A study memo on *Cohomology Theories* by Eldon Dyer

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Preface

A word from the transcriber

This is a study memo of "Cohomology Theories" by Eldon Dyer. It is rather amazing that this succinct account on the subject was published in 1969, more than fifty years ago.

The preface by Dyer

Let $\mathscr{P}$ be the category whose objects are finite cell complexes and whose maps are continuous maps. Let $\mathscr{P}^2$ be the category of pairs in $\mathscr{P}$; i.e., pairs (X, A) where X is a finite cell complex and A is a subcomplex of X , and maps are continuous maps of pairs. Let G be the category of abelian groups and homomorphisms.

Let $T : \mathscr{P}^2 \to \mathscr{P}^2$ be the covariant functor defined by

$$T(X, A) = (A, \varnothing) \text{ for } (X, A) \in \mathscr{P}^2 \,\text{ext}\, \mathit{and}$$
$$T(f) = f|_{(A,\varnothing)} : (A, \varnothing) \to (B, \varnothing) \text{ for}$$

a map $f : (X, A) \to (Y, B)$ in $\mathscr{P}^2$.

A *cohomology theory* on $\mathscr{P}$ is a sequence of contravariant functors

$$H^n : \mathscr{P}^2 \to G$$

and a sequence of natural transformations

$$\delta^n : H^{n-1} \circ T \to H^n$$

subject to the conditions

1) If $f_0, f_1 \in \mathscr{P}$ and $f_0 \simeq f_1$, ($\simeq$ means "is homotopic to"), then $H^n(f_0) = H^n(f_1)$ for all n;

2) If $(X; A, B)$ is a triad in $\mathscr{P}$ (all possible pairs of X, A, B, $A \cup B$ and $A \cap B$ are in $\mathscr{P}$) and $X = A \cup B$, then for the inclusion map $k : (A, A \cap B) \to (X, B)$,

$$H^n(k) : H^n(X, B) \to H^n(A, A \cap B)$$

is an isomorphism for all n; and

3) If $(X, A) \in \mathscr{P}^2$ and $i : (A, \varnothing) \to (X, \varnothing)$ and $j : (X, \varnothing) \to (X, A)$ are the inclusion maps, then the sequence

$$\cdots \to H^{n-1}(A, \varnothing) \xrightarrow{\delta^n} H^n(X, A) \xrightarrow{H^n(j)} H^n(X, \varnothing) \xrightarrow{Hn(i)} H^n(A, \varnothing) \to \cdots$$

of the pair (X, A) is exact.

These conditions are exactly the Eilenberg-Steenrod axioms for cohomology except we *omit* the condition

$$H^i(\mathrm{pt}, 0) = 0 \text{ for } i \neq 0.$$

Recall that with this extra condition, one has the singular cohomology groups with coefficients in $H^0(\mathrm{pt}, 0)$. We shall speak of our cohomology theories as having coefficients in the graded system $\{H^n(\mathrm{pt}, \varnothing)\}$.

In these notes we discuss such cohomology theories. It has been clear for the past two or three years that they furnish a strong tool for analysing manifolds.

The notes are divided into four chapters. The first deals with generalities of such theories: axiomatics, origins, spectral sequence of a fibration, multiplicative structures, orientation of bundles, Poincare duality, and a type of generalised Riemann-Roch theorem. The second studies the unitary group and characteristic classes of complex vector bundles. The third discusses an example: the Grothendieck ring $K^*_{\mathscr{C}}()$. The fourth is concerned with applications: the J-homomorphism, maps of Hopf invariant 1, and properties of stable homotopy.

There are also three appendices, discussing more briefly the cohomology theory $K^*_{\mathscr{R}}$ and the J-groups.

The author wishes to acknowledge here his appreciation of Walter Daum for assistance in preparing these notes.

Contents

Chapter 1

Structure of Cohomology Theories

1.1 Axiomatics

1.1.1 Eilenberg Steenrod axioms for (generalised) cohomology

A *(genealised) cohomology theory* h^* on $\mathscr{Top}^2$ (or any nice subcategory like compact pairs, pairs of CW-complexes, etc,) is a collection of

***contravariant* functors**

$$h^n : \mathscr{Top}^2 \to \mathscr{Ab}, \quad n \in \mathbb{Z}$$

where $\mathscr{Ab}$ denotes the category of abelian groups, and

natural transformations

$$\delta^n : j^n \circ R \to h^{n+1}$$

where $R : \mathscr{Top}^2 \to \mathscr{Top}^2$ is the functor that sends (X, A) to $(A, \varnothing)$ and f to $f|_A$, satisfying the following axioms:

(i) *Homotopy invariance.* If $f \simeq g$, then $h^n(f) = h^n(g)$ for every $n \in \mathbb{Z}$.

(ii) *Excision.* For every pair (X, A) and $U \subset A$ such that the closure $\overline{U}$ is contained in the interior A°, the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces and isomorphism

$$h^n(X \setminus U, A \setminus U) \cong h^n(X, A) \quad \text{for every} \quad n \in \mathbb{Z}.$$

(iii) *Exactness.* For every pair (X, A), consider the inclusions $i : A \to X$ and $j : (X, \varnothing) \to (X, A)$. Then there is a long exact sequence

$$\cdots \to h^{n-1}(A) \xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A)$$
$$\xrightarrow{\delta^n} h^{n+1}(X, A) \xrightarrow{j^*} h^{n+1}(X) \xrightarrow{i^*} h^{n+1}(A) \to \cdots$$

1.1.2 Three Theorems

The three theorems which follow can be proved just as in Eilenberg and Steenrod, but a simpler proof for the third due to M. Barratt and J. H. C. Whitehead is indicated.

Theorem 1.1 (Exact Sequence of a Triple). *If $(X; A, B)$ is a triple, then the sequence*

$$\cdots \to H^{n-1}(A,B) \xrightarrow{\Delta} H^n(X,A) \xrightarrow{\Phi} H^n(X,B) \xrightarrow{\Psi} H^n(A,B) \to \cdots$$

is exact, where Φ and Ψ are induced by the inclusion maps and Δ is the composition

$$H^{n-1}(A,B) \to H^{n-1}(A,\varnothing) \to H^n(X,A).$$

Theorem 1.2 (Exact Sequence of a Triad). *If $(X; A, B)$ is a triad, then the sequence*

$$\cdots \to H^{n-1}(A, A\cap B) \xrightarrow{\Delta} H^n(X, A\cup B) \xrightarrow{\Phi} H^n(X,B) \xrightarrow{\Psi} H^n(A, A\cap B) \to \cdots$$

is exact, where Φ and Ψ are induced by the inclusion maps and Δ is the composition

$$H^{n-1}(A, A\cap B) \xrightarrow{\cong} H^{n-1}(A\cup B, B) \to H^{n-1}(A\cup B, \varnothing) \xrightarrow{\delta} H^n(X, A\cup B)$$

Theorem 1.3 (Mayer-Vietoris Theorem). *Let $(X; A, B)$ be a triad with $X = A\cup B$. Then the sequence*

$$\cdots \to H^{n-1}(A\cap B, \varnothing) \xrightarrow{\Delta} H^n(X,\varnothing) \xrightarrow{\Phi} H^n(A,\varnothing)\oplus H^n(B,\varnothing) \xrightarrow{\Psi} H^n(A\cap B,\varnothing) \to \cdots .$$

is exact, where the homomorphisms are defined as in the proof.

Lemma 1.4 (Barratt-Whitehead lemma). [1] *If the diagramme of groups and homomorphisms*

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \xrightarrow{h_i} & A_{i+1} & \xrightarrow{f_{i+1}} & B_{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \alpha_{i+1}} & & \downarrow{\scriptstyle \beta_{i+1}} & & \\
\cdots & \longrightarrow & A_i' & \xrightarrow[f_i']{} & B_i' & \xrightarrow[g_i']{} & C_i' & \xrightarrow[h_i']{} & A_{i+1}' & \xrightarrow[f_{i+1}']{} & B_{i+1}' & \longrightarrow & \cdots
\end{array}$$

is commutative, the horizontal sequences are exact, and the γ_i are isomorphisms, then the sequence

$$\cdots \to A_i \xrightarrow{(\alpha_i, f_i)} A_i' \oplus B_1 \xrightarrow{f_i - \beta_i} B_i' \xrightarrow{h_i \gamma_i^{-1} g_i'} A_{i+1} \to \cdots .$$

is exact.

[1] See *The First Non-Vanishing Group of an (n+l)-ad* M. Barratt, J. Whitehead Published 1 July 1956 Mathematics Proceedings of The London Mathematical Society

The Mayer-Vietoris Theorem follows by applying this lemma to the following diagramme, in which the isomorphism is condition (ii) in the definition of a cohomology theory (§§1.1.1).

$$\begin{array}{ccccccccccc}
\cdots \to H^{n-1}(X,\varnothing) & \longrightarrow & H^{n-1}(A,\varnothing) & \longrightarrow & H^n(X,A) & \longrightarrow & H^n(X,\varnothing) & \longrightarrow & H^n(A,\varnothing) & \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow\cong & & \downarrow & & \downarrow & \\
\cdots \to H^{n-1}(B,\varnothing) & \to & H^{n-1}(A\cap B,\varnothing) & \to & H^n(B,A\cap B) & \to & H^n(B,\varnothing) & \to & H^n(A\cap B,\varnothing) & \to \cdots
\end{array}$$

1.1.3 Reduced Cohomology Theories

Let $\mathscr{P}_0$ be the category of finite cell complexes with base point and continuous base point preserving maps. A *reduced cohomology theory* is a sequence

$$\widetilde{H}^n : \mathscr{P}_0 \to \mathscr{A}$$

of contravariant functors and a sequence

$$\sigma^n : \widetilde{H}^{n+1} \circ \Sigma \to \widetilde{H}^n$$

of natural transformations subject to three conditions.

Construction of spaces

Before stating these conditions we recall some important constructions. Let (X,x_0) and (Y,y_0) be in $\mathscr{P}_0$.

The *wedge* $(X \vee Y,*) \in \mathscr{P}_0$ is the subspace $(X \times y_0) \cup (x_0 \times Y)$ of $X \times Y$ with $x \times y = *$ as base point.

The *smash* $(X\#Y,*) \in \mathscr{P}_0$ is the factor space $(X \times Y)/(X \vee Y)$ with base point the image of $X \vee Y$.

The *reduced suspension* For $(X,x_0) = (\mathbb{S}^1,1)$, the unit circle $\{z \in \mathbb{C} \| x| = 1\}$, $X\#Y = \Sigma Y$ is the *reduced suspension* of Y. Σ is a covariant functor from $\mathscr{P}_0$ to $\mathscr{P}_0$ with $\Sigma(f) = \mathrm{id}\#f$, $\mathrm{id} : \mathbb{S}^1 \to \mathbb{S}^1$ being the identity map.

We define $p : \mathscr{P}^2 \to \mathscr{P}_0$ by $p(X,A) = (X/A,*)$, where for $A = \varnothing$, $X/\varnothing = X^+$ the disjoint union of X and a distinct point $+$, to be taken as the base point in $p(X,\varnothing) = (X,+)$. The reduced cone $CA \cup X$ of a pair (X,A) with $* \in A \subset X$ consists of X together with a cone over A with the interval from the base point collapsed to the new base point (the "whisker construction"); i.e.,

$$CA \cup X = \{(I,(0))\#(A,*)\} \cup X.$$

Note that $p : (X,A) \to (X/A,*)$ factors through $(CA \cup X,*)$:

$$(X,A) \xrightarrow{C(X,A)} (CA \cup X,*) \xrightarrow{h} (X/A,*)$$

where C is the cone functor and h identifies CA with the base point. As (X,A) is a finite cellular pair, it has the homotopy extension property, so the map $(CA \cup X) \to (X/A)$ is a homotopy equivalence.

Axioms for a reduced cohomology theory

We are now ready to state the conditions for a reduced cohomology theory.

A *reduced cohomology theory* is a sequence

$$\widetilde{H}^n : \mathscr{P}_0 \to \mathscr{A}$$

of contravariant functors and a sequence

$$\sigma^n : \widetilde{H}^{n+1} \circ \Sigma \to \widetilde{H}^n$$

of natural transformations such that

1) If $f_0, f_1 \in \mathscr{P}_0$ and $f_0 \simeq f_1$ in $\mathscr{P}_0$, then $\widetilde{H}^n(f_0) = \widetilde{H}^n(f_1)$ for all n;

2) $\sigma^n(X) : \widetilde{H}^{n+1}(\Sigma X) \to \widetilde{H}^n(X)$ is an isomorphism for all $X \in \mathscr{P}_0$; and

3) If $(X, A) \in \mathscr{P}^2$ and $* \in A$, then the sequence

$$\widetilde{H}^n(X/A) \xrightarrow{\widetilde{H}^n(\overline{p})} \widetilde{H}^n(X) \xrightarrow{\widetilde{H}^n(i)} \widetilde{H}^n(A)$$

is exact at $\widetilde{H}^n(X)$ for each n, where $\overline{p} : X \to X/A$ is the map collapsing A to a point and $i : A \to X$ is the inclusion map.

The condition 3) for reduced theories appears weaker than the corresponding condition for non-reduced theories. However, we may deduce the following condition

3$'$) If $(X, A) \in \mathscr{P}^2$ and $* \in A$, then the sequence

$$\cdots \to \widetilde{H}^{n-1}(A) \xrightarrow{\Delta^{n-1}} \widetilde{H}^n(X/A) \xrightarrow{\widetilde{H}^n(\overline{p})} \widetilde{H}^n(X) \xrightarrow{\widetilde{H}^n(i)} \widetilde{H}^n(A) \to \cdots$$

is exact, where Δ^{n-1} is the composition

$$\widetilde{H}^{n-1}(A) \xrightarrow[\cong]{\sigma^{n-1}(A)^{-1}} \widetilde{H}^n(\Sigma A) \xrightarrow{\widetilde{H}^n(k)} \widetilde{H}^n(CA \cup X) \xrightarrow[\cong]{} \widetilde{H}^n(X/A)$$

and $k : CA \cup X \to \Sigma A$ is the map collapsing X to a point.

To see the exactness at $\widetilde{H}^n(X/A)$ we observe that $\widetilde{H}^{n-1}(A) \to \widetilde{H}^n(CA \cup X) \to \widetilde{H}^n(X)$ is equivalent to

$$\widetilde{H}^n(\Sigma A) \to \widetilde{H}^n(CA \cup X) \to \widetilde{H}^n(X)$$

and 3) applies since $(CA \cup X)/X$ is homeomorphic to ΣA. To see the exactness at $\widetilde{H}^{n-1}(A)$, we observe that $\widetilde{H}^n(X) \to \widetilde{H}^{n-1}(A) \to \widetilde{H}^n(X/A)$ is equivalent to

$$\widetilde{H}^n(\Sigma X) \to \widetilde{H}^n(\Sigma A) \to \widetilde{H}^n(CA \cup X)$$

and 3) applies since ΣA has the homotopy type of $CX \cup (CA \cup X)$ and $(CX \cup (CA \cup X))/(CA \cup X)$ is homeomorphic to ΣX.

Theorem 1.5. *There is a natural* 1-1 *correspondence between theories* $\{H, \delta\}$ *and* $\{\widetilde{H}, \sigma\}$ *given by the commutative diagramme*

$$\begin{array}{ccc} \mathscr{P}^2 & \xrightarrow{\ p\ } & \mathscr{P}_0 \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle \widetilde{H}} & \\ & \mathscr{A} & \end{array}$$

Proof. For $(X, A) \in \mathscr{P}^2$ and a reduced theory $\{\widetilde{H}, \sigma\}$ define $(\alpha\widetilde{H}^n(X, A) = \widetilde{H}^n(X/A)$; define $\delta : (\alpha\widetilde{H}^{n-1})(A, \varnothing) \to (\alpha\widetilde{H}^n)(X, A)$ to be the composition

$$\widetilde{H}^{n-1}(A^+) \underset{\cong}{\xleftarrow{\sigma}} \widetilde{H}^n(\Sigma A^+) \xrightarrow{\widetilde{H}^n(\overline{p})} \widetilde{H}^n(X^+/A^+) \cong \widetilde{H}^n(X/A).$$

This system satisfies the cohomology axioms; in particular, δ is the image of Δ under α.

For $(X, *) \in \mathscr{P}_0$ and a theory $\{H, \delta\}$ define $(\beta H^n)(X) \equiv H^n(X, \{*\})$ and define σ to be composition of the isomorphisms

$$H^n(X), \{*\}) \cong H^{n+1}(CX, X) \cong H^{n+1}(\Sigma X, \{*\}),$$

where the first isomorphism is given by the exact sequence of the triple $(CX, X, \{*\})$ and the second is a special case of more general isomorphism

$$H^n(X, A) \xrightarrow{\cong} H^n(X/A, \{*\});$$

this is given by the composition

$$H^n(X/A, \{*\}) \to H^n(CA \cup X, CA) \to H^n(X, A),$$

where the first homomorphism is an isomorphism since the pairs have the same homotopy type and the second is an isomorphism by the exact sequence of the triad $(CA \cup X; X, CA)$. The system $\{\beta H, \sigma\}$ satisfies the axioms for a reduced cohomology theory; in particular, axiom 3 is a consequence of the exact sequence of a triple.

It is clear from the constructions that $\beta\alpha\{\widetilde{H}, \sigma\} = \{\widetilde{H}, \sigma\}$ and $\alpha\beta\{H, \delta\} = \{H, \delta\}$. □

1.1.4 Spectra

A *spectrum* $\mathscr{X}$ is an indexed family $\{X_i\}_{i \in \mathbb{Z}}$ of spaces with base point together with a family of base point preserving maps $f_i : \Sigma X_i \to X_{i+1}$.

Example 1.6. a) $\mathscr{S}$ is the spectrum with $X_i = \mathbb{S}^i$, the i-sphere and $f_i : \Sigma\mathbb{S}^i \to \mathbb{S}^{i+1}$ the identity.

b) For an abelian group G, $\mathscr{K}(G)$ is the spectrum whose i-th space is

$$\begin{cases} K(G, i) & i > 0 \\ G & i = 0 \\ \text{pt} & i < 0 \end{cases}$$

From the canonical homotopy equivalence $K(G,i) \xrightarrow{\simeq} \Omega K(G,i+1)$, we obtain the maps $\Sigma K(G,i) \longrightarrow K(G,i+1)$.

c) For X a space with base point, $\mathscr{S}(X)$ is the spectrum with i-th term ΣX. Thus, $\mathscr{S}(\mathbb{S}^0) = \mathscr{S}$.

d) For X a space with base point, and $\mathscr{Y}$ a spectrum, the spectra $X\#\mathscr{Y}$ and $\mathscr{Y}\#X$ are given by $\{X\#Y_i\}$ and $\{Y_i\#X\}$. In particular, we have the spectrum $\Sigma\mathscr{Y} = \mathbb{S}^1\#\mathscr{Y}$.

e) For $\mathscr{X}$ a spectrum, $\mathscr{X}^q$ is the spectrum with i-th term $(X^q)_i = X_{q+1}$.

We shall see that spectra define cohomology theories. Let $[A,B]$ denote the set of homotopy classes of base point preserving maps. For $A \in \mathscr{P}_0$ define $\{A,\mathscr{X}\}$ to be

$$\varinjlim([\Sigma^n A, X_n], i^n),$$

where for $\alpha \in [\Sigma^n A, X_n]$, $i^n(\alpha) \in [\Sigma^{n+1}A, X_{n+1}]$ is the composition

$$\Sigma\Sigma^n A \xrightarrow{\Sigma\alpha} \Sigma X \xrightarrow{f_n} X_{n+1}.$$

For $n \geq 1$, $[\Sigma^n A, X]$ is a group and i_n is a homomorphism; for $n \geq 2$, $[\Sigma^n A, X]$ is an abelian group. Thus, $\{A,\mathscr{X}\}$ is an abelian group.

Define the homomorphism $\sigma : \{A,\mathscr{X}\} \longrightarrow \{\Sigma A, \mathscr{X}^1\}$ to be the direct limit of $\sigma : [\Sigma^n A, X_n] \longrightarrow [\Sigma^n \Sigma A, X_{n+1}]$, where $\sigma^n = i^n$. We can take this limit since σ_n and the i^n commute. σ only shifts $\{A,\mathscr{X}\}$ over one in itself; so we have clearly

Lemma 1.7. $\sigma : \{A,\mathscr{X}\} \xrightarrow{\cong} \{\Sigma A, \mathscr{X}^1\}$.

For $g : A \longrightarrow B$ in $\mathscr{P}_0$, composition defines a homomorphism $g^* : \{B,\mathscr{X}\} \longrightarrow \{A,\mathscr{X}\}$.

Lemma 1.8. $(\mathrm{id})^* = \mathrm{id}$*;* $(g \circ f)^* = f^* \circ g^*$*; if* $f \simeq g$ *in* $\mathscr{P}_0$*, then* $f^* = g^*$.

Lemma 1.9. *For* $* \in B \subset Y$*,* $\Sigma^n(CB \cup Y)$ *is homeomorphic to* $C(\Sigma^n B) \cup (\Sigma^n Y)$.

It suffices to prove this for $n = 1$, but that is clear from the definitions.

Lemma 1.10. *For* $* \in B \subset Y$*, and* $B \xrightarrow{i} Y \xrightarrow{p} Y/B$ *the inclusion and collapsing maps, the sequence* $\{Y/B,\mathscr{X}\} \xrightarrow{p^*} \{Y,\mathscr{X}\} \xrightarrow{i^*} \{B,\mathscr{X}\}$ *is exact at* $\{Y,\mathscr{X}\}$.

Proof. For X a space with base point

$$[CB \cup Y, X] \to [Y,X] \to [B,X]$$

is an exact sequence of sets. Thus, by Lemma 1.9

$$[\Sigma^n(CB \cup Y), X_n] \to [\Sigma^n Y, X_n] \to [\Sigma^n B, X_n]$$

is an exact sequence of groups; but exactness commutes with direct limits and $CB \cup Y$ and Y/B have the same homotopy type. The lemma follows. □

Theorem 1.11. *For a spectrum and a space $Y \in \mathscr{P}_0$, $\widetilde{H}^q(Y;\mathscr{X}) = \{Y, \mathscr{X}_q\}$ and $\sigma^n : \widetilde{H}^{n+1}(\Sigma Y;\mathscr{X}) \to \widetilde{H}^n(Y;\mathscr{X})$, the inverse of the isomorphism of Lemma 1.7, define a reduced cohomology theory.*

A map $\mathscr{F} : \mathscr{X} \to \mathscr{Y}$ of spectra is a collection $f_i : X_i \to Y_i$ of maps of the terms of the spectra which for i sufficiently large homotopy commute with the defining maps of the spectra:

$$\begin{array}{ccc} \Sigma X_i & \xrightarrow{\Sigma} & \Sigma Y_i \\ \downarrow & & \downarrow \\ X_{i+1} & \longrightarrow & Y_{i+1} \end{array}$$

Spectra have the role of coefficients in these cohomology theories and maps of them induced "coefficient homomorphisms"; i.e., natural transformations of theories.

We cite without proof the following result of E. Brown [18]:

Theorem 1.12. *If $\widetilde{H}$ is a reduced cohomology theory on $\mathscr{P}_0$ and $\widetilde{H}^q(\mathbb{S}^0)$ is countable for all q, then there is a spectrum $\mathscr{Y}$ such that $\widetilde{H}^*(-)$ and $\widetilde{H}^*(-;\mathscr{Y})$ are naturally equivalent. Furthermore, $\mathscr{Y}$ can be found so that the map $Y_n \to \Omega Y_{n+1}$ induced by $\Sigma Y_n \to Y_{n+1}$, is a homotopy equivalence ($\mathscr{Y}$ is an Ω-spectrum).*

1.2 Spectral Sequence of a Fibration

1.2.1 Exact Couples

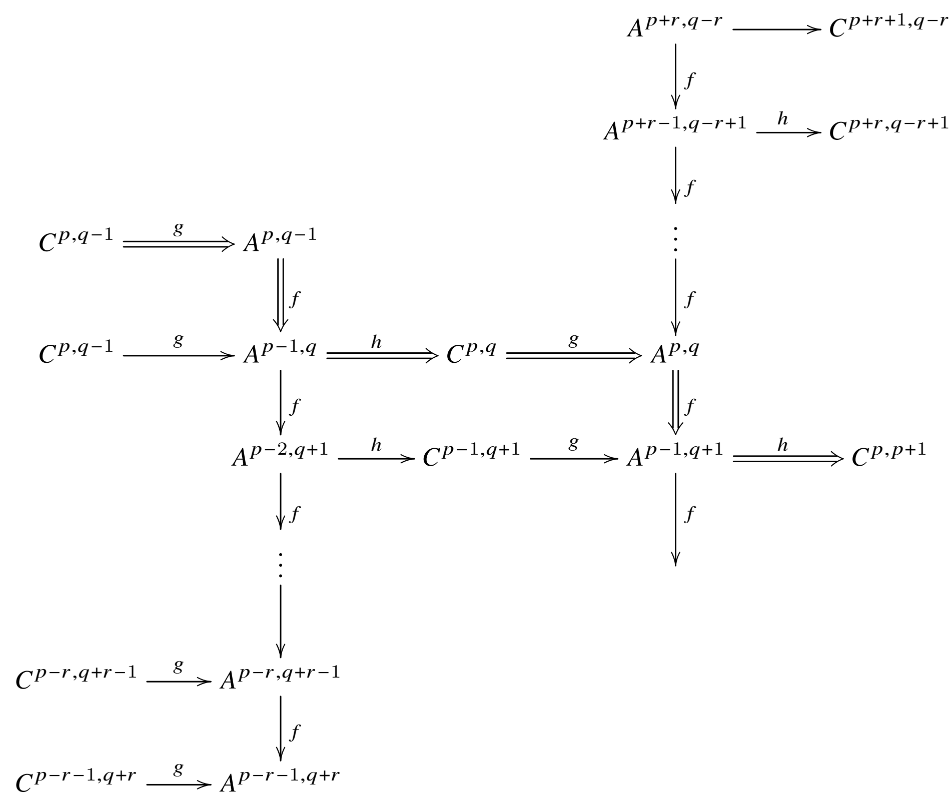
Some familiarity with exact couples is assumed and we only review briefly the basic definitions, notation and propositions.

For each pair (p, q) of integers we are given abelian groups $A^{p,q}$ and $C^{p,q}$ and homomorphisms f, g and h so that the "p-sequence"

$$\cdots \to A^{p-1,q-1} \xrightarrow{h} C^{p,q-1} \xrightarrow{g} A^{p,q-1} \xrightarrow{f} A^{p-1,q} \xrightarrow{h} C^{p,q} \xrightarrow{g} A^{p,q} \to \cdots$$

is exact.

We arrange these sequences in the following diagramme:



The sequence in the bold aroows is the p-sequence and the sum of the indices in each column is constant.

We define

$$
\begin{aligned}
Z_r^{p,q} &= g^{-1}(\operatorname{im} f^{(r-1)}) \subset C^{p,q} \quad \text{and} \\
B_r^{p,q} &= h(\ker f^{(r-1)}) \subset C^{p,q}.
\end{aligned}
$$

Then we can define $\theta_r : Z_r^{p,q} \to C^{p+r,q-r+1}/B_r^{p+r,q-r+1}$ to be $h \circ (f^{(r-1)})^{-1} \circ g$. Letting $E_r^{p,q} \equiv Z_r^{p,q}/B_r^{p,q}$, we see that θ_r induces a homomorphism

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

Furthermore, $d_r \circ d_r = 0$ and so the homology groups $\mathscr{H}(E_r, d_r)$ are defined as usual. Exactness in the diagramme at $A^{p+r-1,q-r+1}$ and $A^{p-r,q+r-1}$ imply that $\mathscr{H}(E_r, d_r) \cong E_{r+1}^{p,q}$.

The sequence of groups $\{E_r^{p,q}\}$ and differentials d_r is the spectral sequence of the exact couple (A, C), $A = \oplus_{p,q} A^{p,q}$ and $C = \oplus_{p,q} C^{p,q}$.

Define

$$
\begin{aligned}
Z_\infty^{p,q} &= g^{-1}(\cap_\nu \Im f^{(\nu)}) \subset C^{p,q}, \\
B_\infty^{p,q} &= h(\cup_\nu \ker f^{(\nu)}) C^{p,q}, \\
E_\infty^{p,q} &= Z_\infty^{p,q} / B_\infty^{p,q}.
\end{aligned}
$$

For $p+q=n$, define $A^n = \varprojlim(A^{p,q}, f)$ and $F_pA^n = \ker(A^n \to A^{p,q})$. Then $F_p(A^n) \supset F_{p+1}(A^n)$.

If the following two convergence conditions hold

1) $A^{p-r,q+r} \xrightarrow{f} A^{p-r-1,q+r+1}$ is 0 for $r > \nu(p,q)$, and

2) $A^{p+r,q-r} \xrightarrow{f} A^{p+r-1,q-r+1}$ is $\cong$ for $r > \widetilde{\nu}(p,q)$,

then $Z_r^{p,q} \cong Z_\infty^{p,q}$ and $B_r^{p,q} \cong B_\infty^{P,q}$ for large r and $E_r^{p,q} \cong E_\infty^{p,q} \cong F_{p-1}(A^n)/F_p(A^n)$, $n = p+q$.

We note that the spectral sequence is a covariant functor on exact couples; i.e., if $F : (A, C) \to (A', C')$ is a map of exact couples, then there are induced maps

$$
E_r(F) : E_r \to E_r'
$$

of spectral sequences with the appropriate functorial properties. If the convergence conditions hold and some $E_r(F)$ is an isomorphism, then all succeeding ones are and $A_n \cong A_n'$.

1.2.2 Spectral Sequence of a h^*-fibration

Let $\pi : E \to B$ be a continuous map of spaces in $\mathscr{P}$; assume B is connected. It suffices to consider the case in which E and B are finite simplicial complexes and π is simplicial, for this is equivalent to the general case up to homotopy.

If $\{h^n\}$ is a cohomology theory, define $A^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}))$, where $B^{(p)}$ is the $p-$skeleton of B, and $C^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}), \pi^{-1}(B^{(p-1)}))$. These groups form an exact couple satisfying the convergence condition. Since E is finite dimensional, $A^n \cong h^n(E)$.

We shall say *π is an h^*-fibration* if for each simplex Δ f B and each vertex v of Δ, the homomorphism $h^*(\pi^{-1}(\Delta)) \to h^*(\pi^{-1}(v))$ induced by inclusion is an isomorphism. A Serre fibration is an H^*-fibration, where H^* is ordinary singular cohomology, and we will see later that an H^*-fibration is an h^*-fibration for any cohomology theory h^*.

To simplify the notation in what follows, we will write $\{X, Y\}^i$ for $h^i(\pi^{-1}(X), \pi^1(Y))$ when $Y \subset X \subset B$. We assume throughout the remainder of this section that π is an h^*-fibration.

Lemma 1.13. *For simplices $\Delta' \subset \Delta \subset B$, the homomorphism $\{\Delta\}^* \to \{\Delta'\}^*$ induced by inclusion is an isomorphism.*

Proof. The diagramme

$$\begin{array}{ccc} \{\Delta\}^* & \longrightarrow & \{\Delta'\}^* \\ & \searrow \qquad \swarrow & \\ & \{v\}^* & \end{array}$$

commutes because the homomorphisms are induced by inclusion, and the two lower homomorphisms are isomorphisms. □

Define a *k-box* to be either a k-simplex or the union of members of a non-empty collection of $(k-1)$-simplices of a k-simplex which does not include all of its $(k-1)$-simplices.

Lemma 1.14. *If $D' \subset D$ are boxes in B , the homomorphism $\{D\}^* \to \{D'\}^*$ induced by inclusion is an isomorphism.*

Proof. The argument is by induction on the dimension of the smaller box D'. It suffices to give the proof when D is a simplex, because each box is contained in a simplex. By Lemma 1.13 the conclusion follows if D' is also a simplex. Suppose $D' = D_1 \cup D_2$, where D_1 is a box and D_2 is a simplex. hen $D1 \cap D_2$ is a box of lower dimension than that of D'. Hence, by the induction assumption and Lemma 1.13, the indicated homomorphisms in the diagramme

$$\begin{array}{ccc} \{D\}^* & \longrightarrow & \{D'\}^* \\ \cong\downarrow & & \downarrow\cong \\ \{D_2\}^* & \xrightarrow[\cong]{} & \{D_1 \cap D_2\}^* \end{array}$$

are isomorphisms, and the diagramme commutes since all homomorphisms are induced by inclusions. □

Lemma 1.15. *If v_p is the last vertex of the p-simplex Δ_p, then $\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{v_p\}^q$.*

Proof. Let $\angle i$ denote the box containing all $(p-1)$-simplices of Δ_p except Δ_p^i, the face opposite the i-th vertex. By the cohomology sequence of the triple $\{\Delta_p, \dot{\Delta}_p, \angle 0\}$ and Lemma 1.14,

$$\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{\dot{\Delta}_p, \angle 0\}^{p+q-1}.$$

By excision $\{\dot{\Delta}_p, \angle 0\}^{p+q-1} \cong \{\Delta_p^0, \dot{\Delta}_p^0\}^{p+q-1}$. Continuing this procedure, we get

$$\{\Delta_p, \dot{\Delta}_p\}^{p+q} \cong \{\dot{\Delta}_p, \angle 0\}^{p+q-1} \cong \cdots \cong \{v_p, \varnothing\}^q.$$

□

A system $\mathscr{G}$ of *local coefficients* over B is a function from simplices of B to abelian groups, $\Delta \mapsto G_\Delta$, together with a function from pairs (Δ, Δ'), $\Delta' \subset \Delta$, of simplices of B to homomorphisms

$$\eta_{\Delta\Delta'} : G_{\Delta'} \to G_\Delta$$

such that

1) $\eta_{\Delta\Delta'} = 1$ if $\Delta = \Delta'$ and

2) $\eta_{\Delta\Delta'} \circ \eta_{\Delta'\Delta''} = \eta_{\Delta\Delta''}$ if $\Delta'' \subset \Delta' \subset \Delta$.

A *p-cochain f^p with coefficients in $\mathscr{G}$* is a function which associates with each p-simplex Δ an element of G_Δ; the p-cochains form an abelian group by coordinate addition, $C^p(B;\mathscr{G})$. There is a homomorphism

$$\delta : C^p(B;\mathscr{G}) \to C^{p+1}(B;\mathscr{G})$$

defined by

$$(\delta f^p)(\Delta_{p+1}) = \sum_{j=0}^{p+1} \eta_{\Delta_{p+1}\Delta^j_{p+1}} f^p(\Delta^j_{p+1}).$$

As usual one computes that $\delta \circ \delta = 0$ and defines cohomology $H^p(B;\mathscr{G})$ with local coefficients as $\ker\delta/\operatorname{im}\delta$ in dimension p.

In the spectral sequence arising from the exact couple of an h^*-fibration one has

$$\begin{aligned} E_1^{p,q} \equiv C^{p,q} &\equiv \{B^{(p)}, B^{(p-1)}\}^{p+q} \\ &\cong \oplus_{\Delta_p \subset B}\{\Delta_p, \dot{\Delta}_p\}^{p+q}, \quad \text{by excision} \\ &\cong \oplus_{\Delta_p \subset B}\{v_{\Delta_p}\}^p, \quad \text{by Lemma 1.15,} \\ &\cong C^p(B; h^q(F)), \end{aligned}$$

where $h^q(F)$ is the local coefficient system defined by the function $\Delta_p \to \{v_{\Delta_p}\}^q$ with v_{Δ_p} the last vertex of Δ_p. For $\Delta_s \subset \Delta_p$ the homomorphism (in fact isomorphism)

$$\eta_{\Delta_p\Delta_s} : \{v_{\Delta_s}\}^q \to \{v_{\Delta_p}\}^q$$

is defined via the inclusion isomorphisms with $\{[v_{\Delta_s}, v_{\Delta_p}]\}^q$, where $[v_{\Delta_s}, v_{\Delta_p}]$ is the 1-simplex from v_{Δ_s} to v_{Δ_p}. That this is a local coefficient system then follows from the first axiom in 1.1.1 for the cohomology system $\{h^q\}$. Finally, we note that since B is connected, $\{v^q\}$ is the same group $h^q(F)$, $F = \pi^{-1}(v)$, for all vertices v of B. Thus, we have defined an isomorphism

$$\lambda_{p,q} : E_1^{p,q} \xrightarrow{\cong} C^p(B; h^q(F)).$$

Theorem 1.16. *$\lambda_{p+1,q} \circ d_1 = \delta\lambda_{p,q}$: that is, the diagramme*

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{d_1} & E_1^{p+1,q} \\ \downarrow{\scriptstyle \lambda_{p,q}} & & \downarrow{\scriptstyle \lambda_{p+1,q}} \\ C^p(B; h^q(F)) & \xrightarrow[\delta]{} & C^{p+1}(B; h^q(F)) \end{array}$$

is commutative.

Proof. By naturality of the spectral sequence of an exact couple and of the isomorphisms $\lambda_{p,q}$, it suffices to prove the assertion in the case $B = \Delta_{p+1}$.

Consider the composition φ^i_{p+1} of

$$\{\Delta^i_{p+1}, \dot{\Delta}^i_{p+1}\}^{p+q} \xrightarrow{j_1^*} \{\dot{\Delta}_{p+1}, \Delta^{p-1}_{p+1}\}^{p+q} \xrightarrow{i^*} \{\dot{\Delta}_{p+1}\} \xrightarrow{\delta} \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1}.$$

By the cohomology sequence of the triple $\{\Delta_{p+1}, \dot{\Delta}_{p+1}, \angle i\}$ and Lemma 1.14, the homomorphism $\Delta : \{\dot{\Delta}_{p+1}, \angle i\}^{p+q} \to \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1}$ is an isomorphism. By excision, $\{\Delta^i_{p+1}, \dot{\Delta}^i_{p+1}\}^{p+q} \cong \{\dot{\Delta}_{p+1}, \angle i\}^{p+q}$. From the commutative diagramme

$$\begin{array}{ccccc}
\{\dot{\Delta}_{p+1}, \Delta^{p-1}_{p+1}\}^{p+q} & \xrightarrow{i^*} & \{\dot{\Delta}_{p+1}\} & & \\
& & & \searrow^{\delta} & \\
\uparrow{\scriptstyle j_1^*} & & & & \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1} \\
& & & \nearrow_{\Delta} & \\
\{\Delta^i_{p+1}, \dot{\Delta}^i_{p+1}\}^{p+q} & \longrightarrow & \{\dot{\Delta}_{p+1}, \angle i\}^{p+q} & &
\end{array}$$

it follows then that φ^i_{p+1} is an isomorphism.

The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is the composition $\delta \circ i^*$. For $f^p \in C^p(B; h^q(F))$,

$$\lambda_{p+1,q} d_1 \lambda^{-1}_{p,q}(f^p)(\Delta_{p+1}) = \sum_{i=0}^{p+1} \varphi^i_{p+1} f_p(\Delta^i_{p+1}).$$

Thus, we wish to show that $\varphi^i_{p+1} = (-1)^i \eta_{\Delta_{p+1}\Delta^i_{p+1}}$.

For $i = 0$ consider the triad $\{\Delta_{p+1}, \Delta^0_{p+1}, \angle 0\}$.

$$\begin{array}{ccccccc}
0 \longrightarrow & \{\Delta^0_{p+1}, \dot{\Delta}^0_{p+1}\}^{p+q} & \underset{\cong}{\xrightarrow{\alpha}} & \{\Delta_{p+1}, \dot{\Delta}_{p+1}\}^{p+q+1} & \longrightarrow 0 \\
& {\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle\delta} & \\
& \{\dot{\Delta}_{p+1}, \angle 0\}^{p+q} & \longrightarrow & \{\dot{\Delta}_{p+1}\}^{p+q} &
\end{array}$$

By the proof of Lemma 1.15, α is $\eta_{\Delta_{p+1}\Delta^0_{p+1}}$; it is also φ^0_{p+1}.

Assume $\varphi^i_j = (-1)^i \eta_{\Delta_j \Delta^i_j}$ for $j < p+1$ and for $j = p+1$ and $i < k$. Let $\Delta^{k,k-1}_{p+1}$ denote the $(p-1)$-simplex opposite the edge $(k, k-1)$. Let $f^p \in C^{p-1}(\Delta^{k,k-1}_{p+1})$. Then

$$\lambda_{p,q} d_1 \lambda^{-1}_{p-1,q} f^{p-1} = \delta f^{p-1} = \begin{cases} (-1)^{k-1} \eta_{\Delta^{k-1}_{p+1}\Delta^{k,k-1}_{p+1}} f^{p-1}(\Delta^{k,k-1}_{p+1}) & \text{on} \quad \Delta^{k-1}_{p+1} \\ (-1)^{k-1} \eta_{\Delta^{k}_{p+1}\Delta^{k,k-1}_{p+1}} f^{p-1}(\Delta^{k,k-1}_{p+1}) & \text{on} \quad \Delta^{k}_{p+1} \end{cases}$$

Since $d_1 \circ d_1 = 0$,

$$\begin{aligned} 0 &= \lambda_{p+1,q} d_1 \lambda_{p,q}^{-1} \lambda_{p,q} d_1 \lambda_{p-1,q}^{-1} (f^{p-1})(\Delta_{p+1}^{k,k-1}) \\ &= \varphi_{p+1}^k ((-1)^{k-1} \eta_{\Delta_{p+1}^{k-1} \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) + \varphi_{p+1}^k ((-1)^{k-1} \eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) \\ &= \eta_{\Delta_{p+1} \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) + \varphi_{p+1}^k ((-1)^{k-1} \eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) \end{aligned}$$

Thus,

$$\varphi_{p+1}^k (\eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})) = (-1)^k \eta_{\Delta_{p+1} \Delta_{p+1}^k} (\eta_{\Delta_{p+1}^k \Delta_{p+1}^{k,k-1}} f^{p-1}(\Delta_{p+1}^{k,k-1})).$$

Hence,

$$\varphi_{p+1}^k = (-1)^k \eta_{\Delta_{p+1} \Delta_{p+1}^k}.$$

□

Corollary 1.17. *The chain complexes $E_1^{p,q}$, d_1 and $\{C^p(B; h^q(F)), \delta\}$ are naturally equivalent.*

Corollary 1.18. *In the spectral sequence of the h^*-fibration $\pi : E \to B$, the term $E_2^{p,q}$ is naturally isomorphic with $H^p(B; h^q(F))$, where the latter is ordinary cohomology with local coefficients.*

To summarise the results of this section, there is a functor from the category of h^*-fibrations ($f \to E \xrightarrow{\pi} B$) to the category of exact couples; by composition, then to the category of spectral sequences. There is a natural isomorphism $E_2^{p,q} \cong H^p(B; h^q(F))$, in which $h^q(F)$ is a system of local coordinates. Furthermore, $\{E_\infty^{p,q}\}$ is the associated graded system to a filtration of $h^*(E)$. The spectral sequence is also natural with respect to transformations of cohomology theories.

In particular, if $\pi : X \to X$ is the identity map, then $E_2^{p,q} \cong H^p(X; h^q(\mathrm{pt}))$ and $E_\infty^{p,q}$ is associated to a filtration of $h^*(X)$. This relation can sometimes be exploited to compute or to deduce properties of $h^*(X)$.

1.2.3 Applications of the Spectral Sequence

Proposition 1.19. *If $\tau : h^* \to k^*$ is a natural transformation of cohomology theories and $\tau : h^*(\mathrm{pt}) \xrightarrow{\cong} k^*(\mathrm{pt})$ is an isomorphism, then $\tau : h^*(X, A) \to k^*(X, A)$ is an isomorphism for all pairs (X, A).*

Proof. The transformation τ induces a homomorphism of the spectral sequence of $\mathrm{id} : X \to X$ in the h^*-theory into that in the k^*-theory. The hypothesis implies this is an isomorphism at the E_2 level and thus for $h^*(X)$ into $k^*(X)$. For a pair (X, A), we consider $CA \cup X$. □

Proposition 1.20. *If $f : B \to B'$ so that $f : H^*(B') \to H^*(B)$ is an isomorphism for ordinary singular cohomology (integer coefficients), then $f_h^* : h^*(B') \to h^*(B)$ is an isomorphism for any cohomology theory h^*.*

Proof. The map f induces a map f of the identity fibration $B \to B$ into $B' \to B'$ and thus a map of the spectral sequences of these fibrations. In particular, the naturality of the universal coefficient sequence and the five lemma imply

$$E_2^{p,q}(f) : H^p(B, h^q(\mathrm{pt})) \to H^p(B'; h^q(\mathrm{pt}))$$

is an isomorphism. The conclusion follows. □

Note that this implies our earlier remark that an H^*-fibration is an h^*-fibration for any cohomology theory h^*.

Proposition 1.21. *If $h^i(\mathrm{pt}) = 0$ for $i \neq 0$, then the cohomology theory h^* is naturally equivalent to singular cohomology with coefficients $h^0(\mathrm{pt})$, $H^*(-; h^0(\mathrm{pt}))$.*

Proof. The spectral sequence collapses and we have

$$H^p(B; h^0(\mathrm{pt})) \cong E_2^{p,q} \cong E_2^{p,\infty} \cong F_{p-1}h^p(B)/F_p h^p(B) \cong h^p(B).$$

□

Proposition 1.22. $H^n(B; G) \cong [B, K(G, n)]$.

Proof. The groups $\{B, \mathcal{K}(G)\}$ give a cohomology theory k^* in which $k^0(\mathrm{pt}) \cong \varinjlim[\mathbb{S}^n, K(G, n)]$. The composition

$$[\mathbb{S}^n, K(G, n)] \xrightarrow{\Sigma} [\mathbb{S}^{n+1}, \Sigma K(G, n+1)] \to [\Sigma\mathbb{S}^{n+1}, K(G, n+1)]$$

is the isomorphism $[\mathbb{S}^n, \Omega K(G, n+1)] \to [\Sigma\mathbb{S}^n, K(G, n++1)]$. By definition each of these groups is G. In other dimensions $k^i(\mathrm{pt}) = 0$. Thus $\{B, \mathcal{K}(G)^n\} \cong H^n(B; G)$. The object on the left is $\varinjlim[\Sigma^i B, K(G, n+i)]$. As seen above, these groups are isomorphic and are mapped isomorphically. Thus $[B, K(G, n)] \cong \{B, \mathcal{K}(G)^n\} \cong H^n(B; G)$. □

It is interesting to note this proof does not use obstruction theory.

1.2.4 The "Universal Cohomology Theory"

For pairs (X, A) and (Y, B), define

$$\pi_j^i(X, A; Y, B) = \varinjlim_n [\Sigma^{j+n}(X/A), \Sigma^{i+n}(Y/B)].$$

For fixed j and (Y, B), this is a cohomology theory.

For an arbitrary cohomology theory h define

$$\tau : \pi_j^i(X, A; Y, B) \times h^k(Y/B) \to h^{k+1-j}(X/A)$$

as follows:

for $f \in [f] \in [\Sigma^{j+N}(X/A), \Sigma^{i+N}(Y/B)]$, the composition

$$h^k(Y/B) \to h^{k+i+N}(\Sigma^{i+N}(Y/B)) \to h^{k+i+N}(\Sigma^{j+N}(X/A)) \to h^{k+i-j}(X/A)$$

defines the pairing τ since it is independent of the representative f in the class of the limit. The pairing is bilinear; and so, induces

$$\tau : \pi^i_j(X, A; Y, B) \otimes h^k(Y/B) \to h^{k+i-j}(X/A).$$

In particular, for $j = 0$ and $(Y, B) = (\mathbb{S}^0, \mathrm{pt})$, $\pi^i_j(X, A; Y, B) = \pi^i_{\mathcal{S}}(X, A)$, the cohomology theory called *stable cohomotopy.* Then for each n

$$\tau \otimes 1 : \sum_{i+j=n} \pi^i_{\mathcal{S}}(X, A) \otimes h^j(\mathrm{pt}) \otimes \mathbb{Q} \to h^n(X, A) \otimes \mathbb{Q}$$

is a natural transformation of cohomology theories.

Theorem 1.23. *The transformation*

$$\tau \otimes 1 : \sum_{i+j=n} \pi^i_{\mathcal{S}}(X, A) \otimes h^j(\mathrm{pt}) \otimes \mathbb{Q} \to h^n(X, A) \otimes \mathbb{Q}$$

is a natural equivalence of cohomology theories, where $\mathbb{Q}$ denotes the rationals. Thus, up to torsion $\pi^i_{\mathcal{S}}$ is a "universal cohomology theory".

Proof. By Proposition 1.19 it suffices to verify the conclusion for $(X, A) = (\mathbb{S}^0, \mathrm{pt})$. But $\pi^i_{\mathcal{S}}(\mathbb{S}^0, \mathrm{pt})$ is a torsion group for $i \neq 0$ and is the integers $\mathbb{Z}$ for $i = 0$ by a theorem of Serre since the stable cohomotopy groups of spheres are isomorphic to the stable homotopy groups of spheres. Then $\tau \otimes 1$ is an isomorphism for $(X, A) = (\mathbb{S}^0, \mathrm{pt})$. □

The maps $\mathbb{S}^n \to K(\mathbb{Z}, n)$ generating the homotopy of $K(\mathbb{Z}, n)$ induce a map of spectra $\mathcal{S} \to \mathcal{K}(\mathbb{Z})$. By Proposition 1.22 this induces a natural transformation of cohomology theories

$$\pi^i_{\mathcal{S}}(-1) \to H^i(-).$$

For a cohomology theory h^* define the "generalised Chern character"

$$Ch : h^n(X, A) \to \sum_{i+j=n} H^i(X, A; h^j(\mathrm{pt}) \otimes \mathbb{Q})$$

to be the composition

$$h^n(X, A) \to h^n(X, A)\otimes\mathbb{Q} \xrightarrow{(\tau\otimes 1)^{-1}} \sum_{i+j=n} \pi^i_{\mathcal{S}}(X, A)\otimes h^j(\mathrm{pt}) \to \sum_{i+j=n} H^i(X, A; h^j(\mathrm{pt})\otimes\mathbb{Q}).$$

Corollary 1.24. *The transformation*

$$Ch \otimes 1 : h^n(X, A) \otimes \mathbb{Q} \to \sum_{i+j=n} H^i(X, A; h^j(\mathrm{pt}) \otimes \mathbb{Q})$$

is a natural equivalence of cohomology theories. It is also natural under transformations of cohomology theories $h^* \to k^*$.

Proof. Proposition 1.19 implies the result since it is clearly true for $(X, A) = (\mathbb{S}^0, \mathrm{pt})$. ☐

Corollary 1.25. *Let h and k be cohomology theories, with k a $\mathbb{Q}$-module valued functor. Then for*

$$\varphi : h^*(\mathrm{pt}) \to k^*(\mathrm{pt}),$$

there is a unique natural transformation

$$\Phi : h^*(-) \to k^*(-)$$

of cohomology theories extending φ.

Proof. Consider the diagramme

$$\begin{array}{ccccc} h^*(X,A) & \longrightarrow & h^*(X,A)\otimes\mathbb{Q} & \xrightarrow{\cong} & \sum H^*(X,A;h^*(\mathrm{pt})\otimes\mathbb{Q}) \\ \downarrow{\scriptstyle\Phi} & & & & \downarrow{\scriptstyle\varphi} \\ k^*(X,A) & \longrightarrow & k^*(X,A)\otimes\mathbb{Q} & \xrightarrow[\cong]{} & \sum H^*(X,A;k^*(\mathrm{pt})\otimes\mathbb{Q}) \end{array}$$

The homomorphism Φ is defined by means of the diagramme; it is unique since any such homomorphism must factor through $h^* \otimes \mathbb{Q}$. ☐

1.3 Multiplicative Cohomology Theories

1.3.1 Preliminaries

A cohomology theory h^* is *multiplicative* if for each (X, A) and (Y, B), there is a homomorphism

$$\otimes : h^i(X,A) \otimes h^j(Y,B) \to h^{i+j}(X \times Y, X \times B \cup A \times Y)$$

which is associative, (graded) commutative (i.e., $u^i \otimes v^j = (-1)^{ij} v^j \otimes u^i$) , natural under maps of pairs, has a unit $1 \in h^0(\mathbb{S}^0, pt)$, and makes the following diagramme commute:

$$\begin{array}{ccc} h^i(A)\otimes h^j(Y,B) & \xrightarrow{\otimes} & h^{i+j}(A\times Y, A\times B) \\ & & \downarrow{\scriptstyle\gamma} \\ \downarrow{\scriptstyle\delta\times 1} & & h^{i+j}(A\times Y\cup X\times B, X\times B) \\ & & \downarrow{\scriptstyle\delta} \\ h^{i+1}(X,A)\otimes h^j(Y,B) & \xrightarrow[\otimes]{} & h^{i+j+1}(X\times X\times B\cup A\times Y) \end{array}$$

The homomorphism γ is excision and the δ on the right side is that of the cohomology sequence of the triple $(X \times Y, A \times Y \cup X \times B)$.

If $(X; A, B)$ is a triad, the external pairing just described defines an internal pairing
$$h^i(X, A) \otimes h^j(X, B) \to h^{i+j}(X, A \cup B)$$
sending (u, v) into $u \cup v$. This pairing is the composition
$$h^i(X, A) \otimes h^j(X, B) \xrightarrow{\otimes} h^{i+j}(X \times X, X \times B \cup A \times Y) \xrightarrow{\Delta^*}, h^{i+j}(X, A \cup B),$$
where $\Delta : X \to X \times X$ is the diagonal map.

This internal pairing has the properties

1. If $A = B$, $h^*(X, A)$ is a ring;

2. If $A = B = \varnothing$, $h^*(X)$ is a ring with unit;

3. If $B = \varnothing$, $h(X, A)$ is an $h^*(X)$-module; and

4. $h^*(A) \xrightarrow{\delta} h^*(X, A)$ is an $h^*(X)$-module homomorphism.

Conversely, given an internal pairing $\cup$, there is an external pairing $\otimes$ defined by
$$x \otimes y = p^*x \cup q^*y,$$
where $x \in h^*(X, A)$, $y \in h^j(Y, B)$ and p and q are the projections of $(X\times X, A\times Y)$ onto (X, A) and $(X \times Y, X \times B)$ onto (Y, B), respectively.

Further, if we are given a natural internal pairing $\cup$ on $h^*(X, A)$ for all pairs $(X, A) \in \mathscr{P}^2$ which satisfies 1, 2, 3 and 4, we can define an external pairing by the following device.

Let i_1 and i_2 denote the injections of $(X, *)$ and $(Y, *)$ in $(X \times Y, *)$ and p_1 and p_2 the projections of $(X \times Y, *)$ onto $(X, *)$ and $(Y, *)$. For $u \in h^i(X, *)$ and $v \in h^j(Y, *)$, the element $p_1^*u \cup p_2^*v \in h^{i+j}(X \times Y, *)$. Since $p_2 \circ i_1 : (X, *) \to (*, *) \subset (Y, *)$,
$$i_1^*(p_1^*u \cup p_2^*v) = i_1^*p_1^*u \cup (0) = 0.$$
Similarly, $i_2^*(p_1^*u \cup p_2^*v) = 0$. Thus, the element $p_1^*u \cup p_2^*v$ lies in $h^{i+j}(X\#Y, *)$. We define $u \otimes v \equiv p_1^*u \cup p_2^*v \in h^{i+j}(X \times Y, X \times * \cup * \times Y)$. For (X, A) and (Y, B) we make the same construction on $(X/A, *)$ and $(Y/B, *)$ and use the homeomorphism $(X/A)\#(Y/B) \cong X \times Y/(X \times B \cup A \times Y)$. It can be checked that this is an external pairing.

We note that if $X = Y$ and $\Delta : X \to X \times X$ is the diagonal map then since $p_1\Delta = p_2\Delta = \mathrm{id}$,
$$\Delta^*(u \otimes v) = \Delta^*p_1^*u \cup \Delta^*Ap_2^*v = u \cup v;$$
and so, the previous process for obtaining an internal pairing from an external one when applied to this external pairing yields the internal pairing we started with.

In $h^n(\mathbb{S}^n, *)$ define a class γ^n to be $(\Sigma^n)^*(1)$ where 1 is the unit of $h^0(\mathbb{S}^0, *)$ and $(\Sigma^n)^*$ is the suspension isomorphism $h^0(\mathbb{S}^0, *) \to h^n(\mathbb{S}^n, *)$. In cohomology with the dimension axiom, γ is a generator of $h^n(\mathbb{S}^n, *)$.

Proposition 1.26. *The diagramme*

$$\begin{array}{ccc}
 & & h^{i+n}(\mathbb{S}^n \times X, \mathbb{S}^n \times A \cup (*) \times X) \\
 & \overset{\gamma^n \times}{\nearrow} & \Big\downarrow \cong \\
h^i(X,A) & \underset{(\Sigma^n))^*}{\searrow} & \\
 & & h^{i+n}(\Sigma^n(X/A), *)
\end{array}$$

is commutative, where $\gamma^n \times$ denotes external multiplication by γ^n and the vertical homomorphism is an excision isomorphism.

Proof. It suffices to prove this for $n = 1$ for it will then follow that $\gamma^n = \gamma^1 \times \gamma^{n-1}$.

Consider the following diagramme.

$$\begin{array}{ccccc}
h^0(\mathbb{S}^0, *) \otimes h^i(X,A) & \xrightarrow{\cong} & h^i(\mathbb{S}^0 \times X, (*) \times X \cup \mathbb{S}^0 \times A) & \longrightarrow & h^i(X/A, *) \\
\cong \Big\downarrow & & \Big\downarrow \cong & & \Big\downarrow (\Sigma)^* \\
h^1(C\mathbb{S}^0, \mathbb{S}^0) \otimes h^i(X,A) & \longrightarrow & h^{i+1}(C\mathbb{S}^0 \times X, \mathbb{S}^0 \times X \cup C\mathbb{S}^0 \times A) & \longrightarrow & h^{i+1}(\Sigma(X/A), *) \\
\cong \Big\downarrow & & \Big\downarrow \cong & \nearrow & \\
h^i(\mathbb{S}^1, *) \otimes h^i(X,A) & \longrightarrow & h^{i+1}(\mathbb{S}^1 \times X, (*) \times X \cup \mathbb{S}^1 \times A) & &
\end{array}$$

The upper line gives $(\Sigma^1)^*$ and the lower line $\gamma^1(-)$. Commutativity is clear except possibly in the upper left-hand corner, where it holds by definition of the external product. □

Proposition 1.27. *Let h^* and k^* be multiplicative cohomology theories. If $k^0(\mathrm{pt})$ is a $\mathbb{Q}$-module, then k^* can be made a $\mathbb{Q}$-module valued functor. If $\varphi : h^*(\mathrm{pt}) \to k^*(\mathrm{pt})$ and Φ is the unique extension of φ as in Corollary 1.25 of Theorem 1.23 and if φ is a multiplicative homomorphism, then Φ is also multiplicative.*

Proof. k^* can be made a $\mathbb{Q}$-module valued functor because $k^i(X,A) \otimes k^0(\mathrm{pt}) \xrightarrow{\cong} k^i(X,A)$.

In the diagramme

$$\begin{array}{ccc}
h^*(X,A) \otimes h^*(Y,B) & \longrightarrow & h^*(X \times Y, X \times B \cup A \times Y) \\
\Big\downarrow & & \Big\downarrow \\
h^*(X,A) \otimes h^*(Y,B) \otimes \mathbb{Q} & \longrightarrow & h^*(X \times Y, X \times B \cup A \times Y) \otimes \mathbb{Q} \\
\Phi \otimes \Phi \Big\downarrow & & \Big\downarrow \Phi \\
k^*(X,A) \otimes k^*(Y,B) & \longrightarrow & k^*(X \times Y, X \times B \cup A \times Y),
\end{array}$$

for a fixed pair (Y, B) all terms in the lower square are cohomology theories in (X, A) and all homomorphisms are natural. By Corollary 1.25 the two composities around the square will agree if they agree for $(X, A) = (\mathbb{S}^0, *)$. But with (X, A) fixed, all terms are cohomology theories in (Y, B), and the same argument holds. To prove commutativity it then suffices to prove it in the case (Y, B) is also $(\mathbb{S}^0, *)$, but this is just the hypothesis that φ is multiplicative. □

1.3.2 Dold-Thom-Gysin Theorem

Theorem 1.28. *Let h^* be a multiplicative cohomology theory and let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be an h^*-fibration. Suppose there are elements $a_1, \ldots, a_r$ in $h^*(E)$ such that $(i^*a_1, \ldots, i^*a_r)$ is an $h^*(\mathrm{pt})$-base for $h^*(F)$ as an $h^*(\mathrm{pt})$-module, then $(a_1, \ldots, a_r)$ is an $h^*(B)$-base for $h^*(E)$ as an $h^*(B)$-module.*

Proof. The $h^*(B)$-module structure of $h^*(E)$ is defined by π^* as the composition

$$h^*(B) \otimes h^*(E) \xrightarrow{\pi^* \otimes 1} h^*(E) \otimes h^*(E) \xrightarrow{\cup} h^*(E).$$

Let C and $C^{'}$ be the exact couples of the fibrations $\mathrm{id} : B \to B$ and $\pi : E \to B$, respectively. The homomorphism

$$\rho : C \oplus C \oplus C \oplus \cdots \oplus C \to C^{'}$$

defined by $\rho(\lambda_1 \oplus \cdots \oplus \lambda_r) = \sum_{i=1}^{r} \pi^*(\lambda_i) a_i$ is a map of exact couples since π is a fibre preserving map of fibrations

$$\begin{array}{ccc} E & \xrightarrow{\pi} & B \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ B & \longrightarrow & B \end{array}$$

and $\rho$is an $h^*(B)$-module homorphism commuting with the coboundary operator in the two couples.

Since $(i^*a_1, \ldots, i^*a_r)$ is an $h^*(\mathrm{pt})$-base for $h^*(F)$, at the E_1-level of the spectral sequences, the homomorphism

$$C^p(B; h^*(\mathrm{pt})) \oplus \cdots \oplus C^p(B; h^*(\mathrm{pt})) \to C^p(B; h^*(F))$$

induced by ρ is an isomorphism. Thus, ρ induces isomorphisms on E_∞ terms of the spectral sequence and consequently we have the isomorphism

$$h^*(B) \oplus \cdots \oplus h^*(B) \to h^*(E),$$

defined by linear combination with the a_i. Thus, $(a_1, \ldots, a_r)$ is an $h^*(B)$-base for $h^*(E)$. □

The Theorem 1.28 has a relative version. Conceptually, it involves nothing new except complicated notation. Thus, we will only define the relevant exact couples and state without proof the theorem for the relative case.

Suppose $E^{'} \subset E$ and $\pi : E \to B$ such that for each vector v of a simplex Δ of B

$$h^*(\pi^{-1}(\Delta), pi^{-1}(\Delta) \cap E^{'}) \to h^*(\pi^{-1}(v), \pi^{-1}(v) \cap E^{'})$$

is an isomorphism. Then $(E, E^{'}, \pi, B)$ is a *relative h^*-fibration with relative fibre* $(F, F^{'}) = (\pi^{-1}(v), pi^{-1}(v) \cap E^{'})$. Let $B^{'}$ be a subcomplex of B. We can then define an exact couple as follows:

$$A^{p,q} = h^{p+q}([\pi^{-1}(B^{(p)}) \cup \pi^{-1}(B^{'})]/[(\pi^{-1}(B^{(p)}) \cap E^{'}) \cup \pi^{-1}(B^{'})])$$

with $C^{p,q}$ equal h^{p+q} of the corresponding pair based on the pth and $(p-1)$th skeletons. As before this leads to the *spectral sequence of the relative fibration* and we have

Corollary 1.29. *In the spectral sequence of the relative fibration $(E, E^{'}, \pi, B, B^{'})$, the term $E_2^{p,q}$ is naturally isomorphic with $H^p(B, B^{'}; h^q(F, F^{'}))$ and $E_\infty^{p,q}$ is the system of graded groups associated with a filtration of $h^n(E, E^{'} \cup \pi^1(B^{'}))$, $p + q = n$.*

In an analogous way we have the following relative version of the Dold-Thom-Gysin isomorphism:

Theorem 1.30. *Let h^* be a multiplicative cohomology theory and let $(E, E^{'}, \pi, B, B^{'})$ be a relative f^*-fibration. Suppose there are elements $a_1, \ldots, a_r$ in $h^*(E, E^{'} \cup \pi^{-1}(B^{'}))$ such that $(i^*a_1, \ldots, i^*a_r)$ is an $h^*(\mathrm{pt})$-base for $h^*(F, F^{'})$ as an $h^*(\mathrm{pt})$-module. Then $\{a_1, \ldots, a_r\}$ is an $h^*(B, B^{'})$-base for $h^*(E, E^{'} \cup \pi^{-1}(B^{'}))$ as an $h^*(B, B^{'})$-module.*

The $h^*(B, B^{'})$-module structure of $h^*(E, E^{'} \cup \pi^{-1}(B^{'}))$ is defined by the composition

$$\begin{aligned} h^*(B, B^{'}) \otimes h^*(E, E^{'} \cup \pi^{-1}(B^{'})) &\to h^*(E, \pi^{-1}(B^{'})) \otimes h^*(E, E^{'} \cup \pi^{-1}(B^{'})) \\ &\xrightarrow{\cup} h^*(E, E^{'} \cup \pi^{-1}(B^{'})). \end{aligned}$$

Actually, a slightly weaker hypothesis describes $h^*(B, B^{'})$: if the classes $a_1, \ldots, a_r$ in $h^*(E, E^{'})$, then

$$(\pi^* h^*(B, B^{'})) \cup a_1 \oplus \cdots \oplus (\pi^* h^*(B, B^{'})) \cup a_r \cong h^*(E, E^{'} \cup \pi^{-1}(B^{'})).$$

1.3.3 Orientability of Bundles

Let $\pi : E \to B$ be an n-plane bundle.

Remark 1.31. We shall usually assume π is vector bundle, but in the following Corollary 1.33 to Theorem 1.30 it is sufficient for the structure group to be group of origin preserving onto-homeomorphisms of $\mathbb{R}^n$, Euclidean n-space.

Let $E^{'}$ denote the complement of the zero cross-section. Then for any cohomology theory h^*, $(E, E^{'}, \pi, B)$ is a relative h^*-fibration with fibre $(F, F^{'}) = (\mathbb{R}^n, \mathbb{R}^n \backslash 0)$, which has the cohomology of $(\mathbb{S}^n, \mathrm{pt})$. Thus, $h^{i+n}(F, F^{'}) \cong h^{i+n}(\mathbb{S}^n, \mathrm{pt}) \cong h^i(\mathrm{pt})$; so $h^*(F, F^{'})$ is a free $h^*(\mathrm{pt})$-module with one generator u_n of degree n corresponding to $1 \in h^0(\mathrm{pt})$.

Definition 1.32. The n-plane bundle π is *h^*-orientable* if there is a class $u \in h^n(E, E')$ such that $i^*u = u_n \in h^n(F, F')$ for each fibre inclusion $i : (F, F') \to (E, E')$. Of course, if B is connected and $i^*u = u_n$ for one fibre, then it is so for all fibres. A choice of such a class u is an *h^*-orientation* of π and π is said to be *h^*-oriented by u.*

Corollary 1.33. *[to Theorem 1.30] If the n-plane bundle π is h^*-oriented by the class $u \in h^n(E, E')$, then*

$$\varphi : h^i(B) \to h^{i+n}(E, E')$$

defined by $\varphi(z) = \pi^* z \cup u$ is an isomorphism.

If the structure group of the bundle π is a subgroup of the orthogonal group, or more generally of the group of auto-homeomorphisms on $\mathbb{R}^n$ preserving distance from the origin, then associated with π are the unit disc bundle $\mathbb{D}(\pi)$ and the unit sphere bundle $\mathbb{S}(\pi)$ of π. The factor space $\mathbb{D}(\pi))/\mathbb{S}(\pi)$ has a well determined base point and is called the *Thom space* of the bundle π, herein denoted B^π.

The inclusion $(\mathbb{D}(\pi), \mathbb{S}(\pi)) \to (E, E')$ induces an isomorphism of cohomology; thus,

$$h^*(E, E') \xrightarrow{\cong} f^*(B^\pi, *).$$

Corresponding to each fibre inclusion $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to (E, E')$ is the disc and sphere inclusion

$$\begin{array}{ccc} (\mathbb{D}^n, \mathbb{S}^{n-1}) & \longrightarrow & (\mathbb{D}(\pi), \mathbb{S}(\pi)) \\ \downarrow & & \downarrow \\ (\mathbb{R}^n, \mathbb{R}^n \setminus 0) & \longrightarrow & (E, E') \end{array}$$

and $\mathbb{D}^n/\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{S}^n$. Thus, we can equivalently define notions of h^*-orientation in terms of a class $u \in h^n(B^\pi, *)$ which restricts to the class $\gamma^n \in h^n(\mathbb{S}^n, *)$ and the isomorphism of the Corollary becomes $\varphi : h^i(B) \to h^{i+n}(B^\pi, *)$ defined by the composition

$$\begin{array}{ccccc} h^i(B) & \xrightarrow{\pi^*} & h^i(\mathbb{D}(\pi)) & \longrightarrow & h^{i+n}(B^\pi, *) \\ & & & \searrow^{\cup u} & \downarrow \cong \\ & & & & h^{i+n}(\mathbb{D}(\pi), \mathbb{S}(\pi)). \end{array}$$

Proposition 1.34. *If α and β are vector bundles over X and Y, respectively, then $X^\alpha\ Y^\beta$ is homeomorphic to $(X \times Y)^{\alpha \times \beta}$, where $\alpha \times \beta$ is the bundle with fibre $\alpha x + \beta_y$ over the point (x, y).*

Proof. The obvious map $X^\alpha \times Y^\beta \to (X \times Y)^{\alpha \times \beta}$ is a homeomorphism except on the wedge, which goes to the base point. □

Proposition 1.35. *If $u \in h^m(X^\alpha, *)$ and $v \in h^n(X^\beta, *)$ are h^*-orientations of the vector bundles α and β over the space X, then $\Delta^*(u \times v)$ is an h^*-orientation of the Whitney sum $\alpha \oplus \beta$ over X, where Δ^* is induced by the map $X^{\alpha\oplus\beta} \xrightarrow{\Delta} (X \times X)^{\alpha\times\beta}$.*

Proof. The Whitney sum $\alpha \oplus \beta$ is the bundle over X induced from $\alpha \times \beta$ over $X \times X$ by the diagonal map. The conclusion then follows from the commutative diagramme below, and the fact that $\gamma^m \times \gamma^n = \gamma^{m+n}$.

$$\begin{array}{ccccc}
h^m(X^\alpha, *) \otimes h^n(X^\beta, *) & \xrightarrow{\times} & h^{m+n}(X^\alpha \# X^\beta, *) & \longrightarrow & h^{m+n}(X^{\alpha\otimes\beta}, *) \\
\downarrow & & \downarrow & & \downarrow \\
h^m(\mathbb{S}^m, *) \otimes h^n(\mathbb{S}^n, *) & \xrightarrow[\times]{} & h^{m+n}(\mathbb{S}^m \# \mathbb{S}^n, *) & \xrightarrow[\cong]{} & h^{m+n}(\mathbb{S}^{m+n}, *)
\end{array}$$

□

Proposition 1.36. *If γ is a vector bundle over X oriented by the class $u \in h^n(X\gamma, *)$ and A is a subcomplex of X, then*

$$\varphi :^i (X, A) \to h^{i+n}(\mathbb{D}(\gamma), \mathbb{S}(\gamma) \cup \mathbb{D}(\gamma|_A))$$

defined by $\varphi(z) = \pi^*_{\mathbb{D}}(z) \cup u$, where $\pi_{\mathbb{D}} : \mathbb{D}(\gamma) \to X$ is the projection map of the associated disc bundle $\mathbb{D}(\gamma)$, is an isomorphism.

Proof. This is an immediate consequence of Theorem 1.30. □

Theorem 1.37. *If α and β are vector bundles over X such that β is h^*-oriented by $v \in h^n(X^\beta, *)$ and $\alpha \oplus \beta$ is h^*-oriented by $w \in h^{n+m}(X^{\alpha+\beta}, *)$, then α is h^*-oriented by a class $u \in h^m(X^\alpha, *)$ characterised by the equation $\rho^* u \cup \pi^* v = w$ (ρ and π as defined in the proof) and $w = \Delta^*(u \times v)$.*

Proof. Let $\pi : \mathbb{D}(\alpha) \to X$, $\rho : \mathbb{D}(\pi^*\beta) \to \mathbb{D}(\alpha)$ and $\sigma : \mathbb{D}(\alpha \oplus \beta) \to X$ be disc bundle projections. Note that there is a disc-preserving homotopy equivalence

$$(\mathbb{D}(\alpha \oplus \beta), \mathbb{S}(\alpha \oplus \beta)) \cong (\mathbb{D}(\pi^*\beta), \mathbb{S}(\pi^*\beta) \cup \mathbb{D}(\pi^*\beta|_{\mathbb{S}(\alpha)})),$$

obtained by deforming the box-like discs on the right radially into the spherical ones on the left. Since β is h-oriented by $v \in h^n(X, *)$, $\pi^*\beta$ is h-oriented by $\pi^* v \in h^n(\mathbb{D}(\alpha)^{\pi^*\beta}, *)$. But this implies

$$\phi_{\pi^*:\beta} h^{m+i}(\mathbb{D}(\alpha), \mathbb{S}(\alpha)) \to h^{m+n+i}(\mathbb{D}(\pi^*\beta), \mathbb{S}(\pi^*\beta) \cup \mathbb{D}(\pi^*\beta|_{\mathbb{S}(\alpha)}))$$

as in Proposition 1.36 is an isomorphism. Also

$$\varphi_{\alpha\oplus\beta} : h^i(X) \to h^{m+n+i}(\mathbb{D}(\alpha \oplus \beta), \mathbb{S}(\alpha \oplus \beta))$$

is an isomorphism. Identifying cohomologies by the homotopy equivalence observed, we then have determined a class $u \in h^m(\mathbb{D}(\alpha), \mathbb{S}(\alpha))$ by

$$u = \varphi^{-1}_{\pi^*\beta}\{\varphi_{\alpha\oplus\beta(1)}\};$$

i.e., u is the unique class of $h^m(\mathbb{D}(\alpha), \mathbb{S}(\alpha))$ such that

$$\rho^* u \cup \pi^* v = w.$$

Restricting u to $h^m(\mathbb{D}^m, \mathbb{S}^{m-1})$, we have that its product with γ^n is γ^{m+n}. Thus, u is an h^*-orientation for α.

By definition of the internal product, the class $u \times v$ in $h^{n+m}((X \times X)^{\alpha\times\beta}, *)$ restricts to $\rho^* u \cup \pi^* v = w$ on the diagonal $X^{\alpha\oplus\beta}$. □

Remark 1.38. Because of the previous results it is clear that if one representative of a stable class of vector bundles ($\alpha \underset{s}{\tilde{\to}} \beta$ if there are trivial bundles n and m such that $n \oplus \alpha \sim m \oplus \beta$) is orientable in a given cohomology theory, then all representatives are. However, there is no uniqueness to this construction, for just as there may be a number of inequivalent homotopy equivalences between two spaces, there also can be a number of inequivalent bundle equivalences. Each of these yields a homotopy equivalence between the corresponding Thorn spaces and consequent transport of orientation, but different equivalences may yield different orientations. For purposes of naturality in some constructions, this point will cause us to take some care in the next section.

1.4 Applications to Differential Manifolds

The object of this section is to establish a type of Riemann-Roch theorem for general cohomology theories.

1.4.1 Orientations of Manifolds and the "Umkehr" Homomorphism

First, we recall some results from differential topology. If $f : M^m \to \mathbb{R}^{m+r}$ is a differentiable imbedding of a differentiable m-manifold into Euclidean space, let $\nu(f)$ denote the *normal* r-plane bundle of f. If also $g : M^m \to \mathbb{R}^{m+r}$ is a differentiable imbedding and $r > m + 1$, then f and g are homotopic by a differentiable homotopy through differentiable imbeddings; hence $\nu(g)$ is equivalent to $\nu(f)$. If, moreover, $r > m + 3$, then two such regular homotopies are themselves regularly homotopic through regular homotopies, and the two resulting bundle equivalences are homotopic through bundle equivalences. Thus, we have

Proposition 1.39. *If f and g are differentiable imbeddings of M^m in $\mathbb{R}^{m+r}$, $r > m + 3$, then there is a disc preserving homeomorphism of $M^{\nu(f)}$ onto $M^{\nu(g)}$ which is unique up to the isotopy class of disc-preserving homeomorphism.*

Proposition 1.40. *If $f : M^m \to \mathbb{R}^{m+r}$ and $f' : M^m \to \mathbb{R}^{m+r'}$ are differentiable imbeddings, then for some pair of integers (s, s') there is a uniquely determined isotopy class of homeoorphisms*

$$M^{s\oplus\nu(f)} \to M^{s'\oplus\nu(f')}.$$

Proof. By Proposition 1.34, with Y a point and $\beta = n$, we have $\Sigma^n(X^\alpha) \cong X$ (for $\cong$ read "is homeomorphic to"). Let $i : \mathbb{R}^{m+r} \to \mathbb{R}^{m+r+1}$ be the usual inclusion; then

$$\Sigma M^{\nu(f)} \cong M^{1 \oplus \nu(f)} \cong M^{\nu(I \circ f)}$$

since $\nu(i \circ f) = 1 \oplus \nu(f)$. Choose s and s' such that $r + s = r' + s' > m + 3$. Then there are well determined isotopy classes of homeomorphisms

$$\Sigma^s M^{\nu(f)} \cong M^{\nu(i^s \circ f)} \cong M^{\nu(i^{s'} \circ f')} \cong \Sigma^{s'} M^{\nu(f')},$$

where i^s stands for the iterated inclusion

$$\mathbb{R}^{m+r} \xrightarrow{i} \mathbb{R}^{m+r+1} \xrightarrow{i} \cdots \xrightarrow{i} \mathbb{R}^{m+r+s};$$

the second homeomorphism in the series is given by Proposition 1.39. Thus, we have determined a unique isotopy class of homeomorphisms $M^{s \oplus \nu(f)} \to M^{s' \oplus \nu(f')}$. □

Proposition 1.41. *Each h^*-orientation $f : M^m \to \mathbb{R}^{m+r}$ and $u \in h^r(M^{\nu(f)}, *)$ determines for any differentiable imbedding $f' : M^m \to \mathbb{R}^{m+r'}$ a unique orientation class $u' \in h^{r'}(M^{\nu(f')}, *)$.*

Proof. The unique isotopy class of homeomorphisms of Proposition 1.40 together with the s'-desuspension from $h^{r+s}(\Sigma^{s'} M^{\nu(f')}, *)$ of the s-suspension of u into $h^{r+s}(\Sigma^s M^{\nu(f)}, *)$ determines the orientation class u'. □

Proposition 1.42. *If M is h^*-oriented by f and u, then*

1) *$\varphi_f : h^j(M) \to h^{j+r}(M^{\nu(f)}, *)$ is an isomorphism defined by $\varphi_f(z) = \pi^*_{\nu(f)}(z) \cup u$,*

2) *the diagramme*

$$\begin{array}{ccc} h^j(M) & \xrightarrow{\varphi_f} & h^{j+r}(M^{\nu(f)}, *) \\ & \searrow & \downarrow \cong \\ & & h^{j+r+1}(\Sigma M^{\nu(f)}, *) \\ & {\scriptstyle (-1)^j \varphi_{i \circ f}} \searrow & \downarrow \cong \\ & & h^{j+r+1}(M^{\nu(i \circ f)}, *) \end{array}$$

is commutative, and

3) *the isomorphism*

$$\varphi_{f'} : h^j(M) \to h^{j+r'}(M^{\nu(f')}, *)$$

is well determined for any differentiable imbedding $f' : M^m \to \mathbb{R}^{m+r'}$.

Proof. (of 2)) By Proposition 1.26 we have the equalities

$$\begin{aligned}\Sigma\varphi_f(z) &= \gamma \cup \varphi_f(z) = \gamma \cup (\pi^*(z) \cup u) = (-1)^j \pi^+(z) \cup (\gamma \cup u)\\ &= (-1)^j \pi^*(z) \cup u' = (-1)^j \varphi_{i\circ f}(z).\end{aligned}$$

□

We have seen in Proposition 1.41 that the possible h-orientations of a manifold are determined by those of any one of its normal bundles. In fact, they are also determined by its tangent bundle τ.

Proposition 1.43. *For any differentiable imbedding $f : M^m \to \mathbb{R}^{m+r}$ there is a well defined $1-1$ correspondence between the h^*-orientations of $\nu(f)$ and the h^* -orientations of $\tau(M)$.*

Proof. By Proposition 1.41 it suffices to prove this for any f and r; we choose $r > m+3$. Then the triviality of the total space $(m+r)$-bundle over M is assured for all imbeddings. The conclusion then follows from Theorem 1.37. □

Definition 1.44. Let M^m and N^n be differentiable manifolds and $f : M \to N$ be a continuous map. For $d > 2m + 3$ there is a differentiable imbedding $i : M \to \mathbb{D}^d$ into the interior of the d-dimensional disc such that the imbedding $\widetilde{f} = f \times i : M \to N \times \mathbb{D}$ is homotopic to a differentiable imbedding. If the normal bundle $\nu(\widetilde{f})$ is h-orientable, then i together with a choice of orientation class $u \in h^{n+d-m}(M^{\nu(\widetilde{f})}, *)$ is said to determine an *h^*-orientation of the map f*. The usual considerations of isotopies of imbeddings show the determination of u is in fact independent of the imbedding and homotopy.

Proposition 1.45. *Let M^m and N^n be differentiable manifolds and $f : M \to N$ be a continuous map. The h^*-orientations of f are in a well defined $1-1$ correspondence with the h^*-orientations of the bundle $\nu(M) \oplus f^*\tau(N)$. In particular, if M and N are h-oriented and f is any continuous map, then f has a well defined h^*-orientation.*

Proof. Let j be a differentiable imbedding of $N \times \mathbb{D}$ in a Euclidean space $\mathbb{R}^r$ of dimension $r > 2(n + d) + 3$ and let $\pi : N \times \mathbb{D} \to N$ denote the projection map. We shall write a number of bundle equivalences; note that in each case they are uniquely defined.

$$\begin{aligned}\tau(N \times \mathbb{D}) &\simeq \pi^*(\tau(N)) \oplus d.\\ \widetilde{f}^*(N \times \mathbb{D}) &\simeq \widetilde{f}^*\pi^*(\tau(N)) \oplus \widetilde{f}^*(d) \simeq f^*(\tau(N)) \oplus d.\end{aligned}$$

$$\begin{aligned}\nu(M) &= \nu(j \circ \widetilde{f}) \simeq \nu(\widetilde{f}) \oplus \widetilde{f}^*\nu(j)\\ \nu(M) \oplus f^*\tau(N) \oplus d &\simeq \nu(\widetilde{f}) \oplus \widetilde{f}^*\nu(j) \oplus \widetilde{f}^*(\tau(N \times \mathbb{D}))\\ &\simeq \nu(\widetilde{f}) \oplus \widetilde{f}^*(r)\\ &\simeq \nu(\widetilde{f}) \oplus r.\end{aligned}$$

The first conclusion follows by suspension and desuspension. The second conclusion is then obvious by Proposition 1.43. $\square$

Definition 1.46. Suppose that $f : M^m \to N^n$ is h^*-oriented by $\widetilde{f} : M^m \to N^n \times \mathbb{D}^d$ and $u \in h^{n+d-m}(M^{\nu(\widetilde{f})}, *)$. The collapsing map ("Thom construction")

$$N^d = (N \times \mathbb{D}^d)/(N \times \mathbb{S}^{d-1}) \xrightarrow{\widehat{f}} \mathbb{D}(\nu(\widetilde{f}))/\mathbb{S}(\nu(\widetilde{f})) = M^{\nu(\widetilde{f})}$$

can be used to define a homomorphism

$$\begin{array}{ccc} h^i(M) & \xrightarrow[\cong]{\varphi} & h^{i+n+d-m}(M^{\nu(\widetilde{f})}, *) & & \\ & & \downarrow \widehat{f}^* & & \\ & & h^{i+n+d-m}(N^d, *) & \xrightarrow{(\Sigma^d)^{*-1}} & h^{i+n-m}(N); \end{array}$$

this homomorphism is denoted

$$f_! : h^i(M) \to h^{i+n-m}(N)$$

and is determined by f and the h^*-orientation of f.

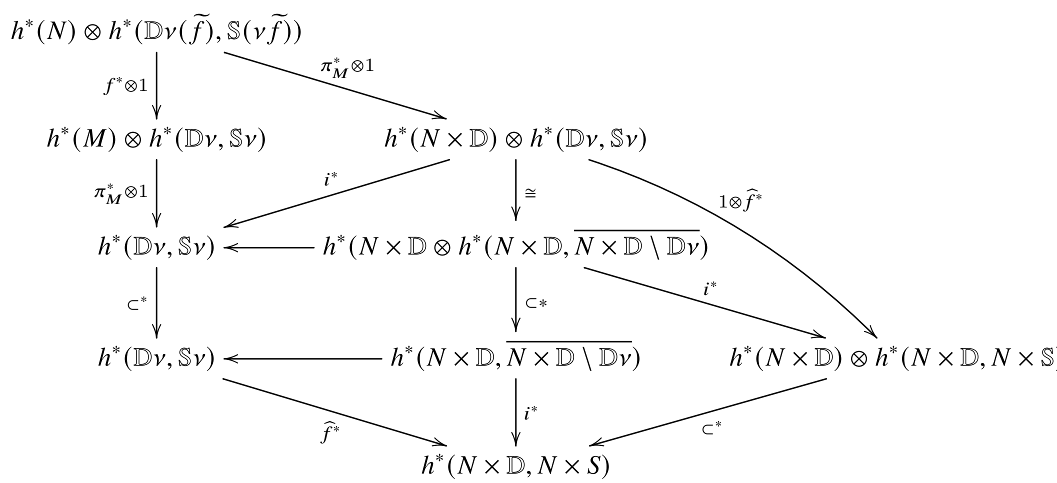
Theorem 1.47. *If $f : M \to N$ is an h^*-oriented map, then for classes $x \in h^*(M)$ and $y \in h^*(N)$,*

$$f_!(f^*(y) \cup x) = y \cup f_!(x).$$

Proof. We first prove the identity

$$\widehat{f}^*(\pi_M^* f^*(y) \cup \varphi_M(x)) = \pi_N^*(y) \cup \widehat{f}^* \varphi_M(x).$$

Consider the diagramme:



The upper quadrilateral commutes because $f = \pi_N \widetilde{f}$; the remainder commutes because all maps are excisions, inclusions, or products. Thus, the diagramme commutes. An element $y \otimes \varphi_M x$ at the top is mapped into $\widehat{f}^*(\pi_M^* f^* y \cup \varphi_x)$ following the arrows on the left and into $\pi_N^* y \cup \widehat{f}^* \varphi_N x$ following the arrows on the right: thus these elements are equal as we asserted.

Since

$$\begin{aligned}\widehat{f}^*(\pi_M^* f^* y \cup \varphi_M x) &= \widehat{f}^*(\pi_M^* f^* y \cup \pi_M^* x \cup \varphi_M(1)) \\ &= \widehat{f}^*(\pi_M^*(f^* y \cup x) \cup \varphi_M(1)) \\ &= \widehat{f}^* \varphi_M(f^* y \cup x),\end{aligned}$$

we have

$$\widehat{f}^* \varphi_M(f^* y \cup x) = \pi_N^* y \cup \widehat{f}^* \varphi_M(x).$$

Thus,

$$\begin{aligned}f_!(f^* y \cup x) &= \varphi_N^{-1} \widehat{f}^* \varphi_M(f^* y \cup x) = \varphi_N^{-1}(\pi_N^* y \cup \widehat{f}^* \varphi_M(x)) \\ &= \varphi_N^{-1}(\pi_N^* y \cup \pi_N^*(\varphi_N^* \widehat{f}^* \varphi_M(x)) \cup \varphi_N(1)) \\ &= y \cup \widehat{f}^* \varphi_M(x) \\ &= y \cup f_!(x).\end{aligned}$$

($\varphi_N = \Sigma^d$ since the bundle over N is trivial.) □

Next, we establish a functorial property of the "umkehr homomorphism" $f_!$. Given $K \xrightarrow{f} L \xrightarrow{g} M$, we seek a theorem of the sort $g_! \circ f_! = (g \circ f)_!$. Since these homomorphisms clearly depend on the h^*-orientations used, in order for there to be such equality the three orientations must be chosen in some compatible way. If we assume f and g have specified orientations, we can then orient $g \circ f$ as follows:

by Proposition 1.45, both

$$\nu(K) \oplus f^* \tau(L), \quad \nu(L) \oplus g^* \tau(M)$$

are h^*-oriented; thus, by Proposition 1.35

$$\nu(K) \oplus f^* \tau(L) \oplus f^*(\nu(L) \oplus g^* \tau(M)) \simeq \nu(K) \oplus f^* \circ g^* \tau(M) \oplus (\text{trivial bundle})$$

is h^*-oriented. This determines an orientation of $g \circ f$.

Equivalently, given $\widetilde{f} : K \to L \times \mathbb{D}^d$ and $\widetilde{g} : L \to M \times \mathbb{D}^{d'}$, the composition

$$K \xrightarrow{\widetilde{f}} L \times \mathbb{D}^d \xrightarrow{\widetilde{g} \times 1} M \times \mathbb{D}^{d'} \times \mathbb{D}^d = M \times \mathbb{D}^{d+d'}$$

is $\widetilde{g \circ f}$. Then

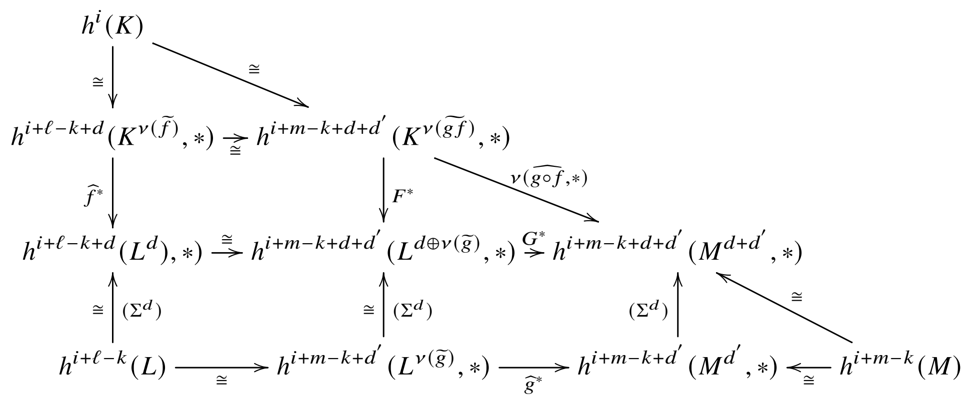
$$\nu(\widetilde{g \circ f}) = \nu((\widetilde{g} \times 1) \circ \widetilde{f}) = \nu(f) \oplus \widetilde{f}^*(\nu(\widetilde{g} \times 1)).$$

This defines the same h^*-orientation for $g \circ f$.

Note, in particular, that if K, L, and M are h^*-oriented, and f, g, and $g \circ f$ are given the induced h^*-orientations, then the two h^*-orientations, i.e., the composition orientation and the induced orientation, of $g \circ f$ agree. This is clear from the first definition of the composition orientation.

Theorem 1.48. *If $f : K \to L$ and $g : LtoM$ are h^*-oriented maps, the composition $g \circ f : toM$ has an induced h^*-orientation and with this orientation $g_! \circ f_! = (g \circ f)_!$. Further, if K, L, and M are h^*-oriented and f, g, and $g \circ f$ are oriented by Proposition 1.45, then the orientation of $g \circ f$ induced by those of f and g agrees with that given by Proposition 1.45.*

Proof. By the comments preceding the statement of the theorem we need only prove that $g_! \circ f_! = (g \circ f)_!$. To this end consider the diagramme below.



All isomorphisms are given by suspensions or Proposition 1.42. The two corner triangles and the corner square commute by Theorem 1.37. The lower right hand square commutes because $G = \Sigma^d \circ \widehat{g}$. The map F is $\Sigma^{d'} \circ \widehat{f}$; and so, the upper left hand square commutes. The collapsing map $M^{d+d'} \to K^{\nu(\widetilde{g \circ f})}$ factors through $F^{d \oplus \nu(\widetilde{g})}$, which implies the commutativity of the central triangle. The left hand column defines $f_!$, the bottom row $g_!$ and the diagonal $(g \circ f)_!$; thus, $g! \circ f_! = (g \circ f)_!$. □

1.4.2 Multiplicative Transformations; Riemann-Roch Theorem

Let h^* be a multiplicative cohomology theory. For $(X, A) \in \mathscr{P}^2$, $h^{**}(X, A)$ is the set of formal Laurent series

$$\sum_{i=-\infty}^{+\infty} \lambda_i, \quad \lambda_i \in h^i(X, A)., \quad \lambda_i = 0 \text{ for all } i < \text{ some } q.$$

Define addition and multiplication in $h^{**}(X,A)$ as follows: if $\lambda = \sum \lambda_i$ and $\mu = \sum \mu_i$, then $\lambda + \mu = \sum(\lambda_i + \mu_i)$ and $\lambda \cup \mu$ is the power series product

$$(\lambda \cup \mu)_k = \sum_{i+j=k} \lambda_i \cup \mu_j.$$

$h^{**}(X,A)$ is a (non-commutative) ring under these operations, and a map $f : (X,A) \to (Y,B)$ induces a ring homomorphism $f^{**} : h^{**}(Y,B) \to h^{**}(X,A)$ by taking f^* on each coordinate. Similarly, we define

$$\delta^{**} : h^{**}(A) \to h^{**}(X,A)$$

coordinate-wise.

Suppose h^* and k^* are multiplicative theorems. Then $t : h^{**} \to k^{**}$ is a (normed) multiplicative transformation if

1. t is a natural, additive transformation between the functors h^{**} and k^{**} with respect to maps f^{**},
2. $t(\mu \cup \lambda) = t(\mu) \cup t(\lambda)$, and
3. If $1_n \in h^{**}(\mathbb{S}^0,*)$ and $1_k \in k^{**}(\mathbb{S}^0,*)$ are the units in h^{**} and k^{**}, respectively, and $\alpha = \Sigma^{**}1_h \in h^{**}(\mathbb{S}^1,*)$ and $\beta = \Sigma^{**}1_k \in k^{**}(\mathbb{S}^1,*)$, then $t(\alpha) = \beta$.

Proposition 1.49. *For classes $u_1 \in h^{**}(X_1,A_1)$ and $u_2 \in h^{**}(X_2,A_2)$, the product $u_1 \times u_2$ is in $h^{**}(X_1 \times X_2, X_1 \times A_2 \cup A_1 \times X_2)$ and $t(u_1 \times u_2) = tu_1 \times tu_2$.*

Proof. As in §1.3,

$$\begin{aligned} u_1 \times u_2 &= p_1^{**}u_1 \cup p_2^{**}u_2 \quad \text{and} \\ t(u_1 \times u_2) &= t(p_1^{**}u_1 \cup p_2^{**}u_2) = t(p_1^{**}u_1) \cup t(p_2^{**}u_2) \\ &= p_1^{**}(tu_1) \cup p_2^{**}(tu_2) = tu_1 \times tu_2. \end{aligned}$$

□

Proposition 1.50. *If $u_1 \in h^{**}(X,A_1)$ and $u_2 \in h^{**}(X,A_2)$, then $u_1 \cup u_2 \in h^{**}(X, A_1 \cup A_2)$ and $t(u_1 \cup u_2) = tu_1 \cup tu_2$.*

Proof. In the following diagramme,

$$\begin{array}{ccccc} h^{**}(X,A_1) \otimes h^{**}(X,A_2) & \xrightarrow{\otimes} & h^{**}(X \times X, X \times A_2 \cup A_1 \times X) & \xrightarrow{\Delta^{**}} & h^{**}(X, A_1 \cup A_2) \\ \downarrow{\scriptstyle t\otimes t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ k^{**}(X,A_1) \otimes k^{**}(X,A_2) & \xrightarrow[\otimes]{} & k^{**}(X \times X, X \times A_2 \cup A_1 \times X) & \xrightarrow[\Delta^{**}]{} & k^{**}(X, A_1 \cup A_2) \end{array}$$

the first square commutes by Proposition 1.49 and the second by naturality. □

Proposition 1.51. *$t \circ \Sigma_h = \Sigma_k \circ k$; i.e., t is stable,* *Proof.* By Proposition 1.26, $\Sigma_h(u) = \alpha \otimes u$ and $\Sigma_k(tu) = \beta \otimes tu$. Thus, $t\Sigma_h(u) = t\alpha \otimes tu = \beta \otimes tu = \Sigma_k(tu)$. □

Proposition 1.52. $t \circ \delta_h^{**} = \delta_k^{**} \circ t$.

Proof. The homomorphism δ_k^{**} is the composition

$$\begin{array}{ccccccc}
h^{**}(A, \varnothing) & \underset{\cong}{\xrightarrow{\Sigma_h}} & h^{**}(\Sigma A^+, +) & \xrightarrow{p^{**}} & h^{**}(CA^+ \cup X^+) & \xrightarrow{\cong} & h^{**}(X, A) \\
 & & & & \downarrow \cong & \nearrow \cong & \\
 & & & & h^{**}(CA \cup X, *) & &
\end{array}$$

All these homomorphisms commute with t.

Suppose α is a vector bundle over X with orientations $u \in h^n(X^\alpha, *)$ and $v \in k^n(X^\alpha, *)$; we can regard u and v as lying in $h^{**}(X^\alpha, *)$ and $k^{**}(X^\alpha, *)$ respectively. Then u and v induce isomorphisms

$$\varphi_h : h^{**}(X) \to h^{**}(X^\alpha, *) \quad \text{and} \quad \varphi_k : k^{**}(X) \to k^{**}(X^\alpha, *)$$

For $x \in h^{**}(X)$, define $t(\alpha, x) \equiv \varphi_k^{-1} t \varphi_h(x) \in k^{**}(X)$. □

Lemma 1.53. *Let α be an h- and k-oriented vector bundle over X. Then for $x \in h^{**}(X)$, $t(x) \cup t(\alpha, 1) = t(\alpha, x)$.*

Proof.

$$\begin{aligned}
\varphi_k(t(x) \cup t(\alpha, 1) &= \varphi_k(t(x) \cup \varphi_k^{-1} t \varphi_h(1) \\
&= \pi_k^{**}(t(x) \cup \varphi_k^{-1} t \varphi_h(1)) \cup v \\
&= \pi_k^{**}(t(x)) \cup (\pi_k^{**} \varphi_k^{-1} t \varphi_h(1) \cup v) \\
&= t\pi_k^{**}(x) \cup t\varphi_h(1) \\
&= t(\pi_h^{**}(x) \cup \varphi_h(1)) \\
&= t\varphi_x.
\end{aligned}$$

Thus, $t(x) \cup t(\alpha, 1) = \varphi_k^{-1} t \varphi_h(x) = t(\alpha, x)$. □

Theorem 1.54 (A generalised Riemann-Roch Theorem)**.** *If $f : X \to Y$ is a h^*-oriented map of compact, closed manifolds and $t : h^{**} \to k^{**}$ is a multiplicative transformation of cohomology theories, then for $x \in h^{**}(X)$,*

$$tf_!^h(x) = f_!^k(t(x) \cup t(\nu(\widetilde{f}), 1)).$$

Proof. We note first that an h^*-orientation $u \in h^n(X^\alpha, *)$ of a vector bundle determines a k^*-orientation $(tu)_n \in k^n(X^\alpha, *)$ since

$$i^* u = \alpha^n \in h^n(\mathbb{S}^n, *) \quad \text{and} \quad t\alpha^n = \beta^n \in k^n(\mathbb{S}^n, *);$$

thus, $\beta^n = i^*(tu)_n$ and α is k^*-oriented. Hence we need not assume additionally that $f : X \to Y$ is k^*-orientable; however, in the statement and proof of this theorem the specific k^*-orientation of f is irrelevant.

The right-hand side of the asserted equation is

$$(\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**}\varphi_k(t(x) \cup t(\nu(\widetilde{f}), 1))$$

which equals

$$(\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**} t\varphi_k(x)$$

by the Lemma. By the naturality of t this equals

$$t((\Sigma_k^d)^{-1} \circ \widehat{f}_k^{**}\varphi_h(x)) = tf_!^h(x).$$

□

Theorem 1.55. *Let $f : X \to Y$ be a continuous map of h^*- and k^*-oriented manifolds. Let $t(X) \equiv t(\nu(X), 1)$ and $t(Y) \equiv t(\nu(Y), 1)$. Then for $x \in h^{**}(X)$,*

$$tf_!^h(x) \cup t(Y) = f_!^k(t(x) \cup t(X)).$$

Proof.

$$\begin{aligned} f_!^k(t(x) \cup t(X)) &= \varphi_k^{-1}\widehat{f}_k^{**}\varphi_k(t(x) \cup t(X)) \\ &= \varphi_k^{-1}\widehat{f}_k^{**} t\varphi_h(x) \\ &= \varphi_k^{-1} t\widehat{f}_h^{**}\varphi_h(x) \\ &= \varphi_k^{-1}[t\varphi_h(\varphi_h^{-1}\widehat{f}_h^{**}\varphi_h(x))] \\ &= \varphi_k^{-1}[\varphi_k(tf_!^h(x) \cup t(Y))] \\ &= tf_!^h(x) \cup t(Y). \end{aligned}$$

□

1.4.3 Wu Formulae

As applications of the above theorems we present here a proof of the Wu formulae; in a later section we will derive the Atiyah-Hirzebruch differential Riemann-Roch Theorem.

Let $h^{**} = k^{**} = H^{**}(-;\mathbb{Z}/p\mathbb{Z})$ and let $t = \mathscr{P} = \sum \mathscr{P}^i$, the sum of the Steenrod reduced power operations. For $p = 2$, $t = \mathrm{Sq}$. The Cartan formula implies t is a multiplicative transformation. For a vector bundle α over X, $\mathrm{Sq}(\alpha, 1)$ is the total Stiefel-Whitney class $W(\alpha)$ of the bundle α.

Lemma 1.56. *Every vector bundle is $H^*(-;\mathbb{Z}/2\mathbb{Z})$-orientable; a vector bundle α is $H^*(-;\mathbb{Z})$- or $H^*(-;\mathbb{Z}/p\mathbb{Z})$-orientable if and only if the first Stiefel-Whitney class $W_1(\alpha) = 0$.*

Proof. If α is an n-plane bundle, consider the n-sphere bundle $\mathbb{S}(\alpha \oplus 1) \xrightarrow{\pi} X$ and the Gysin sequence with $\mathbb{Z}/2\mathbb{Z}$-coefficients:

$$\cdots \to H^r(X) \xrightarrow{\pi^*} H^r(\mathbb{S}(\alpha \oplus 1)) \xrightarrow{\lambda^*} H^{r-n}(X) \to H^{r+1}(X) \to .$$

$\mathbb{S}(\alpha \oplus 1)$ has a cross-section $i : X \to \mathbb{S}(\alpha \oplus 1)$; and so, the Gysin sequence reduces to

$$0 \to H^r(X) \xrightarrow{\pi^*} H^r(\mathbb{S}(\alpha \oplus 1)) \xrightarrow{\lambda^*} H^{r-n}(X) \to 0.$$

There is a class $v \in H^n(\mathbb{S}(\alpha \oplus 1))$ such that $\lambda^* v = 1 \in H^0(X)$. Let $w = v - \pi * i^* v$. Then $\lambda^* w = 1$ and $i^* w = 0$.

The sequence $X \xrightarrow{i} \mathbb{S}(\alpha \oplus 1) \xrightarrow{c} X$ is coexact and so

$$0 \longrightarrow H^*(X^\alpha, *) \xrightarrow{c^*} H^*(\mathbb{S}(\alpha \oplus 1)) \underset{\pi^*}{\overset{i_*}{\rightleftarrows}} H^*(X) \longrightarrow 0.$$

is exact. There is a unique $u_n \in H^n(X^\alpha, *)$ such that $c^* u_n = w$.

Since the diagramme

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & \mathbb{S}(\alpha \oplus 1)) \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & X \end{array}$$

is commutative, so also is

$$\begin{array}{ccc} H^n(X^\alpha, *) & & \\ \downarrow & & \\ H^n(\mathbb{S}(\alpha \oplus 1)) & \xrightarrow{\lambda *} & H^0(X) \\ \downarrow & & \downarrow \\ H^n(\mathbb{S}^n) & \longrightarrow & H^0(X), \end{array}$$

and it follows that u_n restricts to the generator of $H^n(\mathbb{S}^n, *)$.

To prove the second part of the lemma, note that $w_1(\alpha) = \varphi^{-1} \beta un$, where β is the Bockstein homomorphism associated with

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Thus,

$$\begin{aligned} w_1(\alpha) = 0 &\Leftrightarrow u_n \text{ is the reduction of a mod 4 class} \\ &\Leftrightarrow \alpha \text{ is } H^*(-;\mathbb{Z}/4\mathbb{Z}) - \text{orientable.} \end{aligned}$$

This is equivalent to $\pi_1(X.x_0)$ operating trivially on $H^*(\mathbb{S}^n; \mathbb{Z})$ which then by the Gysin sequence implies the $H^*(-;\mathbb{Z})$- and $H^*(-;\mathbb{Z}/p\mathbb{Z})$-orientability of α. The converse is clear. □

The Wu formula asserts there is a class $V \in H^{**}(X)$ such that $\mathrm{Sq}\, V = W(X)(= \mathrm{Sq}(\tau(X), 1))$ and $[\alpha \cup V](X) = [\mathrm{Sq}\, \alpha](X)$ for all $\alpha \in H^*(X)$. Let $\mathrm{Sq}^{-1} = \frac{1}{1+(\mathrm{Sq}^{-1})}$

as formal power series; then $\mathrm{Sq}\,\mathrm{Sq}^{-1} = \mathrm{Sq}^{-1}\,\mathrm{Sq} = 1$ and Sq is multiplicative, Let $f : X to\, \mathrm{pt}$. By Theorem 1.55,

$$f_! : H^*(\mathrm{Sq}^{-1}(x) \cup \mathrm{Sq}^{-1}(\nu(X), 1) = \mathrm{Sq}^- 1(f_!(x))$$

where $f_! : H^*(X) \longrightarrow H^*(\mathrm{pt})$ is given by $f_!(x) = [x](X)$, the value of the cohomology class x on the top homology class of the manifold X (with $\mathbb{Z}/2\mathbb{Z}$ coefficients). For $x = \mathrm{Sq}\,\alpha$, we then have

$$[\alpha \cup \mathrm{Sq}^{-1}(\nu(X), 1)](X) = [\mathrm{Sq}\,\alpha](X).$$

We claim that $\mathrm{Sq}(\mathrm{Sq}^{-1}(\nu(X), 1)) = \mathrm{Sq}(\tau(X), 1) = W(X)$. Then $V = \mathrm{Sq}^1(\nu(X), 1)$ is the Wu class and satisfies the formulae above.

Since $\mathrm{Sq}(\tau, 1) \cup \mathrm{Sq}(\nu, \nu) = \mathrm{Sq}(\tau \oplus \nu, 1) = 1$ [Appendix B], to verify the claim we need only prove that

$$\mathrm{Sq}(\mathrm{Sq}^{-1}(\nu(X), 1)) \cup \mathrm{Sq}(\nu, 1) = 1.$$

To this end we compute

$$\begin{aligned} \varphi(\mathrm{Sq}(\mathrm{Sq}^{-1}(\nu(X), 1)) \cup \mathrm{Sq}(\nu, 1)) &= \pi^*\,\mathrm{Sq}\,\varphi^{-1}\,\mathrm{Sq}^{-1}\,\varphi(1) \cup \pi^*\varphi^{-1}\,\mathrm{Sq}\,\varphi(1) \cup (1) \\ &= \pi^*\,\mathrm{Sq}\,\varphi^{-1}\,\mathrm{Sq}^{-1}\,\varphi(1) \cup \mathrm{Sq}\,\varphi(1) \\ &= \mathrm{Sq}(\varphi\varphi^{-1}\,\mathrm{Sq}^{-1}\,\varphi(1)) \\ &= \varphi(1). \end{aligned}$$

But φ is an isomorphism.

Chapter 2

Complex Vector Bundles and the Bott Periodicity Theorem

2.1 Bott Periodicity Theorem

2.1.1 Homology of the Unitary Groups

For the complex n-dimensional space $\mathbb{C}^n$ equipped with a positive definite Hermitian inner product, the unitary group $U(n)$ is the group of linear transformations preserving this inner product. Consider $\mathbb{C}^{n-1}$ as a linear subspace of $\mathbb{C}^n$ and choose a vector $e_1 \in \mathbb{C}^n$ such that $(e_1, e_1) = 1$ and $e_1 \perp \mathbb{C}^{n-1}$ ($(e_1, x) = 0$ for all $x \in \mathbb{C}^{n-1}$). This defines an inclusion $U(n-1) \to U(n)$ whereby the elements of $U(n-1)$ are those transformations of $U(n)$ which leave e_1 fixed.

The group $U(n)$ acts effectively on the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{C}^n$ and $U(n-1)$ is the subgroup of $U(n)$ whose elements leave $e_1 \in \mathbb{S}^{2n-1}$ fixed. Thus, $U(n)/U(n-1)$ is homeomorphic to $\mathbb{S}^{2n-1}$ and the fibration $U(n-1) \xrightarrow{i} U(n) \xrightarrow{\pi} \mathbb{S}^{2n-1}$ is a principal $U(n-1)$-fibration.

2.1.2 The Universal Base Spaces $BU(n)$

2.1.3 Bott Periodicity Theorem for BU

2.2 Complex Vector Bundles

2.2.1 Characteristic Classes

2.2.2 Complex Vector Bundles over Spheres

Chapter 3

The Cohomology Theory $K_{\mathbb{C}}^*$

3.1 Basic Properties of $K_{\mathbb{C}}^*$

3.1.1 Definition of $K_{\mathbb{C}}^*$

The groups $K_{\mathbb{C}}^0$ are a special case of the universal enveloping abelian group of a monoid. Let G be a set of objects with an associative operation $\oplus$ and a unit 0. Let A be the free abelian group on the objects of G modulo the subgroup generated by elements of the form

$$[\alpha \oplus \beta] - [\alpha] - [\beta]$$

Then A is an abelian group and the the transformation

$$e : G \to A$$

defined by $e(g) = [g] \in A$, preserves sums and unit element.

3.1.2 The Multiplicative Transformation $ch : K_{\mathbb{C}}^* \to H^{**}(-; \mathbb{Q})$

3.1.3 Cohomology Operations in $K_{\mathbb{C}}^*$

3.1.4 $K_{\mathbb{C}}^*$-orientation of Complex Vector Bundles

Chapter 4

Some Geometric Applications

4.1 Vector Bundles over Cell Complexes $\mathbb{S}^n \cup e^m$

4.1.1 Two technical lemmas

4.1.2 Divisors of Orders of Stable Homotopy Classes, J-homomorphisms

4.1.3 Maps of Hopf Invariant One

4.2 Toda Brackets

Appendices

Appendix A

A.1 The Cohomology Theory $K^*_{\mathbb{R}}$

Appendix B

B.1 A Multiplication Formula

This appendix presents a proof of the following technical proposition which was used a number of times throughout the text:

Appendix C

C.1 Fiber Homotopy Equivalence of Bundles, The Groups $J(X)$

Appendix D

Historical Comments

The general idea of obtaining cohomology theories by maps into spectra can be traced to Barratt [12], in that the track groups of maps from a cofibration yield an exact sequence. This was made explicit by Puppe [28]. The converse that cohomology theories are representable in this way was established by Brown [18].

The material in Chapter 1 on reduced and non-reduced cohomology theories is described in the fashion of Whitehead [30], while the spectral sequence of a fibration is presented along lines suggested by Dold [19]. The proof given of the Dold-Thom-Gysin Theorem also follows Dold (ibid). The approach to orientability of vector bundles, multiplicative transformations and the Riemann-Roch Theorem for differentiable manifolds is essentially folk-lore; it was first presented in [20] but its ideas lie in the work of Atiyah and Hirzebruch as expounded, for instance, by Hirzebruch in [24]. The proof given here of the Wu formulas is essentially that of Atiyah and Hirzebruch in [10].

Through the use of Spanier-Whitehead duality [29] one can view the cohomology of the Thom space of a normal bundle as being the homology of the manifold under consideration. An alternate procedure is developed by Whitehead in [30] Various of the constructions of Chapter 1 then appear as versions of Poincare duality, umkehr homomorphism, etc.

The development of the topology of the Unitary Group and associated spaces presented in Chapter 2 follows that given by the authour in a course at the University of Chicago in the Winter of 1960. A similar but substantially more complicated approach works for the Orthogonal and Symplectic Groups [22]. The explicit description of the maps generating $\pi_{2n+1}(\mathrm{SU})$ given in Theorem 17 is due to Toda.

Atiyah is credited with initiating study of the theory $X^*_{\mathbb{C}}$, as a cohomology theory. The material of Chapter 3, Section A, parts 1 and 2 is taken from a paper by Atiyah and Hirzebruch [10].

The technical lemmas of Section A of Chapter 4 are found in Atiyah and Hirzebruch [10] and Hirzebruch [23]. The material of Sections 2 and 3 of Chapter 4 follows the authour's exposition in [21], although it must be noted that Adams found ccitee03 a proof similar in spirit to that given here of Theorem 23 some time earlier than the author; its differences are that he uses results from [1] while the proof given here uses results on complex cobordism [25].

The Appendix A on the K^*-theory essentially follows Atiyah and Hirzebruch as in [24]. The proposition of Appendix B is very basic, leading in various roles to the theorems on Stiefel-Whitney and Chern classes of sums of vector bundles and to the theory of multiplicative sequences. The Appendix C is very incomplete and is a presentation of material presented by Adams in various lectures.

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