X \to Y$ we will denote by the symbol $[f] \operatorname{rel} A$, and the set of all such classes - by the symbol $[X, Y] \operatorname{rel} A$.

The properties of relative homotopy classes are similar to the properties of the "absolute" classes discussed above. For example, any continuous map $f : Y \to Z$ defines by the formula

$$f_*([\varphi] \operatorname{rel} A) = [f \circ \varphi] \operatorname{rel} A, \quad \varphi : X \to Y,$$

some map

$$f_* : [X, Y] \operatorname{rel} A \to [X, Z] \operatorname{rel} A.$$

If the space Y is Hausdorff, then any homotopy $f_t : X \to Y$ that is stationary on A will obviously also be stationary on the closure $\overline{A}$ of the subspace A. Consequently, in this case we can, without loss of generality, consider the subspace A to be closed.

The problem of constructing a homotopy relative to a closed subspace A is equivalent to the problem of extending to the entire space $X \times I$ the map

$$F : (X \times 0) \cup (A \times I) \to (X \times 1) \to Y,$$

defined by the formula

$$F(x, t) = \begin{cases} f(x), & \text{if} \quad t = 0 \text{ for } x \in A, \\ g(x), & \text{if} \quad t = 1. \end{cases}$$

In what follows, to simplify the formulae, for any pair (X, A) we will denote the subspace $(X \times 0) \cup (A \times I) \cup (X \times 1)$ of the space $X \times I$ (see Fig. 2.1) by the symbol $I(X, A)$. When A is closed, it is closed.

Note that if $f|_A = g|_A$ and $f \sim g$, then, generally speaking, it cannot be asserted that $f \sim \operatorname{rel} A$. However,

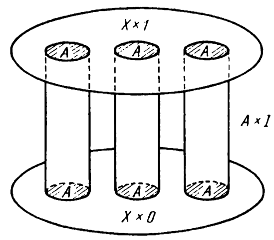


Figure 2.1:

Proposition 2.14. *if the pair $(X \times I, I(X, A))$ satisfies the axiom of homotopy extension and if*

$$H_{A\times I} \sim F\,\mathrm{rel}(A \times \cup A \times 1),$$

where H is the map $X \times I \to Y$ corresponding to the homotopy $h_t : X \times I \to Y$, connecting the maps f and g, and F is the map $A \times I \to Y$ corresponding to the stationary homotopy $(1_f)_t : A \to Y$, then

$$f \sim g\,\mathrm{rel}\,A$$

Proof. Indeed, we can extend the homotopy relative to $A \times 0 \cup A \times 1$ connecting the maps $H|_{A\times I}$ and F to some homotopy

$$H_t : I(X, A) \to Y \quad 0 \le t \le 1,$$

assuming that

$$H_t(x, 0) = f(x), \quad H_t(x, 1) = g(x)$$

for all $t \in I$. It is clear that

$$H_0|_{A\times I} = H|_{A\times I}.$$

Therefore, since the pair $X \times I, I(X, A))$ satisfies, by condition, the axiom of homotopy extension, there exists a homotopy

$$F_t : X \times I \to Y, \quad 0 \le t \le 1,$$

such that

$$F_0 = H, \quad F_t|_{I(X,A)} = H_t, \quad 0 \le t \le 1.$$

Therefore, the map

$$F_1 : X \times I \to Y,$$

defines (by the formula $f_t(x) = F_1(x, t)$) a homotopy

$$f_t : X \to Y, \quad 0 \le t \le 1,$$

relative to A, such that $f_0 = f$ and $f_1 = g$. Consequently, $f \sim g\,\mathrm{rel}\,A$. □

In what follows, continuous maps into a given space X of the unit n-dimensional sphere $\mathbb{S}^n$, $n \geq 0$ will be of great importance. In particular, we will be interested in the conditions under which any such map is homotopic to a constant map, i.e., a map to one point. In this connection, we first note that

Proposition 2.15. *a map $f : \mathbb{S}^n \to x$ is homotopic to a constant map if and only if the map f can be extended to some map $\mathbb{E}^{n+1} \to X$.*

Proof. Indeed, any homotopy

$$f_t : \mathbb{S}^n \to X,$$

for which $f_0 = f$ and $f_1(\boldsymbol{u}) = x_0$ for any point $\boldsymbol{u} \in \mathbb{S}^n$, where x_0 is some fixed point of the space X defines by the formula

$$F(v\boldsymbol{u}) = f_{1-v}(\boldsymbol{u}), \quad 0 \leq v \leq 1, \ \boldsymbol{u} \in \mathbb{S}^n,$$

a continuous map $F : \mathbb{E}^{n+1} \to X$, for which

$$F|_{\mathbb{S}^n} = f.$$

Conversely, any such map $F : \mathbb{E}^{n+1} \to X$ defines by the formula

$$f_t(\boldsymbol{u})F((1-t)\boldsymbol{u}), \quad 0 \leq t \leq 1, \ \boldsymbol{u} \in \mathbb{S}^n,$$

a homotopy $f_t : \mathbb{S}^n \to X$ for which $f_0 = f$ and $f_1(\boldsymbol{u}) = x_0$, where $x_0 = F(\mathbf{0})$. □

Thus,

Proposition 2.16. *for any space X the following properties are equivalent:*

1) any map $f : \mathbb{S}^n \to X$ is homotopic to a constant map;

2) any map $f : \mathbb{S}^n \to X$ can be extended to some map $F : \mathbb{E}^{n+1} \to X$.

Spaces with these properties we will call *n-aspherical*. Property 2) for $n = 0$ is obviously equivalent to the path-connectedness. Thus,

Proposition 2.17. *a space X is 0-aspherical if and only if it is path-connected.*

Spaces that are n-aspherical for all non-negative $n \leq m$ will be called *m-connected*.

It is easy to see that

Proposition 2.18. *the open unit interval $(0, 1)$ (as well as the closed segment $I = [0, 1]$) is an m-connected space for any $m \geq 0$.*

This statement follows immediately from Tietze's theorem (see §1.3), since the ball $\mathbb{E}^{n+1}$ is a normal space. However, it is easy to see that for each $n \geq 0$ the extension $F : \mathbb{E}^{n+1} \to (0, 1)$ of an arbitrary map $f : \mathbb{S}^n \to (0, 1)$ can be defined by the formula

$$F(v\boldsymbol{u}) = \frac{1 + (2f(\boldsymbol{u}) - 1)v}{2}, \quad 0 \leq v \leq 1, \ \boldsymbol{u} \in \mathbb{S}^n.$$

It is equally easy to see that

Proposition 2.19. *the topological product $X \times Y$ of two m-connected spaces X and Y is also an m-connected space.*

Proof. Indeed, let
$$f : \mathbb{S}^n \to X \times Y$$
be an arbitrary map of the n-dimensional $(n \leq m)$ sphere $\mathbb{S}^n$ into the product $X \times Y$. Assuming for any point $\boldsymbol{u} \in \mathbb{S}^n$,

$$f(\boldsymbol{u}) = (f_1(\boldsymbol{u}), f_2(\boldsymbol{u}), \quad f_1(\boldsymbol{u}) \in X,\ f_2(\boldsymbol{u}) \in Y,$$

we obtain two (obviously continuous) maps

$$f_1 : \mathbb{S}^n \to X, \quad f_2 : \mathbb{S}^n \to Y.$$

By the condition, these maps can be extended to maps

$$F_1 : \mathbb{E}^{n+1} \to X, \quad F_2 : \mathbb{E}^{n+1} \to Y.$$

It is clear that the formula

$$F(\boldsymbol{v}) = (F_1(\boldsymbol{v}), F_2(\boldsymbol{v})), \quad \boldsymbol{v} \in \mathbb{E}^{n+1},$$

then defines a continuous map

$$F : \mathbb{E}^{n+1} \to X \times Y,$$

which is an extension of the map f. □

Comparing the proved statements, we obtain, in particular, that

Proposition 2.20. *for any m-connected space X the space $X \times (0, 1)$ is also m-connected.*

2.2 Homotopy equivalences and deformation retracts

Let X be an arbitrary space. Each homotopy

$$\xi_t : X \to X, \quad 0 \leq t \leq 1,$$

for which $\xi_0 = 1_X$, we will call the *deformation* of the space X. A continuous map
$$h : X \to X$$
we will call *homotopically identical* if it is homotopic to the identity map 1_X of the space X, i.e. if there exists a deformation $\xi_t : X \to X$ of the space X such that $\xi_1 = h$. A continuous map

$$f : X \to Y$$

will be called a *homotopy equivalence* if there exists a continuous map

$$g : Y \to X,$$

such that both maps

$$g \circ f : X \to X, \quad f \circ g : Y \to Y$$

are homotopically identical. In this case, the map g is also a homotopy equivalence. We will call it the homotopy equivalence *inverse* to the equivalence f. Since the equivalence f is in turn inverse to the equivalence g, we will sometimes call the equivalences f and g *mutually inverse*. It is clear that any map that is homotopic to a homotopy equivalence is also a homotopy equivalence. We will call spaces X and Y *homotopically equivalent* if there exists at least one homotopy equivalence $f : X \to Y$.

It is obvious that the composition

$$f_2 \circ f_1 : X \to Z$$

of two homotopy equivalences $f_1 : X \to Y$ and $f_2 : Y \to Z$ is also a homotopy equivalence. Therefore, the relation of homotopy equivalence of spaces is transitive. Since it is obviously reflexive and symmetric, the totality of all topological spaces decomposes into *homotopy types* of pairwise homotopy equivalent spaces.

A continuous map

$$f : X \to Y$$

we will call *homotopically injective* (resp. *homotopically surjective*) if there exists a continuous map such that the composition $g \circ f : X \to X$ (resp. the composition $f\circ : Y \to Y$) is homotopically identical. It is clear that any homotopy equivalence is a map that is both homotopy injective and homotopy surjective. It turns out that the converse is also true, i.e.

Proposition 2.21. *a map $f : X \to Y$ is a homotopy equivalence if and only if it is homotopy injective and homotopy surjective.*

Proof. Indeed, let the map f be homotopically injective and simultaneously homotopically surjective, i.e. let there exist maps

$$g_1 : Y \to X, \quad g_2 : Y \to X,$$

such that

$$g_1 \circ f \sim 1_X, \quad f \circ g_2 \sim 1_Y.$$

Then

$$f \circ g_1 \sim (f \circ g_1) \circ (f \circ g_2) = f \circ (g_1 \circ f) \circ g_2 \sim f \circ g_2 \sim 1_Y$$

and similarly

$$g_2 \circ f \sim 1_X.$$

Therefore, the map f is a homotopy equivalence and each of the maps g_1 and g_2 is a homotopy equivalence inverse to the equivalence f. □

Note that the above argument has a general, "purely categorical" character. Similar general considerations show that if for the map $f : X \to Y$ there exist a space Z and a map $h : Z \to X$ such that the composition $f \circ h$ is a homotopy equivalence, then the map f is homotopy surjective. Similarly, if there exist a space Z_1 and a map $h_1 : Y \to Z_1$ such that the composition $h_1 \circ f$ is a homotopy equivalence, then the map f is homotopy injective. Thus,

Proof. if for the map $f : X \to Y$ there exist maps $h : Z \to X$ and $h_1 : Y \to Z_1$ such that the compositions $f \circ h$ and $h_1 \circ f$ are homotopy equivalences, then the map f is also a homotopy equivalence. □

Let the spaces X and Y be homotopically equivalent to the spaces X' and Y', respectively. Continuous maps

$$f : X \to Y, \quad f' : X' \to Y'$$

we will call ***homotopically equivalent*** if there exist homotopic equivalences

$$\varphi : X \to X', \quad \psi : Y \to Y',$$

such that

$$\psi \circ f \sim f' \circ \varphi$$

i.e. if the diagramme

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

is homotopically commutative. Denoting by

$$\varphi' : X' \to X, \quad \psi' : Y' \to Y$$

the homotopy equivalences inverse to the equivalences φ and ψ, respectively, we immediately obtain that this condition is equivalent to both the condition

$$f \sim \psi' \circ f' \circ \varphi$$

and the condition

$$\psi \circ f \circ \varphi' \sim f'$$

It is clear that a map that is homotopically equivalent to a homotopically injective (resp. surjective) map is also homotopically injective (resp. surjective).

A trivial example of homotopy equivalence is an arbitrary homeomorphic map $f : X \to Y$. This shows that all the concepts introduced above are topologically invariant.

Less trivial make-ups of homotopy equivalences arise when considering retractive maps

$$f : X \supset A.$$

A subspace A is called a *deformation retract* of the space X if there exists a retractive map $r : X \supset A$ that is a homotopy equivalence. Since, by definition, $r \circ i = 1_A$, the retractive map r is necessarily homotopy injective. Therefore, it is a homotopy equivalence if and only if it is homotopy injective, i.e., when there exists a map $k : A \to X$ such that $k \circ r \sim 1_X$. As we have seen, in this case both maps i and k are necessarily homotopy equivalences, inverse to the homotopy equivalence r. In particular, the condition $i \circ r \sim 1_X$ will be satisfied. Thus,

Proposition 2.22. *for a retracting map r to be a homotopy equivalence, it is necessary and sufficient that the composite map*

$$i \circ r : X \to X$$

be homotopy identical.

Moreover,

Proposition 2.23. *for any deformation retract A the inclusion map $i : A \subset X$ is a homotopy equivalence.*

Note that the converse is generally not true.

We obtain an important example of a deformation retract by considering (see §1.6) the cylinder Z_f of an arbitrary continuous map

$$f : X \to Y.$$

Namely, as we will now show,

Proposition 2.24. *the subspace Y of the space Z_f is its deformation retract.*

Proof. To this end, for any $\tau \in I$ we define the map

$$p_\tau : Z_f \to Z_f, \quad 0 \le \tau \le 1,$$

of the space Z_f into itself, putting

$$p_\tau(x, t) = (x, t + \tau - \tau t), \quad x \in X, \ 0 \le t \le 1,$$
$$p_\tau(y) = y, \quad y \in Y.$$

(In the first of these formulae, as in similar cases below, the symbol $(x, 1)$ is understood to mean the point $f(x) \in Y$.) The maps p_τ constructed in this way obviously constitute a homotopy (i.e. the corresponding map $Z_f \times I \to Z_f$ is unique and continuous). In addition, $p_0 = 1_{Z_f}$, i.e. this homotopy is a deformation of the space Z_f. Finally, for the map

$$p_1 : Z_f \to Z_f$$

the formula

$$p_1 = j \circ p,$$

holds, where

$$j : Y \subset Z_f,$$

is the inclusion map, and

$$p : Z_f \supset Y$$

is the retracting map of the space Z_f onto its subspace Y, defined by the formulae

$$p(x,t) = f(x), \quad x \in X,\ 0 \le t \le 1,$$
$$p(y) = y, \quad y \in Y.$$

Since $p \circ j = 1_Y$, the above statement is thus completely proven. □

Another remarkable property of the space Z_f is that

Proposition 2.25. *the map*

$$j \circ f : X \to Z_f$$

is homotopic to the map

$$i : X \to Zf,$$

defined by the formula

$$i(x) = (x, 0), \quad x \in X.$$

Proof. Indeed, the formula

$$i_t(x) = (x, 0), \quad x \in X,\ 0 \le t \le 1,$$

defines, as is easy to see, a homotopy

$$i_t : X \to Z_f,$$

connecting the map f with the map $j \circ f$. □

Since the inclusion map j is, as proved, a homotopy equivalence, this proposition means that the maps i and f are homotopy equivalent. On the other hand, identifying each point $x \in X$ with the corresponding point $i(x) = (x, 0) \in Z_f$, we can assume that

$$i : X \subset Z_f.$$

Thus, it is proved that

Proposition 2.26. *for any map $f : X \to Y$ there exists a space Z_f homotopically equivalent to the space Y and containing the space X such that the map $f : X \to Y$ is homotopically equivalent to the inlclusion map $i : X \subset Z_f$.*

Since a map homotopically equivalent to a homotopy equivalence is itself a homotopy equivalence, it follows from this statement that

Proposition 2.27. *if a subspace X of Z_f is its deformation retract, then the map f is a homotopy equivalence.*

It turns out that the converse is also true, i.e.

Proposition 2.28. *for any homotopy equivalence $f : X \to Y$ the space X is a deformation retract of the space Z_f.*

Proof. Indeed, let

$$g : Y \to X$$

be a homotopy equivalence inverse to the homotopy equivalence f, and let

$$\xi_t : X \to X, \quad \eta_t : Y \to Y$$

be deformations of the spaces X and Y, respectively, such that

$$\xi_1 = g \circ f, \quad \eta_1 = f \circ g.$$

For any point $z \in Z_f$ and any $\tau \in I$ we set

$$q_\tau(z) = \begin{cases} (x, t + 4\tau(1-t)) & \text{for} \quad 0 \le \tau \le 1/4, \\ \eta_{4\tau-1}(f(x)) & \text{for} \quad 1/4 \le \tau \le 1/2, \\ (g(f(x)), 3 - 4\tau) & \text{for} \quad 1/2 \le \tau \le 3/4, \\ \xi_{1+(1-t)(3-4\tau)} & \text{for} \quad 3/4 \le \tau \le 1, \end{cases}$$

if $z = (x, t)$, $xinX$, $0 \le t \le 1$, and

$$q_\tau(z) = \begin{cases} y & \text{for} \quad 0 \le \tau \le 1/4, \\ \eta_{4\tau-1}(y) & \text{for} \quad 1/4 \le \tau \le 1/2, \\ (g(y), 3 - 4\tau) & \text{for} \quad 1/2 \le \tau \le 3/4, \\ g(y) & \text{for} \quad 3/4 \le \tau \le 1, \end{cases}$$

if $z = y \in Y$.

It is easy to verify that the family

$$q_\tau ; Z_f \to Z_f, \quad 0 \le \tau \le 1,$$

defined in this way is a homotopy (recall that according to the results of §1.6, the map $\alpha \times 1_I : [(X \times I \cup Y] \times I \to Z_f \times I$, where $\alpha : (X \times I) \cup Y \to Z_f$ is the natural projection, is an identification map). Since, in addition,

$$q_0 = 1_{Z_f},$$

thus the homotopy q_τ is a deformation of the space Z_f.

On the other hand, the map

$$q_1 : Z_f \to Z_f$$

obviously has the form

$$q_1 = i \circ q,$$

where q is the map $Z_f \to X$ defined by the formulae

$$q(x, t) = \xi_t(x), \quad x \in X, \ 0 \le t \le 1,$$
$$q(y) = g(y), \quad y \in Y.$$

Since $q \circ i = 1_X$ the above statement is thus completely proven. □

2.3 Homotopy type of glued spaces

Let X and Y be topological spaces, A be a closed subspace of X and $f : A \to Y$ be a continuous map of the subspace A into the space Y. Then (see §1.6) the space $X \cup_f Y$ is defined, obtained by gluing the space X along the subspace A to the space Y by means of the map f. In this section we will show that for a sufficiently "good" pair (X, A) the homotopy type of the space $X \cup_f Y$ depends only on the homotopy class of the map f and the homotopy type of the space Y. In other words, for any map $g : A \to Y$ homotopic to the map f, the space $X \cup_g Y$ is homotopically equivalent to the space $X \cup_f Y$, and for ny space Z homotopically equivalent to the space Y, the space $X \cup_{h\circ f} Z$, where h is an arbitrary homotopy equivalence $Y \to Z$, is homotopically equivalent to the space $X \cup_f Y$.

First of all, we will show that

Proposition 2.29. *if a pair (X, A) has the property that both it and the pair $(X\times I, I(X, A))$ satisfy the axiom of homotopy extension, then for any two homotopic maps*

$$f, g : A \to Y$$

the spaces $X \cup_f Y$ and $X \cup_g Y$ are homotopy equivalent, and the homotopy equivalence $X \cup_f Y \to X \cup_g Y$ can be chosen in such a way that on the space Y it is the identity map.

Proof. Let

$$f_t : A \to Y, \quad 0 \le t \le 1,$$

be an arbitrary homotopy connecting the map f with the map g. First, we extend this homotopy to some homotopy of the space $\cup Y$ into the space $X \cup_f Y$. Since the natural projection

$$\alpha : X \cup Y \to X \cup_f Y$$

on the subspace $A \subset Y$ coincides with the map $f : A \to Y$ and since the pair (X, A) satisfies, by hypothesis, the axiom of homotopy extension, then there exists a homotopy

$$\alpha_t^* : X \to X \cup_f Y, \quad 0 \le t \le 1,$$

such that

$$\alpha_0^* = \alpha|_X, \quad \alpha_t^* = f_t.$$

Putting

$$\alpha_t = \begin{cases} \alpha_t & \text{on} \quad X, \\ 1_Y & \text{on} \quad Y, \end{cases} \quad 0 \le t \le 1,$$

we obviously obtain a homotopy

$$\alpha_t : X \cup Y \to X \cup_f Y, \quad 0 \le t \le 1,$$

for which

$$\alpha_0 = \alpha, \quad \alpha_t|_A = f_t.$$

Since the homotopy α_t has the property that

$$\alpha_1|_A = g,$$

the map

$$\eta = \alpha_1 \circ \beta^{-1} : X \cup_g Y \to X \cup_f Y,$$

where

$$\beta : X \cup Y \to X \cup_g Y$$

is the natural projection, is a single-valued map. Moreover, since the map

$$\alpha_1 = \eta \circ \beta$$

is continuous, and the map β is an identification map, the map η is continuous (see §1.5). On the space Y this map is identical:

$$\eta|_Y = 1_Y.$$

Let us now construct the "inverse" map $X \cup_g Y \to X \cup_f Y$. By the same considerations as above, there exists a homotopy

$$\beta_t : X \cup Y \to X \cup_g Y, \quad 0 \le t \le 1,$$

such that

$$\beta_0 = \beta, \quad \beta_t|_A = f_{1-t},$$

and

$$\beta_t|_Y = 1_Y.$$

Since $\beta_1|_A = f$, then, setting

$$\xi = \beta_1 \circ \alpha^{-1},$$

we obtain a single-valued continuous map

$$\xi : X \cup_f Y \to X \cup_g Y,$$

for which, as for η,

$$\xi|_Y = 1_Y.$$

The above statement will obviously be proved if we show that the maps ξ and η are mutually inverse homotopy equivalences. With this in mind, for any number $t \in I$ we set

$$\gamma_t = \begin{cases} \alpha_{2t}, & \text{if} \quad 0 \le t \le 1/2, \\ \eta \circ \beta_{2t-1}, & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

Since $\alpha_1 = \eta \circ \beta_0$, then we thereby obtain some homotopy

$$\gamma_t : X \cup Y \to X \cup_f Y, \quad 0 \le t \le 1,$$

for which

$$\gamma_0 = \alpha, \quad \gamma_1 = \eta \circ \xi \circ \alpha.$$

The homotopy γ_t is *not*, generally speaking, a homotopy relative to A, because

$$\gamma_t|_A = \begin{cases} f_{2t}, & \text{if} \quad 0 \leq t \leq 1/2, \\ f_{2t-1}, & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

Nevertheless, the map $A \times I \to X \cup_f Y$ corresponding to the homotopy $\gamma_t|_A$, as is easy to see, is homotopic relative to $A \times 0 \cup A \times 1$ to the map $A \times I \to X \cup_f Y$, which maps each point $(a, t) \in A \times I$ to the point $\alpha(a) \in X \cup_f Y$, i.e., which defines a stationary homotopy of the map $\alpha|_A$. The corresponding homotopy

$$H_\tau : A \times I \to X \cup_f Y, \quad 0 \leq \tau \leq 1,$$

can, for example, be defined by the formula

$$H_\tau(a, t) = \begin{cases} f_{2t(1-\tau)}(a), & \text{if} \quad 0 \leq t \leq 1/2, \\ f_{2(1-t)(1-\tau)}(a), & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

where (a, t) is an arbitrary point in space $A \times I$. Therefore, according to the statement proved in 2.14 (the conditions of applicability of which are fulfilled), the maps α and $\eta \circ \xi\alpha$ are also homotopic relative to A. The corresponding homotopy

$$\gamma_t^* : X \cup Y \to X \cup_f Y, \quad 0 \leq t \leq 1,$$

relative to A has the property that

$$\gamma_t^*|_A = f$$

for any $t \in I$. Therefore, the formula

$$h_t = \gamma_t^* \circ \alpha^{-1}, \quad 0 \leq t \leq 1,$$

uniquely defines a certain homotopy

$$h_t : X \cup_f Y \to X \cup_f Y$$

(recall that by the results of §1.5 the map

$$\alpha \times 1_I : (X \cup Y) \times I \to (X \cup_f Y) \times I$$

is an identification map), which, obviously, has the property that

$$h_0 = 1_{X \cup_f Y}, \quad h_1 = \eta \circ \xi.$$

Therefore, the map

$$\eta \circ \xi : X \cup_f Y \to X \cup_f Y$$

is homotopy identical.

Since the maps f and g are completely equivalent, then, by symmetry considerations, the map

$$\xi \circ \eta : X \cup_g Y \to X \cup_g Y$$

is also homotopy identical.

Thus, the proposition formulated above is completely proven. □

Let us now prove that the homotopy type of the space $X \cup_f Y$ does not change even when the space Y is replaced by a space homotopically equivalent to it. More precisely, we will prove that

Proposition 2.30. *under the same assumptions on the pair (X, A) as above, any homotopy equivalence $f : Y \to Z$ can be extend to some homotopy equivalence*

$$H : X \cup_f Y \to X \cup_{h\circ f} Z.$$

Proof. Let

$$g : Z \to Y$$

be the homotopy equivalence inverse to h, and let

$$H = \beta^{-1} \circ \overline{h} \circ \alpha : X \cup_f Y \to X \cup_{h\circ f} Z,$$
$$G = \alpha' \circ \overline{g} \circ \beta^{-1} : X \cup_{h\circ f} Z \to X \cup_{g\circ h\circ f} Y,$$

where, respectively,

$$\overline{h} = 1_X \cup h : X \cup Y \to X \cup Z,$$
$$\overline{g} = 1_X \cup g : X \cup Z \to X \cup Y,$$
$$\alpha : X \cup Y \to X \cup_f Y,$$
$$\alpha' : X \cup Y \to X \cup_{g\circ h\circ f} Y,$$
$$\beta : X \cup Z \to X \cup_{h\circ f} Z,$$

are the natural projections. Since

$$\beta \circ \overline{h}|_A = \beta|_A = h \circ f, \quad \alpha' \circ \overline{g}|_A = \alpha'|_A = g \circ h \circ f,$$

then the maps H and G are single-valued and continuous.

Since the map

$$g \circ h : Y \to Y$$

is homotopically identical (i.e., has the form σ_1 where

$$\sigma_t : Y \to Y$$

is some deformation of the space Y), the map $f \circ h \circ f : A \to Y$ is homotopic to the map $f : A \to Y$, and therefore, according to the previous proposition, the spaces $X \cup_f Y$ and $X \cup_{g\circ h\circ f} Y$ are homotopically equivalent. Moreover, from the construction imposed in the proof of the last proposition it follows that the homotopy equivalence

$$\eta : X \cup_{g\circ h\circ f} Y \to X \cup_f Y$$

connecting the spaces $X \cup_{g\circ h\circ f} Y$ and $X \cup_f Y$, can be defined by the formula

$$\eta = \alpha_1 \circ (\alpha^{'})^{-1},$$

where α_1 is a finite map of some homotopy

$$\alpha_t : X \cup Y \to X \cup_f Y, \quad 0 \leq t \leq 1,$$

for which

$$\alpha_0 = \alpha, \quad \alpha_t|_A = \sigma_t \circ f,$$

and

$$\alpha_t|_Y = 1_Y.$$

Therefore, the formula

$$k_t = \begin{cases} \alpha \circ \overline{\sigma}_{2t}, & \text{if} \quad 0 \leq t \leq 1/2, \\ \alpha_{2t-1} \circ \overline{g} \circ h), & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

where

$$\overline{\sigma}_t = 1_X \cup \sigma_t : X \cup Y \to X \cup Y, \quad 0 \leq t \leq 1,$$

defines some homotopy

$$k_t : X \cup Y \to X \cup_f Y, \quad 0 \leq t \leq 1,$$

connecting the map

$$k_0 = \alpha \circ \overline{\sigma}_0 = \alpha$$

with the map

$$k_1 = \alpha_1 \circ \overline{g} \circ \overline{h} = \eta \circ G \circ H \circ \alpha.$$

In this case

$$k_t|_A = \begin{cases} \alpha, & \text{if} \quad 0 \leq t \leq 1/2, \\ \alpha_{2t-1}, & \text{if} \quad 1/2 \leq t \leq 1, \end{cases}$$

i. e,

$$k_t|_A = \begin{cases} f, & \text{if} \quad 0 \leq t \leq 1/2, \\ \sigma_{2t-1} \circ f, & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

Therefore, assuming

$$k_t^* = k_t \circ \alpha^{-1}, \quad 0 \leq t \leq 1,$$

we obtain a deformation

$$k_t^* : X \cup_f Y \to X \cup_f Y, \quad 0 \leq t \leq 1,$$

of the space $X \cup_f Y$, connecting the identity map of this space with the map $\eta \circ G \circ H$.

Thus, the last map is homotopy identical and therefore the map H is homotopy injective, and the map $\eta \circ G$, and therefore the map G, is homotopy surjective.

Similarly (taking the maps $h \circ f$ and g as the maps f and h respectively), we obtain that the map G is homotopy injective. Consequently, the map G, and therefore the map $\eta \circ G$, is a homotopy equivalence. But then the map H will also be a homotopy equivalence. To complete the proof, it remains to note that

$$H|_Y = h.$$

□

2.4 Homotopy groups and weak homotopy equivalences

Let $n \geq 0$ and let $\boldsymbol{u}_0$ be a point $(1, 0, \dots, 0)$ of the unit n-dimensional sphere $\mathbb{S}^n$. Let, further, x_0 be an arbitrary point of the topological space X. The subject of our study in this section will be continuous maps

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X, x_0)$$

of the pair $(\mathbb{S}^n, \boldsymbol{u}_0)$ into the pair (X, x_0). The set $[(\mathbb{S}^n, \boldsymbol{u}_0), (X, x_0)]$ of all homotopy classes of such maps relative to x_0 will be denoted by the symbol

$$\pi_n(X; x_0).$$

The class of the constant map $\mathbb{S}^n \to x_0$ we will denote by the symbol 0_{x_0} (or simply 0) and we will call it the *zero* of the set $\pi_n(X; x_0)$.

It is clear that the set $\pi_0(X; x_0)$ is naturally identified with the set $\pi_0(X)$ of the path-connected components of the space X. With this identification, the zero 0_{x_0} of the set $\pi_0(X; x_0)$ corresponds to the component of the space X containing the point x_0.

We will call a map of one set of the form $\pi_n(X; x_0)$ into another such set a *homomorphism* if it maps zero to zero. We will call an injective homomorphism a *monomorphism*, a surjective homomorphism an *epimorphism*, and a bijective homomorphism an *isomorphism*.

Remark 2.31. For $n > 0$, an algebraic operation can be introduced into the set $\pi_n(x; x_0)$, with respect to which this set turns out to be a group (for $n > 1$, even an Abelian group) with zero element 0_{x_0}. This group is called the *n-th homotopy group of the space X at the point x_0*. In what follows, we will also adhere to this terminology (even for $n = 0$), although we do not need the mentioned operation, and we will neither define nor consider it. All homomorphisms of homotopy groups considered below will in fact be homomorphisms in the usual group-theoretical sense. We will also ignore this fact.

According to what was said in §2.1, any continuous map

$$f : X \to Y$$

defines by the formula

$$f_*([\varphi] \operatorname{rel} \boldsymbol{u}_0) = [f \circ \varphi] \operatorname{rel} \boldsymbol{u}_0,$$

where φ is an arbitrary map $(\mathbb{S}^n, \boldsymbol{U}_0) \to (X; x_0)$, some map

$$f_* : \pi_n(X; x_0) \to \pi_n(Y; y_0), \quad y_0 = f(x_0).$$

This maps zero 0_{x_0} to zero 0_{y_0}, i.e. is a homomorphism (in the sense indicated above). In those cases where it is necessary to explicitly indicate the point x_0, we will denote this homomorphism by the symbol f_{*,x_0}.

It is clear that if $f = 1_X$, then $f_* = 1_{\pi_n(X;x_0)}$, and that for any maps $f : X \to Y$ and $g : Y \to Z$ we have

$$(g \circ f)_* = g_* \circ f_*.$$

In the language of category theory, this means that the pair $(\pi_n(X; x_0), f_*)$ is a covariant functor.

We will not present here all the numerous properties of homotopy groups - this would take us too far from the main topic. We will limit ourselves to only those properties of these groups that will be needed later. In this case, for any $n \geq 0$ we will represent the sphere $\mathbb{S}^n$ as a union of two hemispheres - the "lower" hemisphere $\mathbb{E}^n_-$ consisting of all points $\boldsymbol{u} = (u^1, \dots, u^{n+1}) \in \mathbb{S}^n$, for which $u^{n+1} \leq 0$, and the "upper" hemisphere $\mathbb{E}^n_+$, consisting of all points $\boldsymbol{u} = (u^1, \dots, u^{n+1}) \in \mathbb{S}^n$, for which $u^{n+1} \geq 0$. The intersection $\mathbb{E}^n_- \cap \mathbb{E}^n_+$ of these hemispheres is the *equator* $u^{n+1} = 0$ of the sphere $\mathbb{S}^n$, which we will identify with the sphere $\mathbb{S}^{n-1}$. The projection

$$\omega : \mathbb{S}^n \to \mathbb{E}^n,$$

defined by the formula

$$\omega(u^1, \dots, u^{n+1}) = (u^1, \dots, u^n),$$

is, obviously, a homeomorphism on each of the hemispheres $\mathbb{E}^n_-$ and $\mathbb{E}^n_+$. We will denote these homeomorphisms by the symbols ω_- and ω_+, respectively.

In what follows we will constantly use the fact that

Proposition 2.32. *pairs $(\mathbb{S}^n, \boldsymbol{u}_0)$, $(\mathbb{S}^n, \mathbb{E}^n_+)$, $(\mathbb{S}^n \times I(\mathbb{S}^n, \boldsymbol{u}_0))$ and $(\mathbb{S}^n \times I(\mathbb{S}^n, \mathbb{E}^n_+))$ satisfy the axiom of homotopy extension.*

This fact is a special case of one general statement, which we will prove in §3.4; see Remark 3.56. Therefore, we will leave it here without proof. We will not use the results of this section until §3.6.

We also note that any vector $\boldsymbol{v} \in \mathbb{E}^n$ can be "related to a point $\boldsymbol{u}_0$", i.e. represented in the form

$$\boldsymbol{v} = \boldsymbol{u}_0 + (\boldsymbol{u} - \boldsymbol{u}_0)r,$$

where $\boldsymbol{u} \in \mathbb{S}^{n-1}$ and $0 \le r \le 1$. In this case, the number r is determined by the vector $\boldsymbol{v}$ uniquely. The vector $\boldsymbol{u}$ is also determined uniquely, if only $\boldsymbol{v} \neq \boldsymbol{u}_0$. In addition, for $\boldsymbol{v} \neq \boldsymbol{u}_0$ the number r is nonzero (and for $\boldsymbol{v} = \boldsymbol{u}_0$ it is zero). To simplify the formulae, we will henceforth denote the vector $\boldsymbol{v} = \boldsymbol{u}_0 + (\boldsymbol{u} - \boldsymbol{u}_0) r \in \mathbb{E}^n$ by the symbol $[\boldsymbol{u}, r]$, and the vectors $\omega_+^{-1}(\boldsymbol{v}) \in \mathbb{E}^n_+$ and $\omega_-^{-1}(\boldsymbol{v}) \in \mathbb{E}^n_-$ by the symbols $[\boldsymbol{u}, r]_+$ and $[\boldsymbol{u}, r]_-$, respectively.

First of all, we will find out under what conditions the two maps

$$f, g : \mathbb{E}^n \to X$$

of the ball $\mathbb{E}^n$ into the space X, coinciding on its boundary $\mathbb{S}^{n-1}$, are homotopic relative to $\mathbb{S}^{n-1}$.

For this purpose, to any two such maps f and g we assign a map

$$\varphi : \mathbb{S}^n \to X$$

of the sphere $\mathbb{S}^n$ into the space X, "glued" from maps f and g, considered as maps of the hemispheres $\mathbb{E}^n_+$ and $\mathbb{E}^n_-$, i.e. defined by the equalities

$$\varphi_{\mathbb{E}^n_-} = f \circ \omega_-, \quad \varphi_{\mathbb{E}^n_+} = f \circ \omega_+.$$

By hypothesis, $f|_{\mathbb{S}^{n-1}} = g|_{\mathbb{S}^{n-1}}$ and, therefore, the map φ is uniquely defined and continuous. The element $[\varphi]$ of the group $\pi_n(X; x_0)$, where $x_0 = \varphi(\boldsymbol{u}_0)(= f(\boldsymbol{u}_0) = g(\boldsymbol{u}_0))$, defined by the map φ, we will call the element *distinguishing* the maps f and g, and denote it by the symbol $\delta(f, g)$.

It is easy to see that any element of the set $\pi_n(X; x_0)$ can serve as an element that distinguishes some map from a given one. Namely,

Proposition 2.33. *for any map*

$$g : \mathbb{E}^n \to X$$

and any element $\alpha \in \pi_n(x; x_0)$*, where* $x_0 = g(\boldsymbol{u}_0)$*, there exists a map*

$$f : \mathbb{E}^n \to X,$$

such that

$$f|_{\mathbb{S}^n} = g|_{\mathbb{S}^n}$$

and

$$\delta(f, g) = \alpha.$$

Proof. Indeed, let

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X, x_0)$$

be an arbitrary map of class α. Obviously, on the hemisphere $\mathbb{E}^n_+$ the map φ is homotopic relatve to $\boldsymbol{u}_0$ to the map $\boldsymbol{g} \circ \omega_+$. And the corresponding homotopy

$$\varphi_t : \mathbb{E}^n_+ \to X$$

can be defined, for example, by the formula

$$\varphi_t([\boldsymbol{u}, r]_+) = \begin{cases} \varphi([\boldsymbol{u}, (1-2t)r]_+), & \text{if} \quad 0 \le t \le 1/2, \\ f([\boldsymbol{u}, (2t-1)r]), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

where $[\boldsymbol{u}, r]_+$ is an arbitrary point of the hemisphere $\mathbb{E}^n_+$; the essence of the matter here is that the hemisphere $\mathbb{E}^n_+$ can be contracted in itself to the point $\boldsymbol{u}_0$ and therefore any map of this hemisphere is homotopic to the constant map, so that any two maps are homotopic. Since the pair $(\mathbb{S}^n, \mathbb{E}^n_+)$ satisfies, as was said, the axiom of homotopy extension, it follows that the map φ is homotopic relative to $\boldsymbol{u}_0$ to a map

$$\psi : \mathbb{S}^n \to X$$

such that $\psi|_{\mathbb{E}^n_+} = g \circ \omega_+$. But then it is clear that the map

$$f = \psi|_{\mathbb{E}^n_-} \circ \omega_-^{-1} : \mathbb{E}^n \to X$$

has all the required properties. □

The significance of distinguishing elements $\delta(f, g)$ for the problem of homotopy relative to $\mathbb{S}^{n-1}$ of maps f and g is determined by the fact that

Proposition 2.34. *if for maps coinciding on* $\mathbb{S}^{n-1}$

$$f, g : \mathbb{E}^n \to X,$$

there exists a map

$$h : \mathbb{E}^n \to X, \quad h|_{\mathbb{S}^{n-1}} = f|_{\mathbb{S}^{n-1}} = g|_{\mathbb{S}^{n-1}},$$

such that

$$\delta(f, h) = \delta(g, h),$$

then the maps f and g are homotopic relative to $\mathbb{S}^n$.

Proof. To prove this statement, we consider the maps

$$\varphi_0, \varphi_1 : \mathbb{S}^n \to X,$$

defined by the formulae

$$\varphi_0|_{\mathbb{E}^n_-} = f \circ \omega_-, \quad \varphi_1|_{\mathbb{E}^n_-} = g \circ \omega_-, \quad \varphi_0|_{\mathbb{E}^n_+} = \varphi_1|_{\mathbb{E}^n_+} = h \circ \omega_+.$$

As is easy to see, it is enough for us to prove that

$$\varphi_0 \sim \varphi_1 \operatorname{rel} \mathbb{E}^n_+. \tag{2.35}$$

Indeed, for any homotopy $\varphi_t : \varphi_0 \sim \varphi_1 \operatorname{rel} \mathbb{E}^n_+$ the family of maps

$$\varphi_t|_{\mathbb{E}^n_-} \circ \omega^{-1} : \mathbb{E}^n \to X, \quad 0 \le t \le 1,$$

will be a homotopy relatiive to $\mathbb{S}^{n-1}$, connecting the map f with the map g.

By hypothesis, $\delta(f, h) = \delta(g, h)$, so that the maps φ_0 and φ_1 are homotopic relative to $\boldsymbol{u}_0$. Having chosen some homotopy relative to $\boldsymbol{u}_0$ that connects the maps φ_0 and φ_1, we consider the map

$$\Phi : \mathbb{S}^n \times I \to X$$

of the product $\mathbb{E}^n_+ \times I$ into the space X corresponding to this homotopy. Let, in addition,

$$\Psi : \mathbb{E}^n_+ \times I \to X$$

be a map of the product $\mathbb{E}^n_+ \times I$ into the space X defined by the formula

$$\Psi(\boldsymbol{u}, t) = \varphi_0(\boldsymbol{u}) = (h \circ \omega_+)(\boldsymbol{u}), \quad (\boldsymbol{u}) \in \mathbb{E}^n_-, 0 \leq t \leq 1.$$

According to Proposition 2.14, it is applicable, since the pair $(\mathbb{S}^n \times I, I(\mathbb{S}^n, \mathbb{E}^n_+))$ satisfies the axiom of homotopy extension, to prove relation (2.35) it is sufficient to show that

$$\Phi_0|_{\mathbb{E}^n_+ \times I} \sim \Psi \operatorname{rel}(\mathbb{E}^n_+ \times 0 \cup \mathbb{E}^n_+ \times 1).$$

For any $\tau \in I$ and any point $([\boldsymbol{u}, r]_+, t) \in \mathbb{E}^n_+ \times I$, where $\boldsymbol{u} \in \mathbb{S}^{n-1}$, $r, t \in I$, we set (see Fig. 2.2)

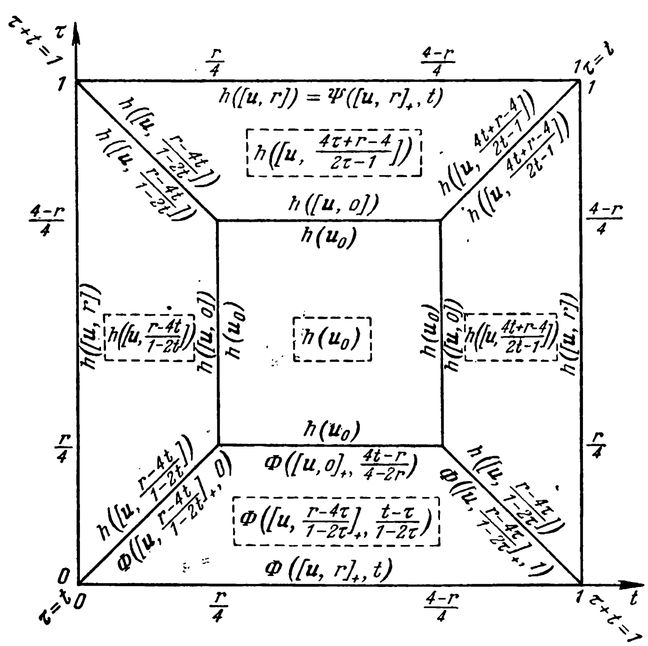


Figure 2.2:

$$\Phi_\tau([\boldsymbol{u},r]_+,t)=\begin{cases}\Phi\left([\boldsymbol{u},\frac{r-4\tau}{1-2\tau}]_+,\frac{t-\tau}{1-2\tau}\right), & \text{if } 0\le\tau\le\frac{r}{4},\tau\le t\le 1-\tau,\\ h\left([\boldsymbol{u},\frac{r-4t}{1-2t}]\right), & \text{if } t\le\tau\le 1-t, 0\le t\le\frac{r}{4},\\ h(\boldsymbol{u}_0), & \text{if } \frac{r}{4}\le\tau\le\frac{4-r}{4},\frac{r}{4}\le t\le\frac{4-r}{4},\\ h\left([\boldsymbol{u},\frac{4t+r-4}{2t-2}]\right), & \text{if } 1-t\le\tau\le t,\frac{4-r}{4}\le t\le 1,\\ h\left([\boldsymbol{u},\frac{4\tau+r-4}{2\tau-1}]\right), & \text{if } \frac{4-r}{4}\le\tau\le 1, 1-\tau\le t\le\tau.\end{cases}$$

It is easy to verify (Fig. 2.2) that we thereby obtain a certain homotopy

$$\Phi_\tau:\mathbb{E}^n_+\times I\to X$$

relative to $\mathbb{E}^n_+\times 0\cup\mathbb{E}^n_+\times 1$, for which

$$\Phi_0=\Phi|_{\mathbb{E}^n_+\times I},\quad \Phi_1=\Psi.$$

The statement formulated above is thus completely proven. □

Remark 2.36. For the existence of a homotopy Φ_τ it is essential that the map φ_0 is homotopic to the map φ_1 relative to $\boldsymbol{u}_0$. If φ_0 is simply homotopic to φ_1, then the homotopy Φ_τ may not exist.

Let us now consider the question of the dependence of the group $\pi_n(X;x_0)$ on the point x_0.

Let x_0 and x_1 be two arbitrary points of the space X, which can be connected in X by some path $u:I\to X$. Since the pair $(\mathbb{S}^n,\boldsymbol{u}_0)$ satisfies the axiom of homotopy extension, then for any map

$$\varphi:(\mathbb{S}^n,\boldsymbol{u}_0)\to(X,x_0)$$

there exists a homotopy

$$\varphi_t:\mathbb{S}^n\to X,\quad 0\le t\le 1,$$

such that $\varphi_0=\varphi$ and $\varphi_t(\boldsymbol{u}_0)=u(t)$ for each $t\in I$. We will call this homotopy a *homotopy of the map φ along the path u.*

We will call two paths u and v, connecting a point x_0 with a point x_1 *equivalent* if they are homotopic relative to the points 0 and 1, i.e. if there exists a homotopy

$$u_\tau:I\to X,\quad 0\le\tau\le 1,$$

consisting of paths connecting the points x_0 and x_1 such that $u_0=u$ and $u_1=v$.

It turns out that

Proposition 2.37. *if the maps*

$$\varphi,\psi:(\mathbb{S}^n,\boldsymbol{u}_0)\to(X,x_0)$$

are homotopic relative to $\boldsymbol{u}_0$, then for any two equivalent paths u and v connecting a point x_0 with a point x_1, and any two homotopies

$$\varphi_t,\psi_t:\mathbb{S}^n\to X$$

of the maps φ and ψ along, respectively, the paths u and v the pointed maps

$$\varphi_1, \psi_1 : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X, x_1)$$

are homotopic relative to $\boldsymbol{u}_0$.

Proof. Indeed, let

$$\xi_t : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X, x_0), \quad 0 \le t \le 1,$$

be a homotopy relative to $\boldsymbol{u}_0$, connecting the map φ with the map ψ. It is clear that the formula

$$\omega_t = \begin{cases} \varphi_{1-3t}, & \text{if} \quad 0 \le t \le 1/3, \\ \xi_{3t-1}, & \text{if} \quad 1/3 \le t \le 2/3, \\ \psi_{3t-2}, & \text{if} \quad 2/3 \le t \le 1, \end{cases}$$

defines a certain homotopy

$$\omega_t : \mathbb{S}^n \to X, \quad 0 \le t \le 1,$$

connecting the map φ_1 with the map ψ_1. Let

$$\Omega : \mathbb{S}^n \times I \to X$$

be the map of the product $\mathbb{S}^n \times I$ into the space X corresponding to this homotopy and let

$$\Omega_1 : \boldsymbol{u}_0 \times I \to X$$

be the constant map of the segment $\boldsymbol{u}_0 \times I$ into the point $x_1 \in X$. Since the pair $(\mathbb{S}^n \times I, I(\mathbb{S}^n, \boldsymbol{u}_0))$ satisfies the axiom of homotopy extension, then, according to the proposition proved in §point 2.1, to prove the relation

$$\varphi_1 \sim \psi_1 \operatorname{rel} \boldsymbol{u}_0$$

it suffices to prove that

$$\Omega|_{\boldsymbol{u}_0 \times I} \sim \Omega_1 \operatorname{rel}(\boldsymbol{u}_0 \times 0 \cup \boldsymbol{u}_0 \times 1),$$

i.e., the path

$$w : I \to X, \quad w(0) = w(1) = x_1,$$

defined by the formula

$$w(t) = \Omega(\boldsymbol{u}_0, t) = \begin{cases} u(1-3t), & \text{if} \quad 0 \le t \le 1/3, \\ x_0, & \text{if} \quad 1/3 \le t \le 2/3, \\ v(3t-2), & \text{if} \quad 2/3 \le t \le 1, \end{cases}$$

is equivalent to the degenerate path u_{x_1}, which is a constant map of the segment I to the point x_1.

For this purpose, for any $t, \tau \in I$ we set (see Fig. 2.3)

$$w_\tau(t) = \begin{cases} u\left(\frac{3\tau-6t+2}{3\tau+2}\right), & \text{if} \quad 0 \le \tau \le 1/3, 0 \le t \le \frac{3\tau+2}{6}, \\ x_0, & \text{if} \quad 0 \le \tau \le 1/3, \frac{3\tau+2}{6} \le t \le \frac{4-3\tau}{6}, \\ v\left(\frac{3\tau+6t-4}{3\tau+2}\right), & \text{if} 0 \le \tau \le 1/3, \frac{4-3\tau}{6} \le t \le 1, \\ u(1-2t), & \text{if} \quad 1/3 \le \tau \le 2/3, 0 \le t \le 1/2, \\ u_{2-3\tau}(2t-1), & \text{if} \quad 1/3 \le \tau \le 2/3, 1/2 \le t \le 1, \\ u(1-6t(1-\tau)), & \text{if} \quad 2/3 \le \tau \le 1, 0 \le t \le 1/2, \\ u(1-6(1-t)(1-\tau)), & \text{if} \quad 2/3 \le \tau \le 1, 1/2 \le t \le 1, \end{cases}$$

where $u_\tau : I \to X$, $0 \le \tau \le 1$, is a homotopy relative to the points 0 and 1, connecting the path u with the path v. It is easy to verify that we thereby

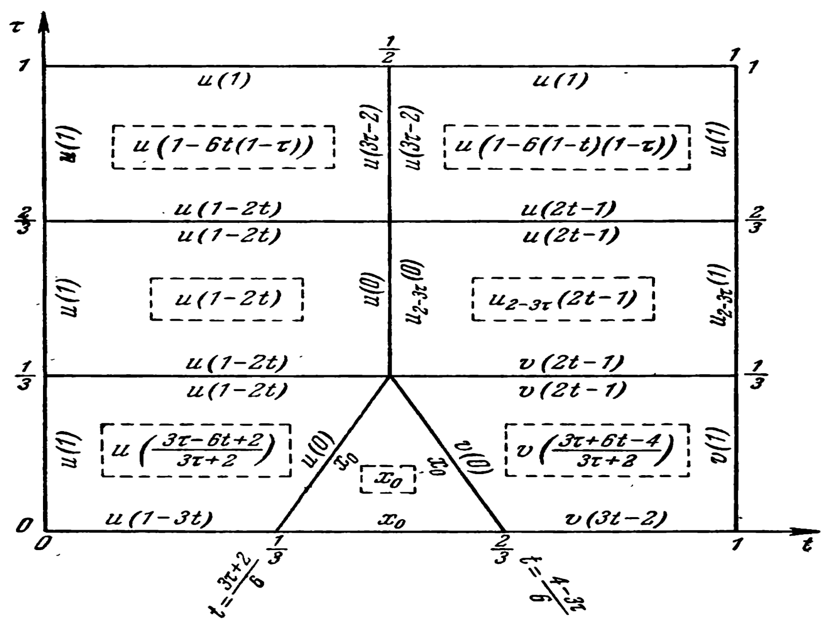


Figure 2.3:

obtain a homotopy

$$w_t : I \to X, \quad 0 \le \tau 1,$$

relative to the points 0 and 1, connecting the path w with the path u_{x_1}.

The above statement is thus completely proven. □

It follows directly from this statement that for any path u, connecting points x_0 and x_1 the formula

$$u^{\#}(\alpha) = [\varphi_1] \operatorname{rel} \boldsymbol{u}_0,$$

where $\alpha \in \pi_n(X;x_0)$, and φ_1 is a pointed map of an arbitrary homotopy φ_t along the path u of some map

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X;x_0)$$

of class α uniquely determines some map

$$u^{\#} : \pi_n(X;x_0) \to \pi_n(X;x_1),$$

mapping zero 0_{x_0} to zero 0_{x_1}, i.e. being a homomorphism. At the same time,

Proposition 2.38. *the homomorphism $u^{\#}$ depends only on the equivalence class of the path u, i.e. for any two equivalent paths u and v the homomorphisms $u^{\#}$ and $v^{\#}$ coincide.*

Now let u and v be paths of the space X such that

$$u(1) = v(0).$$

Then the formula

$$w(t) = \begin{cases} u(2t), & \text{if} \quad 0 \le t \le 1/2, \\ v(2t-1), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

defines, as is easy to see, a certain path

$$w : I \to X,$$

connecting the point $x_0 = u(0)$ with the point $x_1 = v(1)$. We will call this path the *product* of paths u and v and will denote it by the symbol uv. It is easy to see that

Proposition 2.39. *the homomorphism*

$$(uv)^{\#} : \pi_n(X;x_0) \to \pi_n(X;x_2),$$

corresponding to the path uv, is a composition of $v^{\#} \cdot u^{\#}$ of homomorphisms

$$u^{\#} : \pi_n(X;x_0) \to \pi_n(X;x_1), \quad v^{\#} : \pi_n(X;x_1) \to \pi_n(X;x_2),$$

corresponding to the paths u and v.

Proof. Indeed, for any homotopy φ_t along the path u of an arbitrary map $\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X;x_0)$ and any homotopy ψ_t, along the path v of an arbitrary map

$$\psi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X;x_1)$$

the formula

$$\omega_t = \begin{cases} \varphi(2t), & \text{if} \quad 0 \le t \le 1/2, \\ \psi(2t-1), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

defines a homotopy along the path uv for which $\omega_0 = \varphi$ and $\omega_1 = \psi_1$. □

To each path $u : I \to X$ connecting a point x_0 with a point x_1 we assign a path $u^{'} : I \to X$ connecting the point $x_1 x$ with the point x_0, defined by the formula

$$u^{'}(t) = u(1-t), \quad 0 \le t \le 1.$$

It is easy to see that

Proposition 2.40. *for any path u connecting points x_0 and x_1 the paths $uu^{'}$ and $u^{'}u$ are equivalent to the corresponding degenerate paths u_{x_0} and u_{x_1}.*

Proof. Indeed, a homotopy $v_\tau : I \to X$, $0 \le \tau \le 1$, relative to the points 0 and 1, connecting, say, the path u_{x_0} with the path $uu^{'}$ can, for example, be defined by the formula

$$v_\tau(t) = \begin{cases} u(2t\tau), & \text{if} \quad 0 \le t \le 1/2, \\ u(2(1-t)\tau), & \text{if} \quad 1/2 \le t \le 1, \end{cases} \quad 0 \le \tau \le 1.$$

□

Since the homomorphisms $(u_{x_0})^{\#}$ and $(u_{x_1})^{\#}$ corresponding to the degenerate paths u_{x_0} and u_{x_1} are, as is easy to see, identity maps of the groups $\pi_n(X; x_0)$ and $\pi_n(X; x_1)$ respectively, it follows directly from the last two statements that

Proposition 2.41. *for any path $u :\to X$ connecting the point x_0 with the point x_1 the homomorphisms $u^{\#}$ and $(u^{'})^{\#}$ are mutually inverse isomorphisms between the groups $\pi_n(X; x_0)$ and $\pi_n(X; x_1)$.*

In particular,

Proposition 2.42. *for any path u connecting a point x_0 with a point x_1, the homomorphism*

$$u^{\#} : \pi_n(X; x_0) \to \pi_n(X; x_1)$$

is an isomorphism.

Thus, for any points x_0 and x_1 belonging to the same path component of the space X, the groups $\pi_n(X; x_0)$ and $\pi_n(X; x_1)$ are essentially the same. For points belonging to different path components of the space X, these groups, generally speaking, are not connected with each other in any way.

In order to clarify the geometric meaning of the isomorphisms $u^{\#}$, we will show that

Proposition 2.43. *for elements $\alpha \in pi_n(X; x_0)$ and $\beta \in \pi_n(X; x_1)$ if and only if there exists a path $u : I \to X$, $u(0) = x_0$, $u(1) = x_1$, such that*

$$\beta = u^{\#}(\alpha),$$

if and only if when the maps

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X; x_0), \quad \psi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X; x_1),$$

belonging to the classes α and β are homotopic to each other.

Proof. Indeed, if $\beta = u^{\#}(\alpha)$, then the map ψ is homotopic (relative to $\boldsymbol{u}_0$) to a pointed map φ_1 of some homotopy φ_t of the map φ along the path u. Thus, $\varphi_t : \varphi \sim \psi$ and, consequently, $\psi \sim \varphi$. Conversely, any homotopy $\varphi_t : \varphi \sim \psi$ can be considered as a homotopy of the map φ along the path

$$u(t) = \varphi_t(\boldsymbol{u}_0), \quad 0 \le t \le 1.$$

Therefore, if $\varphi_t : \varphi \sim \psi$, then $\beta = u^{\#}(\alpha)$. □

Since for each path u the map $u^{\#}$ is an isomorphism, the only element $\alpha \in \pi_n(X; x_0)$ for which $u^{\#}(\alpha) = 0_{x_1}$ is the element 0_{x_0}. This means that

Proposition 2.44. *the map*

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X; x_0)$$

is homotopic to some constant map if and only if it is homotopic relative to $\boldsymbol{u}_0$ *to the constant map to the point* x_0.

In other words (see §2.1),

Proposition 2.45. *the map*

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X; x_0)$$

belongs to the class 0_{x_0} *if and only if when it can be extended to some map* $\mathbb{E}^n \to X$.

Therefore,

Proposition 2.46. *the space* X *is* m*-connected if and only if for some (and therefore for any) point* $x_0 \in X$ *and any* $n \le m$ *the equality*

$$\pi_n(X; x_0) = 0$$

holds.

Let us now consider an arbitrary deformation.

$$\xi_t : X \to X$$

of the space X. For any point $x_0 \in X$ this deformation defines a path

$$u(t) = \xi_t(x_0), \quad 0 \le t \le 1,$$

connecting the point x_0 with the point $x_1 = \xi_1(x_0)$. On the other hand, it is clear that for any map

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (X; x_0)$$

the family of maps

$$\xi_t \circ \varphi : \mathbb{S}^n \to X$$

represents a homotopy of the map φ along the path u. Therefore, for any element $\alpha \in \pi_n(X; x_0)$, the element $u^{\#}(\alpha) \in \pi_n(X; x_1)$ is the class of the map $\xi \circ \varphi$, where $\xi = \xi_1$, and φ is an arbitrary map of class α. This means that

Proposition 2.47. *the homomorphism*

$$\xi_* : \pi_n(X; x_0) \to \pi_n(X; x_1)$$

coincides with the homomorphism

$$u^\# : \pi_n(X; x_0) \to \pi_n(X; x_1)$$

Since the last homomorphism is, as we know, an isomorphism, it is thus proved that

Proposition 2.48. *for any homotopy identity map*

$$\xi : X \to X$$

and any point $x_0 \in X$, the homomorphism

$$\xi_* : \pi_n(X; x_0) \to \pi_n(X; x_1), \quad x_1 = \xi(x_0),$$

is an isomorphism.

A map

$$f : X \to Y$$

of a topological space X into a topological space Y will be called a *weak homotopy equivalence* if for any $n \geq 0$ and any point $x_0 \in X$ the homomorphism

$$f_* : \pi_n(X; x_0) \to \pi_n(Y; y_0), \quad y_0 = f(x_0),$$

is an isomorphism. From the proposition just proved it easily follows that

Proposition 2.49. *any homotopy equivalence*

$$f : X \to Y$$

is a weak homotopy equivalence.

Proof. Indeed, let

$$g : Y \to X$$

be the homotopy equivalence inverse to the equivalence f, and let x_0 be an arbitrary point in X. Setting

$$y_0 = f(x_0), \quad x_1 = g(y_0), \quad y_1 = f(x_1),$$

for any $n \geq 0$ we consider the homomorphisms

$$\begin{aligned} f_* &= f_{*,x_0} : \pi_n(X; x_0) \to \pi_n(Y; y_0), \\ f_*' &= f_{*,x_1} : \pi_n(X; x_1) \to \pi_n(Y; y_1), \\ g_* &= g_{*,y_0} : \pi_n(Y; y_0) \to \pi_n(X; x_1), \end{aligned}$$

induced by the maps f and g, as well as the homomorphsims

$$(g \circ f)_* : \pi_n(X; x_0) \to \pi_n(X; x_1),$$
$$(f^{'} \circ g)_* : \pi_n(Y; y_0) \to \pi_n(Y; y_1),$$

induced by composite maps

$$g \circ f : X \to X, \quad f^{'} \circ g : Y \to Y.$$

As we know,

$$(g \circ f)_* = g_* \circ g_*, \quad (f^{'} \circ g)_* = f^{'} \circ g_*.$$

On the other hand, since the maps $g \circ f$ and $f^{'} \circ g$ are, by hypothesis, homotopically identical, the homomorphisms $(g \circ f)_*$ and $(f^{'} \circ g)_*$ are isomorphisms. Therefore, from the equality $(g \circ f)_* = g_* \circ f*$ it follows that the map g_* is epimorphic, and from the equality $(f^{'} \circ g)_* = f^{'}_* \circ g_*$ it follows that the map g_* is monomorphic. Consequently, the map g_*, and therefore the map f_*, is an isomorphism. □

Remark 2.50. The converse is generally not true: there are weak homotopy equivalences that are not homotopy equivalences. However, as we shall see below, for sufficiently "good" spaces (namely cellular spaces), any weak homotopy equivalence is a homotopy equivalence.

It is clear that the composition of weak homotopy equivalences is also a weak homotopy equivalence. Therefore, it follows immediately from the previous statement that

Proposition 2.51. *any map that is homotopy equivalent to a weak homotopy equivalence is also a weak homotopy equivalence.*

Remark 2.52. By analogy with the relation of homotopy equivalence of spaces, one could introduce the relation of their weak homotopy equivalence. This relation is obviously reflexive and transitive, but, generally speaking, it is not symmetric. We will not consider it.

We will call a subspace A of a space X *representative* if the inclusioon map $i : A \subset X$ is a weak homotopy equivalence, i.e. if for any $n \geq 0$ and any point $x_0 \in A$ the homomorphism

$$i_* : \pi_n(A; x_0) \to \pi_n(X; x_0)$$

is an isomorphism. It follows directly from the proposition just proved that

Proposition 2.53. *the map $f : X \to Y$ is a weak homotopy equivalence if and only if the space X is a representative subspace of the cylinder Z_f of the map f.*

In connection with this proposition, it is useful to note that

Proposition 2.54. *if a subspace A of X is representative, then for any $n \geq 0$ each map*

$$f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, A)$$

is homotopic relative to $\mathbb{S}^{n-1}$ to some map $g : \mathbb{E}^n \to A$ (i. e, more precisely, to the map $i \circ g$, where $i : A \subset X$).

Proof. Indeed, assuming that $n > 0$ (for $n = 0$ the reasoning is only simplified), we assign to the map f the element $\alpha^{'}$ of the group $\pi_{n-1}(A; x_0$, where $x_0 = f(\boldsymbol{u}_0)$, defined by the map

$$f|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to A.$$

Since the map $i \circ f|_{\mathbb{S}^{n-1}}$ is a restriction of the map $f : \mathbb{E}^n \to X$, then, as proved above, $i_*(\alpha^{'}) = 0$, and therefore $\alpha^{'} = 0$ (since the homomorphism i_* is, by assumption, an isomorphism). Therefore, there exists a map

$$h : \mathbb{E}^n \to A,$$

such that

$$h|_{\mathbb{S}^{n-1}} = f|_{\mathbb{S}^{n-1}}.$$

Let

$$\beta = \delta(f, i \circ h)$$

be an element of the group $\pi_n(X; x_0)$ that distinguishes the maps f and $i \circ h$. Since the subspace A is representative, in the group $\pi_n(A; x_0)$ there exists an element α such that $i*(\alpha) = \beta$. Let

$$g : \mathbb{E}^n \to A$$

be a map of the ball $\mathbb{E}^n$ into the space A that coincides on $\mathbb{S}^{n-1}$ with the map h such that

$$\delta(g, h) = \alpha.$$

Then, as is easy to see,

$$\delta(i \circ g, i \circ h) = i_*(\alpha),$$

i.e,

$$\delta(i \circ g, i \circ h) = \delta(f, i \circ h).$$

Therefore, the maps f and $i \circ g$ are homotopic relative to $\mathbb{S}^{n-1}$. □

Remark 2.55. The converse is also true: if for each $n \geq 0$ any map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, A)$ is homotopic relative to $\mathbb{S}^{n-1}$ to some map $\mathbb{E}^n \to A$ then the subspace A is representative. We will not need this fact, and we will leave its proof to the reader as a simple exercise.

2.5 Homotopy limits

Let

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \tag{2.56}$$

be an increasing sequence of subspaces of a topological space X, the union of which is the entire space X. Consider the inclusion maps

$$i_m^n : X_n \longrightarrow X_m, \quad i^n : X_n \longrightarrow X, \quad 0 \le n \le m \le \infty,$$

and for any point $x \in X$ and any number $n \ge n_x$, where n_x is the smallest n for which $x \in X_n$, the homomorphisms induced by these maps

$$\begin{aligned} (i_m^n)_* : \pi_k(X_n; x) \longrightarrow \pi_k(X_m; x), \\ (i^n)_* : \pi_k(X_n; x) \longrightarrow \pi_k(X; x), \end{aligned} \quad k \ge 0,$$

of homotopy groups.

We will say that the space X is a *weak homotopy limit* of subspaces (2.56) if

1) for any number $k \ge 0$, any point $x \in X$, and any element $\alpha \in \pi_k(X; x)$, there exist a number $n \ge n_x$ and an element $\alpha_n \in \pi_k(X_n; x)$ such that

$$\alpha = (i^n)_*(\alpha_n);$$

2) for any number $k \ge 0$, any number $n \ge 0$, any point $x \in X$, and any elements $\alpha, \alpha' \in \pi_k(X; x)$, with the property that

$$(i^n)_*(\alpha_n) = (i^n)_*(\alpha'_n),$$

there exists a number $m \ge n$ such that

$$(i_m^n)_*(\alpha_n) = (i_m^n)_*(\alpha'_n),$$

It is clear that if the number $n \ge n_x$ satisfies condition 1), then any number $n' \ge n$ also satisfies this condition, i.e. there exists an element $\alpha_{n'} \in \pi_k(X_{n'}; x)$ such that $(i^{n'})_*(\alpha_{n'}) = \alpha$ (at least the element $\alpha_{n'} = (i_{n'}^n)_*(\alpha n)$ has this property). Similarly, any number $m' \ge m$ satisfies condition 2) together with the number m.

Remark 2.57. The reader familiar with the concept of a spectrum of groups will immediately discover that for any number $k \ge 0$ and any point $x \in X$ the groups $\pi k(X_n; x)$, $n \ge m$, and the homomorphisms $(i_m^n)_*$, $n_x \le n \le m$ constitute a spectrum and that conditions 1) and 2) are equivalent to the fact that the homomorphism of the limit group of this spectrum into the group $\pi_k(X; x)$, induced by the homomorphisms $(i^n)_*$, is an isomorphism.

One can specify simple set-theoretic conditions under which the space X is a weak homotopy limit of the subspaces (2.56). For example,

Proposition 2.58. *if for any compact set $C \subset X$ there exists a number $n \geq 0$ such that $C \subset X_n$, then the space X is a weak homotopy limit of the subspaces X_n.*

Proof. Indeed, let $k \geq 0$, $x \in X$, and let α be an arbitrary element of the group $\pi_k(X;x)$. Consider an arbitrary map

$$f : (\mathbb{S}^k, \boldsymbol{u}_0) \to (X;x)$$

of class α. Since the sphere $\mathbb{S}^k$ is compact, the set $f(\mathbb{S}^k)$ is also compact, and, consequently, there exists a number $n \geq 0$ such that $f(\mathbb{S}^k) \subset X_n$. Therefore, we can consider the map f as a map $f : (\mathbb{S}^k, \boldsymbol{u}_0) \to (X_n;x)$. It is clear that the element $\alpha_n \in \pi_k(X_n;x)$ defined by this map has the property that $(ii^n)_*(\alpha_n) = \alpha$.

Condition 2 is verified similarly (only instead of the compactness of the sphere $\mathbb{S}^k$, we have to use the compactness of the product $\mathbb{S}^k \times I$). □

In the case of an arbitrary sequence (2.56), the space X is not, generally speaking, a weak homotopy limit of the subspaces X_n. However, it can be argued that

Proposition 2.59. *For any increasing sequence*

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \tag{2.60}$$

of subspaces of X, there exists a space X^Σ and a sequence

$$X_0^\Sigma \subset X_1^\Sigma \subset \cdots \subset X_n^\Sigma \subset \cdots$$

of subspaces of X^Σ such that

1) for any $n \geq 0$, the space X_n^Σ is homotopy equivalent to the space X_n;

2) the space X^Σ is a weak homotopy limit of the subspaces X_n^Σ.

Proof. Indeed, let X^Σ be the subspace of the product $X \times \mathbb{R}$ consisting of all points (x,t), $x \in X$, $t \in \mathbb{R}$, for which $t \geq n+1$ for $x \notin X_n$,

$$X^\Sigma = \cup_{k=0}^\infty (X_k \times [k, \infty]) = \cup_{k=0}^\infty (X_k \times [k, k+1])$$

and let X_n^Σ be the subspace of X^Σ consisting of all points $(x,t) \in X^\Sigma$ for which $t \leq n$, i.e.,

$$X_n^\Sigma = (X_0 \times [0,1]) \cup (X_1 \times [1,2]) \cup \cdots \cup (X_{n-1} \times [n-1, n]) \cup (X_n \times n).$$

Since the coordinate t of a point (x,t) of an arbitrary compact set $C \subset x^\Sigma$, being a continuous function on C, is bounded, then $C \subset X_n^\Sigma$ for some $n \geq 0$. Consequently, according to the statement proved above, the space X^Σ is a weak homotopy limit of the subspaces X_n^Σ.

Let us further consider the natural map

$$p^\Sigma : X^\Sigma \to X,$$

defined by the formula

$$p^{\Sigma}(x,t) = x, \quad (x,t) \in X^{\Sigma}.$$

It is clear that for any $n \geq 00$ the inclusion

$$p^{\Sigma}(X_n^{\Sigma}) \subset X_n,$$

holds and therefore the map p^{Σ} defines a map

$$p_n^{\Sigma} : X_n^{\Sigma} \longrightarrow X_n.$$

(This map is defined by the same formula as the map p^{Σ}, with the only difference that now $(x,t) \in X_n^{\Sigma}$.) Setting for any point $x \in X$

$$q_n^{\Sigma}(x) = (x,n),$$

we obviously obtain a continuous map

$$q_n^{\Sigma} : X_n \longrightarrow X_n^{\Sigma}.$$

for which $p_n^{\Sigma} \circ q_n^{\Sigma} = 1_{X_n}$. On the other hand, setting

$$\xi_{\tau}(x,t) = (x, t + \tau(n-t)), \quad (x,t) \in X_n^{\Sigma}, 0 \leq \tau \leq 1,$$

we obtain a deformation

$$\xi_{\tau} : X_n^{\Sigma} \longrightarrow X_n^{\Sigma}, \quad 0 \leq \tau \leq 1,$$

of the space X_n^{Σ} such that $\xi_1 = q^{\Sigma} \circ p_n^{\Sigma}$. Therefore, the maps p_n^{Σ} and q_n^{Σ} are mutually inverse homotopy equivalences.

Thus, the proposition formulated above is completely proven. □

Generally speaking, it is impossible to claim that the map p^{Σ} is a homotopy equivalence (even a weak one). Specifically, as we will now show,

Proposition 2.61. *the map*

$$p^{\Sigma} : X^{\Sigma} \to X$$

is a weak homotopy equivalence if and only if the space X is a weak homotopy limit of the subspaces X_n.

We will prove an even more general proposition, which applies to the situation where we are given an arbitrary space X, which is a weak homotopy limit of subspaces

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

an arbitrary space Y, which is the union of an increasing sequence of subspaces

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n \subset \cdots$$

and a continuous map

$$f : X \longrightarrow Y$$

such that

$$f(X_n) \subset Y_n$$

for any $n \geq 0$. Let

$$f_n : X_n \longrightarrow Y_n, \quad n \geq 0,$$

be the map defined by the map f. This map is related to the map f by the formula

$$f \circ i^n = j^n \circ f_n,$$

where i^n is, as above, the inclusion map $X_n \subset X$, and j^n is the inclusion map $Y_n \subset Y$. Moreover, for $n \leq m$, the maps f_n and f_m are related by the formula

$$f_m \circ i^n = j^n_m \circ f_n,$$

where $i^n_m : X_n \subset X_m$ and $j^n_m : Y_n \subset Y_m$.

We will show that

Proposition 2.62. *if for any $n \geq 0$ the map f_n is a weak homotopy equivalence, then the following two statements are equivalent:*

1) the space Y is a weak homotopy limit of subspaces Y_n;

2) the map f is a weak homotopy equivalence.

Proof. Indeed, let Statement 1) be true. Consider an arbitrary number $k \geq 0$ and an arbitrary point $xinX$. Since the space Y is a weak homotopy limit of subspaces Y_n, then for any element $\beta \in \pi_k(Y; y)$, where $y = f(x)$, there exist a number $n \geq n_y n$ and an element $\beta_n \in \pi_k(Y_n; y)$ such that $\beta = (j^n)_*(\beta_n)$. In this case, without loss of generality, we can assume that $n \geq n_x$, i.e., that the group $\pi_k(X_n; x)$ is meaningful. Since the corresponding homomorphism

$$(f_n)_* : \pi_k(X_n; x) \longrightarrow \pi_k(Y_n; y)$$

is, by assumption, an isomorphism, there exists an element $\alpha_n \in \pi_k(X_n; x)$ such that $\beta_n = (f_n)_*(\alpha_n)$. Consequently, assuming $\alpha = (i^n)_*(\alpha_n)$, we get that

$$f_*(\alpha) = (f \circ i^n)_*(\alpha_n) = (j^n \circ f_n)_*\alpha_n) = (i^n)_*(\beta_n) = \beta.$$

Thus, for any number $k \geq 0$ and any point $x \in X$, the homomorphism

$$f_* : \pi_k(X; x) \longrightarrow \pi_k(Y; f(x)) \tag{2.63}$$

is an epimorphism.

Now let α and α' be elements of the group $\pi_k(X; x)$, such that $f_*(\alpha) = f_*(\alpha')$. Since the space X is a weak homotopy limit of subspaces X_n, there exist a number

$n \geq n_x$ and elements $\alpha_n, \alpha_n^{'} \in \pi_k(X;x)$ such that $\alpha = (i^n)_*(\alpha_n)$ and $\alpha^{'} = (i^n)_*(\alpha_n^{'})$. Let $\beta_n = (f_n)_*(\alpha_n)$ and $\beta_n^{'} = (f_n)_*(\alpha_n^{'})$. Since

$$\begin{aligned}(j^n)_*(\beta_n) &= (j^n \circ f_n)_*(\alpha_n) = (f \circ i^n)_*(\alpha_n)) = f_*(\alpha)\\ &= f_*(\alpha^{'}) = (f \circ i^n)_*(\alpha_n^{'}) = (j^n \circ fn)_*(\alpha_n^{'}) = (j^n)_*(\beta_n^{'}),\end{aligned}$$

then there exists a number $m \geq n$ such that

$$(j_m^n)_*(\beta_n) = (j_m^n)_*(\beta_n^{'}).$$

Therefore, putting $\alpha_m = (i_m^n)_*(\alpha_n)$ and $\alpha_m^{'} = (i_m^n)_*(\alpha_n^{'})$, we get that

$$\begin{aligned}(f_m)_*(\alpha_m) &= (f_m \circ i_m^n)_*(\alpha_n) = (j_m^n \circ f_n)_*(\alpha_n)) = (j_m^n)_*(\beta_n)\\ &= (j_m^n)_*(\beta_n^{'}) = (j_m^n \circ f_n)_*(\alpha_n^{'}) = (f_m \circ i_m^n)_*(\alpha_n^{'}) = (f_m)_*(\alpha_n^{'}).\end{aligned}$$

Since the map $(f_m)_*$ is, by assumption, isomorphic, it follows that $\alpha_m = \alpha_m^{'}$, and therefore

$$\alpha = (i^n)_*(\alpha_n) = (i^m)_*(\alpha_m) = (i^m)_*(\alpha_m^{'}) = (i^m \circ i_m^n)*(\alpha_n^{'}) = \alpha^{'}.$$

Thus, homomorphism (2.63) is also a monomorphism. Therefore, the implication 1) $\Rightarrow$ 2) is completely proved.

Now let assertion 2) be true. We will prove that condition 1) of the definition of a weak homotopy limit is satisfied for the subspaces Y_n, i.e., that for any number $k \geq 0$, any point $y \in Y$, and any element $\beta \in \pi_k(Y;y)$, there exists a number $n \geq n_y$ such that $\beta = (j^n)_*(\beta)$. Since the map f induces, by assumption, a one-to-one correspondence between the path connected components of the space X and the path connected components of the space Y, we can, without loss of generality, assume that $y = f(x)$, where x is some point of the space X. Consider the corresponding homomorphism

$$f_* : \pi_k(X;x) \longrightarrow \pi_k(Y;y).$$

Since this homomorphism is, by assumption, an isomorphism, there exists an element $\alpha \in \pi_k(X;x)$ such that $\beta = f_*(\alpha)$. Since the space X is a weak homotopy limit of subspaces X_n, there exists a number $n \geq n_x \geq n_y$ and an element $\alpha_n \in \pi_k(X;x)$ such that $\alpha = (i^n)_*(\alpha_n)$. Therefore, setting $\beta_n = (f_n)_*(\alpha_n)$, we obtain that:

$$\beta = (f \circ i^n)_*(\alpha_n) = (j^n \circ f_n)_*(\alpha_n) = (j^n)_*(\beta_n).$$

Condition 2) is checked in a completely similar way.

Thus, the above statement is fully proven. □

We will say that the space X is a *homotopy limit* of the sequence of subspaces

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \tag{2.64}$$

if the map

$$p^\Sigma : X^\Sigma \to X$$

is a homotopy equivalence. By the proposition just proved,

Proposition 2.65. *any space X that is a homotopy limit of subspaces X_n is also their weak homotopy limit.*

We will begin our study of homotopy limits by proving that

Proposition 2.66. *if*

1) every point $x \in X$ is an interior point of some subspace X_n, i.e., the family $\{\operatorname{int} X_n; n \geq 0\}$ is an open covering of X;

2) the covering $\{\operatorname{int} X_n; n \geq 0\}$ can be refined into a locally finite open covering $\Gamma = \{U_\alpha; \alpha \in A\}$, for which there exists a subordinate partition of unity $\{f_\alpha; \alpha \in A\}$,

then the space X is a homotopy limit of the subspaces X_n.

Proof. Indeed, denoting for each element $\alpha \in A$ by the symbol $b(\alpha)$ the smallest n for which $U_\alpha \in \operatorname{int} X_n$, and setting

$$f(x) = \sum_{\alpha \in A} n(\alpha) f_\alpha(x), \quad x \in X,$$

we define on the space X a certain numerical function $f(x)$. This function, as is easy to see, is continuous (since the covering Γ, by assumption, is locally finite). Furthermore, it has the property that $f(x) \geq n+1$ for $x \notin X_n$, since if $x \notin X_n$, then $f(x) = 0$ for all $\alpha \in A$ for which $n(\alpha) \leq n$, and therefore

$$f(x) \geq (n+1) \sum_{\alpha \in A} f_\alpha(x) = n+1.$$

Consequently, the formula

$$q(x) = (x, f(x)), \quad x \in X,$$

defines a continuous map

$$q : X \to X^\Sigma.$$

Clearly, $p \circ q = 1_X$. Moreover, the formula

$$\xi_\tau(x,t) = (x, (1-\tau)t + \tau f(x)), \quad (x,t) \in X^\Sigma, 0 \leq \tau \leq 1,$$

defines, as is easy to see, a deformation $\xi_\tau : X^\Sigma \to X^\Sigma$, for which $\xi_1 = q \circ p$. Thus, the map p is, as stated, a homotopy equivalence.

For the existence of a partition of unity $\{f_\alpha; \alpha \in A\}$ subordinate to a locally finite covering Γ, it is sufficient, as we know, that the space X be normal. On the other hand, for a covering Γ to exist, it is sufficient, for example, to require that the space X be paracompact or (assuming that X is normal) that for any $n \geq 0$ the inclusion $\overline{\operatorname{int} X_n} \subset \operatorname{int} X_{n+1}$ holds. (Indeed, in the latter case, the locally finite covering inscribed in the covering $\{\operatorname{int} X_n; x \geq 0\}$ is the covering $\{\operatorname{int} X_1, \operatorname{int} X_{n+1} \setminus \overline{V}_{n-1}; n \geq 0\}$, where V_{n-1} is a neighbourhood of the set $\overline{\operatorname{int} X_{n-1}}$ such that $\overline{V}_{n-1} \subset \operatorname{int} X_n$). Thus,

Proposition 2.67. *if condition 1) of the previous proposition is satisfied, and the space X is normal and, in addition, either this space is paracompact or*

$$\overline{\operatorname{int} X_n} \subset \operatorname{int} X_{n+1}$$

for any $n \geq 0$, then the space X is a homotopy limit of the subspaces X_n.

These conditions are by no means necessary.

Let us now consider (for an arbitrary space X and arbitrary subspaces X_n), the subspaces

$$X_0^\Sigma \subset X_1^\Sigma \subset \cdots \subset X_n^\Sigma \subset \cdots$$

of the space X^Σ. It is clear that $\operatorname{int} X_n^\Sigma$ is the set of all points $(x,t) \in X^\Sigma$ for which $t < n$. Therefore,

$$X^\Sigma = \sum_{n=0}^{\infty} \operatorname{int} X_n^\Sigma,$$

i.e., for the subspaces X_n^Σ, condition 1) of the proposition proved above is satisfied. Moreover, it is easy to see that the sets

$$\operatorname{int} X_1^\Sigma, \quad \operatorname{int} X_{n+1}^\Sigma \setminus \operatorname{int} X_{n-1}^\Sigma \qquad n \geq 1, \tag{2.68}$$

form a locally finite covering refining the covering $\{\operatorname{int} X_n^\Sigma; n \geq 0\}$, and the functions f_{n+1}, $n \geq 0$, defined by the formula

$$f_{n+1}(x,t) = \begin{cases} 0, & \text{if} \quad t \leq n - \frac{3}{4} \text{ or } t \geq n + \frac{3}{4}, \\ \frac{4t-4n+3}{2}, & \text{if} \quad n - \frac{3}{4} \leq t \leq n - \frac{1}{4}, \\ 1, & \text{if} \quad n - \frac{1}{4} \leq t \leq n + \frac{1}{4}, \\ \frac{3+4n-4t}{2}, & \text{if} \quad n - +\frac{1}{4} \leq t \leq n + \frac{3}{4}, \end{cases}$$

constitute a decomposition of the unit subordinate to covering (2.68). Hence,

Proposition 2.69. *for any space X and any of its subspaces*

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

the space X^Σ is the homotopy limit of the subspaces

$$X_n^\Sigma = (X_0 \times [0,1]) \cup \cdots \cup (X_{n-1} \times [n-1,n]) \cup (X_n \times n).$$

This proposition is a strengthening of the property of subspaces X_n^Σ proved above. Using it, we can extend the basic properties of weak homotopy limits to the case of homotopy limits.

Let, for example, X be an arbitrary space that is a homotopy limit of the subspaces

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

Y be an arbitrary space in which an increasing sequence of subspaces

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n \subset \cdots$$

is defined, and f be a continuous map

$$f : X \to Y$$

such that

$$f(X_n) \subset Y_n$$

for any $n \geq 0$. Then

Proposition 2.70. *if for each $n \geq 0$ the map f defined by*

$$f_n : X_n \to Y_n$$

is a homotopy equivalence, then the following two conditions are equivalent:

1) the space Y is a homotopy limit of the subspaces Y_n;

2) the map f is a homotopy equivalence.

Proof. Indeed, let, as above,

$$X^{\Sigma} = \cup_{k=0}^{\infty}(X_k \times [k, k+1])$$

and let similarly

$$Y^{\Sigma} = \cup_{k=0}^{\infty}(Y_k \times [k, k+1]).$$

Since $f(X_k) \subset Y_k$, the formula

$$f^{\Sigma}(x,t) = (f(x), t), \quad (x,t) \in X^{\Sigma},$$

defines a continuous map

$$f^{\Sigma} : X^{\Sigma} \to Y^{\Sigma}$$

The map f is obviously related to the map f^{Σ} by the formula

$$f \circ p^{\Sigma} = \overline{p}^{\Sigma} \circ f^{\Sigma},$$

where

$$p^{\Sigma} : X^{\Sigma} \to X, \quad \overline{p}^{\Sigma} : Y^{\Sigma} \to Y$$

are natural maps. Therefore, if statement 1) is true, then the map f^{Σ} is homotopically equivalent to the map f, and if statement 2) is true, then the composition of the maps f^{Σ} and $\overline{p}^{\Sigma}$ is a homotopy equivalence. Therefore, to prove the equivalence of statements 1) and 2), it suffices to prove that

Proposition 2.71. *the map f^{Σ} is a homotopy equivalence.*

Let us first consider the case when $X = Y$ and $X_n = Y_n$ for all $n \geq 0$. Let, in addition, each map

$$f_n : X_n \to X_n, \quad n \geq 0,$$

be homotopically identical and let

$$\xi_{n,t} : X_n \to X_n$$

be a deformation of the space X_n that connects the identity map 1_{X_n} with the map f_n. For any point $x \in X_n$, $n \geq 0$, and any $t, \tau \in I$, we set

$$h_\tau(x, n+t) = \begin{cases} (f(x), n + t(2\tau + 1)), & \text{if } \ 0 \leq \tau \leq 1/2,\ 0 \leq t \leq 1/2, \\ (f(x), n + 2(1-t)\tau + t), & \text{if } \ 0 \leq \tau \leq 1/2,\ 1/2 \leq t \leq 1, \\ (\xi_{n,2-2\tau}(x), n + 2t), & \text{if } \ 1/2 \leq \tau \leq 1,\ 0 \leq t \leq 1/2, \\ (\xi_{n,1-(3-4t)(2\tau-1)}(x), n+1), & \text{if } \ 1/2 \leq \tau \leq 1,\ 1/2 \leq t \leq 3/4, \\ (\xi_{n+1,1-(4t-3)(2\tau-1)}(x), n+1), & \text{if } \ 1/2 \leq \tau \leq 1,\ 3/4 \leq t \leq 1. \end{cases}$$

It is easy to see (Fig. 2.4) that we thereby obtain a certain homotopy

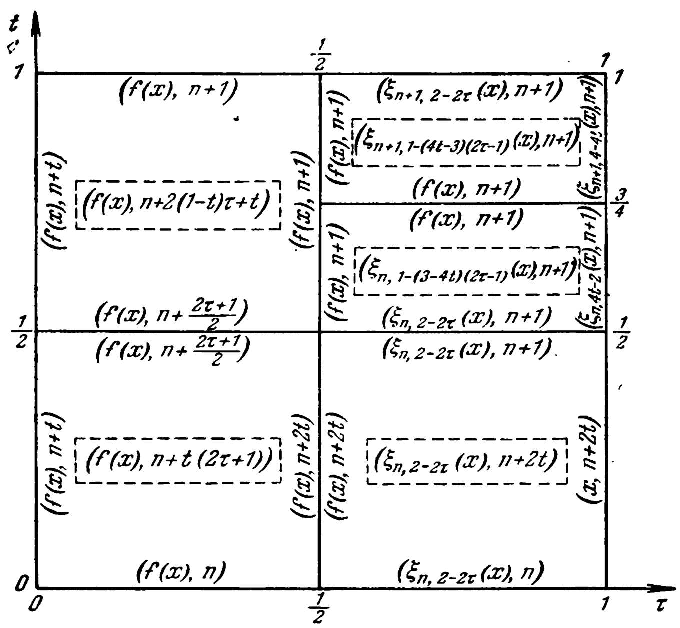


Figure 2.4:

$$h_\tau : X^\Sigma \to X^\Sigma$$

connecting the map f^Σ with the map h defined by the formula

$$h(x, n+t) = \begin{cases} (x, n + 2t), & \text{if } \ 0 \leq t \leq 1/2, \\ (\xi_{n,4t-2}(x), n+1), & \text{if } \ 1/2 \leq t \leq 3/4, \\ (\xi_{n+1,4-4t}(x), n+1), & \text{if } \ 3/4 \leq t \leq 1. \end{cases}$$

Therefore, it is sufficient for us to prove that the map h is a homotopy equivalence. With this in mind, we note that by the equality

$$h\left(x, n + \frac{1}{2}\right) = h(x, n+1) = h(x, n+1)$$

the formula

$$g(x, n+t) = \begin{cases} (x, n+2t), & \text{if} \quad 0 \le t \le 1/2, \\ h\left(x, n + \frac{3-2t}{2}\right), & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

uniquely determines some continuous map

$$g : X^\Sigma \longrightarrow X^\Sigma.$$

Let

$$\eta_\tau(x, n+t) = \begin{cases} (x, n+(1+6\tau)t), & \text{if} \quad 0 \le \tau \le 1/2,\ 0 \le t \le 1/4, \\ (x, n+2(1-t)\tau + t), & \text{if} \quad 0 \le \tau \le 1/2,\ 1/4 \le t \le 1, \\ (h \circ g)(x, n+t), & \text{if} \quad 1/2 \le \tau \le 1,\ 0 \le t \le \tau/2, \\ h(x, n+\tau), & \text{if} \quad 1/2 \le \tau \le 1,\ \tau/2 \le t \le \frac{3-2\tau}{2}, \\ (h \circ g)(x, n+t), & \text{if} \quad 1/2 \le \tau \le 1,\ \frac{3-2\tau}{2} \le t \le 1. \end{cases}$$

Since

$$(h \circ g)\left(x, n + \frac{\tau}{2}\right) = h(x, n+\tau), \quad 0 \le \tau \le 1,$$

we have

$$\begin{aligned} (h \circ g)\left(x, n + \frac{\tau}{2}\right) &= (h \circ h)\left(x, n + \frac{3 - 2\frac{3-2\tau}{2}}{2}\right) \\ &= (h \circ h)(n+\tau) = \begin{cases} h(\xi_{n,\tau-2}(x), n+1) & = (\xi_{n,\tau-2}(x), n+1) \\ h(\xi_{n+1,4-4\tau}(x), n+1) & = (\xi_{n+1,4-4\tau}(x), n+1) \end{cases} \\ &= h(x, n+\tau), \quad 1/2 \le \tau \le 1, \end{aligned}$$

and

$$(h \circ g)(x, n+t) = (x, n+4t), \quad 0 \le t \le 1/4,$$

this formula defines (Fig. 2.5) a certain homotopy,

$$\eta_\tau : X^\Sigma \to X^\Sigma, \quad 0 \le \tau \le 1,$$

connecting the identity map 1_{X^Σ} with the map

$$h \circ g : X^\Sigma \longrightarrow X^\Sigma.$$

Similarly, in view of the equalities

$$(g \circ h)(x, n + \frac{\tau}{2}) = g(x, n+\tau), \quad 0 \le \tau \le 1,$$

we have

$$\begin{aligned} (g \circ h)\left(x, n + \frac{3-2\tau}{2}\right) &= h\left(x, n + \frac{3-2\tau}{2}\right) \\ &= g(n+\tau), \quad 1/2 \le \tau \le 1, \end{aligned}$$

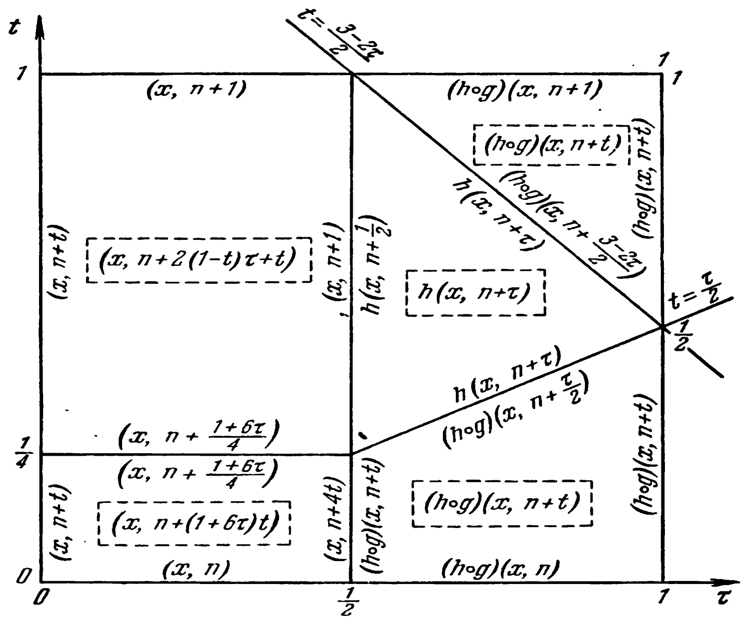


Figure 2.5:

and

$$(g \circ h)(x, n+t) = (x, n+4t), \quad 0 \le t \le 1/4,$$

the formula

$$\overline{\eta}_\tau(x, n+t) = \begin{cases} (x, n+(1+6\tau)t), & \text{if} \quad 0 \le \tau \le 1/2,\ 0 \le t \le 1/4, \\ (x, n+(1-t)\tau + t), & \text{if} \quad 0 \le \tau \le 1/2,\ 1/4 \le t \le 1, \\ (g \circ h)(x, n+t), & \text{if} \quad 1/2 \le \tau \le 1,\ 0 \le t \le \tau/2, \\ g(x, n+\tau), & \text{if} \quad 1/2 \le \tau \le 1,\ \tau/2 \le t \le \frac{3-2\tau}{2}, \\ (g \circ h)(x, n+t), & \text{if} \quad 1/2 \le \tau \le 1,\ \frac{3-2\tau}{2} \le t \le 1. \end{cases}$$

defines some homotopy

$$\overline{\eta}_\tau : X^\Sigma \to X^\Sigma, \quad 0 \le \tau \le 1,$$

connecting the identity map 1_{X^Σ} with the map

$$g \circ h : X^\Sigma \to X^\Sigma.$$

Thus, in the particular case under consideration, the map f^Σ is indeed a homotopy equivalence.

Let us now consider the general case.

Let

$$g_n : Y_n \to X_n, \quad n \ge 0,$$

be homotopy equivalences inverse to the homotopy equivalences of f_n, and let

$$i_n : X_n \subset X_{n+1}, \quad j_n : Y_n \subset Y_{n+1}$$

be inclusion maps. Since $f_{n+1} \circ i_n = j_n \circ f_n$, then

$$i_n \circ g_n \sim g_{n+1} \circ f_{n+1} \circ i_n \circ g_n = g_{n+1} \circ j_n \circ f_n \circ g_n \sim g_{n+1} \circ j_n.$$

Let

$$h_{n,\tau} : Y_n \longrightarrow X_{n+1}, \quad 0 \leq \tau \leq 1,$$

be an arbitrary homotopy connecting the map $i_n \circ g_n$ with the map $g_{n+1} \circ j_n$. It is easy to see that the formula

$$h(y, n+t) = \begin{cases} (g_n(y), n+2t), & \text{if} \quad 0 \leq t \leq 1/2, \\ (h_{n,2t-1}(y), n+1), & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

where $y \in Y_n$, $n \geq t$ and $0 \leq t \leq 1$, uniquely determines some continuous map

$$h : Y^\Sigma \longrightarrow X^\Sigma.$$

Consider the map

$$h \circ f^\Sigma : X^\Sigma \longrightarrow X^\Sigma.$$

It is clear that

$$(h \circ f^\Sigma)(X_n^\Sigma) \subset X_n^\Sigma, \quad n \geq 0,$$

and therefore for any $n \geq 0$ the map $f \circ f^\Sigma$ defines a certain map

$$(h \circ f^\Sigma)_n : X_n^\Sigma \longrightarrow X_n^\Sigma.$$

Moreover, for any point $x \in X$ the equality

$$(h \circ f^\Sigma)_n(x, n) = ((g_n \circ f_n)(x), n).$$

holds. In other words,

$$(h \circ f^\Sigma)_n \circ q_n^\Sigma = q_n^\Sigma \circ (g_n \circ f_n).$$

where q_n^Σ is the homotopy equivalence $q_n^\Sigma : X_n \longrightarrow X_n^\Sigma$ constructed above, which is inverse to the homotopy equivalence $p_n^\Sigma : X_n^\Sigma \longrightarrow X_n$. Therefore,

$$(h \circ f^\Sigma)_n \sim (h \circ f^\Sigma)_n \circ (q_n^\Sigma \circ p_n^\Sigma) = q_n^\Sigma \circ ((g_n \circ f_n) \circ p_n^\Sigma \sim q_n^\Sigma \sim q_n^\Sigma \circ p_n^\Sigma \sim 1_{X_n^\Sigma}.$$

Thus, the map $h \circ f^\Sigma$ has (with respect to the subspaces X_n^Σ) the property that we required of the map f in the first part of the proof. Therefore, by what has already been proved, the map $h \circ f^\Sigma$ is a homotopy equivalence. Consequently, the map f^Σ is homotopy injective.

Similarly, the map

$$f^\Sigma \circ h : Y^\Sigma \longrightarrow Y^\Sigma$$

has the property that

$$(f^\Sigma \circ h)(Y_n^\Sigma) \subset Y_n^\Sigma, \quad n \geq 0,$$

and therefore defines maps

$$(f^\Sigma \circ h)_n : (Y_n^\Sigma) \longrightarrow Y_n^\Sigma, \quad n \geq 0,$$

satisfying the relation

$$(f^\Sigma \circ h)_n \circ \overline{q}_n^\Sigma = \overline{q}_n^\Sigma \circ (f_n \circ g_n).$$

where $\overline{q}_n^\Sigma$ is the map q_n^Σ constructed for the spaces Y_n. From the last relation it follows that the maps $(f^\Sigma \circ h)_n$ are homotopy identities. Therefore, the map $f^\Sigma \circ h$ is also a homotopy equivalence, and therefore the map f^Σ is homotopy sujective. □

Being both homotopically injective and homotopically surjective, the map f^Σ is a homotopy equivalence. □

Chapter 3

Cellular decompositions

General cellular decompositions seem to provide us with the most natural stock of objects for constructing a homotopy theory. They have all the basic geometric properties of classical simplicial decompositions (triangulations), and at the same time their theory compares favourably with the theory of simplicial decompositions in its generality, structure, and internal integrity. However, until now no one has apparently attempted to give a coherent and independent exposition of the basic properties of cellular decompositions. This chapter is the first attempt in this direction.

The definition and simplest properties of cellular decompositions are presented in §3.1 and §3.2.

In §3.3 it is proved that any cellularly decomposed space is paracompact (and, therefore, normal). Here it is also proved that the topological product of two cellular decompositions is a cellular decomposition if at least one of the given decompositions is locally finite or if both of them are locally countable.

In §3.4, after a number of simple remarks on continuous maps of cellularly decomposed spaces, it is proved that any cellular pair satisfies the axiom of homotopy extension, and the main theorem on cellular maps is formulated (along with some corollaries). §3.5 is devoted to the proof of this last theorem.

In the final §3.6, a remarkable theorem of Whitehead is proved that any weak homotopy equivalence connecting cellular spaces is a homotopy equivalence. In this section, several simple remarks are placed on quasi-polyhedra (i.e., spaces homotopy equivalent to cellularly decomposed spaces).

3.1 Cellular pre-decompositions

Let X be an arbitrary Hausdorff space. We call a subset e of X an *(open) cell* if there exists a continuous map

$$\chi : \mathbb{E}^n \to X \tag{3.1}$$

of the unit Euclidean ball $\mathbb{E}^n$ into X that homeomorphically maps the open ball $\dot{\mathbb{E}}^n$ onto the set e and has the property that $\chi(\mathbb{S}^{n-1}) \cap e = \varnothing$ where $\mathbb{S}^{n-1} = \mathbb{E}^n \setminus \dot{\mathbb{E}}^n$. We will call the dimension n of the ball $\mathbb{E}^n$ the *dimension* $\dim e$ of the cell e. Clearly, it is uniquely determined by the cell e (since the balls $\mathbb{E}^n$ are not homeomorphic for different n). When it is necessary to specifically indicate the dimension n of the cell e, we will denote this cell by the symbol e^n.

The image $\chi(\mathbb{E}^n)$ of the ball $\mathbb{E}^n$ under map (3.1) is compact and therefore a closed subset of the Hausdorff space X. Consequently, the closure $\overline{e} = \overline{\chi(\mathbb{E}^n)}$ of the cell e is contained in the set $\chi(\mathbb{E}^n)$ and therefore coincides with it (since the continuity of the map χ implies the inclusion $\chi(\mathbb{E}^n) \subset \overline{\chi(\dot{E}^n)}$ (see the properties of continuous maps in §1.5). Thus,

Proposition 3.2. *the image of the ball $\mathbb{E}^n$ under the map (3.1) is the closure $\overline{e}$ of the cell e.*

The cells of the space X need not be open in X. For example, for $n = 0$, the ball $\mathbb{E}^0 = \dot{E}^n$ is a one-point space, so that any point of the space X is its zero-dimensional cell.

For $n > 0$, the points of the ball $\mathbb{E}^n$ have the form $v\boldsymbol{u}$, where $0 \le v \le 1$, and $\boldsymbol{u}$ is the unit vector, i.e., a point of the unit sphere $\mathbb{S}^{n-1}$. Thus, any point of the set $\overline{e}$ (after the map (3.1) is chosen) has the form $\chi(v\boldsymbol{u})$, where $0 \le v \le 1$, and $|\boldsymbol{u}| = 1$. Moreover, the number v is uniquely determined by a given point. The vector $\boldsymbol{u}$ is also uniquely determined if only $0 < v < 1$.

We will call map (3.1) *characteristic* for the cell e. Clearly, every map of the form $\chi \circ \alpha$ is also characteristic, where $\alpha : \mathbb{E}^n \to \mathbb{E}^n$ is an arbitrary homeomorphism of the ball $\mathbb{E}^n$ onto itself. However, there may exist characteristic maps $\mathbb{E}^n \to X$ for the cell e that are different from maps of the form $\chi \circ \alpha$.

Thus, for example, for a disk $\mathbb{E}^2$ in a plane, along with all possible homeomorphisms $\mathbb{E}^2 \to \mathbb{E}^2$, its characteristic maps are identification maps, under which some arc of the boundary circle $\mathbb{S}^1$ is contracted to a point.

Sometimes it will be convenient for us to imagine the set $\overline{e}$ as a continuous image not of the unit ball $\mathbb{E}^n$, but of some space $\mathbb{E}^n_e$ homeomorphic to the ball $\mathbb{E}^n$ (for example, the unit cube or the product $\mathbb{E}^p \times \mathbb{E}^q$, where $p+q = n$). In this case, the maps $\mathbb{E}^n_e \to X$, which are the composition of some homeomorphism $\mathbb{E}^n_e \to \mathbb{E}^n$ and an arbitrary characteristic map $\mathbb{E}^n \to X$, we will also call *characteristic maps.*

Since the space X is, by assumption, Hausdorff and the ball $\mathbb{E}^n$ is compact, the characteristic map χ (considered as a map of the ball $\mathbb{E}^n$ onto the closure $\overline{e}$ of the cell e) is an identification map and therefore

Proposition 3.3. *the topology of the set $\overline{e}$ is the identification topology.*

In particular, we see that the topology of the set $\overline{e}$ does not depend on the topology of the space X in the sense that any other topology in X in which the set e is a cell (with characteristic map χ) induces the same identification topology on $\overline{e}$.

Note that, being a continuous image of a compact and path-connected set $\mathbb{E}^n$,

Proposition 3.4. *the set $\overline{e}$ is compact and path-connected.*

The set

$$\dot{e} = \overline{e} \setminus e$$

will be called the *boundary* of the cell e. For any choice of the characteristic map $\chi : \mathbb{E}^n \to X$, it coincides with the set $\chi(\mathbb{S}^{n-1})$. Therefore, for $n > 1$, the set $\dot{e}$ is path-connected, and for $n = 1$, it consists of at most two points. For $n = 0$, the set $\dot{e}$ is empty.

The main object of our study will be certain families K of pairwise disjoint cells $e \subset X$. For each such family, we will denote by $|K|$ its *body*, i.e., the subspace of X that is the union of all cells $e \in K$, and by K^n, where n is some non-negative integer, its n-th skeleton, i.e., the subfamily of the family K consisting of all cells $e \in K$ whose dimension $\dim e$ does not exceed n. It is convenient to add to the number of skeletons the empty subfamily of the family K, considering it the (-1)-th skeleton of K^{-1}, as well as the family K itself, considering it the ∞-th skeleton of K^∞.

By choosing characteristic maps $\chi_e : \mathbb{E}_e \to X$ for all cells e of the family K such that $\mathbb{E}_{e_1} \cap \mathbb{E}_{e_2} = \varnothing$ for $e_1 \neq e_2$, we can define a topological sum

$$P_K = \cup_{e \in K} \mathbb{E}_e$$

of the spaces $\mathbb{E}_e$ and a map

$$\chi : P_k \to X,$$

coinciding on each of the summands of $\mathbb{E}_e$ with the corresponding map χ_e. We will call the (obviously continuous) map $\chi : P_k \to X$ constructed in this way *characteristic* for the family K.

A family K of pairwise disjoint cells $e \subset K$ will be called a *cellular pre-decomposition* of the space X if $|K| = X$ and for any $n > 0$ the boundary $\dot{e}^n$ of each n-dimensional cell $e^n \in K$ belongs to the subspace $|K^n|$ (the body of the $n-1$-th skeleton).

Remark 3.5. Usually, the condition $\chi(\mathbb{S}^{n-1} \cap e) = \varnothing$ is not included in the definition of a cell, since only cells that make up pre-decompositions are of interest, and for such cells this condition is satisfied automatically.

It is clear that

Proposition 3.6. *any cellular sub-decomposition K contains at least one zero-dimensional cell, i.e., its 0-th skeleton K^0 is not empty (unless, of course, the space X is empty).*

In what follows, we will call the zero-dimensional cells of a cellular pre-decomposition K its *vertices*.

Since

$$K^m \subset K^n$$

for $m \leq n$ and, in particular, $K^0 \subset K^n$ for $n \geq 0$, then

Proposition 3.7. *any skeleton K^n of an arbitrary cellular pre-decomposition K is not empty.*

A pre-decomposition K is called *finite-dimensional* if there exists $n \geq 0$ such that $K = K^n$. The smallest of these n is called the *dimension* $\dim K$ of the pre-decomposition K. Clearly, $K^m = K$ for any $m \geq \dim K$. Moreover, $\dim K^m \leq m$ for any $m \geq 0$ (the strict inequality $\dim K^m < m$ holds if the decomposition K has no cells of dimension m).

Remark 3.8. Thus, for unequal n and m, the equality $K^n = K^m$ is possible (even for n and m less than $\dim K$). This is possible when the pre-decomposition has no cells of intermediate dimensions.

Example 3.9. (of pre-decompositions.)

1) Every Hausdorff space X can be "scattered" into a family K consisting of all its points. This family will be a cellular pre-decomposition of X containing only zero-dimensional cells.

2) For any $n \geq 0$, the family $K = \{e^0, e^n\}$ consisting of the point $e^0 = \boldsymbol{u}_0 = (1, 0, \ldots, 0)$ of the sphere $\mathbb{S}^n$ and its complement $e^n = \mathbb{S}^n \setminus \boldsymbol{u}_0$ is a cellular pre-decomposition of the sphere $\mathbb{S}^n$. Here $\dot{e}^n = e^0$ (if $n > 0$), $k^m = \{e^0\}$ for $0 \leq m < n$, and $K^m = K$ for $m \geq n$.

3) For any $n \geq 0$, the family $k = \{e^0, e^n, e^{n+1}\}$ consisting of the cell $e^{n+1} = \dot{\mathbb{E}}^n$ and the pre-decomposition $\{e^0, e^n\}$ of the sphere $\mathbb{S}^n$ constructed in Example 2) is a cellular pre-decomposition of the ball $\mathbb{E}^{n+1}$. Similarly, the family consisting of two points 0 and 1 and the open interval $(0, 1)$ is a cellular pre-decomposition of the segment $I = [0, 1]$. Taking liberties, we will denote this pre-decomposition by the same symbol I.

4) For any $n \geq 0$, the family K consisting of the cell $e^{n+1} = \dot{\mathbb{E}}^n$ and all points of the sphere $\mathbb{S}^n$ is also a cellular pre-decomposition of the ball $\mathbb{E}^{n+1}$. Its 0-th skeleton K^0 contains (for $n > 0$) an uncountable number of cells and is a pre-decomposition of 1), constructed for the space $X = \mathbb{S}^n$.

Remark 3.10. A reader familiar with the concept of a (finitely) triangulated space will immediately recognise that any triangulation of such a space is a cellular pre-decomposition of it. In the following discussion, cellular pre-decomposition of this kind are not used.

We will call a subfamily L of a cellular pre-decomposition K its *sub-pre-decomposition* if $\overline{e} \subset |L|$ for any cell $e \in L$. It is clear that

1) any sub-pre-decomposition L is a cellular pre-decomposition of the subspace $|L|$;

2) any subfamily L of a cellular pre-decomposition K, for which the subspace $|L|$ is closed, is a sub-pre-decomposition;

3) for any $n \geq 0$, the skeleton K^n is a sub-pre-decomposition of the pre-decomposition K;

4) the union and intersection of any (finite or infinite) system $\{L_\alpha\}$ of sub-pre-decomposition of the pre-decomposition K is also a sub-pre-decomposition, and
$$|\cap L_\alpha| = \cap |L_\alpha|, \quad |\cup L_\alpha| = \cup |L_\alpha|.$$

In connection with assertion 2), we note that there may well exist sub-pre-decompositions of L for which the subspace $|L|$ is not closed. (For example, any subset of the space X is a sub-pre-decomposition of the pre-decomposition of this space specified in Example 3.9 - 1)). However,

Proposition 3.11. *for any finite (i.e., consisting of a finite number of cells) sub-pre-decomposition of L, the subspace $|L|$ is closed and even compact.*

Proof. Indeed, in this case the subspace $|L|$ is the union of a finite number of compact sets $\overline{e}$, $e \in L$. □

From property 4) it follows, in particular, that

Proposition 3.12. *for any subset $A \subset X$ there exists a smallest sub-pre-decomposition $L \subset K$ for which $|L| \supset A$.*

Such a sub-pre-decomposition is the intersection of all sub-pre-decompositions of $L \subset K$ for which $|L| \supset A$. We will denote it by the symbol $K(A)$.

It is clear that

1) for any point $x \in X$, the equality
$$K(x) = K(e) = K(\overline{e})$$
holds, where e is a cell of the pre-decomposition K containing the point x;

2) for any cell $e \in K$, the sub-pre-decomposition $K(e)$ consists of the sub-pre-decomposition $K(\dot{e})$ and the cell e:
$$K(e) = \{K(\dot{e}), e\};$$

3) for any cell e belonging to a sub-pre-decomposition L of a pre-decomposition K, the equality
$$L(e) = K(e)$$
holds.

We will call a cellular pre-decomposition K of a space X *point finite* if any point $x \in X$ belongs to the body $|L|$ of some finite sub-pre-decomposition $L \subset K$ or, in other words, if for any point $x \in X$ the sub-pre-decomposition $K(x)$ is finite. Similarly, if any point $x \in X$ is an interior point of the body $|L|$ of some finite (resp. countable) sub-pre-decomposition $L \subset K$, then we will call the sub-pre-decomposition K *locally finite* (respectively, *locally countable*). It is clear that

Proposition 3.13. *any locally finite pre-decomposition of K is point finite.*

Moreover,

Proposition 3.14. *any sub-pre-decomposition L of a point finite (resp. locally finite or locally countable) pre-decomposition K is point finite (respectively, locally finite and locally countable).*

Further, it is easy to see that

Proposition 3.15. *for any locally finite (resp. locally countable) pre-decomposition K of X, the characteristic map*

$$\chi : P_K \to X$$

is locally compact (respectively, locally countably compact).

Proof. Indeed, let p be an arbitrary point in the space P_K. By the condition, the point $\chi(p) \in X$ has in the space X a neighbourhood U contained in the body $|L|$ of some finite (resp. countable) sub-pre-decomposition $L \subset K$. Let $S = \chi^{-1}(U)$ and $C = \chi^{-1}(|L|)$. It is clear that the set S is a saturated (with respect to the map χ) neighbourhood of the point p in the space P_K, and the set C is the union of a finite (respectively, countable) number of terms $\mathbb{E}_e$ of the topological sum P_K and is therefore closed and compact (respectively, countably compact). Moreover, the set C is saturated with respect to the map χ and contains the closure $\overline{S}$ of the neighbourhood S. Thus, each point in the space P_K has a saturated neighbourhood, the closure of which is contained in a saturated, closed, and compact (resp. countably compact) set. But this, by definition, means that the map χ is locally compact (resp. locally countably compact). □

Remark 3.16. It is easy to see that if a pre-decomposition K is point finite, then the converse is also true, i.e., any point finite pre-decomposition K for which the characteristic map $P_K \to X$ is locally compact (reps. locally countably compact) is a locally finite (resp. locally countable) pre-decomposition. We will not need this fact.

For any closed set $A \subset X$, all sets of the form $A \cap \overline{e}$, $e \in K$, are, of course, closed (in X). If the converse is true, i.e. if every set $A \subset X$ for which the family $\{A \cap \overline{e}; e \in K\}$ consists of closed sets is itself closed (in X), then we will say that the topology of the space X is a *weak topology* with respect to the pre-decomposition K. In other words, the topology of a space X is a weak topology with respect to the pre-decomposition K if this space is a free union of subspaces $\overline{e} \subset X$, $e \in K$. Clearly, this is the case if and only if the characteristic map

$$\chi : P_K \to X$$

is an identification map.

Similarly, for any closed set $A \subset X$ and any finite sub-pre-decomposition $L \subset K$ the intersection $A \cap |L|$ is closed in X. It is easy to see that

Proposition 3.17. *if the converse is true, i.e., if a set $A \subset X$ is closed in X when it has the property that for any finite sub-pre-decomposition L of a pre-decomposition K the intersection $A \cap |L|$ is closed in X, then the topology of the space X is a weak topology with respect to the pre-decomposition K.*

Proof. Indeed, if for any cell $e \in K$ the intersection $A \cap \overline{e}$ is closed, then for any finite sub-pre-decomposition $L \subset K$ the intersection $A \cap |L|$ is also closed (since the space $|L|$ is the union of a finite number of sets of the form $\overline{e}, e \in L$), and therefore, according to the condition, the set A is closed. □

However, in the space X, whose topology is a weak topology with respect to the pre-decomposition K, there may exist non-closed subsets A possessing the property that for any finite sub-pre-decomposition $L \subset K$, the intersection $A \cap |L|$ is closed (this property is possessed, for example, by any subset A of the sphere $\mathbb{S}^{n-1}$ in Example 3.9 - 4)).

It follows directly from the proven statement, in particular, that

Proposition 3.18. *for any finite pre-decomposition K of the space X (when such a pre-decomposition exists), the topology of the space X is a weak topology with respect to the pre-decomposition K.*

Moreover, it is easy to see that

Proposition 3.19. *the last statement holds not only for finite but also for any locally finite pre-decompositions.*

Proof. Indeed, for any locally finite pre-decomposition K of the space X, the family $\{\overline{e} : e \in K\}$ is obviously a locally finite family of closed subspaces of the space X, and therefore this space is their free union (see §1.1). □

Let K and L be arbitrary cellular pre-decompositions of Hausdorff spaces X and Y, respectively. It is clear that in the case when the spaces X and Y do not intersect, the union $K \cup L$ of the cellular pre-decompositions K and L is a cellular pre-decomposition of the topological sum $X \cup Y$. We will call the pre-decomposition $K \cup L$ the *sum* of the pre-decompositions K and L. According to what has just been said,

$$|K \cup L| = |K| \cup |L|.$$

Similar statements hold for any number of summands.

Furthermore, since for any $n \geq 0$ and $m \geq 0$, the ball $\mathbb{E}^{n+m}$ is homeomorphic to the product $\mathbb{E}^n \times \mathbb{E}^n$, then for any two cells $e^n \in K$ and $e^m \in L$ (with characteristic maps $\chi_1 : \mathbb{E}^n \to X$ and $\chi_2 : \mathbb{E}^n \to Y$), the product $e^n \times e^m$ represents an $(n+m)$-dimensional cell of the space $X \times Y$ (for the corresponding characteristic map, one can take, for example, the map $\chi_1 \times \chi_2 : \mathbb{E}^n \times \mathbb{E}^m \to X \times Y$), where

$$(e^n \times e^m)^{\cdot} = \dot{e}^n \times e^m \cup e^n \times \dot{e}^m.$$

Consequently, the family $K \times L$ of all cells of the form $e^1 \times e^2$, where $e^1 \in K$ and $e^2 \in L$, constitutes a cellular pre-decomposition of the space $X \times Y$, denoted as $|K \times L| = |K| \times |L|$. We will call the pre-decomposition $K \times L$ the *product* of the pre-decompositions K and L. The product of any finite number of pre-decompositions is defined similarly.

It is clear that

Proposition 3.20. *the product of point finite (respectively, locally finite and locally countable) pre-decompositions is point finite (respectively, locally finite and locally countable).*

In the special case where the pre-decomposition L is the pre-decomposition $I = \{0, 1, (0, 1)\}$ of the interval $I = [0, 1]$, we obtain a pre-decomposition $\times I$ of the product $X \times I$. The cells of this pre-decomposition are of the form $e \times 0$, $e \times 1$ and $e \times (0, 1)$, where $e \in K$. In this case, the dimension of the cells $e \times 0$ and $e \times 1$ is equal to the dimension n of the cell e, and the dimension of the cell $e \times (0, 1)$ is equal to $n + 1$.

Let

$$\chi_K : P_K \to X, \quad \chi_L : P_L \to Y$$

be the characteristic maps for the pre-decompositions K and L respectively. It is clear that in the case when the spaces P_K and P_L are chosen to be disjoint, we can take their topological sum $P_K \cup P_L$ as the space $P_{K \cup L}$, and the corresponding characteristic map $P_{K \cup L} \to X \cup Y$ can then be considered as the map

$$\chi_K \cup \chi_L : P_K \cup P_Y \to X \cup Y.$$

Similarly, we can consider the space $P_K \times P_L$ as the space $P_{X \times Y}$ and the map

$$\chi_K \times \chi_L : P_K \times P_Y \to X \times Y.$$

as the characteristic map of the pre-decomposition $K \times L$.

3.2 Cellular decompositions

We will call a cellular pre-decomposition K of a space X a *cellular decomposition* if

- it is point finite, and
- the topology of the space X is a weak topology with respect to the pre-decomposition K.

It follows directly from the results of the previous section that

Proposition 3.21. *any locally finite (in particular, any finite) pre-decomposition of K is a decomposition.*

Thus, the pre-decompositions indicated in examples 2) and 3) of Example 3.9 in §3.1 are decompositions. Conversely, the pre-decomposition in example 1) is a decomposition only when the space X is discrete (because otherwise its topology will not be a weak topology). In particular, when $X = \mathbb{S}^n$, $n > 0$, this pre-decomposition is not a decomposition. Similarly, the pre-decomposition in example 4) of Example 3.9 in §3.1 is not a decomposition either, because it is not point finite (note that the condition of weakness of the topology is satisfied in this example).

The last pre-decomposition also possesses sub-pre-decompositions that are not decompositions (these are arbitrary subsets of the sphere $\mathbb{S}^{n-1}$). At the same time,

Proposition 3.22. *any sub-pre-decomposition of an arbitrary cell decomposition K is itself a decomposition.*

Before proving the last statement, we note that the sufficient condition for the weakness of the topology of the space X indicated in §3.1 is also necessary in the case of decompositions, so that

Proposition 3.23. *A subset A of a space $X = |K|$ is closed if for any finite sub-decomposition L of the decomposition K the intersection $A \cap |L|$ is closed.*

Proof. Indeed, since for any cell $e \in K$ the intersection $A \cap |K(e)|$ is closed (since the sub-decomposition $K(e)$ is finite) and since $\overline{e} \subset |K(e)|$, the intersection $A \cap \overline{e} = (A \cap |K(e)|) \cap \overline{e}$ is closed. Therefore, the subset A is also closed. □

It follows from this that, in contrast to the case of arbitrary sub-decompositions,

Proposition 3.24. *the body $|L|$ of each sub-pre-decomposition L of the decomposition K is closed in the space $X = |K|$.*

Proof. Indeed, for any finite sub-decomposition of $N \subset K$ the intersection $L \cap N$ is also finite and therefore the intersection $|L| \cap |N| = |L \cap N|$ is closed. □

Now let A be an arbitrary subset of $|L|$ such that, for any finite sub-decomposition $N \subset L$, the intersection $A \cap |N|$ is closed (in $|L|$, and therefore, by what has been proved, in X). Then, for any finite sub-decomposition $N \subset K$, the intersection

$$A \cap |N| = A \cap |L \cap N|$$

is also closed, and, consequently, the set A is closed (in X, and therefore in $|L|$). This proves that the topology of the space $|L|$ is a weak topology with respect to the pre-decomposition L. Since the pre-decomposition L is, moreover, point finite (being a sub-pre-decomposition of the point finite decomposition K), this pre-decomosition is, as stated above, a decomposition.

In what follows, we will call sub-decompositions L of an arbitrary cellular decomposition K its sub-decompositions.

When studying the properties of cellular decompositions, it is useful to keep in mind that

Proposition 3.25. *for any compact subset C of X and any cellular decomposition K of X, the sub-decomposition $K(C)$ is finite.*

Proof. To prove this statement, it is sufficient to prove that the subset C intersects only a finite number of cells $e \in K$, since then it will be contained in a finite sub-decomposition that is the union of sub-decompositions $K(e)$ for which $e \cap C \neq \varnothing$.

Let there exist an infinite system $\{e_i\}$ of different cells $e_i \in K$ for which $e_i \cap C \neq \varnothing$ for each i, and let $x_i \in e_i \cap C$. Consider an arbitrary finite sub-decomposition L of the decomposition K. It contains only a finite number of cells e_i and $e_i \cap |L| = \varnothing$ if $e_i \notin L$. Therefore, the subspace $|L|$ contains only finitely many points x_i, so the intersection $\{x_i\} \cap |L|$ is closed. Consequently, the set Xi is closed. By similar arguments, any subset of the set $\{x_i\}$ is also closed, so the set $\{x_i\}$ is discrete. But this is impossible, since any discrete subset of a compact set must be finite. Consequently, the set C actually intersects only a finite number of cells of the decomposition K. □

It easily follows from the proven statement, for example, that

Proposition 3.26. *any pre-decomposition K, for which, for each $n = 0, 1, \dots, \infty$, the topology of the space K^n is a weak topology with respect to the pre-decomposition K^n, constitutes a decomposition.*

Proof. Indeed, since $K = K^\infty$, we only need to prove that the pre-decomposition K is point finite, for which it is sufficient in turn to prove that for any finite n, the pre-decomposition K^n is point finite (because if $\dim e = n$, then $K(e) = K^n(e)$). We will prove this by induction on n, considering that the pre-decomposition K^0 is obviously point finite.

Let it already be proven for some $n > 0$ that the pre-decomposition K^{n-1} is point inite. Since, by assumption, the topology of the space $|K^{n-1}|$ is a weak topology with respect to this pre-decomposition, the pre-decomposition K^{n-1} is a decomposition, and therefore, according to the statement just proven, for any compact subset $C \subset |K^{n-1}|$ the sub-pre-decomposition $K^{n-1}(C) = K(C)$ is finite. In particular, for any n-dimensional cell $e^n \in K$, the sub-pre-decomposition $K(\dot{e}^{n-1})$ is finite. But, as we know, $K(e^n) = \{K(e^n), e^n\}$. Therefore, the sub-pre-decomposition $K(e^n)$ is also finite. □

In the study of cellular decompositions, an important role is also played by the fact that

Proposition 3.27. *for any increasing sequence*

$$K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$$

of sub-decompositions of the decomposition K, whose union is the entire decomposition K, the space $X = |K|$ is a free union of subspaces

$$|K_0| \subset |K_1| \subset \cdots \subset |K_n| \subset \cdots$$

Proof. We must show that any subset A of the space X for which the intersection $A \cap |K^n|$ is closed for any n is itself closed. Since the space X is a free union of the sets $\overline{e}$, $e \in K$, it suffices to prove that for any cell $e \in K$, the intersection $A \cap e$ is closed.

Let n be a number such that $e \in K^n$. Then

$$A \cap \overline{e} = (A \cap \{K^n\}) \cap \overline{e},$$

and therefore, this intersection is closed. □

In particular,

Proposition 3.28. *for any decomposition K of the space X, this space is a free union of subspaces*

$$|K^0| \subset |K^1| \subset \cdots \subset |K^n| \subset \cdots$$

Note that the converse of the proposition proven above also holds, i.e.,

Proposition 3.29. *if a cellular pre-decomposition K of the space X is the union of an increasing sequence*

$$K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$$

of sub-decompositions K^n, each of which is a cellular decomposition (of the subspace K^n), and if the space X is a free union of subspaces

$$|K_0| \subset |K_1| \subset \cdots \subset |K_n| \subset \cdots$$

then the pre-decomposition K is a decomposition.

Proof. Indeed, since for any cell $e \in K^n$ the equality $K(e) = K_n(e)$ holds, the pre-decomposition K is point finite. Therefore, we only need to prove that the topology of the space X is a weak topology with respect to the pre-decomposition K. Let A be an arbitrary subset of the space X for which the family $\{A \cap \overline{e}; e \in K\}$ consists of closed sets. Since

$$(A \cap |K^n|) \cap \overline{e} = A \cap \overline{e}$$

for $e \in K^n$ and since the topology of the space $|K^n|$ is a weak topology with respect to the decomposition K^n, for any $n \geq 0$ the set $A \cap |K^n|$ is closed in $|K^n|$. Therefore, the set A is closed in X. Consequently, the topology of the space X is indeed a weak topology with respect to the pre-decomposition K. □

The existence of a cellular decomposition K for a space X imposes rather strong restrictions on the topology of that space. For example, it is clear that

Proposition 3.30. *Any space X that admits a cellular decomposition K is a canonical space.*

Furthermore, since the closure $\overline{e}$ of each cell e of an arbitrary cell decomposition K is connected, any connected component A of the space $X = |K|$ either contains such a closure entirely, $A \cap \overline{e} = \overline{e}$, or does not intersect with it, $A \cap \overline{e} = \varnothing$. Since in both cases the intersection $A \cap \overline{e}$ is closed and open in $\overline{e}$, the connected component A is closed and open in X. This means that

Proposition 3.31. *any space X that admits a cellular decomposition K is a topological sum of its connected components.*

Furthermore, for each connected component $A \subset X$, the set L of all cells $e \in K$ whose closures are contained in A (i.e., for which $A \cap \overline{e} = \overline{e}$) constitutes a sub-decomposition of the decomposition K, and the body of this sub-decomposition coincides with $A : |L| = A$.

Thus,

Proposition 3.32. *any connected component A of the space X, which admits a cell decomposition K, serves as the body of some sub-decomposition $L \subset K$.*

On the other hand, since the sets $\overline{e}$, $e \in K$, are not only connected but even path-connected, the path-connected components A of the space X also possess the property that for any cell $e \in K$, either $A \cap \overline{e} = \varnothing$ or $A \cap \overline{e} = \overline{e}$, and therefore each path-connected component A is simultaneously closed and open in X. Consequently,

Proposition 3.33. *every connected space X that admits a cellular decomposition K is path-connected.*

Furthermore, it is easy to see that

Proposition 3.34. *for every space X admitting a cell decomposition K, any of its connected components is a path-connected component, and, conversely, any of its path-connected components is a connected component.*

Proof. Indeed, any connected component of the space X, being the body of some sub-decomposition of the decomposition K, is path-connected and therefore is a path-connected component. Conversely, since any path-connected component is both closed and open, it coincides with the connected component containing it. □

Since each connected component of the space X, being the body of some sub-decomposition, contains at least one vertex of the decomposition K, and since his component is path-connected, it follows that

Proposition 3.35. *any point in the space $X = |K|$ can be connected by a path to at least one of the vertices of the decomposition.*

Therefore,

Proposition 3.36. *the space $X = |K|$ is connected if its subspace K^1 is connected.*

It turns out that the converse statement is also true, i.e.,

Proposition 3.37. *if the space $X = |K|$ is connected, then its subspace K^1 is also connected.*

Proof. Indeed, let A be an arbitrary connected component of the space $|K^1|$. As we know, $A = |L|$, where L is some sub-decomposition of the decomposition of the skeleton K^1. We will show that there exists a sequence of sub-decompositions $L1 \subset L_2 \subset \cdots \subset L_n \subset \cdots$ of the form $L_n \subset K^n$, such that

1) $L_1 = L$;

2) $(L_{n+1})^n = L_n$;

3) For any cell $e \in K^n$, either $\overline{e} \cap |L_n| = \varnothing$ or $\overline{e} \cap |L_n| = \overline{e}$.

Let a sub-decomposition L_n be already constructed for some $n > 0$. From condition 3) it immediately follows that the set $|L_n|$ of this sub-decomposition is both closed and open in the space $|K^n|$. Therefore, for any $(n+1)$-dimensional cell $e^{n+1} \in K$, either $\dot{e}^{n+1} \cap |L_n| = \varnothing$ or $\dot{e}^{n+1} \cap |L_n| = \dot{e}^{n+1}$ (recall that the set $\dot{e}^{n+1}$ is connected). Therefore, the collection L_{n+1} of all cells $e \in K^{n+1}$ for which $\overline{e} \cap |L_n| \neq \varnothing$ is a sub-decomposition of the decomposition K^{n+1}. It is clear that this sub-decomposition satisfies conditions 1) - 3).

Now let us consider the sub-decomposition

$$L_\infty = \cup_{n=1}^{\infty} L_n$$

of the partition K. It is clear that its body $|L_\infty| \subset X$ has the property that for any cell $e \in K$, either $\overline{e} \cap |L_\infty| = \varnothing$ or $\overline{e} \cap |L_\infty| = \overline{e}$. Therefore, the subspace $|L_\infty|$ is closed and simultaneously open in X. Consequently, $|L_\infty| = X$, i.e., $L_\infty = K$. But it is clear that $(L_\infty)^1 = L_1 = L$. Therefore, $L = K^1$ and, consequently, $A = |L| = |K^1|$. Therefore, the subspace $|K^1|$ of the space X is connected. □

Since $(K^n)^1 = K^1$ for any $n \geq 1$, it immediately follows from the last two statements that

Proposition 3.38. *if the space $X = |K|$ is connected, then for any $n \geq 1$ its subspace $|K^n|$ is also connected; if at least for one $n \geq 1$ the subspace $|K^n|$ is connected, then the entire space X is also connected.*

As for the subspace $|K^0|$, it is easy to see that

Proposition 3.39. *for any cell decomposition K of the space X, the subspace $|K^0|$ is discrete.*

Therefore, the subspace $|K^0|$ is connected if and only if the decomposition K contains only one zero-dimensional cell (such decompositions are called *single-vertex* decompositions).

Remark 3.40 (Terminology Convention). Further, for the sake of brevity, we will not distinguish between cellular decompositions of K and the corresponding spaces $|K|$. Accordingly, we will allow expressions such as

"connected cellular decomposition of K,"

"continuous map of a decomposition of K into a decomposition of L," and so on, meaning, respectively,

"a cellular decomposition of K for which the space $|K|$ is connected,"

"a continuous map of the space $|K|$ into the space $|L|$,"

and so on. In particular, the formula $e \in K$ will, as above, mean that the cell e belongs to the cellular decomposition of K, and the formula $e \subset K$, means that the cell e lies in the space $|K|$.

In cases where this convention could lead to misunderstandings, we will, naturally, continue to distinguish between decompositions of K and spaces $|K|$. In this case, spaces of the form $|K|$, i.e., spaces that admit cellular decompositions, will be called *cellular polyhedra*.

3.3 Theorem on paracompactness

In this section, we prove that any cellular decomposition (i.e., the space $|K|$; see the terminology convention at the end of the previous section) is paracompact, i.e., any open covering Γ of it can be refined into a locally finite covering Δ. We will construct the covering structure Δ by "ascending" step by step along the skeletons K^n of the decomposition K. To describe this construction, it is convenient for us to introduce the following general definition:

Let $X \subset Y$ and let $\{U_\alpha; \alpha \in A\}$ and $\{V_\beta; \beta \in B\}$ be families of open sets in spaces X and Y respectively. The family $\{V_\beta; \beta \in B\}$ will be called an *extension of* the family $\{U_\alpha; \alpha \in A\}$ if $A \subset B$ and

$$U_\alpha = V_\alpha \cap X$$

for every $\alpha \in A$.

In order to construct an open covering Δ of a decomposition K, it is obviously sufficient to construct a sequence of open coverings

$$\Delta^n = \{V^n_\beta; \beta \in B^n\}$$

of the skeletons K^n, having the property that for any $n > 0$ the covering Δ^n" is an extension of the covering Δ^{n-1}. Indeed, then the family Δ of sets

$$V_\beta = \cup_n V^n_\beta$$

where the summation is extended to all n for which $\beta \in B^n$, will, as is easily seen, be an open covering of the decomposition K (with the index set B being the union of the sets $B^0 \subset B^1 \subset \cdots \subset B^n \subset \cdots$).

For this covering to be refined in the given covering

$$\Gamma = \{U_\alpha; \alpha \in A\}$$

of the decomposition K, it is necessary to ensure that there exists a map α

$$\alpha : B \to A,$$

such that for any $n > 0$ the inclusion holds

$$V^n_\beta \subset U_{\alpha(\beta)} \cap K^n.$$

To carry out the proof, it will be more convenient for us to require more, namely, that the inclusion take place

$$\overline{V}^n_\beta \subset U_{\alpha(\beta)} \cap K^n.$$

Let us now consider the question of conditions that ensure local finiteness of the covering

$$\Delta = \{V_\beta; \beta \in B\}$$

Let x be an arbitrary point in the space K. To construct a neighbourhood of a point x in the space K, it is sufficient for any x for which $x \in K^n$ to define in the space K^n a neighbourhood W^n_x of the point x with the property that for $x \in K^n$

$$W^n_x \cap K^{n-1} = W^{n-1}_x.$$

Indeed, then the formula

$$W_x = \cup_n W^n_x$$

(the summation is extended to all n for which $x \in K^n$) will define for us some neighbourhood of the point x in the entire space K. This method can obviously be used to obtain any neighbourhood W_x (it suffices to set $W^n_x = W_x \cap K^n$).

Note that for $W_x \cap V_\beta \neq \varnothing$ there exists a number $n \geq 0$ such that $x \in K^n$ and

$$W^n_x \cap V^n_\beta \neq \varnothing.$$

Now suppose that the neighbourhoods W^n_x have the following two properties:

i) the intersection $W^n_x \cap V^n_\beta$ for $x \in K^m$, where $m \leq n$, is non-empty if and only if $\beta \in B^m$ and the intersection $W^m_x \cap V^m_\beta$ is non-empty;

ii) for $x \in K^m \setminus K^{n-1}$ the intersection $W^n_x \cap V^n_\beta$ is non-empty only for a finite number of indices β.

From what has just been said, it follows directly that, if conditions i) and ii) are satisfied, the intersection $W_x \cap V_\beta$ will be non-empty only for a finite number of indices β. In other words, the existence of neighborhoods W^n_x satisfying conditions i) and ii) ensures the local finiteness of the covering Δ.

Summarising all that has been said, we see that to construct a locally finite open covering Δ refined in a given open covering Γ, it is sufficient to construct

a) a sequence of open coverings $\Delta^n = \{V^n_\beta; \beta \in B^n\}$, $n = 0, 1, 2, \ldots$, each of which is an extension of the previous covering;

b) a map

$$\alpha : B \to A,$$

where $B = \cup^\infty_{n=0} B^n$, having the property that for any $n \geq 0$ and any $\beta \in B^n$, the inclusion

$$\overline{V}^n_\beta \subset U_{\alpha(\beta)} \cap K^n$$

holds;

c) for any $n \geq 0$ and any point $x \in K^n$ in the neighbourhood of W_x^n, with the above properties i) and ii).

We will construct these objects by induction on the number n.

For $n = 0$, we take the 0-th skeleton K^0 of the decomposition K for the set B^0. We define the map α on the set B^0 by choosing, for each point $\beta \in B^0 = K^0$, the element of the covering Γ that contains it and taking $\alpha(\beta)$ as the index of this element. For the set V_β^0 corresponding to the point $\beta \in B^0$, we take this point itself. Similarly, for the set W_x^0 corresponding to the point $x \in K^0$, we also take this point itself. Since the space K^0 is discrete, it is easy to see that all our conditions are satisfied.

Let for some $n \geq 0$ we have already constructed a set B, a map α of this set into a set A, open sets $V_\beta^n \subset K^n$ and neighbourhoods $W_x^n \subset K^n$, $x \in K^n$, satisfying conditions a), b), c).

In the trivial case $K^{n+1} =^n$ we set

$$B^{n+1} = B^n, \quad V_\beta^{n+1} = V_\beta^n, \quad W_x^{n+1} = W_x^n.$$

It is clear that the index sets B^{n+1}, open sets V_β^{n+1} and neighbourhoods W_x^{n+1} constructed in this way still satisfy conditions a), b), c).

Thus, we need to consider only the case when $K^{n+1} \neq K^n$, i.e., when the decomposition K contains $n+1$-dimensional cells $^{n+1}$. For each such cell, we, having chosen some characteristic map

$$\chi : \mathbb{E}^{n+1} \to X,$$

wee put

$$U'_\alpha = \chi^{-1}(U_\alpha) = \chi^{-1}(U_\alpha \cap \bar{e}^{n+1}), \quad \alpha \in A,$$
$$V'_\beta = \chi^{-1}(V_\beta) = \chi^{-1}(V_\beta^n \cap \dot{e}^{n+1}), \quad \beta \in B.$$

The sets U'_α (many of these sets are, generally speaking, empty) constitute an open covering Γ' of the ball $\mathbb{E}^{n+1}$, and the sets V'_β (among which there may also be empty sets) constitute an open covering of its boundary sphere $\mathbb{S}^n$. Moreover, in view of condition b), for any element $\beta \in B^n$, the inclusion holds

$$\overline{V}'_\beta \subset U'_{\alpha(\beta)} \cap \mathbb{S}^n$$

(from which, in particular, it follows that the covering $\{V'_\beta\}$ is refined in the covering $\Gamma' \cap \mathbb{S}^n = \{U'_{\alpha(\beta)} \cap \mathbb{S}^n; \alpha \in A\}$). Moreover, in view of condition c) (by which the covering $\Delta^n = \{V_\beta^n; \beta \in B^n\}$ of the space K^n is locally finite), the covering $\{V'_\beta; \beta \in B^n\}$ of the sphere $\mathbb{S}^n$ is locally finite. (However, it is easy to see by induction that in fact only finitely many elements of this covering are non-empty.)

Next, for each point $x \in K^n$ we set

$$W'_x = \chi^{-1}(W_x^n) = \chi^{-1}(W_x^n \cap \dot{e}^{n+1}).$$

Clearly, the (possibly empty) sets W'_x are open in the sphere $\mathbb{S}^n$ and have the property that for each point x there are only finitely many indices β for which $W'_x \cap V'_\beta \neq \varnothing$.

For any set $G \subset \mathbb{S}^n$ and any positive $\varepsilon < 1$, we will denote by G_ε the "radial extension to ε" of the set G into the ball, i.e., the subset of the ball $\mathbb{E}^{n+1}$ consisting of all points $\boldsymbol{v} = v\boldsymbol{u} \in \mathbb{E}^{n+1}$, $0 < v < 1$, $|\boldsymbol{u}| = 1$, for which

$$1 - \varepsilon < v \leq 1, \quad \boldsymbol{u} \in G.$$

It is clear that if G is open in the sphere $\mathbb{S}^n$, then the set G_ε is open in the ball $\mathbb{E}^{n+1}$.

The sets G_ε have the property that $G_\varepsilon \cap \mathbb{S}^n = G$. Moreover, for any $\varepsilon < \varepsilon'$, the inclusion

$$G_{\varepsilon'} \subset G_\varepsilon$$

holds. Moreover, for any sets $G, H \subset \mathbb{S}^n$ and any positive $\varepsilon < 1$ and $\eta < 1$, the intersection $G_\varepsilon \cap H_\eta$ of their "radial extensions" is non-empty if and only if the intersection of the sets $G \cap H$ is non-empty. In particular, the set G_ε is non-empty if and only if the set G is non-empty.

Now, choosing for each element $\beta \in B^n$ some positive number $\varepsilon_\beta < 1$, we set

$$V^*_\beta = (V'_\beta)_{\varepsilon_\beta}.$$

Since $\overline{V}'_\beta \subset U'_{\alpha(\beta)} \cap \mathbb{S}^n$, for sufficiently small ε_β the inclusion

$$V^*_\beta \subset U'_{\alpha(\beta)}$$

(and even the inclusion $\overline{V}^*_n \subset U'_{\alpha^0(\beta)}$) holds, so that all sets V^*_β, $\beta \in B^n$, form an open (and obviously locally finite) covering "adjacent" to the sphere $\mathbb{S}^n$ of the set

$$V^* = \cup_{\beta \in B^n} V^*_\beta,$$

refined in the covering $\Gamma' \cap V^* = \{U'_\alpha \cap V^*\}$.

Since the set $\mathbb{E}_{n+1} \setminus V^*$ is closed in the ball $\mathbb{E}^{n+1}$, it is compact, and therefore it can be covered by a finite number of non-empty sets of the covering Γ'. Let $U'_{\alpha_1}, \dots, U'_{\alpha_s}$ be these sets. Since V^* is open and contains the sphere $\mathbb{S}^n$, it is possible to refine the open covering $U'_{\alpha_1}, \dots, U'_{\alpha_s}$ of the set $\mathbb{E}^{n+1} \setminus V^*$ into an open covering $T_1, \dots, T_s$ such that

$$\overline{T}_i \subset U'_{\alpha_i}, \quad \overline{T}_i \cap \mathbb{S}^n = \varnothing.$$

For consistency of notation, we will write $V^*_{\alpha_i}$ instead of T_i. Here, the superscript $*$, unlike in the previous case, of course, does not mean that $V^*_{\alpha_i}$ is obtained by radial extension. Furthermore, this notation implies that the sets B^n and A do not intersect; clearly, this last assumption does not restrict generality. It is obvious that the sets V^*_β, $\beta \in B^n$, and $V^*_{\alpha_i}$, $i = 1, \dots, s$, provide an open locally finite (essentially, even finite) covering of Δ^* of the ball $\mathbb{E}^{n+1}$, refined in the covering Γ.

Since the set $\cup_{i=1}^{s}\overline{V}^*_{\alpha_i}$ is closed and does not intersect the sphere $\mathbb{S}^n$, then for any point $x \in K^n$ there exists a positive $\varepsilon_x^{'} < 1$ such that the set

$$W_x^* = (W_x^{'})_{\varepsilon_x^{'}}$$

does not intersect with the set $\cup_{i=1}^{s}\overline{V}^*_{\alpha_i}$. The sets W_x^* are open in the ball $\mathbb{E}^{n+1}$ and have the property that

$$W_x^* \cap \mathbb{S}^n = W_x^{'}.$$

Furthermore, each of the sets W_x^* intersects with only a finite number of sets from the covering Δ^*.

Let v be an arbitrary interior point of the ball $\mathbb{E}^{n+1}$. Since the covering Δ^* is locally finite, this point has neighbourhoods that consist entirely of interior points of the ball $\mathbb{E}^{n+1}$ and intersect with only a finite number of sets of this covering. Choosing one of these neighbourhoods, we denote it by W_x^*, where $x \in e^{n+1}$ is the image of the point v under the characteristic map χ. Here, the asterisk again does not denote radial extension, but is introduced only for uniformity of notation.

All the constructed objects will have to be considered below for all cells $e^{n+1} \in K$ simultaneously. Therefore, we will introduce an additional index e^{n+1} into all notations. Thus, we will consider maps $\chi_{e^{n+1}}$, numbers $s_{e^{n+1}}$, indices $\alpha_{i,e^{n+1}}$, sets $V^*_{e^{n+1}}$, etc.

Now we have everything ready to construct the objects we need for $n+1$. We obtain the set B^{n+1} by adding to the set B^n all possible pairs of the form (i, e^{n+1}), where $e^{n+1} \in K$, $i = 1, \ldots, s_{e^{n+1}}$. We define the map α on the set B^{n+1} by the formula

$$\alpha(\beta) = \begin{cases} \alpha(\beta) & \text{if} \quad \beta \in B^n, \\ \alpha_{i,e^{n+1}} & \text{if} \quad \beta = (i, e^{n+1}), \end{cases}$$

of the set $V_\beta^{n+1} \subset K^{n+1}$, $\beta \in B^{n+1}$, by the formula

$$V_\beta^{n+1} = \begin{cases} V_\beta^n \cup_{e^{n+1}\in K} \chi_{e^{n+1}}(V^*_{\beta,e^{n+1}}), & \text{if} \quad \beta \in B^n, \\ \chi_{e^{n+1}}(V_{\alpha_i,e^{n+1}}), & \text{if} \quad \beta = (i, e^{n+1}), \end{cases}$$

and, finally, on the neighbourhood W_x^{n+1}, $x \in K^{n+1}$, by the formula

$$W_x^{n+1} = \begin{cases} W_x^n \cup_{e^{n+1}\in K} \chi_{e^{n+1}}(W^*_{x,e^{n+1}}), & \text{if} \quad x \in K^n, \\ \chi_{e^{n+1}}(W^{n+1}_{x,e^{n+1}}), & \text{if} \quad x \in e^{n+1}. \end{cases}$$

Since for any cell $e^{n+1} \in K$ the characteristic map

$$\chi_{e^{n+1}} : \mathbb{E}^{n+1} \to X$$

is homeomorphic on the open ball $\dot{\mathbb{E}}^{n+1} = \mathbb{E}^{n+1} \setminus \mathbb{S}^n$, and $V^*_{\alpha_{i,e^{n+1}}} \subset \dot{\mathbb{E}}^{n+1}$ for each $i = 1, \ldots, s_{e^{n+1}}$, then for any $\beta = (i, e^{n+1})$ the set $V_\beta^{n+1} = \chi_{e^{n+1}}(V^*_{\alpha,e^{n+1}})$ is open in

$\overline{e}^{n+1}$. Consequently, this set is also open in K^{n+1}, because for any cell $e \in K^{n+1}$ different from the cell e^{n+1}, the intersection $V_\beta^{n+1} \cap \overline{e}$ is obviously empty (and therefore open in $\overline{e}$).

Furthermore, since for any $\beta \in B^n$ and any cell $e^{n+1} \in K$ the following equality holds:

$$V^*_{\beta, e^{n+1}} \cap \mathbb{S}^n = V'_{\beta, e^{n+1}} = \quad \chi^{-1}_{e^{n+1}}(V^n_\beta),$$

so the set $V^*_{e^{n+1}}$ is saturated with respect to the map $\chi_{e^{n+1}}$ (considered as a map from the ball $\mathbb{E}^{n+1}$ to the closure of the cell $\overline{e}^{n+1}$) and therefore the set $\chi_{e^{n+1}}(V^*_{e^{n+1}})$ is open in $\overline{e}^{n+1}$. On the other hand, for any cell $e \in Kn+1$ either $V_\beta n + 1 \cap \overline{e} = V^n_\beta \cap \overline{e}$ (if $\dim e \leq n$) or $V^{n+1}_\beta \cap \overline{e} = \chi_{e^{n+1}}(V^*_{\beta, e^{n+1}}$ (if $e =^{n+1}$). Since in both cases the intersection $V_\beta n + 1 \cap \overline{e}$ is open in $\overline{e}$, it is thus proved that the set V^{n+1}_β is open in the space K^{n+1} even for $\beta i n n B^n$.

It is similarly proved that all sets of the form W^{n+1}_x are also open in the space K^{n+1}.

Let us now check conditions i) and ii) for these sets. First, we will consider condition ii).

Let $x \in K^{n+1} \setminus K^n$ and $\beta \in B^{n+1}$. Consider a cell e^{n+1} of the decomposition K containing the point x. By definition,

$$W^{n+1}_x = \chi_{e^{n+1}}(W^*_{x, e^{n+1}}),$$

and therefore the intersection $W^{n+1}_x \cap V^{n+1}_\beta$ can be non-empty, only when $\beta \in B^n$ or when $\beta = (i, e^{n+1})$, where $i = 1, \ldots, s_{e^{n+1}}$. Since the number of indices β of the form (i, e^{n+1}) is finite, it suffices to prove that the number of those $\beta \in B^n$ for which this intersection is non-empty is also finite. But it is clear that for $\beta \in B^n$

$$W^{n+1}_x \cap V^{n+1}_\beta = \chi_{e^{n+1}}(W^*_{x, e^{n+1}} \cap V^*_{\beta, e^{n+1}}),$$

and therefore, according to the choice of the set $W^*_{x, e^{n+1}}$, the intersection $W^{n+1}_x \cap V^{n+1}_\beta$ is non-empty only for a finite number of indices $\beta \in B^n$.

Let us now check condition i). Let $m < n+1$, $\beta \in B^{n+1}$ and $x \in K^m$. Since $m < n+1$, then

$$W^{n+1}_x = W^n_x \cup \cup_{e^{n+1} \in K} \chi_{e^{n+1}}(W^*_{x, e^{n+1}})$$

Therefore if $\beta = (i, e^{n+1})$, then

$$W^{n+1}_x \cap V^{n+1}_\beta = \chi_{e^{n+1}}(W^*_{x, e^{n+1}} \cap V^*_{\beta, e^{n+1}}) = \varnothing.$$

If $\beta \in B^n$, then due to inclusions

$$\chi_{e^{n+1}}(W^*_{x, e^{n+1}}) \cap K^n \subset W^n_x,$$
$$\chi_{e^{n+1}}(V^*_{\beta, e^{n+1}}) \cap K^n \subset V^n_\beta$$

the equality will hold

$$W^{n+1}_x \cap V^{n+1}_\beta = (W^n_x \cap V^n_\beta) \cup \cup_{e^{n+1} \in K} \chi_{e^{n+1}}(W^*_{x, e^{n+1}} \cap V^*_{\beta, e^{n+1}}).$$

Let for some cell $e^{n+1} \in K$ the intersection

$$W^*_{x,e^{n+1}} \cap V^*_{\beta,e^{n+1}}$$

is non-empty. Then the intersection is non-empty

$$W'_{x,e^{n+1}} \cap V'_{\beta,e^{n+1}} = \chi^{-1}_{e^{n+1}}(W^n_x \cap V^n_\beta),$$

and the intersection as well

$$W^n_x \cap V^n_\beta.$$

It follows from this that the intersection $W^{n+1}_x \cap V^{n+1}_\beta$, $\beta \in B^n$, is non-empty if and only if the intersection $W^n_x \cap V^n_\beta$ is non-empty, i.e., by the induction hypothesis, when $\beta \in B^m$ and the intersection $W^m_x \cap V^m_\beta$ is non-empty. Thus, the construction of coverings Δ^n and neighbourhoods W^n_x by induction is accomplished for all n. According to the above, it is thus proved that

Proposition 3.41. *any open covering Γ of a cellular decomposition K can be refined into a locally finite open covering Δ.*

In other words, we have proved that

Proposition 3.42. *any cellular decomposition of K is a paracompact space.*

Since the space K is, by assumption, Hausdorff, it follows directly from this statement that

Proposition 3.43. *any cellular decomposition of K is a normal (and even stably normal) space.*

Let us now consider the characteristic map $\chi_{K\times L} : P_{K\times L} \to K \times L$ for the pre-decomposition $K\times L$ which is the product of two cellular decompositions K and L. By what has just been proved, the spaces K and L are regular. Moreover, as we know, if the decomposition K is locally finite (resp. locally countable), then the characteristic map $\chi_K : P_K \to K$ is locally compact (resp. locally countable). On the other hand, as noted at the end of §3.1, we can assume that $P_{K\times L} = P_K \times P_L$ and $\chi_{K\times L} = \chi_K \times \chi_L$. Therefore, by the theorem proved in §1.5, the map $\chi_{K\times L}$ is an identification map if at least one of the decompositions K and L is locally finite or if both these decompositions are locally countable. Since the pre-decomposition $K\times L$ is obviously point-wise finite, it is thus proved that

Proposition 3.44. *The product $K \times L$ of cellular decompositions K and L is a cellular decomposition if at least one of the decompositions K and L is locally finite or if both these decompositions are locally countable.*

In particular,

Proposition 3.45. *for any cellular decomposition K, the product $K\times I$ is also a cellular decomposition.*

Remark 3.46. As Dowker showed, there exist cellular decompositions K and L such that their product $K \times L$ is not a cellular decomposition.

3.4 Continuous maps of cellular decomposition

Let K be an arbitrary cellular decomposition and Y be an arbitrary topological space.

Since the decomposition K is a free union of sets $\overline{e}$, $e \in K$, then

Proposition 3.47. *the single-valued map*

$$f : K \to Y$$

is continuous if and only if for any cell $e \in K$ the map

$$f|_{\overline{e}} : \overline{e} \to Y$$

is continuous.

Similarly, since a cellular decomposition K is a free union of its skeletons K^n, then

Proposition 3.48. *the map*

$$f : K \to Y$$

is continuous if and only if for any finite $n \geq 0$ the map

$$f|_{K^n} : K^n \to Y.$$

is continuous.

In what follows, we will repeatedly construct maps

$$f : K \to Y.$$

by constructing, for any $n \geq 0$, "consistent" continuous maps f^n, i.e., continuous maps

$$f^n : K^n \to Y.$$

such that

$$f^{n+1}|_{K^n} = f^n$$

for any $n \geq 0$, and setting for any point $x \in K$

$$f(x) = f^n(x), \quad \text{if} \quad x \in K^n.$$

According to the previous assertion,

Proposition 3.49. *the map*

$$f : K \to Y.$$

constructed in this way is continuous.

All these statements are applicable, in particular, to the decomposition $K \times I$ and to continuous maps $K \times I \to Y$, i.e., to homotopies $f_t : K \to Y$, $0 \leq t \leq 1$. Since the decomposition $K \times I$ is obviously a free union of sets $\overline{e} \times I$, we obtain from this that

Proposition 3.50. *the family of maps*

$$f_t : K \to Y, \quad 0 \le t \le 1,$$

is a homotopy if and only if for any cell $e \in K$ the homotopy is the family

$$f_t|_{\overline{e}} : \overline{e} \to Y, \quad 0 \le t \le 1,$$

and also if and only if for any $n \ge 0$ the homotopy is the family

$$f_t|_{K^n} : K^n \to Y.$$

Moreover,

Proposition 3.51. *if for all $n \ge 0$ we are given "consistent" skeleton homotopies, i.e., homotopies*

$$fn_t : K^n \to Y, \quad 0 \le t \le 1,$$

such that

$$f_t^{n+1}|_{K^n} : \overline{e} \to Y, \quad 0 \le t \le 1,$$

for any $n \ge 0$, then, setting for each point $x \in K$ and any $t \in I$

$$f_t(x) = f_t^n(x), \quad \text{if} \quad x \in K^n,$$

we obtain some homotopy

$$f_t : K \to Y, \quad 0 \le t \le 1.$$

Sometimes we will consider not the sequence $K^0, K^1, \dots, K^n, \dots$ of skeletons, but an arbitrary increasing sequence

$$K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$$

of sub-decompositions of the decomposition K, the union of which is the entire decomposition K. Since the decomposition K is the free union of any such sequence, then

Proposition 3.52. *All the assertions proved above remain valid even if the sequence of skeletons is replaced by an arbitrary increasing sequence of decompositions of the decomposition K, the union of which is this entire decomposition.*

On the other hand, since any closed (or open) subspace X of the decomposition K is a free union of sets $X \cap \overline{e}$, $e \in K$, then everything said above is true (with obvious modifications) for any such space. Thus, for example,

Proposition 3.53. *the map*

$$f : X \to Y$$

is continuous if and only if for any cell $e \in K$ the map

$$f|_{X \cap \overline{e}} : X \cap \overline{e} \to Y$$

is continuous, and also if and only if for any $n \geq 0$ the map

$$f|_{X\cap K^n} : X \cap K^n \to Y$$

is continuous.

A pair (K, L) consisting of a cellular decomposition K and an arbitrary sub-decomposition L of it will be called a *cellular pair.* Using the above remarks, one can easily prove that the extension property of maps of spheres, which underlies the definition of m-connected spaces (see §2.1), holds for any cellular pairs (K, L) for which $\dim(K \setminus L) \leq m+1$, i.e., for which $\dim e \leq m+1$ for each cell $e \in K \setminus L$. Namely,

Proposition 3.54. *if $\dim(K \setminus L) \leq m + 1$, then any map $f : L \to Y$ of a sub-decomposition L into an m-connected ($m \geq 0$) space Y can be extended to some map $g : K \to Y$.*

Proof. Indeed, let

$$K_n = K^n \cup L.$$

For any $n = 0, 1, \dots, m + 1$, we construct a continuous map

$$g_n : K_n \to Y,$$

such that $g_n|_L = f$, $g_{n+1}|_{K_n} = g_n$. Then the map g_{m+1} will be the desired map g (since, according to the condition, $K_{m+1} = K$).

We will construct the mapping g_0 by arbitrarily defining it on the vertices of the decomposition K that do not belong to the sub-decomposition L (on L, it must, of course, coincide with f). Clearly, this map is continuous.

Let g_n already be constructed for some non-negative $n \leq m$. If $K_{n+} = K_n$, then we set ${}_{n+1} = g_n$. Let ${}_{n+1} \neq K_n$ and let e^{n+1} be an arbitrary $n+1$-dimensional cell of the decomposition K that does not belong to the sub-decomposition L. Having chosen a characteristic mapping for each such cell,

$$\chi_{e^{n+1}} : \mathbb{E}^{n+1} \to K,$$

we consider the composition

$$g_n \circ \chi_{e^{n+1}}|_{\mathbb{S}^n} : \mathbb{S}^n \to Y,$$

of the restriction $\chi_{e^{n+1}}|_{\mathbb{S}^n}$ of the map $\chi_{e^{n+1}}$ to the sphere $\mathbb{S}^n$ and the ma g_n. Since the space Y is, by assumption, m-connected, and $n \leq m$, we can extend this composition to some map

$$\chi_{e^{n+1}} : \mathbb{E}^{n+1} \to Y.$$

For any point $x \in K_{n+1}$, we now set

$$g_{n+1}(x) = \begin{cases} g_n(x), & \text{if} \quad x \in K_n, \\ \chi_{e^{n+1}}(\chi_{e^{n+1}}^{-1}(x)) & \text{if} \quad x \notin K_n, \end{cases}$$

where e^{n+1} is the cell of the decomposition K containing the point x. It is clear that we thereby obtain a single-valued continuous map

$$g_{n+1} : K_{n+1} \longrightarrow Y,$$

possessing all the required properties. □

Let us now show that

Proposition 3.55. *any cellular pair (K, L) satisfies the axiom of homotopy extension.*

Proof. Indeed, let Y be an arbitrary space, $f : K \to Y$ an arbitrary map, and $g_t : L \to Y$, $0 \le t \le 1$, be a homotopy such that $g_0 = f|_L$. We must construct a homotopy $f_t : K \to Y$ for which $f_0 = f$ and $f_t|_L = g_t$ for any $t \in I$. Let $K_n = Kn \cup L$. According to the above, to construct a homotopy f_t, it is sufficient to construct for any $n \ge 0$ a homotopy

$$f_t^n : K_n \longrightarrow Y,$$

such that

$$f_0^n = f|_{K_n}, \quad f_t^n|_L = g_t, \quad \text{and} \quad f_t^{n+1}|_{K_n} = f_t^n.$$

For any point $x \in K_0$ we define a homotopy f_t^0 by the formula

$$f_t^0 = \begin{cases} g_t(x), & \text{if} \quad x \in L, \\ f(x), & \text{if} \quad x \in K^0. \end{cases}$$

It is clear that this formula indeed defines a homotopy, for which $f_0^0 = |_{K_0}$ and $f_t^0 = g_t$ for any $t \in I$.

Let for $n \ge 0$ we have already constructed a homotopy f_t^n. If $K_{n+1} = K_n$, then we set $f_t^{n+1} = f_t^n$, $0 \le t \le 1$. Let $K_{n+1} \ne K_n$ and let e^{n+1} be an arbitrary $n+1$-dimensional cell of K that does not belong to L. Having chosen for each such cell e^{n+1} a characteristic map

$$\chi_{e^{n+1}} : \mathbb{E}^n \longrightarrow K,$$

For each point $x = \chi_{e^{n+1}}(v\boldsymbol{u})$, $0 \le v \le 1$, $|\boldsymbol{u}|$, of the set $\overline{e}^{n+1}$ and any number $t \in I$ we set

$$g_{e^{n+1},t}(x) = \begin{cases} f\left(\chi_e^{n+1}\left(\frac{2v}{2-t}\boldsymbol{u}\right)\right), & \text{if} \quad 0 \le v \le \frac{2-t}{2}, \\ f_{2v+t-2}^n(\chi_e^{n+1}(\boldsymbol{u})), & \text{if} \quad \frac{2-t}{2} \le v \le 1. \end{cases}$$

It is easy to see (Fig. 3.1) that this formula uniquely defines a certain map

$$g_{e^{n+1},t} : \overline{e}^{n+1} \longrightarrow Y$$

(depending on the choice of the map χ). Moreover, since the family of maps

$$g_{e^{n+1},t} \circ \chi_{e^{n+1}} : \mathbb{E}^{n+1} \longrightarrow Y$$

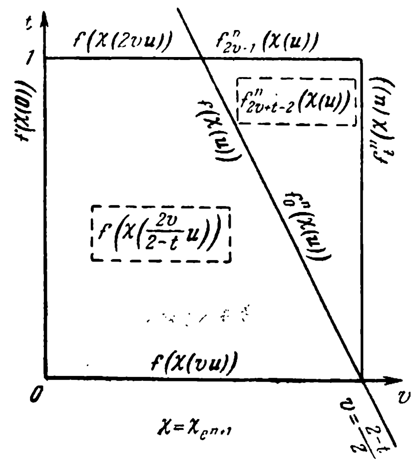


Figure 3.1:

is obviously a homotopy, the family of maps $g_{e^{n+1},t}$ is also a homotopy (since the characteristic map $\chi_{e^{n+1}}$ considered as the map $\mathbb{E}^{n+1} \longrightarrow \overline{e}^{n+1}$ is an identity map, and the segment I is compact, so the map $\chi_{e^{n+1}} \times 1 : \mathbb{E}^{n+1} \times I \longrightarrow \overline{e}^{n+1} \times I$ is also an identity map). This homotopy has the property that

$$g_{e^{n+1},0} = f|_{\overline{e}^{n+1}} \quad \text{and} \quad g_{e^{n+1},t}|_{\dot{e}^{n+1}} = f^n_t|_{\dot{e}^{n+1}}$$

for any $t \in$. Therefore, the formula

$$f^{n+1}_t = \begin{cases} f^n_t(x), & \text{if} \quad x \in K_n, \\ g_{e^{n+1},t}(x) & \text{if} \quad x \in e^{n+1} \in K_{n+1} \setminus K_n, \end{cases}$$

defines a certain homotopy $f^{n+1}_t : K_{n+} \longrightarrow Y$, $0 \leq t \leq 1$, which obviously has all the required properties.

Thus, the homotopies f^n_t are constructed, by induction, for all $n \geq 0$. The proposition formulated above is thus completely proven. □

Remark 3.56. A special case of the proved proposition is the statement formulated in §2.4, since all pairs listed in this statement, as is easy to see, are cellular.

We see, therefore, that all the results of §2.3 are applicable to arbitrary cellular pairs (K, L), since, together with the pair (K, L), the pair $(K \times I, I(K, L))$ is also cellular. Thus, firstly,

Proposition 3.57. *for any cellular pair* (K, L) *of any space* Y *and any two homotopic maps*

$$f, g : L \longrightarrow Y$$

the spaces $K \cup_f Y$ and $K \cup_g Y$ are homotopically equivalent moreover, the corresponding homotopic equivalence can be chosen so that it is the identity map on K,

and, secondly,

Proposition 3.58. *For any cell pair (K, L), any space Y, any continuous map*

$$f : L \to Y$$

and any space Z homotopically equivalent to Y, every homotopy equivalence

$$h : Y \to Z$$

can be extended to some homotopy equivalence

$$H : K \cup_f Y \to K \cup_{h \circ f} Z.$$

Generally speaking, the space $K \cup_f Y$ is not a cellular decomposition even in the case when the space Y is a cellular decomposition. One can only say that

Proposition 3.59. *for any cellular pair (K, L), any cellular decomposition Q and any continuous map*

$$f : L \to Q$$

the space $K \cup_f Q$ is homotopy equivalent to some cellular decomposition $\widetilde{Q}$ that contains the decomposition Q as a sub-decomposition, and the homotopy equivalence $K \cup_f Q \to \widetilde{Q}$ can be chosen such that it is the identity map on Q.

To prove this statement, we introduce continuous maps

$$f : K \to Q$$

of a cellular decomposition K into a cellular decomposition Q, with the property that

$$f(K^n) \subset Q^n$$

for any $n \geq 0$. We will call such maps of cellular decompositions *cellular*. The statement formulated above will obviously be proven if we show, first, that

Proposition 3.60. *any continuous map $K \to Q$ is homotopic to some cellular map $K \to Q$*

and, secondly, that

Proposition 3.61. *For any cellular pair (K, L), any cellular decomposition Q, and any cellular map,*

$$f : L \to Q$$

the space $K \cup_f Q$ is a cellular decomposition containing the decomposition Q as a sub–decomposition.

Let us first consider the second statement. Since the cellular decompositions K and Q are, as we know, normal spaces, the space $K \cup_f Q$ is Hausdorff (§1.6). Moreover, since the natural projection $\alpha : K \cup Q \to K \cup_f Q$ onto $K \setminus L$ and Q is a homeomorphism, then for any cell $e \in (K \setminus L) \cup Q$ (with characteristic map χ) the set $\alpha(e)$ is a cell of the space $K \cup_f Q$ (with characteristic map $\alpha \circ \chi$). Moreover, given that the map f is cellular, all cells $\alpha(e)$, $e \in (K \setminus L) \cup Q$ constitute a cellular pre-decomposition of this space.

Let us show that this subdivision is point-finite, i.e., that any cell $\alpha(e)$ of it belongs to some finite subdivision. For $e \in Q$, this is obvious, since the map $\alpha|_Q$ is homeomorphic, and therefore the image $\alpha(Q(e))$ of the sub-decomposition $Q(e)$ under the map α is a finite sub-decomposition of the pre-decomposition $K \cup_f Q$, containing the cell $\alpha(e)$. Let $e \in K \setminus L$. Since the sub-decomposition $K(e) \cap L$ is finite, it is compact, and therefore its image $f(K(e) \cap L) \subset Q$ under the map f is contained in some finite sub-decomposition Q_e of the decomposition Q. Consider the set $\alpha(K(e) \cup Q_e)$. It is obviously compact. Furthermore, it is the union of cells of the form $\alpha(e)$, $e \in (K(e) \setminus L) \cup Q_e$. Therefore, this set is a finite sub-decomposition of the pre-decomposition $K \cup_f Q$, containing the cell $\alpha(e)$.

Thus, to prove the statement under consideration, we only need to prove that the topology of the space $K \cup_f Q$ is a weak topology. As we know, for this it is sufficient to prove (see §3.1), that the set $A \subset K \cup_f Q$ is closed if its intersection $a \cap P$ with any finite sub-decomposition $P \subset K \cup_f Q$ is closed. At the same time, since the natural projection α is an identification map, and the topological sum $K \cup Q$ is a cellular decomposition, the set $A \subset KK \cup_f Q$ is closed if and only if for any cell $e \in K \cup Q$ the intersection $\alpha^1(A) \cap \overline{e}$ is closed. Thus, we need to prove that if the intersection $A \cap P$ of some set $A \subset K \cup_f Q$ with any finite sub-decomposition $P \subset K \cup_f Q$ is closed, then for any cell $e \in K \cup Q$ the intersection $\alpha^{-1}(A) \cap \overline{e}$ is closed. It is easy to see that this fact will be proved if we show that for any cell $e \in K \cup Q$ the set $\alpha(e)$ is contained in some finite sub-decomposition P_e of the pre-decomposition $K \cup_f Q$.

Indeed, since the subdivision P_e, being finite, is closed, and $\alpha(e) \subset P_e$, then $\alpha(\overline{e}) \subset \overline{\alpha(e)} \subset P_e$, and therefore $\overline{e} \subset \alpha^{-1}(P_e)$. Therefore,

$$\alpha^{-1}(A) \cap \overline{e} \subset \alpha^{-1}(A) \cap \alpha^{-1}(P_e) \cap \overline{e} \subset \alpha^{-1}(A \cap P_e) \subset \alpha^{-1}(A) \cap \overline{e},$$

and therefore

$$\alpha^{-1}(A) \cap \overline{e} = \alpha^{-1}(A \cap P_e) \cap \overline{e}.$$

By assumption, the right-hand side of this equality is closed. Therefore, its left-hand side is also closed.

Thus, we only need to prove that for any cell $e \in K \cup Q$ there exists a finite sub-decomposition $P_e \subset K \cup_f Q$ such that $\alpha(e) \subset P_e$. But if $e \notin L$, then the set $\alpha(e)$ is a cell of the pre-decomposition $K \cup_f Q$ and therefore, as has already been proved, is contained in some finite sub-pre-decomposition P_e of this pre-decomposition. Let $e \in L$. Since the set $f(\overline{e}) \subset Q$ is compact, it is contained in some finite sub-decomposition of the decomposition Q. Therefore, in this case, the set $\alpha(e) = f(e) \subset f(\overline{e})$ (generally speaking, no longer a cell) is contained in some finite sub-decomposition P_e of the pre-decomposition $K \cup_f Q$.

Thus, the statement that the pre-decomposition $K \cup_f Q$ is a decomposition is completely proven.

In particular, we see that

Proposition 3.62. *for any cellular pair (K, L) the space K/L is a cellular decomposition.*

Similarly,

Proposition 3.63. *the cylinder Z_f of any cellular map $f : K \to L$ is a cellular decomposition (with cells of the form $e \in K$, $e \in L$ and $e \times (0, 1)$, $e \in K$).*

Remark 3.64. For the above statement to be true, the requirement that the map f be cellular is not necessary. It is sufficient that this map have the property that $f(L \cap \dot{e} \subset Q^{n-1}$ for any cell e $K \setminus L$, where $n = \dim e$.

As for the first of the above statements, it can even be strengthened somewhat. Namely, it turns out that

Proposition 3.65. *any continuous map $K \to Q$ that is a cellular map on some sub-decomposition $L \subset K$ is homotopic to the cellular map $K \to Q$* rel L.

We will prove this fundamental theorem about cellular maps in the next section. We will dedicate the end of this section to deducing one important consequence from it.

A homotopy

$$f_t : K \to Q, \quad 0 \le t \le 1,$$

will be called *cellular* if the maps f_0 and f_1 are cellular and if

$$f_t(K^n) \subset Q^{n+1} \quad \text{for any} \quad n \ge 0.$$

Clearly, a homotopy $f_t : K \to Q$, $0 \le t \le 1$, is cellular if and only if the corresponding map $K \times I \to Q$ is a cellular map. Cellular maps associated by a cellular homotopy will be called *cellularly homotopic.*

It follows from the theorem on cellular mappings that

Proposition 3.66. *if cellular maps*

$$f, g : K \to Q$$

are homotopic relative to some (possibly empty) sub-decomposition $L \subset K$, then they are also cellularly homotopic rel L.

Proof. Indeed, the statement that the maps f and g are homotopic rel L means that there exists a map

$$F : K \times I \to Q,$$

such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x)$$

for any point $x \in K$ and

$$F(x,t) = f(x) = g(x)$$

for any points $x \in L$ and $t \in I$. It is clear that the map F is cellular on the sub-decomposition

$$I(K,L) = L \times \cup K \times I \cup L \times 1$$

of the decomposition $K \times I$ and therefore, according to the theorem on cellular maps, it is homotopic relative to this sub-decomposition to some cellular map

$$G : K \times I \to Q.$$

The family of maps corresponding to G

$$g_t : K \to Q, \quad 0 \le t \le 1,$$

obviously represents a cellular homotopy rel L, connecting the maps f and g. □

Thus, when studying the homotopy properties of cellular decompositions, we can, without loss of generality, restrict ourselves to considering only cellular maps and their cellular homotopies.

Let us apply, for example, this remark to the study of homotopy groups $\pi_n(X; x_0)$ in the case where the space X is a cellular decomposition of K. Here we can obviously assume that the point x_0 is some vertex e^0 of the decomposition K.

Having agreed to regard the sphere $\mathbb{S}^n$ as a cellular decomposition with a zero-dimensional cell $\boldsymbol{u}_0$ and an n-dimensional cell $\mathbb{S}^n \setminus \boldsymbol{u}_0$, we apply to an arbitrary map

$$\varphi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (K, e^0)$$

the theorem on cellular maps. Since the points $\boldsymbol{u}_0$ and e^0 are vertices, this map is cellular on $\boldsymbol{u}_0$. Therefore, according to the theorem on cellular maps, it is homotopic rel $\boldsymbol{u}_0$ to some cellular map

$$\psi : (\mathbb{S}^n, \boldsymbol{u}_0) \to (K, e^0).$$

But the last map, being cellular, is a map of the sphere $\mathbb{S}^n$ into the n-dimensional skeleton K^n of the decomposition K. Therefore, any map $(\mathbb{S}^n, \boldsymbol{u}_0) \to (K, e^0)$ is homotopic rel $\boldsymbol{u}_0$ to a map of the form $i \circ \psi$, where $i : K^n \to K$ is an inclusion, and ψ is some map $(\mathbb{S}^n, \boldsymbol{u}_0) \to (K^n, e^0)$. This means that

Proposition 3.67. *the inclusion $i : K^n \to K$ defines an epimorphism*

$$i_* : \pi_n(K^n; e^0) \to \pi_n(K; e^0)$$

of the group $\pi_n(K^n; e^0)$ onto the group $\pi_n(K; e^0)$.

Similarly, using the fact that homotopic cellular maps are cellularly homotopic, we obtain that

Proposition 3.68. *the group $\pi_n(K; e^0)$ is isomorphic to the group $\pi_n(K^{n+1}; e^0)$ (and, in general, to any group $\pi_n(K^m; e^0)$ with $m \geq n+1$).*

In particular, it directly follows from this general statement that

Proposition 3.69. *for any $m < n$, the following equality holds:*

$$\pi_n(\mathbb{S}^m, \boldsymbol{u}_0) = 0,$$

i.e., that the sphere $\mathbb{S}^m$ is an $(m-1)$-connected space.

We emphasise, however, that we obtained this statement by relying on the theorem about cellular maps. On the other hand, in the next section, when proving the last theorem, we will make substantial use of this statement. Therefore, to avoid a vicious circle, we will be forced to provide an independent proof of it there.

3.5 Proof of the theorem on cellular maps

Let Q be an arbitrary cellular decomposition and

$$\varnothing = Q_1 \subset Q_0 \subset \cdots \subset Q_n \subset \cdots$$

be an arbitrary increasing sequence of its sub-decompositions (the union of these sub-decompositions may be the decomposition Q or may not). A map

$$f : K \to Q$$

of some cellular decomposition K into a decomposition Q will be called *subordinate to* the sequence $\{Q_n\}$ if

$$f(K^n) \subset Q_n$$

for any $n \geq 0$. Accordingly, we will say that for a sequence $\{Q_n\}$ *the theorem on subordinate maps holds* if for any cellular pair (K, L) each map $f : K \to Q$ whose restriction $f|_L$ is subordinate to $\{Q_n\}$ is homotopic rel L to the subordinate $\{Q_n\}$ map $g : K \to Q$.

In the case where the sequence $\{Q_n\}$ consists of skeletons Q^n of the decomposition Q, the subordination of the mapping f means its cellularity.

Thus, the theorem on cellular maps formulated in the previous section means that

Proposition 3.70. *for the sequence $\{Q_n\}$ of skeletons of the decomposition Q, the theorem on subordinate maps holds.*

The basis of the proof of the theorem on cellular mappings is the fact that

Proposition 3.71. *if the sequence $\{Q_n\}$ is such that for any n^0 every mapping $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q, Q_{n-1})$ is homotopic* rel $\mathbb{S}^{n-1}$ *to some map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q_n, Q_{n-1})$, then the theorem on subordinate maps holds for the sequence $\{Q_n\}$.*

In other words, if the sequence $\{Q_n\}$ has the indicated property, then every continuous map $f : K \to Q$ subordinated on L to the sequence $\{Q_n\}$ is homotopic rel L to some map $g : K \to Q$ subordinated to the sequence $\{Q_n\}$.

Proof. According to the remarks made at the beginning of the previous section, to construct a homotopy

$$f_t : K \to Q \text{ rel } L, \quad 0 \le t \le 1,$$

connecting the map f with the map g, it is sufficient for us to construct for each $n \ge 0$ a homotopy

$$f_t^n : K \to Q, \quad 0 \le t \le 1,$$

such that

$$f_0^n = f|_{K^n}, \quad f_1^n(K^n) \subset Q_n, \quad f_t^n|_{L^n} = f|_{L^n}, \quad f_t^{n+1}|_{K^n} = f_t^n$$

for a $n \ge 0$ and $t \in I$.

By hypothesis, every map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q, Q_{n-1})$ is a homotopy rel $\mathbb{S}^{n-1}l$ to some map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q_n, Q_{n-1})$. For $n = 0$, this means that for any point $y \in Q$, there exists a path $u_y : I \to Q$ such that $u_y(0) = y$ and $u_y(1) \in Q_0$. For every point $x \in K^0$ and any number $t \in I$ we put

$$f_t^0(x) = \begin{cases} u_{f(x)}, & \text{if } \; x \notin L, \\ f(x), & \text{if } \; x \in L. \end{cases}$$

It is clear that this will give us a homotopy

$$f_t^0 : K^0 \to Q, \quad 0 \le t \le 1,$$

for which

$$f_0^0 = f|_{L^0}, \quad f_1^0(K^0) \subset Q_0, \quad f_t^0|_{L^0} = f|_{L^0}.$$

Let for $n \ge 0$ a homotopy f_t^n has already been constructed. If $K^{n+1} = K^n$, then we set $f_t^{n+1} = f_t^n$. Let $K^{n+1} \ne K^n$. Since the pair (K^{n+1}, K^n), being cellular, satisfies the axiom of homotopy extension, there exists a homotopy

$$g_t : K^{n+1} \to Q, \quad 0 \le t \le 1,$$

such that

$$g_0 = f|_{K^{n+1}}, \quad \text{and} \quad g_t|_{K^n} = f_t^n$$

for any $t \in I$.

Let's consider the map

$$g_1 : K^{n+1} \to Q.$$

Since $g_1|_{K^n} = f_t^1$, then

$$g_1(K^n) \subset Q_n.$$

Let e^{n+1} be an arbitrary $n+1$-dimensional cell of the decomposition K and let

$$\chi : \mathbb{E}^{n+1} \to K$$

be an arbitrary characteristic map of this cell. Since $\dot{e}^{n+1} \subset K$ and $g_1(K^n) \subset Q_n$, then

$$g_1 \circ \chi : (\mathbb{E}^{n+1}, \mathbb{S}^n) \to (Q, Q_n)$$

Therefore, according to the condition, there exists a homotopy

$$\xi_t : \mathbb{E}^{n+1} \to Q \ \text{ rel } \ \mathbb{S}^n,$$

such that

$$\xi_0 = g_1 \circ \chi, \quad \xi_1(\mathbb{E}^{n+1}) \subset Q_{n+1}.$$

For any point $\boldsymbol{x} = \chi(v\boldsymbol{u})$, $0 \le v \le 1$, $|\boldsymbol{u}| = 1$, any cell $e^{n+1}1$, and any number $t \in I$, we set

$$h_{e^{n+1},t}(x) = \begin{cases} g_{\frac{2t}{1+v}}(\chi(v\boldsymbol{u})), & \text{if} \quad 0 \le t \le \frac{1+v}{2}, \\ \xi_{\frac{2t-v-1}{1-v}}(v\boldsymbol{u}), & \text{if} \quad \frac{1+v}{2} \le t \le 1. \end{cases}$$

It is easy to verify (Fig. 3.2) that we thereby obtain a certain uniquely defined

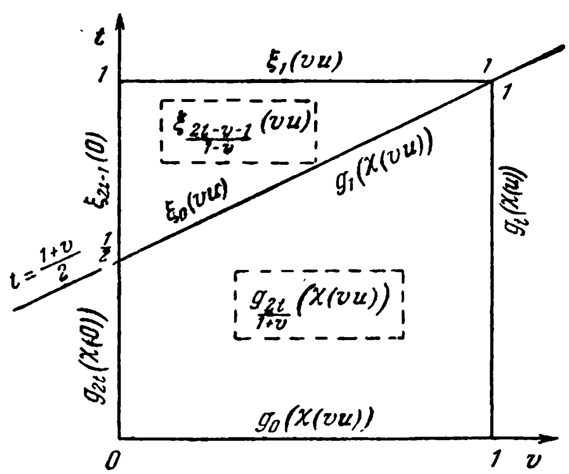


Figure 3.2:

homotopy

$$h_{e^{n+1},t} : \overline{e}^{n+1} \to L, \quad 0 \le t \le 1.$$

To construct a homotopy f_t^{n+1}, we now consider an arbitrary point $x \in K^{n+1}$. In the case where $x \notin K^n$, the point x belongs to some uniquely determined $n+1$-dimensional cell $e_x^{n+1} \in K$. For any number $t \in I$

$$f_t^{n+1}(x) = \begin{cases} f_t^n(x), & \text{if} \quad x \in K^n, \\ f(x), & \text{if} \quad x \notin K^n \quad \text{and} \quad e_x^{n+1} \in L, \\ h_{e_x^{n+1}}(x), & \text{if} \quad x \notin K^n \quad \text{and} \quad e_x^{n+1} \notin L. \end{cases}$$

It is easy to see that this gives us a homotopy

$$f_t^{n+1} : K^{n+1} \to Q,$$

such that

$$f_0^{n+1} = f|_{K^{n+1}}, \quad f_1^{n+1}(K^{n+1}) =\subset Q_{n+1}$$

and

$$f_t^{n+1}|_{L^{n+1}} = f_{K^{n+1}}, \quad f_t^{n+1}|_{K^n} = f_t^n \text{ for any } t \in I.$$

Thus, the homotopies f_t^n are constructed for all $n \geq 0$.

The proposition formulated above is completely proved. □

Now consider the following statement:

Proposition 3.72 (A_n)**.** *Any map*

$$(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q, Q^{n-1})$$

is homotopic rel $\mathbb{S}^{n-1}$) *to some map*

$$(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q^n, Q^{n-1}).$$

According to the proposition just proved,

Proposition 3.73. *To prove the theorem on cellular maps, it suffices to prove statement (A_n) for any $n \geq 0$.*

Remark 3.74. For $n = 0$, the statement (A_n) is true (since in this case it simply states that any point of the partition Q can be connected by a path to some of its vertices). Thus, this statement requires proof only for $n > 0$.

A pair (X, X_0) consisting of some Hausdorff space X and its closed subspace X_0, which has the property that the complement $X \setminus X_0$ is an m-dimensional $(m \geq 0)$ cell e^m, we will call an *m-dimensional relative cell.*

Consider the following statement:

Proposition 3.75 (B_n)**.** *For any $m > n$ and any m-dimensional relative cell (X, X_0), every continuous map f of the ball $\mathbb{E}^n$ into the space X that maps the sphere $\mathbb{S}^{n-1}$ into the subspace X_0, i.e., a map of the pair $(\mathbb{E}^n, \mathbb{S}^{n-1})$ into the pair (X, X_0), is homotopic* rel $\mathbb{S}^{n-1}$ *to some map $\mathbb{E}^n \to X_0$.*

It is easy to see that

Proposition 3.76. *if for some $n \geq 0$ statement B_n is true, then statement A_n is also true (for any cellular decomposition Q and any map $f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q, Q^{n-1})$).*

Proof. Indeed, let assertion (B_n) be true and let f be an arbitrary map of the ball $\mathbb{E}^n$ into some cellular decomposition Q that takes the sphere $\mathbb{S}^{n-1}$ to the $(n-1)$-th skeleton Q^{n-1}. Since the ball $\mathbb{E}^n$ is compact, its image $f(\mathbb{E}^n)$ under the map f is also compact and therefore is contained in some finite sub-decomposition of the decomposition Q. Therefore, without loss of generality, we can assume from the outset that Q is finite.

Using the finiteness of the decomposition Q, we will prove the statement (A_n) for it by induction on its dimension $m = \dim Q$ and on the number $a_m(Q)$ of its m-dimensional cells. If $m = n$, then the assertion (A_n) is trivially true. Assuming that this assertion has already been proven for all finite decompositions of Q for which either $\dim Q < m$, where $m > n$, or the number $a_m(Q)$ is less than some positive integer k, consider an arbitrary decomposition of Q for which $\dim Q = m$ and $a_m(Q) = k$. By choosing an arbitrary m-dimensional cell $e^m \in Q$ and setting $Q_0 = Q \setminus e^m$, we obviously obtain an m-dimensional relative cell (Q, Q_0). Since any map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q, Q^{n-1})$ is automatically a map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (Q, Q_0)$, then, according to the statement (B_n), each such map is homotopic rel $\mathbb{S}^{n-1})$ to some map $g : \mathbb{E}^n \to Q_0$.

On the other hand, it is clear that the subspace Q_0 is a cellular decomposition such that either $\dim Q_0 < m$ or $a_m(Q_0) < k$. Therefore, by the induction hypothesis, the map g is homotopic with rel $\mathbb{S}^{n-1})$ to some map $h : \mathbb{E}^n \to Q_0^n$. Consequently, the map f is also homotopic rel $\mathbb{S}^{n-1})$ to the map h. Thus, assertion (A_n) is completely proved. □

Thus, to prove the theorem on cellular maps, we only need to prove the statement (B_n) for any $n \geq 0$.

Remark 3.77. Like the statement (A_n), the statement (B_n) is trivially true when $n = 0$ (since the set $\overline{e}^m$ is path-connected).

Now consider the following statement:

Proposition 3.78 (C_n)**.** *For any $m > n$ and any m-dimensional relative cell (X, X_0), every map $f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, X_0)$ is homotopic* rel $\mathbb{S}^{n-1}$ *to some map $g : \mathbb{E}^n \to X$ with the property that the image $g(\mathbb{E}^n)$ of the ball $\mathbb{E}^n$ under g does not contain the entire cell $e^m = X \setminus X_0$.*

It is easy to see that

Proposition 3.79. *statement (C_n) implies statement (B_n).*

Proof. Indeed, let the statement (C_n) be true and let x_0 be an arbitrary point of the cell e^m that does not belong to the set $g(\mathbb{E}^n)$. Let us choose a map

$$\chi : \mathbb{E}^m \to \overline{e}^m,$$

characteristic of the cell e^m such that $\chi(\mathbf{0}) = x_0$ (it is clear that such a map can always be found). It is easy to see that, by setting

$$g(x) = \chi : ((v(x), \boldsymbol{u}(x)), \quad x \in G$$

(where g is the map $\mathbb{E}^n \to X$ specified by the statement (C_n)), we uniquely define two continuous maps on the open subset $G = g^{-1}(e^m)$ of the ball $\mathbb{E}^n$:

$$v : G \to I, \quad \boldsymbol{u} : G \to \mathbb{S}^{m-1}$$

(since, by hypothesis, $\chi(\mathbf{0}) \notin g(\mathbb{E}^n)$, then $v(x) \neq 0$ for each point $x \in G$ and therefore the map $\boldsymbol{u} : G \to \mathbb{S}^{m-1}$ is uniquely defined). For any point $x \in \mathbb{E}^n$ and any number $t \in I$, we now set

$$g_t(x) = \begin{cases} \chi(((1-t)v(x) + t)\boldsymbol{u}(x)), & \text{if} \quad x \in G, \\ g(x), & \text{if} \quad x \notin G \end{cases}$$

It is clear that we thereby obtain a homotopy $g_t : \mathbb{E}^n \to X$ rel $\mathbb{S}^{n-1}$, for which $g_0 =$ and $g_1(\mathbb{E}^n) \subset X_0$. Thus, the map g, and hence the map f, is homotopic rel $\mathbb{S}^{n-1}$ to some map $\mathbb{E}^n \to X_0$. Assertion (B_n) is thus completely proved. □

Thus, all that remains for us to prove is statement (C_n) for all $n > 0$.

To this end, we consider the following statement:

Proposition 3.80 (D_n)**.** *For any $m > n$, any n-dimensional relative cell (X, X_0), and any map $f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, X_0)$, there exists a map*

$$g : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, X_0),$$

such that

1) *the image $g(\mathbb{E}^n)$ of the ball $\mathbb{E}^n$ under the map g does not contain the entire cell $e^n = X \setminus X_0$;*

2) *the map g coincides with the map f outside some open set $U \subset \mathbb{E}^n$;*

3) *the image of the set U under each of the maps f and g is contained in the cell e^m.*

It is easy to see that

Proposition 3.81. *the map g provided by statement (D_n) is homotopic* rel $\mathbb{S}^{n-1}$ *to the map f.*

Proof. Indeed, let

$$\chi : \mathbb{E}^m \to X$$

be an arbitrary map characteristic of the cell e^m. Since, by hypothesis, $f(U) \subset e^m$ and $g(U) \subset e^m$, then single-valued continuous maps are defined on the set U

$$f^{'} \chi^{-1} \circ f : U \to \mathbb{E}^m, \quad g^{'} \chi^{-1} \circ g : U \to \mathbb{E}^m$$

For any point $\boldsymbol{v} \in \mathbb{E}^n$ and any number $t \in I$ we put

$$g_t(\boldsymbol{v}) = \begin{cases} \chi(tf^{'}(\boldsymbol{v}) + (1-t)g^{'}(\boldsymbol{v})), & \text{if} \quad \boldsymbol{v} \in U, \\ f(\boldsymbol{v}), & \text{if} \quad \boldsymbol{v} \notin U. \end{cases}$$

Clearly, we thereby obtain a homotopy $g_t : \mathbb{E}^n \to X$ rel $\mathbb{S}^{n-1}$, connecting the map f with the map g. □

By condition 1) of assertion (D_n), the proven proposition means that

Proposition 3.82. *the truth of assertion (D_n) implies the truth of assertion (C_n).*

Thus, to prove the theorem on cellular mappings, we only need to prove statement (D_n) for any $n > 0$. We will carry out this proof by induction on the number n. First, we will consider the case $n = 1$.

Proof. (the case $n = 1$) Let x_0 be an arbitrary interior point of the cell e^m and let F be its complete preimage $f^{-1}(x_0)$ under the map $f : \mathbb{E}^1 \to X$. The set F is a closed subset of the segment $\mathbb{E}^1 = [-1, 1]$ that does not contain its endpoints. Since the map f is continuous and the cell e^m is open in the space X, on the segment $\mathbb{E}^1$ there exists a finite system of open intervals (a_i, b_i), $i = 1, \dots, k$ such that their union U contains the set F and goes under the map f into the cell e^m. Since the set $e^m \setminus x_0$ is linearly connected (since $m > 1$), then for any $i = 1, \dots, k$ there exists a map g_i of the closed segment $[a_i, b_i]$ into the set $e^m \setminus x_0$ such that $g_i(a_i) = f(a_i)$ and $g_i(b_i) = f(b_i)$. But then it is clear that the map $g : \mathbb{E}^1 \to X$ defined by the formula

$$g(\mathbf{v}) = \begin{cases} g_i(\mathbf{v}), & \text{if} \quad \mathbf{v} \in (a_i, b_i), \\ f(\mathbf{v}), & \text{if} \quad \mathbf{v} \notin U, \end{cases}$$

is continuous and satisfies all the conditions of assertion (D_1). Therefore, assertion (D_1) is completely proved. □

Now suppose that for some $n > 1$ the assertions $(D_1, \dots, D_{n-1})$ have already been proved, and we prove the assertion (D_n).

To this end, we first note that

Proposition 3.83. *the validity of the assertions $(D_1, \dots, D_{n-1})$ implies that for any $m > n$ the sphere $\mathbb{S}^m$ is an $n-1$-connected space.*

Since the statement (D_k), $k = 1, \dots, n-1$, implies the statement (B_k), to prove this proposition it suffices to prove that

Proposition 3.84. *from the validity of the statement (B_k) it follows that for any $m > k$ the sphere $\mathbb{S}^m$ is a k-aspherical space.*

Proof. Let g be an arbitrary map $\mathbb{S}^k \to \mathbb{S}^m$. Having chosen some map

$$\chi : \mathbb{E}^k \to \mathbb{S}^k,$$

that maps the sphere $\mathbb{S}^{k-1} \subset \mathbb{E}^k$ to the point $\mathbf{u}_0 = (1, 0, \dots, 0) \in \mathbb{S}^k$ and homeomorphically maps the open ball $\dot{\mathbb{E}}^k = \mathbb{E}^k \setminus \mathbb{S}^{k-1}$ to the set $\mathbb{S}^k \setminus \mathbf{u}_0$ (i.e., a characteristic ma of the cell $\mathbb{S}^k \setminus \mathbf{u}_0$), we consider the map

$$f = g \circ \chi : (\mathbb{E}^k, \mathbb{S}^{k-1}) \to (\mathbb{S}^m, \mathbf{w}_0),$$

where $w_0 = g(u_0) \in \mathbb{S}^m$. Since the pair $(\mathbb{S}^m, w_0)$ is obviously an m-dimensional relative cell and $m > k$, assertion (B_k) applies to this map. Therefore, it is homotopic rel $\mathbb{S}^{k-1}$ to the map of the ball $\mathbb{E}^k$ to the point w_0. Let

$$f_t : \mathbb{E}^k \to \mathbb{S}^m$$

be the corresponding homotopy. Clearly, for any $t \in I$, the map

$$g_t = f_t \circ \chi^{-1} : \mathbb{S}^k \to \mathbb{S}^m$$

is uniquely defined and that all these maps form a homotopy connecting the map g with the constant map $\mathbb{S}^k \to w_0$. Therefore, the sphere $\mathbb{S}^m$ is k-aspherical. □

Using the statements proved in §2.1, we immediately obtain from this that

Proposition 3.85. *from the validity of the statements $(D_1), \dots, (D_n)$ it follows that for any $m > n$ the product $\mathbb{S}^{m-1} \times (0,1)$ is an $n-1$-connected space.*

After these preliminary remarks, we can now proceed directly to the proof of the assertion (D_n) (assuming that the assertions $(D_1), \dots, (D_{n-1})$ have already been proven).

Proof. (of the assertion (D_n)) Let $m > n$ and let

$$f : (\mathbb{E}^n, \mathbb{S}^{n-1}) \to (X, X_0)$$

be an arbitrary map of the pair $(\mathbb{E}^n, \mathbb{S}^{n-1})$ into some m-dimensional relative cell (X, X_0). Let, in addition, x_0 be an arbitrary interior point of the cell $e^m = X \setminus X_0$ and let

$$F = f^{-1}(x_0)$$

be its complete preimage under the map f. Since, by hypothesis, the boundary $\mathbb{S}^{n-1}$ of the ball $\mathbb{E}^n$ is mapped by f into the subspace X_0, the set F (obviously closed) is contained in the open ball $\dot{\mathbb{E}}^n = \mathbb{E}^n \setminus \mathbb{S}^{n-1}$. By similar considerations, the set $G = f^{-1}(e^m)$ (obviously open) is also contained in the ball $\dot{\mathbb{E}}^n$. Since $F \subset G$, there exists a cubillage of the Euclidean space $\mathbb{R}^n$ (i.e., a partition of the space $\mathbb{R}^n$ into cubes by orthogonal systems of parallel hyperplanes) so small that the set F has a neighbourhood U contained in the open set G, which is the union of some open cubes of this cubillage. Let P be the closure of the set U and Q be the boundary of this closure, i.e., its intersection with the union of all closed cubes of the cubillage under consideration that do not belong to it. It is clear that the set P is a cellular decomposition of dimension n (whose cells are the open cubes of the cubillage under consideration and their open faces), and the subset Q is a cellular subdivision of it. On the other hand, it is clear that $F \cap Q = \varnothing$ and therefore

$$f(Q) \subset e^m \setminus x_0,$$

so that we can view the map $f|_Q$ as the map

$$f|_Q : Q \to e^m \setminus x_0,$$

Finally, it is obvious that the set $e^m \setminus x_0$ is homeomorphic to the product $\mathbb{S}^{n-1} \times (0,1)$ and therefore, by the induction hypothesis and the remarks made above, it is $n-1$-connected.

Thus, the mapping $f|_Q$ is a continuous map of the sub-decomposition Q of P into the $n-1$-connected space $e^m \setminus x_0$, where $\dim(P \setminus Q) \leq n$. Therefore, by the theorem proved in the previous section, there exists some extension

$$h : P \to e^m \setminus x_0.$$

for this mapping. We define the map

$$g : \mathbb{E}^n \to X$$

by putting

$$g(v) = \begin{cases} h(v), & \text{if} \quad v \in P, \\ f(v), & \text{if} \quad v \notin P \end{cases}$$

for any point $v \in \mathbb{E}^n$. Clearly, this map is continuous and satisfies all the conditions of assertion (D_n). □

Thus, the assertions (D_n) are proved for all $n > 0$. Along with them, the theorem on cellular maps is also completely proven.

Remark 3.86. Incidentally, we have proven that for any $n > 0$, the sphere $\mathbb{S}^n$ is an $(n-1)$-connected space (see the end of §3.4).

3.6 Whitehead's theorem. Quasi-polyhedra

Let us return to the proposition proved at the beginning of the previous section. Setting in this proposition $Q = K$ and $Q_0 = Q_1 = \cdots = Q_n = \cdots = K_0$, where K_0 is some sub-decomposition of the cellular decomposition K, we immediately obtain that

Proposition 3.87. *if for every $n \geq 0$ any map $(\mathbb{E}^n, \mathbb{S}^{n-1}) \to (K, K_0)$ is homotopic* rel $S\mathbb{S}^{n-1}$ *to some map $\mathbb{E}^n \to K_0$, then every map*

$$f : (K, K_0) \to (K, K_0)$$

is homotopic rel K_0 *to some map*

$$g : K \to K_0.$$

According to the assertion proved at the end of §2.4, the conditions of this proposition are satisfied if the sub-decomposition K_0 is a representative subspace of the space K. On the other hand, in the case where the map f is the identity map 1_K of the decomposition K, the map g provided by this proposition is obviously a retraction $K \supset K_0$. Thus,

Proposition 3.88. *any sub-decomposition K_0 of the decomposition K, which is its representative subspace, is a deformation retract of the space K.*

In other words,

Proposition 3.89. *for sub-decompositions of cellular decompositions, the property of being a representative subspace is equivalent to the property of being a deformation retract.*

It easily follows from this proposition that

Proposition 3.90. *for any cellular decompositions K and L, every weak homotopy equivalence*

$$f : K \to L$$

is a homotopy equivalence.

In other words,

Proposition 3.91. *For cell decomposition maps, the property of "being a weak homotopy equivalence" is equivalent to the property of "being a homotopy equivalence".*

Proof. Indeed, according to the theorem on cellular maps, we can assume without loss of generality that the map f is cellular, and therefore its cylinder Z_f is a cellular decomposition. On the other hand, the fact that the map f is a weak homotopy equivalence means, as we know, that the subspace K of the cylinder Z_f is representative. Therefore, since this subspace is clearly a sub-decomposition of the cylinder Z_f, it is a deformation retract of it, and therefore the map f is a homotopy equivalence. □

The proved proposition is known as *Whitehead's theorem*. It is one of the fundamental tools for studying the homotopy properties of cellular decompositions. For example, this theorem almost immediately implies that

Proposition 3.92. *any cellular decomposition K is the homotopy limit of every increasing sequence*

$$K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$$

of its sub-decomposition, the union of which coincides with the entire decomposition K.

Proof. Indeed, the space K^Σ corresponding to the sequence under consideration is, obviously, a cellular decomposition, and the natural map

$$p^\Sigma : K^\Sigma \to K$$

is weak homotopy equivalence. Therefore, according to Whitehead's theorem, this map is a homotopy equivalence. □

We will call a topological space X a *quasi-polyhedron* if it is homotopically equivalent to some cellular decomposition. Clearly, Whitehead's theorem remains valid for quasi-polyhedra as well, i.e.,

Proposition 3.93. *any continuous map*

$$f : X \to Y$$

of a quasipolyhedron X into a quasipolyhedron Y that is a weak homotopy equivalence is a homotopy equivalence.

Moreover, from what was said at the end of §3.4 it immediately follows that

Proposition 3.94. *for any cell pair (K, L) and any continuous map $f : L \to Y$ of a sub-decomposition of L into an arbitrary quasi-polyhedron Y, the space $K \cup_f Y$ is a quasi-polyhedron.*

Remark 3.95. Quasi-polyhedra constitute a remarkable class of topological spaces, distinguished by the property that practically any "reasonable" operations on topological spaces do not lead outside this class. For example, for any quasi-polyhedra X and Y, the space Y^X of all continuous maps $X \to Y$, considered in the so-called "compact-open topology" (see §9.1), is also a quasi-polyhedron, and every topological space X for which there exists a homotopically injective map $A \to K$ into some quasi-polyhedron K is itself necessarily a quasi-polyhedron. On the other hand, in all homotopy questions one can restrict oneself only to quasi-polyhedra, since for any space X there exists a continuous map from some quasi-polyhedron Z to the space X that is a weak homotopy equivalence. These properties of quasi-polyhedra will not be needed by us, and we will leave them without proof.

Chapter 4

Smooth Manifolds. I

This chapter is mainly devoted to the construction of tensor calculus on arbitrary smooth manifolds.

In the preparatory §4.1, we introduce the concept of a smooth premanifold as a Hausdorff topological space on which a certain sheaf of germs of real-valued functions is defined (to use the currently fashionable terminology).

In §4.2, which also has a preparatory character, we prove the classical theorem on differentiable maps with non-zero Jacobian for Euclidean spaces; in doing so, we specifically emphasise some details that are essential for what follows, related to estimating the diameter of the domain in which the map is diffeomorphic, which are usually left without consideration.

In §4.3, smooth manifolds are defined as smooth premanifolds that are locally diffeomorphic to Euclidean spaces. Here, the concept of a product of smooth manifolds is also introduced.

In §4.4, we prove analogues of Urysohn's lemma and Tietze's theorem for smooth functions. Here, we also prove that any convex open subset of Euclidean space is diffeomorphic to an open ball.

In §4.5, we introduce vector fields as derivations of the algebra of smooth functions and show that in any coordinate neighbourhood, each vector field is a linear combination of partial derivations with respect to local coordinates.

In §4.6, we introduce the concept of a vector at a point, define the manifold M_* of all vectors at all possible points of a given manifold M, and show that the vector fields introduced in §4.5 can be interpreted as smooth maps $M \to M_*$. In this section, we also introduce the concept of a differential of a smooth map and, in connection with this, the concept of a regular map.

In §4.7, linear differential forms are considered in a similar way, and in §4.8, tensor fields of arbitrary type.

In §4.9, an algebra of tensors and tensor fields is constructed. In particular, the operation of convolution is considered in detail here.

In the final §4.10, the concept of a Riemannian space is defined and, based on the results of §4.4, it is proved that on any smooth separable (i.e., with a countable base) manifold there exists a metric tensor field, i.e., that each such manifold can be defined as a Riemannian space.

4.1 Smooth submanifolds

Let M be an arbitrary set and let f, $f^1, \dots, f^r$ be some real functions defined on the space M. We will say that a function f *depends smoothly* on functions $f^1, \dots, f^r$ if there exists an infinitely differentiable function $u(t^1, \dots, t^r)$ of real variables $t^1, \dots, t^r$, defined (and infinitely differentiable) for all values of these variables, such that $f = u(f^1, \dots, f^r)$ on M, i.e. such that

$$f(p) = u(f^1(p), \dots, f^r(p)) \tag{4.1}$$

for any point $p \in M$. If equality (4.1) holds only for points p of some set $U \subset M$, then we will say that the function f depends smoothly on the functions $f^1, \dots f^r$ on the set U.

Remark 4.2. Some authors require that the function u, which establishes the smooth dependence (4.1), be defined (and infinitely differentiable) only in some open set of the arithmetic space $\mathbb{R}^r$, containing all points of the form $(f^1(p), \dots, f^r(p))$, $p \in M$. It is easy to show that this (formally more general) definition essentially coincides with our definition. On the other hand, it is often required not that the function u be infinitely differentiable, but only that it have a finite number of derivatives (up to some fixed order N). It can easily be verified that all the theory developed below remains valid with this definition of smooth dependence, provided that the number N is sufficiently large.

4.2 Inverse function theorem

4.3 Smooth manifolds

4.4 E-manifolds

4.5 Vector fields

4.6 Vectors

4.7 Linear differential forms

4.8 Tensors and tensor fields

4.9 Operations on tensors and tensor fields

4.10 Riemannian spaces

Chapter 5

Smooth Manifolds. II

Chapter 6

Critical points of smooth functions

Chapter 7

Elements of Riemannian Geometry

Chapter 8

Variational theory of geodesics

Chapter 9

Path Space Exploration. Applications

9.1 Path spaces

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