A study memo on *Cohomology Theories* by Eldon Dyer

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Preface

A word from the transcriber

This is a study memo of "Cohomology Theories" by Eldon Dyer. It is rather amazing that this succinct account on the subject was published in 1969, more than fifty years ago.

The transcriber has added some materials in the hope that they would help understanding of the reader.

The preface by Dyer

Let $\mathscr{P}$ be the category whose objects are finite cell complexes and whose maps are continuous maps. Let $\mathscr{P}^2$ be the category of pairs in $\mathscr{P}$; i.e., pairs (X, A) where X is a finite cell complex and A is a subcomplex of X , and maps are continuous maps of pairs. Let G be the category of abelian groups and homomorphisms.

Let $T : \mathscr{P}^2 \to \mathscr{P}^2$ be the covariant functor defined by

$$T(X, A) = (A, \varnothing) \text{ for } (X, A) \in \mathscr{P}^2 \text{ext} \textit{and}$$
$$T(f) = f|_{(A,\varnothing)} : (A, \varnothing) \to (B, \varnothing) \text{ for}$$

a map $f : (X, A) \to (Y, B)$ in $\mathscr{P}^2$.

A *cohomology theory* on $\mathscr{P}$ is a sequence of contravariant functors

$$H^n : \mathscr{P}^2 \to G$$

and a sequence of natural transformations

$$\delta^n : H^{n-1} \circ T \to H^n$$

subject to the conditions

1) If $f_0, f_1 \in \mathscr{P}$ and $f_0 \simeq f_1$, ($\simeq$ means "is homotopic to"), then $H^n(f_0) = H^n(f_1)$ for all n;

2) If $(X; A, B)$ is a triad in $\mathscr{P}$ (all possible pairs of X, A, B, $A \cup B$ and $A \cap B$ are in $\mathscr{P}$) and $X = A \cup B$, then for the inclusion map $k : (A, A \cap B) \to (X, B)$,

$$H^n(k) : H^n(X, B) \longrightarrow H^n(A, A \cap B)$$

is an isomorphism for all n; and

3) If $(X, A) \in \mathscr{P}^2$ and $i : (A, \varnothing) \to (X, \varnothing)$ and $j : (X, \varnothing) \to (X, A)$ are the inclusion maps, then the sequence

$$\cdots \to H^{n-1}(A, \varnothing) \xrightarrow{\delta^n} H^n(X, A) \xrightarrow{H^n(j)} H^n(X, \varnothing) \xrightarrow{Hn(i)} H^n(A, \varnothing) \to \cdots$$

of the pair (X, A) is exact.

These conditions are exactly the Eilenberg-Steenrod axioms for cohomology except we *omit* the condition

$$H^i(\mathrm{pt}, 0) = 0 \text{ for } i \neq 0.$$

Recall that with this extra condition, one has the singular cohomology groups with coefficients in $H^0(\mathrm{pt}, 0)$. We shall speak of our cohomology theories as having coefficients in the graded system $\{H^n(\mathrm{pt}, \varnothing)\}$.

In these notes we discuss such cohomology theories. It has been clear for the past two or three years that they furnish a strong tool for analysing manifolds.

The notes are divided into four chapters. The first deals with generalities of such theories: axiomatics, origins, spectral sequence of a fibration, multiplicative structures, orientation of bundles, Poincare duality, and a type of generalised Riemann-Roch theorem. The second studies the unitary group and characteristic classes of complex vector bundles. The third discusses an example: the Grothendieck ring $K^*_{\mathscr{C}}()$. The fourth is concerned with applications: the J-homomorphism, maps of Hopf invariant 1, and properties of stable homotopy.

There are also three appendices, discussing more briefly the cohomology theory $K^*_{\mathscr{R}}$ and the J-groups.

The author wishes to acknowledge here his appreciation of Walter Daum for assistance in preparing these notes.

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Chapter 1

Structure of Cohomology Theories

1.1 Axiomatics

1.1.1 Eilenberg Steenrod axioms for (generalised) cohomology

A *(genealised) cohomology theory* h^* on $\mathscr{Top}^2$ (or any nice subcategory like compact pairs, pairs of CW-complexes, etc,) is a collection of

***contravariant* functors**

$$h^n : \mathscr{Top}^2 \to \mathscr{Ab}, \quad n \in \mathbb{Z}$$

where $\mathscr{Ab}$ denotes the category of abelian groups, and

natural transformations

$$\delta^n : j^n \circ R \to h^{n+1}$$

where $R : \mathscr{Top}^2 \to \mathscr{Top}^2$ is the functor that sends (X, A) to $(A, \varnothing)$ and f to $f|_A$, satisfying the following axioms:

(i) *Homotopy invariance.* If $f \simeq g$, then $h^n(f) = h^n(g)$ for every $n \in \mathbb{Z}$.

(ii) *Excision.* For every pair (X, A) and $U \subset A$ such that the closure $\overline{U}$ is contained in the interior A°, the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces and isomorphism

$$h^n(X \setminus U, A \setminus U) \cong h^n(X, A) \quad \text{for every} \quad n \in \mathbb{Z}.$$

(iii) *Exactness.* For every pair (X, A), consider the inclusions $i : A \to X$ and $j : (X, \varnothing) \to (X, A)$. Then there is a long exact sequence

$$\cdots \to h^{n-1}(A) \xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A)$$
$$\xrightarrow{\delta^n} h^{n+1}(X, A) \xrightarrow{j^*} h^{n+1}(X) \xrightarrow{i^*} h^{n+1}(A) \to \cdots$$

1.1.2 Three Theorems

The three theorems which follow can be proved just as in Eilenberg and Steenrod, but a simpler proof for the third due to M. Barratt and J. H. C. Whitehead is indicated.

Theorem 1.1 (Exact Sequence of a Triple). *If $(X; A, B)$ is a triple, then the sequence*

$$\cdots \to H^{n-1}(A,B) \xrightarrow{\Delta} H^n(X,A) \xrightarrow{\Phi} H^n(X,B) \xrightarrow{\Psi} H^n(A,B) \to \cdots$$

is exact, where Φ and Ψ are induced by the inclusion maps and Δ is the composition

$$H^{n-1}(A,B) \to H^{n-1}(A,\varnothing) \to H^n(X,A).$$

Theorem 1.2 (Exact Sequence of a Triad). *If $(X; A, B)$ is a triad, then the sequence*

$$\cdots \to H^{n-1}(A, A\cap B) \xrightarrow{\Delta} H^n(X, A\cup B) \xrightarrow{\Phi} H^n(X,B) \xrightarrow{\Psi} H^n(A, A\cap B) \to \cdots$$

is exact, where Φ and Ψ are induced by the inclusion maps and Δ is the composition

$$H^{n-1}(A, A\cap B) \xrightarrow{\cong} H^{n-1}(A\cup B, B) \to H^{n-1}(A\cup B, \varnothing) \xrightarrow{\delta} H^n(X, A\cup B)$$

Theorem 1.3 (Mayer-Vietoris Theorem). *Let $(X; A, B)$ be a triad with $X = A\cup B$. Then the sequence*

$$\cdots \to H^{n-1}(A\cap B, \varnothing) \xrightarrow{\Delta} H^n(X,\varnothing) \xrightarrow{\Phi} H^n(A,\varnothing)\oplus H^n(B,\varnothing) \xrightarrow{\Psi} H^n(A\cap B,\varnothing) \to \cdots.$$

is exact, where the homomorphisms are defined as in the proof.

We quote (without proof)

- M. Barratt, J. Whitehead *The First Non-Vanishing Group of an $(n+1)$-ad* Mathematics Proceedings of The London Mathematical Society. (1956)

for

Lemma 1.4. *If the diagramme of groups and homomorphisms*

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "\alpha_i"'] & B_i \arrow[r, "g_i"] \arrow[d, "\beta_i"] & C_i \arrow[r, "h_i"] \arrow[d, "\gamma_i"] & A_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\alpha_{i+1}"] & B_{i+1} \arrow[r] \arrow[d, "\beta_{i+1}"] & \cdots \\
\cdots \arrow[r] & A_i' \arrow[r, "f_i'"'] & B_i' \arrow[r, "g_i'"'] & C_i' \arrow[r, "h_i'"'] & A_{i+1}' \arrow[r, "f_{i+1}'"'] & B_{i+1}' \arrow[r] & \cdots
\end{tikzcd}$$

is commutative, the horizontal sequences are exact, and the γ_i are isomorphisms, then the sequence

$$\cdots \to A_i \xrightarrow{(\alpha_i, f_i)} A_i' \oplus B_1 \xrightarrow{f_i - \beta_i} B_i' \xrightarrow{h_i\gamma_i^{-1}g_i'} A_{i+1} \to \cdots.$$

is exact.

The Mayer-Vietoris Theorem follows by applying this lemma to the following diagramme, in which the isomorphism is condition (ii) in the definition of a cohomology theory (§§1.1.1).

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(X,\varnothing) \arrow[r] \arrow[d] & H^{n-1}(A,\varnothing) \arrow[r] \arrow[d] & H^n(X,A) \arrow[r] \arrow[d, "\cong"] & H^n(X,\varnothing) \arrow[r] \arrow[d] & H^n(A,\varnothing) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^{n-1}(B,\varnothing) \arrow[r] & H^{n-1}(A\cap B,\varnothing) \arrow[r] & H^n(B,A\cap B) \arrow[r] & H^n(B,\varnothing) \arrow[r] & H^n(A\cap B,\varnothing) \arrow[r] & \cdots
\end{tikzcd}$$

1.1.3 Reduced Cohomology Theories

Let $\mathscr{P}_0$ be the category of finite cell complexes with base point and continuous base point preserving maps. A *reduced cohomology theory* is a sequence

$$\widetilde{H}^n : \mathscr{P}_0 \to \mathscr{A}$$

of contravariant functors and a sequence

$$\sigma^n : \widetilde{H}^{n+1} \circ \Sigma \to \widetilde{H}^n$$

of natural transformations subject to three conditions.

Construction of spaces

Before stating these conditions we recall some important constructions. Let (X, x_0) and (Y, y_0) be in $\mathscr{P}_0$.

The *wedge* $(X \vee Y, *) \in \mathscr{P}_0$ is the subspace $(X \times y_0) \cup (x_0 \times Y)$ of $X \times Y$ with $x \times y = *$ as base point.

The *smash* $(X\#Y, *) \in \mathscr{P}_0$ is the factor space $(X\times Y)/(X \vee Y)$ with base point the image of $X \vee Y$.

The *reduced suspension* For $(X, x_0) = (\mathbb{S}^1, 1)$, the unit circle $\{z \in \mathbb{C} | |x| = 1\}$, $X\#Y = \Sigma Y$ is the *reduced suspension* of Y. Σ is a covariant functor from $\mathscr{P}_0$ to $\mathscr{P}_0$ with $\Sigma(f) = \mathrm{id}\,\#f$, $\mathrm{id} : \mathbb{S}^1 \to \mathbb{S}^1$ being the identity map.

We define $p : \mathscr{P}^2 \to \mathscr{P}_0$ by $p(X, A) = (X/A, *)$, where for $A = \varnothing$, $X/\varnothing = X^+$ the disjoint union of X and a distinct point $+$, to be taken as the base point in $p(X, \varnothing) = (X, +)$. The reduced cone $CA \cup X$ of a pair (X, A) with $* \in A \subset X$ consists of X together with a cone over A with the interval from the base point collapsed to the new base point (the "whisker construction"); i.e.,

$$CA \cup X = \{(I, (0))\#(A, *)\} \cup X.$$

Note that $p : (X, A) \to (X/A, *)$ factors through $(CA \cup X, *)$:

$$(X, A) \xrightarrow{C(X,A)} (CA \cup X, *) \xrightarrow{h} (X/A, *)$$

where C is the cone functor and h identifies CA with the base point. As (X, A) is a finite cellular pair, it has the homotopy extension property, so the map $(CA \cup X) \to (X/A)$ is a homotopy equivalence.

Axioms for a reduced cohomology theory

We are now ready to state the conditions for a reduced cohomology theory.

A *reduced cohomology theory* is a sequence

$$\widetilde{H}^n : \mathscr{P}_0 \to \mathscr{A}$$

of contravariant functors and a sequence

$$\sigma^n : \widetilde{H}^{n+1} \circ \Sigma \to \widetilde{H}^n$$

of natural transformations such that

1) If $f_0, f_1 \in \mathscr{P}_0$ and $f_0 \simeq f_1$ in $\mathscr{P}_0$, then $\widetilde{H}^n(f_0) = \widetilde{H}^n(f_1)$ for all n;

2) $\sigma^n(X) : \widetilde{H}^{n+1}(\Sigma X) \to \widetilde{H}^n(X)$ is an isomorphism for all $X \in \mathscr{P}_0$; and

3) If $(X, A) \in \mathscr{P}^2$ and $* \in A$, then the sequence

$$\widetilde{H}^n(X/A) \xrightarrow{\widetilde{H}^n(\overline{p})} \widetilde{H}^n(X) \xrightarrow{\widetilde{H}^n(i)} \widetilde{H}^n(A)$$

is exact at $\widetilde{H}^n(X)$ for each n, where $\overline{p} : X \to X/A$ is the map collapsing A to a point and $i : A \to X$ is the inclusion map.

The condition 3) for reduced theories appears weaker than the corresponding condition for non-reduced theories. However, we may deduce the following condition

3$'$) If $(X, A) \in \mathscr{P}^2$ and $* \in A$, then the sequence

$$\cdots \to \widetilde{H}^{n-1}(A) \xrightarrow{\Delta^{n-1}} \widetilde{H}^n(X/A) \xrightarrow{\widetilde{H}^n(\overline{p})} \widetilde{H}^n(X) \xrightarrow{\widetilde{H}^n(i)} \widetilde{H}^n(A) \to \cdots$$

is exact, where Δ^{n-1} is the composition

$$\widetilde{H}^{n-1}(A) \xrightarrow[\cong]{\sigma^{n-1}(A)^{-1}} \widetilde{H}^n(\Sigma A) \xrightarrow{\widetilde{H}^n(k)} \widetilde{H}^n(CA \cup X) \underset{\cong}{\to} \widetilde{H}^n(X/A)$$

and $k : CA \cup X \to \Sigma A$ is the map collapsing X to a point.

To see the exactness at $\widetilde{H}^n(X/A)$ we observe that $\widetilde{H}^{n-1}(A) \to \widetilde{H}^n(CA \cup X) \to \widetilde{H}^n(X)$ is equivalent to

$$\widetilde{H}^n(\Sigma A) \to \widetilde{H}^n(CA \cup X) \to \widetilde{H}^n(X)$$

and 3) applies since $(CA \cup X)/X$ is homeomorphic to ΣA. To see the exactness at $\widetilde{H}^{n-1}(A)$, we observe that $\widetilde{H}^n(X) \to \widetilde{H}^{n-1}(A) \to \widetilde{H}^n(X/A)$ is equivalent to

$$\widetilde{H}^n(\Sigma X) \to \widetilde{H}^n(\Sigma A) \to \widetilde{H}^n(CA \cup X)$$

and 3) applies since ΣA has the homotopy type of $CX \cup (CA \cup X)$ and $(CX \cup (CA \cup X))/(CA \cup X)$ is homeomorphic to ΣX.

Theorem 1.5. *There is a natural* 1-1 *correspondence between theories* $\{H, \delta\}$ *and* $\{\widetilde{H}, \sigma\}$ *given by the commutative diagramme*

$$\begin{tikzcd} \mathscr{P}^2 \arrow[rr, "p"] \arrow[dr, "H"'] & & \mathscr{P}_0 \arrow[dl, "\widetilde{H}"] \\ & \mathscr{A} & \end{tikzcd}$$

Proof. For $(X, A) \in \mathscr{P}^2$ and a reduced theory $\{\widetilde{H}, \sigma\}$ define $(\alpha\widetilde{H}^n(X, A) = \widetilde{H}^n(X/A)$; define $\delta : (\alpha\widetilde{H}^{n-1})(A, \varnothing) \to (\alpha\widetilde{H}^n)(X, A)$ to be the composition

$$\widetilde{H}^{n-1}(A^+) \xleftarrow[\cong]{\sigma} \widetilde{H}^n(\Sigma A^+) \xrightarrow{\widetilde{H}^n(\overline{p})} \widetilde{H}^n(X^+/A^+) \cong \widetilde{H}^n(X/A).$$

This system satisfies the cohomology axioms; in particular, δ is the image of Δ under α.

For $(X, *) \in \mathscr{P}_0$ and a theory $\{H, \delta\}$ define $(\beta H^n)(X) \equiv H^n(X, \{*\})$ and define σ to be composition of the isomorphisms

$$H^n(X), \{*\}) \cong H^{n+1}(CX, X) \cong H^{n+1}(\Sigma X, \{*\}),$$

where the first isomorphism is given by the exact sequence of the triple $(CX, X, \{*\})$ and the second is a special case of more general isomorphism

$$H^n(X, A) \xrightarrow{\cong} H^n(X/A, \{*\});$$

this is given by the composition

$$H^n(X/A, \{*\}) \to H^n(CA \cup X, CA) \to H^n(X, A),$$

where the first homomorphism is an isomorphism since the pairs have the same homotopy type and the second is an isomorphism by the exact sequence of the triad $(CA \cup X; X, CA)$. The system $\{\beta H, \sigma\}$ satisfies the axioms for a reduced cohomology theory; in particular, axiom 3 is a consequence of the exact sequence of a triple.

It is clear from the constructions that $\beta\alpha\{\widetilde{H}, \sigma\} = \{\widetilde{H}, \sigma\}$ and $\alpha\beta\{H, \delta\} = \{H, \delta\}$. □

1.1.4 Spectra

A *spectrum* $\mathscr{X}$ is an indexed family $\{X_i\}_{i \in \mathbb{Z}}$ of spaces with base point together with a family of base point preserving maps $f_i : \Sigma X_i \to X_{i+1}$.

Example 1.6. a) $\mathscr{S}$ is the spectrum with $X_i = \mathbb{S}^i$, the i-sphere and $f_i : \Sigma\mathbb{S}^i \to \mathbb{S}^{i+1}$ the identity.

b) For an abelian group G, $\mathscr{K}(G)$ is the spectrum whose i-th space is

$$\begin{cases} K(G, i) & i > 0 \\ G & i = 0 \\ \text{pt} & i < 0 \end{cases}$$

From the canonical homotopy equivalence $K(G,i) \xrightarrow{\simeq} \Omega K(G,i+1)$, we obtain the maps $\Sigma K(G,i) \to K(G,i+1)$.

c) For X a space with base point, $\mathcal{S}(X)$ is the spectrum with i-th term ΣX. Thus, $\mathcal{S}(\mathbb{S}^0) = \mathcal{S}$.

d) For X a space with base point, and $\mathcal{Y}$ a spectrum, the spectra $X\#\mathcal{Y}$ and $\mathcal{Y}\#X$ are given by $\{X\#Y_i\}$ and $\{Y_i\#X\}$. In particular, we have the spectrum $\Sigma\mathcal{Y} = \mathbb{S}^1\#\mathcal{Y}$.

e) For $\mathcal{X}$ a spectrum, $\mathcal{X}^q$ is the spectrum with i-th term $(X^q)_i = X_{q+1}$.

We shall see that spectra define cohomology theories. Let $[A,B]$ denote the set of homotopy classes of base point preserving maps. For $A \in \mathcal{P}_0$ define $\{A,\mathcal{X}\}$ to be

$$\varinjlim([\Sigma^n A, X_n], i^n),$$

where for $\alpha \in [\Sigma^n A, X_n]$, $i^n(\alpha) \in [\Sigma^{n+1}A, X_{n+1}]$ is the composition

$$\Sigma\Sigma^n A \xrightarrow{\Sigma\alpha} \Sigma X \xrightarrow{f_n} X_{n+1}.$$

For $n \geq 1$, $[\Sigma^n A, X]$ is a group and i_n is a homomorphism; for $n \geq 2$, $[\Sigma^n A, X]$ is an abelian group. Thus, $\{A,\mathcal{X}\}$ is an abelian group.

Define the homomorphism $\sigma : \{A,\mathcal{X}\} \to \{\Sigma A, \mathcal{X}^1\}$ to be the direct limit of $\sigma : [\Sigma^n A, X_n] \to [\Sigma^n\Sigma A, X_{n+1}]$, where $\sigma^n = i^n$. We can take this limit since σ_n and the i^n commute. σ only shifts $\{A,\mathcal{X}\}$ over one in itself; so we have clearly

Lemma 1.7. $\sigma : \{A,\mathcal{X}\} \xrightarrow{\cong} \{\Sigma A, \mathcal{X}^1\}$.

For $g : A \to B$ in $\mathcal{P}_0$, composition defines a homomorphism $g^* : \{B,\mathcal{X}\} \to \{A,\mathcal{X}\}$.

Lemma 1.8. $(\mathrm{id})^* = \mathrm{id}$*;* $(g \circ f)^* = f^* \circ g^*$*; if* $f \simeq g$ *in* $\mathcal{P}_0$*, then* $f^* = g^*$.

Lemma 1.9. *For* $* \in B \subset Y$*,* $\Sigma^n(CB \cup Y)$ *is homeomorphic to* $C(\Sigma^n B) \cup (\Sigma^n Y)$.

It suffices to prove this for $n = 1$, but that is clear from the definitions.

Lemma 1.10. *For* $* \in B \subset Y$*, and* $B \xrightarrow{i} Y \xrightarrow{p} Y/B$ *the inclusion and collapsing maps, the sequence* $\{Y/B,\mathcal{X}\} \xrightarrow{p^*} \{Y,\mathcal{X}\} \xrightarrow{i^*} \{B,\mathcal{X}\}$ *is exact at* $\{Y,\mathcal{X}\}$.

Proof. For X a space with base point

$$[CB \cup Y, X] \to [Y,X] \to [B,X]$$

is an exact sequence of sets. Thus, by Lemma 1.9

$$[\Sigma^n(CB \cup Y), X_n] \to [\Sigma^n Y, X_n] \to [\Sigma^n B, X_n]$$

is an exact sequence of groups; but exactness commutes with direct limits and $CB \cup Y$ and Y/B have the same homotopy type. The lemma follows. □

Theorem 1.11. *For a spectrum and a space $Y \in \mathscr{P}_0$, $\widetilde{H}^q(Y;\mathscr{X}) = \{Y, \mathscr{X}_q\}$ and $\sigma^n : \widetilde{H}^{n+1}(\Sigma Y;\mathscr{X}) \to \widetilde{H}^n(Y;\mathscr{X})$, the inverse of the isomorphism of Lemma 1.7, define a reduced cohomology theory.*

A map $\mathscr{F} : \mathscr{X} \to \mathscr{Y}$ of spectra is a collection $f_i : X_i \to Y_i$ of maps of the terms of the spectra which for i sufficiently large homotopy commute with the defining maps of the spectra:

$$\begin{tikzcd}
\Sigma X_i \arrow[r, "\Sigma"] \arrow[d] & \Sigma Y_i \arrow[d] \\
X_{i+1} \arrow[r] & Y_{i+1}
\end{tikzcd}$$

Spectra have the role of coefficients in these cohomology theories and maps of them induced "coefficient homomorphisms"; i.e., natural transformations of theories.

We cite without proof the following result of E. Brown [18]:

Theorem 1.12. *If $\widetilde{H}$ is a reduced cohomology theory on $\mathscr{P}_0$ and $\widetilde{H}^q(\mathbb{S}^0)$ is countable for all q, then there is a spectrum $\mathscr{Y}$ such that $\widetilde{H}^*(-)$ and $\widetilde{H}^*(-;\mathscr{Y})$ are naturally equivalent. Furthermore, $\mathscr{Y}$ can be found so that the map $Y_n \to \Omega Y_{n+1}$ induced by $\Sigma Y_n \to Y_{n+1}$, is a homotopy equivalence ($\mathscr{Y}$ is an Ω-spectrum).*

1.2 Spectral Sequence of a Fibration

1.2.1 Exact Couples

Some familiarity with exact couples is assumed and we only review briefly the basic definitions, notation and propositions.

For each pair (p, q) of integers we are given abelian groups $A^{p,q}$ and $C^{p,q}$ and homomorphisms f, g and h so that the "p-sequence"

$$\cdots \to A^{p-1,q-1} \xrightarrow{h} C^{p,q-1} \xrightarrow{g} A^{p,q-1} \xrightarrow{f} A^{p-1,q} \xrightarrow{h} C^{p,q} \xrightarrow{g} A^{p,q} \to \cdots$$

is exact.

We arrange these sequences in the following diagramme:

$$
\begin{tikzcd}
 & & & A^{p+r,q-r} \arrow[r] \arrow[d, "f"] & C^{p+r+1,q-r} \\
 & & & A^{p+r-1,q-r+1} \arrow[r, "h"] \arrow[d, "f"] & C^{p+r,q-r+1} \\
 & & & \vdots \arrow[dd, "f"] & \\
C^{p,q-1} \arrow[r, Rightarrow, "g"] & A^{p,q-1} \arrow[d, Rightarrow, "f"] & & & \\
C^{p,q-1} \arrow[r, "g"] & A^{p-1,q} \arrow[r, Rightarrow, "h"] \arrow[d, "f"] & C^{p,q} \arrow[r, Rightarrow, "g"] & A^{p,q} \arrow[d, Rightarrow, "f"] & \\
 & A^{p-2,q+1} \arrow[r, "h"] \arrow[d, "f"] & C^{p-1,q+1} \arrow[r, "g"] & A^{p-1,q+1} \arrow[r, Rightarrow, "h"] \arrow[d, "f"] & C^{p,p+1} \\
 & \vdots \arrow[d] & & {} & \\
C^{p-r,q+r-1} \arrow[r, "g"] & A^{p-r,q+r-1} \arrow[d, "f"] & & & \\
C^{p-r-1,q+r} \arrow[r, "g"] & A^{p-r-1,q+r} & & &
\end{tikzcd}
$$

The sequence in the bold arrows is the p-sequence and the sum of the indices in each column is constant.

We define

$$
\begin{aligned}
Z_r^{p,q} &= g^{-1}(\operatorname{im} f^{(r-1)}) \subset C^{p,q} \quad \text{and} \\
B_r^{p,q} &= h(\ker f^{(r-1)}) \subset C^{p,q}.
\end{aligned}
$$

Then we can define $\theta_r : Z_r^{p,q} \to C^{p+r,q-r+1}/B_r^{p+r,q-r+1}$ to be $h \circ (f^{(r-1)})^{-1} \circ g$. Letting $E_r^{p,q} \equiv Z_r^{p,q}/B_r^{p,q}$, we see that θ_r induces a homomorphism

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

Furthermore, $d_r \circ d_r = 0$ and so the homology groups $\mathscr{H}(E_r, d_r)$ are defined as usual. Exactness in the diagramme at $A^{p+r-1,q-r+1}$ and $A^{p-r,q+r-1}$ imply that $\mathscr{H}(E_r, d_r) \cong E_{r+1}^{p,q}$.

The sequence of groups $\{E_r^{p,q}\}$ and differentials d_r is the spectral sequence of the exact couple (A, C), $A = \oplus_{p,q} A^{p,q}$ and $C = \oplus_{p,q} C^{p,q}$.

Define

$$\begin{aligned} Z_\infty^{p,q} &= g^{-1}(\cap_\nu \Im f^{(\nu)}) \subset C^{p,q}, \\ B_\infty^{p,q} &= h(\cup_\nu \ker f^{(\nu)}) C^{p,q}, \\ E_\infty^{p,q} &= Z_\infty^{p,q}/B_\infty^{p,q}. \end{aligned}$$

For $p+q=n$, define $A^n = \varprojlim(A^{p,q}, f)$ and $F_p A^n = \ker(A^n \to A^{p,q})$. Then $F_p(A^n) \supset F_{p+1}(A^n)$.

If the following two convergence conditions hold

1) $A^{p-r,q+r} \xrightarrow{f} A^{p-r-1,q+r+1}$ is 0 for $r > \nu(p,q)$, and

2) $A^{p+r,q-r} \xrightarrow{f} A^{p+r-1,q-r+1}$ is $\cong$ for $r > \widetilde{\nu}(p,q)$,

then $Z_r^{p,q} \cong Z_\infty^{p,q}$ and $B_r^{p,q} \cong B_\infty^{P,q}$ for large r and $E_r^{p,q} \cong E_\infty^{p,q} \cong F_{p-1}(A^n)/F_p(A^n)$, $n = p+q$.

We note that the spectral sequence is a covariant functor on exact couples; i.e., if $F : (A, C) \to (A^{'}, C^{'})$ is a map of exact couples, then there are induced maps

$$E_r(F) : E_r \to E_r^{'}$$

of spectral sequences with the appropriate functorial properties. If the convergence conditions hold and some $E_r(F)$ is an isomorphism, then all succeeding ones are and $A_n \cong A_n^{'}$.

1.2.2 Spectral Sequence of a h^*-fibration

Let $\pi : E \to B$ be a continuous map of spaces in $\mathscr{P}$; assume B is connected. It suffices to consider the case in which E and B are finite simplicial complexes and π is simplicial, for this is equivalent to the general case up to homotopy.

If $\{h^n\}$ is a cohomology theory, define $A^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}))$, where $B^{(p)}$ is the p−skeleton of B, and $C^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}), \pi^{-1}(B^{(p-1)}))$. These groups form an exact couple satisfying the convergence condition. Since E is finite dimensional, $A^n \cong h^n(E)$.

We shall say π *is an* h^**-fibration* if for each simplex Δ f B and each vertex v of Δ, the homomorphism $h^*(\pi^{-1}(\Delta)) \to h^*(\pi^{-1}(v))$ induced by inclusion is an isomorphism. A Serre fibration is an H^*-fibration, where H^* is ordinary singular cohomology, and we will see later that an H^*-fibration is an h^*-fibration for any cohomology theory h^*.

To simplify the notation in what follows, we will write $\{X, Y\}^i$ for $h^i(\pi^{-1}(X), \pi^1(Y))$ when $Y \subset X \subset B$. We assume throughout the remainder of this section that π is an h^*-fibration.

Lemma 1.13. *For simplices $\Delta^{'} \subset \Delta \subset B$, the homomorphism $\{\Delta\}^* \to \{\Delta^{'}\}^*$ induced by inclusion is an isomorphism.*

Proof. The diagramme

$$\begin{tikzcd}
\{\Delta\}^* \arrow[rr] \arrow[dr] & & \{\Delta'\}^* \arrow[dl] \\
& \{v\}^* &
\end{tikzcd}$$

commutes because the homomorphisms are induced by inclusion, and the two lower homomorphisms are isomorphisms. □

Define a *k-box* to be either a k-simplex or the union of members of a non-empty collection of $(k-1)$-simplices of a k-simplex which does not include all of its $(k-1)$-simplices.

Lemma 1.14. *If $D' \subset D$ are boxes in B , the homomorphism $\{D\}^* \to \{D'\}^*$ induced by inclusion is an isomorphism.*

Proof. The argument is by induction on the dimension of the smaller box D'. It suffices to give the proof when D is a simplex, because each box is contained in a simplex. By Lemma 1.13 the conclusion follows if D' is also a simplex. Suppose $D' = D_1 \cup D_2$, where D_1 is a box and D_2 is a simplex. hen $D1 \cap D_2$ is a box of lower dimension than that of D'. Hence, by the induction assumption and Lemma 1.13, the indicated homomorphisms in the diagramme

$$\begin{tikzcd}
\{D\}^* \arrow[r] \arrow[d, "\cong"'] & \{D'\}^* \arrow[d, "\cong"] \\
\{D_2\}^* \arrow[r, "\cong"'] & \{D_1 \cap D_2\}^*
\end{tikzcd}$$

are isomorphisms, and the diagramme commutes since all homomorphisms are induced by inclusions. □

Lemma 1.15. *If v_p is the last vertex of the p-simplex Δ_p, then $\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{v_p\}^q$.*

Proof. Let $\angle i$ denote the box containing all $(p-1)$-simplices of Δ_p except Δ_p^i, the face opposite the i-th vertex. By the cohomology sequence of the triple $\{\Delta_p, \dot{\Delta}_p, \angle 0\}$ and Lemma 1.14,

$$\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{\dot{\Delta}_p, \angle 0\}^{p+q-1}.$$

By excision $\{\dot{\Delta}_p, \angle 0\}^{p+q-1} \cong \{\Delta_p^0, \dot{\Delta}_p^0\}^{p+q-1}$. Continuing this procedure, we get

$$\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{\dot{\Delta}_p, \angle 0\}^{p+q-1} \cong \cdots \cong \{v_p, \varnothing\}^q.$$

□

A system $\mathscr{G}$ of *local coefficients* over B is a function from simplices of B to abelian groups, $\Delta \mapsto G_\Delta$, together with a function from pairs $(\Delta, \Delta^{'})$, $\Delta^{'} \subset \Delta$, of simplices of B to homomorphisms

$$\eta_{\Delta\Delta'} : G_{\Delta'} \to G_\Delta$$

such that

1) $\eta_{\Delta\Delta'} = 1$ if $\Delta = \Delta^{'}$ and

2) $\eta_{\Delta\Delta'} \circ \eta_{\Delta'\Delta''} = \eta_{\Delta\Delta''}$ if $\Delta^{''} \subset \Delta^{'} \subset \Delta$.

A *p-cochain f^p with coefficients in $\mathscr{G}$* is a function which associates with each p-simplex Δ an element of G_Δ; the p-cochains form an abelian group by coordinate addition, $C^p(B; \mathscr{G})$. There is a homomorphism

$$\delta : C^p(B; \mathscr{G}) \to C^{p+1}(B; \mathscr{G})$$

defined by

$$(\delta f^p)(\Delta_{p+1}) = \sum_{j=0}^{p+1} \eta_{\Delta_{p+1}\Delta^j_{p+1}} f^p(\Delta^j_{p+1}).$$

As usual one computes that $\delta \circ \delta = 0$ and defines cohomology $H^p(B; \mathscr{G})$ with local coefficients as $\ker\delta/\operatorname{im}\delta$ in dimension p.

In the spectral sequence arising from the exact couple of an h^*-fibration one has

$$\begin{aligned} E_1^{p,q} \equiv C^{p,q} &\equiv \{B^{(p)}, B^{(p-1)}\}^{p+q} \\ &\cong \oplus_{\Delta_p \subset B}\{\Delta_p, \dot{\Delta}_p\}^{p+q}, \quad \text{by excision} \\ &\cong \oplus_{\Delta_p \subset B}\{v_{\Delta_p}\}^p, \quad \text{by Lemma 1.15,} \\ &\cong C^p(B; h^q(F)), \end{aligned}$$

where $h^q(F)$ is the local coefficient system defined by the function $\Delta_p \to \{v_{\Delta_p}\}^q$ with v_{Δ_p} the last vertex of Δ_p. For $\Delta_s \subset \Delta_p$ the homomorphism (in fact isomorphism)

$$\eta_{\Delta_p\Delta_s} : \{v_{\Delta_s}\}^q \to \{v_{\Delta_p}\}^q$$

is defined via the inclusion isomorphisms with $\{[v_{\Delta_s}, v_{\Delta_p}]\}^q$, where $[v_{\Delta_s}, v_{\Delta_p}]$ is the 1-simplex from v_{Δ_s} to v_{Δ_p}. That this is a local coefficient system then follows from the first axiom in 1.1.1 for the cohomology system $\{h^q\}$. Finally, we note that since B is connected, $\{v^q\}$ is the same group $h^q(F)$, $F = \pi^{-1}(v)$, for all vertices v of B. Thus, we have defined an isomorphism

$$\lambda_{p,q} : E_1^{p,q} \xrightarrow{\cong} C^p(B; h^q(F)).$$

Theorem 1.16. *$\lambda_{p+1,q} \circ d_1 = \delta\lambda_{p,q}$: that is, the diagramme*

$$\begin{tikzcd} E_1^{p,q} \arrow[r, "d_1"] \arrow[d, "\lambda_{p,q}"'] & E_1^{p+1,q} \arrow[d, "\lambda_{p+1,q}"] \\ C^p(B; h^q(F)) \arrow[r, "\delta"'] & C^{p+1}(B; h^q(F)) \end{tikzcd}$$

is commutative.

Proof. By naturality of the spectral sequence of an exact couple and of the isomorphisms $\lambda_{p,q}$, it suffices to prove the assertion in the case $B = \Delta_{p+1}$.

Consider the composition φ^i_{p+1} of

$$\{\Delta^i_{p+1}, \dot{\Delta}^i_{p+1}\}^{p+q} \xrightarrow{j_1^*} \{\dot{\Delta}_{p+1}, \Delta^{p-1}_{p+1}\}^{p+q} \xrightarrow{i^*} \{\dot{\Delta}_{p+1}\} \xrightarrow{\delta} \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1}.$$

By the cohomology sequence of the triple $\{\Delta_{p+1}, \dot{\Delta}_{p+1}, \angle i\}$ and Lemma 1.14, the homomorphism $\Delta : \{\dot{\Delta}_{p+1}, \angle i\}^{p+q} \to \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1}$ is an isomorphism. By excision, $\{\Delta^i_{p+1}, \dot{\Delta}^i_{p+1}\}^{p+q} \cong \{\dot{\Delta}_{p+1}, \angle i\}^{p+q}$. From the commutative diagramme

$$\begin{tikzcd}
\{\dot{\Delta}_{p+1}, \Delta^{p-1}_{p+1}\}^{p+q} \arrow[r, "i^*"] & \{\dot{\Delta}_{p+1}\} \arrow[dr, "\delta"] & \\
& & \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1} \\
\{\Delta^i_{p+1}, \dot{\Delta}^i_{p+1}\}^{p+q} \arrow[uu, "j_1^*"] \arrow[r] & \{\dot{\Delta}_{p+1}, \angle i\}^{p+q} \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

it follows then that φ^i_{p+1} is an isomorphism.

The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is the composition $\delta \circ i^*$. For $f^p \in C^p(B; h^q(F))$,

$$\lambda_{p+1,q} d_1 \lambda^{-1}_{p,q}(f^p)(\Delta_{p+1}) = \sum_{i=0}^{p+1} \varphi^i_{p+1} f_p(\Delta^i_{p+1}).$$

Thus, we wish to show that $\varphi^i_{p+1} = (-1)^i \eta_{\Delta_{p+1}\Delta^i_{p+1}}$.

For $i = 0$ consider the triad $\{\Delta_{p+1}, \Delta^0_{p+1}, \angle 0\}$.

$$\begin{tikzcd}
0 \arrow[r] & \{\Delta^0_{p+1}, \dot{\Delta}^0_{p+1}\}^{p+q} \arrow[r, "\alpha", "\cong"'] & \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1} \arrow[r] & 0 \\
& \{\dot{\Delta}_{p+1}, \angle 0\}^{p+q} \arrow[u, "\cong"] \arrow[r] & \{\dot{\Delta}_{p+1}\}^{p+q} \arrow[u, "\delta"'] &
\end{tikzcd}$$

By the proof of Lemma 1.15, α is $\eta_{\Delta_{p+1}\Delta^0_{p+1}}$; it is also φ^0_{p+1}.

Assume $\varphi^i_j = (-1)^i \eta_{\Delta_j \Delta^i_j}$ for $j < p+1$ and for $j = p+1$ and $i < k$. Let $\Delta^{k,k-1}_{p+1}$ denote the $(p-1)$-simplex opposite the edge $(k, k-1)$. Let $f^p \in C^{p-1}(\Delta^{k,k-1}_{p+1})$. Then

$$\lambda_{p,q} d_1 \lambda^{-1}_{p-1,q} f^{p-1} = \delta f^{p-1} = \begin{cases} (-1)^{k-1} \eta_{\Delta^{k-1}_{p+1}\Delta^{k,k-1}_{p+1}} f^{p-1}(\Delta^{k,k-1}_{p+1}) & \text{on} \quad \Delta^{k-1}_{p+1} \\ (-1)^{k-1} \eta_{\Delta^{k}_{p+1}\Delta^{k,k-1}_{p+1}} f^{p-1}(\Delta^{k,k-1}_{p+1}) & \text{on} \quad \Delta^{k}_{p+1} \end{cases}$$

Since $d_1 \circ d_1 = 0$,

$$\begin{aligned} 0 &= \lambda_{p+1,q} d_1 \lambda_{p,q}^{-1} \lambda_{p,q} d_1 \lambda_{p-1,q}^{-1} (f^{p-1})(\Delta_{p+1}^{k,k-1}) \\ &= \varphi_{p+1}^k((-1)^{k-1}\eta_{\Delta_{p+1}^{k-1}\Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) + \varphi_{p+1}^k((-1)^{k-1}\eta_{\Delta_{p+1}^{k}\Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) \\ &= \eta_{\Delta_{p+1}\Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) + \varphi_{p+1}^k((-1)^{k-1}\eta_{\Delta_{p+1}^{k}\Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) \end{aligned}$$

Thus,

$$\varphi_{p+1}^k(\eta_{\Delta_{p+1}^{k}\Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) = (-1)^k \eta_{\Delta_{p+1}\Delta_{p+1}^{k}} (\eta_{\Delta_{p+1}^{k}\Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})).$$

Hence,

$$\varphi_{p+1}^k = (-1)^k \eta_{\Delta_{p+1}\Delta_{p+1}^{k}}.$$

□

Corollary 1.17. *The chain complexes $E_1^{p,q}$, d_1 and $\{C^p(B; h^q(F)), \delta\}$ are naturally equivalent.*

Corollary 1.18. *In the spectral sequence of the h^*-fibration $\pi : E \to B$, the term $E_2^{p,q}$ is naturally isomorphic with $H^p(B; h^q(F))$, where the latter is ordinary cohomology with local coefficients.*

To summarise the results of this section, there is a functor from the category of h^*-fibrations ($f \to E \xrightarrow{\pi} B$) to the category of exact couples; by composition, then to the category of spectral sequences. There is a natural isomorphism $E_2^{p,q} \cong H^p(B; h^q(F))$, in which $h^q(F)$ is a system of local coordinates. Furthermore, $\{E_\infty^{p,q}\}$ is the associated graded system to a filtration of $h^*(E)$. The spectral sequence is also natural with respect to transformations of cohomology theories.

In particular, if $\pi : X \to X$ is the identity map, then $E_2^{p,q} \cong H^p(X; h^q(\mathrm{pt}))$ and $E_\infty^{p,q}$ is associated to a filtration of $h^*(X)$. This relation can sometimes be exploited to compute or to deduce properties of $h^*(X)$.

1.2.3 Applications of the Spectral Sequence

Proposition 1.19. *If $\tau : h^* \to k^*$ is a natural transformation of cohomology theories and $\tau : h^*(\mathrm{pt}) \xrightarrow{\cong} k^*(\mathrm{pt})$ is an isomorphism, then $\tau : h^*(X, A) \to k^*(X, A)$ is an isomorphism for all pairs (X, A).*

Proof. The transformation τ induces a homomorphism of the spectral sequence of $\mathrm{id} : X \to X$ in the h^*-theory into that in the k^*-theory. The hypothesis implies this is an isomorphism at the E_2 level and thus for $h^*(X)$ into $k^*(X)$. For a pair (X, A), we consider $CA \cup X$. □

Proposition 1.20. *If $f : B \to B'$ so that $f : H^*(B') \to H^*(B)$ is an isomorphism for ordinary singular cohomology (integer coefficients), then $f_h^* : h^*(B') \to h^*(B)$ is an isomorphism for any cohomology theory h^*.*

Proof. The map f induces a map f of the identity fibration $B \to B$ into $B' \to B'$ and thus a map of the spectral sequences of these fibrations. In particular, the naturality of the universal coefficient sequence and the five lemma imply

$$E_2^{p,q}(f) : H^p(B, h^q(\mathrm{pt})) \to H^p(B'; h^q(\mathrm{pt}))$$

is an isomorphism. The conclusion follows. □

Note that this implies our earlier remark that an H^*-fibration is an h^*-fibration for any cohomology theory h^*.

Proposition 1.21. *If $h^i(\mathrm{pt}) = 0$ for $i \neq 0$, then the cohomology theory h^* is naturally equivalent to singular cohomology with coefficients $h^0(\mathrm{pt})$, $H^*(-; h^0(\mathrm{pt}))$.*

Proof. The spectral sequence collapses and we have

$$H^p(B; h^0(\mathrm{pt})) \cong E_2^{p,q} \cong E_2^{p,\infty} \cong F_{p-1}h^p(B)/F_p h^p(B) \cong h^p(B).$$

□

Proposition 1.22. $H^n(B; G) \cong [B, K(G, n)]$.

Proof. The groups $\{B, \mathscr{K}(G)\}$ give a cohomology theory k^* in which $k^0(\mathrm{pt}) \cong \varinjlim[\mathbb{S}^n, K(G, n)]$. The composition

$$[\mathbb{S}^n, K(G, n)] \xrightarrow{\Sigma} [\mathbb{S}^{n+1}, \Sigma K(G, n+1)] \to [\Sigma\mathbb{S}^{n+1}, K(G, n+1)]$$

is the isomorphism $[\mathbb{S}^n, \Omega K(G, n+1)] \to [\Sigma\mathbb{S}^n, K(G, n++1)]$. By definition each of these groups is G. In other dimensions $k^i(\mathrm{pt}) = 0$. Thus $\{B, \mathscr{K}(G)^n\} \cong H^n(B; G)$. The object on the left is $\varinjlim[\Sigma^i B, K(G, n+i)]$. As seen above, these groups are isomorphic and are mapped isomorphically. Thus $[B, K(G, n)] \cong \{B, \mathscr{K}(G)^n\} \cong H^n(B; G)$. □

It is interesting to note this proof does not use obstruction theory.

1.2.4 The "Universal Cohomology Theory"

For pairs (X, A) and (Y, B), define

$$\pi_j^i(X, A; Y, B) = \varinjlim_n[\Sigma^{j+n}(X/A), \Sigma^{i+n}(Y/B)].$$

For fixed j and (Y, B), this is a cohomology theory.

For an arbitrary cohomology theory h define

$$\tau : \pi_j^i(X, A; Y, B) \times h^k(Y/B) \to h^{k+1-j}(X/A)$$

as follows:

for $f \in [f] \in [\Sigma^{j+N}(X/A), \Sigma^{i+N}(Y/B)]$, the composition

$$h^k(Y/B) \to h^{k+i+N}(\Sigma^{i+N}(Y/B)) \to h^{k+i+N}(\Sigma^{j+N}(X/A)) \to h^{k+i-j}(X/A)$$

defines the pairing τ since it is independent of the representative f in the class of the limit. The pairing is bilinear; and so, induces

$$\tau : \pi^i_j(X, A; Y, B) \otimes h^k(Y/B) \to h^{k+i-j}(X/A).$$

In particular, for $j = 0$ and $(Y, B) = (\mathbb{S}^0, \mathrm{pt})$, $\pi^i_j(X, A; Y, B) = \pi^i_{\mathcal{S}}(X, A)$, the cohomology theory called *stable cohomotopy*. Then for each n

$$\tau \otimes 1 : \sum_{i+j=n} \pi^i_{\mathcal{S}}(X, A) \otimes h^j(\mathrm{pt}) \otimes \mathbb{Q} \to h^n(X, A) \otimes \mathbb{Q}$$

is a natural transformation of cohomology theories.

Theorem 1.23. *The transformation*

$$\tau \otimes 1 : \sum_{i+j=n} \pi^i_{\mathcal{S}}(X, A) \otimes h^j(\mathrm{pt}) \otimes \mathbb{Q} \to h^n(X, A) \otimes \mathbb{Q}$$

is a natural equivalence of cohomology theories, where $\mathbb{Q}$ denotes the rationals. Thus, up to torsion $\pi^i_{\mathcal{S}}$ is a "universal cohomology theory".

Proof. By Proposition 1.19 it suffices to verify the conclusion for $(X, A) = (\mathbb{S}^0, \mathrm{pt})$. But $\pi^i_{\mathcal{S}}(\mathbb{S}^0, \mathrm{pt})$ is a torsion group for $i \neq 0$ and is the integers $\mathbb{Z}$ for $i = 0$ by a theorem of Serre since the stable cohomotopy groups of spheres are isomorphic to the stable homotopy groups of spheres. Then $\tau \otimes 1$ is an isomorphism for $(X, A) = (\mathbb{S}^0, \mathrm{pt})$. □

The maps $\mathbb{S}^n \to K(\mathbb{Z}, n)$ generating the homotopy of $K(\mathbb{Z}, n)$ induce a map of spectra $\mathcal{S} \to \mathcal{K}(\mathbb{Z})$. By Proposition 1.22 this induces a natural transformation of cohomology theories

$$\pi^i_{\mathcal{S}}(-1) \to H^i(-).$$

For a cohomology theory h^* define the "generalised Chern character"

$$\mathrm{Ch} : h^n(X, A) \to \sum_{i+j=n} H^i(X, A; h^j(\mathrm{pt}) \otimes \mathbb{Q})$$

to be the composition

$$h^n(X, A) \to h^n(X, A) \otimes \mathbb{Q} \xrightarrow{(\tau\otimes 1)^{-1}} \sum_{i+j=n} \pi^i_{\mathcal{S}}(X, A) \otimes h^j(\mathrm{pt}) \to \sum_{i+j=n} H^i(X, A; h^j(\mathrm{pt}) \otimes \mathbb{Q}).$$

Corollary 1.24. *The transformation*

$$\mathrm{Ch} \otimes 1 : h^n(X, A) \otimes \mathbb{Q} \to \sum_{i+j=n} H^i(X, A; h^j(\mathrm{pt}) \otimes \mathbb{Q})$$

is a natural equivalence of cohomology theories. It is also natural under transformations of cohomology theories $h^* \to k^*$.

Proof. Proposition 1.19 implies the result since it is clearly true for $(X, A) = (\mathbb{S}^0, \mathrm{pt})$. $\square$

Corollary 1.25. *Let h and k be cohomology theories, with k a $\mathbb{Q}$-module valued functor. Then for*

$$\varphi : h^*(\mathrm{pt}) \to k^*(\mathrm{pt}),$$

there is a unique natural transformation

$$\Phi : h^*(-) \to k^*(-)$$

of cohomology theories extending φ.

Proof. Consider the diagramme

$$\begin{tikzcd}
h^*(X,A) \arrow[r] \arrow[d, "\Phi"'] & h^*(X,A) \otimes \mathbb{Q} \arrow[r, "\cong"] & \sum H^*(X, A; h^*(\mathrm{pt}) \otimes \mathbb{Q}) \arrow[d, "\varphi"] \\
k^*(X,A) \arrow[r] & k^*(X,A) \otimes \mathbb{Q} \arrow[r, "\cong"'] & \sum H^*(X, A; k^*(\mathrm{pt}) \otimes \mathbb{Q})
\end{tikzcd}$$

The homomorphism Φ is defined by means of the diagramme; it is unique since any such homomorphism must factor through $h^* \otimes \mathbb{Q}$. $\square$

1.3 Multiplicative Cohomology Theories

1.3.1 Preliminaries

A cohomology theory h^* is *multiplicative* if for each (X, A) and (Y, B), there is a homomorphism

$$\otimes : h^i(X, A) \otimes h^j(Y, B) \to h^{i+j}(X \times Y, X \times B \cup A \times Y)$$

which is associative, (graded) commutative (i.e., $u^i \otimes v^j = (-1)^{ij} v^j \otimes u^i$) , natural under maps of pairs, has a unit $1 \in h^0(\mathbb{S}^0, pt)$, and makes the following diagramme commute:

$$\begin{tikzcd}
h^i(A) \otimes h^j(Y,B) \arrow[r, "\otimes"] \arrow[dd, "\delta\times 1"'] & h^{i+j}(A \times Y, A \times B) \arrow[d, "\gamma"] \\
 & h^{i+j}(A \times Y \cup X \times B, X \times B) \arrow[d, "\delta"] \\
h^{i+1}(X,A) \otimes h^j(Y,B) \arrow[r, "\otimes"'] & h^{i+j+1}(X \times X \times B \cup A \times Y)
\end{tikzcd}$$

The homomorphism γ is excision and the δ on the right side is that of the cohomology sequence of the triple $(X \times Y, A \times Y \cup X \times B)$.

If $(X; A, B)$ is a triad, the external pairing just described defines an internal pairing
$$h^i(X, A) \otimes h^j(X, B) \to h^{i+j}(X, A \cup B)$$
sending (u, v) into $u \cup v$. This pairing is the composition
$$h^i(X, A) \otimes h^j(X, B) \xrightarrow{\otimes} h^{i+j}(X \times X, X \times B \cup A \times Y) \xrightarrow{\Delta^*}, h^{i+j}(X, A \cup B),$$
where $\Delta : X \to X \times X$ is the diagonal map.

This internal pairing has the properties

1. If $A = B$, $h^*(X, A)$ is a ring;
2. If $A = B = \emptyset$, $h^*(X)$ is a ring with unit;
3. If $B = \emptyset$, $h(X, A)$ is an $h^*(X)$-module; and
4. $h^*(A) \xrightarrow{\delta} h^*(X, A)$ is an $h^*(X)$-module homomorphism.

Conversely, given an internal pairing $\cup$, there is an external pairing $\otimes$ defined by
$$x \otimes y = p^* x \cup q^* y,$$
where $x \in h^*(X, A)$, $y \in h^j(Y, B)$ and p and q are the projections of $(X \times X, A \times Y)$ onto (X, A) and $(X \times Y, X \times B)$ onto (Y, B), respectively.

Further, if we are given a natural internal pairing $\cup$ on $h^*(X, A)$ for all pairs $(X, A) \in \mathscr{P}^2$ which satisfies 1, 2, 3 and 4, we can define an external pairing by the following device.

Let i_1 and i_2 denote the injections of $(X, *)$ and $(Y, *)$ in $(X \times Y, *)$ and p_1 and p_2 the projections of $(X \times Y, *)$ onto $(X, *)$ and $(Y, *)$. For $u \in h^i(X, *)$ and $v \in h^j(Y, *)$, the element $p_1^* u \cup p_2^* v \in h^{i+j}(X \times Y, *)$. Since $p_2 \circ i_1 : (X, *) \to (*, *) \subset (Y, *)$,
$$i_1^*(p_1^* u \cup p_2^* v) = i_1^* p_1^* u \cup (0) = 0.$$
Similarly, $i_2^*(p_1^* u \cup p_2^* v) = 0$. Thus, the element $p_1^* u \cup p_2^* v$ lies in $h^{i+j}(X \# Y, *)$. We define $u \otimes v \equiv p_1^* u \cup p_2^* v \in h^{i+j}(X \times Y, X \times * \cup * \times Y)$. For (X, A) and (Y, B) we make the same construction on $(X/A, *)$ and $(Y/B, *)$ and use the homeomorphism $(X/A) \# (Y/B) \cong X \times Y / (X \times B \cup A \times Y)$. It can be checked that this is an external pairing.

We note that if $X = Y$ and $\Delta : X \to X \times X$ is the diagonal map then since $p_1 \Delta = p_2 \Delta = \mathrm{id}$,
$$\Delta^*(u \otimes v) = \Delta^* p_1^* u \cup \Delta^* A p_2^* v = u \cup v;$$
and so, the previous process for obtaining an internal pairing from an external one when applied to this external pairing yields the internal pairing we started with.

In $h^n(\mathbb{S}^n, *)$ define a class γ^n to be $(\Sigma^n)^*(1)$ where 1 is the unit of $h^0(\mathbb{S}^0, *)$ and $(\Sigma^n)^*$ is the suspension isomorphism $h^0(\mathbb{S}^0, *) \to h^n(\mathbb{S}^n, *)$. In cohomology with the dimension axiom, γ is a generator of $h^n(\mathbb{S}^n, *)$.

Proposition 1.26. *The diagramme*

$$\begin{tikzcd}
 & h^{i+n}(\mathbb{S}^n \times X, \mathbb{S}^n \times A \cup (*) \times X) \arrow[dd, "\cong"] \\
h^i(X,A) \arrow[ur, "\gamma^n\times"] \arrow[dr, "(\Sigma^n))^*"'] & \\
 & h^{i+n}(\Sigma^n(X/A), *)
\end{tikzcd}$$

is commutative, where $\gamma^n\times$ denotes external multiplication by γ^n and the vertical homomorphism is an excision isomorphism.

Proof. It suffices to prove this for $n = 1$ for it will then follow that $\gamma^n = \gamma^1 \times \gamma^{n-1}$.

Consider the following diagramme.

$$\begin{tikzcd}
h^0(\mathbb{S}^0, *) \otimes h^i(X,A) \arrow[r, "\cong"] \arrow[d, "\cong"'] & h^i(\mathbb{S}^0 \times X, (*) \times X \cup \mathbb{S}^0 \times A) \arrow[r] \arrow[d, "\cong"] & h^i(X/A, *) \arrow[d, "(\Sigma)^*"] \\
h^1(C\mathbb{S}^0, \mathbb{S}^0) \otimes h^i(X,A) \arrow[r] \arrow[d, "\cong"'] & h^{i+1}(C\mathbb{S}^0 \times X, \mathbb{S}^0 \times X \cup C\mathbb{S}^0 \times A) \arrow[r] \arrow[d, "\cong"] & h^{i+1}(\Sigma(X/A), *) \\
h^i(\mathbb{S}^1, *) \otimes h^i(X,A) \arrow[r] & h^{i+1}(\mathbb{S}^1 \times X, (*) \times X \cup \mathbb{S}^1 \times A) \arrow[ur] &
\end{tikzcd}$$

The upper line gives $(\Sigma^1)^*$ and the lower line $\gamma^1(-)$. Commutativity is clear except possibly in the upper left-hand corner, where it holds by definition of the external product. $\square$

Proposition 1.27. *Let h^* and k^* be multiplicative cohomology theories. If $k^0(\mathrm{pt})$ is a $\mathbb{Q}$-module, then k^* can be made a $\mathbb{Q}$-module valued functor. If $\varphi : h^*(\mathrm{pt}) \to k^*(\mathrm{pt})$ and Φ is the unique extension of φ as in Corollary 1.25 of Theorem 1.23 and if φ is a multiplicative homomorphism, then Φ is also multiplicative.*

Proof. k^* can be made a $\mathbb{Q}$-module valued functor because $k^i(X,A) \otimes k^0(\mathrm{pt}) \xrightarrow{\cong} k^i(X,A)$.

In the diagramme

$$\begin{tikzcd}
h^*(X,A) \otimes h^*(Y,B) \arrow[r] \arrow[d] & h^*(X \times Y, X \times B \cup A \times Y) \arrow[d] \\
h^*(X,A) \otimes h^*(Y,B) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\Phi\otimes\Phi"'] & h^*(X \times Y, X \times B \cup A \times Y) \otimes \mathbb{Q} \arrow[d, "\Phi"] \\
k^*(X,A) \otimes k^*(Y,B) \arrow[r] & k^*(X \times Y, X \times B \cup A \times Y),
\end{tikzcd}$$

for a fixed pair (Y, B) all terms in the lower square are cohomology theories in (X, A) and all homomorphisms are natural. By Corollary 1.25 the two composities around the square will agree if they agree for $(X, A) = (\mathbb{S}^0, *)$. But with (X, A) fixed, all terms are cohomology theories in (Y, B), and the same argument holds. To prove commutativity it then suffices to prove it in the case (Y, B) is also $(\mathbb{S}^0, *)$, but this is just the hypothesis that φ is multiplicative. □

1.3.2 Dold-Thom-Gysin Theorem

Theorem 1.28. *Let h^* be a multiplicative cohomology theory and let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be an h^*-fibration. Suppose there are elements $a_1, \ \ldots, \ a_r$ in $h^*(E)$ such that $(i^*a_1, \ \ldots, \ i^*a_r)$ is an $h^*(\mathrm{pt})$-base for $h^*(F)$ as an $h^*(\mathrm{pt})$-module, then $(a_1, \ \ldots, \ a_r)$ is an $h^*(B)$-base for $h^*(E)$ as an $h^*(B)$-module.*

Proof. The $h^*(B)$-module structure of $h^*(E)$ is defined by π^* as the composition

$$h^*(B) \otimes h^*(E) \xrightarrow{\pi^* \otimes 1} h^*(E) \otimes h^*(E) \xrightarrow{\cup} h^*(E).$$

Let C and $C^{'}$ be the exact couples of the fibrations $\mathrm{id} : B \to B$ and $\pi : E \to B$, respectively. The homomorphism

$$\rho : C \oplus C \oplus C \oplus \cdots \oplus C \to C^{'}$$

defined by $\rho(\lambda_1 \oplus \cdots \oplus \lambda_r) = \sum_{i=1}^{r} \pi^*(\lambda_i) a_i$ is a map of exact couples since π is a fibre preserving map of fibrations

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "\pi"'] & B \arrow[d, "\mathrm{id}"] \\ B \arrow[r] & B \end{tikzcd}$$

and $\rho$is an $h^*(B)$-module homorphism commuting with the coboundary operator in the two couples.

Since $(i^*a_1, \ \ldots, \ i^*a_r)$ is an $h^*(\mathrm{pt})$-base for $h^*(F)$, at the E_1-level of the spectral sequences, the homomorphism

$$C^p(B; h^*(\mathrm{pt})) \oplus \cdots \oplus C^p(B; h^*(\mathrm{pt})) \to C^p(B; h^*(F))$$

induced by ρ is an isomorphism. Thus, ρ induces isomorphisms on E_∞ terms of the spectral sequence and consequently we have the isomorphism

$$h^*(B) \oplus \cdots \oplus h^*(B) \to h^*(E),$$

defined by linear combination with the a_i. Thus, $(a_1, \ \ldots, \ a_r)$ is an $h^*(B)$-base for $h^*(E)$. □

The Theorem 1.28 has a relative version. Conceptually, it involves nothing new except complicated notation. Thus, we will only define the relevant exact couples and state without proof the theorem for the relative case.

Suppose $E^{'} \subset E$ and $\pi : E \to B$ such that for each vector v of a simplex Δ of B

$$h^*(\pi^{-1}(\Delta), pi^{-1}(\Delta) \cap E^{'}) \to h^*(\pi^{-1}(v), \pi^{-1}(v) \cap E^{'})$$

is an isomorphism. Then $(E, E^{'}, \pi, B)$ is a *relative h^*-fibration with relative fibre* $(F, F^{'}) = (\pi^{-1}(v), pi^{-1}(v) \cap E^{'})$. Let $B^{'}$ be a subcomplex of B. We can then define an exact couple as follows:

$$A^{p,q} = h^{p+q}([\pi^{-1}(B^{(p)}) \cup \pi^{-1}(B^{'})]/[(\pi^{-1}(B^{(p)}) \cap E^{'}) \cup \pi^{-1}(B^{'})])$$

with $C^{p,q}$ equal h^{p+q} of the corresponding pair based on the pth and $(p-1)$th skeletons. As before this leads to the *spectral sequence of the relative fibration* and we have

Corollary 1.29. *In the spectral sequence of the relative fibration $(E, E^{'}, \pi, B, B^{'})$, the term $E_2^{p,q}$ is naturally isomorphic with $H^p(B, B^{'}; h^q(F, F^{'}))$ and $E_\infty^{p,q}$ is the system of graded groups associated with a filtration of $h^n(E, E^{'} \cup \pi^1(B^{'}))$, $p + q = n$.*

In an analogous way we have the following relative version of the Dold-Thom-Gysin isomorphism:

Theorem 1.30. *Let h^* be a multiplicative cohomology theory and let $(E, E^{'}, \pi, B, B^{'})$ be a relative f^*-fibration. Suppose there are elements $a_1, \ldots, a_r$ in $h^*(E, E^{'} \cup \pi^{-1}(B^{'}))$ such that $(i^*a_1, \ldots, i^*a_r)$ is an $h^*(\mathrm{pt})$-base for $h^*(F, F^{'})$ as an $h^*(\mathrm{pt})$-module. Then $\{a_1, \ldots, a_r\}$ is an $h^*(B, B^{'})$-base for $h^*(E, E^{'} \cup \pi^{-1}(B^{'}))$ as an $h^*(B, B^{'})$-module.*

The $h^*(B, B^{'})$-module structure of $h^*(E, E^{'} \cup \pi^{-1}(B^{'}))$ is defined by the composition

$$\begin{aligned} h^*(B, B^{'}) \otimes h^*(E, E^{'} \cup \pi^{-1}(B^{'})) &\to h^*(E, \pi^{-1}(B^{'})) \otimes h^*(E, E^{'} \cup \pi^{-1}(B^{'})) \\ &\xrightarrow{\cup} h^*(E, E^{'} \cup \pi^{-1}(B^{'})). \end{aligned}$$

Actually, a slightly weaker hypothesis describes $h^*(B, B^{'})$: if the classes $a_1, \ldots, a_r$ in $h^*(E, E^{'})$, then

$$(\pi^* h^*(B, B^{'})) \cup a_1 \oplus \cdots \oplus (\pi^* h^*(B, B^{'})) \cup a_r \cong h^*(E, E^{'} \cup \pi^{-1}(B^{'})).$$

1.3.3 Orientability of Vector Bundles

Let $\pi : E \to B$ be an n-plane bundle.

Remark 1.31. We shall usually assume π is vector bundle, but in the following Corollary 1.33 to Theorem 1.30 it is sufficient for the structure group to be group of origin preserving onto-homeomorphisms of $\mathbb{R}^n$, Euclidean n-space.

Let $E^{'}$ denote the complement of the zero cross-section. Then for any cohomology theory h^*, $(E, E^{'}, \pi, B)$ is a relative h^*-fibration with fibre $(F, F^{'}) = (\mathbb{R}^n, \mathbb{R}^n \backslash 0)$, which has the cohomology of $(\mathbb{S}^n, \mathrm{pt})$. Thus, $h^{i+n}(F, F^{'}) \cong h^{i+n}(\mathbb{S}^n, \mathrm{pt}) \cong h^i(\mathrm{pt})$; so $h^*(F, F^{'})$ is a free $h^*(\mathrm{pt})$-module with one generator u_n of degree n corresponding to $1 \in h^0(\mathrm{pt})$.

Definition 1.32. The n-plane bundle π is *h^*-orientable* if there is a class $u \in h^n(E, E')$ such that $i^*u = u_n \in h^n(F, F')$ for each fibre inclusion $i : (F, F') \to (E, E')$. Of course, if B is connected and $i^*u = u_n$ for one fibre, then it is so for all fibres. A choice of such a class u is an *h^*-orientation* of π and π is said to be *h^*-oriented by u*.

Corollary 1.33. *[to Theorem 1.30] If the n-plane bundle π is h^*-oriented by the class $u \in h^n(E, E')$, then*

$$\varphi : h^i(B) \to h^{i+n}(E, E')$$

defined by $\varphi(z) = \pi^*z \cup u$ is an isomorphism.

If the structure group of the bundle π is a subgroup of the orthogonal group, or more generally of the group of auto-homeomorphisms on $\mathbb{R}^n$ preserving distance from the origin, then associated with π are the unit disc bundle $\mathbb{D}(\pi)$ and the unit sphere bundle $\mathbb{S}(\pi)$ of π. The factor space $\mathbb{D}(\pi))/\mathbb{S}(\pi)$ has a well determined base point and is called the *Thom space* of the bundle π, herein denoted B^π.

The inclusion $(\mathbb{D}(\pi), \mathbb{S}(\pi)) \to (E, E')$ induces an isomorphism of cohomology; thus,

$$h^*(E, E') \xrightarrow{\cong} f^*(B^\pi, *).$$

Corresponding to each fibre inclusion $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to (E, E')$ is the disc and sphere inclusion

$$\begin{tikzcd}
(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[r] \arrow[d] & (\mathbb{D}(\pi), \mathbb{S}(\pi)) \arrow[d] \\
(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \arrow[r] & (E, E')
\end{tikzcd}$$

and $\mathbb{D}^n/\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{S}^n$. Thus, we can equivalently define notions of h^*-orientation in terms of a class $u \in h^n(B^\pi, *)$ which restricts to the class $\gamma^n \in h^n(\mathbb{S}^n, *)$ and the isomorphism of the Corollary becomes $\varphi : h^i(B) \to h^{i+n}(B^\pi, *)$ defined by the composition

$$\begin{tikzcd}
h^i(B) \arrow[r, "\pi^*"] & h^i(\mathbb{D}(\pi)) \arrow[r] \arrow[dr, "\cup u"'] & h^{i+n}(B^\pi, *) \arrow[d, "\cong"] \\
 & & h^{i+n}(\mathbb{D}(\pi), \mathbb{S}(\pi)).
\end{tikzcd}$$

Proposition 1.34. *If α and β are vector bundles over X and Y, respectively, then $X^\alpha\ Y^\beta$ is homeomorphic to $(X \times Y)^{\alpha\times\beta}$, where $\alpha \times \beta$ is the bundle with fibre $\alpha x + \beta_y$ over the point (x, y).*

Proof. The obvious map $X^\alpha \times Y^\beta \to (X \times Y)^{\alpha\times\beta}$ is a homeomorphism except on the wedge, which goes to the base point. □

Proposition 1.35. *If $u \in h^m(X^\alpha, *)$ and $v \in h^n(X^\beta, *)$ are h^*-orientations of the vector bundles α and β over the space X, then $\Delta^*(u \times v)$ is an h^*-orientation of the Whitney sum $\alpha \oplus \beta$ over X, where Δ^* is induced by the map $X^{\alpha\oplus\beta} \xrightarrow{\Delta} (X \times X)^{\alpha\times\beta}$.*

Proof. The Whitney sum $\alpha \oplus \beta$ is the bundle over X induced from $\alpha \times \beta$ over $X \times X$ by the diagonal map. The conclusion then follows from the commutative diagramme below, and the fact that $\gamma^m \times \gamma^n = \gamma^{m+n}$.

$$\begin{tikzcd}
h^m(X^\alpha,*) \otimes h^n(X^\beta,*) \arrow[r, "\times"] \arrow[d] & h^{m+n}(X^\alpha \# X^\beta,*) \arrow[r] \arrow[d] & h^{m+n}(X^{\alpha\otimes\beta},*) \arrow[d] \\
h^m(\mathbb{S}^m,*) \otimes h^n(\mathbb{S}^n,*) \arrow[r, "\times"'] & h^{m+n}(\mathbb{S}^m \# \mathbb{S}^n,*) \arrow[r, "\cong"'] & h^{m+n}(\mathbb{S}^{m+n},*)
\end{tikzcd}$$

□

Proposition 1.36. *If γ is a vector bundle over X oriented by the class $u \in h^n(X\gamma, *)$ and A is a subcomplex of X, then*

$$\varphi :^i (X, A) \to h^{i+n}(\mathbb{D}(\gamma), \mathbb{S}(\gamma) \cup \mathbb{D}(\gamma|_A))$$

defined by $\varphi(z) = \pi^*_{\mathbb{D}}(z) \cup u$, where $\pi_{\mathbb{D}} : \mathbb{D}(\gamma) \to X$ is the projection map of the associated disc bundle $\mathbb{D}(\gamma)$, is an isomorphism.

Proof. This is an immediate consequence of Theorem 1.30. □

Theorem 1.37. *If α and β are vector bundles over X such that β is h^*-oriented by $v \in h^n(X^\beta, *)$ and $\alpha \oplus \beta$ is h^*-oriented by $w \in h^{n+m}(X^{\alpha+\beta}, *)$, then α is h^*-oriented by a class $u \in h^m(X^\alpha, *)$ characterised by the equation $\rho^* u \cup \pi^* v = w$ (ρ and π as defined in the proof) and $w = \Delta^*(u \times v)$.*

Proof. Let $\pi : \mathbb{D}(\alpha) \to X$, $\rho : \mathbb{D}(\pi^*\beta) \to \mathbb{D}(\alpha)$ and $\sigma : \mathbb{D}(\alpha \oplus \beta) \to X$ be disc bundle projections. Note that there is a disc-preserving homotopy equivalence

$$(\mathbb{D}(\alpha \oplus \beta), \mathbb{S}(\alpha \oplus \beta)) \cong (\mathbb{D}(\pi^*\beta), \mathbb{S}(\pi^*\beta) \cup \mathbb{D}(\pi^*\beta|_{\mathbb{S}(\alpha)})),$$

obtained by deforming the box-like discs on the right radially into the spherical ones on the left. Since β is h-oriented by $v \in h^n(X, *)$, $\pi^*\beta$ is h-oriented by $\pi^* v \in h^n(\mathbb{D}(\alpha)^{\pi^*\beta}, *)$. But this implies

$$\phi_{\pi^*:\beta} h^{m+i}(\mathbb{D}(\alpha), \mathbb{S}(\alpha)) \to h^{m+n+i}(\mathbb{D}(\pi^*\beta), \mathbb{S}(\pi^*\beta) \cup \mathbb{D}(\pi^*\beta|_{\mathbb{S}(\alpha)}))$$

as in Proposition 1.36 is an isomorphism. Also

$$\varphi_{\alpha\oplus\beta} : h^i(X) \to h^{m+n+i}(\mathbb{D}(\alpha \oplus \beta), \mathbb{S}(\alpha \oplus \beta))$$

is an isomorphism. Identifying cohomologies by the homotopy equivalence observed, we then have determined a class $u \in h^m(\mathbb{D}(\alpha), \mathbb{S}(\alpha))$ by

$$u = \varphi^{-1}_{\pi^*\beta}\{\varphi_{\alpha\oplus\beta(1)}\};$$

i.e., u is the unique class of $h^m(\mathbb{D}(\alpha), \mathbb{S}(\alpha))$ such that

$$\rho^* u \cup \pi^* v = w.$$

Restricting u to $h^m(\mathbb{D}^m, \mathbb{S}^{m-1})$, we have that its product with γ^n is γ^{m+n}. Thus, u is an h^*-orientation for α.

By definition of the internal product, the class $u \times v$ in $h^{n+m}((X \times X)^{\alpha \times \beta}, *)$ restricts to $\rho^* u \cup \pi^* v = w$ on the diagonal $X^{\alpha \oplus \beta}$. □

Remark 1.38. Because of the previous results it is clear that if one representative of a stable class of vector bundles ($\alpha \underset{s}{\tilde{\rightarrow}} \beta$ if there are trivial bundles n and m such that $n \oplus \alpha \sim m \oplus \beta$) is orientable in a given cohomology theory, then all representatives are. However, there is no uniqueness to this construction, for just as there may be a number of inequivalent homotopy equivalences between two spaces, there also can be a number of inequivalent bundle equivalences. Each of these yields a homotopy equivalence between the corresponding Thorn spaces and consequent transport of orientation, but different equivalences may yield different orientations. For purposes of naturality in some constructions, this point will cause us to take some care in the next section.

1.4 Applications to Differential Manifolds

The object of this section is to establish a type of Riemann-Roch theorem for general cohomology theories.

1.4.1 Orientations of Manifolds and the "Umkehr" Homomorphism

First, we recall some results from differential topology. If $f : M^m \to \mathbb{R}^{m+r}$ is a differentiable imbedding of a differentiable m-manifold into Euclidean space, let $\nu(f)$ denote the *normal* r-plane bundle of f. If also $g : M^m \to \mathbb{R}^{m+r}$ is a differentiable imbedding and $r > m + 1$, then f and g are homotopic by a differentiable homotopy through differentiable imbeddings; hence $\nu(g)$ is equivalent to $\nu(f)$. If, moreover, $r > m + 3$, then two such regular homotopies are themselves regularly homotopic through regular homotopies, and the two resulting bundle equivalences are homotopic through bundle equivalences. Thus, we have

Proposition 1.39. *If f and g are differentiable imbeddings of M^m in $\mathbb{R}^{m+r}$, $r > m + 3$, then there is a disc preserving homeomorphism of $M^{\nu(f)}$ onto $M^{\nu(g)}$ which is unique up to the isotopy class of disc-preserving homeomorphism.*

Proposition 1.40. *If $f : M^m \to \mathbb{R}^{m+r}$ and $f' : M^m \to \mathbb{R}^{m+r'}$ are differentiable imbeddings, then for some pair of integers (s, s') there is a uniquely determined isotopy class of homeoorphisms*

$$M^{s \oplus \nu(f)} \to M^{s' \oplus \nu(f')}.$$

Proof. By Proposition 1.34, with Y a point and $\beta = n$, we have $\Sigma^n(X^\alpha) \cong X$ (for $\cong$ read "is homeomorphic to"). Let $i : \mathbb{R}^{m+r} \to \mathbb{R}^{m+r+1}$ be the usual inclusion; then

$$\Sigma M^{\nu(f)} \cong M^{1 \oplus \nu(f)} \cong M^{\nu(I \circ f)}$$

since $\nu(i \circ f) = 1 \oplus \nu(f)$. Choose s and s' such that $r + s = r' + s' > m + 3$. Then there are well determined isotopy classes of homeomorphisms

$$\Sigma^s M^{\nu(f)} \cong M^{\nu(i^s \circ f)} \cong M^{\nu(i^{s'} \circ f')} \cong \Sigma^{s'} M^{\nu(f')},$$

where i^s stands for the iterated inclusion

$$\mathbb{R}^{m+r} \xrightarrow{i} \mathbb{R}^{m+r+1} \xrightarrow{i} \cdots \xrightarrow{i} \mathbb{R}^{m+r+s};$$

the second homeomorphism in the series is given by Proposition 1.39. Thus, we have determined a unique isotopy class of homeomorphisms $M^{s \oplus \nu(f)} \to M^{s' \oplus \nu(f')}$. □

Definition 1.41. A differentiable manifold M^m is *h^*-orientable* if for some differentiable imbedding $f : M^m \to \mathbb{R}^{m+r}$, the bundle $\nu(f)$ is h^*-orientable; an *h^*-orientation* of M is determined by such an f and a specific orientation class $u \in h^r(M^{\nu(f)}, *)$.

Proposition 1.42. *Each h^*-orientation $f : M^m \to \mathbb{R}^{m+r}$ and $u \in h^r(M^{\nu(f)}, *)$ determines for any differentiable imbedding $f' : M^m \to \mathbb{R}^{m+r'}$ a unique orientation class $u' \in h^{r'}(M^{\nu(f')}, *)$.*

Proof. The unique isotopy class of homeomorphisms of Proposition 1.40 together with the s'-desuspension from $h^{r+s}(\Sigma^{s'} M^{\nu(f')}, *)$ of the s-suspension of u into $h^{r+s}(\Sigma^s M^{\nu(f)}, *)$ determines the orientation class u'. □

Proposition 1.43. *If M is h^*-oriented by f and u, then*

1) $\varphi_f : h^j(M) \to h^{j+r}(M^{\nu(f)}, *)$ is an isomorphism defined by $\varphi_f(z) = \pi^*_{\nu(f)}(z) \cup u$,

2) the diagramme

$$\begin{tikzcd}
h^j(M) \arrow[r, "\varphi_f"] \arrow[ddr, "(-1)^j \varphi_{i \circ f}"'] & h^{j+r}(M^{\nu(f)}, *) \arrow[d, "\cong"] \\
& h^{j+r+1}(\Sigma M^{\nu(f)}, *) \arrow[d, "\cong"] \\
& h^{j+r+1}(M^{\nu(i \circ f)}, *)
\end{tikzcd}$$

is commutative, and

3) the isomorphism

$$\varphi_{f'} : h^j(M) \to h^{j+r'}(M^{\nu(f')}, *)$$

is well determined for any differentiable imbedding $f' : M^m \to \mathbb{R}^{m+r'}$.

Proof. (of 2)) By Proposition 1.26 we have the equalities

$$\begin{aligned}\Sigma\varphi_f(z) &= \gamma \cup \varphi_f(z) = \gamma \cup (\pi^*(z) \cup u) = (-1)^j \pi^+(z) \cup (\gamma \cup u) \\ &= (-1)^j \pi^*(z) \cup u' = (-1)^j \varphi_{i\circ f}(z).\end{aligned}$$

□

We have seen in Proposition 1.42 that the possible h-orientations of a manifold are determined by those of any one of its normal bundles. In fact, they are also determined by its tangent bundle τ.

Proposition 1.44. *For any differentiable imbedding* $f : M^m \to \mathbb{R}^{m+r}$ *there is a well defined* $1-1$ *correspondence between the* h^**-orientations of* $\nu(f)$ *and the* h^* *-orientations of* $\tau(M)$.

Proof. By Proposition 1.42 it suffices to prove this for any f and r; we choose $r > m+3$. Then the triviality of the total space $(m+r)$-bundle over M is assured for all imbeddings. The conclusion then follows from Theorem 1.37. □

Definition 1.45. Let M^m and N^n be differentiable manifolds and $f : M \to N$ be a continuous map. For $d > 2m+3$ there is a differentiable imbedding $i : M \to \mathbb{D}^d$ into the interior ofinterior of the d-dimensional disc such that the imbedding $\widetilde{f} = f \times i : M \to N \times \mathbb{D}$ is homotopic to a differentiable imbedding. If the normal bundle $\nu(\widetilde{f})$ is h-orientable, then i together with a choice of orientation class $u \in h^{n+d-m}(M^{\nu(\widetilde{f})}, *)$ is said to determine an *h^*-orientation of the map f*. The usual considerations of isotopies of imbeddings show the determination of u is in fact independent of the imbedding and homotopy.

Proposition 1.46. *Let* M^m *and* N^n *be differentiable manifolds and* $f : M \to N$ *be a continuous map. The* h^**-orientations of* f *are in a well defined* $1-1$ *correspondence with the* h^**-orientations of the bundle* $\nu(M) \oplus f^*\tau(N)$. *In particular, if* M *and* N *are* h*-oriented and* f *is any continuous map, then* f *has a well defined* h^**-orientation.*

Proof. Let j be a differentiable imbedding of $N \times \mathbb{D}$ in a Euclidean space $\mathbb{R}^r$ of dimension $r > 2(n+d)+3$ and let $\pi : N \times \mathbb{D} \to N$ denote the projection map. We shall write a number of bundle equivalences; note that in each case they are uniquely defined.

$$\begin{aligned}\tau(N \times \mathbb{D}) &\simeq \pi^*(\tau(N)) \oplus d. \\ \widetilde{f}^*(N \times \mathbb{D}) &\simeq \widetilde{f}^*\pi^*(\tau(N)) \oplus \widetilde{f}^*(d) \simeq f^*(\tau(N)) \oplus d.\end{aligned}$$

$$\begin{aligned}\nu(M) &= \nu(j \circ \widetilde{f}) \simeq \nu(\widetilde{f}) \oplus \widetilde{f}^*\nu(j)\\ \nu(M) \oplus f^*\tau(N) \oplus d &\simeq \nu(\widetilde{f}) \oplus \widetilde{f}^*\nu(j) \oplus \widetilde{f}^*(\tau(N \times \mathbb{D}))\\ &\simeq \nu(\widetilde{f}) \oplus \widetilde{f}^*(r)\\ &\simeq \nu(\widetilde{f}) \oplus r.\end{aligned}$$

The first conclusion follows by suspension and desuspension. The second conclusion is then obvious by Proposition 1.44. □

Definition 1.47. Suppose that $f : M^m \to N^n$ is h^*-oriented by $\widetilde{f} : M^m \to N^n \times \mathbb{D}^d$ and $u \in h^{n+d-m}(M^{\nu(\widetilde{f})}, *)$. The collapsing map ("Thom construction")

$$N^d = (N \times \mathbb{D}^d)/(N \times \mathbb{S}^{d-1}) \xrightarrow{\widehat{f}} \mathbb{D}(\nu(\widetilde{f}))/\mathbb{S}(\nu(\widetilde{f})) = M^{\nu(\widetilde{f})}$$

can be used to define a homomorphism

$$\begin{tikzcd} h^i(M) \arrow[r, "\varphi", "\cong"'] & h^{i+n+d-m}(M^{\nu(\widetilde{f})}, *) \arrow[d, "\widehat{f}^*"] & \\ & h^{i+n+d-m}(N^d, *) \arrow[r, "(\Sigma^d)^{*-1}"] & h^{i+n-m}(N); \end{tikzcd}$$

this homomorphism is denoted

$$f_! : h^i(M) \to h^{i+n-m}(N)$$

and is determined by f and the h^*-orientation of f.

Theorem 1.48. *If $f : M \to N$ is an h^*-oriented map, then for classes $x \in h^*(M)$ and $y \in h^*(N)$,*

$$f_!(f^*(y) \cup x) = y \cup f_!(x).$$

Proof. We first prove the identity

$$\widehat{f}^*(\pi_M^* f^*(y) \cup \varphi_M(x)) = \pi_N^*(y) \cup \widehat{f}^*\varphi_M(x).$$

Consider the diagramme:

$$
\begin{tikzcd}
h^*(N)\otimes h^*(\mathbb{D}\nu(\widetilde{f}),\mathbb{S}(\nu\widetilde{f})) \arrow[d, "f^*\otimes 1"'] \arrow[dr, "\pi_M^*\otimes 1"] & & \\
h^*(M)\otimes h^*(\mathbb{D}\nu,\mathbb{S}\nu) \arrow[d, "\pi_M^*\otimes 1"'] & h^*(N\times\mathbb{D})\otimes h^*(\mathbb{D}\nu,\mathbb{S}\nu) \arrow[dl, "i^*"'] \arrow[d, "\cong"] \arrow[ddr, bend left, "1\otimes\widehat{f}^*"] & \\
h^*(\mathbb{D}\nu,\mathbb{S}\nu) \arrow[d, "\subset^*"'] & h^*(N\times\mathbb{D}\otimes h^*(N\times\mathbb{D},\overline{N\times\mathbb{D}\setminus\mathbb{D}\nu}) \arrow[l] \arrow[d, "\subset *"] \arrow[dr, "i^*"] & \\
h^*(\mathbb{D}\nu,\mathbb{S}\nu) \arrow[dr, "\widehat{f}^*"'] & h^*(N\times\mathbb{D},\overline{N\times\mathbb{D}\setminus\mathbb{D}\nu}) \arrow[l] \arrow[d, "i^*"] & h^*(N\times\mathbb{D})\otimes h^*(N\times\mathbb{D},N\times\mathbb{S}) \arrow[dl, "\subset^*"] \\
& h^*(N\times\mathbb{D},N\times S) &
\end{tikzcd}
$$

The upper quadrilateral commutes because $f=\pi_N\widetilde{f}$; the remainder commutes because all maps are excisions, inclusions, or products. Thus, the diagramme commutes. An element $y\otimes\varphi_M x$ at the top is mapped into $\widehat{f}^*(\pi_M^* f^* y\cup\varphi_x)$ following the arrows on the left and into $\pi_N^* y\cup\widehat{f}^*\varphi_N x$ following the arrows on the right: thus these elements are equal as we asserted. Since

$$
\begin{aligned}
\widehat{f}^*(\pi_M^* f^* y\cup\varphi_M x) &= \widehat{f}^*(\pi_M^* f^* y\cup\pi_M^* x\cup\varphi_M(1)) \\
&= \widehat{f}^*(\pi_M^*(f^* y\cup x)\cup\varphi_M(1)) \\
&= \widehat{f}^*\varphi_M(f^* y\cup x),
\end{aligned}
$$

we have

$$\widehat{f}^*\varphi_M(f^* y\cup x)=\pi_N^* y\cup\widehat{f}^*\varphi_M(x).$$

Thus,

$$
\begin{aligned}
f_!(f^* y\cup x) &= \varphi_N^{-1}\widehat{f}^*\varphi_M(f^* y\cup x)=\varphi_N^{-1}(\pi_N^* y\cup\widehat{f}^*\varphi_M(x)) \\
&= \varphi_N^{-1}(\pi_N^* y\cup\pi_N^*(\varphi_N^*\widehat{f}^*\varphi_M(x))\cup\varphi_N(1)) \\
&= y\cup\widehat{f}^*\varphi_M(x) \\
&= y\cup f_!(x).
\end{aligned}
$$

($\varphi_N=\Sigma^d$ since the bundle over N is trivial.) □

Next, we establish a functorial property of the "umkehr homomorphism" $f_!$. Given $K\xrightarrow{f}L\xrightarrow{g}M$, we seek a theorem of the sort $g_!\circ f_!=(g\circ f)_!$. Since these homomorphisms clearly depend on the h^*-orientations used, in order for there to be such equality the three orientations must be chosen in some compatible way. If we assume f and g have specified orientations, we can then orient $g\circ f$ as follows:

by Proposition 1.46, both

$$\nu(K) \oplus f^*\tau(L), \quad \nu(L) \oplus g^*\tau(M)$$

are h^*-oriented; thus, by Proposition 1.35

$$\nu(K) \oplus f^*\tau(L) \oplus f^*(\nu(L) \oplus g^*\tau(M)) \simeq \nu(K) \oplus f^* \circ g^*\tau(M) \oplus (\text{trivial bundle})$$

is h^*-oriented. This determines an orientation of $g \circ f$.

Equivalently, given $\widetilde{f} : K \to L \times \mathbb{D}^d$ and $\widetilde{g} : L \to M \times \mathbb{D}^{d'}$, the composition

$$K \xrightarrow{\widetilde{f}} L \times \mathbb{D}^d \xrightarrow{\widetilde{g} \times 1} M \times \mathbb{D}^{d'} \times \mathbb{D}^d = M \times \mathbb{D}^{d+d'}$$

is $\widetilde{g \circ f}$. Then

$$\nu(\widetilde{g \circ f}) = \nu((\widetilde{g} \times 1) \circ \widetilde{f}) = \nu(f) \oplus \widetilde{f}^*(\nu(\widetilde{g} \times 1)).$$

This defines the same h^*-orientation for $g \circ f$.

Note, in particular, that if K, L, and M are h^*-oriented, and f, g, and $g \circ f$ are given the induced h^*-orientations, then the two h^*-orientations, i.e., the composition orientation and the induced orientation, of $g \circ f$ agree. This is clear from the first definition of the composition orientation.

Theorem 1.49. *If $f : K \to L$ and $g : LtoM$ are h^*-oriented maps, the composition $g \circ f : toM$ has an induced h^*-orientation and with this orientation $g_! \circ f_! = (g \circ f)_!$. Further, if K, L, and M are h^*-oriented and f, g, and $g \circ f$ are oriented by Proposition 1.46, then the orientation of $g \circ f$ induced by those of f and g agrees with that given by Proposition 1.46.*

Proof. By the comments preceding the statement of the theorem we need only prove that $g_! \circ f_! = (g \circ f)_!$. To this end consider the diagramme below.

$$\begin{tikzcd}
h^i(K) \arrow[d, "\cong"'] \arrow[dr, "\cong"] & & & \\
h^{i+\ell-k+d}(K^{\nu(\widetilde{f})}, *) \arrow[r, "\cong"'] \arrow[d, "\widehat{f}^*"'] & h^{i+m-k+d+d'}(K^{\nu(\widetilde{gf})}, *) \arrow[d, "F^*"] \arrow[dr, "\nu(\widehat{g\circ f}{,}*)"] & & \\
h^{i+\ell-k+d}(L^d), *) \arrow[r, "\cong"] & h^{i+m-k+d+d'}(L^{d\oplus\nu(\widetilde{g})}, *) \arrow[r, "G^*"] & h^{i+m-k+d+d'}(M^{d+d'}, *) & \\
h^{i+\ell-k}(L) \arrow[u, "\cong", "(\Sigma^d)"'] \arrow[r, "\cong"'] & h^{i+m-k+d'}(L^{\nu(\widetilde{g})}, *) \arrow[u, "\cong", "(\Sigma^d)"'] \arrow[r, "\widehat{g}^*"'] & h^{i+m-k+d'}(M^{d'}, *) \arrow[u, "(\Sigma^d)"] & h^{i+m-k}(M) \arrow[l, "\cong"] \arrow[ul, "\cong"']
\end{tikzcd}$$

All isomorphisms are given by suspensions or Proposition 1.43. The two corner triangles and the corner square commute by Theorem 1.37. The lower right hand square commutes because $G = \Sigma^d \circ \widehat{g}$. The map F is $\Sigma^{d'} \circ \widehat{f}$; and so, the upper left hand square commutes. The collapsing map $M^{d+d'} \to K^{\nu(\widetilde{g \circ f})}$ factors through $F^{d \oplus \nu(\widehat{g})}$, which implies the commutativity of the central triangle. The left hand column defines $f_!$, the bottom row $g_!$ and the diagonal $(g \circ f)_!$; thus, $g! \circ f_! = (g \circ f)_!$. □

1.4.2 Multiplicative Transformations; Riemann-Roch Theorem

Let h^* be a multiplicative cohomology theory. For $(X, A) \in \mathscr{P}^2$, $h^{**}(X, A)$ is the set of formal Laurent series

$$\sum_{i=-\infty}^{+\infty} \lambda_i, \quad \lambda_i \in h^i(X, A), \quad \lambda_i = 0 \text{ for all } i < \text{ some } q.$$

Define addition and multiplication in $h^{**}(X, A)$ as follows: if $\lambda = \sum \lambda_i$ and $\mu = \sum \mu_i$, then $\lambda + \mu = \sum(\lambda_i + \mu_i)$ and $\lambda \cup \mu$ is the power series product

$$(\lambda \cup \mu)_k = \sum_{i+j=k} \lambda_i \cup \mu_j.$$

$h^{**}(X, A)$ is a (non-commutative) ring under these operations, and a map $f : (X, A) \to (Y, B)$ induces a ring homomorphism $f^{**} : h^{**}(Y, B) \to h^{**}(X, A)$ by taking f^* on each coordinate. Similarly, we define

$$\delta^{**} : h^{**}(A) \to h^{**}(X, A)$$

coordinate-wise.

Suppose h^* and k^* are multiplicative theorems. Then $t : h^{**} \to k^{**}$ is a (normed) multiplicative transformation if

1. t is a natural, additive transformation between the functors h^{**} and k^{**} with respect to maps f^{**},
2. $t(\mu \cup \lambda) = t(\mu) \cup t(\lambda)$, and
3. If $1_n \in h^{**}(\mathbb{S}^0, *)$ and $1_k \in k^{**}(\mathbb{S}^0, *)$ are the units in h^{**} and k^{**}, respectively, and $\alpha = \Sigma^{**} 1_h \in h^{**}(\mathbb{S}^1, *)$ and $\beta = \Sigma^{**} 1_k \in k^{**}(\mathbb{S}^1, *)$, then $t(\alpha) = \beta$.

Proposition 1.50. *For classes $u_1 \in h^{**}(X_1, A_1)$ and $u_2 \in h^{**}(X_2, A_2)$, the product $u_1 \times u_2$ is in $h^{**}(X_1 \times X_2, X_1 \times A_2 \cup A_1 \times X_2)$ and $t(u_1 \times u_2) = tu_1 \times tu_2$.*

Proof. As in §1.3,

$$\begin{aligned} u_1 \times u_2 &= p_1^{**} u_1 \cup p_2^{**} u_2 \quad \text{and} \\ t(u_1 \times u_2) &= t(p_1^{**} u_1 \cup p_2^{**} u_2) = t(p_1^{**} u_1) \cup t(p_2^{**} u_2) \\ &= p_1^{**}(tu_1) \cup p_2^{**}(tu_2) = tu_1 \times tu_2. \end{aligned}$$

□

Proposition 1.51. *If $u_1 \in h^{**}(X, A_1)$ and $u_2 \in h^{**}(X, A_2)$, then $u_1 \cup u_2 \in h^{**}(X, A_1 \cup A_2)$ and $t(u_1 \cup u_2) = tu_1 \cup tu_2$.*

Proof. In the following diagramme,

$$\begin{tikzcd}
h^{**}(X,A_1)\otimes h^{**}(X,A_2) \arrow[r, "\otimes"] \arrow[d, "t\otimes t"'] & h^{**}(X\times X, X\times A_2\cup A_1\times X) \arrow[r, "\Delta^{**}"] \arrow[d, "t"] & h^{**}(X, A_1\cup A_2) \arrow[d, "t"] \\
k^{**}(X,A_1)\otimes k^{**}(X,A_2) \arrow[r, "\otimes"'] & k^{**}(X\times X, X\times A_2\cup A_1\times X) \arrow[r, "\Delta^{**}"'] & k^{**}(X, A_1\cup A_2)
\end{tikzcd}$$

the first square commutes by Proposition 1.50 and the second by naturality. □

Proposition 1.52. *$t \circ \Sigma_h = \Sigma_k \circ k$; i.e., t is stable,*

Proof. By Proposition 1.26, $\Sigma_h(u) = \alpha \otimes u$ and $\Sigma_k(tu) = \beta \otimes tu$. Thus, $t\Sigma_h(u) = t\alpha \otimes tu = \beta \otimes tu = \Sigma_k(tu)$. □

Proposition 1.53. *$t \circ \delta_h^{**} = \delta_k^{**} \circ t$.*

Proof. The homomorphism δ_k^{**} is the composition

$$\begin{tikzcd}
h^{**}(A,\varnothing) \arrow[r, "\Sigma_h", "\cong"'] & h^{**}(\Sigma A^+, +) \arrow[r, "p^{**}"] & h^{**}(CA^+\cup X^+) \arrow[r, "\cong"] \arrow[d, "\cong"'] & h^{**}(X,A) \\
 & & h^{**}(CA\cup X, *) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

All these homomorphisms commute with t.

Suppose α is a vector bundle over X with orientations $u \in h^n(X^\alpha, *)$ and $v \in k^n(X^\alpha, *)$; we can regard u and v as lying in $h^{**}(X^\alpha, *)$ and $k^{**}(X^\alpha, *)$ respectively. Then u and v induce isomorphisms

$$\varphi_h : h^{**}(X) \to h^{**}(X^\alpha, *) \quad \text{and} \quad \varphi_k : k^{**}(X) \to k^{**}(X^\alpha, *)$$

For $x \in h^{**}(X)$, define $t(\alpha, x) \equiv \varphi_k^{-1} t \varphi_h(x) \in k^{**}(X)$. □

Lemma 1.54. *Let α be an h- and k-oriented vector bundle over X. Then for $x \in h^{**}(X)$, $t(x) \cup t(\alpha, 1) = t(\alpha, x)$.*

Proof.

$$\begin{aligned}
\varphi_k(t(x) \cup t(\alpha,1) &= \varphi_k(t(x) \cup \varphi_k^{-1} t\varphi_h(1) \\
&= \pi_k^{**}(t(x) \cup \varphi_k^{-1} t\varphi_h(1)) \cup v \\
&= \pi_k^{**}(t(x)) \cup (\pi_k^{**}\varphi_k^{-1} t\varphi_h(1) \cup v) \\
&= t\pi_k^{**}(x) \cup t\varphi_h(1) \\
&= t(\pi_h^{**}(x) \cup \varphi_h(1)) \\
&= t\varphi_x.
\end{aligned}$$

Thus, $t(x) \cup t(\alpha, 1) = \varphi_k^{-1} t\varphi_h(x) = t(\alpha, x)$. □

Theorem 1.55 (A generalised Riemann-Roch Theorem). *If $f : X \to Y$ is a h^*-oriented map of compact, closed manifolds and $t : h^{**} \to k^{**}$ is a multiplicative transformation of cohomology theories, then for $x \in h^{**}(X)$,*

$$tf_!^h(x) = f_!^k(t(x) \cup t(\nu(\widetilde{f}), 1)).$$

Proof. We note first that an h^*-orientation $u \in h^n(X^\alpha, *)$ of a vector bundle determines a k^*-orientation $(tu)_n \in k^n(X^\alpha, *)$ since

$$i^*u = \alpha^n \in h^n(\mathbb{S}^n, *) \quad \text{and} \quad t\alpha^n = \beta^n \in k^n(\mathbb{S}^n, *);$$

thus, $\beta^n = i^*(tu)_n$ and α is k^*-oriented. Hence we need not assume additionally that $f : X \to Y$ is k^*-orientable; however, in the statement and proof of this theorem the specific k^*-orientation of f is irrelevant.

The right-hand side of the asserted equation is

$$(\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**}\varphi_k(t(x) \cup t(\nu(\widetilde{f}), 1))$$

which equals

$$(\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**}t\varphi_k(x)$$

by the Lemma. By the naturality of t this equals

$$t((\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**}\varphi_h(x)) = tf_!^h(x).$$

□

Theorem 1.56. *Let $f : X \to Y$ be a continuous map of h^*- and k^*-oriented manifolds. Let $t(X) \equiv t(\nu(X), 1)$ and $t(Y) \equiv t(\nu(Y), 1)$. Then for $x \in h^{**}(X)$,*

$$tf_!^h(x) \cup t(Y) = f_!^k(t(x) \cup t(X)).$$

Proof.

$$\begin{aligned}
f_!^k(t(x) \cup t(X)) &= \varphi_k^{-1}\widehat{f}_k^{**}\varphi_k(t(x) \cup t(X)) \\
&= \varphi_k^{-1}\widehat{f}_k^{**}t\varphi_h(x) \\
&= \varphi_k^{-1}t\widehat{f}_h^{**}\varphi_h(x) \\
&= \varphi_k^{-1}[t\varphi_h(\varphi_h^{-1}\widehat{f}_h^{**}\varphi_h(x))] \\
&= \varphi_k^{-1}[\varphi_k(tf_!^h(x) \cup t(Y))] \\
&= tf_!^h(x) \cup t(Y).
\end{aligned}$$

□

1.4.3 Wu Formulae

As applications of the above theorems we present here a proof of the Wu formulae; in a later section we will derive the Atiyah-Hirzebruch differential Riemann-Roch Theorem.

Let $h^{**} = k^{**} = H^{**}(-;\mathbb{Z}/p\mathbb{Z})$ and let $t = \mathscr{P} = \sum \mathscr{P}^i$, the sum of the Steenrod reduced power operations. For $p = 2$, $t = \mathrm{Sq}$. The Cartan formula implies t is a multiplicative transformation. For a vector bundle α over X, $\mathrm{Sq}(\alpha, 1)$ is the total Stiefel-Whitney class $W(\alpha)$ of the bundle α.

Lemma 1.57. *Every vector bundle is $H^*(-;\mathbb{Z}/2\mathbb{Z})$-orientable; a vector bundle α is $H^*(-;\mathbb{Z})$- or $H^*(-;\mathbb{Z}/p\mathbb{Z})$-orientable if and only if the first Stiefel-Whitney class $W_1(\alpha) = 0$.*

Proof. If α is an n-plane bundle, consider the n-sphere bundle $\mathbb{S}(\alpha \oplus 1) \xrightarrow{\pi} X$ and the Gysin sequence with $\mathbb{Z}/2\mathbb{Z}$-coefficients:

$$\cdots \to H^r(X) \xrightarrow{\pi^*} H^r(\mathbb{S}(\alpha \oplus 1)) \xrightarrow{\lambda^*} H^{r-n}(X) \to H^{r+1}(X) \to .$$

$\mathbb{S}(\alpha\oplus 1)$ has a cross-section $i : X \to \mathbb{S}(\alpha\oplus 1)$; and so, the Gysin sequence reduces to

$$0 \to H^r(X) \xrightarrow{\pi^*} H^r(\mathbb{S}(\alpha \oplus 1)) \xrightarrow{\lambda^*} H^{r-n}(X) \to 0.$$

There is a class $v \in H^n(\mathbb{S}(\alpha\oplus 1))$ such that $\lambda^* v = 1 \in H^0(X)$. Let $w = v - \pi * i^* v$. Then $\lambda^* w = 1$ and $i^* w = 0$.

The sequence $X \xrightarrow{i} \mathbb{S}(\alpha \oplus 1) \xrightarrow{c} X$ is coexact and so

$$0 \longrightarrow H^*(X^\alpha, *) \xrightarrow{c^*} H^*(\mathbb{S}(\alpha \oplus 1)) \underset{\pi^*}{\overset{i*}{\rightleftarrows}} H^*(X) \longrightarrow 0.$$

is exact. There is a unique $u_n \in H^n(X^\alpha, *)$ such that $c^* u_n = w$.

Since the diagramme

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[d] & \mathbb{S}(\alpha \oplus 1)) \arrow[d] \\ \mathrm{pt} \arrow[r] & X \end{tikzcd}$$

is commutative, so also is

$$\begin{tikzcd} H^n(X^\alpha, *) \arrow[d] & \\ H^n(\mathbb{S}(\alpha \oplus 1)) \arrow[r, "\lambda*"] \arrow[d] & H^0(X) \arrow[d] \\ H^n(\mathbb{S}^n) \arrow[r] & H^0(X), \end{tikzcd}$$

and it follows that u_n restricts to the generator of $H^n(\mathbb{S}^n, *)$.

To prove the second part of the lemma, note that $w_1(\alpha) = \varphi^{-1}\beta u_n$, where β is the Bockstein homomorphism associated with

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Thus,

$$\begin{aligned} w_1(\alpha) = 0 &\Leftrightarrow u_n \text{ is the reduction of a mod 4 class} \\ &\Leftrightarrow \alpha \text{ is } H^*(-;\mathbb{Z}/4\mathbb{Z}) - \text{orientable.} \end{aligned}$$

This is equivalent to $\pi_1(X.x_0)$ operating trivially on $H^*(\mathbb{S}^n;\mathbb{Z})$ which then by the Gysin sequence implies the $H^*(-;\mathbb{Z})$- and $H^*(-;\mathbb{Z}/p\mathbb{Z})$-orientability of α. The converse is clear. □

The Wu formula asserts there is a class $V \in H^{**}(X)$ such that $\mathrm{Sq}\, V = W(X)(= \mathrm{Sq}(\tau(X),1))$ and $[\alpha \cup V](X) = [\mathrm{Sq}\,\alpha](X)$ for all $\alpha \in H^*(X)$. Let $\mathrm{Sq}^{-1} = \frac{1}{1+(\mathrm{Sq}^{-1})}$ as formal power series; then $\mathrm{Sq}\,\mathrm{Sq}^{-1} = \mathrm{Sq}^{-1}\,\mathrm{Sq} = 1$ and Sq is multiplicative, Let $f : X \to \mathrm{pt}$. By Theorem 1.56,

$$f_! : H^*(\mathrm{Sq}^{-1}(x) \cup \mathrm{Sq}^{-1}(\nu(X),1) = \mathrm{Sq}^-1(f_!(x))$$

where $f_! : H^*(X) \to H^*(\mathrm{pt})$ is given by $f_!(x) = [x](X)$, the value of the cohomology class x on the top homology class of the manifold X (with $\mathbb{Z}/2\mathbb{Z}$ coefficients). For $x = \mathrm{Sq}\,\alpha$, we then have

$$[\alpha \cup \mathrm{Sq}^{-1}(\nu(X),1)](X) = [\mathrm{Sq}\,\alpha](X).$$

We claim that $\mathrm{Sq}(\mathrm{Sq}^{-1}(\nu(X),1)) = \mathrm{Sq}(\tau(X),1) = W(X)$. Then $V = \mathrm{Sq}^1(\nu(X),1)$ is the Wu class and satisfies the formulae above.

Since $\mathrm{Sq}(\tau,1) \cup \mathrm{Sq}(\nu,\nu) = \mathrm{Sq}(\tau \oplus \nu, 1) = 1$ [Appendix B], to verify the claim we need only prove that

$$\mathrm{Sq}(\mathrm{Sq}^{-1}(\nu(X),1)) \cup \mathrm{Sq}(\nu,1) = 1.$$

To this end we compute

$$\begin{aligned} \varphi(\mathrm{Sq}(\mathrm{Sq}^{-1}(\nu(X),1)) \cup \mathrm{Sq}(\nu,1)) &= \pi^*\,\mathrm{Sq}\,\varphi^{-1}\,\mathrm{Sq}^{-1}\,\varphi(1) \cup \pi^*\varphi^{-1}\,\mathrm{Sq}\,\varphi(1) \cup (1) \\ &= \pi^*\,\mathrm{Sq}\,\varphi^{-1}\,\mathrm{Sq}^{-1}\,\varphi(1) \cup \mathrm{Sq}\,\varphi(1) \\ &= \mathrm{Sq}(\varphi\varphi^{-1}\,\mathrm{Sq}^{-1}\,\varphi(1)) \\ &= \varphi(1). \end{aligned}$$

But φ is an isomorphism.

Chapter 2

Complex Vector Bundles and the Bott Periodicity Theorem

2.1 Bott Periodicity Theorem

2.1.1 Homology of the Unitary Groups

For the complex n-dimensional space $\mathbb{C}^n$ equipped with a positive definite Hermitian inner product, the unitary group $\mathrm{U}(n)$ is the group of linear transformations preserving this inner product. Consider $\mathbb{C}^{n-1}$ as a linear subspace of $\mathbb{C}^n$ and choose a vector $e_1 \in \mathbb{C}^n$ such that $(e_1, e_1) = 1$ and $e_1 \perp \mathbb{C}^{n-1}$ ($(e_1, x) = 0$ for all $x \in \mathbb{C}^{n-1}$). This defines an inclusion $\mathrm{U}(n-1) \to \mathrm{U}(n)$ whereby the elements of $\mathrm{U}(n-1)$ are those transformations of $\mathrm{U}(n)$ which leave e_1 fixed.

The group $\mathrm{U}(n)$ acts effectively on the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{C}^n$ and $\mathrm{U}(n-1)$ is the subgroup of $\mathrm{U}(n)$ whose elements leave $e_1 \in \mathbb{S}^{2n-1}$ fixed. Thus, $\mathrm{U}(n)/\mathrm{U}(n-1)$ is homeomorphic to $\mathbb{S}^{2n-1}$ and the fibration $\mathrm{U}(n-1) \xrightarrow{i} \mathrm{U}(n) \xrightarrow{\pi} \mathbb{S}^{2n-1}$ is a principal $\mathrm{U}(n-1)$-fibration.

The boundary homomorphism $\partial : \pi_{2n-1}(\mathbb{S}^{2n-1}) \to \pi_{2n-2}(\mathrm{U}(n-1))$ of the homotopy sequence of this fibration defines an element $\partial(\iota_{2n-1}) \in \pi_{2n-2}(\mathrm{U}(n-1))$, where ι_{2n-1} is the canonical generator of $\pi_{2n-1}(\mathbb{S}^{2n-1})$, and its Hurewicz image $\nu \in H_{2n-2}(\mathrm{U}(n-1))$, coefficients being $\mathbb{Z}$ throughout this section.

The homology of an H-space has a product structure (the "Pontrjagin product") defined as follows:

if $\mu : X \times X \to X$ is the multiplication of the H-space X, then the composition

$$H_*(X) \otimes H_*(X) \to H_*(X \times X) \xrightarrow{\mu_*} H_*(X)$$

defines a bilinear product with identity which is associative if μ is homotopy associative and is (graded) commutative if μ is homotopy commutative.

Theorem 2.1. *If $G \xrightarrow{i} E \xrightarrow{\pi} \mathbb{S}^k$ is a principal right G-bundle (G is a connected topological group acting freely on E on the right and $\mathbb{S}^k$ is the orbit space) and*

$\nu \in H_{k-1}(G)$ is the image of $\iota_k \in \pi_k(\mathbb{S}^k)$ under the composition $\pi_k(\mathbb{S}^k) \xrightarrow{\partial} \pi_{k-1}(G) \xrightarrow{h} H_{k-1}(G)$, h being the Hurewicz map, then the (Wang) sequence

$$\cdots \to H_i(G) \xrightarrow{i_*} H_i(E) \xrightarrow{\lambda_*} H_{i-k}(G) \xrightarrow{\nu^*} H_{i-1}(G) \to \cdots$$

is exact, where ν^* is Pontrjagin product on the left with ν. If $a \in H_k(E)$ is such that $\pi_*(a)$ generates $H_k(\mathbb{S}^k)$, then $\lambda_*(a)$ is a unit of $H_0(G)$.

Proof. The sphere $\mathbb{S}^k$ is the union of two hemispheres, $\mathbb{S}^k = \mathbb{D}^k_+ \cup \mathbb{D}^k_-$, with $\mathbb{D}^k_+ \cap \mathbb{D}^k_- = \mathbb{S}^{k-1}$. The bundle $\pi|_{\mathbb{D}^k_-}$ is trivial since $\mathbb{D}^k_-$ is contractible; thus, $\pi|_{\mathbb{D}^k_-} \simeq \mathbb{D}^k_- \times G$. For each $p \in \mathbb{S}^{k-1}$ there is a translation $f_p : G \to G$, $f_p(g) = f_p(e) \cdot g$, such that the map $\alpha : (\mathbb{D}^k_+, \mathbb{S}^{k-1}) \times G \to (E, G)$ defined by $\mathbb{D}^k_+ \times G \to \pi^{-1}(\mathbb{D}^k_+)$ and on $\mathbb{S}^{k-1} \times G \to G$ by $(p, g) \mapsto f_p(g)$ is a relative homeomorphism (we have identified $\mathbb{D}^k_+$ to a point and $\pi^{-1}(\mathbb{D}^k_+)$ to G). The map $\mathbb{S}^{k-1} \to G$ defined by $p \mapsto f_p(e)$ is homotopic to the image of i_k in $\pi_{k-1}(G)$.

Consider the diagramme below.

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(G) \arrow[r] & H_i(E) \arrow[r] & H_i(E,G) \arrow[r] & H_{i-1}(G) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_i(\mathbb{S}^{k-1} \times G) \arrow[r] \arrow[u] & H_i(\mathbb{D}^k_+ \times G) \arrow[r] \arrow[u] & H_i((\mathbb{D}^k_+, \mathbb{S}^{k-1}) \times G) \arrow[r] \arrow[u, "\alpha_*", "\cong"'] & H_{i-1}(\mathbb{S}^{k-1} \times G) \arrow[r] \arrow[u, "\beta_*"'] & \cdots
\end{tikzcd}$$

Let ν be the generator of $H_k(\mathbb{D}^k_+, \mathbb{S}^{k-1})$ corresponding to $i_k \in \pi_k(\mathbb{S}^k)$ and $w \in H_k(E, G)$ be $\alpha(\nu \otimes 1)$. Then $\alpha_*(\nu \otimes x) = w * x$ and

$$\partial_*(w * x) = \partial_* \alpha_*(\nu \otimes x) = \beta_* \partial_*(\nu \otimes x) = \beta_*((\partial * \nu) \otimes x) = \nu * x,$$

since $\nu = \partial_* \nu$.

There is an isomorphism $\varphi : H_{i-k}(G) \to H_i((\mathbb{D}^k_+, \mathbb{S}^{k-1}) \times G)$ given by $\varphi(x) = \nu \otimes x$ which yields the Wang sequence

$$\cdots \to H_i(G) \xrightarrow{i_*} H_i(E) \xrightarrow{\lambda_*} H_{i-k}(G) \xrightarrow{\nu^*} H_{i-1}(G) \to \cdots$$

To determine λ_* consider the diagramme

$$\begin{tikzcd}
H_i(E) \arrow[r, "j_*"] \arrow[d, "\pi_*"'] & H_i(E,G) \arrow[dl, "\pi_*"'] & \\
H_i(\mathbb{S}^k, *) \arrow[r, leftrightarrow, "\Phi"'] & H_i((\mathbb{D}^k_+, \mathbb{S}^{k-1}) \times G) \arrow[u, "\alpha_*", "\cong"'] & H_{i-k}(G) \arrow[l, "\varphi"', "\cong"]
\end{tikzcd}$$

$\lambda_* = \varphi^{-1} \circ \alpha_*^{-1} \circ j_*$ If $a \in H_k(E)$ is such that $\pi_*(a)$ generates $H_k(\mathbb{S}^k, *)$, then $\alpha_*^{-1} j_*(a) = \pm \nu \otimes 1$, which is the image of ± 1 under φ. Thus, $\lambda_*(a) = \pm 1$. ◻

Theorem 2.2. *As Hopf algebras*

$$H_*(\mathrm{U}(n); \mathbb{Z}) \cong E(\nu_1, \ \ldots, \ \nu_n),$$

the primitively generated exterior Hopf algebra on generators ν_i of dimension $2i - 1$.

Proof. The theorem is true for $n = 1$ since $\mathrm{U}(1)$ is topologically a circle. The proof is by induction on n; and so, we assume the conclusion for $\mathrm{U}(n-1)$.

Consider the principal $\mathrm{U}(n-1)$-fibration

$$\mathrm{U}(n-1) \xrightarrow{i} \mathrm{U}(n) \xrightarrow{\pi} \mathbb{S}^{2n-1}$$

and its Wang sequence. The element ν is of dimension $2n-2$; it is spherical and hence primitive. By the inductive hypothesis it is thus 0. Thus, the Wang sequence breaks into short exact sequences

$$0 \to H_j(\mathrm{U}(n-1)) \xrightarrow{i*} H_j(\mathrm{U}(n)) \xrightarrow{\lambda_*} H_{j-(2n-1)}(\mathrm{U}(n-1)) \to 0.$$

It follows immediately that $H_*(\mathrm{U}(n))$ is torsion-free and is additively isomorphic to $H_*(\mathrm{U}(n-1)) \otimes H_*(\mathbb{S}^{2n-1})$. Let $\nu_n \in H_{2n-1}(\mathrm{U}(n))$ be in the preimage of $1 \in H_0(\mathrm{U}(n-1))$ under λ_*. Then no multiple of ν_n is decomposable, for if it were, it would be in $\mathrm{im}(i_*)$ and 1 would be of finite order in $H_0(\mathrm{U}(n-1))$.

Since $H^*(\mathrm{U}(n);\mathbb{Q})$ is a connected, associative, commutative, finite dimensional Hopf algebra, it is an exterior algebra and thus has a commutative coproduct; i.e., $H^*(\mathrm{U}(n);\mathbb{Q})$ has a commutative product. Since $H^*(\mathrm{U}(n);\mathbb{Z})$ is torsion-free, $H^*(\mathrm{U}(n);\mathbb{Z}) \to H^*(\mathrm{U}(n);\mathbb{Q})$ is a monomorphism. Thus $H^*(\mathrm{U}(n);\mathbb{Z})$ has a commutative product.

To see that $\nu_1, \ldots, \nu_n$ generate $H_*(\mathrm{U}(n))$ as an algebra, let $x \in H_i(\mathrm{U}(n))$ and $y = \lambda_*(x)$. Also, $\lambda_*(\nu_n \cdot y) = 1 \cdot y = y$. Thus, if $x = x - \nu_n \cdot y$, $\lambda_*(z) = 0$; and so, $z \in H_i(\mathrm{U}(n-1)$. Since both z and y are in $E(\nu_1, \ldots, \nu_{n-1})$, they are polynomials in the variables $\nu_1, \ldots, \nu_{n-1}$, and $x = z + \nu_n \cdot y$ is a polynomial in $\nu_1, \ldots, \nu_n$.

We need only show that a ν_n can be chosen which is primitive. We will then have demonstrated that $H_*(\mathrm{U}(n);\mathbb{Z})$ is a connected, associative, commutative, coassociative, cocomutative, primitively generated, finite dimensional Hopf algebra. The conclusion follows.

To do this we make a geometric construction which will have later applications. Assume the ν_i, $1 \le i < n$ have been chosen primitive (in $H_*(\mathrm{U}(n-1);\mathbb{Z})$). Define $f_n : \sum P_{n-1}(\mathbb{C}) \to \mathrm{SU}(n)$ as follows:

for $0 \neq x \in \mathbb{C}^n$, let (x) denote the equivalence class of x in $P^{n-1}(\mathbb{C})$; for $(\lambda, (x)) \in \mathbb{S}^1 \times P^{n-1}(\mathbb{C})$, a unitary transformation $A \in \mathrm{U}(n)$ is uniquely determined by the conditions

$$Ay = \begin{cases} y & \text{if} \quad y \perp x \\ \lambda y & \text{if} \quad y = \mu x \text{ for some } \mu \in \mathbb{C}; \end{cases}$$

A is unitary since $|\lambda| = 1$. Map $\mathrm{U}(n) \to \mathrm{SU}(n)$ by taking a transformation A into $A \cdot \mathrm{M}(I_{n-1} \oplus \sigma)$, where $\sigma^{-1} = \det(A)$ and I_{n-1} is the $(n-1) \times (n-1)$ unit matrix. The composition $\mathbb{S}^1 \times P^{n-1}(\mathbb{C}) \to \mathrm{U}(n) \to \mathrm{SU}(n)$ carries the wedge $\mathbb{S}^1 \times (e_n) \cup (1) \times P^{n-1}(\mathbb{C})$ into I_n; and so, $f_n : \mathbb{S}^1 \# P^{n-1}(\mathbb{C}) \to \mathrm{SU}(n)$ is defined.

It is a computation to verify that

a)

$$\begin{tikzcd} \mathbb{S}^1 \# P^{k-1}(\mathbb{C}) \arrow[r, "f_k"] \arrow[d] & \mathrm{SU}(k) \arrow[d] \\ \mathbb{S}^1 \# P^{n-1}(\mathbb{C}) \arrow[r, "f_n"'] & \mathrm{SU}(n) \end{tikzcd}$$

commutes for $k < n$, and

b) in the commutative diagramme

$$\begin{tikzcd} \mathbb{S}^1 \# P^{n-1}(\mathbb{C}) \arrow[r] & \mathrm{SU}(n), \mathrm{SU}(n-1) \arrow[r] \arrow[d] & (\mathbb{S}^{2n-1}, *) \\ & (\mathrm{U}(n), \mathrm{U}(n-1)) \arrow[ur] & \end{tikzcd}$$

the composition of the top is a relative homeomorphism.

Let ν_n be the image in $H_{2n-1}(\mathrm{U}(n); \mathbb{Z})$ of the top class of $H_*(\sum^{n-1}(\mathbb{C}); \mathbb{Z})$. Then $\pi_*(\nu_n)$ is a generator of $H_{2n-1}(\mathbb{S}^{2n-1}, *)$, and so, $\lambda_*(\nu_n) = \pm 1 \in H_0(\mathrm{U}(n-1))$. Also, ν_n is primitive since every homology class in a suspension is.

We note this also proves

c)

$$f_{n*} : H_*(\mathbb{S}^1 \# P^{n-1}(\mathbb{C})) \to H_*(\mathrm{SU}(n))$$

is a monomorphism, and $\mathrm{im}(f_{n*})$ generates $H_*(\mathrm{SU}(n))$ as an algebra. □

2.1.2 The Universal Base Spaces $B_{\mathrm{U}(n)}$

Let E be an acyclic $\mathrm{U}(n)$-free space; e.g., $E = \varinjlim_m \mathrm{U}(n+m)/\mathrm{U}(m)$, but the specific method of construction of E is unimportant. Collapsing E under the action of $\mathrm{U}(n)$ produces a principal $\mathrm{U}(n)$-bundle

$$\mathrm{U}(n) \to E \to B_{\mathrm{U}(n)}.$$

Since E is acyclic, there is a homotopy equivalence $\Omega B_{\mathrm{U}(n)} \simeq \mathrm{U}(n)$ preserving the H-space structures.

For an inclusion $\mathbb{C}^{n-1} \to \mathbb{C}^n$ with e_1 an orthogonal unit vector, we have the monomorphism $j_1 : \mathrm{U}(n-1l) \to \mathrm{U}(n)$. There is also the monomorphism $\mathrm{U}(1) \to \mathrm{U}(n)$ obtained by multiplying $e_1 \in \mathbb{C}^n$ by λ, $|\lambda| = 1$. The product $\mathrm{U}(1) \times \mathrm{U}(n-1) \to \mathrm{U}(n)$ is also a monomorphism. These induce maps

$$B_{\mathrm{U}(n-1)} = E/j_1(\mathrm{U}(n-1)) \to E/\mathrm{U}(n) = B_{\mathrm{U}(n)}, \quad B_{\mathrm{U}(1)} \times B_{\mathrm{U}(n-1)} \to B_{\mathrm{U}(n)}.$$

Similarly, for an inclusion $\mathbb{C}^{n-2} \to \mathbb{C}^n$ with unit orthogonal vector e_2, we have

$$B_{\mathrm{U}(1)} \times B_{\mathrm{U}(n-2)} \to B_{\mathrm{U}(n-1)}.$$

A sequence of such inclusions defines monomorphisms and maps:

$$\mathrm{U}(1)^k \times \mathrm{U}(n-k) \to \mathrm{U}(n) \quad B^k_{\mathrm{U}(1)} \times B_{\mathrm{U}(n-k)} \to B_{\mathrm{U}(n)}.$$

The bundle $E \times_{\mathrm{U}(n))} \mathbb{C}^n \to B_{\mathrm{U}(n)}$ is the *universal n-plane bundle*, where for $A \in \mathrm{U}(n)$ pairs (e, Ax) and (eA, x) are identified; and for $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ the unit $(2n-1)$-sphere,

$$E \times_{\mathrm{U}(n)} \mathbb{S}^{2n-1} \to B_{\mathrm{U}(n)}$$

is the *associated sphere bundle*. The map

$$\widetilde{\lambda} : E \to E \times_{\mathrm{U}(n))} \mathbb{S}^{2n-1}$$

defined by $\widetilde{\lambda}(e) = ((e, e_1))$ is onto, and $\widetilde{\lambda}(e) = \widetilde{\lambda}(f)$ if and only if $(e, e_1) \sim (f, e_1)$; i.e., if and only if there exists $B \in \mathrm{U}(n-1)$ such that $e(j, B) =$, since e_1 is left fixed. Thus, $\widetilde{\lambda}$ factors through $B_{\mathrm{U}(n-1)}$:

$$\begin{tikzcd} & E/\mathrm{U}(n-1) \arrow[dr, "\lambda"] & \\ E \arrow[ur] \arrow[rr, "\widetilde{\lambda}"'] & & E \times_{\mathrm{U}(n))} \mathbb{S}^{2n-1}. \end{tikzcd}$$

The map λ is a homeomorphism. Thus, $B_{\mathrm{U}(n-1)}$ is the total space of the associated $(2n-1)$-sphere bundle over $B_{\mathrm{U}(n)}$; we have the fibration

$$\mathbb{S}^{2n-1} \to B_{\mathrm{U}(n-1)} \to B_{\mathrm{U}(n)}.$$

With the loop space identifications $\Omega B_{\mathrm{U}(k)} \simeq \mathrm{U}(k)$, we recover the fibration

$$\mathrm{U}(n-1) \to \mathrm{U}(n) \to \mathbb{S}^{2n-1}.$$

used earlier.

Theorem 2.3. *The integral cohomology ring $H^*(B_{\mathrm{U}(n)}; \mathbb{Z})$ is a polynomial ring $P(C_1, \ldots, C_n)$ on generators C_i of dimension $2i$; the classes C_i can be chosen for all n so that $\pi^*(C_i) = C_i, \leq n-1$, and C_n is the Gysin class of the fibration $\mathbb{S}^{2n-1} \to B_{\mathrm{U}(n-1)} \xrightarrow{\pi} B_{\mathrm{U}(n)}$.*

Proof. The argument is by induction on n. Since $B_{\mathrm{U}(1)} = P^\infty(\mathbb{C}$, the conclusion is true for $n = 1$. Suppose the resut is correct for $B_{\mathrm{U}(n-1)}$. In the homotopy sequence of the fibration π, $\pi_i(\pi)$ is an isomorphism for $i \leq 2n-2$ and is onto for $i = 2n-1$; thus, the same is true in homology by the Whitehead theorem. Since $H_{2n-1}(B_{\mathrm{U}(n-1)}) = 0$ by the inductive hypothesis, $H_i(\pi)$ is an isomorphism for $\leq 2n-1$; and thus, $H^i(\pi)$ is an isomorphism for $i \leq 2n-1$.

The Gysin sequence

$$\cdots \to H^i(B_{\mathrm{U}(n)}) \xrightarrow{\pi^*} H^i(B_{\mathrm{U}(n-1)}) \to H^{i-(2n-1)}(B_{\mathrm{U}(n)}) \xrightarrow{\cup C_n} H^{i+1}(B_{\mathrm{U}(n)}) \to \cdots$$

implies that

(1) $\ker(\pi^*)$ is the ideal generated by C_n, and

(2) for all coefficient rings, $H^{\mathrm{odd}}(B_{\mathrm{U}(n)}) = 0$, and consequently, $H^{\mathrm{even}}(B_{\mathrm{U}(n)})$ is torsion-free integrally.

Let $C_1, \dots, C_{n-1}$ be the classes in $H^*(B_{\mathrm{U}(n)})$ such that $\pi^* C_i = C_i$ in $H^*(B_{\mathrm{U}(n-1)})$. We claim that $H^*(B_{\mathrm{U}(n)})$ is a polynomial ring in $(C_1, \dots, C_n)$. To see that these classes generate $H^*(B_{\mathrm{U}(n)})$ let $y \in H^*(B_{\mathrm{U}(n)})$. Then $\pi^* y$ is a polynomial $w(C_1, \dots, C_n)$. This same polynomial w is in $H^*(B_{\mathrm{U}(n)})$ and $\pi^* w = w$. Thus, $\pi^*(y - w) = 0$ and $y - w$ is in the ideal generated by C_n. The conclusion follows.

Next, suppose there is a polynomial relation $\sum_{i=0}^{k} C_n^i w_i = 0$, where the w_i are polynomials in the $C_1, \dots, C_{n-1}$. $0 = \pi^*\left(\sum_{i=0}^{k} C_n^i w_i\right) = w_0$. Thus, $C_n \cdot \sum_{i=1}^{k} C_n^{i-1} w_i = 0$; but this implies $\sum_{i=1}^{k} C_n^{i-1} w_i = 0$ since π^* is onto, which in turn implies $w_1 = 0$, etc. Finally, there is no relation among the $C_1, \dots, C_{n-1}$ since by assumption there is none among $\pi^* C_1, \dots, \pi^* C_{n-1}$. □

The decomposition $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$ defines a monomorphism $\mathrm{U}(m) \times \mathrm{U}(n) \xrightarrow{j} \mathrm{U}(m+n)$ and a map

$$B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} = E/j(\mathrm{U}(m) \times \mathrm{U}(n)) \xrightarrow{j} E/\mathrm{U}(m+n) = B_{\mathrm{U}(m+n)}.$$

For the bundle $\gamma^{m+n} = E \times_{\mathrm{U}(m+n)} \mathbb{C}^{m+n}$,

$$j^* \gamma^{m+n} = \gamma^m \times \gamma^n \quad \text{over} \quad B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)}.$$

Lemma 2.4. *If $j : B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \to B_{\mathrm{U}(m+n)}$ is the map induced by the decomposition $\mathbb{C}^m \oplus \mathbb{C}^n = \mathbb{C}^{m+n}$, then*

$$j^*(1 + C_1 + \cdots + C_{m+n}) = (1 + C_1, + \cdots + C_m) \times (1 + C_1 + \cdots + C_n).$$

Proof. The Gysin class of γ^{m+n} is C_{m+n} and $j^* C_{m+n} = C_m \times C_n$ by Theorem 1.37.

The proof of the lemma is inductive. $j^*(1 + C_1 + \cdots + C_{m+n})$ is a Polynomial $P_{m,n}(1, C_1, \dots, C_m; 1, C_1, \dots, C_n)$. By the inductive hypothesis and the commutative diagramme

$$\begin{tikzcd}
B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \arrow[r] & B_{\mathrm{U}(m+n)} \\
B_{\mathrm{U}(m-1)} \times B_{\mathrm{U}(n)} \arrow[r] \arrow[u] & B_{\mathrm{U}(m-1+n)} \arrow[u]
\end{tikzcd}$$

we have that

$$P_{m,n} \equiv (1 + C_1 + \cdots + C_m) \times (1 + C_1 + \cdots + C_n) \bmod C_m.$$

Similarly

$$P_{m,n} \equiv (1 + C_1 + \cdots + C_m) \times (1 + C_1 + \cdots + C_n) \bmod C_n.$$

Thus,

$$P_{m,n} \equiv (1 + C_1 + \cdots + C_m) \times (1 + C_1 + \cdots + C_n) \bmod C_m \times C_n.$$

and

$$P_{m,n} = (1 + C_1 + \cdots + C_m) \times (1 + C_1 + \cdots + C_n) + C_m \times C_n.$$

Since $j^* C_{m+n} = C_m \times C_n$, $k = 0$. □

Theorem 2.5. *For $j : B^n_{\mathrm{U}(1))} \to B_{\mathrm{U}(n)}$, $j^*(C_i) = \sigma_i^{(n)}(u_1, \ldots, u_n)$, the i-th elementary symmetric function in $u_1, \ldots, u_n$, where u_j is the generator of $H^2(B_{\mathrm{U}(1)}; \mathbb{Z})$ in the j-th factor.*

Proof. Again the argument is inductive. For the composition

$$B^{n-1}_{\mathrm{U}(1)} \times B_{\mathrm{U}(1)} \xrightarrow{k} B_{\mathrm{U}(n-1)} \times B_{\mathrm{U}(1)} \xrightarrow{\ell} B_{\mathrm{U}(n)},$$

we have $j =$
$ell \circ k$ and

$$\begin{aligned} k^* \circ \mathrm{el}^*(1 + C_1 + \cdots + C_n) &= k^*((1 + C_1 + \cdots + C_{n-1}) \times (1 + u_n)) \\ &= (1 + \sigma^{(n-1)} + \cdots + \sigma^{(n-1)}_{(n-1}) \times (1 + u_n) \\ &= 1 + \sigma_1^{(n))} + \cdots + \sigma_n^{(n)}. \end{aligned}$$

□

2.1.3 Bott Periodicity Theorem for B_{U}

In the proof of Theorem 2.2 we defined maps

$$f_k : \mathbb{S}^1 \# P^{k-1}(\mathbb{C}) \to \mathrm{SU}(k).$$

Passing to the limit using the weak topology, we thus determine a map

$$f : \mathbb{S}^1 \# P^\infty(\mathbb{C}) \to \mathrm{SU},$$

which induces a monomorphism in homology such that $\mathrm{im}(f_*)$ generates $H_*(\mathrm{SU}; \mathbb{Z})$ as an algebra.

The adjoint of f is the composition

$$P^\infty(\mathbb{C}) \xrightarrow{i} \Omega\Sigma P^\infty(\mathbb{C}) \xrightarrow{\Omega f} \Omega\,\mathrm{SU}\,.$$

The map f induces a map of the path-loop fibration of $\Sigma P^\infty(\mathbb{C})$ into that of SU. Let λ_i denote an additive generator of $H_{2i}(P^\infty(\mathbb{C}); \mathbb{Z})$ and σ_* homology suspension in the homology spectral sequences of the fibrations. Then $\Sigma_* \lambda_* = \pm\sigma_* i_*(\lambda_i)$, where $\Sigma_* : H_*(P^\infty(\mathbb{C})) \xrightarrow{\cong} H_*(\Sigma P^\infty(\mathbb{C}))$ is the suspension isomorphism. Since the classes

$$\pm f_* \Sigma_* \lambda_i = f_* \sigma_* i_*(\lambda_i) = \sigma_* \circ (\Omega f)_* \circ i_*(\lambda_i)$$

generate $H_*(\mathrm{SU})$, it is transgressively generated. The comparison theorem for spectral sequences then shows the

Lemma 2.6. *$H_*(\Omega\,\mathrm{SU};\mathbb{Z})$ is a polynomial algebra $P(\lambda_1, \ldots)$, with $\dim \lambda_i = 2i$ and $\lambda_i = (\Omega f)_* i_*(\lambda_i)$.*

We next define a map $b : B_{\mathrm{U}} \to \Omega\,\mathrm{SU}$, which we will later show is a homotopy equivalence. We proceed by defining

$$b_{m,n} : \mathrm{U}(m+n)/(\mathrm{U}(m) \times \mathrm{U}(n)) \to \Omega\,\mathrm{SU}(m+n).$$

As before, let $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$. For $x \in \mathbb{C}^{m+n}$, $x = x_1 + x_2$ with $x_1 \in \mathbb{C}^m$ and $x_2 \in \mathbb{C}^n$. For a real number θ, let $\varphi_{m,n}(\theta)(x) = x_1 e^{i\theta} + x_2 e^{-\theta}$.

For $A \in \mathrm{U}(m+n)$, let $\widetilde{b}_{m,n}(A)$ be the loop in $\mathrm{SU}(m+n)$ defined by the commutator

$$\widetilde{b}_{m,n}(A)(\theta) = [\varphi_{m,n}(\theta), A], \quad 0 \le \theta \le \pi.$$

For $A \in \mathrm{U}(m) \times \mathrm{U}(n)$, $\widetilde{b}_{m,n}(A)$ is the trivial loop. Thus, $\widetilde{b}_{m,n}$ induces

$$b_{m,n} : \mathrm{U}(m+n)/(\mathrm{U}(m) \times \mathrm{U}(n)) \to \Omega\,\mathrm{SU}(m+n).$$

For $(m,n) \le (m', n')$ the inclusion

$$\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n \to \mathbb{C}^{m'} \oplus \mathbb{C}^{n'} = \mathbb{C}^{m'+n'}$$

induces a commutative diagramme

$$\begin{tikzcd}
\mathrm{U}(m+n)/(\mathrm{U}(m) \times \mathrm{U}(n)) \arrow[r, "b_{m,n}"] \arrow[d] & \Omega\,\mathrm{SU}(m+n) \arrow[d] \\
\mathrm{U}(m'+n')/(\mathrm{U}(m') \times \mathrm{U}(n')) \arrow[r, "b_{m',n'}"'] & \Omega\,\mathrm{SU}(m'+n').
\end{tikzcd}$$

The lattice of pairs

$$\begin{tikzcd}
 & (n+1, n+1) \arrow[dl, dash] \arrow[dr, dash] & \\
(n+1, 1) \arrow[dr, dash] & & (n, n) \arrow[dl, dash] \\
 & (n, 1) &
\end{tikzcd}$$

yields a commutative diagramme

$$\begin{tikzcd}
P^n\mathbb{C} \arrow[rr] \arrow[dr] \arrow[dd] & & \Omega\,\mathrm{SU}(n+1) \arrow[dr] \arrow[dd] & \\
& P^{n+1}\mathbb{C} \arrow[rr] \arrow[dd] & & \Omega\,\mathrm{SU}(n+2) \arrow[dd] \\
\mathrm{U}(2n)/(\mathrm{U}(n)\times\mathrm{U}(n)) \arrow[rr] \arrow[dr] & & \Omega\,\mathrm{SU}(n+2) \arrow[dr] & \\
& \mathrm{U}(2n+2)/(\mathrm{U}(n+1)\times\mathrm{U}(n+1)) \arrow[rr] & & \Omega\,\mathrm{SU}(2n+2).
\end{tikzcd}$$

and letting $B_{\mathrm{U}} = \varinjlim \mathrm{U}(2n)/(\mathrm{U}(n)\times\mathrm{U}(n))$ and passing to the limit we obtain the commutative diagramme

$$\begin{tikzcd}
P^\infty(\mathbb{C}) \arrow[r, "b_{\infty,1}"] \arrow[d, "k"'] & \Omega\,\mathrm{SU} \arrow[d, "\mathrm{id}"] \\
B_{\mathrm{U}} \arrow[r, "b"'] & \Omega\,\mathrm{SU}
\end{tikzcd}$$

The map $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \to \mathbb{C}^\infty$ carrying $(x_i, 0)$ into x_{2i} and $(0, x_j)$ into x_{2j+1} induces a map $B_{\mathrm{U}} \times B_{\mathrm{U}} \to B_{\mathrm{U}}$. This product is homotopy commutative and homotopy associative (such operations involve sequences of permutations of coordinates, but the permutation matrices lie in SU and, as it is connected, are homotopic to the identity; furthermore, for any element only finitely many permutations are involved; and so, the operation is continuous in the weak topology). By similar reasoning the map b preserves this product up to homotopy.

Theorem 2.7. $b : B_{\mathrm{U}} \to \Omega\,\mathrm{SU}$ *is a homotopy equivalence.*

Proof. If $b \circ k \simeq \Omega f \circ i : P^\infty(\mathbb{C}) \to \Omega\Sigma P^\infty(\mathbb{C}) \to \Omega\,\mathrm{SU}$, then by the previous lemma and the fact that b is a map of H-spaces, $H_*(b)$ is an epimorphism. Our computations show that $\operatorname{rank} H_{2n}(B_{\mathrm{U}}) = \operatorname{rank} H_{2n}(\Omega\,\mathrm{SU})$ for all n; thus, $H_*(b)$ can have only 0 kernel. Hence $H_*(b)$ is an isomorphism and since B_{U} and $\Omega\,\mathrm{SU}$ are simply connected, the Whitehead theorem then implies b is a homotopy equivalence.

Since $b_{\infty,1} \simeq b \circ k$, it thus suffices to show that $\Omega f \circ i = b_{\infty,1}$. This we will do by direct computation. The map $\overline{b}_{n,1}$ can be regarded as the composition

$$\mathbb{S}^1 \times P^n(\mathbb{C}) \to \mathbb{S}^1 \times \frac{\mathrm{U}(n+1)}{\mathrm{U}(n)\times\mathrm{U}(1)} \xrightarrow{\overline{b}_{n,1}} \mathrm{SU}(n+1)$$

where $P^n(\mathbb{C}) \to \mathrm{U}(n+1/(\mathrm{U}(n)\times\mathrm{U}(1))$ is the homeomorphism taking the class $(x) \in P^n(\mathbb{C})$ into the class of any matrix B, with $Be_1 = x$, x of unit norm, where as before $\mathbb{C}^{n+1} = \mathbb{C}^n \oplus \mathbb{C}^1$ and $e_1 \in \mathbb{C}^1$. This is well defined for if $x = \lambda y$ and $\overline{B}e_1 = y$, then $\lambda\overline{B}e_1 = x$. But $B^{-1} \cdot \lambda\overline{B} \in \mathrm{U}(n) \times \mathrm{U}(1)$. In the above composition

$$(e^{2i\theta}, (x)) \text{ goes into } [\varphi_{n,1}(\theta), B] = T(\theta, x), \quad 0 \le \theta \le \pi.$$

$$\begin{aligned}
T(\theta,x)(x) &= \varphi_{n,1}(\theta) \cdot B \cdot \varphi_{n,1}^{-1}(\theta) \cdot B^{-1}x \\
&= \varphi_{n,1}(\theta) \cdot B \cdot e_1 \cdot e^{i\theta} \\
&= \varphi_{n,1}(\theta) \cdot x \cdot e^{i\theta} \\
&= (e^{i\theta}x_1 + e^{-i\theta}x_2) \cdot e^{i\theta} \\
&= e^{2i\theta}x + (1 - e^{2i\theta})x_2.
\end{aligned}$$

If $y \perp x$, then $y = B \cdot \gamma$ for some $\gamma \perp e_1$. Then

$$\begin{aligned}
T(\theta,x)(y) &= \varphi_{n,1}(\theta) \cdot B \cdot \varphi_{n,1}^{-1}(\theta) \cdot B^{-1}y \\
&= \varphi_{n,1}(\theta) \cdot B \cdot \varphi_{n,1}^{-1}(\theta) \cdot \gamma \\
&= \varphi_{n,1}(\theta) \cdot B \cdot \gamma \cdot e^{i\theta} \\
&= \varphi_{n,1}(\theta) \cdot y \cdot e^{i\theta} \\
&= (e^{i\theta}y_1 + e^{-i\theta}y_2) \cdot e^{i\theta} \\
&= y + (e^{-2i\theta} - 1) \cdot y_2.
\end{aligned}$$

Let $\overline{f}_{n+1}$ denote f_{n+1}, before normalised. Then

$$T(\theta,x)(x) = \overline{f}_{n+1}(\theta,x) \cdot \mathrm{M}(I_n \oplus +e^{2i\theta}).$$

But $T(\theta,x) \in \mathrm{SU}(n+1)$; and so, $\det(\overline{f}) \cdot e^{-2\theta} = 1$. Thus

$$T(\theta,x) = f_{n+1}(\theta,x).$$

We have shown that $f : \Sigma P^\infty(\mathbb{C}) \to \mathrm{SU}$ is the adjoint of $b_{\infty,1} : P^\infty(\mathbb{C}) \to \Omega\,\mathrm{SU}$, which completes the argument. □

2.2 Complex Vector Bundles

2.2.1 Characteristic Classes

As we have seen earlier, the decomposition $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$ induces a map

$$j : B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \to B_{\mathrm{U}(m+n)}.$$

There is a similar decomposition involving tensor products over $\mathbb{C}$

$$\mathbb{C}^{mn} = \mathbb{C}^m \otimes \mathbb{C}^n,$$

which induces a monomorphism $\mathrm{U}(m) \times \mathrm{U}(n) \to \mathrm{U}(mn)$ and a subsequent map

$$k : B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \to B_{\mathrm{U}(mn)}.$$

By the map $i : B^n_{\mathrm{U}(1)} \to B_{\mathrm{U}(n)}$ we define a class $\mathrm{ch}(n) \in H^{**}(B_{\mathrm{U}(n)}; \mathbb{Q})$ by the formula $i^{**}(\mathrm{ch}(n)) = \sum_{\ell=1}^n e^{u_\ell}$, whose u_ℓ is the image in $H^2(B_{\mathrm{U}(1)}; \mathbb{Z})$ of the generator u_ℓ of $H^2(B_{\mathrm{U}(1)}; \mathbb{Z})$ in the ℓ-th factor (strictly, $i^{**}(\mathrm{ch}(n)) = \sum_{\ell=1}^n 1 \times 1 \times \cdots \times 1 \times e^{u_\ell} \times 1 \times \cdots \times 1)$. The class $\mathrm{ch}(n)$ is called the *Universal Chern Character.*

Proposition 2.8. *For* $j : B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \to B_{\mathrm{U}(m+n)}$, $j^{**}\,\mathrm{ch}(m+n) = \mathrm{ch}(m) \times 1 + 1 \times \mathrm{ch}(n)$.

Proof. The composition

$$B^{m+n}_{\mathrm{U}(1)} \twoheadrightarrow B^m_{\mathrm{U}(1)} \times B^n_{\mathrm{U}(1)} \to B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \xrightarrow{j} B_{\mathrm{U}(m+n)}$$

is the map $i : B^{m+n}_{\mathrm{U}(1)} \to$, used to define $\mathrm{ch}(m+n)$. But $\mathrm{ch}(m) \times 1 + 1 \times \mathrm{ch}(n)$ is the unique class in $H^{**}(B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)}, \mathbb{Q})$ which pulls back to

$$\sum_1^{m+n} e^{u_\ell} = \sum_1^m e^{u_\ell} + \sum_1^n e^{u_\ell}.$$

$\square$

Proposition 2.9. *For* $k : B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \to B_{\mathrm{U}(mn)}$, $k^{**}\,\mathrm{ch}(mn) = \mathrm{ch}(m) \times \mathrm{ch}(n)$.

Proof. The following diagramme is commutative:

$$\begin{array}{ccc} B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} & \longrightarrow & B_{\mathrm{U}(mn)} \\ \uparrow & & \uparrow \\ B^m_{\mathrm{U}(1)} \times B^n_{\mathrm{U}(1)} & \xrightarrow[\ell]{} & B^{mn}_{\mathrm{U}(1)} \end{array}$$

where ℓ is the composition

$$B^m_{\mathrm{U}(1)} \times B^n_{\mathrm{U}(1)} \xrightarrow{(\Delta_n)^m \times (\Delta_m)^n} (B^n_{\mathrm{U}(1)})^m \times (B^m_{\mathrm{U}(1)})^n \to (B_{\mathrm{U}(1)} \times B_{\mathrm{U}(1)})^{mn} \xrightarrow{\mu} B^{mn}_{\mathrm{U}(1)}$$

with $\mu : B_{\mathrm{U}(1)} \times B_{\mathrm{U}(1)} \to B_{\mathrm{U}(1)}$ the canonical H-space structure.

For $1 \le i \le mn$, i can be written uniquely as $i = (k-1) \cdot n + j$ with $1 \le k \le m$ and $1 \le j \le n$. The class e^{w_i} in $H^{**}(B^{mn}_{\mathrm{U}(1)}; \mathbb{Q}$ under μ^{**}, maps as follows

$$\mu{**}(e^{w_i} = \mu{**}\left(\sum_{\ell=0}^{\infty} \frac{w_i^\ell}{\ell!}\right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{p=0}^{\ell} \binom{\ell}{p} w_i^p \times w_i^{\ell-p} = e^{w_i} \times e^{w_i}.$$

Thus, $\ell^{**}(e^{w_i}) = e^{u_k} \times e^{v_j}$, with k and j as above. Finally,

$$\ell^{**}(\sum_1^{mn} e^{w_i}) = (\sum_1^m e^{u_k}) \times (\sum_1^n e^{u_j}$$

since all combinations of k and j occur. The commutative diagramme then implies the conclusion. $\square$

For each map $f : X \longrightarrow B_{\mathrm{U}(n)}$, there is defined the induced $\mathrm{U}(n)$-principal bundle $f^*(\gamma^n)$ over X. The spaces $B_{\mathrm{U}(n)}$ are called *universal base spaces* because of the following theorem.

Theorem 2.10 (Bundle Classification Theorem)**.** *Let X be a (normal) paracompact Hausdorff space. There is a* $1-1$ *correspondence between equivalence classes of n-plane complex vector bundles over X and homotopy classes* $[X, B_{\mathrm{U}(n)}]$ *given by the equivalence class of the associated n-plane complex vector bundle of the principal* $\mathrm{U}(n)$*-bundle* $f*(\gamma^n)$ *for* $f : X \longrightarrow B_{\mathrm{U}(n)}$.

If only the equivalence class of a bundle τ over X is relevant, we shall frequently identify τ with its map $\tau : X \longrightarrow B_{\mathrm{U}(n)}$.

Definition 2.11 (Sum and Product of Plane Bundles)**.** The equivalence class $\tau^m + \tau^n$ of the *sum* of m-plane and n-plane complex vector bundles τ^m and τ^n is the associated $(m+n)$-plane complex vector bundle to the principal bundle induced by the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\tau^m \times \tau^n} B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \xrightarrow{j} B_{\mathrm{U}(m+n)}.$$

The equivalence class $\tau^m \otimes \tau^n$ of the *product* of m-plane and n-plane complex vector bundles τ^m and τ^n is the associated $(m \cdot n)$-plane complex vector bundle to the principal bundle induced by the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\tau^m \times \tau^n} B_{\mathrm{U}(m)} \times B_{\mathrm{U}(n)} \xrightarrow{k} B_{\mathrm{U}(mn)}.$$

Definition 2.12. For an n-plane complex vector bundle γ over X, let $\gamma : X \longrightarrow B_{\mathrm{U}(n))}$ be a representative of the homotopy class determined by γ. The *Chern classes*, $\{C_i(\gamma)\}$, of γ are

$$C_i(\gamma) = \gamma^*(C_i) \in H^{2i}(X; \mathbb{Z});$$

the *Chern character*, $\mathrm{ch}(\gamma)$, of γ is

$$\mathrm{ch}(\gamma) = \gamma^{**}\,\mathrm{ch}(n) \in H^{**}(X; \mathbb{Q}).$$

There follows immediately from the Lemma 2.6, and the above Propositions 2.8 and 2.9 and the

Proposition 2.13. *For m-plane and n-plane complex vector bundles γ^m and γ^n over X,*

(a)

$$\begin{aligned}&(1 + C_1(\gamma^m) + \cdots + C_m(\gamma^m)) \cup (1 + C_1(\gamma^n) + \cdots + C_n(\gamma^n)) = \\ &1 + C_1(\gamma^m \oplus (\gamma^n) + \cdots + C_{m+n}(\gamma^m \oplus (\gamma^n),\end{aligned}$$

(b) $\mathrm{ch}(\gamma^m) + \mathrm{ch}(\gamma^n) = \mathrm{ch}(\gamma^m \oplus \gamma^n)$*, and*

(c) $\mathrm{ch}(\gamma^m) \cup \mathrm{ch}(\gamma^n) = \mathrm{ch}(\gamma^m \otimes \gamma^n)$.

From this proposition it appears that if a cohomology theory can be constructed with elements of the groups being complex vector bundles, then ch will furnish a multiplicative transformation of this theory in $H^{**}(-; \mathbb{Q})$. We shall see in the next chapter that this is indeed the case.

2.2.2 Complex Vector Bundles over Spheres

From the homotopy equivalences $B_{\mathrm{U}} \to \Omega\,\mathrm{SU}$ and $\Omega B_{\mathrm{SU}} \to \mathrm{SU}$ and the fact that topologically $\mathbb{S}^1 \times \mathrm{SU}$ is homeomorphic to U, we can immediately deduce that for all $i \geq 1$, $\pi_i(B_{\mathrm{U}}) \cong \pi_{i+2}(B_{\mathrm{U}})$. It is easy to compute that $\pi_1(B_{\mathrm{U}}) = 0$ and $\pi_2(B_{\mathrm{U}}) \cong \mathbb{Z}$.

It is a general result (called the *Eckmann-Hilton argument*) that if X is a co-H-space and Y is an H-space, then the two group structures on $[X, Y]$, due respectively to the co-H-space structure of X and the H-space structure of Y, coincide and are in fact abelian. For a proof, see Appendix D.

If we use the H-space structure on B_{U} defined by the map $j : B_{\mathrm{U}} \times B_{\mathrm{U}} \to B_{\mathrm{U}}$ inducing sums of bundles and the usual co-H-space structure on $\mathbb{S}^{2n}$, we have then that if $\lambda : \mathbb{S}^{2n} \to B_{\mathrm{U}}$ is in the class of a generator of $\pi_{2n}(B_{\mathrm{U}})$, then the sum of k copies of the bundle induced by λ is equivalent to the bundle induced by $k\lambda \in \pi_{2n}(B_{\mathrm{U}})$. It follows then that $\mathrm{ch}(k\lambda) = k\,\mathrm{ch}(\lambda)$. The class $\mathrm{ch}(\lambda)$ has a component $\mathrm{ch}_n(\lambda)$ in dimension $2n$ in $H^{2n}(\mathbb{S}^{2n}; \mathbb{Q})$; this class has value a rational number on the generating homology class of $H_{2n}(\mathbb{S}^{2n}; \mathbb{Z})$. We shall show this number is ± 1, establishing the

Theorem 2.14. *Let γ be an m-plane complex vector bundle over the sphere $\mathbb{S}^{2n}$, then $\langle \mathrm{ch}_n(\gamma), \mathbb{S}^{2n} \rangle$ is an integer; if $m > n$ and $\gamma : \mathbb{S}^{2n} \to B_{\mathrm{U}(m)}$ is a generator of $\pi_{2n}(B_{\mathrm{U}(m)})$, then $\langle \mathrm{ch}_n(\gamma), \mathbb{S}^{2n} \rangle = \pm 1$.*

In the previous discussion we considered only maps into B_{U}; however, the fibrations $\mathbb{S}^{2n-1} \to B_{\mathrm{U}(n-1)} \to B_{\mathrm{U}}$ show the homomorphism $\pi(B_{\mathrm{U}(m)}) \to \pi_n(B_{\mathrm{U}})$ is an isomorphism for $n \leq 2m - 1$. For $m \leq n$, we add enough trivial bundles to get in this "stable" range and the conclusion follows (it is not very interesting in this case, however, as the integer obtained is always 0).

We wish, then, to determine the Chern character of the generating bundle over $\mathbb{S}^{2n}$. To this end we need a good bit of technical preparation, and the remainder of this section is concerned with that.

Proposition 2.15. *Let $A = P[u_1, u_2, \ldots,]$ be a polynomial Hopf algebra over the integers generated by elements u_i of dimension $2i$ with diagonal*

$$\psi(u_n) = \sum_{p+q=n} u_p \otimes u_q.$$

Then the dual A^* of A with respect to $\mathbb{Z}$ is a polynomial Hopf algebra $P[a_1, a_2, \ldots]$ generated by elements a_i of dimension $2i$ with diagonal

$$\psi(a_n) = \sum_{p+q=n} a_p \otimes a_q.$$

Furthermore, the generators a_i may be taken to be $\overline{u}_1^i$, the duals of u_1^i; in this case

$$\overline{u}_k = (-1)^{k-1} k a_k + \text{ decomposable elements.}$$

Proof. Let C be a coalgebra with generators $\{e_1, e_2, \ldots\}$, $\dim e_i = 2i$, and diagonal

$$\psi(e_n) = \sum_{p+q=n} e_p \otimes e_q.$$

Define $f : C \to A$ by $f(e_i) = u_i$. Then f is a morphism of coalgebras. Let g_k be the composition

$$\otimes^k C \xrightarrow{\otimes^k f} \otimes^k A \to A$$

where $\otimes^k X$ denotes the tensor product of k copies of X and the homomorphism on the right is given by the algebra structure of A. Then

$$g_k(e_{i_1}^{(1)} \times \cdots \times e_{i_k}^{(k)}) = u_{i_1} \cdots u_{i_k}.$$

Thus, g_k is epimorphic in dimensions less than $2k+2$.

C^* is the polynomial algebra in one indeterminate of dimension 2, and $(\otimes^k C)^*$ is the polynomial ring in k indeterminates $x_1, \ldots, x_k$, each of dimension 2, which can be chosen so that

$$e_{i_j}^{(\ell)} = \overline{x}_\ell^{i_j}.$$

The dual of g

$$g^* : A^* \to (\otimes^k C)^*$$

is a homomorphism of algebras and is a monomorphism in dimensions less than $2k+2$. In particular,

$$g^*(\overline{u_{i_1} \cdots u_{i_k}}) = \sum_{\sigma \in S_k} x_1^{i_{\sigma(1)}} \otimes \cdots \otimes x_k^{i_{\sigma(k)}},$$

where σ ranges through the symmetric group S_k on k symbols. Then the image of g^* is generated by $\sigma_1, \ldots, \sigma_k$, elementary symmetric functions in $x_1, \ldots, x_k$, in dimensions less than $2k+2$. Define $a_i \in A^*$ by

$$g^*(a_i) = \sigma_i, \quad i \le k.$$

Then $a_i = \overline{u}_1^i$, and the a_i generate A^* as an algebra in dimensions less than or equal to $2k+1$.

The diagonal map is given by

$$\langle \Delta(a_i), u_1^p \otimes u_1^q \rangle = \langle a_i, u_1^{p+q} \rangle = \begin{cases} 1 & \text{if} \quad i = p+q, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\Delta(a_i) = \sum_{p+q=i} a_p \otimes a_q$.

Finally, $g^*(\overline{u}_k) = x_1^k + \cdots + x_k^k = \gamma \cdot \sigma_k + F(\sigma_1, \ldots, \sigma_{k-1})$ for some integer γ and polynomial F over $\mathbb{Z}$. To evaluate γ, let $x_1, \ldots, x_k$ be the k^{th} roots of 1. Then $\sigma_k = (-1)^{k-1}$ and $\sigma_i = 0$ for $i \neq k$. Thus, $k = (-1)^{k-1}\gamma$, as asserted. Since $g*$ is a monomorphism,

$$\overline{u}_k = (-1)^{k-1} k a_k + \text{ decomposable elements.}$$

□

Corollary 2.16. *For $b : B_U \to \Omega SU$,*

$$b^*(\overline{\lambda}_1^i = \pm C_i + \text{ decomposable elements.}$$

The following construction is due to Hopf. Given a base-product preserving map $f : X \times Y \to Z$, there is a map

$$Sf : X \circ Y \to \Sigma Z$$

of the join of X with Y to the suspension of Z given by

$$Sf(x, t, y) = (t, f(x, y)).$$

This carries $(x, 0, y)$, $(x, 1, y)$ and $(*, t, *)$ into the base point; and so, it induces the map asserted.

As examples, two of the Hopf maps $\mathbb{S}^3 \to \mathbb{S}^2$ and $\mathbb{S}^7 \to \mathbb{S}^4$ are given by complex and quatemionic multiplications $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ and $\mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$ of their respective unit spheres.

We note also the homotopy equivalence $X \circ Y \xrightarrow{\simeq} \Sigma(X \# Y)$.

For $X = \mathbb{C}P^\infty$, since X is an H-space with multiplication $m : X \times X \to X$, there is defined

$$\zeta : X \circ X \to \Sigma X.$$

If λ_n denotes the generator of $H_{2n}(X; \mathbb{Z})$, then

$$m^*(\overline{\lambda}_1) = \overline{\lambda}_1 \otimes 1 + 1 \otimes \overline{\lambda}_1; \quad \text{and so}$$
$$m^*(\overline{\lambda}_1^k) = \sum_{i=0}^{k} \binom{k}{i} \overline{\lambda}_1^i \otimes \overline{\lambda}_1^{k-i}.$$

Equivalently,

$$m^*(\overline{\lambda}_k) = \sum_{i=0}^{k} \binom{k}{i} \overline{\lambda}_i \otimes \overline{\lambda}_{k-i}.$$

If we restrict the map ζ to $\mathbb{S}^2 \circ X$, the 2-skeleton of X being $\mathbb{S}^2$, we obtain

$$\zeta : \Sigma^3 X \to \Sigma X$$

and

$$\zeta^*(\Sigma^* \overline{\lambda}_k) = k \cdot \Sigma^3 \overline{\lambda}_{k-1}, \quad \text{for} k \geq 2.$$

Theorem 2.17. *The group $\pi_{2n+1}(\mathrm{SU})$ is generated by the composition*

$$\mathbb{S}^{2n+1} \to \Sigma^{2n-1}\mathbb{C}P^\infty \xrightarrow{\Sigma^{2n-4}\zeta} \Sigma^{2n-3}\mathbb{C}P^\infty \to \cdots \to \Sigma^3\mathbb{C}P^\infty \xrightarrow{\zeta} \Sigma\mathbb{C}P^\infty \xrightarrow{f} \mathrm{SU}.$$

The image of the generator under the Hurewicz homomorphism $\pi_{2n+1}(\mathrm{SU}) \to H_{2n+1}(\mathrm{SU})$ is $n!\nu_{n+1}$.

Proof. Consider first $\Sigma^3\mathbb{C}P^\infty \xrightarrow{\zeta} \Sigma\mathbb{C}P^\infty \xrightarrow{f} \mathrm{SU}$. By Proposition 2.15, $H^*(\Omega\,\mathrm{SU}) = P[\overline{\lambda}_1, \overline{\lambda}_1^2, \ldots]$, and $\overline{\lambda}_k = (-1)^{k-1}k\overline{\lambda}_1^k +$ decomposable elements. Let

$$\xi : \Omega\Sigma^3\mathbb{C}P^\infty \to \Omega\,\mathrm{SU}$$

be the loop map of the composition $f \circ \zeta$ and

$$i : \Sigma^2\mathbb{C}P^\infty \to \Omega\Sigma\Sigma^2\mathbb{C}P^\infty$$

the canonical inclusion of space in $\Omega\Sigma$ of it.

We shall first show that

$$i^* \circ \xi^*(\overline{\lambda}_1^k) = (-1)^{k-1}(\Sigma^2)^*\overline{\lambda}_{k-1}, \quad k = 2, 3, \ldots .$$

Since the λ_i generate $H_*(\Omega\,\mathrm{SU})$,

$$(\xi \circ i)_*\Sigma^2_*\lambda_{k-1} = \gamma \cdot \lambda_k + F(\lambda_1, \ldots, \lambda_{i-1}).$$

Taking homology suspension,

$$\sigma_*(\xi \circ i)_*\Sigma^2_*\lambda_{k-1} = \gamma\sigma_*\lambda_k,$$

since σ_* annihilates decomposable elements. We have seen earlier that

$$\sigma_*\lambda_k = \nu_{k+1} \in H_*(\mathrm{SU}).$$

Also,

$$\begin{aligned} \sigma_*(\xi \circ i)_*\Sigma^2_*\lambda_{k-1} &= f_*\zeta_*\sigma_*(i_*\Sigma^2_*\lambda_{k-1}) \\ &= f_*\zeta_*(\Sigma^3_*\lambda_{k-1}) \\ &= f_*(k\Sigma_*\lambda_k) \\ &= k \cdot \nu_{k+1}. \end{aligned}$$

Thus, $\gamma\nu_{k+1} = k\nu_{k+1}$ and $\gamma = k$. Then dually, we have

$$(\xi \circ i)^*\overline{\lambda}_k = k \cdot (\Sigma^2)^*\overline{\lambda}_{k-1}.$$

Since i^* annihilates decomposable elements,

$$(\xi \circ i)^*((-1)^{k-1}k\overline{\lambda}_1^k) = k(\Sigma^2)^*\overline{\lambda}_{k-1},$$

and as sought

$$i \circ \xi^*(\overline{\lambda}_1^k) = (-1)^{k-1}(\Sigma^2)^*\overline{\lambda}_{k-1}.$$

Taking loops once more, we obtain the following diagramme where to conserve notation we have written the previous $i \circ \xi$ simply as ξ.

$$\begin{tikzcd}
 & H^*(\Omega\Sigma^2\mathbb{C}P^\infty) \arrow[dl, "i^*"'] & H^*(\Omega^2\,\mathrm{SU}) \arrow[l, "(\Omega\xi)^*"'] \\
H^*(\Sigma\mathbb{C}P^\infty) & H^*(\Sigma^2\mathbb{C}P^\infty) \arrow[l, "(\Sigma^*)^{-1}"] \arrow[u, "\sigma^*"'] & H^*(\Omega\,\mathrm{SU}) \arrow[l, "\xi_*"] \arrow[u, "\sigma_*"']
\end{tikzcd}$$

For $i > 1$, let $\sigma^*(\overline{\lambda}_1^{i+1}) = e_i' \in H^*(\Omega^2 \mathrm{SU})$. Then for $f' = \Omega\xi \circ i$, we have

$$f'^* e_i' = (-1)^i \Sigma^* \overline{\lambda}_i.$$

By the comparison theorem for spectral sequences as proved in

- Erik Christopher Zeeman, *A proof of the comparison theorem for spectral sequences*, Proc. Cambridge Philos. Soc., 53: 57-62 (1957)

$H^*(\Omega^2 \mathrm{SU})$ is an exterior algebra generated by e_1' and the e_i'. Thus, $H_*(\Omega\, \mathrm{SU})$ is an exterior algebra generated by $\overline{e}_1'$ and the $\overline{e}_i'$ and

$$f_*'(\Sigma_* \lambda_i) = (-1)^i \overline{e}_i'.$$

Since SU is the simply connected covering space of $\Omega^2 \mathrm{SU}$ (up to homotopy), we have the commutative diagramme

$$\begin{tikzcd} & \mathrm{SU} \arrow[d] \\ \Sigma\mathbb{C}P^\infty \arrow[ur, "f_{(1)}"] \arrow[r, "f'"'] & \Omega^2\,\mathrm{SU} \simeq \mathrm{U} \end{tikzcd}$$

for which $f_{(1)*}$ is a monomorphism in homology and $\mathrm{im}(f_{(1)*})$ generates $H_*(\mathrm{SU})$ as an exterior algebra. Iterating this process, we then see that if the lift in the diagramme

$$\begin{tikzcd} & \mathrm{SU} \arrow[d] \\ \Sigma\mathbb{C}P^\infty \arrow[ur, "f_{(n-1)}"] \arrow[r] & \Omega^{2n-2}\,\mathrm{SU} \simeq \mathrm{U} \end{tikzcd}$$

is a monomorphism in homology with image generating as an algebra, then for

$$\begin{tikzcd} & & \mathrm{SU} \arrow[d] \\ \Sigma^3\mathbb{C}P^\infty \arrow[r] & \Sigma\mathbb{C}P^\infty \arrow[ur, "f_{(n-1)}"] \arrow[r] & \Omega^{2n-2}\,\mathrm{SU}, \end{tikzcd}$$

the construction can be repeated to obtain

$$\begin{tikzcd} & \mathrm{SU} \arrow[d] \\ \Sigma\mathbb{C}P^\infty \arrow[ur, "f_{(n)}"] \arrow[r] & \Omega^{2n}\,\mathrm{SU} \simeq \mathrm{U} \end{tikzcd}$$

with $f_{(n)}$ inducing a monomorphism in homology with image generating.

Then by the Hurewicz Theorem, the composition

$$\mathbb{S}^3 \to \Sigma_* \mathbb{C}P^\infty \to \Omega^{2n-2}\,\mathrm{SU}$$

generates $\pi_3(\Omega^{2n-2}\,\mathrm{SU})$. This composition is adjoint to the one asserted in the theorem; and so, the conclusion follows.

In homology, the class $(\mathbb{S}^{2n+1})$ is mapped successively into

$$\Sigma_*^{2n-1}\lambda_1, \quad \Sigma_*^{2n-3}\lambda_2, \quad \ldots, \quad n!\Sigma_*\lambda_n, \quad n!\nu_{n+1},$$

which is the second part of the conclusion. □

Corollary 2.18.

$$\pi_i(\mathrm{U}(n)) \cong \begin{cases} 0 & \textit{for} \quad i \textit{ even, } i < 2n \\ \mathbb{Z} & \textit{for} \quad i \textit{ odd, } i < 2n \\ \mathbb{Z}/n!\mathbb{Z} & \textit{for} \quad i = 2n. \end{cases}$$

Proof. The computations of the homotopy groups of U by Theorem 2.7 together with the sequence of fibrations $\mathrm{SU}(k-1) \to \mathrm{SU}(k) \to \mathbb{S}^{2k-1}$ establish the first two assertions. To see the third consider the fibration

$$\begin{tikzcd} \mathrm{SU}(n) \arrow[r, "i"] & \mathrm{SU}(n+1) \arrow[d, "\pi"] \\ & \mathbb{S}^{2n+1}. \end{tikzcd}$$

In the proof of Theorem 2.2, the class ν_{n+1} was chosen so that $\pi_*(\nu_{n+1})$ generated $H_{2n+1}(\mathbb{S}^{2n+1};\mathbb{Z})$. Since the generating map $g_n : \mathbb{S}^{2n+1} \to \mathrm{SU}$ of Theorem 2.17 factors through $\mathrm{SU}(n+1)$, by the above comments, and if ι is a generator of $H_{2n+1}(\mathbb{S}^{2n+1};\mathbb{Z})$, $g_{n*}(\iota) = n!\nu_{n+1}$, the composition $\pi \circ g_n : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ has degree $n!$. As $\pi_{2n}(\mathrm{SU}(n+1) = 0$, the homotopy sequence of the above fibration then shows $\pi_{2n}(\mathrm{SU}(n)) \cong \mathbb{Z}/n!\mathbb{Z}$. □

Corollary 2.19. *If $h : \mathbb{S}^{2n} \to B_{\mathrm{U}}$ generates $\pi_{2n}(B_{\mathrm{U}})$, then*

$$\langle h*(C_n), \mathbb{S}^{2n}\rangle = \pm(n-1)!$$

Proof. Consider the commutative diagramme

$$\begin{tikzcd} H_*(\mathbb{S}^{2n}) \arrow[r, "i_*"] \arrow[dr, "\Sigma_*"'] & H_*(\Omega\mathbb{S}^{2n+1}) \arrow[r, "(\Omega g)_*"] \arrow[d, "\sigma_*"] & H_*(\Omega\,\mathrm{SU}) \arrow[d, "\sigma_*"] \\ & H_*(\mathbb{S}^{2n+1}) \arrow[r, "g_*"'] & H_*(\mathrm{SU}), \end{tikzcd}$$

where $g : \mathbb{S}^{2n+1} \to \mathrm{SU}$ generates $\pi_{2n+1}(\mathrm{SU})$ as in Theorem 2.17. Then

$$\sigma_* \circ (\Omega g)_* \circ i_*(\mathbb{S}^{2n}) = g_*(\Sigma_*\mathbb{S}^{2n}) = g_*(\mathbb{S}^{2n+1}) = n!\nu_{n+1}.$$

As we observed in the proof of Theorem 2.17,

$$\sigma_*(\lambda_n) = \nu_{n+1}$$

and σ_* annihilates decomposable elements. The map $h = \Omega g \circ i : \mathbb{S}^{2n} \to \Omega\,\mathrm{SU}$ generates $\pi_{2n}(\Omega\,\mathrm{SU})$ and we have

$$h_*(\mathbb{S}^{2n}) = n!\lambda_n + \text{ decomposable elements.}$$

Thus $\langle h^*\overline{\lambda}_n, \mathbb{S}^{2n}\rangle = n!$, and since $\overline{\lambda}_n = (-1)^{n-1} n \cdot \overline{\lambda}_1^n +$ decomposable elements,

$$\langle h^*\overline{\lambda}_1^n, \mathbb{S}^{2n}\rangle = (-1)^{n-1}(n-1)!.$$

Corollary 2.16 completes the proof. □

To complete the proof of Theorem 2.14 we need to relate $\mathrm{ch}_n(\gamma)$ to the Chern classes of γ. This is an exercise in symmetric function theory. Recall that the i-th Chern class C_i is defined in terms of the elementary symmetric function $\sigma_{i_n}^{(n)}$ in n indeterminates $u_1, \ \ldots, \ u_n$, $i \le n$, and $\mathrm{ch}_i = \sum_{k=1}^n u_k^i / i!$. We derive a formula known as the "Newton formula":

$$\begin{aligned}
\prod_{i=1}^n (1 + u_i t) &= 1 + \sigma_1 t + \sigma 2 t^2 + \cdots + \sigma_n t^n \\
\frac{d}{dt}\prod_{i=1}^n (1 - u_i t) &= \prod_{i=1}^n (1 - u_i t) \cdot \sum_{k=1}^n \frac{-u_k}{1 - u_k t} \\
&= -\prod_{i=1}^n (1 - u_i t) \cdot \sum_{k=1}^n \sum_{j=1}^\infty (u_k^{j+1} t^j) \\
&= -\prod_{i=1}^n (1 - u_i t) \cdot \sum_{j=1}^\infty p_{j+1} t^j,
\end{aligned}$$

where $p = u_1^r + \cdots + u_n^r$. Thus,

$$-\sigma_1 + 2\sigma_2 t - 3\sigma_3 t + \cdots = -(1 - \sigma_1 t + \sigma_2 t^2 - \cdots)(p_1 + p_2 t + p_3 t^2 + \cdots).$$

Equating the coefficients of t^{n-1}, we have

$$\begin{aligned}
&(-1)^n \cdot n \cdot \sigma_n = -[p_n - p_{n-1}\sigma_1 + \cdots + (-1)^{n-1} p_1 \sigma_{n-1}] \quad \text{or} \\
&p_n - p_{n-1}\sigma_1 + \cdots + (-1)^{n-1} p_1 \sigma_{n-1} + (-1)^n \cdot n \cdot \sigma_n = 0.
\end{aligned}$$

Thus, since $n!\,\mathrm{ch}_n = p_n$,

$$(n-1)!\,\mathrm{ch}_n = \pm C_n + \text{ decomposable elements,}$$

By this remark and Corollary 2.19, we see

$$(n-1)!\langle h^*\,\mathrm{ch}_n, \mathbb{S}^{2n}\rangle = \pm\langle h^* C_n, \mathbb{S}^{2n}\rangle = \pm(n-1)!$$

and

$$\langle h^*\,\mathrm{ch}_n, \mathbb{S}^{2n}\rangle = \pm 1.$$

Chapter 3

The Cohomology Theory $K^*_{\mathbb{C}}$

3.1 Basic Properties of $K^*_{\mathbb{C}}$

3.1.1 Definition of $K^*_{\mathbb{C}}$

The groups $K^0_{\mathbb{C}}$ are a special case of the universal enveloping abelian group of a monoid. Let G be a set of objects with an associative operation $\oplus$ and a unit 0. Let A be the free abelian group on the objects of G modulo the subgroup generated by elements of the form

$$[\alpha \oplus \beta] - [\alpha] - [\beta]$$

Then A is an abelian group and the the transformation

$$e : G \to A$$

defined by $e(g) = [g] \in A$, preserves sums and unit element. A is characterised by the property if $f : G \to B$ is a transformation of G to an abelian group B preserving sums and unit, there is a unique homomorphism $\overline{f} : A \to B$ of abelian groups such that the diagramme

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\
& A \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

commutes.

For the case where G is the set of equivalence classes of complex vector bundles over a space X, $\oplus$ is sum of vector bundles, and 0 is the identity map of X into itself, the abelian group A is denoted $K_{\mathbb{C}}(X)$ and called the *Grothendieck group* of X. Throughout much of this chapter we consider only complex vector bundles and will write $K(X)$ instead of $K_{\mathbb{C}}(X)$. (Later we consider also real vector bundles and wish to use both $K_{\mathbb{C}}$ and $K_{\mathbb{R}}$). It is a simple verification that the product of vector bundles induces a commutative ring structure on $K(X)$, with unit the trivial line bundle over X. The construction K is a contravariant functor from spaces to commutative rings with unit via induced bundles.

Proposition 3.1. *$K(X) = [X, \mathbb{Z} \times B_{\mathrm{U}}]$, for $X \in \mathscr{P}$.*

Recall that $\mathscr{P}$ is the category whose objects are finite cell complexes and whose maps are continuous maps.

Proof. It suffices to prove this isomorphism for connected spaces X. For

$$f : X \longrightarrow \mathbb{Z} \times B_{\mathrm{U}},$$

$\mathrm{im}(f) \subset n \times B_{\mathrm{U}}$ since X is connected, and since $X \in \mathscr{P}$, $\mathrm{im}(f) \subset n \times B_{\mathrm{U}(m)} \subset n \times B_{\mathrm{U}}$. Thus, f induces a complex m-plane bundle $\{f\}$ over X. Define

$$\theta : [X, \mathbb{Z} \times B_{\mathrm{U}}] \longrightarrow K(X)$$

by $\theta([f]) = [\{f\}] \oplus [n] \oplus [-m] \in K(X)$.

To define an inverse map for $\xi \in K(X)$, $\xi = [\alpha] - [\beta]$, where α and β are vector bundles over X. By the Lemma which follows, there is a vector bundle γ over X such that $\beta \oplus \gamma$ is the trivial q-plane bundle; i.e., in $K(X)$ we have

$$-[\beta] = [\gamma] - [q], \quad \text{and} \quad \xi = [\alpha \oplus \gamma] - [q].$$

We define the image of ξ in $[X, \mathbb{Z} \times B_{\mathrm{U}}]$ to be the class of a map $X \longrightarrow n \times B_{\mathrm{U}(n+q)}$, which induces the bundle $\alpha \oplus \gamma$ over X, where $\alpha \oplus \gamma$ is an $(n+q)$-bundle. This is inverse to θ. □

Lemma 3.2. *If β is a complex vector bundle over X, there is a complex vector bundle γ over X such that $\beta \oplus \gamma$ is a trivial bundle.*

Proof. The bundle β is induced by a map of X into some $G_{m,n}$, the Grassman manifold of n-planes in $\mathbb{C}$, with the bundle of orthogonal n-frames over it. The orthogonal bundle of m-frames of $G_{m,n}$ induces γ; and as the sum of the two frame bundles is trivial, so is $\beta \oplus \gamma$. □

For $(X, x_0) \in \mathscr{P}_0$, let $\widetilde{K}(X)$ be the kernel of $K(X) \longrightarrow K(x_0)$ induced by the inclusion $x \longrightarrow X$. $K(x_0) \cong \mathbb{Z}$ and $\widetilde{K}(X)$ is an ideal in $K(X)$. For $X, A) \in \mathbb{P}^2$, we define

$$K(X, A) \equiv \widetilde{K}(X/A).$$

Then $K(X, \varnothing) = \widetilde{K}(X/\varnothing) \cong \widetilde{K}(X^+) \cong K(X)$.

Using Proposition 3.1, we now define a cohomology theory $K (= K_{\mathbb{C}}^*)$ such that $K^0(X) = K(X)$. Let $\mathscr{B}_{\mathrm{U}}$ be the spectrum

$$(\mathscr{B}_{\mathrm{U}})_{2n} = \mathbb{Z} \times B_{\mathrm{U}}$$
$$(\mathscr{B}_{\mathrm{U}})_{2n+1} = \mathrm{U},$$

with maps $\Sigma(\mathbb{Z}\times B_{\mathrm{U}}) \to \mathrm{U}$ and $\Sigma U \to \mathbb{Z}\times B_{\mathrm{U}}$ adjoint to the homotopy equivalences $\mathbb{Z}\times B_{\mathrm{U}} \to \Omega\,\mathrm{U}$ and $\mathrm{U} \to \Omega(\mathbb{Z}\times B_{\mathrm{U}})$; the base point of $\mathbb{Z}\times B_{\mathrm{U}}$ is chosen to lie in $(0)\times B_{\mathrm{U}}$. Then

$$\widetilde{K}^q(X) = \{X, \mathbb{Z}\times B_{\mathrm{U}}^{(q)}\}$$

for $(X,*) \in \mathscr{P}$ defines a cohomology theory. For $q = 0$, $\widetilde{K}^0(X) = \varinjlim[\Sigma^i X, (B_{\mathrm{U}})_i]$. Since the compositions

$$[X, \mathbb{Z}\times B_{\mathrm{U}}] \to [\Sigma X, \Sigma(\mathbb{Z}\times B_{\mathrm{U}})] \to [\Sigma X, \mathrm{U}]$$

and

$$[\Sigma X, \mathrm{U}] \to [\Sigma^2 X, \Sigma\,\mathrm{U}] \to [\Sigma^2 X, \mathbb{Z}\times B_{\mathrm{U}}]$$

are adjoint to

$$[X, \mathbb{Z}\times B_{\mathrm{U}}] \xrightarrow{\cong} [X, \Omega\,\mathrm{U}]$$

and

$$[X, \Omega\,\mathrm{U}] \xrightarrow{\cong} [X, \Omega^2(\mathbb{Z}\times B_{\mathrm{U}})],$$

respectively, $\widetilde{K}^0(X) \cong \widetilde{K}(X)$, for connected spaces X and thus for all $(X,*) \in \mathscr{P}_0$. Further, because of Theorem 2.7, $\mathbb{Z}\times B_{\mathrm{U}} \xrightarrow{\cong} \Omega^2(\mathbb{Z}\times B_{\mathrm{U}})$; and so,

$$K^n(X,A) \cong K^{n+2}(X,A)$$

for all $(X,A) \in \mathscr{P}^2$. Thus, all information in this cohomology is contained in

$$\widetilde{K}^0(X) \cong \widetilde{K}(X) \quad \text{and}$$
$$\widetilde{K}^{-1}(X) \cong \widetilde{K}(\Sigma X).$$

We next observe that the theory K^* is a multiplicative cohomology theory. By §1.3 it suffices to define an internal natural pairing $\otimes$ on $K^*(X,A)$ for all pairs $(X,A) \in \mathscr{P}^2$ such that

1) $K^*(X,A)$ is a ring,

2) $K^*(X,\varnothing)$ is a ring with unit,

3) $K^*(X,A)$ is a $K^*(X,\varnothing)$-module, and

4) $\delta : K^*(A,\varnothing) \to K^*(X,A)$ is a $K^*(X,\varnothing)$-module homomorphism.

For α and β in $K(X) = K(X,\varnothing)$, $\alpha\otimes\beta \in K(X)$ has been defined and makes $K(X)$ a ring with unit. If $\mathrm{pt} \xrightarrow{i} X \xrightarrow{p} \mathrm{pt}$ are the inclusion and projection maps on the base point, then for $\gamma \in K(X)$, $\gamma - p^*i^*\gamma \in \ker(i^*) = \widetilde{K}(X)$. $\ker(i^*)$ is an ideal in $K(X)$; i.e., $K(X,*)$ is a ring and is a $K(X,\varnothing)$-module. For $\alpha \in \widetilde{K}(X)$ and $\beta \in \widetilde{K}(Y)$, $\alpha\otimes\beta \in \widetilde{K}(X\#Y)$. For $\alpha \in \widetilde{K}^{-i}(X)$ and $\beta \in \widetilde{K}^{-j}(Y)$, let $\overline{\alpha} \in \widetilde{K}(\Sigma^i X)$ and $\overline{\beta} \in \widetilde{K}(\Sigma^j Y)$ be their suspensions. Then $\overline{\alpha}\otimes\overline{\beta} \in \widetilde{K}(\Sigma^i X\#\Sigma^j Y)$, and $\alpha\otimes\beta \in \widetilde{K}^{-i-j}(X\#Y)$ is defined to be the desuspension of $\overline{\alpha}\otimes\overline{\beta}$. Of course, we need only consider the values 0 and 1 for i and j. The usual permutations of $\mathbb{S}^i\#\mathbb{S}^j$ into

$\mathbb{S}^j \# \mathbb{S}^i$ show $\alpha \otimes \beta = (-1)^{ij} \beta \otimes \alpha$. Finally, to verify condition 4) of the pairing $\otimes$ we note the homotopy commutative diagramme

$$\begin{tikzcd}
\Sigma^{i+j}(X,A) \arrow[r] \arrow[d] & \Sigma^{i+j}(A,*) \arrow[r] & \Sigma^{i+j+1}(A\#A,*) \arrow[d] \\
\Sigma^{i+j}(X \times X, A \times X \cup X \times *) \arrow[d] & & \Sigma^{i+1}(A,*) \times \Sigma^j(A,*) \arrow[d] \\
\Sigma^i(X,A) \times \Sigma^j(X,*) \arrow[rr] & & \Sigma^{i+1}(A,*) \times \Sigma^j(X,*)
\end{tikzcd}$$

Proposition 3.3. *$K^*_{\mathbb{C}}$ is a multiplicative cohomology theory.*

3.1.2 The Multiplicative Transformation $\mathrm{ch} : K^*_{\mathbb{C}} \to H^{**}(-;\mathbb{Q})$

The domain of definition of the Chern character ch can be extended from complex vector bundles over a space X to $K(X)$ by linearity.

For $\alpha \in \widetilde{K}^{2i}(X)$, let $\mathrm{ch}(\alpha)$ equal $\mathrm{ch}(\overline{\alpha})$, where $\overline{\alpha}$ is the element of $\widetilde{K}(X)$ equivalent to α, and for $\alpha \in \widetilde{K}^{2i+1}(X)$, let $\mathrm{ch}(\alpha) = (\Sigma^*)^{-1}\,\mathrm{ch}(\overline{\alpha})$, where $\overline{\alpha}$ is the equivalent element of $\widetilde{K}(\Sigma X)$ and $(\Sigma^*)^{-1} : H^{**}(\Sigma X;\mathbb{Q}) \to H^{**}(X;\mathbb{Q})$ is desuspension.

Proposition 3.4. $\mathrm{ch} : K^*_{\mathbb{C}} \to H^{**}(-;\mathbb{Q})$ *is a multiplicative transformation.*

Proof. Since ch on $(\mathbb{S}^0,*)$ is a multiplicative, $\mathbb{Q}$-module valued functor, by Proposition 1.27, ch has a unique natural extension, which must be then the Chern character as above, and the extension is multiplicative. Furthermore, ch is normed as it preserves the units for $\mathbb{S}^1$. □

Proposition 3.5. $\mathrm{ch} : K^n_{\mathbb{C}}(X,A) \times \mathbb{Q} \to \sum_{i+j=n} H^i(X,A;K^j(\mathrm{pt}) \otimes \mathbb{Q})$ *is a natural equivalence. In particular,*

(a) $\mathrm{ch} : K^n_{\mathbb{C}}(X,A) \otimes \mathbb{Q} \xrightarrow{\cong} H^{even,*}(X,A;\mathbb{Q})$ *and*

(b) $\mathrm{ch} : K^1(X,A) \otimes < \mathbb{Q} \xrightarrow{\cong} H^{odd,*}(X,A;\mathbb{Q})$.

Proof. This is an immediate consequence of Corollaries 1.24 and 1.25 of §1.2. We note that the equivalence (a) is multiplicative. □

Let $E_r^{p,q}$ and $'E_r^{p,q}$ be the spectral sequences of the identity fibration $\mathrm{id} : X \to X$ in the $K^*_{\mathbb{C}}$ -, and $H^*(-;\mathbb{Q})$ - cohomology theories, respectively. Since ch is a ransformation of cohomology theories, it induces

$$\mathrm{ch}_r^{p,q} : E_r^{p,q} \to {}'E_r^{p,q},$$

a morphism of spectral sequences. At the E_2-level,

$$\mathrm{ch}_2^{p,q} : H^p(X;K^q(\mathrm{pt})) \to H^p(X;H^q(\mathrm{pt};\mathbb{Q}))$$

is trivial except for $q = 0$, where it is induced by the coefficient homomorphism $\mathbb{Z} \to \mathbb{Q}$ since there it is a natural transformation of ordinary singular cohomology theories, and does this for $X = \mathrm{pt}$.

The exact couple for the spectral sequence $E_r^{p,q}$ is given by $A^{p,q} = K^{p+q}(X^{(p)})$ and $C^{p+q} = K^{p+q}(X^{(p)}, X^{(p-1)})$. Thus $A^{p,q} \xrightarrow{\cong} A^{p,q+2}$ and $C^{p,q} \xrightarrow{\cong} C^{p,q+2}$; and so, there is an isomorphism of spectral sequences $E_r^{p,q} \xrightarrow{\cong} E_r^{p,q+2}$. Thus, in $\{E_r^{p,q}\}$, $d_r^{p,q} = d_r^{p,q+2}$. We also observe that $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is trivial for all even r since $E_r^{p,q} = 0$ for all odd q and one of q and $q - r + 1$ is odd when r is even.

Lemma 3.6. *If $X \in \mathscr{P}$ and $H^*(X;\mathbb{Z})$ is torsion-free, then the spectral sequence in the $K^*_{\mathbb{C}}$-cohomology theory of the identity fibration* $\mathrm{id} : X \to X$ *collapses; i.e., all differentials d_r are trivial.*

Proof. In this case $\mathrm{ch}_2^{p,0}$ is a monomorphism. The $'E$ sequences obviously collapse. Since $d_2 = 0$, $\mathrm{ch}_2^{p,0} = \mathrm{ch}_3^{p,0} : E_3^{p,0} \to {'E}_3^{p,0}$ is again a monomorphism. The differential $d_3^{p,q}$ is trivial for odd q ($E_2^{p,\mathrm{odd}} = 0$) and for q even, $d_3^{p,q}$ is equivalent to $d_3^{p,2}$. But ${'d}_3^{p,2} = 0$ and $\mathrm{ch}_3^{p+3,0}\, d_3^{p,2} = {'d}_3^{p,2}\,\mathrm{ch}_3^{p,2} = 0$; since $\mathrm{ch}_3^{p+3,0}$ is a monomorphism, $d_3^{p,q} = d_3^{p,2} = 0$. Then $d_4 = 0$ and $\mathrm{ch}_5^{p,0} = \mathrm{ch}_3^{p,0}$ is a monomorphism. $d_5^{p,q} = 0$ for q odd and $d_5^{p,q} = d_5^{p,4}$ for q even, etc. □

By the lemma, $E_2^{p,0} \cong E_\infty^{p,0}$ and

$$H^p(X;\mathbb{Z}) \cong F_{p-1}(K^p(X))/F_p(K^p(X))$$

is mapped monomorphically into $H^p(X;\mathbb{Q})$ by ch, where we recall that

$$F_i(K^p(X)) \equiv \ker(K^p(X) \to K^p(X^{(i)})).$$

This establishes

Theorem 3.7. *Suppose, for $X \in \mathscr{P}$, that $H^*(X;\mathbb{Z})$ is torsion-free. Then*

1) $\eta \in K^*_{\mathbb{C}}(X)$ is in $\ker(K^*_{\mathbb{C}}(X) \to K^*_{\mathbb{C}}(X^{(p-1)}))$ if and only if $\mathrm{ch}(\eta) = 0$ in all dimensions less than p,

2) $K^*_{\mathbb{C}}(X)$ is torsion-free and $\mathrm{ch} : K^*_{\mathbb{C}}(X) \to H^*(X;\mathbb{Q})$ is a monomorphism, and

3) $\mathrm{im}(H^p(X;\mathbb{Z}) \to H^p(X;\mathbb{Q}))$ is the set of Chern characters of elements of $K^*(X)$ in $\ker(K^*(X) \to K^*(X^{(p-1)}))$.

We note that 2) is a special case of a theorem of F. Peterson; we also note that Theorem 2.14 is an immediate consequence of 3) and is not used in its proof.

3.1.3 Cohomology Operations in $K_{\mathbb{C}}^*$

We describe operations in the $K_{\mathbb{C}}^*$-theory first used by Atiyah and Hirzebruch in [9] and developed and applied by Adams in [2]. The most effective way of doing this is by means of unitary representations, and we shall indicate this later on in another connection. For the purpose of the operations in $K_{\mathbb{C}}^*$-theory we have adequate equipment to proceed directly.

Theorem 3.8. *Let $(\mathbb{C}P^n)^k$ denote the Cartesian product of k-copies of $\mathbb{C}P^n$. If $\lambda \in K_{\mathbb{C}}((\mathbb{C}P^n)^k)$ is such that $\mathrm{ch}(\lambda) \in H^{**}((\mathbb{C}P^n)^k;\mathbb{Q})$ is in the ring of symmetric functions (in $u_1, \ldots, u_k$), then λ is the restriction of one and only one element of $K_{\mathbb{C}}(G_{k,kn})$, where*

$$G_{k,kn} = \mathrm{U}(k+kn)/(\mathrm{U}(k)\times\mathrm{U}(kn)) \quad \textit{and} \quad j : (\mathbb{C}P^n)^k \to G_{k,kn}$$

is induced by the decomposition

$$(\mathbb{C}\oplus\mathbb{C}^n)\oplus\cdots\oplus(\mathbb{C}\oplus\mathbb{C}^n) = (\mathbb{C}^k)\oplus(\mathbb{C}^{kn}).$$

Proof. The cofibration $(\mathbb{C}P^n)^k \xrightarrow{j} G_{k,kn} \xrightarrow{p} E$ induces the following ladder of exact sequences

$$\begin{tikzcd}
K^1(E)) \arrow[d, "\mathrm{ch}"'] & K^0((\mathbb{C}P^n)^k) \arrow[l, "\delta^*"'] \arrow[d, "\mathrm{ch}"'] & K^0(G_{k,kn}) \arrow[l, "j^*"'] \arrow[d, "\mathrm{ch}"'] & K^0(E) \arrow[l, "p^*"'] \arrow[d, "\mathrm{ch}"'] \\
H^{\mathrm{odd},*}(E;\mathbb{Q}) & H^{\mathrm{even},*}((\mathbb{C}P^n)^k;\mathbb{Q}) \arrow[l, "\delta^*"] & H^{\mathrm{even},*}(G_{k,kn};\mathbb{Q}) \arrow[l, "j^*"] & H^{\mathrm{even},*}(E;\mathbb{Q}) \arrow[l, "p^*"]
\end{tikzcd}$$

in which all the ch's are monomorphisms by 2) of Theorem 3.7. As $\mathrm{ch}(\lambda)$ is symmetric, by Theorem 2.5 it lies in $\mathrm{im}(j^*)$; and so, $\delta^*\,\mathrm{ch}(\lambda) = 0$. This implies $\mathrm{ch}\circ\delta^*(\lambda) = 0$, but since ch is a monomorphism, $\delta^*\lambda = 0$. Thus, there is a $\mu \in K(G_{k,kn})$ such that $j^*\mu = \lambda$. Since j^* on $H^{\mathrm{even},*}(G_{k,kn})$ is a monomorphism and $H^{\mathrm{odd},*}((\mathbb{C}P^n)^k) = 0$, $H^{\mathrm{even},*}(E;\mathbb{Q}) = 0$. ch being a monomorphism then implies $K^0(E) = 0$; and so, μ is unique. □

We now apply Theorem 3.8 to obtain a system of operations $\{\psi_t\}$ in $K_{\mathbb{C}}^*$ with properties to be stated. For t a positive integer, let $\Delta_t : \mathrm{U}(1) \to \mathrm{U}(1)^t$ be the t-fold diagonal. The composition

$$\mathrm{U}(1)^k \xrightarrow{(\Delta_t)^k} (\mathrm{U}(1)^t)^k \to (\mathrm{U}(t))^k \to \mathrm{U}(kt)$$

is a monomorphism, which induces a map

$$\lambda_t : (\mathbb{C}P^n)^k \to B_{\mathrm{U}(kt)}$$

Let λ_t also denote the element of $K_{\mathbb{C}}((\mathbb{C}P^n)^k)$ determined by the map λ_t. Since $\mathrm{ch}(\lambda_t) = \exp(tu_1)+\cdots+\exp(tu_k)$, there is a well-defined element $\mu_t \in K_{\mathbb{C}}(G_{k,kn})$ with $\mathrm{ch}_i(\mu_t) = t^i\,\mathrm{ch}_i(k)$, $\leq k$.

For $-t$ a negative integer, we let Δ_{-t} be the composition of Δ_t with complex conjugation. Then $\mathrm{ch}(\lambda_{-t}) = \exp(-tu_1) + \cdots + \exp(-tu_k)$, and $\mathrm{ch}_i(\mu_{-t}) = (-t)^i\, \mathrm{ch}_i(k), \leq k$. For the map $G_{k,kn} \times G_{\ell,\ell m} \to G_{k+\ell,(k+\ell)(m+n)}$ the class μ_t pulls back to $\mu_t^{'} \times 1 + 1 \times \mu_t^{''}$, and similarly for tensor products, with μ_t pulling back to $\mu_t^{'} \times \mu_t^{''}$.

For a k-plane complex vector bundle λ over X, there is an n such that λ is induced by a map

$$\lambda : X \to G_{k,kn}.$$

The composition

$$X \xrightarrow{\lambda} G_{k,kn} \xrightarrow{\mu_t} \mathbb{Z} \times B_{\mathrm{U}}$$

determines an element $\psi_t(\lambda) \in K_{\mathbb{C}}(X)$. It follows from the above paragraph that $\psi_t(\lambda \oplus \mu) = \psi_t(\lambda) \oplus \psi_t(\mu)$ and $\psi_t(\lambda \otimes \mu) = \psi_t(\lambda) \otimes \psi_t(\mu)$. For the trivial line bundle 1 over $\mathbb{S}1$, $\psi_t(1) = 1$ for all t. We extend the operations ψ_t to $K_{\mathbb{C}}(X)$ bv linearity and define ψ_t on $K_{\mathbb{C}}^{-1}(X)$ to be ψ_t on $K_{\mathbb{C}}(\Sigma X)$.

Theorem 3.9. *For each $t \in \mathbb{Z}$, there is a multiplicative transformation $\psi_t : K_{\mathbb{C}} \to K_{\mathbb{C}}$, such that*

1) $\mathrm{ch}_i(\psi_t(\lambda)) = t^i\, \mathrm{ch}_i(\lambda)$ *and*

2) $\psi_t(\psi_s) = \psi_{ts}$.

The conclusion 2) is a consequence of $\mathrm{ch}_i(\mu_t) = t^i\, \mathrm{ch}_i(k)$ and the fact that ch is a monomorphism on $K_{\mathbb{C}}(G_{k,kn})$.

3.1.4 $K_{\mathbb{C}}^*$-orientation of Complex Vector Bundles

We have restricted ourselves throughout to consider cohomology only of finite cell complexes; we shall continue to do so, but we frequently write limit complexes in place of skeletal approximations of suitably high dimensions.

Earlier we discussed the fibration

$$\mathbb{S}^{2n-1} \to B_{\mathrm{U}(n-1)} \xrightarrow{\pi} B_{\mathrm{U}(n)},$$

where π is induced by the standard inclusion $\mathrm{U}(n-1) \to \mathrm{U}(n)$. The mapping cylinder $M(\pi)$ of π is equivalent to the unit disc bundle $\mathbb{D}(\gamma^n)$ associated with the Universal n-plane bundle over $B_{\mathrm{U}(n)}$, and $B_{\mathrm{U}(n-1)}$ is the associated boundary sphere bundle $\mathbb{S}(\gamma^n)$. Thus

$$M\pi / B_{\mathrm{U}(n-1)} = \mathbb{D}(\gamma^n)/\mathbb{S}(\gamma^n) = B_{\mathrm{U}(n)}^{\gamma^n}.$$

We seek an orientation "bundle" on $B_{\mathrm{U}(n)}^{\gamma^n}$. The process will be to find an element of $K(B_{\mathrm{U}(1)}^n)$ which restricts trivially in $K(B_{\mathrm{U}(1)}^{n-1})$. Its extension to $K(B_{\mathrm{U}(n)})$, by Theorem 3.8, or equivalently to $K(M(n))$, then restricts trivially in $K(B_{\mathrm{U}(n-1)})$; and so, it comes from an element of $K(B_{\mathrm{U}(n)}^{\gamma^n})$, which in turn is uniquely determined since $K^1(B_{\mathrm{U}(n-1)}) = 0$.

Let $\alpha \in K(B^n_{\mathrm{U}(1)})$ be the element with $\mathrm{ch}(\alpha) = \prod_{i=}^n (1 - \exp(-u_i))$. (The virtual bundle α is the tensor product of a number of copies of the trivial line bundle minus the conjugate canonical line bundle.) Restricted to $B^{n-1}_{\mathrm{U}(1)}$, the symbol $\exp(-u_n)$ becomes 1; hence, this restriction of α has 0 Chern character and is itself the 0 of $K(B^{n-1}_{\mathrm{U}(1)})$.

Let λ be the extension of α to $K^*(B_{\mathrm{U}(n)}) = K^*(\mathbb{D}(\gamma^n))$. Since $\mathrm{ch}(\lambda|_{\mathbb{S}(\gamma^n)}) = 0$, $\lambda|_{\mathbb{S}(\gamma^n)} = 0$. Hence, λ is the image of an element $\mu \in K(B^{\gamma^n}_{\mathrm{U}(n)})$. As stated above, μ is uniquely determined by α.

Let φ_H denote the Thom-Gysin isomorphism in the $H^*(-;\mathbb{Q})$-theory. Then

$$\varphi_H(1) = \pi^*(\prod_{i=1}^n u_i).$$

We note this is a relative class; i.e., it lies in $H^*(B^{\gamma^n}_{\mathrm{U}(n)};\mathbb{Q})$. Also,

$$\mathrm{ch}(\mu) = \pi^*\left(\prod_{i=1}^n (1 - \exp(-u_i))\right).$$

Thus,

$$\mathrm{ch}(\mu) = \varphi_H\left(\prod_{i=1}^n \frac{\exp(-u_i) - 1}{-u_i}\right).$$

We observe that $\mathrm{ch}(\mu|_{\mathbb{S}^{2n}}) = 1 \cdot \overline{\mathbb{S}^{2n}}$; and so, μ is a $K_{\mathbb{C}}^*$-orientation class. Also

$$\mathrm{ch}(\gamma^n, 1) = \prod_{i=1}^n \frac{\exp(-u_i) - 1}{-u_i}$$

with this orientation.

Since γ^n is the Universal n-plane bundle, it follows that for any n-plane *complex* vector bundle α over X, $X \in \mathscr{P}$, α has a canonical $K_{\mathbb{C}}^*$-orientation and

$$\mathrm{ch}(\alpha, 1) = \prod_{i=1}^n \frac{\exp(-u_i) - 1}{-u_i}, \quad \text{where}$$

$$\prod_{i=1}^n (+u_i t) = 1 + C_1(\alpha)t + C_2(\alpha)t^2 + \cdots + C_n(\alpha)t^n,$$

where $C_i(\alpha)$ is the i-th Chern class of the bundle α.

If $\alpha \oplus \beta$ is a trivial bundle, then $\mathrm{ch}(\alpha, 1) \cup \mathrm{ch}(\beta, 1) = \mathrm{ch}(\alpha \oplus \beta, 1) = 1$ and

$$\mathrm{ch}(\beta, 1) = \prod_{i=1}^n \frac{-u_i}{\exp(-u_i) - 1}$$

Define polynomials with rational coefficients

$$T_k(C_1, \ldots, C_k) = \frac{(-1)^k}{k!} B_k^{(n)}(u_1, \ldots, u_n), \quad \text{for} \quad k \leq n,$$

where $C_i = \sigma_i^{(n)}(u_1, \ldots, u_n)$ and

$$\prod_{i=1}^{n} \frac{u_i x}{\exp(u_i x) - 1} = \sum_{j=0}^{\infty} \frac{x^j}{j!} B_j^{(n)}(u_1, \ldots, u_n).$$

Then

$$\mathrm{ch}_i(\beta, 1) = T_i(C_1, \ldots, C_i),$$

where the C_j's are the Chern classes of α. The polynomials $T_k(C_1, \ldots, C_k)$ are called the *Todd polynomials*. As a shorthand we write

$$\mathrm{ch}(\alpha, 1) = 1 + T_1(-\alpha) + T_2(-\alpha) + T_3(-\alpha) + \cdots .$$

Note that

$$(1+T_1(\alpha)+T_2(\alpha)+\cdots) \cup (1+T_1(\beta)+T_2(\beta)+\cdots) = 1+T_1(\alpha\oplus\beta)+T_2(\alpha\oplus\beta)+\cdots .$$

A manifold is *weakly almost complex* if for some trivial bundle n, the structure group of $\tau\oplus n$ has a specific unitary reduction. In particular, almost complex manifolds, i.e., smooth manifolds equipped with smooth linear complex structures on tangent spaces, are weakly almost complex.

If M is weakly almost complex, then it has a canonical $K_{\mathbb{C}}^*$-orientation, as described above, and

$$\begin{aligned} \mathrm{ch}(M, 1) &= 1 + T_1(\tau) + T_2(\tau) + \cdots \\ &\equiv T_*(\tau). \end{aligned}$$

Theorem 3.10. *If $f : M \to N$ is a continuous map of weakly almost complex manifolds, then for $\lambda \in K_{\mathbb{C}}^*(M)$*

$$f_!^{H^*}(\mathrm{ch}(\lambda) \cup T_*(\tau(M)) = \mathrm{ch}(f_!^{K^*}(\lambda)) \cup T_*(\tau(N)).$$

Of course, if $\nu(f)$ is $K_{\mathbb{C}}^*$-orientable, there is a similar result, using Theorem 1.55 instead of Theorem 1.56. We will later describe a more general result involving orienting real vector bundles with $w_1 = w_3 = 0$. This will yield the best form of the above theorem.

Chapter 4

Some Geometric Applications

4.1 Vector Bundles over Cell Complexes $\mathbb{S}^n \cup e^m$

4.1.1 Two technical lemmas

The exponent of the prime p in $k!$ does not exceed $\left[\frac{k-1}{p-1}\right]$ and equals $\frac{k-1}{p-1}$ if and only if k is a power of p. Let $f = p^{1/(p-1)}$; for r a rational number with no p in the denominator, $f^m r \equiv 0 \bmod p$ for $m > 0$. Then

$$\frac{\exp(-fu)-1}{-fu} = \sum_{k=1}^{\infty} \frac{(-fu)^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{(-u)^{k-1} k!^{\frac{k-1}{p-1}}}{p}$$

$$\equiv \sum_{i=0}^{\infty} (-u)^{p^i-1} \frac{p^{\frac{p^i-1}{p-1}}}{(p^i)!} \mod p \equiv \sum_{i=0}^{\infty} (-1) u^{i^i} \mod p,$$

since by Wilson's Theorem

$$(p^i)!/p^{\frac{p^i-1}{p-1}} \equiv (p-1)!^{\frac{p^i-1}{p-1}} \equiv (-1)^{\frac{p^i-1}{p-1}} \equiv (-1)^i \mod p.$$

For the canonical line bundle λ over $B_{\mathrm{U}(1)}$, $\mathscr{P}^{-1}(\lambda, 1) = \frac{1}{u}\mathscr{P}^{-1}(u)$. But $\mathscr{P}(u) = u + u^p$; and so, $\mathscr{P}(u - u^p + u^{p^2} - \ \dots) = u$; i.e, $\mathscr{P}^{-1}(u) = \sum_{i=0}^{\infty} (-1)^i u^{p^i}$. Thus,

$$\mathscr{P}^{-1}(\lambda, 1) \equiv \frac{\exp(-fu)-1}{-fu} \mod p.$$

Over $B^n_{\mathrm{U}(1)}$, we then have

$$\mathscr{P}^{-1}(\lambda ?_1 \times \cdots \times \lambda n, 1) \equiv \prod_{i=1}^{n} \frac{\exp(-fu_i)-1}{-fu_i} \mod p.$$

Hence,

$$\mathscr{P}^{-1}(\gamma^n, 1) \equiv \sum_{j=0}^{\infty} f^j \, \mathrm{ch}_j(\gamma^n, 1) \mod p,$$

where $\mathrm{ch}(\gamma^n, 1)$ is as defined in §§3.1.4. We have seen earlier that if $\gamma + \mu$ is trivial, then

$$\mathscr{P}^{-1}(\gamma, 1) = \mathscr{P}^{-1}(\mathscr{P}(\mu, 1)).$$

Since $\mathrm{ch}_j(\gamma, 1) = T_j(\mu)$, we have for any complex vector bundle the

Lemma 4.1.

$$\mathscr{P}^{-1}(\mathscr{P}(\mu, 1)) \equiv \sum_{j=0}^{\infty} T_j(C_1(\mu), \ \ldots, \ C_j(\mu)) \mod p.$$

We cite the following result from Hirzebruch ([10], p. 13 of [23]):

Proposition 4.2. *For the power series $f(u)$, the sum of terms of degree k, $k \le n$, of $\prod_{i=1}^n f(u_i)$ is a symmetric function in $u_1, \ \ldots, \ u_n$ and thus equals a polynomial $F_k(\sigma_1^{(n)}, \ldots, \ \sigma_k^{(n)})$. The coefficient s_k, of $\sigma_k^{(n)}$, in F_k, is given by the formula*

$$f(u)\frac{d}{du}\left(\frac{u}{f(u)}\right) = \sum_k (-1)^k s_k u^k.$$

As an application of the Proposition consider

$$f(u) = \frac{-u}{\exp(-u) - 1} = 1 + \frac{u}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} u^{2k}.$$

Then $F_k(\sigma_1^{(n)}, \ \ldots, \ \sigma_k^{(n)}) = T_k(C_1 \ \ldots, \ C_k)$ for $k \le n$ and

$$\frac{-u}{\exp(-u) - 1}\frac{d}{du}\left(u/\frac{-u}{\exp(-u) - 1}\right) = \frac{u}{\exp(u) - 1} = f(-u).$$

Thus,

Lemma 4.3. *The coefficient of C_{2k} in $T_{2k}(C_1 \ \ldots, \ C_{2k})$ is $(-1)^{k-1}\frac{B_k}{(2k)!}$.*

4.1.2 Divisors of Orders of Stable Homotopy Classes, J-homomorphisms

Suppose $H^{2n}(X; \mathbb{Z}) = 0$ and $f : \mathbb{S}^{2n-1} \to X$. Let r denote the order of the class $(f) \in \pi_{2n-1}(X)$ and also let $r : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ denote a map of degree r. Then $f \circ r \simeq 0$ and extends to $\overline{f} : C_-\mathbb{S}^{2n-1} \to X$. Then the map

$$g = \overline{f} \cup C_+ r : \mathbb{S}^{2n} = (C_-\mathbb{S}^{2n-1} \cup C_+\mathbb{S}^{2n-1}) \to X \cup_f C_+\mathbb{S}^{2n-1} = X \cup_f e^{2n}$$

composed with the collapsing map $p : X \cup_f e^{2n} \to \mathbb{S}^{2n}$ has degree r.

Consider the diagram below.

$$\begin{tikzcd}
K(\mathbb{S}^{2n}) \arrow[r, "p^*"] \arrow[d, "\mathrm{ch}"'] & K(X \cup_f e^{2n}) \arrow[r, "i^*"] \arrow[d, "\mathrm{ch}"] \arrow[rr, bend left, "g^*"] & K(X) \arrow[d, "\mathrm{ch}"] & K(\mathbb{S}^{2n}) \arrow[d, "\mathrm{ch}"] \\
H(\mathbb{S}^{2n};\mathbb{Q}) \arrow[r, "p^*"'] & H(X \cup_f e^{2n};\mathbb{Q}) \arrow[r, "i^*"'] \arrow[rr, bend right, "g^*"'] & H(X;\mathbb{Q}) & H(\mathbb{S}^{2n};\mathbb{Q})
\end{tikzcd}$$

For $\lambda \in K(X \cup_f e^{2n})$, $i^* \operatorname{ch}(\lambda) = 0$ and so $\operatorname{ch}(\lambda) = \frac{\alpha}{\beta} p^*(\mathbb{S}^{2n})$, where $\alpha/beta$ is a rational number and $p^*(\mathbb{S}^{2n})$ is the image of the integral class $p^*(\mathbb{S}^{2n})$ injected in the rational cohomology. Since $g^* \circ p^*$ has degree r in dimension $2n$ and p^* has degree 1, $\operatorname{ch}(g^*\lambda) = r \cdot \frac{\alpha}{\beta} \cdot (\mathbb{S}^{2n})$, which is an integral multiple of $(\mathbb{S}^{2n})$ since $g^*\lambda$ is a virtual bundle over $\mathbb{S}^{2n}$

Proposition 4.4. *If $H^{2n}(X;\mathbb{Z}) = 0$, the order of $(f) \in \pi_{2n-1}(X)$ is r, $p : X \cup_f e^{2n} \to \mathbb{S}^{2n}$ is the collapsing map, and $\lambda \in K(X \cup_f e^{2n})$, then $\operatorname{ch}_n(\lambda) = \frac{\alpha}{\beta} p^*(\mathbb{S}^{2n})$ and if α and β are relatively prime, β divides r. Furthermore, this is stably true.*

If, moreover, λ is an even-dimensional sphere $\mathbb{S}^{2m}$ of lower dimension, the diagram shows that for bundles λ with $\operatorname{ch}(\lambda) = 1 \cdot (\mathbb{S}^{2n})$, the number $\frac{\alpha}{\beta}$ is uniquely determined mod 1. This defines a homomorphism

$$e : \pi_{2n-1}(\mathbb{S}^{2n}) \to \mathbb{Q}/\mathbb{Z}$$

which has a number of important applications due to Frank Adams.

For $f : \mathbb{S}^{4k-1} \to \mathrm{U}(n)$, $4k < 2n-1$, the composition

$$\mathbb{S}^{4k-1} \xrightarrow{f} \mathrm{U}(n) \hookrightarrow M(\mathbb{S}^{2n-1}, \mathbb{S}^{2n-1}) \hookrightarrow \Omega^{2n}\mathbb{S}^{2n},$$

where $M(\mathbb{S}^{2n-1}, \mathbb{S}^{2n-1})$ is the space of maps of $\mathbb{S}^{2n-1}$ into itself of degree 1, defines an element $J_C((f)) \in \pi_{2n+4k-1}(\mathbb{S}^{2n})$, $J_C : \pi_{4k-1}(\mathrm{U}(n)) \to \pi_{2n+4k-1}(\mathbb{S}^{2n})$ is the complex J-homomorphism.

Lemma 4.5. *In stable dimension $4k-1$, the order of the image of the complex J-homomorphism is divisible by the denominator of $\frac{B_k}{2k}$ expressed in lowest form.*

Proof. If $g : \mathbb{S}^{4k} \to B_{\mathrm{U}(n)}$ is the bundle defined by f, then $(\mathbb{S}^{4k})^g$ is a cell-complex with cells e^0, e^{2n}, e^{2n+4k}, and e^{2n+4k} is attached to $\mathbb{S}^{2n}$ by the map $J_C((f))$.

Let $f : \mathbb{S}^{4k-1} \to \mathrm{U}(n)$ generate $\pi_{4k-1}(\mathrm{U}(n)) \cong \mathbb{Z}$. Then $\langle < \mathbb{S}^{4k}, \operatorname{ch}_{2k}(g)\rangle = \pm 1$. Since $\operatorname{ch}_{2k} = \frac{p_{2k}}{(k))!}$ and $p_{2k} - p_{2k-1}C_1 + \cdots + (2k)C_{2k} = 0$ (see the proof of Theorem 2.14),

$$\langle \mathbb{S}^{4k}, \operatorname{ch}_{2k}(g)\rangle = \pm(2k-1)!.$$

For the map $\widetilde{f} : (\mathbb{S}^{4k})^g \to B^{\gamma^n}_{\mathrm{U}(n)}$,

$$\langle e^{2n+4k}, \widetilde{f}^*(\varphi(C_{2k}))\rangle = \pm(2k-1)!.$$

(Recall that $\langle e^{2n+4k}, \varphi(\overline{\mathbb{S}^{2n}})\rangle = 1$.) There is an element μ in $K(B^{\gamma^n}_{\mathrm{U}(n)})$ with

$$\mathrm{ch}(\mu) = \varphi(T_*(C_1, \ \dots)).$$

Thus,

$$\begin{aligned}\langle e^{2n+4k}, \mathrm{ch}_{n+2k}(\widetilde{f}^*\mu)\rangle &= \langle e^{2n+4k}, \widetilde{f}^*\varphi T_{2k}(C_1, \ \dots, \ C_{2k}))\rangle \\ &= \pm\langle e^{2n+4k}, \widetilde{f}^*\varphi(\frac{B_k}{(2k)!}C_{2k}))\rangle \\ &= \pm\frac{B_k}{(2k)}.\end{aligned}$$

□

In fact the complex J-homomorphism factors, since $\mathrm{U}(n) \xrightarrow{i} \mathrm{SO}(2n) \to M(\mathbb{S}^{2n-1}, \mathbb{S}^{2n-1})$; and so, for $J_R : \pi_{4k-1}(\mathrm{SO}(2n)) \to \pi_{2n+4k-1}(\mathbb{S}^{2n})$ the real J-homomorphism, we have $J_C = i\# \circ J_R$.

By somewhat more involved techniques than those which sufficed to establish periodicity for the unitary group, it may be shown that the orthogonal group is also periodic, of period 8 with $\Omega^8 B_{\mathrm{O}} \simeq B_{\mathrm{O}}$, and the first eight homotopy groups of O are $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$. (See [22]). The proof also shows that

$$i_* : \pi_{4k-1}(\mathrm{U}) \to \pi_{4k-1}(\mathrm{O})$$

is onto for k odd and onto $\mathbb{Z}$ for k even. Thus we have

Theorem 4.6. *In stable dimensions $4k-1$, the order of the image of the real J-homomorphism is divisible by the denominator of $\frac{a_k B_k}{4k}$ expressed in lowest form, where a_k, is 1 for k even and is 2 for k odd.*

4.1.3 Maps of Hopf Invariant One

The map $f : \mathbb{S}^{2n+2k-1} \to \mathbb{S}^{2n}$ is said to have mod p *Hopf invariant* 1 if in the cell-complex $X(f) = \mathbb{S}^{2n} \cup_f e^{2n+2k}$ the Steenrod operation $\mathscr{P}^i$ acts non-trivially, where $k = i(p-1)$. For $p = 2$ we understand by $\mathscr{P}^i$ the operation Sq^{2i}.

Theorem 4.7. *If f has* mod *p Hopf invariant* 1*, then*

$$k = \begin{cases} 1, 2, \ \text{or } 4 & \text{if } p = 2, \ \text{and} \\ p-1 & \text{if } p \text{ is an odd prime.}\end{cases}$$

We note, but do not prove, that the converse is also true. There are known several proofs of this theorem. The one given here uses part of a theorem of Milnor's [25] on the stable homotopy of the Thom complexes $G^{\gamma^n}_{n,\ell}$. The only proof of this theorem is too far from the spirit of these notes to be included here; it would be nice indeed to have a better proof.

Under the inclusion $G_{n,\ell} \to G_{n+1,\ell+1}$, the bundle γ^{n+1} pulls back to $1_{\mathbb{C}} \oplus \gamma^n$; thus, a map $\Sigma^2 G_{n,\ell}^{\gamma^n} \to G_{n+1,\ell+1}^{\gamma^{n+1}}$ induced. This defines a spectrum MU. Milnor proves in [25] that

$$H^{-*}(\mathrm{pt}; MU) = \begin{cases} 0, & \text{in odd dimensions} \\ \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}, & \pi(n) \text{ copies in dimension } 2n. \end{cases}$$

These groups are the complex cobordism groups. We need only the result that they are 0 in odd dimensions.

Proof. (of Theorem 4.7) $f : \mathbb{S}^{2n+2k-1} \to \mathbb{S}^{2n}$ has mod p *Hopf invariant* 1. We may assume $k < n$ since $\Sigma^2 X(f) \simeq X(\Sigma^2 f)$ and Steenrod operations commute with suspension.

By Milnors theorem, for ℓ large the homotopy groups of $G_{n,\ell}^{\gamma^n}$ are 0 in odd dimensions less than $2n-2$. Thus, the map $\mathbb{S}^{2n} \to G_{n,\ell}^{\gamma^n}$ of degree 1 can be extended to a map $g : X(f) \to G_{n,\ell}^{\gamma^n}$. (The only obstruction lies in $H^{2n+2k}(X(f); \pi_{2n+2k-}(G_{n,\ell}^{\gamma^n}) = 0.)$

In the proof of Lemma 4.1 it was shown that in $G_{n,\ell}^{\gamma^n}$,

$$\mathscr{P}^{-1}\rho_*\varphi_H(1) = \varphi_H\{\rho_* \sum_j f^j \operatorname{ch}_j(\gamma^n, 1)\},$$

where ρ_* is mod p coefficient reduction and $f = p^{\frac{1}{p-1}}$. Denote the component of $\mathscr{P}^{-1}$ in dimension $2k = 2i(p-1)$ by $\mathscr{X}(\mathscr{P}^i)$. $\mathscr{X}(\mathscr{P}^i) = -\mathscr{P}^i +$ decomposable elements. Then

$$\begin{aligned} \mathscr{X}(\mathscr{P}^i)\rho_*\varphi_H(1) &= \varphi_H \rho_*\{p^i \operatorname{ch}_k(\gamma^n, 1)\} \\ &= \rho_*\{p^i \operatorname{ch}_{n+k}(\varphi_K(1))\}. \end{aligned}$$

The virtual bundle $\varphi_K(1)$ over $G_{n,\ell}^{\gamma^n}$ pulls back to a virtual bundle $\mu = g^*\varphi_K(1)$ over $X(f)$. Then

$$\begin{aligned} \rho_*\{p^i \operatorname{ch}_{n+k}(\mu)\} &= \rho_*\{p^i \operatorname{ch}_{n+k}(g^*\varphi_K(1))\} \\ &= g^*\rho_*\{p^i \operatorname{ch}_{n+k}(\varphi_K(1))\} \\ &= g^*\mathscr{X}(\mathscr{P}^i)\rho_*\varphi_H(1) \\ &= \mathscr{X}(\mathscr{P}^i)\rho_* g^*\varphi_H(1) \\ &= \mathscr{X}(\mathscr{P}^i)\rho_*(\overline{\mathbb{S}^{2n}}) \\ &= -\mathscr{P}^i\rho_*(\overline{\mathbb{S}^{2n}}) \\ &\neq 0. \end{aligned}$$

But $\operatorname{ch}_{n+k}(\mu) = q \cdot \overline{g}_{2n+2k}$, where q is a rational number; and so, $p^i \cdot q \not\equiv 0 \mod p$; i.e., $q = a/(p^i \cdot b)$, where a and b are prime to p. Thus,

$$\begin{aligned} \operatorname{ch}_n(\mu) &= \overline{\mathbb{S}^{2n}} \\ \operatorname{ch}_{n+k}(\mu) &= (a/(p^i b))\overline{g}_{2n+2k}, \quad \mu \in K(X(f)). \end{aligned}$$

For t a positive integer,

$$\mathrm{ch}_n(\psi_t(\mu) - t^n\mu) = t^n\,\mathrm{ch}_n(\mu) - t^n\,\mathrm{ch}_n(\mu) = 0.$$

Thus, $\mathrm{ch}_{n+k}(\psi_t(\mu) - t^n\mu)$ is an integral multiple of $\overline{g}_{2n+2k}$.

$$\begin{aligned}\mathrm{ch}_{n+k}(\psi_t(\mu) - t^n\mu) &= (t^{n+k} - t^n)\,\mathrm{ch}_{n+k}(\mu)\\ &= \frac{t^n(t^k-1)\cdot a}{p^i\cdot b}\overline{g}_{2n+2k}\end{aligned}$$

Recalling that $k = i(p-1)$, the above expression is integral for t prime to p only if

$$t^{i(p-1)} \equiv 1 \bmod p^i.$$

Thus, if f has mod p Hopf invariant 1, then

$$t^{i(p-1)} \equiv 1 \bmod p^i.$$

for all integers t prime to p. It is a theorem of elementary number theory [27] that this is true only in the cases stated. □

More explicitly, for odd p there are numbers t prime to p such that

$$t^{\varphi(p^i)} \equiv 1 \bmod p^i.$$

but for no smaller exponent does this congruence hold. φ is the Euler totient function and in particular $\varphi(p^i) = p^{i-1}(p-1)$. For $i = 1$, $\varphi(p) = p-1$ and $t^{p-1} \equiv 1 \bmod p$ is a congruence of Fermat. For $i > 1$, $i\cdot(p-1) < p^{i-1}(p-1)$ and the necessary congruence does not hold. For $p = 2$, if t is odd,

$$\begin{array}{ll} t^1 \equiv 1 \bmod 2^1 & t^3 \not\equiv 1 \bmod 2^3\\ t^2 \equiv 1 \bmod 2^2 & t^4 \equiv \bmod 2^4\end{array}$$

For $i \geq 3$, $3^{2^{i-2}} \equiv 1 \bmod 2$, but this is the least such exponent. Thus, $3^i \not\equiv 1 \bmod 2^i$ if $3 \leq i < 2^{i-2}$; i.e., if $5 \leq i$.

4.2 Toda Brackets

In homotopy theory, as in cohomology theory, higher order operations play an important role. In homotopy, composition is the primary operation. Secondary operations are "Toda brackets" defined when in the diagramme

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D,$$

$g\circ f \simeq 0$ and $h\circ g \simeq 0$. For each $F : CB \to D$ extending $h\circ g$ and $G : CA \to C$ extending $g\circ f$, $\circ Tf$ and $h\circ G$ define a map $\Sigma A \to D$. The totality of homotopy classes of maps $\Sigma A \to D$ obtained by all choices of null homotopies F and G of $h\circ g$ and $g\circ f$ in this way is the Toda bracket $\{h, g, f\} \subset [\Sigma A, D]$. If $[\Sigma A, D]$ is abelian, $\{h, g, f\}$ is a coset determined by $\{h\}\circ[\Sigma A, C] + [\Sigma B, D]\circ\Sigma\{f\}$.

The next proposition was discovered by J. F. Adams.

Proposition 4.8. *Suppose $f : \mathbb{S}^a \to \mathbb{S}^b$ and $g : \mathbb{S}^b \to \mathbb{S}^c$ are maps for which there is an integer m with $(m\iota) \circ f \simeq 0$ and $g \circ (m\iota) \simeq 0$. Then*

$$e(\{g, m\iota, f\}) = me(f)e(g) \bmod 1.$$

Proof. Let $F : C\mathbb{S}^a \to \mathbb{S}^b$ and $G : C\mathbb{S}^b \to \mathbb{S}^c$ be homotopies of $(m\iota) \circ f \simeq 0$ and $g \circ (m\iota) \simeq 0$, respectively. These define maps $\widetilde{F} : \mathbb{S}^b \cup_f e^{a+1} \to \mathbb{S}^b$ and $\widetilde{G} : \mathbb{S}^b \cup_f e^{b+1} \to \mathbb{S}^c$ so that the diagrammes

$$\begin{tikzcd} \mathbb{S}^b \arrow[d] \arrow[dr, "m\iota"] & \\ \mathbb{S}^b \cup_f e^{a+1} \arrow[r, "\widetilde{F}"'] & \mathbb{S}^b \\ C\mathbb{S}^a \arrow[u] \arrow[ur, "F"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{S}^b \arrow[d] \arrow[dr, "g"] & \\ \mathbb{S}^b \cup_m e^{b+1} \arrow[r, "\widetilde{G}"'] & \mathbb{S}^c \\ C\mathbb{S}^b \arrow[u] \arrow[ur, "G"'] & \end{tikzcd}$$

commute. These maps induce

$$\overline{F} : \mathbb{S}^{a+1} \to \mathbb{S}^b \cup_m e^{b+1} \quad \text{and} \quad \overline{G} : \mathbb{S}^{b+1} \to \mathbb{S}^c \cup_g e^{b+1}.$$

There are maps

$$\alpha : \mathbb{S}^c \cup_{\widetilde{G}\overline{F}} e^{a+2} \to (\mathbb{S}^c \cup_g e^{b+1}) \cup_{\overline{G}\Sigma f} e^{a+2}$$

induced by the inclusion $\mathbb{S}^c \to \mathbb{S}^c \cup_g e^{b+1}$, (the attachment $\widetilde{G}\overline{F}$ goes into $\overline{G}\Sigma f$ under this inclusion), and

$$\beta : \mathbb{S}^{b+1} \cup_{\Sigma f} e^{a+2} \to (\mathbb{S}^c \cup_g e^{b+1}) \cup_{\overline{G}\Sigma f} e^{a+2}$$

induced by $\overline{G}$. Since $\mathbb{S}^c \cup_g e^{b+1} \to \mathbb{S}^{b+1}$ has degree 1 in dimension $b+1$, $\overline{G}$ has degree m in dimension $b+1$, as then does β. Also, β has degree 1 in dimension $a+2$.

Let τ be a virtual bundle over $(\mathbb{S}^c \cup_g e^{b+1}) \cup_{\overline{G}\Sigma f} e^{a+2}$ with $\mathrm{ch}(\tau) = 1 \cdot \overline{\mathbb{S}^c}$. By restricting τ to $\mathbb{S}^c \cup_g e^{b+1}$, we see that $\mathrm{ch}(\tau) = e(g)$. The map α has degree 1 in dimensions c and $a+2$, so $\mathrm{ch}_{a+2}(\alpha^*\tau) = e(\{g, m\iota, f\})$. Finally, we have then that

$$\mathrm{ch}_{b+1}(\beta^*\tau) = me(g) \quad \text{and} \quad \mathrm{ch}_{a+2}(\beta^*\tau) = e(\{g, m\iota, f\}).$$

There is a virtual bundle μ on $\mathbb{S}^{b+1} \cup_{\Sigma f} e^{a+2}$ with

$$\mathrm{ch}_{b+1}(\mu) = 1 \quad \text{and} \quad \mathrm{ch}_{a+2}(\mu) = e(\Sigma f).$$

Then $\mathrm{ch}_{b+1}(me(g)\mu - \beta^*\tau) = 0$; and so,

$$\mathrm{ch}_{a+2}(me(g)\mu - \beta^*\tau) = me(\Sigma f)e(g) - e(\{g, m\iota, f\})$$

is an integer; i.e.,

$$e(\{g, m\iota, f\}) \equiv me(\Sigma f)e(g) \bmod 1.$$

□

This result does not depend on the choices of F and G and thus it does not depend on the representative of the Toda bracket.

As an example of the utility of result, there is an element α_1 of order p, an odd prime, in the image of the J-homomorphism in the stable stem $2p-3$ with $e(\alpha_1) = \frac{1}{p}$. Defining inductively

$$\begin{aligned} \alpha_k &= \{\alpha_1, p\iota, \alpha_{k-1}\}, \quad \text{we have} \\ e(\alpha_k) &\equiv pe(\alpha_1)e(\alpha_{k-1}) \bmod 1 \\ &\equiv p \cdot \frac{1}{p}\frac{1}{p} \bmod 1 \\ &\equiv \frac{1}{p} \bmod 1. \end{aligned}$$

This then defines a recurrent family of elements in the stable stems. M. Barratt has used a refinement of this process to determine completely the image of the real J-homomorphism [to appear].

Appendices

The appendices A - C and E are provided by Eldon Dyer, the original authour of these notes.

Other appendices are provided by the transcriber for the reader's convenience.

Appendix A

The object of this appendix is to indicate briefly properties of the cohomology theory $K^*_{\mathbb{R}}$ derived from stable equivalence classes of real vector bundles, relations between this theory and the $K^*_{\mathbb{C}}$-theory, and conditions for $K^*_{\mathbb{R}}$-orientability of bundles. To discuss the latter it is necessary to recall various theorems on representations of Lie groups.

A.1 The Cohomology Theory $K^*_{\mathbb{R}}$

As in §3.1, but using equivalence classes of real vector bundles instead of complex ones, we define the commutative ring with unit $K_{\mathbb{R}}(X)$ for each space X. Replacing real for complex throughout that section, we have

$$K_{\mathbb{R}}(X) \cong [X, \mathbb{Z} \times B_{\mathrm{O}}] \quad \text{for} \quad X \in \mathscr{P}$$

and for $(X, x_0) \in \mathscr{P}_0$ the reduced group

$$\widetilde{K}_{\mathbb{R}}(X)$$

is the kernel of $K_{\mathbb{R}}(X) \longrightarrow K_{\mathbb{R}}(x_0)$ induced by the $x_0 \longrightarrow X$. The cohomology theory $K^*_{\mathbb{R}}(-)$ is then defined using the spectrum B_{O}. This spectrum is somewhat more complicated than B_{U}, having period 8 instead of 2. The following sequence of homotopy equivalences is valid:

$$\begin{array}{cc}
B_{\mathrm{O}} \simeq \Omega(\mathrm{U}/\mathrm{O}) & B_{\mathrm{Sp}} \simeq \Omega(\mathrm{U}/\mathrm{Sp}) \\
\mathrm{U}/\mathrm{O} \simeq \Omega(\mathrm{Sp}/\mathrm{O}) & \mathrm{U}/\mathrm{Sp} \simeq \Omega(\mathrm{O}/\mathrm{U}) \\
\mathrm{Sp}/\mathrm{O} \simeq \Omega(\mathrm{Sp}) & \mathrm{O}/\mathrm{U} \simeq \Omega(\mathrm{O}) \\
\mathrm{Sp} \simeq \Omega(B_{\mathrm{Sp}}) & \mathrm{O} \simeq \Omega B_{\mathrm{O}}
\end{array}$$

This was proved by R. Bott in his celebrated paper

- Raoul Bott, *The Stable Homotopy of the Classical Groups*, Raoul Bott, Ann of Math. Second Series, Vol. 70, No. 2 (Sep., 1959), pp. 313-337.

This proof used deep results in Morse theory. A slightly more accessible account is

- J. W. Minor, *Morse Theory*, Princeton University Press, 1963.

From the viewpoint of the algebraic topologist, more ordinary proofs have been given by J. C. Moore and the author and R. K. Lashof [22].

In terms of this sequence, one defines the spectrum B_{O} and obtains the cohomology theory $K^*_{\mathbb{R}}(-)$. Clearly, this theory is periodic of period 8 and $\widetilde{K}^0_{\mathbb{R}}(-) = \widetilde{K}_{\mathbb{R}}(-)$. This theory is also multiplicative, and this proved in much the same way as it was shown that $K^*_{\mathbb{C}}$ is multiplicative. The "coefficient groups" are given by the homotopy of B_{O}, which are periodic of period 8 and are

i	0	1	2	3	4	5	6	7
$\pi_1(B_{\mathrm{O}})$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0

To determine the induced maps in homotopy of the various inclusions, etc: $\mathrm{O} \to \mathrm{U}$, $\mathrm{U} \to \mathrm{O}$, $\mathrm{U} \to \mathrm{Sp}$ and $\mathrm{Sp} \to \mathrm{U}$, it suffices to recall only that the quotients lie among the loop spaces of B_{O}. For example:

$$
\begin{array}{llll}
 & \mathrm{U} \longrightarrow & \mathrm{O} \longrightarrow & \mathrm{O/U} \\
1 & \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\
2 & 0 & 0 & \mathbb{Z} \\
3 & \mathbb{Z} & \mathbb{Z} & \\
4 & 0 & 0 & \text{(either 0 or } \mathbb{Z}/2\mathbb{Z}, \text{ but } \mathbb{Z}/2\mathbb{Z} \text{ is impossible)} \\
5 & \mathbb{Z} & 0 & 0 \\
6 & 0 & 0 & \mathbb{Z} \\
7 & \mathbb{Z} & \mathbb{Z} & \\
8 & \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} &
\end{array}
$$

The table of groups and homomorphisms then is completed from the homotopy exact sequence and the knowledge that the groups of O/U are those of O shifted some degree.

Complexification of real bundles ($\mathbb{R}^n \otimes \mathbb{C} \to \mathbb{C}^n$) induces the inclusion $\mathrm{O} \to \mathrm{U}$ and

$$c : B_{\mathrm{O}} \to B_{\mathrm{U}}.$$

This defines an additive homomorphism $c : K_{\mathbb{R}}(-) \to K_{\mathbb{C}}(-)$, and a map $C : \mathscr{B}_{\mathrm{O}} \to \mathscr{B}_{\mathrm{U}}$ of spectra. Complexification in fact commutes with tensor products of bundles; and so,

$$c : K_{\mathbb{R}}(-) \to K_{\mathbb{C}}(-)$$

is a multiplicative transformation of cohomology theories.

Conjugation of the unit circle has degree -1; this implies that for λ the canonical line bundle over $\mathbb{C}P^\infty$, $C_1(\overline{\lambda}) = -C_1(\lambda)$. Since Chern classes are defined in terms of elementary symmetric functions on $C_1(\lambda)$ (Theorem 2.5), for $\lambda \in K^*_{\mathbb{C}}(X)$,

$$C_i(\overline{\lambda}) = (-1)^i C_i(\lambda).$$

For $\lambda \in K^*_{\mathbb{R}}(X)$, $\overline{c(\lambda)} = c(\lambda)$.

For $\lambda \in K_{\mathbb{R}}(X)$, let $\widehat{P}(\lambda) = C(c(\lambda))$, where C is the total Chern class. For i odd,

$$C_i(c(\lambda)) = C_i(\overline{c(\lambda)}) = (-1))^i C_i(c(\lambda)) = -C_i(c(\lambda));$$

thus, $C_{\mathrm{odd}}(c(\lambda))$ is at most of order two. Define

$$p_i(\lambda) = (-1)^i C_{2i}(c(\lambda)).$$

p_i is the i-th *Pontrjagin class* of λ; it is an element of $H^{4i}(X;\mathbb{Z})$.

Complex vector bundles can be made real simply by neglecting the complex structure. More precisely, there is a homomorphism $\mathrm{U}(n) \to \mathrm{O}(2n)$ defined by sending a unitary matrix $A = B + iC$ into the orthogonal matrix $\begin{bmatrix} B & -C \\ C & B \end{bmatrix}$. This induces a map

$$r : B_{\mathrm{U}(n)} \to B_{\mathrm{O}(2n)}$$

and a natural transformation of cohomology theories

$$r : K^*_{\mathbb{C}}(-) \to K^*_{\mathbb{R}}(-).$$

The composition $c \circ r(A)$ in $\mathrm{U}(2n)$ is $\begin{bmatrix} B & -C \\ C & B \end{bmatrix}$. Note that

$$\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} I & iI \\ iI & I \end{bmatrix}\right\}\left\{\begin{bmatrix} B & -C \\ C & B \end{bmatrix}\right\}\cdot\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} I & -iI \\ -iI & I \end{bmatrix}\right\} = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}$$

Since $\mathrm{U}(2n)$ is connected, there is a path from $\begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$ to the identity. Thus,

$$c \circ r : \mathrm{U}(n) \to \mathrm{U}(2n)$$

is homotopic through homomorphisms to the map

$$\mathrm{U}(n) \to \mathrm{U}(2n)$$

defined by $A \to \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}$. This implies the first part of

Proposition A.1. *(a) For $\lambda \in K^*_{\mathbb{C}}(X)$, $c \circ r(\lambda) = \lambda \oplus \lambda$.*

(b) For $\mu \in K^*_{\mathbb{R}}(X)$, $r \circ c(\mu) = \mu \oplus \mu$.

Part (b) is more easily seen by similar considerations.

Proposition A.2. *For $\lambda \in K^*_{\mathbb{R}}(X)$,*

$$\widehat{P}(r(\lambda)) = C(\lambda) \cup C(\overline{\lambda}) \quad \textit{and}$$
$$\sum_{i=0}^{\infty}(-1)^i p_i(r(\lambda)) = \left(\sum_{i=0}^{\infty} C_i(\lambda)\right) \cap \left(\sum_{j=0}^{\infty}(-1)^j C_j(\lambda)\right).$$

Corollary A.3. *If ${}_i(\lambda) = \sigma_i(x_1, \ldots, x_n)$, then $p_i(r(\lambda)) = \sigma_i(x_1^2, \ldots, x_n^2)$.*

Proof. This is seen by equating terms of degree $2i$ in the equality

$$\prod_{j=1}^{n}(1 - u_j^2 t^2) = \prod_{j=1}^{n}(1 + u_j t) \cdot \prod_{j=1}^{n}(1 - u_j t).$$

□

For $\lambda \in K_{\mathbb{R}}(X)$, we can formally factor

$$1 + p_1(\lambda)z + \cdots + p_n(\lambda)z^n = \prod_{i=1}^{n}(1 + \beta_i z),$$

where βi are complex numbers. Then $p_k = \sigma_k^n(\beta_1, \ldots, \beta_n)$. For a power series $f(u)$, the terms of $\prod_{i=1}^n f(u)$ of degree k are symmetric in the u_i and are thus polynomials

$$F_k(\sigma_1^{(n)}(u_1, \ldots, u_n), \ldots, \sigma_k{}^{(n)}(u_1, \ldots, u_n))$$

in the elementary symmetric functions. For $u_i = \beta_i$, they are polynomials $F_k(p_1(\lambda), \ldots, p_k(\lambda))$ in the Pontrjagin classes of λ.

In particular, for the power series $\frac{\frac{1}{2}\sqrt{u}}{\sinh \frac{1}{2}\sqrt{u}}$ we obtain polynomials $\widehat{A}(p_1, \ldots, p_k)$.

Theorem A.4. *For $\lambda \in K_{\mathbb{C}}(X)$ and $p_i = p_i(r(\lambda))$,*

$$T_*(C(\lambda)) = \exp\left(\frac{c_1(\lambda)}{2}\right) \cup \widehat{A}_*(p(r(\lambda))).$$

Proof. This follows from Corollary A.3 and the facts that the Todd polynomials for $C(\lambda)$ are determined by

$$\frac{u}{1 - \exp(-u)} = \exp\left(\frac{u}{2}\right)\frac{\frac{u}{2}}{\sinh \frac{u}{2}}$$

and the polynomials for $\widehat{A}$ for $p(r(\lambda))$ are determined by $\frac{\frac{1}{2}\sqrt{u^2}}{\sinh \frac{1}{2}\sqrt{u^2}}$. □

A.2 Representation of Lie groups

To discuss more completely orientability in $K^*_{\mathbb{C}}$ and in $K^*_{\mathbb{R}}$ it is necessary to review some basic properties of representations of Lie groups.

Every compact, connected, abelian Lie group is a torus; i.e., a product of a finite number of unit circles. If G is a compact Lie group and T is a maximal connected, abelian subgroup of G , then T is closed and so it is a torus. The torus T is called a *maximal torus* of G. The basic theorem (E. Cartan) is that all maximal tori of G are conjugate and every point of G is contained in a maximal torus.

In terms of an orthonormal basis for $\mathbb{C}^n$, the unitary group $\mathrm{U}(n)$ can be considered as the group of $n \times n$ unitary matrices. A maximal torus is the subgroup of diagonal matrices

$$T = \mathbb{S}^{1 \times n} \subset \mathrm{U}(n).$$

As seen earlier this inclusion induces a map

$$B_{\mathbb{S}^{1\times n}} \longrightarrow B_{\mathrm{U}(n)}.$$

The cohomology of $B_{\mathbb{S}^{1\times n}}$ is a polynomial ring $P(\alpha_1, \ldots, \alpha_n)$ with $\dim \alpha_i = 2$, $1 \le i \le n$. We have seen earlier that the cohomology of $B_{\mathrm{U}(n)}$ restricts isomorphically onto the ring of symmetric polynomials in $P(\alpha_1, \ldots, \alpha_n)$; i.e., it restricts isomorphically onto the subring of $P(\alpha_1, \ldots, \alpha_n)$ invariant under the action of the symmetric group on n letters acting as permutation group on the α's.

More generally, the Weyl group $w(G)$ is the group of all automorphisms of T which come from inner automorphisms of G This group is independent of choice of maximal torus. As $w(G)$ acts on T, it induces an action on $H^*(B_T; \mathbb{Q})$. It is not difficult to see that $\mathrm{im}(H^*(B_G; \mathbb{Q}) \to H^*(B_T; \mathbb{Q})$ lies in the subring of $H^*(B_T; \mathbb{Q})$ invariant under the action of $w(G)$. It is also true, but substantially more difficult to prove, that $H^*(B_G; \mathbb{Q})$ is mapped isomorphically onto the subring of $H^*(B_T; \mathbb{Q})$ invariant under $w(G)$. For the unitary group $\mathrm{U}(n)$, the Weyl group is the symmetric group on n letters.

We earlier defined the universal Chern character $\mathrm{ch} \in h^{*\ \mathrm{even}}(B_{\mathrm{U}(n)}; \mathbb{Q})$ to be $\mathrm{ch} = \exp(\alpha_1) + \cdots + \exp(\alpha_n)$.

A unitary representation of the Lie group G is a homomorphism

$$\rho : G \longrightarrow \mathrm{U}(n).$$

We can assume (taking a conjugate of ρ if necessary) that $\rho|_T : T \to \mathbb{S}^{1\times n}$. Let $\alpha_1, \ldots \alpha_n$ be the usual base for $H^1(\mathbb{S}^{1\times n}; \mathbb{Z})$ and $_i = (\rho|T)^*\alpha_i$. The classes w_i can be identified with the *weights of the representation* ρ in the classical sense [14], [15], [16]. Identify α_i with its negative transgression in $H^2(B_{\mathbb{S}^{1\times n}}; \mathbb{Z}) \subset H^2(B_{\mathbb{S}^{1\times n}}; \mathbb{Q})$, and let $\mathrm{ch}(\rho) = \sum \exp(w_i)$. This class can be identified with the character of the representation ρ in the classical sense. If $\rho : B_G \to B_{\mathrm{U}(n)}$ is the map induced by the representation ρ, then $\mathrm{ch}(\rho) = \rho^* \mathrm{ch}$. Hence, we can identify the two notions of character for unitary representations.

As we saw in discussing complex vector bundles, there are homomorphisms $\mathrm{U}(m)\times\mathrm{U}(n) \to \mathrm{U}(m+n)$ and $\mathrm{U}(m)\times\mathrm{U}(n) \to \mathrm{U}(m\cdot n)$ induced by sums and tensor products of vector spaces. The set of equivalence classes of unitary representations of a Lie group G is a monoid with sum induced by $\mathrm{U}(m)\times\mathrm{U}(n) \to \mathrm{U}(m+n)$. As in §3.1, we form the universal enveloping abelian group of this monoid, $RU(G)$. The tensor product map $\mathrm{U}(m)\times\mathrm{U}(n) \to \mathrm{U}(m\cdot n)$ induces a ring structure on $RU(G)$, making it a commutative ring with unit. The ring $RU(G)$ is called the *virtual representation ring* of G.

Since $\mathrm{U}(n)$ is connected, equivalent representations induce homotopic maps $B_G \to B_{\mathrm{U}(n)}$ (see the discussion before Proposition A.1). Thus, there is a transformation of the monoid of equivalence classes of unitary representations into the monoid $[B_G, \mathbb{Z}\times B_{\mathrm{U}}]$, the factor $\mathbb{Z}$ determined by the degree n of the representation. This induces a multiplicative transformation $\alpha : RU(G) \to K^0_{\mathbb{C}}(B_G)$. If $h : H \to G$ is a homomorphism of Lie groups, $h^* : RU(G) \to RU(H)$ is defined by composition $H \xrightarrow{h} G \xrightarrow{\rho} \mathrm{U}(n)$, and the diagramme

$$\begin{tikzcd} RU(G) \arrow[r, "\alpha"] \arrow[d, "h^*"'] & K^0_{\mathbb{C}}(B_G) \arrow[d, "h^*"] \\ RU(H) \arrow[r, "\alpha"'] & K^0_{\mathbb{C}}(B_H) \end{tikzcd}$$

commutes, where the vertical homomorphism on the right is induced by $h : B_H \to B_G$.

The specific facts we need are

1) a virtual representation is trivial if and only if its character is zero in positive dimensions, and

2) there exist representations

$$\Delta_+ : \mathrm{Spin}(2n) \to \mathrm{U}(2^n) \quad \text{and} \quad \Delta_- : \mathrm{Spin}(2n) \to \mathrm{U}(2^n)$$

such that

$$\mathrm{ch}(\Delta_+) = \sum_{(\varepsilon_1,\ \ldots,\ \varepsilon_n)} \exp[(\varepsilon_1\beta_1+\cdots+\varepsilon_n\beta_n)/2] \quad \text{for} \quad \varepsilon_i = \pm 1 \text{ and } \varepsilon_1\cdot\cdots\cdot\varepsilon_n = 1$$

and

$$\mathrm{ch}(\Delta_-) = \sum_{(\varepsilon_1,\ \ldots,\ \varepsilon_n)} \exp[(\varepsilon_1\beta_1+\cdots+\varepsilon_n\beta_n)/2] \quad \text{for} \quad \varepsilon_i = \pm 1 \text{ and } \varepsilon_1\cdot\cdots\cdot\varepsilon_n = -1$$

where the β_i are generators of $H^2(B_{T^n})$, T^n a maximal torus of $\mathrm{SO}(2n)$, and $\beta_i/2$ are their lifts in the corresponding maximal torus $\overline{T}$ in $\mathrm{Spin}(2n)$ under the double covering $\mathrm{Spin}(2n) \to \mathrm{SO}(2n)$.

The virtual representation $\Delta_+ - \Delta_-$ has character given by

$$\begin{aligned}\mathrm{ch}(\Delta_+ - \Delta_-) &= \prod_{i=1}^{n}(\exp(\beta_i/2) - \exp(-\beta_i/2)) \\ &= \beta_1 \cdots \beta_n \prod_{i=1}^{n} \frac{\sinh(\beta_i/2)}{(\beta_i/2)}.\end{aligned}$$

Under the inclusion $i : \mathrm{Spin}(2n-1) \to \mathrm{Spin}(2n)$ the restriction of $\Delta_+ - \Delta_-$ to $\mathrm{Spin}(2n-1)$ is trivial since $\mathrm{ch}(i^*(\Delta_+ - \Delta_-)) = 0$.

Thus, in the sphere bundle

$$\begin{tikzcd} \mathbb{S}^{2n-1} \arrow[r] & B_{\mathrm{Spin}(2n-1)} \arrow[d, "i"] \\ & B_{\mathrm{Spin}(2n)}, \end{tikzcd}$$

$i_*(\alpha(\Delta_+ - \Delta_-)) = 0$. Letting ν^{2n} denote the universal $\mathrm{Spin}(2n)$-bundle, this implies there is an element

$$\gamma \in K_{\mathbb{C}}(M(i)/B_{\mathrm{Spin}(2n-1)}) \cong K_{\mathbb{C}}(B_{\mathrm{Spin}(2n)}^{\nu^{2n}})$$

whose restriction to $B_{\mathrm{Spin}(2n)}$ ($\simeq M(i)$) has Chern character

$$\beta_1 \cdots \beta_n \prod_{i=1}^{n} \frac{\sinh(\beta_i/2)}{(\beta_i/2)}.$$

The bundle ν^{2n} is orientable in integral cohomology with orienting class $\beta_1 \cdots \beta_n$. Thus,

$$\mathrm{ch}(\gamma) = \varphi\left(\prod_{i=1}^{n} \frac{\sinh(\beta_i/2)}{(\beta_i/2)}\right).$$

It follows that the restriction of γ to $\mathbb{S}^{2n}$ has ch equal 1 in dimension $2n$; and so, γ restricted to $\mathbb{S}^{2n}$ generates $K_{\mathbb{C}}^*(\mathbb{S}^{2n})$ as a $K_{\mathbb{C}}^*(\mathrm{pt})$-module. Thus, γ is a $K_{\mathbb{C}}^*$-orientation of ν^{2n}.

By a theorem of Cartan-Malcev [24], for $n = 4k$ the representations Δ_+ and Δ_- in fact factor through $\mathrm{SO}(2^{4k-1})$; thus, γ is the image under c of an element $\rho \in K_{\mathbb{R}}(B_{\mathrm{Spin}(8k)}^{\nu^{8k}})$. Further, since

$$\pi_{8k}(B_{\mathrm{O}}) \xrightarrow{c_*} \pi_{8k}(B_{\mathrm{U}})$$

is an isomorphism and $c(\rho) = \gamma$ restricted to $\mathbb{S}^{8k}$ generates $\pi_{8k}(B_{\mathrm{U}})$, ρ restricted to $\mathbb{S}^{8k}$ generates $\pi_{8k}(B_{\mathrm{O}})$; and so ρ is a $K_{\mathbb{R}}^*$-orientation of ν^{8k}, and

$$\mathrm{ch}(c(\rho)) = \varphi\left(\prod_{i=1}^{n} \frac{\sinh(\beta_i/2)}{(\beta_i/2)}\right) = \varphi(\widehat{A}_*(p_*(\eta^{8k}))^{-1}),$$

η^{8k} being the universal SO$(8k)$-bundle.

Let $X \in \mathscr{P}$ and suppose X is a $(2n)$-dimensional real vector bundle over X with $w_1(\lambda) = 0$ and $w_2(\lambda)$ being the mod 2 reduction of an integral cohomology class c_1. (These conditions are equivalent to $w_1(\lambda) = w_3(\lambda) = 0$.) Let

$$\mu : X \to K(\mathbb{Z}, 2) = B_{\mathrm{U}(1)} = B_{\mathrm{SO}(2)}$$

be a bundle map such that $C_1(\mu) = -c_1$. Then $\mu \oplus \lambda$ is a $(2n+2)$-dimensional real vector bundle over X with $w_1 = w_2 = 0$; hence, its structure group can be reduced to Spin$(2n+2)$ and it has a $K^*_{\mathbb{C}}$-orientation with

$$\mathrm{ch}(\mu \oplus \lambda) = \widehat{A}(p_*(\mu \oplus \lambda))^{-1}.$$

The bundle $-\mu$ has a $K^*_{\mathbb{C}}$-orientation with

$$\mathrm{ch}(\mu, -1) = \frac{\exp(c_1)}{c_1} = \exp(c_1/2) \cdot \frac{\sinh(c_1/2)}{(c_1/2)(c_1/2)} = \exp(c_1/2)\widehat{A}(p_*(-\mu))^{-1}.$$

Thus, λ has $K^*_{\mathbb{C}}$-orientation with

$$\begin{aligned}
\mathrm{ch}(\lambda, 1) &= \mathrm{ch}(-\mu \oplus \mu \oplus \lambda, 1) \\
&= \mathrm{ch}(\mu, 1) \cup \mathrm{ch}(\mu \oplus \lambda, 1) \\
&= \exp(c_1/2)\widehat{A}(p_*(-\mu))^{-1} \cdot \widehat{A}(p_*(\mu \oplus \lambda))^{-1} \\
&= \exp(c_1/2)\widehat{A}(p_*(\lambda))^{-1}.
\end{aligned}$$

Thus, we have

Theorem A.5. *A real vector bundle λ over $X \in \mathscr{P}$ is $K^*_{\mathbb{C}}$-orientable if $w_1(\lambda) = 0$ and $w_2(\lambda)$ is the reduction* mod 2 *of an integral cohomology class c_1 (equivalently, $w_1(\lambda) = w_3(\lambda) = 0$). There is a $K^*_{\mathbb{C}}$-orientation such that*

$$\mathrm{ch}(\lambda, 1) = \exp(c_1/2)\widehat{A}(p_*(\lambda))^{-1}.$$

Appendix B

B.1 A Multiplication Formula

This appendix presents a proof of the following technical proposition which was used a number of times throughout the text:

Proposition B.1. *Let h^* and k^* be multiplicative cohomology theories, t : $h^{**} \to^{**}$ be a multiplicative transformation, and α and β be h^*- and k^*-oriented vector bundles over a space $X \in \mathscr{P}$. If $\varphi_\alpha^k(1)$ commutes with $\pi_\beta^*(t(\beta, 1))$, then*

$$t(\alpha \oplus \beta, 1) = t(\alpha, 1) \cup t(\beta, 1).$$

In all cases of interest herein either $\varphi_\alpha^k(1)$ lies in even-dimensional cohomology or coefficients are $\mathbb{Z}/2\mathbb{Z}$; thus, the commutation condition is satisfied. Examples of applications of the Proposition are

i) $w(\alpha \oplus \beta) = \mathrm{Sq}(\alpha \oplus \beta, 1) = \mathrm{Sq}(\alpha, 1) \cup \mathrm{Sq}(\beta, 1) = w(\alpha) \cup (\beta)$,

ii) $T_*(C(\lambda \oplus \mu)) = \mathrm{ch}(\lambda \oplus \mu, 1) = \mathrm{ch}(\lambda, 1) \cup \mathrm{ch}(\mu, 1) = T_*(C(\lambda)) \cup T_*(C(\mu))$,

iii) $\widehat{A}_*(p(\lambda \oplus \mu)) = \mathrm{ch}(\lambda \oplus \mu, 1)^{-1} = \mathrm{ch}(\lambda, 1)^{-1} \cup \mathrm{ch}(\mu, 1)^{-1} = \widehat{A}_*(p(\lambda)) \cup \widehat{A}_*(p(\mu))$.

Proof. (of Proposition) Referring back to Proposition 1.35, we recall that if $u \in h^m(X^\alpha, *)$ and $v \in h^n(X^\beta, *)$ are h^* -orientations of the vector bundles α and β over the space X, then $\Delta^*(u \times v)$ is an h^*-orientation of the Whitney sum $\alpha \oplus \beta$, where Δ^* is induced by the map

$$\Delta : X^{\alpha \oplus \beta} \to (X \times X)^{\alpha \times \beta}.$$

We have

$$\begin{aligned}
\varphi_{\alpha\times\beta}^k(t(\alpha, 1) \times t(\beta, 1)) &= \pi_{\alpha\times\beta}^*(t(\alpha, 1) \times t(\beta, 1)) \cup \varphi_{\alpha\times\beta}^k(1) \\
&= \{\pi_\alpha^*(t(\alpha, 1)\} \times \{\pi_\beta^*(t(\beta, 1))\} \cup \{\varphi_\alpha^k(1) \times \varphi_\beta^k(1)\} \\
&= \{\pi_\alpha^*(t(\alpha, 1) \cup \varphi_\alpha^k(1)\} \times \{\pi_\beta^*(t(\beta, 1)) \cup \varphi_\beta^k(1)\} \\
&= \varphi_\alpha^k(t(\alpha, 1)) \times \varphi_\beta^k(t(\beta, 1))
\end{aligned}$$

and

$$\begin{aligned}
\varphi^k_{\alpha\oplus\beta}(t(\alpha,1)\cup t(\beta,1)) &= \varphi^k_{\alpha\oplus\beta}\Delta^*(t(\alpha,1)\times t(\beta,1)) \\
&= \Delta^*\varphi^k_{\alpha\oplus\beta}(t(\alpha,1)\times t(\beta,1)) \\
&= \Delta^*\{\varphi^k_\alpha(1)\times\varphi^k_\beta(1)\} \\
&= \Delta^*\{t\varphi^h_\alpha(1)\times t\varphi^h_\beta(1)\} \\
&= t\Delta^*\{\varphi^h_\alpha(1)\times\varphi^h_\beta(1)\} \\
&= t\varphi^k_{\alpha\oplus\beta}(1)
\end{aligned}$$

Since $\varphi^k_{\alpha\oplus\beta}$ is an isomorphism, $t(\alpha,1)\cup t(\beta,1) = t(\alpha\oplus\beta,1)$. □

Appendix C

C.1 Fibre Homotopy Equivalence of Bundles, The Groups $J(X)$

This appendix presents a very incomplete sketch of parts of the theory and applications of fibre homotopy equivalence of sphere bundles.

Definition C.1. Fiber bundles E and $E^{'}$ over X with common fibre F are of the *same fibre homotopy type*, $E \simeq E^{'}$, if there are fibre preserving maps $f : E \to E^{'}$ and $f^{'} : E^{'} \to E$ and fibre preserving homotopies $H_t : E \to E$ and $H_t^{'} : E^{'} \to E^{'}$ such that $H_0 = \mathrm{id}_E$, $H_1 = f^{'} \circ f$, $H_0^{'} = \mathrm{id}_{E^{'}}$, $H'_1 = f \circ f^{'}$. (See [7].)

For λ a vector bundle over X, we write (λ) to denote the associated sphere bundle of one real dimension lower.

Recall that $K_\Lambda(X)$, $\Lambda = \mathbb{R}$ or $\mathbb{C}$, is the factor group of the free group $F_\Lambda(X)$ generated by equivalence classes of Λ-vector bundles by the relation $R_\Lambda(X)$ generated by

$$\{\lambda \oplus \mu\} - \{\lambda\} - \{\mu\}.$$

Let $U_\Lambda(X)$ be the subgroup of $F_\Lambda(X)$ generated by $R_\Lambda(X)$ and by $\{\{\lambda\} - \{\mu\} | (\lambda) \simeq (\mu)\}$. Of course, $R_\Lambda(X) \subset (X)$.

Definition C.2.

$$J_\Lambda(X) = F_\Lambda(X)/U_\Lambda(X).$$

There is a natural epimorphism $h : K_\Lambda(X) \to J_\Lambda(X)$. (See [3] and [5].)

The groups $J_\Lambda(X)$ are not exact on short coexact sequences of spaces and do not lead to a cohomology theory.

$$\begin{tikzcd}
K_\Lambda(X) \arrow[d, leftrightarrow, "h"'] & \cong & \widetilde{K}_\Lambda(X) \arrow[d, leftrightarrow, "h"] & + & K_\Lambda(\mathrm{pt}) \arrow[d, leftrightarrow, "h"] \\
J_\Lambda(X) & \cong & \widetilde{J}_\Lambda(X) & + & J_\Lambda(\mathrm{pt})
\end{tikzcd}$$

The groups $\widetilde{J}_\Lambda(X)$ are determined by identifying the classes of λ and μ if $(\lambda \oplus m) \simeq (\mu \oplus n)$ for some trivial bundles m and n.

The associative H-space H_n of homotopy equivalences of $\mathbb{S}^n$ into itself has a universal base space B_{H_n}. There are inclusions via suspension $H_n \to H_{n+1}$ which induce maps $B_{H_n} \to B_{H_{n+1}}$. Let $B_H = \cup B_{H_n}$. Maps into B_H do define a cohomology theory. In particular,

$$[\mathbb{S}^r, B_{H_n}] \cong [\mathbb{S}^{r-1}, H_n] \cong \pi_{n-1+r-1}(\mathbb{S}^{n-1}) \quad \text{for} \quad \leq r \leq n-2.$$

Thus, $[\mathbb{S}^r, B_{H_n}] \cong \mathcal{G}_{r-1}$, the $(r-1)$-stable stem of homotopy groups of spheres.

The inclusion $\mathrm{O}(n) \to H_n$ induces $B_{\mathrm{O}(n)} \to B_{H_n}$ and a transformation

$$J : \widetilde{K}_{\mathbb{R}}(X) \to \widetilde{H}^0(X; H).$$

$\mathrm{im}(J) = \widetilde{J}(X)$.

For a finite cell-complex X, $\widetilde{H}^0(X;H)$ is finite since $\widetilde{H}^0(\mathbb{S}^r;H)$ is finite. As $\widetilde{J}(X) \subset J(X)\widetilde{H}^0(X;H)$, $\widetilde{J}(X)$ is finite. Further since

$$\widetilde{K}_{\mathbb{C}}(X) \to \widetilde{K}_{\mathbb{R}}(X) \to \widetilde{H}^0(X;H),$$

$\widetilde{J}_{\mathbb{C}}(X) \subset \widetilde{J}_{\mathbb{R}}(X)$.

In particular, we have the

Proposition C.3. *$\widetilde{J}_{\mathbb{R}}(\mathbb{S}^r)$ is isomorphic to the image of the real J-homomorphism in the stem $\mathcal{G}_{r-1}$.*

We mention one further application of the J-groups. Let ζ be the canonical line bundle over $\mathbb{R}P^k$ (resp. $\mathbb{C}P^k$). Note that $\mathbb{S}^{n-1}$ has r linearly independent vector fields if and only if the fibration

$$\begin{array}{c} \mathrm{O}(n)/\mathrm{O}(n-r-1) \\ \downarrow \\ \mathrm{O}(n)/\mathrm{O}(n-1) = \mathbb{S}^{n-1} \end{array}$$

has a cross-section.

Theorem C.4. $\mathrm{O}(n)/\mathrm{O}_{n-k} \to \mathbb{S}^{n-1}$ *has a cross-section if and only if $n(\geq 2k)$ is a multiple of q_k, where q_k is the order of $J_{\mathbb{R}}(\zeta - 1)$ in $J_{\mathbb{R}}(\mathbb{R}P^k)$, and* $\mathrm{U}(n)/\mathrm{U}(n-k) \to \mathbb{S}^{2n-1}$*) has a cross-section if and only if n is a multiple of q_k, where q_k is the order of $J_{\mathbb{C}}(\zeta - 1)$ in $J_{\mathbb{C}}(\mathbb{C}P^k)$.*

Proof. See [7]. □

These two results indicate reasons for interest in the J-groups. A number of people have investigated them; J. F. Adams has essentially made a determination of the relevant orders in the above cited Proposition and Theorem. (See [4].) The remainder of the appendix is concerned with a discussion of some of his techniques.

To determine whether a bundle is fibre homotopically trivial we have the criterion of Dold:

Proposition C.5. *$(\lambda) \simeq (n)$ if and only if there exists a map $f : E_{(\lambda)} \to \mathbb{S}^{n-1}$ such that the following diagramme is homotopy commutative:*

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "\subset", "fibre"'] \arrow[dr, "\mathrm{id}"'] & E(\lambda) \arrow[d, "f"] \\ & \mathbb{S}^{n-1}. \end{tikzcd}$$

Since $(\alpha) \simeq (\beta)$ implies $X^\alpha \simeq X^\beta$ (has the same homotopy type), we can restate Dold's criterion as follows:

Proposition C.6. *$(\alpha) \simeq (n)$ if and only if there exists a map $X^\alpha \to \mathbb{S}^n$ such that the composition $\mathbb{S}^n \to X^\alpha \to \mathbb{S}^n$ has degree 1.*

Thus, a bundle is stably fibre homotopically trivial if and only if it is orientable in the stable cohomotopy theory and thus orientable in all cohomology theories.

Certain identities hold for bundles having the same fibre homotopy type. For example,

Proposition C.7. *If $(\alpha) \simeq (\beta)$, h^* and k^* are multiplicative cohomology theories, and α is both h^*- and k^*-orientable, then*

a) β is both h^*- and k^*-orientable; and

b) if φ_h^α, φ_k^α, φ_h^β and φ_k^β be the corresponding orientations, then there exist classes $v \in \widetilde{h}^*(X)$ and $v' \in \widetilde{k}^*(X)$ such that for any multiplicative transformation $t : h^* \to k^*$,

$$t(\alpha, 1 + v) = t(\beta, 1) \cup (1 + v').$$

If $h^* = k^*$, then $v = v'$.

Proof. Let $f : X^\alpha \to X^\beta$ be the homotopy equivalence induced by the fibre homotopy equivalence. The class v is defined as follows:

$$\begin{aligned} f^*\varphi_h^\beta(1) &= \varphi_h^\alpha(1) + u \\ &= \varphi_h^\alpha(1) + \varphi_h^\alpha(v) \\ &= \varphi_h^\alpha(1 + v). \end{aligned}$$

The class v' is defined similarly in the k^*-theory. We have the following equations:

$$\begin{aligned} f^*\varphi_h^\beta(w) &= f^*(\pi_\beta w \cup \varphi_h^\beta(1)) \\ &= \pi_\alpha w \cup f^*\varphi_h^\beta(1) \\ &= \pi_\alpha w \cup \varphi_h^\alpha(1 + v) \\ &= \varphi_h^\alpha(w \cup (1 + v)). \end{aligned}$$

Similar equations hold for $v^{'}$ is defined in the k^*-theory.. Thus, we have

$$\begin{aligned}\varphi_k^\alpha(t(\alpha, 1+v)) &= t\varphi_k^\alpha(1+v)\\ &= tf^*\varphi_h^\beta(1)\\ &= f^*t\varphi_h^\beta(1)\\ &= f^*\varphi_k^\beta t(\beta,1)\\ &= \varphi_k^\alpha(t(\beta,1)\cup(1+v^{'})).\end{aligned}$$

Since φ_k^α is an isomorphism, the conclusion follows. □

Consider this Proposition for the case $h^* = k^* = K_{\mathbb{C}}^*$, and $t = \psi_k$. Then for $(\alpha) \simeq (\beta)$, $v = v^{'} \in \widetilde{K}_{\mathbb{C}}(X)$ and

$$\psi_k(\alpha, 1+v) = \psi_k(\beta, 1) \cup (1+v).$$

Furthermore, this equation in α, β, and v is valid for all k; i.e., v does not depend on k.

Following Adams, we define groups $J_{\mathbb{C}}^{'}(X)$ as $K_{\mathbb{C}}(X)/V_{\mathbb{C}}(X)$, where $V_{\mathbb{C}}(X)$ is generated by classes of complex vector bundles $(\{\alpha\} - \{\beta\})$ such that for some $v \in \widetilde{K}_{\mathbb{C}}(X)$,

$$\psi_k(\alpha, 1+v) = \psi_k(\beta, 1) \cup (1+v).$$

for all k. If $(\alpha) \simeq (\beta)$, then $(\{\alpha\} - \{\beta\}) \in V_{\mathbb{C}}(X)$; and so

$$J_{\mathbb{C}}(X) \xrightarrow{\text{onto}} J_{\mathbb{C}}^{'}(X).$$

Since $\psi_k(n, 1) = k^n \cdot 1$ if $(\alpha) \simeq (n)$, then for some v

$$\psi_k(\alpha, 1) \cup (1+v) = k^n \cdot 1 \cup \psi_k(1+v).$$

This gives an effective means of computing $J_{\mathbb{C}}^{'}(X)$ since the classes $\psi_k(\alpha, 1)$ can be computed. This is done in the universal case as follows:

$$\begin{tikzcd} K_{\mathbb{C}}(X^\alpha) \arrow[r, "\psi_k"] & K_{\mathbb{C}}(X^\alpha) \arrow[r, "\mathrm{ch}"] & H(X^\alpha) \\ K_{\mathbb{C}}(X) \arrow[u, "\varphi_K"'] & K_{\mathbb{C}}(X) \arrow[u, "\varphi_K"'] & H(X) \arrow[u, "\varphi_H"'] \end{tikzcd}$$

$$(\mathrm{ch} \circ \psi_k)(\alpha, 1) = \mathrm{ch}(\alpha, \psi_k(\alpha, 1)) = \mathrm{ch}(\alpha, 1) \cup \mathrm{ch}(\psi_k(\alpha, 1)).$$

We have previously computed $\mathrm{ch}(\alpha, 1)$. Also,

$$\begin{aligned}(\mathrm{ch} \circ \psi_k)(\alpha, 1) &= \varphi_h^{-1}(\mathrm{ch} \circ \psi_k(\varphi_k(1)))\\ &= \varphi_h^{-1}\left(\sum_t t^k \,\mathrm{ch}_t(\varphi_k(1))\right)\\ &= \sum_t t^k \,\mathrm{ch}_t(\alpha, 1).\end{aligned}$$

Thus, $(\mathrm{ch} \circ \psi_k)(\alpha, 1)$ is determined; and so, $(\mathrm{ch}(\psi_k(\alpha, 1))$ is determined. But in the universal case this determines $\psi_k(\alpha, 1)$.

By the method above we can compute $J'_{\mathbb{C}}(X)$ and thus find a lower bound for $J_{\mathbb{C}}(X)$. To find upper bounds is more difficult. Adams proceeds by first generalising Dold's theorem to the

Theorem C.8. *If α and β are vector bundles over $X \in \mathscr{P}$ and there is a fibre preserving map $f : E_{(\alpha)} \to E_{(\beta)}$ of degree k on each fibre, then there is an integer e such that*

$$(|k|^e \alpha) \simeq (|k|^e \beta)$$

A consequence of this theorem is the

Theorem C.9. *If $k \in \mathbb{Z}$ and $y \in K_{\mathbb{C}}(X)$ is such that either y is a linear combination of line bundles over X or $X = \mathbb{S}^{2n}$, then there is an integer e such that*

$$(k^e \psi_k(y)) \simeq (k^e y).$$

This suggests the following definition: let $W_{\mathbb{C}}(X) \subset K_{\mathbb{C}}(X)$ be generated by $\{k^e(\psi_k - 1)y \mid k \in \mathbb{Z}, e$ is a positive integer, and y is a complex vector bundle$\}$.

Let $J''_{\mathbb{C}}(X) = K_{\mathbb{C}}(X)/W_{\mathbb{C}}(X)$.

The identities

$$\psi_\ell(\psi_k(y,1)) \cup \psi_k(y,1) = \psi_{k\ell}(y,1) = \psi_\ell(\psi_k(y,1)) \cup \psi_\ell(y,1)$$

imply that $W_{\mathbb{C}}(X) \subset V_{\mathbb{C}}(X)$; and so, we have

$$\begin{tikzcd}
& J_{\mathbb{C}}(X) \arrow[dr] & \\
K_{\mathbb{C}}(X) \arrow[ur] \arrow[dr] & & J_{\mathbb{C}'}(X) \\
& J''_{\mathbb{C}}(X) \arrow[ur] &
\end{tikzcd}$$

The second theorem stated gives $J''_{\mathbb{C}}(X) \to J_{\mathbb{C}}(X)$ in some cases. Further, Adams has asserted that $J''_{\mathbb{C}}(X) \xrightarrow{\cong} J'_{\mathbb{C}}(X)$ for X any finite cell-complex. At the time of this writing none of the details of this section have appeared. They are to appear in a forthcoming series of papers "On the groups $J(X)$" in the journal "Topology".

Transcriber's comment

For $J(X)$, see the following articles by Adams:

- John Adams, *On the groups* $J(X)$ I, Topology 2 (3) (1963)
- John Adams, *On the groups* $J(X)$ II, Topology 3 (2) (1965)

- John Adams, *On the groups $J(X)$ III*, Topology 3 (3) (1965)
- John Adams, *On the groups $J(X)$ IV*, Topology 5: 21,(1966) + Correction, Topology 7 (3): 331 (1968)

For the proof of Adams' conjecture, see

- Daniel Quillen, *The Adams conjecture*, Topology. an International Journal of Mathematics 10: 67-80 (1971)

A related discussion:

- Dennis Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. 100 (1974), 1-79.

Appendix D

This appendix gives two proofs for the Eckmann-Hilton argument referred to in §§2.2.2.

D.1 The Eckmann-Hilton argument

The Eckmann-Hilton argument shows that a monoid or group object in the category of monoids or groups is commutative. In other words, if a set is equipped with two monoid structures, such that one is a homomorphism for the other, then the two structures coincide and the resulting monoid is commutative.

Proposition D.1. *Let X be a set equipped with two binary operations, which we will write $\circ$ and $\otimes$, and suppose:*

1. *$\circ$ and $\otimes$ are both unital.*
2. *$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$ for all $a, b, c, d \in X$.*

Then $\circ$ and $\otimes$ are the same and in fact commutative and associative.

Remark D.2. The operations $\otimes$ and $\circ$ are often referred to as *multiplications*, but this might imply they are associative, a property which is *not* required for the proof. In fact, associativity follows. Likewise, we do not have to require that the two operations have the same neutral element; this is a consequence.

First proof

Proof. First, observe that the units of the two operations coincide:

$$\begin{aligned} 1_\circ &= 1_\circ \circ 1_\circ = (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) \\ &= (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) = 1_\otimes \otimes 1_\otimes = 1_\otimes . \end{aligned}$$

Now, let $a, b \in X$. Then

$$\begin{aligned} a \circ b &= (1 \otimes a) \circ (b \otimes 1) = (1 \circ b) \otimes (a \circ 1) = b \otimes a \\ &= (b \circ 1) \otimes (1 \circ a) = (b \otimes 1) \circ (1 \otimes a) = b \circ a . \end{aligned}$$

This establishes that the two operations coincide and are commutative.

For associativity,

$$\begin{aligned}(a \otimes b) \otimes c &= (a \otimes b) \otimes (1 \otimes c) \\ &= (a \otimes 1) \otimes (b \otimes c) = a \otimes (b \otimes c).\end{aligned}$$

□

Alternate proof

Proof. For this version of the proof, we write the two operations as vertical and horizontal juxtaposition, i.e., $\begin{array}{c} a \\ b \end{array}$ and $a\ b$. The interchange property can then be expressed as follows:

For all $a, b, c, d \in X$, $\begin{array}{c} (a\ b) \\ (c\ d) \end{array} = \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$, so we can write $\begin{array}{cc} a & b \\ c & d \end{array}$ without ambiguity.

Let $\bullet$ and $\circ$ be the units for vertical and horizontal composition respectively. Then $\bullet = \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{cc} \bullet & \circ \\ \circ & \bullet \end{array} = \circ\ \circ = \circ$, so both units are equal.

Now, for all $a, b \in X$, $a\ b = \begin{array}{cc} a & \bullet \\ \bullet & b \end{array} = \begin{array}{c} a \\ b \end{array} = \begin{array}{cc} \bullet & a \\ b & \bullet \end{array} = b\ a = \begin{array}{cc} b & \bullet \\ \bullet & a \end{array} = \begin{array}{c} b \\ a \end{array}$, so horizontal composition is the same as vertical composition and both operations are commutative. Finally, for all $a, b, c \in X$, $a\ (b\ c) = a\ \begin{pmatrix} b \\ c \end{pmatrix} = \begin{array}{cc} a & b \\ \bullet & c \end{array} = (a\ b)\ c$, so composition is associative. □

Appendix E

Historical Comments

The general idea of obtaining cohomology theories by maps into spectra can be traced to Barratt [12], in that the track groups of maps from a cofibration yield an exact sequence. This was made explicit by Puppe [28]. The converse that cohomology theories are representable in this way was established by Brown [18].

The material in Chapter 1 on reduced and non-reduced cohomology theories is described in the fashion of Whitehead [30], while the spectral sequence of a fibration is presented along lines suggested by Dold [19]. The proof given of the Dold-Thom-Gysin Theorem also follows Dold (ibid). The approach to orientability of vector bundles, multiplicative transformations and the Riemann-Roch Theorem for differentiable manifolds is essentially folk-lore; it was first presented in [20] but its ideas lie in the work of Atiyah and Hirzebruch as expounded, for instance, by Hirzebruch in [24]. The proof given here of the Wu formulae is essentially that of Atiyah and Hirzebruch in [10].

Through the use of Spanier-Whitehead duality [29] one can view the cohomology of the Thom space of a normal bundle as being the homology of the manifold under consideration. An alternate procedure is developed by Whitehead in [30]. Various of the constructions of Chapter 1 then appear as versions of Poincare duality, umkehr homomorphism, etc.

The development of the topology of the Unitary Group and associated spaces presented in Chapter 2 follows that given by the authour in a course at the University of Chicago in the Winter of 1960. A similar but substantially more complicated approach works for the Orthogonal and Symplectic Groups [22]. The explicit description of the maps generating $\pi_{2n+1}(\mathrm{SU})$ given in Theorem 2.17 is due to Toda.

Atiyah is credited with initiating study of the theory $K^*_{\mathbb{C}}$, as a cohomology theory. The material of §§3.1.1 and §§3.1.2 is taken from a paper by Atiyah and Hirzebruch [10].

The technical lemmas of §4.1 are found in Atiyah and Hirzebruch [10] and Hirzebruch [23]. The material of §§4.1.2 and §§subsect:4-A-3 follows the authour's exposition in [21], although it must be noted that Adams found [3] a proof similar in spirit to that given here of Theorem 4.7 some time earlier than the authour; its differences are that he uses results from [1] while the proof given here uses results on complex cobordism [25].

The Appendix A on the K^*-theory essentially follows Atiyah and Hirzebruch as in [24]. The proposition of Appendix B is very basic, leading in various roles to the theorems on Stiefel-Whitney and Chern classes of sums of vector bundles and to the theory of multiplicative sequences. The Appendix C is very incomplete and is a presentation of material presented by Adams in various lectures.

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