

A study memo on *Cohomology Theories* by
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Preface

A word from the transcriber

This is a study memo of “Cohomology Theories” by Eldon Dyer. It is rather amazing that this succinct account on the subject was published in 1969, more than fifty years ago.

The preface by Dyer

Let \mathcal{P} be the category whose objects are finite cell complexes and whose maps are continuous maps. Let \mathcal{P}^2 be the category of pairs in \mathcal{P} ; i.e., pairs (X, A) where X is a finite cell complex and A is a subcomplex of X , and maps are continuous maps of pairs. Let G be the category of abelian groups and homomorphisms.

Let $T : \mathcal{P}^2 \rightarrow \mathcal{P}^2$ be the covariant functor defined by

$$T(X, A) = (A, \emptyset) \text{ for } (X, A) \in \mathcal{P}^2 \text{ ext and}$$
$$T(f) = f|_{(A, \emptyset)} : (A, \emptyset) \rightarrow (B, \emptyset) \text{ for}$$

a map $f : (X, A) \rightarrow (Y, B)$ in \mathcal{P}^2 .

A *cohomology theory* on \mathcal{P} is a sequence of contravariant functors

$$H^n : \mathcal{P}^2 \rightarrow G$$

and a sequence of natural transformations

$$\delta^n : H^{n-1} \circ T \rightarrow H^n$$

subject to the conditions

- 1) If $f_0, f_1 \in \mathcal{P}$ and $f_0 \simeq f_1$, (\simeq means “is homotopic to”), then $H^n(f_0) = H^n(f_1)$ for all n ;
- 2) If $(X; A, B)$ is a triad in \mathcal{P} (all possible pairs of $X, A, B, A \cup B$ and $A \cap B$ are in \mathcal{P}) and $X = A \cup B$, then for the inclusion map $k : (A, A \cap B) \rightarrow (X, B)$,

$$H^n(k) : H^n(X, B) \rightarrow H^n(A, A \cap B)$$

is an isomorphism for all n ; and

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3) If $(X, A) \in \mathcal{P}^2$ and $i : (A, \emptyset) \rightarrow (X, \emptyset)$ and $j : (X, \emptyset) \rightarrow (X, A)$ are the inclusion maps, then the sequence

$$\dots \rightarrow H^{n-1}(A, \emptyset) \xrightarrow{\delta^n} H^n(X, A) \xrightarrow{H^n(j)} H^n(X, \emptyset) \xrightarrow{H^n(i)} H^n(A, \emptyset) \rightarrow \dots$$

of the pair (X, A) is exact.

These conditions are exactly the Eilenberg-Steenrod axioms for cohomology except we *omit* the condition

$$H^i(\text{pt}, 0) = 0 \text{ for } i \neq 0.$$

Recall that with this extra condition, one has the singular cohomology groups with coefficients in $H^0(\text{pt}, 0)$. We shall speak of our cohomology theories as having coefficients in the graded system $\{H^n(\text{pt}, \emptyset)\}$.

In these notes we discuss such cohomology theories. It has been clear for the past two or three years that they furnish a strong tool for analysing manifolds.

The notes are divided into four chapters. The first deals with generalities of such theories: axiomatics, origins, spectral sequence of a fibration, multiplicative structures, orientation of bundles, Poincare duality, and a type of generalised Riemann-Roch theorem. The second studies the unitary group and characteristic classes of complex vector bundles. The third discusses an example: the Grothendieck ring $K_{\mathcal{Q}}^*$. The fourth is concerned with applications: the J -homomorphism, maps of Hopf invariant 1, and properties of stable homotopy.

There are also three appendices, discussing more briefly the cohomology theory $K_{\mathcal{Q}}^*$ and the J -groups.

The author wishes to acknowledge here his appreciation of Walter Daum for assistance in preparing these notes.

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Chapter 1

Structure of Cohomology Theories

1.1 Axiomatics

1.1.1 Eilenberg Steenrod axioms for (generalised) cohomology

A (generalised) cohomology theory h^* on $\mathcal{T}\text{-}\mathcal{P}^2$ (or any nice subcategory like compact pairs, pairs of CW-complexes, etc.) is a collection of **contravariant functors**

$$h^n : \mathcal{T}\text{-}\mathcal{P}^2 \rightarrow \mathcal{A}\mathcal{B}, \quad n \in \mathbb{Z}$$

where $\mathcal{A}\mathcal{B}$ denotes the category of abelian groups, and

natural transformations

$$\delta^n : j^n \circ R \rightarrow h^{n+1}$$

where $R : \mathcal{T}\text{-}\mathcal{P}^2 \rightarrow \mathcal{T}\text{-}\mathcal{P}^2$ is the functor that sends (X, A) to (A, \emptyset) and f to $f|_A$, satisfying the following axioms:

- (i) *Homotopy invariance.* If $f \simeq g$, then $h^n(f) = h^n(g)$ for every $n \in \mathbb{Z}$.
- (ii) *Excision.* For every pair (X, A) and $U \subset A$ such that the closure \bar{U} is contained in the interior A° , the inclusion $(X \setminus U, A \setminus U) \rightarrow (X, A)$ induces an isomorphism

$$h^n(X \setminus U, A \setminus U) \cong h^n(X, A) \quad \text{for every } n \in \mathbb{Z}.$$

- (iii) *Exactness.* For every pair (X, A) , consider the inclusions $i : A \rightarrow X$ and $j : (X, \emptyset) \rightarrow (X, A)$. Then there is a long exact sequence

$$\begin{aligned} \cdots \rightarrow h^{n-1}(A) &\xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A) \\ &\xrightarrow{\delta^n} h^{n+1}(X, A) \xrightarrow{j^*} h^{n+1}(X) \xrightarrow{i^*} h^{n+1}(A) \rightarrow \cdots \end{aligned}$$

1.1.2 Three Theorems

The three theorems which follow can be proved just as in Eilenberg and Steenrod, but a simpler proof for the third due to M. Barratt and J. H. C. Whitehead is indicated.

Theorem 1.1 (Exact Sequence of a Triple). *If $(X; A, B)$ is a triple, then the sequence*

$$\cdots \rightarrow H^{n-1}(A, B) \xrightarrow{\Delta} H^n(X, A) \xrightarrow{\Phi} H^n(X, B) \xrightarrow{\Psi} H^n(A, B) \rightarrow \cdots$$

is exact, where Φ and Ψ are induced by the inclusion maps and Δ is the composition

$$H^{n-1}(A, B) \rightarrow H^{n-1}(A, \emptyset) \rightarrow H^n(X, A).$$

Theorem 1.2 (Exact Sequence of a Triad). *If $(X; A, B)$ is a triad, then the sequence*

$$\cdots \rightarrow H^{n-1}(A, A \cap B) \xrightarrow{\Delta} H^n(X, A \cup B) \xrightarrow{\Phi} H^n(X, B) \xrightarrow{\Psi} H^n(A, A \cap B) \rightarrow \cdots$$

is exact, where Φ and Ψ are induced by the inclusion maps and Δ is the composition

$$H^{n-1}(A, A \cap B) \xrightarrow{\cong} H^{n-1}(A \cup B, B) \rightarrow H^{n-1}(A \cup B, \emptyset) \xrightarrow{\delta} H^n(X, A \cup B)$$

Theorem 1.3 (Mayer-Vietoris Theorem). *Let $(X; A, B)$ be a triad with $X = A \cup B$. Then the sequence*

$$\cdots \rightarrow H^{n-1}(A \cap B, \emptyset) \xrightarrow{\Delta} H^n(X, \emptyset) \xrightarrow{\Phi} H^n(A, \emptyset) \oplus H^n(B, \emptyset) \xrightarrow{\Psi} H^n(A \cap B, \emptyset) \rightarrow \cdots$$

is exact, where the homomorphisms are defined as in the proof.

Lemma 1.4 (Barratt-Whitehead lemma). ¹ *If the diagramme of groups and homomorphisms*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \xrightarrow{h_i} & A_{i+1} & \xrightarrow{f_{i+1}} & B_{i+1} & \longrightarrow & \cdots \\ & & \alpha_i \downarrow & & \downarrow \beta_i & & \downarrow \gamma_i & & \downarrow \alpha_{i+1} & & \downarrow \beta_{i+1} & & \\ \cdots & \longrightarrow & A'_i & \xrightarrow{f'_i} & B'_i & \xrightarrow{g'_i} & C'_i & \xrightarrow{h'_i} & A'_{i+1} & \xrightarrow{f'_{i+1}} & B'_{i+1} & \longrightarrow & \cdots \end{array}$$

is commutative, the horizontal sequences are exact, and the γ_i are isomorphisms, then the sequence

$$\cdots \rightarrow A_i \xrightarrow{(\alpha_i, f_i)} A'_i \oplus B_i \xrightarrow{f_i - \beta_i} B'_i \xrightarrow{h_i \gamma_i^{-1} g'_i} A_{i+1} \rightarrow \cdots$$

is exact.

¹See *The First Non-Vanishing Group of an $(n+1)$ -ad* M. Barratt, J. Whitehead Published 1 July 1956 Mathematics Proceedings of The London Mathematical Society

The Mayer-Vietoris Theorem follows by applying this lemma to the following diagramme, in which the isomorphism is condition (ii) in the definition of a cohomology theory (§§1.1.1).

$$\begin{array}{ccccccccccc} \dots & \rightarrow & H^{n-1}(X, \emptyset) & \longrightarrow & H^{n-1}(A, \emptyset) & \longrightarrow & H^n(X, A) & \longrightarrow & H^n(X, \emptyset) & \longrightarrow & H^n(A, \emptyset) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow & & \downarrow & & \\ \dots & \rightarrow & H^{n-1}(B, \emptyset) & \rightarrow & H^{n-1}(A \cap B, \emptyset) & \rightarrow & H^n(B, A \cap B) & \rightarrow & H^n(B, \emptyset) & \rightarrow & H^n(A \cap B, \emptyset) & \rightarrow & \dots \end{array}$$

1.1.3 Reduced Cohomology Theories

Let \mathcal{P}_0 be the category of finite cell complexes with base point and continuous base point preserving maps. A *reduced cohomology theory* is a sequence

$$\tilde{H}^n : \mathcal{P}_0 \rightarrow \mathcal{A}$$

of contravariant functors and a sequence

$$\sigma^n : \tilde{H}^{n+1} \circ \Sigma \rightarrow \tilde{H}^n$$

of natural transformations subject to three conditions.

Construction of spaces

Before stating these conditions we recall some important constructions. Let (X, x_0) and (Y, y_0) be in \mathcal{P}_0 .

The wedge $(X \vee Y, *) \in \mathcal{P}_0$ is the subspace $(X \times y_0) \cup (x_0 \times Y)$ of $X \times Y$ with $x \times y = *$ as base point.

The smash $(X \# Y, *) \in \mathcal{P}_0$ is the factor space $(X \times Y)/(X \vee Y)$ with base point the image of $X \vee Y$.

The reduced suspension For $(X, x_0) = (\mathbb{S}^1, 1)$, the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, $X \# Y = \Sigma Y$ is the *reduced suspension* of Y . Σ is a covariant functor from \mathcal{P}_0 to \mathcal{P}_0 with $\Sigma(f) = \text{id} \# f$, $\text{id} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ being the identity map.

We define $p : \mathcal{P}^2 \rightarrow \mathcal{P}_0$ by $p(X, A) = (X/A, *)$, where for $A = \emptyset$, $X/\emptyset = X^+$ the disjoint union of X and a distinct point $+$, to be taken as the base point in $p(X, \emptyset) = (X, +)$. The reduced cone $CA \cup X$ of a pair (X, A) with $* \in A \subset X$ consists of X together with a cone over A with the interval from the base point collapsed to the new base point (the “whisker construction”); i.e.,

$$CA \cup X = \{(I, (0)) \# (A, *)\} \cup X.$$

Note that $p : (X, A) \rightarrow (X/A, *)$ factors through $(CA \cup X, *)$:

$$(X, A) \xrightarrow{C(X, A)} (CA \cup X, *) \xrightarrow{h} (X/A, *)$$

where C is the cone functor and h identifies CA with the base point. As (X, A) is a finite cellular pair, it has the homotopy extension property, so the map $(CA \cup X) \rightarrow (X/A)$ is a homotopy equivalence.

Axioms for a reduced cohomology theory

We are now ready to state the conditions for a reduced cohomology theory.

A *reduced cohomology theory* is a sequence

$$\tilde{H}^n : \mathcal{P}_0 \rightarrow \mathcal{A}$$

of contravariant functors and a sequence

$$\sigma^n : \tilde{H}^{n+1} \circ \Sigma \rightarrow \tilde{H}^n$$

of natural transformations such that

- 1) If $f_0, f_1 \in \mathcal{P}_0$ and $f_0 \simeq f_1$ in \mathcal{P}_0 , then $\tilde{H}^n(f_0) = \tilde{H}^n(f_1)$ for all n ;
- 2) $\sigma^n(X) : \tilde{H}^{n+1}(\Sigma X) \rightarrow \tilde{H}^n(X)$ is an isomorphism for all $X \in \mathcal{P}_0$; and
- 3) If $(X, A) \in \mathcal{P}^2$ and $* \in A$, then the sequence

$$\tilde{H}^n(X/A) \xrightarrow{\tilde{H}^n(\bar{p})} \tilde{H}^n(X) \xrightarrow{\tilde{H}^n(i)} \tilde{H}^n(A)$$

is exact at $\tilde{H}^n(X)$ for each n , where $\bar{p} : X \rightarrow X/A$ is the map collapsing A to a point and $i : A \rightarrow X$ is the inclusion map.

The condition 3) for reduced theories appears weaker than the corresponding condition for non-reduced theories. However, we may deduce the following condition

- 3') If $(X, A) \in \mathcal{P}^2$ and $* \in A$, then the sequence

$$\dots \rightarrow \tilde{H}^{n-1}(A) \xrightarrow{\Delta^{n-1}} \tilde{H}^n(X/A) \xrightarrow{\tilde{H}^n(\bar{p})} \tilde{H}^n(X) \xrightarrow{\tilde{H}^n(i)} \tilde{H}^n(A) \rightarrow \dots$$

is exact, where Δ^{n-1} is the composition

$$\tilde{H}^{n-1}(A) \xrightarrow[\cong]{\sigma^{n-1}(A)^{-1}} \tilde{H}^n(\Sigma A) \xrightarrow{\tilde{H}^n(k)} \tilde{H}^n(CA \cup X) \xrightarrow[\cong]{} \tilde{H}^n(X/A)$$

and $k : CA \cup X \rightarrow \Sigma A$ is the map collapsing X to a point.

To see the exactness at $\tilde{H}^n(X/A)$ we observe that $\tilde{H}^{n-1}(A) \rightarrow \tilde{H}^n(CA \cup X) \rightarrow \tilde{H}^n(X)$ is equivalent to

$$\tilde{H}^n(\Sigma A) \rightarrow \tilde{H}^n(CA \cup X) \rightarrow \tilde{H}^n(X)$$

and 3) applies since $(CA \cup X)/X$ is homeomorphic to ΣA . To see the exactness at $\tilde{H}^{n-1}(A)$, we observe that $\tilde{H}^n(X) \rightarrow \tilde{H}^{n-1}(A) \rightarrow \tilde{H}^n(X/A)$ is equivalent to

$$\tilde{H}^n(\Sigma X) \rightarrow \tilde{H}^n(\Sigma A) \rightarrow \tilde{H}^n(CA \cup X)$$

and 3) applies since ΣA has the homotopy type of $CX \cup (CA \cup X)$ and $(CX \cup (CA \cup X))/(CA \cup X)$ is homeomorphic to ΣX .

Theorem 1.5. *There is a natural 1-1 correspondence between theories $\{H, \delta\}$ and $\{\tilde{H}, \sigma\}$ given by the commutative diagramme*

$$\begin{array}{ccc} \mathcal{P}^2 & \xrightarrow{P} & \mathcal{P}_0 \\ & \searrow H & \swarrow \tilde{H} \\ & \mathcal{A} & \end{array}$$

Proof. For $(X, A) \in \mathcal{P}^2$ and a reduced theory $\{\tilde{H}, \sigma\}$ define $(\alpha\tilde{H}^n(X, A) = \tilde{H}^n(X/A)$; define $\delta : (\alpha\tilde{H}^{n-1})(A, \emptyset) \rightarrow (\alpha\tilde{H}^n)(X, A)$ to be the composition

$$\tilde{H}^{n-1}(A^+) \xleftarrow{\sigma} \tilde{H}^n(\Sigma A^+) \xrightarrow{\tilde{H}^n(\bar{p})} \tilde{H}^n(X^+/A^+) \cong \tilde{H}^n(X/A).$$

This system satisfies the cohomology axioms; in particular, δ is the image of Δ under α .

For $(X, *) \in \mathcal{P}_0$ and a theory $\{H, \delta\}$ define $(\beta H^n)(X) \equiv H^n(X, \{*\})$ and define σ to be composition of the isomorphisms

$$H^n(X, \{*\}) \cong H^{n+1}(CX, X) \cong H^{n+1}(\Sigma X, \{*\}),$$

where the first isomorphism is given by the exact sequence of the triple $(CX, X, \{*\})$ and the second is a special case of more general isomorphism

$$H^n(X, A) \xrightarrow{\cong} H^n(X/A, \{*\});$$

this is given by the composition

$$H^n(X/A, \{*\}) \rightarrow H^n(CA \cup X, CA) \rightarrow H^n(X, A),$$

where the first homomorphism is an isomorphism since the pairs have the same homotopy type and the second is an isomorphism by the exact sequence of the triad $(CA \cup X; X, CA)$. The system $\{\beta H, \sigma\}$ satisfies the axioms for a reduced cohomology theory; in particular, axiom 3 is a consequence of the exact sequence of a triple.

It is clear from the constructions that $\beta\alpha\{\tilde{H}, \sigma\} = \{\tilde{H}, \sigma\}$ and $\alpha\beta\{H, \delta\} = \{H, \delta\}$. \square

1.1.4 Spectra

A *spectrum* \mathcal{X} is an indexed family $\{X_i\}_{i \in \mathbb{Z}}$ of spaces with base point together with a family of base point preserving maps $f_i : \Sigma X_i \rightarrow X_{i+1}$.

Example 1.6. a) \mathcal{S} is the spectrum with $X_i = \mathbb{S}^i$, the i -sphere and $f_i : \Sigma \mathbb{S}^i \rightarrow \mathbb{S}^{i+1}$ the identity.

b) For an abelian group G , $\mathcal{K}(G)$ is the spectrum whose i -th space is

$$\begin{cases} K(G, i) & i > 0 \\ G & i = 0 \\ \text{pt} & i < 0 \end{cases}$$

From the canonical homotopy equivalence $K(G, i) \xrightarrow{\cong} \Omega K(G, i+1)$, we obtain the maps $\Sigma K(G, i) \rightarrow K(G, i+1)$.

- c) For X a space with base point, $\mathcal{S}(X)$ is the spectrum with i -th term ΣX . Thus, $\mathcal{S}(\mathbb{S}^0) = \mathcal{S}$.
- d) For X a space with base point, and \mathcal{Y} a spectrum, the spectra $X\#\mathcal{Y}$ and $\mathcal{Y}\#X$ are given by $\{X\#Y_i\}$ and $\{Y_i\#X\}$. In particular, we have the spectrum $\Sigma\mathcal{Y} = \mathbb{S}^1\#\mathcal{Y}$.
- e) For \mathcal{X} a spectrum, \mathcal{X}^q is the spectrum with i -th term $(X^q)_i = X_{q+1}$.

We shall see that spectra define cohomology theories. Let $[A, B]$ denote the set of homotopy classes of base point preserving maps. For $A \in \mathcal{P}_0$ define $\{A, \mathcal{X}\}$ to be

$$\varinjlim([\Sigma^n A, X_n], i^n),$$

where for $\alpha \in [\Sigma^n A, X_n]$, $i^n(\alpha) \in [\Sigma^{n+1} A, X_{n+1}]$ is the composition

$$\Sigma\Sigma^n A \xrightarrow{\Sigma\alpha} \Sigma X \xrightarrow{f_n} X_{n+1}.$$

For $n \geq 1$, $[\Sigma^n A, X]$ is a group and i_n is a homomorphism; for $n \geq 2$, $[\Sigma^n A, X]$ is an abelian group. Thus, $\{A, \mathcal{X}\}$ is an abelian group.

Define the homomorphism $\sigma : \{A, \mathcal{X}\} \rightarrow \{\Sigma A, \mathcal{X}^1\}$ to be the direct limit of $\sigma : [\Sigma^n A, X_n] \rightarrow [\Sigma^n \Sigma A, X_{n+1}]$, where $\sigma^n = i^n$. We can take this limit since σ_n and the i^n commute. σ only shifts $\{A, \mathcal{X}\}$ over one in itself; so we have clearly

Lemma 1.7. $\sigma : \{A, \mathcal{X}\} \xrightarrow{\cong} \{\Sigma A, \mathcal{X}^1\}$.

For $g : A \rightarrow B$ in \mathcal{P}_0 , composition defines a homomorphism $g^* : \{B, \mathcal{X}\} \rightarrow \{A, \mathcal{X}\}$.

Lemma 1.8. $(\text{id})^* = \text{id}$; $(g \circ f)^* = f^* \circ g^*$; if $f \simeq g$ in \mathcal{P}_0 , then $f^* = g^*$.

Lemma 1.9. For $* \in B \subset Y$, $\Sigma^n(CB \cup Y)$ is homeomorphic to $C(\Sigma^n B) \cup (\Sigma^n Y)$.

It suffices to prove this for $n = 1$, but that is clear from the definitions.

Lemma 1.10. For $* \in B \subset Y$, and $B \xrightarrow{i} Y \xrightarrow{p} Y/B$ the inclusion and collapsing maps, the sequence $\{Y/B, \mathcal{X}\} \xrightarrow{p^*} \{Y, \mathcal{X}\} \xrightarrow{i^*} \{B, \mathcal{X}\}$ is exact at $\{Y, \mathcal{X}\}$.

Proof. For X a space with base point

$$[CB \cup Y, X] \rightarrow [Y, X] \rightarrow [B, X]$$

is an exact sequence of sets. Thus, by Lemma 1.9

$$[\Sigma^n(CB \cup Y), X_n] \rightarrow [\Sigma^n Y, X_n] \rightarrow [\Sigma^n B, X_n]$$

is an exact sequence of groups; but exactness commutes with direct limits and $CB \cup Y$ and Y/B have the same homotopy type. The lemma follows. \square

Theorem 1.11. *For a spectrum and a space $Y \in \mathcal{P}_0$, $\tilde{H}^q(Y; \mathcal{X}) = \{Y, \mathcal{X}_q\}$ and $\sigma^n : \tilde{H}^{n+1}(\Sigma Y; \mathcal{X}) \rightarrow \tilde{H}^n(Y; \mathcal{X})$, the inverse of the isomorphism of Lemma 1.7, define a reduced cohomology theory.*

A map $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{Y}$ of spectra is a collection $f_i : X_i \rightarrow Y_i$ of maps of the terms of the spectra which for i sufficiently large homotopy commute with the defining maps of the spectra:

$$\begin{array}{ccc} \Sigma X_i & \xrightarrow{\Sigma} & \Sigma Y_i \\ \downarrow & & \downarrow \\ X_{i+1} & \longrightarrow & Y_{i+1} \end{array}$$

Spectra have the role of coefficients in these cohomology theories and maps of them induced “coefficient homomorphisms”; i.e., natural transformations of theories.

We cite without proof the following result of E. Brown [18]:

Theorem 1.12. *If \tilde{H} is a reduced cohomology theory on \mathcal{P}_0 and $\tilde{H}^q(\mathbb{S}^0)$ is countable for all q , then there is a spectrum \mathcal{Y} such that $\tilde{H}^*(-)$ and $\tilde{H}^*(-; \mathcal{Y})$ are naturally equivalent. Furthermore, \mathcal{Y} can be found so that the map $Y_n \rightarrow \Omega Y_{n+1}$ induced by $\Sigma Y_n \rightarrow Y_{n+1}$, is a homotopy equivalence (\mathcal{Y} is an Ω -spectrum).*

1.2 Spectral Sequence of a Fibration

1.2.1 Exact Couples

Some familiarity with exact couples is assumed and we only review briefly the basic definitions, notation and propositions.

For each pair (p, q) of integers we are given abelian groups $A^{p,q}$ and $C^{p,q}$ and homomorphisms f , g and h so that the “ p -sequence”

$$\dots \rightarrow A^{p-1, q-1} \xrightarrow{h} C^{p, q-1} \xrightarrow{g} A^{p, q-1} \xrightarrow{f} A^{p-1, q} \xrightarrow{h} C^{p, q} \xrightarrow{g} A^{p, q} \rightarrow \dots$$

is exact.

We arrange these sequences in the following diagramme:

$$\begin{array}{ccccccc}
 & & & & & & A^{p+r, q-r} \longrightarrow C^{p+r+1, q-r} \\
 & & & & & & \downarrow f \\
 & & & & & & A^{p+r-1, q-r+1} \xrightarrow{h} C^{p+r, q-r+1} \\
 & & & & & & \downarrow f \\
 & & & & & & \vdots \\
 & & & & & & \downarrow f \\
 C^{p, q-1} & \xrightarrow{g} & A^{p, q-1} & & & & A^{p, q} \\
 & & \downarrow f & & & & \downarrow f \\
 C^{p, q-1} & \xrightarrow{g} & A^{p-1, q} & \xrightarrow{h} & C^{p, q} & \xrightarrow{g} & A^{p, q} \\
 & & \downarrow f & & & & \downarrow f \\
 & & A^{p-2, q+1} & \xrightarrow{h} & C^{p-1, q+1} & \xrightarrow{g} & A^{p-1, q+1} & \xrightarrow{h} & C^{p, p+1} \\
 & & \downarrow f & & & & \downarrow f \\
 & & \vdots & & & & \downarrow f \\
 & & \downarrow f & & & & \downarrow f \\
 C^{p-r, q+r-1} & \xrightarrow{g} & A^{p-r, q+r-1} & & & & \\
 & & \downarrow f & & & & \\
 C^{p-r-1, q+r} & \xrightarrow{g} & A^{p-r-1, q+r} & & & &
 \end{array}$$

The sequence in the bold arrows is the p -sequence and the sum of the indices in each column is constant.

We define

$$\begin{aligned}
 Z_r^{p, q} &= g^{-1}(\text{im } f^{(r-1)}) \subset C^{p, q} \quad \text{and} \\
 B_r^{p, q} &= h(\ker f^{(r-1)}) \subset C^{p, q}.
 \end{aligned}$$

Then we can define $\theta_r : Z_r^{p, q} \rightarrow C^{p+r, q-r+1}/B_r^{p+r, q-r+1}$ to be $h \circ (f^{(r-1)})^{-1} \circ g$. Letting $E_r^{p, q} \equiv Z_r^{p, q}/B_r^{p, q}$, we see that θ_r induces a homomorphism

$$d_r : E_r^{p, q} \rightarrow E_r^{p+r, q-r+1}.$$

Furthermore, $d_r \circ d_r = 0$ and so the homology groups $\mathcal{H}(E_r, d_r)$ are defined as usual. Exactness in the diagramme at $A^{p+r-1, q-r+1}$ and $A^{p-r, q+r-1}$ imply that $\mathcal{H}(E_r, d_r) \cong E_{r+1}^{p, q}$.

The sequence of groups $\{E_r^{p, q}\}$ and differentials d_r is the spectral sequence of the exact couple (A, C) , $A = \bigoplus_{p, q} A^{p, q}$ and $C = \bigoplus_{p, q} C^{p, q}$.

Define

$$\begin{aligned} Z_\infty^{p,q} &= g^{-1}(\cap_v \mathfrak{S} f^{(v)}) \subset C^{p,q}, \\ B_\infty^{p,q} &= h(\cup_v \ker f^{(v)}) C^{p,q}, \\ E_\infty^{p,q} &= Z_\infty^{p,q} / B_\infty^{p,q}. \end{aligned}$$

For $p + q = n$, define $A^n = \varprojlim(A^{p,q}, f)$ and $F_p A^n = \ker(A^n \rightarrow A^{p,q})$. Then $F_p(A^n) \supset F_{p+1}(A^n)$.

If the following two convergence conditions hold

- 1) $A^{p-r, q+r} \xrightarrow{f} A^{p-r-1, q+r+1}$ is 0 for $r > \nu(p, q)$, and
- 2) $A^{p+r, q-r} \xrightarrow{f} A^{p+r-1, q-r+1}$ is \cong for $r > \tilde{\nu}(p, q)$,

then $Z_r^{p,q} \cong Z_\infty^{p,q}$ and $B_r^{p,q} \cong B_\infty^{p,q}$ for large r and $E_r^{p,q} \cong E_\infty^{p,q} \cong F_{p-1}(A^n)/F_p(A^n)$, $n = p + q$.

We note that the spectral sequence is a covariant functor on exact couples; i.e., if $F : (A, C) \rightarrow (A', C')$ is a map of exact couples, then there are induced maps

$$E_r(F) : E_r \rightarrow E'_r$$

of spectral sequences with the appropriate functorial properties. If the convergence conditions hold and some $E_r(F)$ is an isomorphism, then all succeeding ones are and $A_n \cong A'_n$.

1.2.2 Spectral Sequence of a h^* -fibration

Let $\pi : E \rightarrow B$ be a continuous map of spaces in \mathcal{P} ; assume B is connected. It suffices to consider the case in which E and B are finite simplicial complexes and π is simplicial, for this is equivalent to the general case up to homotopy.

If $\{h^n\}$ is a cohomology theory, define $A^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}))$, where $B^{(p)}$ is the p -skeleton of B , and $C^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}), \pi^{-1}(B^{(p-1)}))$. These groups form an exact couple satisfying the convergence condition. Since E is finite dimensional, $A^n \cong h^n(E)$.

We shall say π is an h^* -fibration if for each simplex Δ of B and each vertex v of Δ , the homomorphism $h^*(\pi^{-1}(\Delta)) \rightarrow h^*(\pi^{-1}(v))$ induced by inclusion is an isomorphism. A Serre fibration is an H^* -fibration, where H^* is ordinary singular cohomology, and we will see later that an H^* -fibration is an h^* -fibration for any cohomology theory h^* .

To simplify the notation in what follows, we will write $\{X, Y\}^i$ for $h^i(\pi^{-1}(X), \pi^{-1}(Y))$ when $Y \subset X \subset B$. We assume throughout the remainder of this section that π is an h^* -fibration.

Lemma 1.13. *For simplices $\Delta' \subset \Delta \subset B$, the homomorphism $\{\Delta\}^* \rightarrow \{\Delta'\}^*$ induced by inclusion is an isomorphism.*

Proof. The diagramme

$$\begin{array}{ccc} \{\Delta\}^* & \longrightarrow & \{\Delta'\}^* \\ & \searrow & \swarrow \\ & \{v\}^* & \end{array}$$

commutes because the homomorphisms are induced by inclusion, and the two lower homomorphisms are isomorphisms. \square

Define a k -box to be either a k -simplex or the union of members of a non-empty collection of $(k-1)$ -simplices of a k -simplex which does not include all of its $(k-1)$ -simplices.

Lemma 1.14. *If $D' \subset D$ are boxes in B , the homomorphism $\{D\}^* \rightarrow \{D'\}^*$ induced by inclusion is an isomorphism.*

Proof. The argument is by induction on the dimension of the smaller box D' . It suffices to give the proof when D is a simplex, because each box is contained in a simplex. By Lemma 1.13 the conclusion follows if D' is also a simplex. Suppose $D' = D_1 \cup D_2$, where D_1 is a box and D_2 is a simplex. Then $D_1 \cap D_2$ is a box of lower dimension than that of D' . Hence, by the induction assumption and Lemma 1.13, the indicated homomorphisms in the diagramme

$$\begin{array}{ccc} \{D\}^* & \longrightarrow & \{D'\}^* \\ \cong \downarrow & & \downarrow \cong \\ \{D_2\}^* & \xrightarrow{\cong} & \{D_1 \cap D_2\}^* \end{array}$$

are isomorphisms, and the diagramme commutes since all homomorphisms are induced by inclusions. \square

Lemma 1.15. *If v_p is the last vertex of the p -simplex Δ_p , then $\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{v_p\}^q$.*

Proof. Let $\angle i$ denote the box containing all $(p-1)$ -simplices of Δ_p except Δ_p^i , the face opposite the i -th vertex. By the cohomology sequence of the triple $\{\Delta_p, \dot{\Delta}_p, \angle 0\}$ and Lemma 1.14,

$$\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{\dot{\Delta}_p, \angle 0\}^{p+q-1}.$$

By excision $\{\dot{\Delta}_p, \angle 0\}^{p+q-1} \cong \{\Delta_p^0, \dot{\Delta}_p^0\}^{p+q-1}$. Continuing this procedure, we get

$$\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{\dot{\Delta}_p, \angle 0\}^{p+q-1} \cong \dots \cong \{v_p, \emptyset\}^q.$$

\square

A system \mathcal{G} of *local coefficients* over B is a function from simplices of B to abelian groups, $\Delta \mapsto G_\Delta$, together with a function from pairs (Δ, Δ') , $\Delta' \subset \Delta$, of simplices of B to homomorphisms

$$\eta_{\Delta\Delta'} : G_{\Delta'} \rightarrow G_\Delta$$

such that

- 1) $\eta_{\Delta\Delta'} = 1$ if $\Delta = \Delta'$ and
- 2) $\eta_{\Delta\Delta'} \circ \eta_{\Delta'\Delta''} = \eta_{\Delta\Delta''}$ if $\Delta'' \subset \Delta' \subset \Delta$.

A *p-cochain f^p with coefficients in \mathcal{G}* is a function which associates with each p -simplex Δ an element of G_Δ ; the p -cochains form an abelian group by coordinate addition, $C^p(B; \mathcal{G})$. There is a homomorphism

$$\delta : C^p(B; \mathcal{G}) \rightarrow C^{p+1}(B; \mathcal{G})$$

defined by

$$(\delta f^p)(\Delta_{p+1}) = \sum_{j=0}^{p+1} \eta_{\Delta_{p+1}\Delta_{p+1}^j} f^p(\Delta_{p+1}^j).$$

As usual one computes that $\delta \circ \delta = 0$ and defines cohomology $H^p(B; \mathcal{G})$ with local coefficients as $\ker \delta / \text{im } \delta$ in dimension p .

In the spectral sequence arising from the exact couple of an h^* -fibration one has

$$\begin{aligned} E_1^{p,q} &\equiv C^{p,q} \equiv \{B^{(p)}, B^{(p-1)}\}^{p+q} \\ &\cong \oplus_{\Delta_p \subset B} \{\Delta_p, \dot{\Delta}_p\}^{p+q}, \quad \text{by excision} \\ &\cong \oplus_{\Delta_p \subset B} \{v_{\Delta_p}\}^p, \quad \text{by Lemma 1.15,} \\ &\cong C^p(B; h^q(F)), \end{aligned}$$

where $h^q(F)$ is the local coefficient system defined by the function $\Delta_p \rightarrow \{v_{\Delta_p}\}^q$ with v_{Δ_p} the last vertex of Δ_p . For $\Delta_s \subset \Delta_p$ the homomorphism (in fact isomorphism)

$$\eta_{\Delta_p\Delta_s} : \{v_{\Delta_s}\}^q \rightarrow \{v_{\Delta_p}\}^q$$

is defined via the inclusion isomorphisms with $\{[v_{\Delta_s}, v_{\Delta_p}]\}^q$, where $[v_{\Delta_s}, v_{\Delta_p}]$ is the 1-simplex from v_{Δ_s} to v_{Δ_p} . That this is a local coefficient system then follows from the first axiom in 1.1.1 for the cohomology system $\{h^q\}$. Finally, we note that since B is connected, $\{v^q\}$ is the same group $h^q(F)$, $F = \pi^{-1}(v)$, for all vertices v of B . Thus, we have defined an isomorphism

$$\lambda_{p,q} : E_1^{p,q} \xrightarrow{\cong} C^p(B; h^q(F)).$$

Theorem 1.16. $\lambda_{p+1,q} \circ d_1 = \delta \lambda_{p,q}$: that is, the diagramme

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{d_1} & E_1^{p+1,q} \\ \lambda_{p,q} \downarrow & & \downarrow \lambda_{p+1,q} \\ C^p(B; h^q(F)) & \xrightarrow{\delta} & C^{p+1}(B; h^q(F)) \end{array}$$

is commutative.

Proof. By naturality of the spectral sequence of an exact couple and of the isomorphisms $\lambda_{p,q}$, it suffices to prove the assertion in the case $B = \Delta_{p+1}$.

Consider the composition φ_{p+1}^i of

$$\{\Delta_{p+1}^i, \dot{\Delta}_{p+1}^i\}^{p+q} \xrightarrow{J_1^*} \{\dot{\Delta}_{p+1}, \Delta_{p+1}^{p-1}\}^{p+q} \xrightarrow{i^*} \{\dot{\Delta}_{p+1}\} \xrightarrow{\delta} \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1}.$$

By the cohomology sequence of the triple $\{\Delta_{p+1}, \dot{\Delta}_{p+1}, \mathcal{L}i\}$ and Lemma 1.14, the homomorphism $\Delta : \{\dot{\Delta}_{p+1}, \mathcal{L}i\}^{p+q} \rightarrow \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1}$ is an isomorphism. By excision, $\{\Delta_{p+1}^i, \dot{\Delta}_{p+1}^i\}^{p+q} \cong \{\dot{\Delta}_{p+1}, \mathcal{L}i\}^{p+q}$. From the commutative diagramme

$$\begin{array}{ccc} \{\dot{\Delta}_{p+1}, \Delta_{p+1}^{p-1}\}^{p+q} & \xrightarrow{i^*} & \{\dot{\Delta}_{p+1}\} \\ \uparrow J_1^* & & \searrow \delta \\ \{\Delta_{p+1}^i, \dot{\Delta}_{p+1}^i\}^{p+q} & \longrightarrow & \{\dot{\Delta}_{p+1}, \mathcal{L}i\}^{p+q} \\ & & \nearrow \Delta \\ & & \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1} \end{array}$$

it follows then that φ_{p+1}^i is an isomorphism.

The differential $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$ is the composition $\delta \circ i^*$. For $f^p \in C^p(B; h^q(F))$,

$$\lambda_{p+1,q} d_1 \lambda_{p,q}^{-1} (f^p)(\Delta_{p+1}) = \sum_{i=0}^{p+1} \varphi_{p+1}^i f^p(\Delta_{p+1}^i).$$

Thus, we wish to show that $\varphi_{p+1}^i = (-1)^i \eta_{\Delta_{p+1} \Delta_{p+1}^i}$.

For $i = 0$ consider the triad $\{\Delta_{p+1}, \Delta_{p+1}^0, \mathcal{L}0\}$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \{\Delta_{p+1}^0, \dot{\Delta}_{p+1}^0\}^{p+q} & \xrightarrow[\cong]{\alpha} & \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1} & \longrightarrow & 0 \\ & & \cong \uparrow & & \uparrow \delta & & \\ & & \{\dot{\Delta}_{p+1}, \mathcal{L}0\}^{p+q} & \longrightarrow & \{\dot{\Delta}_{p+1}\}^{p+q} & & \end{array}$$

By the proof of Lemma 1.15, α is $\eta_{\Delta_{p+1} \Delta_{p+1}^0}$; it is also φ_{p+1}^0 .

Assume $\varphi_j^i = (-1)^i \eta_{\Delta_j \Delta_j^i}$ for $j < p+1$ and for $j = p+1$ and $i < k$. Let $\Delta_{p+1}^{k,k-1}$ denote the $(p-1)$ -simplex opposite the edge $(k, k-1)$. Let $f^p \in C^{p-1}(\Delta_{p+1}^{k,k-1})$. Then

$$\lambda_{p,q} d_1 \lambda_{p-1,q}^{-1} f^{p-1} = \delta f^{p-1} = \begin{cases} (-1)^{k-1} \eta_{\Delta_{p+1}^{k-1} \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1}) & \text{on } \Delta_{p+1}^{k-1} \\ (-1)^{k-1} \eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1}) & \text{on } \Delta_{p+1}^k \end{cases}$$

Since $d_1 \circ d_1 = 0$,

$$\begin{aligned} 0 &= \lambda_{p+1,q} d_1 \lambda_{p,q}^{-1} \lambda_{p,q} d_1 \lambda_{p-1,q}^{-1} (f^{p-1})(\Delta_{p+1}^{k,k-1}) \\ &= \varphi_{p+1}^k ((-1)^{k-1} \eta_{\Delta_{p+1}^{k,k-1} \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) + \varphi_{p+1}^k ((-1)^{k-1} \eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) \\ &= \eta_{\Delta_{p+1} \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1}) + \varphi_{p+1}^k ((-1)^{k-1} \eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) \end{aligned}$$

Thus,

$$\varphi_{p+1}^k (\eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) = (-1)^k \eta_{\Delta_{p+1} \Delta_{p+1}^k} (\eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})).$$

Hence,

$$\varphi_{p+1}^k = (-1)^k \eta_{\Delta_{p+1} \Delta_{p+1}^k}.$$

□

Corollary 1.17. *The chain complexes $E_1^{P,q}$, d_1 and $\{C^p(B; h^q(F)), \delta\}$ are naturally equivalent.*

Corollary 1.18. *In the spectral sequence of the h^* -fibration $\pi : E \rightarrow B$, the term $E_2^{P,q}$ is naturally isomorphic with $H^p(B; h^q(F))$, where the latter is ordinary cohomology with local coefficients.*

To summarise the results of this section, there is a functor from the category of h^* -fibrations ($f \rightarrow E \xrightarrow{\pi} B$) to the category of exact couples; by composition, then to the category of spectral sequences. There is a natural isomorphism $E_2^{P,q} \cong H^p(B; h^q(F))$, in which $h^q(F)$ is a system of local coordinates. Furthermore, $\{E_\infty^{P,q}\}$ is the associated graded system to a filtration of $h^*(E)$. The spectral sequence is also natural with respect to transformations of cohomology theories.

In particular, if $\pi : X \rightarrow X$ is the identity map, then $E_2^{P,q} \cong H^p(X; h^q(\text{pt}))$ and $E_\infty^{P,q}$ is associated to a filtration of $h^*(X)$. This relation can sometimes be exploited to compute or to deduce properties of $h^*(X)$.

1.2.3 Applications of the Spectral Sequence

Proposition 1.19. *If $\tau : h^* \rightarrow k^*$ is a natural transformation of cohomology theories and $\tau : h^*(\text{pt}) \xrightarrow{\cong} k^*(\text{pt})$ is an isomorphism, then $\tau : h^*(X, A) \rightarrow k^*(X, A)$ is an isomorphism for all pairs (X, A) .*

Proof. The transformation τ induces a homomorphism of the spectral sequence of $\text{id} : X \rightarrow X$ in the h^* -theory into that in the k^* -theory. The hypothesis implies this is an isomorphism at the E_2 level and thus for $h^*(X)$ into $k^*(X)$. For a pair (X, A) , we consider $CA \cup X$. □

Proposition 1.20. *If $f : B \rightarrow B'$ so that $f : H^*(B') \rightarrow H^*(B)$ is an isomorphism for ordinary singular cohomology (integer coefficients), then $f_h^* : h^*(B') \rightarrow h^*(B)$ is an isomorphism for any cohomology theory h^* .*

Proof. The map f induces a map f of the identity fibration $B \rightarrow B$ into $B' \rightarrow B'$ and thus a map of the spectral sequences of these fibrations. In particular, the naturality of the universal coefficient sequence and the five lemma imply

$$E_2^{P,q}(f) : H^P(B, h^q(\text{pt})) \rightarrow H^P(B'; h^q(\text{pt}))$$

is an isomorphism. The conclusion follows. \square

Note that this implies our earlier remark that an H^* -fibration is an h^* -fibration for any cohomology theory h^* .

Proposition 1.21. *If $h^i(\text{pt}) = 0$ for $i \neq 0$, then the cohomology theory h^* is naturally equivalent to singular cohomology with coefficients $h^0(\text{pt})$, $H^*(-; h^0(\text{pt}))$.*

Proof. The spectral sequence collapses and we have

$$H^P(B; h^0(\text{pt})) \cong E_2^{P,q} \cong E_2^{P,\infty} \cong F_{p-1}h^P(B)/F_p h^P(B) \cong h^P(B).$$

\square

Proposition 1.22. $H^n(B; G) \cong [B, K(G, n)]$.

Proof. The groups $\{B, \mathcal{K}(G)\}$ give a cohomology theory k^* in which $k^0(\text{pt}) \cong \varinjlim [\mathbb{S}^n, K(G, n)]$. The composition

$$[\mathbb{S}^n, K(G, n)] \xrightarrow{\Sigma} [\mathbb{S}^{n+1}, \Sigma K(G, n+1)] \rightarrow [\Sigma \mathbb{S}^{n+1}, K(G, n+1)]$$

is the isomorphism $[\mathbb{S}^n, \Omega K(G, n+1)] \rightarrow [\Sigma \mathbb{S}^n, K(G, n+1)]$. By definition each of these groups is G . In other dimensions $k^i(\text{pt}) = 0$. Thus $\{B, \mathcal{K}(G)^n\} \cong H^n(B; G)$. The object on the left is $\varinjlim [\Sigma^i B, K(G, n+i)]$. As seen above, these groups are isomorphic and are mapped isomorphically. Thus $[B, K(G, n)] \cong \{B, \mathcal{K}(G)^n\} \cong H^n(B; G)$. \square

It is interesting to note this proof does not use obstruction theory.

1.2.4 The “Universal Cohomology Theory”

For pairs (X, A) and (Y, B) , define

$$\pi_j^i(X, A; Y, B) = \varinjlim_n [\Sigma^{j+n}(X/A), \Sigma^{i+n}(Y/B)].$$

For fixed j and (Y, B) , this is a cohomology theory.

For an arbitrary cohomology theory h define

$$\tau : \pi_j^i(X, A; Y, B) \times h^k(Y/B) \rightarrow h^{k+1-j}(X/A)$$

as follows:

for $f \in [f] \in [\Sigma^{j+N}(X/A), \Sigma^{i+N}(Y/B)]$, the composition

$$h^k(Y/B) \rightarrow h^{k+i+N}(\Sigma^{i+N}(Y/B)) \rightarrow h^{k+i+N}(\Sigma^{j+N}(X/A)) \rightarrow h^{k+i-j}(X/A)$$

defines the pairing τ since it is independent of the representative f in the class of the limit. The pairing is bilinear; and so, induces

$$\tau : \pi_j^i(X, A; Y, B) \otimes h^k(Y/B) \rightarrow h^{k+i-j}(X/A).$$

In particular, for $j = 0$ and $(Y, B) = (\mathbb{S}^0, \text{pt})$, $\pi_j^i(X, A; Y, B) = \pi_{\mathcal{S}}^i(X, A)$, the cohomology theory called *stable cohomotopy*. Then for each n

$$\tau \otimes 1 : \sum_{i+j=n} \pi_{\mathcal{S}}^i(X, A) \otimes h^j(\text{pt}) \otimes \mathbb{Q} \rightarrow h^n(X, A) \otimes \mathbb{Q}$$

is a natural transformation of cohomology theories.

Theorem 1.23. *The transformation*

$$\tau \otimes 1 : \sum_{i+j=n} \pi_{\mathcal{S}}^i(X, A) \otimes h^j(\text{pt}) \otimes \mathbb{Q} \rightarrow h^n(X, A) \otimes \mathbb{Q}$$

is a natural equivalence of cohomology theories, where \mathbb{Q} denotes the rationals. Thus, up to torsion $\pi_{\mathcal{S}}^i$ is a “universal cohomology theory”.

Proof. By Proposition 1.19 it suffices to verify the conclusion for $(X, A) = (\mathbb{S}^0, \text{pt})$. But $\pi_{\mathcal{S}}^i(\mathbb{S}^0, \text{pt})$ is a torsion group for $i \neq 0$ and is the integers \mathbb{Z} for $i = 0$ by a theorem of Serre since the stable cohomotopy groups of spheres are isomorphic to the stable homotopy groups of spheres. Then $\tau \otimes 1$ is an isomorphism for $(X, A) = (\mathbb{S}^0, \text{pt})$. \square

The maps $\mathbb{S}^n \rightarrow K(\mathbb{Z}, n)$ generating the homotopy of $K(\mathbb{Z}, n)$ induce a map of spectra $\mathcal{S} \rightarrow \mathcal{K}(\mathbb{Z})$. By Proposition 1.22 this induces a natural transformation of cohomology theories

$$\pi_{\mathcal{S}}^i(-1) \rightarrow H^i(-).$$

For a cohomology theory h^* define the “generalised Chern character”

$$\text{Ch} : h^n(X, A) \rightarrow \sum_{i+j=n} H^i(X, A; h^j(\text{pt}) \otimes \mathbb{Q})$$

to be the composition

$$h^n(X, A) \rightarrow h^n(X, A) \otimes \mathbb{Q} \xrightarrow{(\tau \otimes 1)^{-1}} \sum_{i+j=n} \pi_{\mathcal{S}}^i(X, A) \otimes h^j(\text{pt}) \rightarrow \sum_{i+j=n} H^i(X, A; h^j(\text{pt}) \otimes \mathbb{Q}).$$

Corollary 1.24. *The transformation*

$$\text{Ch} \otimes 1 : h^n(X, A) \otimes \mathbb{Q} \rightarrow \sum_{i+j=n} H^i(X, A; h^j(\text{pt}) \otimes \mathbb{Q})$$

is a natural equivalence of cohomology theories. It is also natural under transformations of cohomology theories $h^* \rightarrow k^*$.

Proof. Proposition 1.19 implies the result since it is clearly true for $(X, A) = (\mathbb{S}^0, \text{pt})$. \square

Corollary 1.25. *Let h and k be cohomology theories, with k a \mathbb{Q} -module valued functor. Then for*

$$\varphi : h^*(\text{pt}) \rightarrow k^*(\text{pt}),$$

there is a unique natural transformation

$$\Phi : h^*(-) \rightarrow k^*(-)$$

of cohomology theories extending φ .

Proof. Consider the diagramme

$$\begin{array}{ccc} h^*(X, A) & \longrightarrow & h^*(X, A) \otimes \mathbb{Q} \xrightarrow{\cong} \sum H^*(X, A; h^*(\text{pt}) \otimes \mathbb{Q}) \\ \Phi \downarrow & & \downarrow \varphi \\ k^*(X, A) & \longrightarrow & k^*(X, A) \otimes \mathbb{Q} \xrightarrow{\cong} \sum H^*(X, A; k^*(\text{pt}) \otimes \mathbb{Q}) \end{array}$$

The homomorphism Φ is defined by means of the diagramme; it is unique since any such homomorphism must factor through $h^* \otimes \mathbb{Q}$. \square

1.3 Multiplicative Cohomology Theories

1.3.1 Preliminaries

A cohomology theory h^* is *multiplicative* if for each (X, A) and (Y, B) , there is a homomorphism

$$\otimes : h^i(X, A) \otimes h^j(Y, B) \rightarrow h^{i+j}(X \times Y, X \times B \cup A \times Y)$$

which is associative, (graded) commutative (i.e., $u^i \otimes v^j = (-1)^{ij} v^j \otimes u^i$), natural under maps of pairs, has a unit $1 \in h^0(\mathbb{S}^0, \text{pt})$, and makes the following diagramme commute:

$$\begin{array}{ccc} h^i(A) \otimes h^j(Y, B) & \xrightarrow{\otimes} & h^{i+j}(A \times Y, A \times B) \\ \delta \times 1 \downarrow & & \downarrow \gamma \\ h^{i+1}(X, A) \otimes h^j(Y, B) & \xrightarrow{\otimes} & h^{i+j+1}(A \times Y \cup X \times B, X \times B) \\ & & \downarrow \delta \\ h^{i+1}(X, A) \otimes h^j(Y, B) & \xrightarrow{\otimes} & h^{i+j+1}(X \times X \times B \cup A \times Y) \end{array}$$

The homomorphism γ is excision and the δ on the right side is that of the cohomology sequence of the triple $(X \times Y, A \times Y \cup X \times B)$.

If $(X; A, B)$ is a triad, the external pairing just described defines an internal pairing

$$h^i(X, A) \otimes h^j(X, B) \rightarrow h^{i+j}(X, A \cup B)$$

sending (u, v) into $u \cup v$. This pairing is the composition

$$h^i(X, A) \otimes h^j(X, B) \xrightarrow{\otimes} h^{i+j}(X \times X, X \times B \cup A \times Y) \xrightarrow{\Delta^*} h^{i+j}(X, A \cup B),$$

where $\Delta : X \rightarrow X \times X$ is the diagonal map.

This internal pairing has the properties

1. If $A = B$, $h^*(X, A)$ is a ring;
2. If $A = B = \emptyset$, $h^*(X)$ is a ring with unit;
3. If $B = \emptyset$, $h(X, A)$ is an $h^*(X)$ -module; and
4. $h^*(A) \xrightarrow{\delta} h^*(X, A)$ is an $h^*(X)$ -module homomorphism.

Conversely, given an internal pairing \cup , there is an external pairing \otimes defined by

$$x \otimes y = p^*x \cup q^*y,$$

where $x \in h^*(X, A)$, $y \in h^j(Y, B)$ and p and q are the projections of $(X \times X, A \times Y)$ onto (X, A) and $(X \times Y, X \times B)$ onto (Y, B) , respectively.

Further, if we are given a natural internal pairing \cup on $h^*(X, A)$ for all pairs $(X, A) \in \mathcal{P}^2$ which satisfies 1, 2, 3 and 4, we can define an external pairing by the following device.

Let i_1 and i_2 denote the injections of $(X, *)$ and $(Y, *)$ in $(X \times Y, *)$ and p_1 and p_2 the projections of $(X \times Y, *)$ onto $(X, *)$ and $(Y, *)$. For $u \in h^i(X, *)$ and $v \in h^j(Y, *)$, the element $p_1^*u \cup p_2^*v \in h^{i+j}(X \times Y, *)$. Since $p_2 \circ i_1 : (X, *) \rightarrow (*, *) \subset (Y, *)$,

$$i_1^*(p_1^*u \cup p_2^*v) = i_1^*p_1^*u \cup (0) = 0.$$

Similarly, $i_2^*(p_1^*u \cup p_2^*v) = 0$. Thus, the element $p_1^*u \cup p_2^*v$ lies in $h^{i+j}(X \# Y, *)$. We define $u \otimes v \equiv p_1^*u \cup p_2^*v \in h^{i+j}(X \times Y, X \times * \cup * \times Y)$. For (X, A) and (Y, B) we make the same construction on $(X/A, *)$ and $(Y/B, *)$ and use the homeomorphism $(X/A) \# (Y/B) \cong X \times Y / (X \times B \cup A \times Y)$. It can be checked that this is an external pairing.

We note that if $X = Y$ and $\Delta : X \rightarrow X \times X$ is the diagonal map then since $p_1\Delta = p_2\Delta = \text{id}$,

$$\Delta^*(u \otimes v) = \Delta^*p_1^*u \cup \Delta^*p_2^*v = u \cup v;$$

and so, the previous process for obtaining an internal pairing from an external one when applied to this external pairing yields the internal pairing we started with.

In $h^n(\mathbb{S}^n, *)$ define a class γ^n to be $(\Sigma^n)^*(1)$ where 1 is the unit of $h^0(\mathbb{S}^0, *)$ and $(\Sigma^n)^*$ is the suspension isomorphism $h^0(\mathbb{S}^0, *) \rightarrow h^n(\mathbb{S}^n, *)$. In cohomology with the dimension axiom, γ is a generator of $h^n(\mathbb{S}^n, *)$.

Proposition 1.26. *The diagramme*

$$\begin{array}{ccc}
 & h^{i+n}(\mathbb{S}^n \times X, \mathbb{S}^n \times A \cup (*) \times X) & \\
 \nearrow^{\gamma^n \times} & \downarrow \cong & \\
 h^i(X, A) & & \\
 \searrow_{(\Sigma^n)^*} & & \\
 & h^{i+n}(\Sigma^n(X/A), *) &
 \end{array}$$

is commutative, where $\gamma^n \times$ denotes external multiplication by γ^n and the vertical homomorphism is an excision isomorphism.

Proof. It suffices to prove this for $n = 1$ for it will then follow that $\gamma^n = \gamma^1 \times \gamma^{n-1}$.

Consider the following diagramme.

$$\begin{array}{ccccc}
 h^0(\mathbb{S}^0, *) \otimes h^i(X, A) & \xrightarrow{\cong} & h^i(\mathbb{S}^0 \times X, (*) \times X \cup \mathbb{S}^0 \times A) & \longrightarrow & h^i(X/A, *) \\
 \cong \downarrow & & \downarrow \cong & & \downarrow (\Sigma^1)^* \\
 h^1(C\mathbb{S}^0, \mathbb{S}^0) \otimes h^i(X, A) & \longrightarrow & h^{i+1}(C\mathbb{S}^0 \times X, \mathbb{S}^0 \times X \cup C\mathbb{S}^0 \times A) & \longrightarrow & h^{i+1}(\Sigma(X/A), *) \\
 \cong \downarrow & & \downarrow \cong & \nearrow & \\
 h^i(\mathbb{S}^1, *) \otimes h^i(X, A) & \longrightarrow & h^{i+1}(\mathbb{S}^1 \times X, (*) \times X \cup \mathbb{S}^1 \times A) & &
 \end{array}$$

The upper line gives $(\Sigma^1)^*$ and the lower line $\gamma^1(-)$. Commutativity is clear except possibly in the upper left-hand corner, where it holds by definition of the external product. \square

Proposition 1.27. *Let h^* and k^* be multiplicative cohomology theories. If $k^0(\text{pt})$ is a \mathbb{Q} -module, then k^* can be made a \mathbb{Q} -module valued functor. If $\varphi : h^*(\text{pt}) \rightarrow k^*(\text{pt})$ and Φ is the unique extension of φ as in Corollary 1.25 of Theorem 1.23 and if φ is a multiplicative homomorphism, then Φ is also multiplicative.*

Proof. k^* can be made a \mathbb{Q} -module valued functor because $k^i(X, A) \otimes k^0(\text{pt}) \xrightarrow{\cong} k^i(X, A)$.

In the diagramme

$$\begin{array}{ccc}
 h^*(X, A) \otimes h^*(Y, B) & \longrightarrow & h^*(X \times Y, X \times B \cup A \times Y) \\
 \downarrow & & \downarrow \\
 h^*(X, A) \otimes h^*(Y, B) \otimes \mathbb{Q} & \longrightarrow & h^*(X \times Y, X \times B \cup A \times Y) \otimes \mathbb{Q} \\
 \Phi \otimes \Phi \downarrow & & \downarrow \Phi \\
 k^*(X, A) \otimes k^*(Y, B) & \longrightarrow & k^*(X \times Y, X \times B \cup A \times Y),
 \end{array}$$

for a fixed pair (Y, B) all terms in the lower square are cohomology theories in (X, A) and all homomorphisms are natural. By Corollary 1.25 the two compositions around the square will agree if they agree for $(X, A) = (\mathbb{S}^0, *)$. But with (X, A) fixed, all terms are cohomology theories in (Y, B) , and the same argument holds. To prove commutativity it then suffices to prove it in the case (Y, B) is also $(\mathbb{S}^0, *)$, but this is just the hypothesis that φ is multiplicative. \square

1.3.2 Dold-Thom-Gysin Theorem

Theorem 1.28. *Let h^* be a multiplicative cohomology theory and let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be an h^* -fibration. Suppose there are elements a_1, \dots, a_r in $h^*(E)$ such that (i^*a_1, \dots, i^*a_r) is an $h^*(\text{pt})$ -base for $h^*(F)$ as an $h^*(\text{pt})$ -module, then (a_1, \dots, a_r) is an $h^*(B)$ -base for $h^*(E)$ as an $h^*(B)$ -module.*

Proof. The $h^*(B)$ -module structure of $h^*(E)$ is defined by π^* as the composition

$$h^*(B) \otimes h^*(E) \xrightarrow{\pi^* \otimes 1} h^*(E) \otimes h^*(E) \xrightarrow{\cup} h^*(E).$$

Let C and C' be the exact couples of the fibrations $\text{id} : B \rightarrow B$ and $\pi : E \rightarrow B$, respectively. The homomorphism

$$\rho : C \oplus C \oplus C \oplus \dots \oplus C \rightarrow C'$$

defined by $\rho(\lambda_1 \oplus \dots \oplus \lambda_r) = \sum_{i=1}^r \pi^*(\lambda_i) a_i$ is a map of exact couples since π is a fibre preserving map of fibrations

$$\begin{array}{ccc} E & \xrightarrow{\pi} & B \\ \pi \downarrow & & \downarrow \text{id} \\ B & \longrightarrow & B \end{array}$$

and ρ is an $h^*(B)$ -module homomorphism commuting with the coboundary operator in the two couples.

Since (i^*a_1, \dots, i^*a_r) is an $h^*(\text{pt})$ -base for $h^*(F)$, at the E_1 -level of the spectral sequences, the homomorphism

$$C^P(B; h^*(\text{pt})) \oplus \dots \oplus C^P(B; h^*(\text{pt})) \rightarrow C^P(B; h^*(F))$$

induced by ρ is an isomorphism. Thus, ρ induces isomorphisms on E_∞ terms of the spectral sequence and consequently we have the isomorphism

$$h^*(B) \oplus \dots \oplus h^*(B) \rightarrow h^*(E),$$

defined by linear combination with the a_i . Thus, (a_1, \dots, a_r) is an $h^*(B)$ -base for $h^*(E)$. \square

The Theorem 1.28 has a relative version. Conceptually, it involves nothing new except complicated notation. Thus, we will only define the relevant exact couples and state without proof the theorem for the relative case.

Suppose $E' \subset E$ and $\pi : E \rightarrow B$ such that for each vector v of a simplex Δ of B

$$h^*(\pi^{-1}(\Delta), \pi^{-1}(\Delta) \cap E') \rightarrow h^*(\pi^{-1}(v), \pi^{-1}(v) \cap E')$$

is an isomorphism. Then (E, E', π, B) is a *relative h^* -fibration with relative fibre $(F, F') = (\pi^{-1}(v), \pi^{-1}(v) \cap E')$* . Let B' be a subcomplex of B . We can then define an exact couple as follows:

$$A^{p,q} = h^{p+q}([\pi^{-1}(B^{(p)}) \cup \pi^{-1}(B')] / [(\pi^{-1}(B^{(p)}) \cap E') \cup \pi^{-1}(B')])$$

with $C^{p,q}$ equal h^{p+q} of the corresponding pair based on the p th and $(p-1)$ th skeletons. As before this leads to the *spectral sequence of the relative fibration* and we have

Corollary 1.29. *In the spectral sequence of the relative fibration (E, E', π, B, B') , the term $E_2^{p,q}$ is naturally isomorphic with $H^p(B, B'; h^q(F, F'))$ and $E_\infty^{p,q}$ is the system of graded groups associated with a filtration of $h^n(E, E' \cup \pi^{-1}(B'))$, $p + q = n$.*

In an analogous way we have the following relative version of the Dold-Thom-Gysin isomorphism:

Theorem 1.30. *Let h^* be a multiplicative cohomology theory and let (E, E', π, B, B') be a relative f^* -fibration. Suppose there are elements a_1, \dots, a_r in $h^*(E, E' \cup \pi^{-1}(B'))$ such that (i^*a_1, \dots, i^*a_r) is an $h^*(\text{pt})$ -base for $h^*(F, F')$ as an $h^*(\text{pt})$ -module. Then $\{a_1, \dots, a_r\}$ is an $h^*(B, B')$ -base for $h^*(E, E' \cup \pi^{-1}(B'))$ as an $h^*(B, B')$ -module.*

The $h^(B, B')$ -module structure of $h^*(E, E' \cup \pi^{-1}(B'))$ is defined by the composition*

$$\begin{aligned} h^*(B, B') \otimes h^*(E, E' \cup \pi^{-1}(B')) &\rightarrow h^*(E, \pi^{-1}(B')) \otimes h^*(E, E' \cup \pi^{-1}(B')) \\ &\xrightarrow{\cup} h^*(E, E' \cup \pi^{-1}(B')). \end{aligned}$$

Actually, a slightly weaker hypothesis describes $h^*(B, B')$: if the classes a_1, \dots, a_r in $h^*(E, E')$, then

$$(\pi^*h^*(B, B')) \cup a_1 \oplus \dots \oplus (\pi^*h^*(B, B')) \cup a_r \cong h^*(E, E' \cup \pi^{-1}(B')).$$

1.3.3 Orientability of Vector Bundles

Let $\pi : E \rightarrow B$ be an n -plane bundle.

Remark 1.31. We shall usually assume π is vector bundle, but in the following Corollary 1.33 to Theorem 1.30 it is sufficient for the structure group to be group of origin preserving onto-homeomorphisms of \mathbb{R}^n , Euclidean n -space.

Let E' denote the complement of the zero cross-section. Then for any cohomology theory h^* , (E, E', π, B) is a relative h^* -fibration with fibre $(F, F') = (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, which has the cohomology of $(\mathbb{S}^n, \text{pt})$. Thus, $h^{i+n}(F, F') \cong h^{i+n}(\mathbb{S}^n, \text{pt}) \cong h^i(\text{pt})$; so $h^*(F, F')$ is a free $h^*(\text{pt})$ -module with one generator u_n of degree n corresponding to $1 \in h^0(\text{pt})$.

Definition 1.32. The n -plane bundle π is h^* -orientable if there is a class $u \in h^n(E, E')$ such that $i^*u = u_n \in h^n(F, F')$ for each fibre inclusion $i : (F, F') \rightarrow (E, E')$. Of course, if B is connected and $i^*u = u_n$ for one fibre, then it is so for all fibres. A choice of such a class u is an h^* -orientation of π and π is said to be h^* -oriented by u .

Corollary 1.33. [to Theorem 1.30] If the n -plane bundle π is h^* -oriented by the class $u \in h^n(E, E')$, then

$$\varphi : h^i(B) \rightarrow h^{i+n}(E, E')$$

defined by $\varphi(z) = \pi^*z \cup u$ is an isomorphism.

If the structure group of the bundle π is a subgroup of the orthogonal group, or more generally of the group of auto-homeomorphisms on \mathbb{R}^n preserving distance from the origin, then associated with π are the unit disc bundle $\mathbb{D}(\pi)$ and the unit sphere bundle $\mathbb{S}(\pi)$ of π . The factor space $\mathbb{D}(\pi)/\mathbb{S}(\pi)$ has a well determined base point and is called the *Thom space* of the bundle π , herein denoted B^π .

The inclusion $(\mathbb{D}(\pi), \mathbb{S}(\pi)) \rightarrow (E, E')$ induces an isomorphism of cohomology; thus,

$$h^*(E, E') \xrightarrow{\cong} f^*(B^\pi, *).$$

Corresponding to each fibre inclusion $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \rightarrow (E, E')$ is the disc and sphere inclusion

$$\begin{array}{ccc} (\mathbb{D}^n, \mathbb{S}^{n-1}) & \longrightarrow & (\mathbb{D}(\pi), \mathbb{S}(\pi)) \\ \downarrow & & \downarrow \\ (\mathbb{R}^n, \mathbb{R}^n \setminus 0) & \longrightarrow & (E, E') \end{array}$$

and $\mathbb{D}^n/\mathbb{S}^{n-1}$ is homeomorphic to \mathbb{S}^n . Thus, we can equivalently define notions of h^* -orientation in terms of a class $u \in h^n(B^\pi, *)$ which restricts to the class $\gamma^n \in h^n(\mathbb{S}^n, *)$ and the isomorphism of the Corollary becomes $\varphi : h^i(B) \rightarrow h^{i+n}(B^\pi, *)$ defined by the composition

$$\begin{array}{ccccc} h^i(B) & \xrightarrow{\pi^*} & h^i(\mathbb{D}(\pi)) & \longrightarrow & h^{i+n}(B^\pi, *) \\ & & \searrow \cup u & & \downarrow \cong \\ & & & & h^{i+n}(\mathbb{D}(\pi), \mathbb{S}(\pi)). \end{array}$$

Proposition 1.34. If α and β are vector bundles over X and Y , respectively, then $X^\alpha \times Y^\beta$ is homeomorphic to $(X \times Y)^{\alpha \times \beta}$, where $\alpha \times \beta$ is the bundle with fibre $\alpha_x + \beta_y$ over the point (x, y) .

Proof. The obvious map $X^\alpha \times Y^\beta \rightarrow (X \times Y)^{\alpha \times \beta}$ is a homeomorphism except on the wedge, which goes to the base point. \square

Proposition 1.35. *If $u \in h^m(X^\alpha, *)$ and $v \in h^n(X^\beta, *)$ are h^* -orientations of the vector bundles α and β over the space X , then $\Delta^*(u \times v)$ is an h^* -orientation of the Whitney sum $\alpha \oplus \beta$ over X , where Δ^* is induced by the map $X^{\alpha \oplus \beta} \xrightarrow{\Delta} (X \times X)^{\alpha \times \beta}$.*

Proof. The Whitney sum $\alpha \oplus \beta$ is the bundle over X induced from $\alpha \times \beta$ over $X \times X$ by the diagonal map. The conclusion then follows from the commutative diagramme below, and the fact that $\gamma^m \times \gamma^n = \gamma^{m+n}$.

$$\begin{array}{ccccc} h^m(X^\alpha, *) \otimes h^n(X^\beta, *) & \xrightarrow{\times} & h^{m+n}(X^\alpha \# X^\beta, *) & \longrightarrow & h^{m+n}(X^{\alpha \oplus \beta}, *) \\ \downarrow & & \downarrow & & \downarrow \\ h^m(\mathbb{S}^m, *) \otimes h^n(\mathbb{S}^n, *) & \xrightarrow{\times} & h^{m+n}(\mathbb{S}^m \# \mathbb{S}^n, *) & \xrightarrow{\cong} & h^{m+n}(\mathbb{S}^{m+n}, *) \end{array}$$

□

Proposition 1.36. *If γ is a vector bundle over X oriented by the class $u \in h^n(X\gamma, *)$ and A is a subcomplex of X , then*

$$\varphi : {}^i(X, A) \rightarrow h^{i+n}(\mathbb{D}(\gamma), \mathbb{S}(\gamma) \cup \mathbb{D}(\gamma|_A))$$

defined by $\varphi(z) = \pi_{\mathbb{D}}^(z) \cup u$, where $\pi_{\mathbb{D}} : \mathbb{D}(\gamma) \rightarrow X$ is the projection map of the associated disc bundle $\mathbb{D}(\gamma)$, is an isomorphism.*

Proof. This is an immediate consequence of Theorem 1.30. □

Theorem 1.37. *If α and β are vector bundles over X such that β is h^* -oriented by $v \in h^n(X^\beta, *)$ and $\alpha \oplus \beta$ is h^* -oriented by $w \in h^{n+m}(X^{\alpha \oplus \beta}, *)$, then α is h^* -oriented by a class $u \in h^m(X^\alpha, *)$ characterised by the equation $\rho^*u \cup \pi^*v = w$ (ρ and π as defined in the proof) and $w = \Delta^*(u \times v)$.*

Proof. Let $\pi : \mathbb{D}(\alpha) \rightarrow X$, $\rho : \mathbb{D}(\pi^*\beta) \rightarrow \mathbb{D}(\alpha)$ and $\sigma : \mathbb{D}(\alpha \oplus \beta) \rightarrow X$ be disc bundle projections. Note that there is a disc-preserving homotopy equivalence

$$(\mathbb{D}(\alpha \oplus \beta), \mathbb{S}(\alpha \oplus \beta)) \cong (\mathbb{D}(\pi^*\beta), \mathbb{S}(\pi^*\beta) \cup \mathbb{D}(\pi^*\beta|_{\mathbb{S}(\alpha)})),$$

obtained by deforming the box-like discs on the right radially into the spherical ones on the left. Since β is h -oriented by $v \in h^n(X, *)$, $\pi^*\beta$ is h -oriented by $\pi^*v \in h^n(\mathbb{D}(\alpha)\pi^*\beta, *)$. But this implies

$$\varphi_{\pi^*\beta} h^{m+i}(\mathbb{D}(\alpha), \mathbb{S}(\alpha)) \rightarrow h^{m+n+i}(\mathbb{D}(\pi^*\beta), \mathbb{S}(\pi^*\beta) \cup \mathbb{D}(\pi^*\beta|_{\mathbb{S}(\alpha)}))$$

as in Proposition 1.36 is an isomorphism. Also

$$\varphi_{\alpha \oplus \beta} : h^i(X) \rightarrow h^{m+n+i}(\mathbb{D}(\alpha \oplus \beta), \mathbb{S}(\alpha \oplus \beta))$$

is an isomorphism. Identifying cohomologies by the homotopy equivalence observed, we then have determined a class $u \in h^m(\mathbb{D}(\alpha), \mathbb{S}(\alpha))$ by

$$u = \varphi_{\pi^*\beta}^{-1}\{\varphi_{\alpha \oplus \beta}(1)\};$$

i.e., u is the unique class of $h^m(\mathbb{D}(\alpha), \mathbb{S}(\alpha))$ such that

$$\rho^*u \cup \pi^*v = w.$$

Restricting u to $h^m(\mathbb{D}^m, \mathbb{S}^{m-1})$, we have that its product with γ^n is γ^{m+n} . Thus, u is an h^* -orientation for α .

By definition of the internal product, the class $u \times v$ in $h^{n+m}((X \times X)^{\alpha \times \beta}, *)$ restricts to $\rho^*u \cup \pi^*v = w$ on the diagonal $X^{\alpha \oplus \beta}$. \square

Remark 1.38. Because of the previous results it is clear that if one representative of a stable class of vector bundles ($\alpha \xrightarrow[s]{\sim} \beta$ if there are trivial bundles n and m such that $n \oplus \alpha \sim m \oplus \beta$) is orientable in a given cohomology theory, then all representatives are. However, there is no uniqueness to this construction, for just as there may be a number of inequivalent homotopy equivalences between two spaces, there also can be a number of inequivalent bundle equivalences. Each of these yields a homotopy equivalence between the corresponding Thorn spaces and consequent transport of orientation, but different equivalences may yield different orientations. For purposes of naturality in some constructions, this point will cause us to take some care in the next section.

1.4 Applications to Differential Manifolds

The object of this section is to establish a type of Riemann-Roch theorem for general cohomology theories.

1.4.1 Orientations of Manifolds and the “Umkehr” Homomorphism

First, we recall some results from differential topology. If $f : M^m \rightarrow \mathbb{R}^{m+r}$ is a differentiable imbedding of a differentiable m -manifold into Euclidean space, let $\nu(f)$ denote the *normal* r -plane bundle of f . If also $g : M^m \rightarrow \mathbb{R}^{m+r}$ is a differentiable imbedding and $r > m + 1$, then f and g are homotopic by a differentiable homotopy through differentiable imbeddings; hence $\nu(g)$ is equivalent to $\nu(f)$. If, moreover, $r > m + 3$, then two such regular homotopies are themselves regularly homotopic through regular homotopies, and the two resulting bundle equivalences are homotopic through bundle equivalences. Thus, we have

Proposition 1.39. *If f and g are differentiable imbeddings of M^m in \mathbb{R}^{m+r} , $r > m + 3$, then there is a disc preserving homeomorphism of $M^{\nu(f)}$ onto $M^{\nu(g)}$ which is unique up to the isotopy class of disc-preserving homeomorphism.*

Proposition 1.40. *If $f : M^m \rightarrow \mathbb{R}^{m+r}$ and $f' : M^m \rightarrow \mathbb{R}^{m+r'}$ are differentiable imbeddings, then for some pair of integers (s, s') there is a uniquely determined isotopy class of homeomorphisms*

$$M^{s \oplus \nu(f)} \rightarrow M^{s' \oplus \nu(f')}.$$

Proof. By Proposition 1.34, with Y a point and $\beta = n$, we have $\Sigma^n(X^\alpha) \cong X$ (for \cong read “is homeomorphic to”). Let $i : \mathbb{R}^{m+r} \rightarrow \mathbb{R}^{m+r+1}$ be the usual inclusion; then

$$\Sigma M^{\nu(f)} \cong M^{1 \oplus \nu(f)} \cong M^{\nu(i \circ f)}$$

since $\nu(i \circ f) = 1 \oplus \nu(f)$. Choose s and s' such that $r + s = r' + s' > m + 3$. Then there are well determined isotopy classes of homeomorphisms

$$\Sigma^s M^{\nu(f)} \cong M^{\nu(i^s \circ f)} \cong M^{\nu(i^{s'} \circ f')} \cong \Sigma^{s'} M^{\nu(f')},$$

where i^s stands for the iterated inclusion

$$\mathbb{R}^{m+r} \xrightarrow{i} \mathbb{R}^{m+r+1} \xrightarrow{i} \dots \xrightarrow{i} \mathbb{R}^{m+r+s};$$

the second homeomorphism in the series is given by Proposition 1.39. Thus, we have determined a unique isotopy class of homeomorphisms $M^{s \oplus \nu(f)} \rightarrow M^{s' \oplus \nu(f')}$. \square

Definition 1.41. A differentiable manifold M^m is *h^* -orientable* if for some differentiable imbedding $f : M^m \rightarrow \mathbb{R}^{m+r}$, the bundle $\nu(f)$ is h^* -orientable; an *h^* -orientation* of M is determined by such an f and a specific orientation class $u \in h^r(M^{\nu(f)}, *)$.

Proposition 1.42. Each h^* -orientation $f : M^m \rightarrow \mathbb{R}^{m+r}$ and $u \in h^r(M^{\nu(f)}, *)$ determines for any differentiable imbedding $f' : M^m \rightarrow \mathbb{R}^{m+r'}$ a unique orientation class $u' \in h^{r'}(M^{\nu(f')}, *)$.

Proof. The unique isotopy class of homeomorphisms of Proposition 1.40 together with the s' -desuspension from $h^{r+s}(\Sigma^{s'} M^{\nu(f')}, *)$ of the s -suspension of u into $h^{r+s}(\Sigma^s M^{\nu(f)}, *)$ determines the orientation class u' . \square

Proposition 1.43. If M is h^* -oriented by f and u , then

1) $\varphi_f : h^j(M) \rightarrow h^{j+r}(M^{\nu(f)}, *)$ is an isomorphism defined by $\varphi_f(z) = \pi_{\nu(f)}^*(z) \cup u$,

2) the diagramme

$$\begin{array}{ccc} h^j(M) & \xrightarrow{\varphi_f} & h^{j+r}(M^{\nu(f)}, *) \\ & \searrow & \downarrow \cong \\ & & h^{j+r+1}(\Sigma M^{\nu(f)}, *) \\ & \searrow^{(-1)^j \varphi_{i \circ f}} & \downarrow \cong \\ & & h^{j+r+1}(M^{\nu(i \circ f)}, *) \end{array}$$

is commutative, and

3) the isomorphism

$$\varphi_{f'} : h^j(M) \rightarrow h^{j+r'}(M^{\nu(f')}, *)$$

is well determined for any differentiable imbedding $f' : M^m \rightarrow \mathbb{R}^{m+r'}$.

Proof. (of 2)) By Proposition 1.26 we have the equalities

$$\begin{aligned} \Sigma\varphi_f(z) &= \gamma \cup \varphi_f(z) = \gamma \cup (\pi^*(z) \cup u) = (-1)^j \pi^+(z) \cup (\gamma \cup u) \\ &= (-1)^j \pi^*(z) \cup u' = (-1)^j \varphi_{i \circ f}(z). \end{aligned}$$

□

We have seen in Proposition 1.42 that the possible h -orientations of a manifold are determined by those of any one of its normal bundles. In fact, they are also determined by its tangent bundle τ .

Proposition 1.44. *For any differentiable imbedding $f : M^m \rightarrow \mathbb{R}^{m+r}$ there is a well defined 1 – 1 correspondence between the h^* -orientations of $\nu(f)$ and the h^* -orientations of $\tau(M)$.*

Proof. By Proposition 1.42 it suffices to prove this for any f and r ; we choose $r > m + 3$. Then the triviality of the total space $(m + r)$ -bundle over M is assured for all imbeddings. The conclusion then follows from Theorem 1.37. □

Definition 1.45. Let M^m and N^n be differentiable manifolds and $f : M \rightarrow N$ be a continuous map. For $d > 2m + 3$ there is a differentiable imbedding $i : M \rightarrow \mathbb{D}^d$ into the interior of the d -dimensional disc such that the imbedding $\tilde{f} = f \times i : M \rightarrow N \times \mathbb{D}$ is homotopic to a differentiable imbedding. If the normal bundle $\nu(\tilde{f})$ is h -orientable, then i together with a choice of orientation class $u \in h^{n+d-m}(M^{\nu(\tilde{f})}, *)$ is said to determine an h^* -orientation of the map f . The usual considerations of isotopies of imbeddings show the determination of u is in fact independent of the imbedding and homotopy.

Proposition 1.46. *Let M^m and N^n be differentiable manifolds and $f : M \rightarrow N$ be a continuous map. The h^* -orientations of f are in a well defined 1 – 1 correspondence with the h^* -orientations of the bundle $\nu(M) \oplus f^*\tau(N)$. In particular, if M and N are h -oriented and f is any continuous map, then f has a well defined h^* -orientation.*

Proof. Let j be a differentiable imbedding of $N \times \mathbb{D}$ in a Euclidean space \mathbb{R}^r of dimension $r > 2(n + d) + 3$ and let $\pi : N \times \mathbb{D} \rightarrow N$ denote the projection map. We shall write a number of bundle equivalences; note that in each case they are uniquely defined.

$$\begin{aligned} \tau(N \times \mathbb{D}) &\simeq \pi^*(\tau(N)) \oplus d. \\ \tilde{f}^*(N \times \mathbb{D}) &\simeq \tilde{f}^*\pi^*(\tau(N)) \oplus \tilde{f}^*(d) \simeq f^*(\tau(N)) \oplus d. \end{aligned}$$

$$\begin{aligned}
v(M) &= v(j \circ \tilde{f}) \simeq v(\tilde{f}) \oplus \tilde{f}^* v(j) \\
v(M) \oplus f^* \tau(N) \oplus d &\simeq v(\tilde{f}) \oplus \tilde{f}^* v(j) \oplus \tilde{f}^* (\tau(N \times \mathbb{D})) \\
&\simeq v(\tilde{f}) \oplus \tilde{f}^* (r) \\
&\simeq v(\tilde{f}) \oplus r.
\end{aligned}$$

The first conclusion follows by suspension and desuspension. The second conclusion is then obvious by Proposition 1.44. \square

Definition 1.47. Suppose that $f : M^m \rightarrow N^n$ is h^* -oriented by $\tilde{f} : M^m \rightarrow N^n \times \mathbb{D}^d$ and $u \in h^{n+d-m}(M^{\nu(\tilde{f})}, *)$. The collapsing map (“Thom construction”)

$$N^d = (N \times \mathbb{D}^d) / (N \times \mathbb{S}^{d-1}) \xrightarrow{\tilde{f}} \mathbb{D}(\nu(\tilde{f})) / \mathbb{S}(\nu(\tilde{f})) = M^{\nu(\tilde{f})}$$

can be used to define a homomorphism

$$\begin{array}{ccc}
h^i(M) & \xrightarrow[\cong]{\varphi} & h^{i+n+d-m}(M^{\nu(\tilde{f})}, *) \\
& & \downarrow \tilde{f}^* \\
& & h^{i+n+d-m}(N^d, *) \xrightarrow{(\Sigma^d)^{s-1}} h^{i+n-m}(N);
\end{array}$$

this homomorphism is denoted

$$f_! : h^i(M) \rightarrow h^{i+n-m}(N)$$

and is determined by f and the h^* -orientation of f .

Theorem 1.48. *If $f : M \rightarrow N$ is an h^* -oriented map, then for classes $x \in h^*(M)$ and $y \in h^*(N)$,*

$$f_!(f^*(y) \cup x) = y \cup f_!(x).$$

Proof. We first prove the identity

$$\widehat{f}^*(\pi_M^* f^*(y) \cup \varphi_M(x)) = \pi_N^*(y) \cup \widehat{f}^* \varphi_M(x).$$

Consider the diagramme:

$$\begin{array}{ccccc}
h^*(N) \otimes h^*(\mathbb{D}\nu(\tilde{f}), \mathbb{S}(\nu\tilde{f})) & & & & \\
\downarrow f^* \otimes 1 & \searrow \pi_M^* \otimes 1 & & & \\
h^*(M) \otimes h^*(\mathbb{D}\nu, \mathbb{S}\nu) & & h^*(N \times \mathbb{D}) \otimes h^*(\mathbb{D}\nu, \mathbb{S}\nu) & & \\
\downarrow \pi_M^* \otimes 1 & \swarrow i^* & \downarrow \cong & \searrow 1 \otimes \widehat{f}^* & \\
h^*(\mathbb{D}\nu, \mathbb{S}\nu) & \longleftarrow h^*(N \times \mathbb{D} \otimes h^*(N \times \mathbb{D}, N \times \mathbb{D} \setminus \mathbb{D}\nu)) & \downarrow \subset^* & \xrightarrow{i^*} & h^*(N \times \mathbb{D}) \otimes h^*(N \times \mathbb{D}, N \times \mathbb{S}) \\
\downarrow \subset^* & \longleftarrow h^*(N \times \mathbb{D}, N \times \mathbb{D} \setminus \mathbb{D}\nu) & \downarrow \subset^* & \xrightarrow{i^*} & \\
h^*(\mathbb{D}\nu, \mathbb{S}\nu) & \longleftarrow h^*(N \times \mathbb{D}, N \times \mathbb{D} \setminus \mathbb{D}\nu) & \downarrow i^* & \xrightarrow{\subset^*} & \\
& \searrow \widehat{f}^* & & \swarrow \subset^* & \\
& & h^*(N \times \mathbb{D}, N \times \mathbb{S}) & &
\end{array}$$

The upper quadrilateral commutes because $f = \pi_N \tilde{f}$; the remainder commutes because all maps are excisions, inclusions, or products. Thus, the diagram commutes. An element $y \otimes \varphi_M x$ at the top is mapped into $\widehat{f}^*(\pi_M^* f^* y \cup \varphi_M x)$ following the arrows on the left and into $\pi_N^* y \cup \widehat{f}^* \varphi_N x$ following the arrows on the right: thus these elements are equal as we asserted. Since

$$\begin{aligned}
\widehat{f}^*(\pi_M^* f^* y \cup \varphi_M x) &= \widehat{f}^*(\pi_M^* f^* y \cup \pi_M^* x \cup \varphi_M(1)) \\
&= \widehat{f}^*(\pi_M^*(f^* y \cup x) \cup \varphi_M(1)) \\
&= \widehat{f}^* \varphi_M(f^* y \cup x),
\end{aligned}$$

we have

$$\widehat{f}^* \varphi_M(f^* y \cup x) = \pi_N^* y \cup \widehat{f}^* \varphi_M(x).$$

Thus,

$$\begin{aligned}
f_! (f^* y \cup x) &= \varphi_N^{-1} \widehat{f}^* \varphi_M(f^* y \cup x) = \varphi_N^{-1} (\pi_N^* y \cup \widehat{f}^* \varphi_M(x)) \\
&= \varphi_N^{-1} (\pi_N^* y \cup \pi_N^* (\varphi_N^* \widehat{f}^* \varphi_M(x)) \cup \varphi_N(1)) \\
&= y \cup \widehat{f}^* \varphi_M(x) \\
&= y \cup f_! (x).
\end{aligned}$$

($\varphi_N = \Sigma^d$ since the bundle over N is trivial.) \square

Next, we establish a functorial property of the “umkehr homomorphism” $f_!$. Given $K \xrightarrow{f} L \xrightarrow{g} M$, we seek a theorem of the sort $g_! \circ f_! = (g \circ f)_!$. Since these homomorphisms clearly depend on the h^* -orientations used, in order for there to be such equality the three orientations must be chosen in some compatible way. If we assume f and g have specified orientations, we can then orient $g \circ f$ as follows:

by Proposition 1.46, both

$$\nu(K) \oplus f^* \tau(L), \quad \nu(L) \oplus g^* \tau(M)$$

are h^* -oriented; thus, by Proposition 1.35

$$\nu(K) \oplus f^* \tau(L) \oplus f^*(\nu(L) \oplus g^* \tau(M)) \simeq \nu(K) \oplus f^* \circ g^* \tau(M) \oplus (\text{trivial bundle})$$

is h^* -oriented. This determines an orientation of $g \circ f$.

Equivalently, given $\tilde{f} : K \rightarrow L \times \mathbb{D}^d$ and $\tilde{g} : L \rightarrow M \times \mathbb{D}^{d'}$, the composition

$$K \xrightarrow{\tilde{f}} L \times \mathbb{D}^d \xrightarrow{\tilde{g} \times 1} M \times \mathbb{D}^{d'} \times \mathbb{D}^d = M \times \mathbb{D}^{d+d'}$$

is $\widetilde{g \circ f}$. Then

$$\nu(\widetilde{g \circ f}) = \nu((\tilde{g} \times 1) \circ \tilde{f}) = \nu(f) \oplus \tilde{f}^*(\nu(\tilde{g} \times 1)).$$

This defines the same h^* -orientation for $g \circ f$.

Note, in particular, that if K , L , and M are h^* -oriented, and f , g , and $g \circ f$ are given the induced h^* -orientations, then the two h^* -orientations, i.e., the composition orientation and the induced orientation, of $g \circ f$ agree. This is clear from the first definition of the composition orientation.

Theorem 1.49. *If $f : K \rightarrow L$ and $g : L \rightarrow M$ are h^* -oriented maps, the composition $g \circ f : K \rightarrow M$ has an induced h^* -orientation and with this orientation $g_! \circ f_! = (g \circ f)_!$. Further, if K , L , and M are h^* -oriented and f , g , and $g \circ f$ are oriented by Proposition 1.46, then the orientation of $g \circ f$ induced by those of f and g agrees with that given by Proposition 1.46.*

Proof. By the comments preceding the statement of the theorem we need only prove that $g_! \circ f_! = (g \circ f)_!$. To this end consider the diagramme below.

$$\begin{array}{ccccc}
h^i(K) & & & & \\
\downarrow \cong & \searrow \cong & & & \\
h^{i+\ell-k+d}(K^{\nu(\tilde{f})}, *) & \xrightarrow{\cong} & h^{i+m-k+d+d'}(K^{\nu(\tilde{g}f)}, *) & & \\
\downarrow \tilde{f}^* & & \downarrow F^* & \searrow \nu(g \circ f, *) & \\
h^{i+\ell-k+d}(L^d, *) & \xrightarrow{\cong} & h^{i+m-k+d+d'}(L^{d \oplus \nu(\tilde{g})}, *) & \xrightarrow{G^*} & h^{i+m-k+d+d'}(M^{d+d'}, *) \\
\uparrow \cong (\Sigma^d) & & \uparrow \cong (\Sigma^d) & & \uparrow \cong (\Sigma^d) \\
h^{i+\ell-k}(L) & \xrightarrow{\cong} & h^{i+m-k+d'}(L^{\nu(\tilde{g})}, *) & \xrightarrow{\tilde{g}^*} & h^{i+m-k+d'}(M^{d'}, *) \xleftarrow{\cong} h^{i+m-k}(M)
\end{array}$$

All isomorphisms are given by suspensions or Proposition 1.43. The two corner triangles and the corner square commute by Theorem 1.37. The lower right

hand square commutes because $G = \Sigma^d \circ \widehat{g}$. The map F is $\Sigma^{d'} \circ \widehat{f}$; and so, the upper left hand square commutes. The collapsing map $M^{d+d'} \rightarrow K^{\nu(\widehat{g \circ f})}$ factors through $F^{d \oplus \nu(\widehat{g})}$, which implies the commutativity of the central triangle. The left hand column defines $f_!$, the bottom row $g_!$ and the diagonal $(g \circ f)_!$; thus, $g_! \circ f_! = (g \circ f)_!$. \square

1.4.2 Multiplicative Transformations; Riemann-Roch Theorem

Let h^* be a multiplicative cohomology theory. For $(X, A) \in \mathcal{P}^2$, $h^{**}(X, A)$ is the set of formal Laurent series

$$\sum_{i=-\infty}^{+\infty} \lambda_i, \quad \lambda_i \in h^i(X, A), \quad \lambda_i = 0 \text{ for all } i < \text{some } q.$$

Define addition and multiplication in $h^{**}(X, A)$ as follows: if $\lambda = \sum \lambda_i$ and $\mu = \sum \mu_i$, then $\lambda + \mu = \sum(\lambda_i + \mu_i)$ and $\lambda \cup \mu$ is the power series product

$$(\lambda \cup \mu)_k = \sum_{i+j=k} \lambda_i \cup \mu_j.$$

$h^{**}(X, A)$ is a (non-commutative) ring under these operations, and a map $f : (X, A) \rightarrow (Y, B)$ induces a ring homomorphism $f^{**} : h^{**}(Y, B) \rightarrow h^{**}(X, A)$ by taking f^* on each coordinate. Similarly, we define

$$\delta^{**} : h^{**}(A) \rightarrow h^{**}(X, A)$$

coordinate-wise.

Suppose h^* and k^* are multiplicative theories. Then $t : h^{**} \rightarrow k^{**}$ is a (normed) multiplicative transformation if

1. t is a natural, additive transformation between the functors h^{**} and k^{**} with respect to maps f^{**} ,
2. $t(\mu \cup \lambda) = t(\mu) \cup t(\lambda)$, and
3. If $1_n \in h^{**}(\mathbb{S}^0, *)$ and $1_k \in k^{**}(\mathbb{S}^0, *)$ are the units in h^{**} and k^{**} , respectively, and $\alpha = \Sigma^{**} 1_n \in h^{**}(\mathbb{S}^1, *)$ and $\beta = \Sigma^{**} 1_k \in k^{**}(\mathbb{S}^1, *)$, then $t(\alpha) = \beta$.

Proposition 1.50. *For classes $u_1 \in h^{**}(X_1, A_1)$ and $u_2 \in h^{**}(X_2, A_2)$, the product $u_1 \times u_2$ is in $h^{**}(X_1 \times X_2, X_1 \times A_2 \cup A_1 \times X_2)$ and $t(u_1 \times u_2) = tu_1 \times tu_2$.*

Proof. As in §1.3,

$$\begin{aligned} u_1 \times u_2 &= p_1^{**} u_1 \cup p_2^{**} u_2 \quad \text{and} \\ t(u_1 \times u_2) &= t(p_1^{**} u_1 \cup p_2^{**} u_2) = t(p_1^{**} u_1) \cup t(p_2^{**} u_2) \\ &= p_1^{**}(tu_1) \cup p_2^{**}(tu_2) = tu_1 \times tu_2. \end{aligned}$$

\square

Proposition 1.51. *If $u_1 \in h^{**}(X, A_1)$ and $u_2 \in h^{**}(X, A_2)$, then $u_1 \cup u_2 \in h^{**}(X, A_1 \cup A_2)$ and $t(u_1 \cup u_2) = tu_1 \cup tu_2$.*

Proof. In the following diagramme,

$$\begin{array}{ccccc} h^{**}(X, A_1) \otimes h^{**}(X, A_2) & \xrightarrow{\otimes} & h^{**}(X \times X, X \times A_2 \cup A_1 \times X) & \xrightarrow{\Delta^{**}} & h^{**}(X, A_1 \cup A_2) \\ \downarrow t \otimes t & & \downarrow t & & \downarrow t \\ k^{**}(X, A_1) \otimes k^{**}(X, A_2) & \xrightarrow{\otimes} & k^{**}(X \times X, X \times A_2 \cup A_1 \times X) & \xrightarrow{\Delta^{**}} & k^{**}(X, A_1 \cup A_2) \end{array}$$

the first square commutes by Proposition 1.50 and the second by naturality. \square

Proposition 1.52. *$t \circ \Sigma_h = \Sigma_k \circ k$; i. e., t is stable,*

Proof. By Proposition 1.26, $\Sigma_h(u) = \alpha \otimes u$ and $\Sigma_k(tu) = \beta \otimes tu$. Thus, $t\Sigma_h(u) = t\alpha \otimes tu = \beta \otimes tu = \Sigma_k(tu)$. \square

Proposition 1.53. *$t \circ \delta_h^{**} = \delta_k^{**} \circ t$.*

Proof. The homomorphism δ_k^{**} is the composition

$$\begin{array}{ccccc} h^{**}(A, \emptyset) & \xrightarrow[\cong]{\Sigma_h} & h^{**}(\Sigma A^+, +) & \xrightarrow{P^{**}} & h^{**}(CA^+ \cup X^+) & \xrightarrow[\cong]{} & h^{**}(X, A) \\ & & & & \downarrow \cong & \nearrow \cong & \\ & & & & h^{**}(CA \cup X, *) & & \end{array}$$

All these homomorphisms commute with t .

Suppose α is a vector bundle over X with orientations $u \in h^n(X^\alpha, *)$ and $v \in k^n(X^\alpha, *)$; we can regard u and v as lying in $h^{**}(X^\alpha, *)$ and $k^{**}(X^\alpha, *)$ respectively. Then u and v induce isomorphisms

$$\varphi_h : h^{**}(X) \rightarrow h^{**}(X^\alpha, *) \quad \text{and} \quad \varphi_k : k^{**}(X) \rightarrow k^{**}(X^\alpha, *)$$

For $x \in h^{**}(X)$, define $t(\alpha, x) \equiv \varphi_k^{-1} t\varphi_h(x) \in k^{**}(X)$. \square

Lemma 1.54. *Let α be an h - and k -oriented vector bundle over X . Then for $x \in h^{**}(X)$, $t(x) \cup t(\alpha, 1) = t(\alpha, x)$.*

Proof.

$$\begin{aligned} \varphi_k(t(x) \cup t(\alpha, 1)) &= \varphi_k(t(x) \cup \varphi_k^{-1} t\varphi_h(1)) \\ &= \pi_k^{**}(t(x) \cup \varphi_k^{-1} t\varphi_h(1)) \cup v \\ &= \pi_k^{**}(t(x)) \cup (\pi_k^{**} \varphi_k^{-1} t\varphi_h(1) \cup v) \\ &= t\pi_k^{**}(x) \cup t\varphi_h(1) \\ &= t(\pi_h^{**}(x) \cup \varphi_h(1)) \\ &= t\varphi_x. \end{aligned}$$

Thus, $t(x) \cup t(\alpha, 1) = \varphi_k^{-1} t\varphi_h(x) = t(\alpha, x)$. \square

Theorem 1.55 (A generalised Riemann-Roch Theorem). *If $f : X \rightarrow Y$ is a h^* -oriented map of compact, closed manifolds and $t : h^{**} \rightarrow k^{**}$ is a multiplicative transformation of cohomology theories, then for $x \in h^{**}(X)$,*

$$t f_!^h(x) = f_!^k(t(x) \cup t(v(\tilde{f}), 1)).$$

Proof. We note first that an h^* -orientation $u \in h^n(X^\alpha, *)$ of a vector bundle determines a k^* -orientation $(tu)_n \in k^n(X^\alpha, *)$ since

$$i^*u = \alpha^n \in h^n(\mathbb{S}^n, *) \quad \text{and} \quad t\alpha^n = \beta^n \in k^n(\mathbb{S}^n, *);$$

thus, $\beta^n = i^*(tu)_n$ and α is k^* -oriented. Hence we need not assume additionally that $f : X \rightarrow Y$ is k^* -orientable; however, in the statement and proof of this theorem the specific k^* -orientation of f is irrelevant.

The right-hand side of the asserted equation is

$$(\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**} \varphi_k(t(x) \cup t(v(\tilde{f}), 1))$$

which equals

$$(\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**} t\varphi_k(x)$$

by the Lemma. By the naturality of t this equals

$$t((\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**} \varphi_h(x)) = t f_!^h(x).$$

□

Theorem 1.56. *Let $f : X \rightarrow Y$ be a continuous map of h^* - and k^* -oriented manifolds. Let $t(X) \equiv t(v(X), 1)$ and $t(Y) \equiv t(v(Y), 1)$. Then for $x \in h^{**}(X)$,*

$$t f_!^h(x) \cup t(Y) = f_!^k(t(x) \cup t(X)).$$

Proof.

$$\begin{aligned} f_!^k(t(x) \cup t(X)) &= \varphi_k^{-1} \widehat{f}_k^{**} \varphi_k(t(x) \cup t(X)) \\ &= \varphi_k^{-1} \widehat{f}_k^{**} t\varphi_h(x) \\ &= \varphi_k^{-1} t \widehat{f}_h^{**} \varphi_h(x) \\ &= \varphi_k^{-1} [t\varphi_h(\varphi_h^{-1} \widehat{f}_h^{**} \varphi_h(x))] \\ &= \varphi_k^{-1} [\varphi_k(t f_!^h(x) \cup t(Y))] \\ &= t f_!^h(x) \cup t(Y). \end{aligned}$$

□

1.4.3 Wu Formulae

As applications of the above theorems we present here a proof of the Wu formulae; in a later section we will derive the Atiyah-Hirzebruch differential Riemann-Roch Theorem.

Let $h^{**} = k^{**} = H^{**}(-; \mathbb{Z}/p\mathbb{Z})$ and let $t = \mathcal{P} = \sum \mathcal{P}^i$, the sum of the Steenrod reduced power operations. For $p = 2$, $t = \text{Sq}$. The Cartan formula implies t is a multiplicative transformation. For a vector bundle α over X , $\text{Sq}(\alpha, 1)$ is the total Stiefel-Whitney class $W(\alpha)$ of the bundle α .

Lemma 1.57. *Every vector bundle is $H^*(-; \mathbb{Z}/2\mathbb{Z})$ -orientable; a vector bundle α is $H^*(-; \mathbb{Z})$ - or $H^*(-; \mathbb{Z}/p\mathbb{Z})$ -orientable if and only if the first Stiefel-Whitney class $W_1(\alpha) = 0$.*

Proof. If α is an n -plane bundle, consider the n -sphere bundle $\mathbb{S}(\alpha \oplus 1) \xrightarrow{\pi} X$ and the Gysin sequence with $\mathbb{Z}/2\mathbb{Z}$ -coefficients:

$$\cdots \rightarrow H^r(X) \xrightarrow{\pi^*} H^r(\mathbb{S}(\alpha \oplus 1)) \xrightarrow{\lambda^*} H^{r-n}(X) \rightarrow H^{r+1}(X) \rightarrow \cdots$$

$\mathbb{S}(\alpha \oplus 1)$ has a cross-section $i : X \rightarrow \mathbb{S}(\alpha \oplus 1)$; and so, the Gysin sequence reduces to

$$0 \rightarrow H^r(X) \xrightarrow{\pi^*} H^r(\mathbb{S}(\alpha \oplus 1)) \xrightarrow{\lambda^*} H^{r-n}(X) \rightarrow 0.$$

There is a class $v \in H^n(\mathbb{S}(\alpha \oplus 1))$ such that $\lambda^*v = 1 \in H^0(X)$. Let $w = v - \pi_*i^*v$. Then $\lambda^*w = 1$ and $i^*w = 0$.

The sequence $X \xrightarrow{i} \mathbb{S}(\alpha \oplus 1) \xrightarrow{c} X$ is coexact and so

$$0 \longrightarrow H^*(X^\alpha, *) \xrightarrow{c^*} H^*(\mathbb{S}(\alpha \oplus 1)) \xrightleftharpoons[\pi^*]{i^*} H^*(X) \longrightarrow 0.$$

is exact. There is a unique $u_n \in H^n(X^\alpha, *)$ such that $c^*u_n = w$.

Since the diagramme

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & \mathbb{S}(\alpha \oplus 1) \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & X \end{array}$$

is commutative, so also is

$$\begin{array}{ccc} H^n(X^\alpha, *) & & \\ \downarrow & & \\ H^n(\mathbb{S}(\alpha \oplus 1)) & \xrightarrow{\lambda^*} & H^0(X) \\ \downarrow & & \downarrow \\ H^n(\mathbb{S}^n) & \longrightarrow & H^0(X), \end{array}$$

and it follows that u_n restricts to the generator of $H^n(\mathbb{S}^n, *)$.

To prove the second part of the lemma, note that $w_1(\alpha) = \varphi^{-1}\beta u_n$, where β is the Bockstein homomorphism associated with

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

Thus,

$$\begin{aligned} w_1(\alpha) = 0 &\Leftrightarrow u_n \text{ is the reduction of a mod 4 class} \\ &\Leftrightarrow \alpha \text{ is } H^*(-; \mathbb{Z}/4\mathbb{Z}) \text{ - orientable.} \end{aligned}$$

This is equivalent to $\pi_1(X.x_0)$ operating trivially on $H^*(\mathbb{S}^n; \mathbb{Z})$ which then by the Gysin sequence implies the $H^*(-; \mathbb{Z})$ - and $H^*(-; \mathbb{Z}/p\mathbb{Z})$ -orientability of α . The converse is clear. \square

The Wu formula asserts there is a class $V \in H^{**}(X)$ such that $\text{Sq} V = W(X) (= \text{Sq}(\tau(X), 1))$ and $[\alpha \cup V](X) = [\text{Sq} \alpha](X)$ for all $\alpha \in H^*(X)$. Let $\text{Sq}^{-1} = \frac{1}{1+(\text{Sq}^{-1})}$ as formal power series; then $\text{Sq} \text{Sq}^{-1} = \text{Sq}^{-1} \text{Sq} = 1$ and Sq is multiplicative, Let $f : X \rightarrow \text{pt}$. By Theorem 1.56,

$$f_! : H^*(\text{Sq}^{-1}(x) \cup \text{Sq}^{-1}(v(X), 1)) = \text{Sq}^{-1}(f_!(x))$$

where $f_! : H^*(X) \rightarrow H^*(\text{pt})$ is given by $f_!(x) = [x](X)$, the value of the cohomology class x on the top homology class of the manifold X (with $\mathbb{Z}/2\mathbb{Z}$ coefficients). For $x = \text{Sq} \alpha$, we then have

$$[\alpha \cup \text{Sq}^{-1}(v(X), 1)](X) = [\text{Sq} \alpha](X).$$

We claim that $\text{Sq}(\text{Sq}^{-1}(v(X), 1)) = \text{Sq}(\tau(X), 1) = W(X)$. Then $V = \text{Sq}^{-1}(v(X), 1)$ is the Wu class and satisfies the formulae above.

Since $\text{Sq}(\tau, 1) \cup \text{Sq}(v, v) = \text{Sq}(\tau \oplus v, 1) = 1$ [Appendix B], to verify the claim we need only prove that

$$\text{Sq}(\text{Sq}^{-1}(v(X), 1)) \cup \text{Sq}(v, 1) = 1.$$

To this end we compute

$$\begin{aligned} \varphi(\text{Sq}(\text{Sq}^{-1}(v(X), 1)) \cup \text{Sq}(v, 1)) &= \pi^* \text{Sq} \varphi^{-1} \text{Sq}^{-1} \varphi(1) \cup \pi^* \varphi^{-1} \text{Sq} \varphi(1) \cup (1) \\ &= \pi^* \text{Sq} \varphi^{-1} \text{Sq}^{-1} \varphi(1) \cup \text{Sq} \varphi(1) \\ &= \text{Sq}(\varphi \varphi^{-1} \text{Sq}^{-1} \varphi(1)) \\ &= \varphi(1). \end{aligned}$$

But φ is an isomorphism.

Chapter 2

Complex Vector Bundles and the Bott Periodicity Theorem

2.1 Bott Periodicity Theorem

2.1.1 Homology of the Unitary Groups

For the complex n -dimensional space \mathbb{C}^n equipped with a positive definite Hermitian inner product, the unitary group $U(n)$ is the group of linear transformations preserving this inner product. Consider \mathbb{C}^{n-1} as a linear subspace of \mathbb{C}^n and choose a vector $e_1 \in \mathbb{C}^n$ such that $(e_1, e_1) = 1$ and $e_1 \perp \mathbb{C}^{n-1}$ ($(e_1, x) = 0$ for all $x \in \mathbb{C}^{n-1}$). This defines an inclusion $U(n-1) \rightarrow U(n)$ whereby the elements of $U(n-1)$ are those transformations of $U(n)$ which leave e_1 fixed.

The group $U(n)$ acts effectively on the unit sphere \mathbb{S}^{2n-1} of \mathbb{C}^n and $U(n-1)$ is the subgroup of $U(n)$ whose elements leave $e_1 \in \mathbb{S}^{2n-1}$ fixed. Thus, $U(n)/U(n-1)$ is homeomorphic to \mathbb{S}^{2n-1} and the fibration $U(n-1) \xrightarrow{i} U(n) \xrightarrow{\pi} \mathbb{S}^{2n-1}$ is a principal $U(n-1)$ -fibration.

The boundary homomorphism $\partial : \pi_{2n-1}(\mathbb{S}^{2n-1}) \rightarrow \pi_{2n-2}(U(n-1))$ of the homotopy sequence of this fibration defines an element $\partial(\iota_{2n-1}) \in \pi_{2n-2}(U(n-1))$, where ι_{2n-1} is the canonical generator of $\pi_{2n-1}(\mathbb{S}^{2n-1})$, and its Hurewicz image $\nu \in H_{2n-2}(U(n-1))$, coefficients being \mathbb{Z} throughout this section.

The homology of an H-space has a product structure (the ‘‘Pontrjagin product’’) defined as follows:

if $\mu : X \times X \rightarrow X$ is the multiplication of the H-space X , then the composition

$$H_*(X) \otimes H_*(X) \rightarrow H_*(X \times X) \xrightarrow{\mu_*} H_*(X)$$

defines a bilinear product with identity which is associative if μ is homotopy associative and is (graded) commutative if μ is homotopy commutative.

Theorem 2.1. *If $G \xrightarrow{i} E \xrightarrow{\pi} \mathbb{S}^k$ is a principal right G -bundle (G is a connected topological group acting freely on E on the right and \mathbb{S}^k is the orbit space) and*

$v \in H_{k-1}(G)$ is the image of $\iota_k \in \pi_k(\mathbb{S}^k)$ under the composition $\pi_k(\mathbb{S}^k) \xrightarrow{\partial} \pi_{k-1}(G) \xrightarrow{h} H_{k-1}(G)$, h being the Hurewicz map, then the (Wang) sequence

$$\cdots \rightarrow H_i(G) \xrightarrow{i_*} H_i(E) \xrightarrow{\lambda_*} H_{i-k}(G) \xrightarrow{v^*} H_{i-1}(G) \rightarrow \cdots$$

is exact, where v^* is Pontrjagin product on the left with v . If $a \in H_k(E)$ is such that $\pi_*(a)$ generates $H_k(\mathbb{S}^k)$, then $\lambda_*(a)$ is a unit of $H_0(G)$.

Proof. The sphere \mathbb{S}^k is the union of two hemispheres, $\mathbb{S}^k = \mathbb{D}_+^k \cup \mathbb{D}_-^k$, with $\mathbb{D}_+^k \cap \mathbb{D}_-^k = \mathbb{S}^{k-1}$. The bundle $\pi|_{\mathbb{D}_-^k}$ is trivial since \mathbb{D}_-^k is contractible; thus, $\pi|_{\mathbb{D}_-^k} \simeq \mathbb{D}_-^k \times G$. For each $p \in \mathbb{S}^{k-1}$ there is a translation $f_p : G \rightarrow G$, $f_p(g) = f_p(e) \cdot g$, such that the map $\alpha : (\mathbb{D}_+^k, \mathbb{S}^{k-1}) \times G \rightarrow (E, G)$ defined by $\mathbb{D}_+^k \times G \rightarrow \pi^{-1}(\mathbb{D}_+^k)$ and on $\mathbb{S}^{k-1} \times G \rightarrow G$ by $(p, g) \mapsto f_p(g)$ is a relative homeomorphism (we have identified \mathbb{D}_+^k to a point and $\pi^{-1}(\mathbb{D}_+^k)$ to G). The map $\mathbb{S}^{k-1} \rightarrow G$ defined by $p \mapsto f_p(e)$ is homotopic to the image of i_k in $\pi_{k-1}(G)$.

Consider the diagramme below.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_i(G) & \longrightarrow & H_i(E) & \longrightarrow & H_i(E, G) & \longrightarrow & H_{i-1}(G) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \cong \uparrow \alpha_* & & \beta_* \uparrow & & \\ \cdots & \longrightarrow & H_i(\mathbb{S}^{k-1} \times G) & \longrightarrow & H_i(\mathbb{D}_+^k \times G) & \longrightarrow & H_i((\mathbb{D}_+^k, \mathbb{S}^{k-1}) \times G) & \longrightarrow & H_{i-1}(\mathbb{S}^{k-1} \times G) & \longrightarrow & \cdots \end{array}$$

Let v be the generator of $H_k(\mathbb{D}_+^k, \mathbb{S}^{k-1})$ corresponding to $i_k \in \pi_k(\mathbb{S}^k)$ and $w \in H_k(E, G)$ be $\alpha(v \otimes 1)$. Then $\alpha_*(v \otimes x) = w * x$ and

$$\partial_*(w * x) = \partial_* \alpha_*(v \otimes x) = \beta_* \partial_*(v \otimes x) = \beta_*((\partial * v) \otimes x) = v * x,$$

since $v = \partial_* v$.

There is an isomorphism $\varphi : H_{i-k}(G) \rightarrow H_i((\mathbb{D}_+^k, \mathbb{S}^{k-1}) \times G)$ given by $\varphi(x) = v \otimes x$ which yields the Wang sequence

$$\cdots \rightarrow H_i(G) \xrightarrow{i_*} H_i(E) \xrightarrow{\lambda_*} H_{i-k}(G) \xrightarrow{v^*} H_{i-1}(G) \rightarrow \cdots$$

To determine λ_* consider the diagramme

$$\begin{array}{ccc} H_i(E) & \xrightarrow{j_*} & H_i(E, G) \\ \pi_* \downarrow & \swarrow \pi_* & \uparrow \alpha_* \cong \\ H_i(\mathbb{S}^k, *) & \xleftarrow{\Phi} & H_i((\mathbb{D}_+^k, \mathbb{S}^{k-1}) \times G) \xleftarrow[\cong]{\varphi} H_{i-k}(G) \end{array}$$

$\lambda_* = \varphi^{-1} \circ \alpha_*^{-1} \circ j_*$ If $a \in H_k(E)$ is such that $\pi_*(a)$ generates $H_k(\mathbb{S}^k, *)$, then $\alpha_*^{-1} j_*(a) = \pm v \otimes 1$, which is the image of ± 1 under φ . Thus, $\lambda_*(a) = \pm 1$. \square

Theorem 2.2. *As Hopf algebras*

$$H_*(U(n); \mathbb{Z}) \cong E(v_1, \dots, v_n),$$

the primitively generated exterior Hopf algebra on generators v_i of dimension $2i - 1$.

Proof. The theorem is true for $n = 1$ since $U(1)$ is topologically a circle. The proof is by induction on n ; and so, we assume the conclusion for $U(n - 1)$.

Consider the principal $U(n - 1)$ -fibration

$$U(n - 1) \xrightarrow{i} U(n) \xrightarrow{\pi} \mathbb{S}^{2n-1}$$

and its Wang sequence. The element ν is of dimension $2n - 2$; it is spherical and hence primitive. By the inductive hypothesis it is thus 0. Thus, the Wang sequence breaks into short exact sequences

$$0 \rightarrow H_j(U(n - 1)) \xrightarrow{i_*} H_j(U(n)) \xrightarrow{\lambda_*} H_{j-(2n-1)}(U(n - 1)) \rightarrow 0.$$

It follows immediately that $H_*(U(n))$ is torsion-free and is additively isomorphic to $H_*(U(n - 1)) \otimes H_*(\mathbb{S}^{2n-1})$. Let $\nu_n \in H_{2n-1}(U(n))$ be in the preimage of $1 \in H_0(U(n - 1))$ under λ_* . Then no multiple of ν_n is decomposable, for if it were, it would be in $\text{im}(i_*)$ and 1 would be of finite order in $H_0(U(n - 1))$.

Since $H^*(U(n); \mathbb{Q})$ is a connected, associative, commutative, finite dimensional Hopf algebra, it is an exterior algebra and thus has a commutative co-product; i.e., $H^*(U(n); \mathbb{Q})$ has a commutative product. Since $H^*(U(n); \mathbb{Z})$ is torsion-free, $H^*(U(n); \mathbb{Z}) \rightarrow H^*(U(n); \mathbb{Q})$ is a monomorphism. Thus $H^*(U(n); \mathbb{Z})$ has a commutative product.

To see that ν_1, \dots, ν_n generate $H_*(U(n))$ as an algebra, let $x \in H_i(U(n))$ and $y = \lambda_*(x)$. Also, $\lambda_*(\nu_n \cdot y) = 1 \cdot y = y$. Thus, if $x = x - \nu_n \cdot y$, $\lambda_*(z) = 0$; and so, $z \in H_i(U(n - 1))$. Since both z and y are in $E(\nu_1, \dots, \nu_{n-1})$, they are polynomials in the variables ν_1, \dots, ν_{n-1} , and $x = z + \nu_n \cdot y$ is a polynomial in ν_1, \dots, ν_n .

We need only show that a ν_n can be chosen which is primitive. We will then have demonstrated that $H_*(U(n); \mathbb{Z})$ is a connected, associative, commutative, coassociative, cocommutative, primitively generated, finite dimensional Hopf algebra. The conclusion follows.

To do this we make a geometric construction which will have later applications. Assume the ν_i , $1 \leq i < n$ have been chosen primitive (in $H_*(U(n - 1); \mathbb{Z})$). Define $f_n : \sum P_{n-1}(\mathbb{C}) \rightarrow SU(n)$ as follows:

for $0 \neq x \in \mathbb{C}^n$, let (x) denote the equivalence class of x in $P^{n-1}(\mathbb{C})$; for $(\lambda, (x)) \in \mathbb{S}^1 \times P^{n-1}(\mathbb{C})$, a unitary transformation $A \in U(n)$ is uniquely determined by the conditions

$$Ay = \begin{cases} y & \text{if } y \perp x \\ \lambda y & \text{if } y = \mu x \text{ for some } \mu \in \mathbb{C}; \end{cases}$$

A is unitary since $|\lambda| = 1$. Map $U(n) \rightarrow SU(n)$ by taking a transformation A into $A \cdot M(I_{n-1} \oplus \sigma)$, where $\sigma^{-1} = \det(A)$ and I_{n-1} is the $(n - 1) \times (n - 1)$ unit matrix. The composition $\mathbb{S}^1 \times P^{n-1}(\mathbb{C}) \rightarrow U(n) \rightarrow SU(n)$ carries the wedge $\mathbb{S}^1 \times (e_n) \cup (1) \times P^{n-1}(\mathbb{C})$ into I_n ; and so, $f_n : \mathbb{S}^1 \# P^{n-1}(\mathbb{C}) \rightarrow SU(n)$ is defined.

It is a computation to verify that

a)

$$\begin{array}{ccc} \mathbb{S}^1 \# P^{k-1}(\mathbb{C}) & \xrightarrow{f_k} & \mathrm{SU}(k) \\ \downarrow & & \downarrow \\ \mathbb{S}^1 \# P^{n-1}(\mathbb{C}) & \xrightarrow{f_n} & \mathrm{SU}(n) \end{array}$$

commutes for $k < n$, and

b) in the commutative diagramme

$$\begin{array}{ccc} \mathbb{S}^1 \# P^{n-1}(\mathbb{C}) & \longrightarrow & \mathrm{SU}(n), \mathrm{SU}(n-1) \longrightarrow (\mathbb{S}^{2n-1}, *) \\ & & \downarrow \nearrow \\ & & (\mathrm{U}(n), \mathrm{U}(n-1)) \end{array}$$

the composition of the top is a relative homeomorphism.

Let ν_n be the image in $H_{2n-1}(\mathrm{U}(n); \mathbb{Z})$ of the top class of $H_*(\Sigma^{n-1}(\mathbb{C}); \mathbb{Z})$. Then $\pi_*(\nu_n)$ is a generator of $H_{2n-1}(\mathbb{S}^{2n-1}, *)$, and so, $\lambda_*(\nu_n) = \pm 1 \in H_0(\mathrm{U}(n-1))$. Also, ν_n is primitive since every homology class in a suspension is.

We note this also proves

c)

$$f_{n*} : H_*(\mathbb{S}^1 \# P^{n-1}(\mathbb{C})) \rightarrow H_*(\mathrm{SU}(n))$$

is a monomorphism, and $\mathrm{im}(f_{n*})$ generates $H_*(\mathrm{SU}(n))$ as an algebra. \square

2.1.2 The Universal Base Spaces $B_{\mathrm{U}(n)}$

Let E be an acyclic $\mathrm{U}(n)$ -free space; e.g., $E = \varinjlim_m \mathrm{U}(n+m)/\mathrm{U}(m)$, but the specific method of construction of E is unimportant. Collapsing E under the action of $\mathrm{U}(n)$ produces a principal $\mathrm{U}(n)$ -bundle

$$\mathrm{U}(n) \rightarrow E \rightarrow B_{\mathrm{U}(n)}.$$

Since E is acyclic, there is a homotopy equivalence $\Omega B_{\mathrm{U}(n)} \simeq \mathrm{U}(n)$ preserving the \mathbb{H} -space structures.

For an inclusion $\mathbb{C}^{n-1} \rightarrow \mathbb{C}^n$ with e_1 an orthogonal unit vector, we have the monomorphism $j_1 : \mathrm{U}(n-1) \rightarrow \mathrm{U}(n)$. There is also the monomorphism $\mathrm{U}(1) \rightarrow \mathrm{U}(n)$ obtained by multiplying $e_1 \in \mathbb{C}^n$ by λ , $|\lambda| = 1$. The product $\mathrm{U}(1) \times \mathrm{U}(n-1) \rightarrow \mathrm{U}(n)$ is also a monomorphism. These induce maps

$$B_{\mathrm{U}(n-1)} = E/j_1(\mathrm{U}(n-1)) \rightarrow E/\mathrm{U}(n) = B_{\mathrm{U}(n)}, \quad B_{\mathrm{U}(1)} \times B_{\mathrm{U}(n-1)} \rightarrow B_{\mathrm{U}(n)}.$$

Similarly, for an inclusion $\mathbb{C}^{n-2} \rightarrow \mathbb{C}^n$ with unit orthogonal vector e_2 , we have

$$B_{\mathrm{U}(1)} \times B_{\mathrm{U}(n-2)} \rightarrow B_{\mathrm{U}(n-1)}.$$

A sequence of such inclusions defines monomorphisms and maps:

$$U(1)^k \times U(n-k) \rightarrow U(n) \quad B_{U(1)}^k \times B_{U(n-k)} \rightarrow B_{U(n)}.$$

The bundle $E \times_{U(n)} \mathbb{C}^n \rightarrow B_{U(n)}$ is the *universal n -plane bundle*, where for $A \in U(n)$ pairs (e, Ax) and (eA, x) are identified; and for $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ the unit $(2n-1)$ -sphere,

$$E \times_{U(n)} \mathbb{S}^{2n-1} \rightarrow B_{U(n)}$$

is the *associated sphere bundle*. The map

$$\tilde{\lambda} : E \rightarrow E \times_{U(n)} \mathbb{S}^{2n-1}$$

defined by $\tilde{\lambda}(e) = ((e, e_1))$ is onto, and $\tilde{\lambda}(e) = \tilde{\lambda}(f)$ if and only if $(e, e_1) \sim (f, e_1)$; i.e., if and only if there exists $B \in U(n-1)$ such that $e(j, B) = f$, since e_1 is left fixed. Thus, $\tilde{\lambda}$ factors through $B_{U(n-1)}$:

$$\begin{array}{ccc} & E/U(n-1) & \\ & \nearrow & \searrow \lambda \\ E & \xrightarrow{\tilde{\lambda}} & E \times_{U(n)} \mathbb{S}^{2n-1}. \end{array}$$

The map λ is a homeomorphism. Thus, $B_{U(n-1)}$ is the total space of the associated $(2n-1)$ -sphere bundle over $B_{U(n)}$; we have the fibration

$$\mathbb{S}^{2n-1} \rightarrow B_{U(n-1)} \rightarrow B_{U(n)}.$$

With the loop space identifications $\Omega B_{U(k)} \simeq U(k)$, we recover the fibration

$$U(n-1) \rightarrow U(n) \rightarrow \mathbb{S}^{2n-1}.$$

used earlier.

Theorem 2.3. *The integral cohomology ring $H^*(B_{U(n)}; \mathbb{Z})$ is a polynomial ring $P(C_1, \dots, C_n)$ on generators C_i of dimension $2i$; the classes C_i can be chosen for all n so that $\pi^*(C_i) = C_i$, $i \leq n-1$, and C_n is the Gysin class of the fibration $\mathbb{S}^{2n-1} \rightarrow B_{U(n-1)} \xrightarrow{\pi} B_{U(n)}$.*

Proof. The argument is by induction on n . Since $B_{U(1)} = P^\infty(\mathbb{C})$, the conclusion is true for $n = 1$. Suppose the result is correct for $B_{U(n-1)}$. In the homotopy sequence of the fibration π , $\pi_i(\pi)$ is an isomorphism for $i \leq 2n-2$ and is onto for $i = 2n-1$; thus, the same is true in homology by the Whitehead theorem. Since $H_{2n-1}(B_{U(n-1)}) = 0$ by the inductive hypothesis, $H_i(\pi)$ is an isomorphism for $i \leq 2n-1$; and thus, $H^i(\pi)$ is an isomorphism for $i \leq 2n-1$.

The Gysin sequence

$$\dots \rightarrow H^i(B_{U(n)}) \xrightarrow{\pi^*} H^i(B_{U(n-1)}) \rightarrow H^{i-(2n-1)}(B_{U(n)}) \xrightarrow{\cup C_n} H^{i+1}(B_{U(n)}) \rightarrow \dots$$

implies that

- (1) $\ker(\pi^*)$ is the ideal generated by C_n , and
- (2) for all coefficient rings, $H^{\text{odd}}(B_{U(n)}) = 0$, and consequently, $H^{\text{even}}(B_{U(n)})$ is torsion-free integrally.

Let C_1, \dots, C_{n-1} be the classes in $H^*(B_{U(n)})$ such that $\pi^*C_i = C_i$ in $H^*(B_{U(n-1)})$. We claim that $H^*(B_{U(n)})$ is a polynomial ring in (C_1, \dots, C_n) . To see that these classes generate $H^*(B_{U(n)})$ let $y \in H^*(B_{U(n)})$. Then π^*y is a polynomial $w(C_1, \dots, C_n)$. This same polynomial w is in $H^*(B_{U(n)})$ and $\pi^*w = w$. Thus, $\pi^*(y - w) = 0$ and $y - w$ is in the ideal generated by C_n . The conclusion follows.

Next, suppose there is a polynomial relation $\sum_{i=0}^k C_n^i w_i = 0$, where the w_i are polynomials in the C_1, \dots, C_{n-1} . $0 = \pi^*\left(\sum_{i=0}^k C_n^i w_i\right) = w_0$. Thus, $C_n \cdot \sum_{i=1}^k C_n^{i-1} w_i = 0$; but this implies $\sum_{i=1}^k C_n^{i-1} w_i = 0$ since π^* is onto, which in turn implies $w_1 = 0$, etc. Finally, there is no relation among the C_1, \dots, C_{n-1} since by assumption there is none among $\pi^*C_1, \dots, \pi^*C_{n-1}$. \square

The decomposition $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$ defines a monomorphism $U(m) \times U(n) \xrightarrow{j} U(m+n)$ and a map

$$B_{U(m)} \times B_{U(n)} = E/j(U(m) \times U(n)) \xrightarrow{j} E/U(m+n) = B_{U(m+n)}.$$

For the bundle $\gamma^{m+n} = E \times_{U(m+n)} \mathbb{C}^{m+n}$,

$$j^*\gamma^{m+n} = \gamma^m \times \gamma^n \quad \text{over} \quad B_{U(m)} \times B_{U(n)}.$$

Lemma 2.4. *If $j : B_{U(m)} \times B_{U(n)} \rightarrow B_{U(m+n)}$ is the map induced by the decomposition $\mathbb{C}^m \oplus \mathbb{C}^n = \mathbb{C}^{m+n}$, then*

$$j^*(1 + C_1 + \dots + C_{m+n}) = (1 + C_1 + \dots + C_m) \times (1 + C_1 + \dots + C_n).$$

Proof. The Gysin class of γ^{m+n} is C_{m+n} and $j^*C_{m+n} = C_m \times C_n$ by Theorem 1.37.

The proof of the lemma is inductive. $j^*(1 + C_1 + \dots + C_{m+n})$ is a Polynomial $P_{m,n}(1, C_1, \dots, C_m; 1, C_1, \dots, C_n)$. By the inductive hypothesis and the commutative diagramme

$$\begin{array}{ccc} B_{U(m)} \times B_{U(n)} & \longrightarrow & B_{U(m+n)} \\ \uparrow & & \uparrow \\ B_{U(m-1)} \times B_{U(n)} & \longrightarrow & B_{U(m-1+n)} \end{array}$$

we have that

$$P_{m,n} \equiv (1 + C_1 + \dots + C_m) \times (1 + C_1 + \dots + C_n) \pmod{C_m}.$$

Similarly

$$P_{m,n} \equiv (1 + C_1 + \dots + C_m) \times (1 + C_1 + \dots + C_n) \pmod{C_n}.$$

Thus,

$$P_{m,n} \equiv (1 + C_1 + \cdots + C_m) \times (1 + C_1 + \cdots + C_n) \pmod{C_m \times C_n}.$$

and

$$P_{m,n} = (1 + C_1 + \cdots + C_m) \times (1 + C_1 + \cdots + C_n) + C_m \times C_n.$$

Since $j^*C_{m+n} = C_m \times C_n$, $k = 0$. □

Theorem 2.5. For $j : B_{\mathbb{U}(1)}^n \rightarrow B_{\mathbb{U}(n)}$, $j^*(C_i) = \sigma_i^{(n)}(u_1, \dots, u_n)$, the i -th elementary symmetric function in u_1, \dots, u_n , where u_j is the generator of $H^2(B_{\mathbb{U}(1)}; \mathbb{Z})$ in the j -th factor.

Proof. Again the argument is inductive. For the composition

$$B_{\mathbb{U}(1)}^{n-1} \times B_{\mathbb{U}(1)} \xrightarrow{k} B_{\mathbb{U}(n-1)} \times B_{\mathbb{U}(1)} \xrightarrow{\ell} B_{\mathbb{U}(n)},$$

we have $j = \ell \circ k$ and

$$\begin{aligned} k^* \circ \ell^*(1 + C_1 + \cdots + C_n) &= k^*((1 + C_1 + \cdots + C_{n-1}) \times (1 + u_n)) \\ &= (1 + \sigma^{(n-1)} + \cdots + \sigma_{(n-1)}^{(n-1)}) \times (1 + u_n) \\ &= 1 + \sigma_1^{(n)} + \cdots + \sigma_n^{(n)}. \end{aligned}$$

□

2.1.3 Bott Periodicity Theorem for $B_{\mathbb{U}}$

In the proof of Theorem 2.2 we defined maps

$$f_k : \mathbb{S}^1 \# P^{k-1}(\mathbb{C}) \rightarrow \mathrm{SU}(k).$$

Passing to the limit using the weak topology, we thus determine a map

$$f : \mathbb{S}^1 \# P^\infty(\mathbb{C}) \rightarrow \mathrm{SU},$$

which induces a monomorphism in homology such that $\mathrm{im}(f_*)$ generates $H_*(\mathrm{SU}; \mathbb{Z})$ as an algebra.

The adjoint of f is the composition

$$P^\infty(\mathbb{C}) \xrightarrow{i} \Omega \Sigma P^\infty(\mathbb{C}) \xrightarrow{\Omega f} \Omega \mathrm{SU}.$$

The map f induces a map of the path-loop fibration of $\Sigma P^\infty(\mathbb{C})$ into that of SU . Let λ_i denote an additive generator of $H_{2i}(P^\infty(\mathbb{C}); \mathbb{Z})$ and σ_* homology suspension in the homology spectral sequences of the fibrations. Then $\Sigma_* \lambda_* = \pm \sigma_* i_*(\lambda_i)$, where $\Sigma_* : H_*(P^\infty(\mathbb{C})) \xrightarrow{\cong} H_*(\Sigma P^\infty(\mathbb{C}))$ is the suspension isomorphism. Since the classes

$$\pm f_* \Sigma_* \lambda_i = f_* \sigma_* i_*(\lambda_i) = \sigma_* \circ (\Omega f)_* \circ i_*(\lambda_i)$$

generate $H_*(\mathrm{SU})$, it is transgressively generated. The comparison theorem for spectral sequences then shows the

Lemma 2.6. $H_*(\Omega \text{SU}; \mathbb{Z})$ is a polynomial algebra $P(\lambda_1, \dots)$, with $\dim \lambda_i = 2i$ and $\lambda_i = (\Omega f)_* i_*(\lambda_i)$.

We next define a map $b : B_U \rightarrow \Omega \text{SU}$, which we will later show is a homotopy equivalence. We proceed by defining

$$b_{m,n} : \text{U}(m+n)/(\text{U}(m) \times \text{U}(n)) \rightarrow \Omega \text{SU}(m+n).$$

As before, let $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$. For $x \in \mathbb{C}^{m+n}$, $x = x_1 + x_2$ with $x_1 \in \mathbb{C}^m$ and $x_2 \in \mathbb{C}^n$. For a real number θ , let $\varphi_{m,n}(\theta)(x) = x_1 e^{i\theta} + x_2 e^{-\theta}$.

For $A \in \text{U}(m+n)$, let $\tilde{b}_{m,n}(A)$ be the loop in $\text{SU}(m+n)$ defined by the commutator

$$\tilde{b}_{m,n}(A)(\theta) = [\varphi_{m,n}(\theta), A], \quad 0 \leq \theta \leq \pi.$$

For $A \in \text{U}(m) \times \text{U}(n)$, $\tilde{b}_{m,n}(A)$ is the trivial loop. Thus, $\tilde{b}_{m,n}$ induces

$$b_{m,n} : \text{U}(m+n)/(\text{U}(m) \times \text{U}(n)) \rightarrow \Omega \text{SU}(m+n).$$

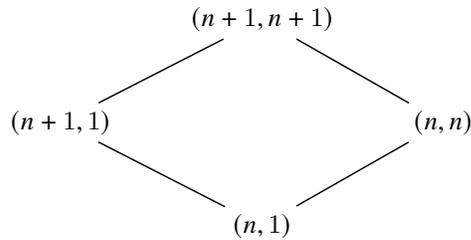
For $(m, n) \leq (m', n')$ the inclusion

$$\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n \rightarrow \mathbb{C}^{m'} \oplus \mathbb{C}^{n'} = \mathbb{C}^{m'+n'}$$

induces a commutative diagramme

$$\begin{array}{ccc} \text{U}(m+n)/(\text{U}(m) \times \text{U}(n)) & \xrightarrow{b_{m,n}} & \Omega \text{SU}(m+n) \\ \downarrow & & \downarrow \\ \text{U}(m'+n')/(\text{U}(m') \times \text{U}(n')) & \xrightarrow{b_{m',n'}} & \Omega \text{SU}(m'+n'). \end{array}$$

The lattice of pairs



yields a commutative diagramme

$$\begin{array}{ccccc}
 P^n \mathbb{C} & \xrightarrow{\quad\quad\quad} & \Omega \mathrm{SU}(n+1) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & P^{n+1} \mathbb{C} & \xrightarrow{\quad\quad\quad} & \Omega \mathrm{SU}(n+2) \\
 & & \downarrow & & \downarrow \\
 \mathrm{U}(2n)/(\mathrm{U}(n) \times \mathrm{U}(n)) & \xrightarrow{\quad\quad\quad} & \Omega \mathrm{SU}(n+2) & & \\
 & \searrow & \downarrow & \searrow & \\
 & & \mathrm{U}(2n+2)/(\mathrm{U}(n+1) \times \mathrm{U}(n+1)) & \xrightarrow{\quad\quad\quad} & \Omega \mathrm{SU}(2n+2).
 \end{array}$$

and letting $B_{\mathrm{U}} = \varinjlim \mathrm{U}(2n)/(\mathrm{U}(n) \times \mathrm{U}(n))$ and passing to the limit we obtain the commutative diagramme

$$\begin{array}{ccc}
 P^\infty(\mathbb{C}) & \xrightarrow{b_{\infty,1}} & \Omega \mathrm{SU} \\
 k \downarrow & & \downarrow \mathrm{id} \\
 B_{\mathrm{U}} & \xrightarrow{b} & \Omega \mathrm{SU}
 \end{array}$$

The map $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$ carrying $(x_i, 0)$ into x_{2i} and $(0, x_j)$ into x_{2j+1} induces a map $B_{\mathrm{U}} \times B_{\mathrm{U}} \rightarrow B_{\mathrm{U}}$. This product is homotopy commutative and homotopy associative (such operations involve sequences of permutations of coordinates, but the permutation matrices lie in SU and, as it is connected, are homotopic to the identity; furthermore, for any element only finitely many permutations are involved; and so, the operation is continuous in the weak topology). By similar reasoning the map b preserves this product up to homotopy.

Theorem 2.7. $b : B_{\mathrm{U}} \rightarrow \Omega \mathrm{SU}$ is a homotopy equivalence.

Proof. If $b \circ k \simeq \Omega f \circ i : P^\infty(\mathbb{C}) \rightarrow \Omega \Sigma P^\infty(\mathbb{C}) \rightarrow \Omega \mathrm{SU}$, then by the previous lemma and the fact that b is a map of H-spaces, $H_*(b)$ is an epimorphism. Our computations show that $\mathrm{rank} H_{2n}(B_{\mathrm{U}}) = \mathrm{rank} H_{2n}(\Omega \mathrm{SU})$ for all n ; thus, $H_*(b)$ can have only 0 kernel. Hence $H_*(b)$ is an isomorphism and since B_{U} and $\Omega \mathrm{SU}$ are simply connected, the Whitehead theorem then implies b is a homotopy equivalence.

Since $b_{\infty,1} \simeq b \circ k$, it thus suffices to show that $\Omega f \circ i = b_{\infty,1}$. This we will do by direct computation. The map $\bar{b}_{n,1}$ can be regarded as the composition

$$\mathbb{S}^1 \times P^n(\mathbb{C}) \rightarrow \mathbb{S}^1 \times \frac{\mathrm{U}(n+1)}{\mathrm{U}(n) \times \mathrm{U}(1)} \xrightarrow{\bar{b}_{n,1}} \mathrm{SU}(n+1)$$

where $P^n(\mathbb{C}) \rightarrow \mathrm{U}(n+1)/(\mathrm{U}(n) \times \mathrm{U}(1))$ is the homeomorphism taking the class $(x) \in P^n(\mathbb{C})$ into the class of any matrix B , with $Be_1 = x$, x of unit norm, where

as before $\mathbb{C}^{n+1} = \mathbb{C}^n \oplus \mathbb{C}^1$ and $e_1 \in \mathbb{C}^1$. This is well defined for if $x = \lambda y$ and $\overline{\lambda}e_1 = y$, then $\lambda\overline{\lambda}e_1 = x$. But $B^{-1} \cdot \lambda\overline{\lambda} \in U(n) \times U(1)$. In the above composition

$$(e^{2i\theta}, (x)) \text{ goes into } [\varphi_{n,1}(\theta), B] = T(\theta, x), \quad 0 \leq \theta \leq \pi.$$

$$\begin{aligned} T(\theta, x)(x) &= \varphi_{n,1}(\theta) \cdot B \cdot \varphi_{n,1}^{-1}(\theta) \cdot B^{-1}x \\ &= \varphi_{n,1}(\theta) \cdot B \cdot e_1 \cdot e^{i\theta} \\ &= \varphi_{n,1}(\theta) \cdot x \cdot e^{i\theta} \\ &= (e^{i\theta}x_1 + e^{-i\theta}x_2) \cdot e^{i\theta} \\ &= e^{2i\theta}x + (1 - e^{2i\theta})x_2. \end{aligned}$$

If $y \perp x$, then $y = B \cdot \gamma$ for some $\gamma \perp e_1$. Then

$$\begin{aligned} T(\theta, x)(y) &= \varphi_{n,1}(\theta) \cdot B \cdot \varphi_{n,1}^{-1}(\theta) \cdot B^{-1}y \\ &= \varphi_{n,1}(\theta) \cdot B \cdot \varphi_{n,1}^{-1}(\theta) \cdot \gamma \\ &= \varphi_{n,1}(\theta) \cdot B \cdot \gamma \cdot e^{i\theta} \\ &= \varphi_{n,1}(\theta) \cdot y \cdot e^{i\theta} \\ &= (e^{i\theta}y_1 + e^{-i\theta}y_2) \cdot e^{i\theta} \\ &= y + (e^{-2i\theta} - 1) \cdot y_2. \end{aligned}$$

Let \overline{f}_{n+1} denote f_{n+1} , before normalised. Then

$$T(\theta, x)(x) = \overline{f}_{n+1}(\theta, x) \cdot M(I_n \oplus +e^{2i\theta}).$$

But $T(\theta, x) \in \text{SU}(n+1)$; and so, $\det(\overline{f}) \cdot e^{-2\theta} = 1$. Thus

$$T(\theta, x) = f_{n+1}(\theta, x).$$

We have shown that $f : \Sigma P^\infty(\mathbb{C}) \rightarrow \text{SU}$ is the adjoint of $b_{\infty,1} : P^\infty(\mathbb{C}) \rightarrow \Omega \text{SU}$, which completes the argument. \square

2.2 Complex Vector Bundles

2.2.1 Characteristic Classes

As we have seen earlier, the decomposition $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$ induces a map

$$j : B_{U(m)} \times B_{U(n)} \rightarrow B_{U(m+n)}.$$

There is a similar decomposition involving tensor products over \mathbb{C}

$$\mathbb{C}^{mn} = \mathbb{C}^m \otimes \mathbb{C}^n,$$

which induces a monomorphism $U(m) \times U(n) \rightarrow U(mn)$ and a subsequent map

$$k : B_{U(m)} \times B_{U(n)} \rightarrow B_{U(mn)}.$$

By the map $i : B_{\mathbb{U}(1)}^n \rightarrow B_{\mathbb{U}(n)}$ we define a class $\text{ch}(n) \in H^{**}(B_{\mathbb{U}(n)}; \mathbb{Q})$ by the formula $i^{**}(\text{ch}(n)) = \sum_{\ell=1}^n e^{u_\ell}$, whose u_ℓ is the image in $H^2(B_{\mathbb{U}(1)}; \mathbb{Z})$ of the generator u_ℓ of $H^2(B_{\mathbb{U}(1)}; \mathbb{Z})$ in the ℓ -th factor (strictly, $i^{**}(\text{ch}(n)) = \sum_{\ell=1}^n 1 \times 1 \times \cdots \times 1 \times e^{u_\ell} \times 1 \times \cdots \times 1$). The class $\text{ch}(n)$ is called the *Universal Chern Character*.

Proposition 2.8. For $j : B_{\mathbb{U}(m)} \times B_{\mathbb{U}(n)} \rightarrow B_{\mathbb{U}(m+n)}$, $j^{**} \text{ch}(m+n) = \text{ch}(m) \times 1 + 1 \times \text{ch}(n)$.

Proof. The composition

$$B_{\mathbb{U}(1)}^{m+n} \twoheadrightarrow B_{\mathbb{U}(1)}^m \times B_{\mathbb{U}(1)}^n \rightarrow B_{\mathbb{U}(m)} \times B_{\mathbb{U}(n)} \xrightarrow{j} B_{\mathbb{U}(m+n)}$$

is the map $i : B_{\mathbb{U}(1)}^{m+n} \rightarrow B_{\mathbb{U}(m+n)}$, used to define $\text{ch}(m+n)$. But $\text{ch}(m) \times 1 + 1 \times \text{ch}(n)$ is the unique class in $H^{**}(B_{\mathbb{U}(m)} \times B_{\mathbb{U}(n)}; \mathbb{Q})$ which pulls back to

$$\sum_1^{m+n} e^{u_\ell} = \sum_1^m e^{u_\ell} + \sum_1^n e^{u_\ell}.$$

□

Proposition 2.9. For $k : B_{\mathbb{U}(m)} \times B_{\mathbb{U}(n)} \rightarrow B_{\mathbb{U}(mn)}$, $k^{**} \text{ch}(mn) = \text{ch}(m) \times \text{ch}(n)$.

Proof. The following diagramme is commutative:

$$\begin{array}{ccc} B_{\mathbb{U}(m)} \times B_{\mathbb{U}(n)} & \longrightarrow & B_{\mathbb{U}(mn)} \\ \uparrow & & \uparrow \\ B_{\mathbb{U}(1)}^m \times B_{\mathbb{U}(1)}^n & \xrightarrow{\ell} & B_{\mathbb{U}(1)}^{mn} \end{array}$$

where ℓ is the composition

$$B_{\mathbb{U}(1)}^m \times B_{\mathbb{U}(1)}^n \xrightarrow{(\Delta_n)^m \times (\Delta_m)^n} (B_{\mathbb{U}(1)}^n)^m \times (B_{\mathbb{U}(1)}^m)^n \rightarrow (B_{\mathbb{U}(1)} \times B_{\mathbb{U}(1)})^{mn} \xrightarrow{\mu} B_{\mathbb{U}(1)}^{mn}$$

with $\mu : B_{\mathbb{U}(1)} \times B_{\mathbb{U}(1)} \rightarrow B_{\mathbb{U}(1)}$ the canonical H-space structure.

For $1 \leq i \leq mn$, i can be written uniquely as $i = (k-1) \cdot n + j$ with $1 \leq k \leq m$ and $1 \leq j \leq n$. The class e^{w_i} in $H^{**}(B_{\mathbb{U}(1)}^{mn}; \mathbb{Q})$ under μ^{**} , maps as follows

$$\mu^{**}(e^{w_i}) = \mu^{**} \left(\sum_{\ell=0}^{\infty} \frac{w_i^\ell}{\ell!} \right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{p=0}^{\ell} \binom{\ell}{p} w_i^p \times w_i^{\ell-p} = e^{w_i} \times e^{w_i}.$$

Thus, $\ell^{**}(e^{w_i}) = e^{u_k} \times e^{v_j}$, with k and j as above. Finally,

$$\ell^{**} \left(\sum_1^{mn} e^{w_i} \right) = \left(\sum_1^m e^{u_k} \right) \times \left(\sum_1^n e^{v_j} \right)$$

since all combinations of k and j occur. The commutative diagramme then implies the conclusion. □

For each map $f : X \rightarrow B_{U(n)}$, there is defined the induced $U(n)$ -principal bundle $f^*(\gamma^n)$ over X . The spaces $B_{U(n)}$ are called *universal base spaces* because of the following theorem.

Theorem 2.10 (Bundle Classification Theorem). *Let X be a (normal) paracompact Hausdorff space. There is a 1–1 correspondence between equivalence classes of n -plane complex vector bundles over X and homotopy classes $[X, B_{U(n)}]$ given by the equivalence class of the associated n -plane complex vector bundle of the principal $U(n)$ -bundle $f^*(\gamma^n)$ for $f : X \rightarrow B_{U(n)}$.*

If only the equivalence class of a bundle τ over X is relevant, we shall frequently identify τ with its map $\tau : X \rightarrow B_{U(n)}$.

Definition 2.11 (Sum and Product of Plane Bundles). The equivalence class $\tau^m + \tau^n$ of the *sum* of m -plane and n -plane complex vector bundles τ^m and τ^n is the associated $(m+n)$ -plane complex vector bundle to the principal bundle induced by the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\tau^m \times \tau^n} B_{U(m)} \times B_{U(n)} \xrightarrow{j} B_{U(m+n)}.$$

The equivalence class $\tau^m \otimes \tau^n$ of the *product* of m -plane and n -plane complex vector bundles τ^m and τ^n is the associated $(m \cdot n)$ -plane complex vector bundle to the principal bundle induced by the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\tau^m \times \tau^n} B_{U(m)} \times B_{U(n)} \xrightarrow{k} B_{U(mn)}.$$

Definition 2.12. For an n -plane complex vector bundle γ over X , let $\gamma : X \rightarrow B_{U(n)}$ be a representative of the homotopy class determined by γ . The *Chern classes*, $\{C_i(\gamma)\}$, of γ are

$$C_i(\gamma) = \gamma^*(C_i) \in H^{2i}(X; \mathbb{Z});$$

the *Chern character*, $\text{ch}(\gamma)$, of γ is

$$\text{ch}(\gamma) = \gamma^{**} \text{ch}(n) \in H^{**}(X; \mathbb{Q}).$$

There follows immediately from the Lemma 2.6, and the above Propositions 2.8 and 2.9 and the

Proposition 2.13. *For m -plane and n -plane complex vector bundles γ^m and γ^n over X ,*

(a)

$$(1 + C_1(\gamma^m) + \cdots + C_m(\gamma^m)) \cup (1 + C_1(\gamma^n) + \cdots + C_n(\gamma^n)) = 1 + C_1(\gamma^m \oplus \gamma^n) + \cdots + C_{m+n}(\gamma^m \oplus \gamma^n),$$

(b) $\text{ch}(\gamma^m) + \text{ch}(\gamma^n) = \text{ch}(\gamma^m \oplus \gamma^n)$, and

(c) $\text{ch}(\gamma^m) \cup \text{ch}(\gamma^n) = \text{ch}(\gamma^m \otimes \gamma^n)$.

From this proposition it appears that if a cohomology theory can be constructed with elements of the groups being complex vector bundles, then ch will furnish a multiplicative transformation of this theory in $H^{**}(-; \mathbb{Q})$. We shall see in the next chapter that this is indeed the case.

2.2.2 Complex Vector Bundles over Spheres

From the homotopy equivalences $B_U \rightarrow \Omega SU$ and $\Omega B_{SU} \rightarrow SU$ and the fact that topologically $\mathbb{S}^1 \times SU$ is homeomorphic to U , we can immediately deduce that for all $i \geq 1$, $\pi_i(B_U) \cong \pi_{i+2}(B_U)$. It is easy to compute that $\pi_1(B_U) = 0$ and $\pi_2(B_U) \cong \mathbb{Z}$.

It is a general result (called the *Eckmann-Hilton argument*) that if X is a co-H-space and Y is an H-space, then the two group structures on $[X, Y]$, due respectively to the co-H-space structure of X and the H-space structure of Y , coincide and are in fact abelian. For a proof, see Appendix D.

If we use the H-space structure on B_U defined by the map $j : B_U \times B_U \rightarrow B_U$ inducing sums of bundles and the usual co-H-space structure on \mathbb{S}^{2n} , we have then that if $\lambda : \mathbb{S}^{2n} \rightarrow B_U$ is in the class of a generator of $\pi_{2n}(B_U)$, then the sum of k copies of the bundle induced by λ is equivalent to the bundle induced by $k\lambda \in \pi_{2n}(B_U)$. It follows then that $\text{ch}(k\lambda) = k \text{ch}(\lambda)$. The class $\text{ch}(\lambda)$ has a component $\text{ch}_n(\lambda)$ in dimension $2n$ in $H^{2n}(\mathbb{S}^{2n}; \mathbb{Q})$; this class has value a rational number on the generating homology class of $H_{2n}(\mathbb{S}^{2n}; \mathbb{Z})$. We shall show this number is ± 1 , establishing the

Theorem 2.14. *Let γ be an m -plane complex vector bundle over the sphere \mathbb{S}^{2n} , then $\langle \text{ch}_n(\gamma), \mathbb{S}^{2n} \rangle$ is an integer; if $m > n$ and $\gamma : \mathbb{S}^{2n} \rightarrow B_{U(m)}$ is a generator of $\pi_{2n}(B_{U(m)})$, then $\langle \text{ch}_n(\gamma), \mathbb{S}^{2n} \rangle = \pm 1$.*

In the previous discussion we considered only maps into B_U ; however, the fibrations $\mathbb{S}^{2n-1} \rightarrow B_{U(n-1)} \rightarrow B_U$ show the homomorphism $\pi(B_{U(m)}) \rightarrow \pi_n(B_U)$ is an isomorphism for $n \leq 2m - 1$. For $m \leq n$, we add enough trivial bundles to get in this “stable” range and the conclusion follows (it is not very interesting in this case, however, as the integer obtained is always 0).

We wish, then, to determine the Chern character of the generating bundle over \mathbb{S}^{2n} . To this end we need a good bit of technical preparation, and the remainder of this section is concerned with that.

Proposition 2.15. *Let $A = P[u_1, u_2, \dots]$ be a polynomial Hopf algebra over the integers generated by elements u_i of dimension $2i$ with diagonal*

$$\psi(u_n) = \sum_{p+q=n} u_p \otimes u_q.$$

Then the dual A^ of A with respect to \mathbb{Z} is a polynomial Hopf algebra $P[a_1, a_2, \dots]$ generated by elements a_i of dimension $2i$ with diagonal*

$$\psi(a_n) = \sum_{p+q=n} a_p \otimes a_q.$$

Furthermore, the generators a_i may be taken to be \bar{u}_1^i , the duals of u_1^i ; in this case

$$\bar{u}_k = (-1)^{k-1} k a_k + \text{decomposable elements.}$$

Proof. Let C be a coalgebra with generators $\{e_1, e_2, \dots\}$, $\dim e_i = 2i$, and diagonal

$$\psi(e_n) = \sum_{p+q=n} e_p \otimes e_q.$$

Define $f : C \rightarrow A$ by $f(e_i) = u_i$. Then f is a morphism of coalgebras. Let g_k be the composition

$$\otimes^k C \xrightarrow{\otimes^k f} \otimes^k A \rightarrow A$$

where $\otimes^k X$ denotes the tensor product of k copies of X and the homomorphism on the right is given by the algebra structure of A . Then

$$g_k(e_{i_1}^{(1)} \times \dots \times e_{i_k}^{(k)}) = u_{i_1} \cdots u_{i_k}.$$

Thus, g_k is epimorphic in dimensions less than $2k + 2$.

C^* is the polynomial algebra in one indeterminate of dimension 2, and $(\otimes^k C)^*$ is the polynomial ring in k indeterminates x_1, \dots, x_k , each of dimension 2, which can be chosen so that

$$e_{ij}^{(\ell)} = \bar{x}_\ell^{ij}.$$

The dual of g

$$g^* : A^* \rightarrow (\otimes^k C)^*$$

is a homomorphism of algebras and is a monomorphism in dimensions less than $2k + 2$. In particular,

$$g^*(\overline{u_{i_1} \cdots u_{i_k}}) = \sum_{\sigma \in S_k} x_1^{i_{\sigma(1)}} \otimes \cdots \otimes x_k^{i_{\sigma(k)}},$$

where σ ranges through the symmetric group S_k on k symbols. Then the image of g^* is generated by $\sigma_1, \dots, \sigma_k$, elementary symmetric functions in x_1, \dots, x_k , in dimensions less than $2k + 2$. Define $a_i \in A^*$ by

$$g^*(a_i) = \sigma_i, \quad i \leq k.$$

Then $a_i = \bar{u}_1^i$, and the a_i generate A^* as an algebra in dimensions less than or equal to $2k + 1$.

The diagonal map is given by

$$\langle \Delta(a_i), u_1^p \otimes u_1^q \rangle = \langle a_i, u_1^{p+q} \rangle = \begin{cases} 1 & \text{if } i = p + q, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\Delta(a_i) = \sum_{p+q=i} a_p \otimes a_q$.

Finally, $g^*(\bar{u}_k) = x_1^k + \cdots + x_k^k = \gamma \cdot \sigma_k + F(\sigma_1, \dots, \sigma_{k-1})$ for some integer γ and polynomial F over \mathbb{Z} . To evaluate γ , let x_1, \dots, x_k be the k^{th} roots of 1. Then $\sigma_k = (-1)^{k-1}$ and $\sigma_i = 0$ for $i \neq k$. Thus, $k = (-1)^{k-1} \gamma$, as asserted. Since g^* is a monomorphism,

$$\bar{u}_k = (-1)^{k-1} k a_k + \text{decomposable elements.}$$

□

Corollary 2.16. For $b : B_U \rightarrow \Omega SU$,

$$b^*(\bar{\lambda}_1^i = \pm C_i + \text{decomposable elements.}$$

The following construction is due to Hopf. Given a base-product preserving map $f : X \times Y \rightarrow Z$, there is a map

$$Sf : X \circ Y \rightarrow \Sigma Z$$

of the join of X with Y to the suspension of Z given by

$$Sf(x, t, y) = (t, f(x, y)).$$

This carries $(x, 0, y)$, $(x, 1, y)$ and $(*, t, *)$ into the base point; and so, it induces the map asserted.

As examples, two of the Hopf maps $\mathbb{S}^3 \rightarrow \mathbb{S}^2$ and $\mathbb{S}^7 \rightarrow \mathbb{S}^4$ are given by complex and quaternionic multiplications $\mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and $\mathbb{S}^3 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3$ of their respective unit spheres.

We note also the homotopy equivalence $X \circ Y \xrightarrow{\cong} \Sigma(X \# Y)$.

For $X = \mathbb{C}P^\infty$, since X is an H-space with multiplication $m : X \times X \rightarrow X$, there is defined

$$\zeta : X \circ X \rightarrow \Sigma X.$$

If λ_n denotes the generator of $H_{2n}(X; \mathbb{Z})$, then

$$\begin{aligned} m^*(\bar{\lambda}_1) &= \bar{\lambda}_1 \otimes 1 + 1 \otimes \bar{\lambda}_1; \quad \text{and so} \\ m^*(\bar{\lambda}_1^k) &= \sum_{i=0}^k \binom{k}{i} \bar{\lambda}_1^i \otimes \bar{\lambda}_1^{k-i}. \end{aligned}$$

Equivalently,

$$m^*(\bar{\lambda}_k) = \sum_{i=0}^k \binom{k}{i} \bar{\lambda}_i \otimes \bar{\lambda}_{k-i}.$$

If we restrict the map ζ to $\mathbb{S}^2 \circ X$, the 2-skeleton of X being \mathbb{S}^2 , we obtain

$$\zeta : \Sigma^3 X \rightarrow \Sigma X$$

and

$$\zeta^*(\Sigma^* \bar{\lambda}_k) = k \cdot \Sigma^3 \bar{\lambda}_{k-1}, \quad \text{for } k \geq 2.$$

Theorem 2.17. The group $\pi_{2n+1}(\text{SU})$ is generated by the composition

$$\mathbb{S}^{2n+1} \rightarrow \Sigma^{2n-1} \mathbb{C}P^\infty \xrightarrow{\Sigma^{2n-4} \zeta} \Sigma^{2n-3} \mathbb{C}P^\infty \rightarrow \dots \rightarrow \Sigma^3 \mathbb{C}P^\infty \xrightarrow{\zeta} \Sigma \mathbb{C}P^\infty \xrightarrow{f} \text{SU}.$$

The image of the generator under the Hurewicz homomorphism $\pi_{2n+1}(\text{SU}) \rightarrow H_{2n+1}(\text{SU})$ is $n!v_{n+1}$.

Proof. Consider first $\Sigma^3 \mathbb{C}P^\infty \xrightarrow{\zeta} \Sigma \mathbb{C}P^\infty \xrightarrow{f} \text{SU}$. By Proposition 2.15, $H^*(\Omega \text{SU}) = P[\bar{\lambda}_1, \bar{\lambda}_1^2, \dots]$, and $\bar{\lambda}_k = (-1)^{k-1} k \bar{\lambda}_1^k +$ decomposable elements. Let

$$\xi : \Omega \Sigma^3 \mathbb{C}P^\infty \rightarrow \Omega \text{SU}$$

be the loop map of the composition $f \circ \zeta$ and

$$i : \Sigma^2 \mathbb{C}P^\infty \rightarrow \Omega \Sigma \Sigma^2 \mathbb{C}P^\infty$$

the canonical inclusion of space in $\Omega \Sigma$ of it.

We shall first show that

$$i^* \circ \xi^*(\bar{\lambda}_1^k) = (-1)^{k-1} (\Sigma^2)^* \bar{\lambda}_{k-1}, \quad k = 2, 3, \dots$$

Since the λ_i generate $H_*(\Omega \text{SU})$,

$$(\xi \circ i)_* \Sigma_*^2 \lambda_{k-1} = \gamma \cdot \lambda_k + F(\lambda_1, \dots, \lambda_{i-1}).$$

Taking homology suspension,

$$\sigma_*(\xi \circ i)_* \Sigma_*^2 \lambda_{k-1} = \gamma \sigma_* \lambda_k,$$

since σ_* annihilates decomposable elements. We have seen earlier that

$$\sigma_* \lambda_k = \nu_{k+1} \in H_*(\text{SU}).$$

Also,

$$\begin{aligned} \sigma_*(\xi \circ i)_* \Sigma_*^2 \lambda_{k-1} &= f_* \zeta_* \sigma_*(i_* \Sigma_*^2 \lambda_{k-1}) \\ &= f_* \zeta_*(\Sigma_*^3 \lambda_{k-1}) \\ &= f_*(k \Sigma_* \lambda_k) \\ &= k \cdot \nu_{k+1}. \end{aligned}$$

Thus, $\gamma \nu_{k+1} = k \nu_{k+1}$ and $\gamma = k$. Then dually, we have

$$(\xi \circ i)^* \bar{\lambda}_k = k \cdot (\Sigma^2)^* \bar{\lambda}_{k-1}.$$

Since i^* annihilates decomposable elements,

$$(\xi \circ i)^*((-1)^{k-1} k \bar{\lambda}_1^k) = k (\Sigma^2)^* \bar{\lambda}_{k-1},$$

and as sought

$$i \circ \xi^*(\bar{\lambda}_1^k) = (-1)^{k-1} (\Sigma^2)^* \bar{\lambda}_{k-1}.$$

Taking loops once more, we obtain the following diagramme where to conserve notation we have written the previous $i \circ \xi$ simply as ξ .

$$\begin{array}{ccccc} & & H^*(\Omega \Sigma^2 \mathbb{C}P^\infty) & \xleftarrow{(\Omega \xi)^*} & H^*(\Omega^2 \text{SU}) \\ & \swarrow i^* & \uparrow \sigma^* & & \uparrow \sigma^* \\ H^*(\Sigma \mathbb{C}P^\infty) & \xleftarrow{(\Sigma^*)^{-1}} & H^*(\Sigma^2 \mathbb{C}P^\infty) & \xleftarrow{\xi_*} & H^*(\Omega \text{SU}) \end{array}$$

For $i > 1$, let $\sigma^*(\bar{\lambda}_1^{i+1}) = e'_i \in H^*(\Omega^2 \text{SU})$. Then for $f' = \Omega\xi \circ i$, we have

$$f'^* e'_i = (-1)^i \Sigma^* \bar{\lambda}_i.$$

By the comparison theorem for spectral sequences¹, $H^*(\Omega^2 \text{SU})$ is an exterior algebra generated by e'_1 and the e'_i . Thus, $H_*(\Omega \text{SU})$ is an exterior algebra generated by \bar{e}'_1 and the \bar{e}'_i and

$$f'_*(\Sigma_* \lambda_i) = (-1)^i \bar{e}'_i.$$

Since SU is the simply connected covering space of $\Omega^2 \text{SU}$ (up to homotopy), we have the commutative diagramme

$$\begin{array}{ccc} & & \text{SU} \\ & \nearrow f_{(1)} & \downarrow \\ \Sigma \mathbb{C}P^\infty & \xrightarrow{f'} & \Omega^2 \text{SU} \simeq \text{U} \end{array}$$

for which $f_{(1)*}$ is a monomorphism in homology and $\text{im}(f_{(1)*})$ generates $H_*(\text{SU})$ as an exterior algebra. Iterating this process, we then see that if the lift in the diagramme

$$\begin{array}{ccc} & & \text{SU} \\ & \nearrow f_{(n-1)} & \downarrow \\ \Sigma \mathbb{C}P^\infty & \longrightarrow & \Omega^{2n-2} \text{SU} \simeq \text{U} \end{array}$$

is a monomorphism in homology with image generating as an algebra, then for

$$\Sigma^3 \mathbb{C}P^\infty \longrightarrow \Sigma \mathbb{C}P^\infty \xrightarrow{f_{(n-1)}} \Omega^{2n-2} \text{SU},$$

the construction can be repeated to obtain

$$\begin{array}{ccc} & & \text{SU} \\ & \nearrow f_{(n)} & \downarrow \\ \Sigma \mathbb{C}P^\infty & \longrightarrow & \Omega^{2n} \text{SU} \simeq \text{U} \end{array}$$

with $f_{(n)}$ inducing a monomorphism in homology with image generating.

Then by the Hurewicz Theorem, the composition

$$\mathbb{S}^3 \rightarrow \Sigma_* \mathbb{C}P^\infty \rightarrow \Omega^{2n-2} \text{SU}$$

¹Zeeman, Erik Christopher (1957), "A proof of the comparison theorem for spectral sequences", Proc. Cambridge Philos. Soc., 53: 57-62

generates $\pi_3(\Omega^{2n-2} \text{SU})$. This composition is adjoint to the one asserted in the theorem; and so, the conclusion follows.

In homology, the class (\mathbb{S}^{2n+1}) is mapped successively into

$$\Sigma_*^{2n-1} \lambda_1, \quad \Sigma_*^{2n-3} \lambda_2, \quad \dots, \quad n! \Sigma_* \lambda_n, \quad n! \nu_{n+1},$$

which is the second part of the conclusion. \square

Corollary 2.18.

$$\pi_i(\text{U}(n)) \cong \begin{cases} 0 & \text{for } i \text{ even, } i < 2n \\ \mathbb{Z} & \text{for } i \text{ odd, } i < 2n \\ \mathbb{Z}/n!\mathbb{Z} & \text{for } i = 2n. \end{cases}$$

Proof. The computations of the homotopy groups of U by Theorem 2.7 together with the sequence of fibrations $\text{SU}(k-1) \rightarrow \text{SU}(k) \rightarrow \mathbb{S}^{2k-1}$ establish the first two assertions. To see the third consider the fibration

$$\begin{array}{ccc} \text{SU}(n) & \xrightarrow{i} & \text{SU}(n+1) \\ & & \downarrow \pi \\ & & \mathbb{S}^{2n+1}. \end{array}$$

In the proof of Theorem 2.2, the class ν_{n+1} was chosen so that $\pi_*(\nu_{n+1})$ generated $H_{2n+1}(\mathbb{S}^{2n+1}; \mathbb{Z})$. Since the generating map $g_n : \mathbb{S}^{2n+1} \rightarrow \text{SU}$ of Theorem 2.17 factors through $\text{SU}(n+1)$, by the above comments, and if ι is a generator of $H_{2n+1}(\mathbb{S}^{2n+1}; \mathbb{Z})$, $g_{n*}(\iota) = n! \nu_{n+1}$, the composition $\pi \circ g_n : \mathbb{S}^{2n+1} \rightarrow \mathbb{S}^{2n+1}$ has degree $n!$. As $\pi_{2n}(\text{SU}(n+1)) = 0$, the homotopy sequence of the above fibration then shows $\pi_{2n}(\text{SU}(n)) \cong \mathbb{Z}/n!\mathbb{Z}$. \square

Corollary 2.19. *If $h : \mathbb{S}^{2n} \rightarrow B_{\text{U}}$ generates $\pi_{2n}(B_{\text{U}})$, then*

$$\langle h_*(C_n), \mathbb{S}^{2n} \rangle = \pm(n-1)!$$

Proof. Consider the commutative diagramme

$$\begin{array}{ccccc} H_*(\mathbb{S}^{2n}) & \xrightarrow{i_*} & H_*(\Omega \mathbb{S}^{2n+1}) & \xrightarrow{(\Omega g)_*} & H_*(\Omega \text{SU}) \\ & \searrow \Sigma_* & \downarrow \sigma_* & & \downarrow \sigma_* \\ & & H_*(\mathbb{S}^{2n+1}) & \xrightarrow{g_*} & H_*(\text{SU}), \end{array}$$

where $g : \mathbb{S}^{2n+1} \rightarrow \text{SU}$ generates $\pi_{2n+1}(\text{SU})$ as in Theorem 2.17. Then

$$\sigma_* \circ (\Omega g)_* \circ i_*(\mathbb{S}^{2n}) = g_*(\Sigma_* \mathbb{S}^{2n}) = g_*(\mathbb{S}^{2n+1}) = n! \nu_{n+1}.$$

As we observed in the proof of Theorem 2.17,

$$\sigma_*(\lambda_n) = \nu_{n+1}$$

and σ_* annihilates decomposable elements. The map $h = \Omega g \circ i : \mathbb{S}^{2n} \rightarrow \Omega \text{SU}$ generates $\pi_{2n}(\Omega \text{SU})$ and we have

$$h_*(\mathbb{S}^{2n}) = n! \lambda_n + \text{decomposable elements.}$$

Thus $\langle h^* \bar{\lambda}_n, \mathbb{S}^{2n} \rangle = n!$, and since $\bar{\lambda}_n = (-1)^{n-1} n \cdot \bar{\lambda}_1^n + \text{decomposable elements}$,

$$\langle h^* \bar{\lambda}_1^n, \mathbb{S}^{2n} \rangle = (-1)^{n-1} (n-1)!.$$

Corollary 2.16 completes the proof. \square

To complete the proof of Theorem 2.14 we need to relate $\text{ch}_n(\gamma)$ to the Chern classes of γ . This is an exercise in symmetric function theory. Recall that the i -th Chern class C_i is defined in terms of the elementary symmetric function $\sigma_i^{(n)}$ in n indeterminates u_1, \dots, u_n , $i \leq n$, and $\text{ch}_i = \sum_{k=1}^n u_k^i / i!$. We derive a formula known as the ‘‘Newton formula’’:

$$\begin{aligned} \prod_{i=1}^n (1 + u_i t) &= 1 + \sigma_1 t + \sigma_2 t^2 + \cdots + \sigma_n t^n \\ \frac{d}{dt} \prod_{i=1}^n (1 - u_i t) &= \prod_{i=1}^n (1 - u_i t) \cdot \sum_{k=1}^n \frac{-u_k}{1 - u_k t} \\ &= - \prod_{i=1}^n (1 - u_i t) \cdot \sum_{k=1}^n \sum_{j=1}^{\infty} (u_k^{j+1} t^j) \\ &= - \prod_{i=1}^n (1 - u_i t) \cdot \sum_{j=1}^{\infty} p_{j+1} t^j, \end{aligned}$$

where $p = u_1^r + \cdots + u_n^r$. Thus,

$$-\sigma_1 + 2\sigma_2 t - 3\sigma_3 t^2 + \cdots = -(1 - \sigma_1 t + \sigma_2 t^2 - \cdots)(p_1 + p_2 t + p_3 t^2 + \cdots).$$

Equating the coefficients of t^{n-1} , we have

$$\begin{aligned} (-1)^n \cdot n \cdot \sigma_n &= -[p_n - p_{n-1} \sigma_1 + \cdots + (-1)^{n-1} p_1 \sigma_{n-1}] \quad \text{or} \\ p_n - p_{n-1} \sigma_1 + \cdots + (-1)^{n-1} p_1 \sigma_{n-1} + (-1)^n \cdot n \cdot \sigma_n &= 0. \end{aligned}$$

Thus, since $n! \text{ch}_n = p_n$,

$$(n-1)! \text{ch}_n = \pm C_n + \text{decomposable elements,}$$

By this remark and Corollary 2.19, we see

$$(n-1)! \langle h^* \text{ch}_n, \mathbb{S}^{2n} \rangle = \pm \langle h^* C_n, \mathbb{S}^{2n} \rangle = \pm (n-1)!$$

and

$$\langle h^* \text{ch}_n, \mathbb{S}^{2n} \rangle = \pm 1.$$

Chapter 3

The Cohomology Theory $K_{\mathbb{C}}^*$

3.1 Basic Properties of $K_{\mathbb{C}}^*$

3.1.1 Definition of $K_{\mathbb{C}}^*$

The groups $K_{\mathbb{C}}^0$ are a special case of the universal enveloping abelian group of a monoid. Let G be a set of objects with an associative operation \oplus and a unit 0 . Let A be the free abelian group on the objects of G modulo the subgroup generated by elements of the form

$$[\alpha \oplus \beta] - [\alpha] - [\beta]$$

Then A is an abelian group and the transformation

$$e : G \rightarrow A$$

defined by $e(g) = [g] \in A$, preserves sums and unit element. A is characterised by the property if $f : G \rightarrow B$ is a transformation of G to an abelian group B preserving sums and unit, there is a unique homomorphism $\bar{f} : A \rightarrow B$ of abelian groups such that the diagramme

$$\begin{array}{ccc} G & \xrightarrow{f} & B \\ & \searrow e & \nearrow \bar{f} \\ & A & \end{array}$$

commutes.

For the case where G is the set of equivalence classes of complex vector bundles over a space X , \oplus is sum of vector bundles, and 0 is the identity map of X into itself, the abelian group A is denoted $K_{\mathbb{C}}(X)$ and called the *Grothendieck group* of X . Throughout much of this chapter we consider only complex vector bundles and will write $K(X)$ instead of $K_{\mathbb{C}}(X)$. (Later we consider also real vector bundles and wish to use both $K_{\mathbb{C}}$ and $K_{\mathbb{R}}$). It is a simple verification that

the product of vector bundles induces a commutative ring structure on $K(X)$, with unit the trivial line bundle over X . The construction K is a contravariant functor from spaces to commutative rings with unit via induced bundles.

Proposition 3.1. $K(X) = [X, \mathbb{Z} \times B_U]$, for $X \in \mathcal{P}$.

Recall that \mathcal{P} is the category whose objects are finite cell complexes and whose maps are continuous maps.

Proof. It suffices to prove this isomorphism for connected spaces X . For

$$f : X \rightarrow \mathbb{Z} \times B_U,$$

$\text{im}(f) \subset n \times B_U$ since X is connected, and since $X \in \mathcal{P}$, $\text{im}(f) \subset n \times B_{U(m)} \subset n \times B_U$. Thus, f induces a complex m -plane bundle $\{f\}$ over X . Define

$$\theta : [X, \mathbb{Z} \times B_U] \rightarrow K(X)$$

by $\theta(\{f\}) = [\{f\}] \oplus [n] \oplus [-m] \in K(X)$.

To define an inverse map for $\xi \in K(X)$, $\xi = [\alpha] - [\beta]$, where α and β are vector bundles over X . By the Lemma which follows, there is a vector bundle γ over X such that $\beta \oplus \gamma$ is the trivial q -plane bundle; i.e., in $K(X)$ we have

$$-[\beta] = [\gamma] - [q], \quad \text{and} \quad \xi = [\alpha \oplus \gamma] - [q].$$

We define the image of ξ in $[X, \mathbb{Z} \times B_U]$ to be the class of a map $X \rightarrow n \times B_{U(n+q)}$, which induces the bundle $\alpha \oplus \gamma$ over X , where $\alpha \oplus \gamma$ is an $(n+q)$ -bundle. This is inverse to θ . \square

Lemma 3.2. *If β is a complex vector bundle over X , there is a complex vector bundle γ over X such that $\beta \oplus \gamma$ is a trivial bundle.*

Proof. The bundle β is induced by a map of X into some $G_{m,n}$, the Grassman manifold of n -planes in \mathbb{C} , with the bundle of orthogonal n -frames over it. The orthogonal bundle of m -frames of $G_{m,n}$ induces γ ; and as the sum of the two frame bundles is trivial, so is $\beta \oplus \gamma$. \square

For $(X, x_0) \in \mathcal{P}_0$, let $\tilde{K}(X)$ be the kernel of $K(X) \rightarrow K(x_0)$ induced by the inclusion $x \rightarrow X$. $K(x_0) \cong \mathbb{Z}$ and $\tilde{K}(X)$ is an ideal in $K(X)$. For $(X, A) \in \mathbb{P}^2$, we define

$$K(X, A) \equiv \tilde{K}(X/A).$$

Then $K(X, \emptyset) = \tilde{K}(X/\emptyset) \cong \tilde{K}(X^+) \cong K(X)$.

Using Proposition 3.1, we now define a cohomology theory $K(= K_{\mathbb{C}}^*)$ such that $K^0(X) = K(X)$. Let \mathcal{B}_U be the spectrum

$$\begin{aligned} (\mathcal{B}_U)_{2n} &= \mathbb{Z} \times B_U \\ (\mathcal{B}_U)_{2n+1} &= U, \end{aligned}$$

with maps $\Sigma(\mathbb{Z} \times B_U) \rightarrow U$ and $\Sigma U \rightarrow \mathbb{Z} \times B_U$ adjoint to the homotopy equivalences $\mathbb{Z} \times B_U \rightarrow \Omega U$ and $U \rightarrow \Omega(\mathbb{Z} \times B_U)$; the base point of $\mathbb{Z} \times B_U$ is chosen to lie in $(0) \times B_U$. Then

$$\tilde{K}^q(X) = \{X, \mathbb{Z} \times B_U^{(q)}\}$$

for $(X, *) \in \mathcal{P}$ defines a cohomology theory. For $q = 0$, $\tilde{K}^0(X) = \varinjlim[\Sigma^i X, (B_U)_i]$. Since the compositions

$$[X, \mathbb{Z} \times B_U] \rightarrow [\Sigma X, \Sigma(\mathbb{Z} \times B_U)] \rightarrow [\Sigma X, U]$$

and

$$[\Sigma X, U] \rightarrow [\Sigma^2 X, \Sigma U] \rightarrow [\Sigma^2 X, \mathbb{Z} \times B_U]$$

are adjoint to

$$[X, \mathbb{Z} \times B_U] \xrightarrow{\cong} [X, \Omega U]$$

and

$$[X, \Omega U] \xrightarrow{\cong} [X, \Omega^2(\mathbb{Z} \times B_U)],$$

respectively, $\tilde{K}^0(X) \cong \tilde{K}(X)$, for connected spaces X and thus for all $(X, *) \in \mathcal{P}_0$. Further, because of Theorem 2.7, $\mathbb{Z} \times B_U \xrightarrow{\cong} \Omega^2(\mathbb{Z} \times B_U)$; and so,

$$K^n(X, A) \cong K^{n+2}(X, A)$$

for all $(X, A) \in \mathcal{P}^2$. Thus, all information in this cohomology is contained in

$$\begin{aligned} \tilde{K}^0(X) &\cong \tilde{K}(X) \quad \text{and} \\ \tilde{K}^{-1}(X) &\cong \tilde{K}(\Sigma X). \end{aligned}$$

We next observe that the theory K^* is a multiplicative cohomology theory. By §1.3 it suffices to define an internal natural pairing \otimes on $K^*(X, A)$ for all pairs $(X, A) \in \mathcal{P}^2$ such that

- 1) $K^*(X, A)$ is a ring,
- 2) $K^*(X, \emptyset)$ is a ring with unit,
- 3) $K^*(X, A)$ is a $K^*(X, \emptyset)$ -module, and
- 4) $\delta : K^*(A, \emptyset) \rightarrow K^*(X, A)$ is a $K^*(X, \emptyset)$ -module homomorphism.

For α and β in $K(X) = K(X, \emptyset)$, $\alpha \otimes \beta \in K(X)$ has been defined and makes $K(X)$ a ring with unit. If $\text{pt} \xrightarrow{i} X \xrightarrow{p} \text{pt}$ are the inclusion and projection maps on the base point, then for $\gamma \in K(X)$, $\gamma - p^*i^*\gamma \in \ker(i^*) = \tilde{K}(X)$. $\ker(i^*)$ is an ideal in $K(X)$; i.e., $K(X, *)$ is a ring and is a $K(X, \emptyset)$ -module. For $\alpha \in \tilde{K}(X)$ and $\beta \in \tilde{K}(Y)$, $\alpha \otimes \beta \in \tilde{K}(X \# Y)$. For $\alpha \in \tilde{K}^{-i}(X)$ and $\beta \in \tilde{K}^{-j}(Y)$, let $\bar{\alpha} \in \tilde{K}(\Sigma^i X)$ and $\bar{\beta} \in \tilde{K}(\Sigma^j Y)$ be their suspensions. Then $\bar{\alpha} \otimes \bar{\beta} \in \tilde{K}(\Sigma^i X \# \Sigma^j Y)$, and $\alpha \otimes \beta \in \tilde{K}^{-i-j}(X \# Y)$ is defined to be the desuspension of $\bar{\alpha} \otimes \bar{\beta}$. Of course, we need only consider the values 0 and 1 for i and j . The usual permutations of $\mathbb{S}^i \# \mathbb{S}^j$ into

$\mathbb{S}^j \# \mathbb{S}^i$ show $\alpha \otimes \beta = (-1)^{ij} \beta \otimes \alpha$. Finally, to verify condition 4) of the pairing \otimes we note the homotopy commutative diagramme

$$\begin{array}{ccccc}
 \Sigma^{i+j}(X, A) & \longrightarrow & \Sigma^{i+j}(A, *) & \longrightarrow & \Sigma^{i+j+1}(A \# A, *) \\
 \downarrow & & & & \downarrow \\
 \Sigma^{i+j}(X \times X, A \times X \cup X \times *) & & & & \Sigma^{i+1}(A, *) \times \Sigma^j(A, *) \\
 \downarrow & & & & \downarrow \\
 \Sigma^i(X, A) \times \Sigma^j(X, *) & \longrightarrow & & \longrightarrow & \Sigma^{i+1}(A, *) \times \Sigma^j(X, *)
 \end{array}$$

Proposition 3.3. $K_{\mathbb{C}}^*$ is a multiplicative cohomology theory.

3.1.2 The Multiplicative Transformation $\text{ch} : K_{\mathbb{C}}^* \rightarrow H^{**}(-; \mathbb{Q})$

The domain of definition of the Chern character ch can be extended from complex vector bundles over a space X to $K(X)$ by linearity.

For $\alpha \in \widetilde{K}^{2i}(X)$, let $\text{ch}(\alpha)$ equal $\text{ch}(\bar{\alpha})$, where $\bar{\alpha}$ is the element of $\widetilde{K}(X)$ equivalent to α , and for $\alpha \in \widetilde{K}^{2i+1}(X)$, let $\text{ch}(\alpha) = (\Sigma^*)^{-1} \text{ch}(\bar{\alpha})$, where $\bar{\alpha}$ is the equivalent element of $\widetilde{K}(\Sigma X)$ and $(\Sigma^*)^{-1} : H^{**}(\Sigma X; \mathbb{Q}) \rightarrow H^{**}(X; \mathbb{Q})$ is desuspension.

Proposition 3.4. $\text{ch} : K_{\mathbb{C}}^* \rightarrow H^{**}(-; \mathbb{Q})$ is a multiplicative transformation.

Proof. Since ch on $(\mathbb{S}^0, *)$ is a multiplicative, \mathbb{Q} -module valued functor, by Proposition 1.27, ch has a unique natural extension, which must be then the Chern character as above, and the extension is multiplicative. Furthermore, ch is normed as it preserves the units for \mathbb{S}^1 . \square

Proposition 3.5. $\text{ch} : K_{\mathbb{C}}^n(X, A) \times \mathbb{Q} \rightarrow \sum_{i+j=n} H^i(X, A; K^j(\text{pt}) \otimes \mathbb{Q})$ is a natural equivalence. In particular,

$$(a) \text{ ch} : K_{\mathbb{C}}^n(X, A) \otimes \mathbb{Q} \xrightarrow{\cong} H^{\text{even},*}(X, A; \mathbb{Q}) \text{ and}$$

$$(b) \text{ ch} : K^1(X, A) \otimes \mathbb{Q} \xrightarrow{\cong} H^{\text{odd},*}(X, A; \mathbb{Q}).$$

Proof. This is an immediate consequence of Corollaries 1.24 and 1.25 of §1.2. We note that the equivalence (a) is multiplicative. \square

Let $E_r^{p,q}$ and $'E_r^{p,q}$ be the spectral sequences of the identity fibration $\text{id} : X \rightarrow X$ in the $K_{\mathbb{C}}^*$ -, and $H^*(-; \mathbb{Q})$ - cohomology theories, respectively. Since ch is a transformation of cohomology theories, it induces

$$\text{ch}_r^{p,q} : E_r^{p,q} \rightarrow 'E_r^{p,q},$$

a morphism of spectral sequences. At the E_2 -level,

$$\text{ch}_2^{p,q} : H^p(X; K^q(\text{pt})) \rightarrow H^p(X; H^q(\text{pt}; \mathbb{Q}))$$

is trivial except for $q = 0$, where it is induced by the coefficient homomorphism $\mathbb{Z} \rightarrow \mathbb{Q}$ since there it is a natural transformation of ordinary singular cohomology theories, and does this for $X = \text{pt}$.

The exact couple for the spectral sequence $E_r^{p,q}$ is given by $A^{p,q} = K^{p+q}(X^{(p)})$ and $C^{p,q} = K^{p+q}(X^{(p)}, X^{(p-1)})$. Thus $A^{p,q} \xrightarrow{\cong} A^{p,q+2}$ and $C^{p,q} \xrightarrow{\cong} C^{p,q+2}$; and so, there is an isomorphism of spectral sequences $E_r^{p,q} \xrightarrow{\cong} E_r^{p,q+2}$. Thus, in $\{E_r^{p,q}\}$, $d_r^{p,q} = d_r^{p,q+2}$. We also observe that $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$ is trivial for all even r since $E_r^{p,q} = 0$ for all odd q and one of q and $q-r+1$ is odd when r is even.

Lemma 3.6. *If $X \in \mathcal{P}$ and $H^*(X; \mathbb{Z})$ is torsion-free, then the spectral sequence in the $K_{\mathbb{C}}^*$ -cohomology theory of the identity fibration $\text{id} : X \rightarrow X$ collapses; i.e., all differentials d_r are trivial.*

Proof. In this case $\text{ch}_2^{p,0}$ is a monomorphism. The $'E$ sequences obviously collapse. Since $d_2 = 0$, $\text{ch}_2^{p,0} = \text{ch}_3^{p,0} : E_3^{p,0} \rightarrow 'E_3^{p,0}$ is again a monomorphism. The differential $d_3^{p,q}$ is trivial for odd q ($E_2^{p,\text{odd}} = 0$) and for q even, $d_3^{p,q}$ is equivalent to $d_3^{p,2}$. But $'d_3^{p,2} = 0$ and $\text{ch}_3^{p+3,0} d_3^{p,2} = 'd_3^{p,2} \text{ch}_3^{p,2} = 0$; since $\text{ch}_3^{p+3,0}$ is a monomorphism, $d_3^{p,q} = d_3^{p,2} = 0$. Then $d_4 = 0$ and $\text{ch}_5^{p,0} = \text{ch}_3^{p,0}$ is a monomorphism. $d_5^{p,q} = 0$ for q odd and $d_5^{p,q} = d_5^{p,4}$ for q even, etc. \square

By the lemma, $E_2^{p,0} \cong E_{\infty}^{p,0}$ and

$$H^p(X; \mathbb{Z}) \cong F_{p-1}(K^p(X))/F_p(K^p(X))$$

is mapped monomorphically into $H^p(X; \mathbb{Q})$ by ch , where we recall that

$$F_i(K^p(X)) \equiv \ker(K^p(X) \rightarrow K^p(X^{(i)})).$$

This establishes

Theorem 3.7. *Suppose, for $X \in \mathcal{P}$, that $H^*(X; \mathbb{Z})$ is torsion-free. Then*

- 1) $\eta \in K_{\mathbb{C}}^*(X)$ is in $\ker(K_{\mathbb{C}}^*(X) \rightarrow K_{\mathbb{C}}^*(X^{(p-1)}))$ if and only if $\text{ch}(\eta) = 0$ in all dimensions less than p ,
- 2) $K_{\mathbb{C}}^*(X)$ is torsion-free and $\text{ch} : K_{\mathbb{C}}^*(X) \rightarrow H^*(X; \mathbb{Q})$ is a monomorphism, and
- 3) $\text{im}(H^p(X; \mathbb{Z}) \rightarrow H^p(X; \mathbb{Q}))$ is the set of Chern characters of elements of $K^*(X)$ in $\ker(K^*(X) \rightarrow K^*(X^{(p-1)}))$.

We note that 2) is a special case of a theorem of F. Peterson; we also note that Theorem 2.14 is an immediate consequence of 3) and is not used in its proof.

3.1.3 Cohomology Operations in $K_{\mathbb{C}}^*$

We describe operations in the $K_{\mathbb{C}}^*$ -theory first used by Atiyah and Hirzebruch in [9] and developed and applied by Adams in [2]. The most effective way of doing this is by means of unitary representations, and we shall indicate this later on in another connection. For the purpose of the operations in $K_{\mathbb{C}}^*$ -theory we have adequate equipment to proceed directly.

Theorem 3.8. *Let $(\mathbb{C}P^n)^k$ denote the Cartesian product of k -copies of $\mathbb{C}P^n$. If $\lambda \in K_{\mathbb{C}}((\mathbb{C}P^n)^k)$ is such that $\text{ch}(\lambda) \in H^{**}((\mathbb{C}P^n)^k; \mathbb{Q})$ is in the ring of symmetric functions (in u_1, \dots, u_k), then λ is the restriction of one and only one element of $K_{\mathbb{C}}(G_{k, kn})$, where*

$$G_{k, kn} = U(k + kn)/(U(k) \times U(kn)) \quad \text{and} \quad j : (\mathbb{C}P^n)^k \rightarrow G_{k, kn}$$

is induced by the decomposition

$$(\mathbb{C} \oplus \mathbb{C}^n) \oplus \cdots \oplus (\mathbb{C} \oplus \mathbb{C}^n) = (\mathbb{C}^k) \oplus (\mathbb{C}^{kn}).$$

Proof. The cofibration $(\mathbb{C}P^n)^k \xrightarrow{j} G_{k, kn} \xrightarrow{p} E$ induces the following ladder of exact sequences

$$\begin{array}{ccccccc} K^1(E) & \xleftarrow{\delta^*} & K^0((\mathbb{C}P^n)^k) & \xleftarrow{j^*} & K^0(G_{k, kn}) & \xleftarrow{p^*} & K^0(E) \\ \text{ch} \downarrow & & \text{ch} \downarrow & & \text{ch} \downarrow & & \text{ch} \downarrow \\ H^{\text{odd},*}(E; \mathbb{Q}) & \xleftarrow{\delta^*} & H^{\text{even},*}((\mathbb{C}P^n)^k; \mathbb{Q}) & \xleftarrow{j^*} & H^{\text{even},*}(G_{k, kn}; \mathbb{Q}) & \xleftarrow{p^*} & H^{\text{even},*}(E; \mathbb{Q}) \end{array}$$

in which all the ch 's are monomorphisms by 2) of Theorem 3.7. As $\text{ch}(\lambda)$ is symmetric, by Theorem 2.5 it lies in $\text{im}(j^*)$; and so, $\delta^* \text{ch}(\lambda) = 0$. This implies $\text{ch} \circ \delta^*(\lambda) = 0$, but since ch is a monomorphism, $\delta^* \lambda = 0$. Thus, there is a $\mu \in K(G_{k, kn})$ such that $j^* \mu = \lambda$. Since j^* on $H^{\text{even},*}(G_{k, kn})$ is a monomorphism and $H^{\text{odd},*}((\mathbb{C}P^n)^k) = 0$, $H^{\text{even},*}(E; \mathbb{Q}) = 0$. ch being a monomorphism then implies $K^0(E) = 0$; and so, μ is unique. \square

We now apply Theorem 3.8 to obtain a system of operations $\{\psi_t\}$ in $K_{\mathbb{C}}^*$ with properties to be stated. For t a positive integer, let $\Delta_t : U(1) \rightarrow U(1)^t$ be the t -fold diagonal. The composition

$$U(1)^k \xrightarrow{(\Delta_t)^k} (U(1)^t)^k \rightarrow (U(t))^k \rightarrow U(kt)$$

is a monomorphism, which induces a map

$$\lambda_t : (\mathbb{C}P^n)^k \rightarrow B_{U(kt)}$$

Let λ_t also denote the element of $K_{\mathbb{C}}((\mathbb{C}P^n)^k)$ determined by the map λ_t . Since $\text{ch}(\lambda_t) = \exp(tu_1) + \cdots + \exp(tu_k)$, there is a well-defined element $\mu_t \in K_{\mathbb{C}}(G_{k, kn})$ with $\text{ch}_i(\mu_t) = t^i \text{ch}_i(k)$, $\leq k$.

For $-t$ a negative integer, we let Δ_{-t} be the composition of Δ_t with complex conjugation. Then $\text{ch}(\lambda_{-t}) = \exp(-tu_1) + \cdots + \exp(-tu_k)$, and $\text{ch}_i(\mu_{-t}) = (-t)^i \text{ch}_i(k)$, $\leq k$. For the map $G_{k,kn} \times G_{\ell,\ell m} \rightarrow G_{k+\ell,(k+\ell)(m+n)}$ the class μ_t pulls back to $\mu_t' \times 1 + 1 \times \mu_t''$, and similarly for tensor products, with μ_t pulling back to $\mu_t' \times \mu_t''$.

For a k -plane complex vector bundle λ over X , there is an n such that λ is induced by a map

$$\lambda : X \rightarrow G_{k,kn}.$$

The composition

$$X \xrightarrow{\lambda} G_{k,kn} \xrightarrow{\mu_t} \mathbb{Z} \times B_U$$

determines an element $\psi_t(\lambda) \in K_{\mathbb{C}}(X)$. It follows from the above paragraph that $\psi_t(\lambda \oplus \mu) = \psi_t(\lambda) \oplus \psi_t(\mu)$ and $\psi_t(\lambda \otimes \mu) = \psi_t(\lambda) \otimes \psi_t(\mu)$. For the trivial line bundle 1 over \mathbb{S}^1 , $\psi_t(1) = 1$ for all t . We extend the operations ψ_t to $K_{\mathbb{C}}(X)$ by linearity and define ψ_t on $K_{\mathbb{C}}^{-1}(X)$ to be ψ_t on $K_{\mathbb{C}}(\Sigma X)$.

Theorem 3.9. *For each $t \in \mathbb{Z}$, there is a multiplicative transformation $\psi_t : K_{\mathbb{C}} \rightarrow K_{\mathbb{C}}$, such that*

$$1) \text{ch}_i(\psi_t(\lambda)) = t^i \text{ch}_i(\lambda) \text{ and}$$

$$2) \psi_t(\psi_s) = \psi_{ts}.$$

The conclusion 2) is a consequence of $\text{ch}_i(\mu_t) = t^i \text{ch}_i(k)$ and the fact that ch is a monomorphism on $K_{\mathbb{C}}(G_{k,kn})$.

3.1.4 $K_{\mathbb{C}}^*$ -orientation of Complex Vector Bundles

We have restricted ourselves throughout to consider cohomology only of finite cell complexes; we shall continue to do so, but we frequently write limit complexes in place of skeletal approximations of suitably high dimensions.

Earlier we discussed the fibration

$$\mathbb{S}^{2n-1} \rightarrow B_{U(n-1)} \xrightarrow{\pi} B_{U(n)},$$

where π is induced by the standard inclusion $U(n-1) \rightarrow U(n)$. The mapping cylinder $M(\pi)$ of π is equivalent to the unit disc bundle $\mathbb{D}(\gamma^n)$ associated with the Universal n -plane bundle over $B_{U(n)}$, and $B_{U(n-1)}$ is the associated boundary sphere bundle $\mathbb{S}(\gamma^n)$. Thus

$$M\pi/B_{U(n-1)} = \mathbb{D}(\gamma^n)/\mathbb{S}(\gamma^n) = B_{U(n)}^{\gamma^n}.$$

We seek an orientation “bundle” on $B_{U(n)}^{\gamma^n}$. The process will be to find an element of $K(B_{U(1)}^n)$ which restricts trivially in $K(B_{U(1)}^{n-1})$. Its extension to $K(B_{U(n)})$, by Theorem 3.8, or equivalently to $K(M(n))$, then restricts trivially in $K(B_{U(n-1)})$; and so, it comes from an element of $K(B_{U(n)}^{\gamma^n})$, which in turn is uniquely determined since $K^1(B_{U(n-1)}) = 0$.

Let $\alpha \in K(B_{U(1)}^n)$ be the element with $\text{ch}(\alpha) = \prod_{i=1}^n (1 - \exp(-u_i))$. (The virtual bundle α is the tensor product of a number of copies of the trivial line bundle minus the conjugate canonical line bundle.) Restricted to $B_{U(1)}^{n-1}$, the symbol $\exp(-u_n)$ becomes 1; hence, this restriction of α has 0 Chern character and is itself the 0 of $K(B_{U(1)}^{n-1})$.

Let λ be the extension of α to $K^*(B_{U(n)}) = K^*(\mathbb{D}(\gamma^n))$. Since $\text{ch}(\lambda|_{\mathbb{S}(\gamma^n)}) = 0$, $\lambda|_{\mathbb{S}(\gamma^n)} = 0$. Hence, λ is the image of an element $\mu \in K(B_{U(n)}^{\gamma^n})$. As stated above, μ is uniquely determined by α .

Let φ_H denote the Thom-Gysin isomorphism in the $H^*(-; \mathbb{Q})$ -theory. Then

$$\varphi_H(1) = \pi^* \left(\prod_{i=1}^n u_i \right).$$

We note this is a relative class; i.e., it lies in $H^*(B_{U(n)}^{\gamma^n}; \mathbb{Q})$. Also,

$$\text{ch}(\mu) = \pi^* \left(\prod_{i=1}^n (1 - \exp(-u_i)) \right).$$

Thus,

$$\text{ch}(\mu) = \varphi_H \left(\prod_{i=1}^n \frac{\exp(-u_i) - 1}{-u_i} \right).$$

We observe that $\text{ch}(\mu|_{\mathbb{S}^{2n}}) = 1 \cdot \overline{\mathbb{S}^{2n}}$; and so, μ is a $K_{\mathbb{C}}^*$ -orientation class. Also

$$\text{ch}(\gamma^n, 1) = \prod_{i=1}^n \frac{\exp(-u_i) - 1}{-u_i}$$

with this orientation.

Since γ^n is the Universal n -plane bundle, it follows that for any n -plane complex vector bundle α over X , $X \in \mathcal{P}$, α has a canonical $K_{\mathbb{C}}^*$ -orientation and

$$\text{ch}(\alpha, 1) = \prod_{i=1}^n \frac{\exp(-u_i) - 1}{-u_i}, \quad \text{where}$$

$$\prod_{i=1}^n (+u_i t) = 1 + C_1(\alpha)t + C_2(\alpha)t^2 + \cdots + C_n(\alpha)t^n,$$

where $C_i(\alpha)$ is the i -th Chern class of the bundle α .

If $\alpha \oplus \beta$ is a trivial bundle, then $\text{ch}(\alpha, 1) \cup \text{ch}(\beta, 1) = \text{ch}(\alpha \oplus \beta, 1) = 1$ and

$$\text{ch}(\beta, 1) = \prod_{i=1}^n \frac{-u_i}{\exp(-u_i) - 1}$$

Define polynomials with rational coefficients

$$T_k(C_1, \dots, C_k) = \frac{(-1)^k}{k!} B_k^{(n)}(u_1, \dots, u_n), \quad \text{for } k \leq n,$$

where $C_i = \sigma_i^{(n)}(u_1, \dots, u_n)$ and

$$\prod_{i=1}^n \frac{u_i x}{\exp(u_i x) - 1} = \sum_{j=0}^{\infty} \frac{x^j}{j!} B_j^{(n)}(u_1, \dots, u_n).$$

Then

$$\text{ch}_i(\beta, 1) = T_i(C_1, \dots, C_i),$$

where the C_j 's are the Chern classes of α . The polynomials $T_k(C_1, \dots, C_k)$ are called the *Todd polynomials*. As a shorthand we write

$$\text{ch}(\alpha, 1) = 1 + T_1(-\alpha) + T_2(-\alpha) + T_3(-\alpha) + \dots$$

Note that

$$(1 + T_1(\alpha) + T_2(\alpha) + \dots) \cup (1 + T_1(\beta) + T_2(\beta) + \dots) = 1 + T_1(\alpha \oplus \beta) + T_2(\alpha \oplus \beta) + \dots$$

A manifold is *weakly almost complex* if for some trivial bundle n , the structure group of $\tau \oplus n$ has a specific unitary reduction. In particular, almost complex manifolds, i.e., smooth manifolds equipped with smooth linear complex structures on tangent spaces, are weakly almost complex.

If M is weakly almost complex, then it has a canonical $K_{\mathbb{C}}^*$ -orientation, as described above, and

$$\begin{aligned} \text{ch}(M, 1) &= 1 + T_1(\tau) + T_2(\tau) + \dots \\ &\equiv T_*(\tau). \end{aligned}$$

Theorem 3.10. *If $f : M \rightarrow N$ is a continuous map of weakly almost complex manifolds, then for $\lambda \in K_{\mathbb{C}}^*(M)$*

$$f_!^{H^*}(\text{ch}(\lambda) \cup T_*(\tau(M))) = \text{ch}(f_!^{K^*}(\lambda)) \cup T_*(\tau(N)).$$

Of course, if $\nu(f)$ is $K_{\mathbb{C}}^*$ -orientable, there is a similar result, using Theorem 1.55 instead of Theorem thm:1-D-10. We will later describe a more general result involving orienting real vector bundles with $w_1 = w_3 = 0$. This will yield the best form of the above theorem.

Chapter 4

Some Geometric Applications

4.1 Vector Bundles over Cell Complexes $\mathbb{S}^n \cup e^m$

4.1.1 Two technical lemmas

4.1.2 Divisors of Orders of Stable Homotopy Classes, J -homomorphisms

4.1.3 Maps of Hopf Invariant One

Theorem 4.1.

4.2 Toda Brackets

In homotopy theory, as in cohomology theory, higher order operations play an important role. In homotopy, composition is the primary operation. Secondary operations are “Toda brackets” defined when in the diagramme

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D,$$

$g \circ f \simeq 0$ and $h \circ g \simeq 0$.

This then defines a recurrent family of elements in the stable stems. M. Barratt has used a refinement of this process to determine completely the image of the real J -homomorphism [to appear].

Appendices

Appendix A

A.1 The Cohomology Theory $K_{\mathbb{R}}^*$

Appendix B

B.1 A Multiplication Formula

This appendix presents a proof of the following technical proposition which was used a number of times throughout the text:

Appendix C

C.1 Fibre Homotopy Equivalence of Bundles, The Groups $J(X)$

Appendix D

D.1 The Eckmann-Hilton argument

The Eckmann-Hilton argument shows that a monoid or group object in the category of monoids or groups is commutative. In other words, if a set is equipped with two monoid structures, such that one is a homomorphism for the other, then the two structures coincide and the resulting monoid is commutative.

Proposition D.1. *Let X be a set equipped with two binary operations, which we will write \circ and \otimes , and suppose:*

1. \circ and \otimes are both unital.
2. $(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$ for all $a, b, c, d \in X$.

Then \circ and \otimes are the same and in fact commutative and associative.

Remark D.2. The operations \otimes and \circ are often referred to as *multiplications*, but this might imply they are associative, a property which is *not* required for the proof. In fact, associativity follows. Likewise, we do not have to require that the two operations have the same neutral element; this is a consequence.

First proof

Proof. First, observe that the units of the two operations coincide:

$$\begin{aligned} 1_\circ &= 1_\circ \circ 1_\circ = (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) \\ &= (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) = 1_\otimes \otimes 1_\otimes = 1_\otimes. \end{aligned}$$

Now, let $a, b \in X$. Then

$$\begin{aligned} a \circ b &= (1 \otimes a) \circ (b \otimes 1) = (1 \circ b) \otimes (a \circ 1) = b \otimes a \\ &= (b \circ 1) \otimes (1 \circ a) = (b \otimes 1) \circ (1 \otimes a) = b \circ a. \end{aligned}$$

This establishes that the two operations coincide and are commutative.

For associativity,

$$\begin{aligned}(a \otimes b) \otimes c &= (a \otimes b) \otimes (1 \otimes c) \\ &= (a \otimes 1) \otimes (b \otimes c) = a \otimes (b \otimes c).\end{aligned}$$

□

Alternate proof

Proof. For this version of the proof, we write the two operations as vertical and horizontal juxtaposition, i.e., $\begin{smallmatrix} a \\ b \end{smallmatrix}$ and $a \cdot b$. The interchange property can then be expressed as follows:

For all $a, b, c, d \in X$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$, so we can write $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$ without ambiguity.

Let \bullet and \circ be the units for vertical and horizontal composition respectively.

Then $\bullet = \begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix} = \begin{smallmatrix} \bullet & \bullet \\ \circ & \bullet \end{smallmatrix} = \circ \circ = \circ$, so both units are equal.

Now, for all $a, b \in X$, $a \cdot b = \begin{smallmatrix} a & \bullet \\ \bullet & b \end{smallmatrix} = \begin{smallmatrix} a \\ \bullet \end{smallmatrix} \begin{smallmatrix} \bullet & a \\ b & \bullet \end{smallmatrix} = b \cdot a = \begin{smallmatrix} b & \bullet \\ \bullet & a \end{smallmatrix} = \begin{smallmatrix} b \\ \bullet \end{smallmatrix} \begin{smallmatrix} \bullet & b \\ a & \bullet \end{smallmatrix}$, so horizontal composition is the same as vertical composition and both operations are commutative. Finally, for all $a, b, c \in X$, $a \cdot (b \cdot c) = a \cdot \begin{pmatrix} b \\ c \end{pmatrix} = \begin{smallmatrix} a & b \\ \bullet & c \end{smallmatrix} = (a \cdot b) \cdot c$, so composition is associative. □

Appendix E

Historical Comments

The general idea of obtaining cohomology theories by maps into spectra can be traced to Barratt [12], in that the track groups of maps from a cofibration yield an exact sequence. This was made explicit by Puppe [28]. The converse that cohomology theories are representable in this way was established by Brown [18].

The material in Chapter 1 on reduced and non-reduced cohomology theories is described in the fashion of Whitehead [30], while the spectral sequence of a fibration is presented along lines suggested by Dold [19]. The proof given of the Dold-Thom-Gysin Theorem also follows Dold (ibid). The approach to orientability of vector bundles, multiplicative transformations and the Riemann-Roch Theorem for differentiable manifolds is essentially folk-lore; it was first presented in [20] but its ideas lie in the work of Atiyah and Hirzebruch as expounded, for instance, by Hirzebruch in [24]. The proof given here of the Wu formulae is essentially that of Atiyah and Hirzebruch in [10].

Through the use of Spanier-Whitehead duality [29] one can view the cohomology of the Thom space of a normal bundle as being the homology of the manifold under consideration. An alternate procedure is developed by Whitehead in [30]. Various of the constructions of Chapter 1 then appear as versions of Poincare duality, umkehr homomorphism, etc.

The development of the topology of the Unitary Group and associated spaces presented in Chapter 2 follows that given by the author in a course at the University of Chicago in the Winter of 1960. A similar but substantially more complicated approach works for the Orthogonal and Symplectic Groups [22]. The explicit description of the maps generating $\pi_{2n+1}(\mathrm{SU})$ given in Theorem 2.17 is due to Toda.

Atiyah is credited with initiating study of the theory $X_{\mathbb{C}}^*$, as a cohomology theory. The material of §§3.1.1 and §§3.1.2 is taken from a paper by Atiyah and Hirzebruch [10].

The technical lemmas of §4.1 are found in Atiyah and Hirzebruch [10] and Hirzebruch [23]. The material of §§4.1.2 and §§subsect:4-A-3 follows the author's exposition in [21], although it must be noted that Adams found [3] a

proof similar in spirit to that given here of Theorem 4.1 some time earlier than the author; its differences are that he uses results from [1] while the proof given here uses results on complex cobordism [25].

The Appendix A on the K^* -theory essentially follows Atiyah and Hirzebruch as in [24]. The proposition of Appendix B is very basic, leading in various roles to the theorems on Stiefel-Whitney and Chern classes of sums of vector bundles and to the theory of multiplicative sequences. The Appendix C is very incomplete and is a presentation of material presented by Adams in various lectures.

Spring 1963.

Bibliography

- [1] J. F. Adams, *On Chern characters and the structure of the Unitary group*, Proc. Camb. Phil. Soc. **57** (1961), 189-199.
- [2] ———, *Vector fields on spheres*, Ann. of Math. **75** (1962), no. 3, 603-632.
- [3] ———, *On $K(X)$* , Manchester University, 1962.
- [4] ———, *Applications of the Grothendieck-Atiyah-Hirzebruch functor $K(X)$* , Colloquium on Algebraic Topology (1962), 104-113.
- [5] ———, *On the groups $J(X)$ - I*, Topology **2** (1963), 181-195.
- [6] M. F. Atiyah, *Bordism and cobordism*, Proc. Camb. Phil. Soc. **57** (1961), 200-208.
- [7] ———, *Thom complexes*, Proc. Lond. Math. Soc. **11** (1961), 291-310.
- [8] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. A.M.S. **65** (1959), 276-281.
- [9] ———, *Quelques théorèmes de non-plongement pour les variétés différentiables*, Bull. Soc. Math. France **87** (1959), 383-396.
- [10] ———, *Cohomologie-Operationen und charakteristische klassen*, Math. Z. **7** (1961), 149-187.
- [11] ———, *Vector bundles and homogeneous spaces*, Proc. Symposia in Pure Math., A.M.S. **III-Diff. Geom.** (1961), 7-38.
- [12] M. G. Barratt, *Track Groups I*, Proc. London Math. Soc. **5** (1955), 71-106.
- [13] ———, *Track Groups II*, Proc. London Math. Soc. **5** (1955), 285-329.
- [14] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458-538.
- [15] ———, *Characteristic classes and homogeneous spaces II*, Amer. J. Math. **81** (1959), 315-382.
- [16] ———, *Characteristic classes and homogeneous spaces III*, Amer. J. Math. **82** (1960), 491-504.
- [17] R. Bott, *Lectures on $K(X)$* , W. A. Benjamin, New York, 1969.
- [18] E. H. Brown Jr., *Cohomology theories*, Ann. of Math. **75** (1962), 467-484.
- [19] A. Dold, *Relations between ordinary and extraordinary homology*, Colloquium on Algebraic Topology (1962), 2-9.
- [20] E. Dyer, *Relations between cohomology theories*, Colloquium on Algebraic Topology (1962), 89-93.
- [21] ———, *Chern characters of certain complexes*, Math. Z. **80** (196), 363-373.
- [22] E. Dyer and R. K. Lashof, *A topological proof of the Bott periodicity theorems*, Annali di Math. **54** (1961), 231-254.

- [23] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, Berlin, New York, 1982.
- [24] ———, *A Riemann-Roch theorem for differentiable manifolds*, Sem. Bourbaki **177** (1959).
- [25] J. Milnor, *On the cobordism ring Ω^* and a complex analogue*, Amer. J. Math. **82** (1960), 505-521.
- [26] B. Morin, *Champs de vecteurs sur les spheres d'après J. F. Adams*, Sem. Bourbaki **233** (1962).
- [27] O. Ore, *Number theory and its history*, McGraw-Hill, New York, 1948.
- [28] D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen I*, Math. Zeitschrift **69** (1958), 299-344.
- [29] E. H. Spanier, *Duality and S-theory*, Bull. A.M.S. **62** (1956), 194-203.
- [30] G. W. Whitehead, *Generalized homology theories*, Trans. A. M. S. **102** (1962), 227-283.